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THE SYMPLECTIC NATURE OF THE SPACE OF DORMANT INDIGENOUS BUNDLES ON ALGEBRAIC CURVES

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Abstract. We study the symplectic nature of the moduli stack of dormant indigenous bundles on proper hyperbolic curves. Our aim of the present paper is to consider the positive characteristic analogue of work of S. Kawai (in the paper entitled "*The symplectic nature of the space of projective connections on Riemann surfaces*"), P. Arés-Gastesi, and I. Biswas. The main result asserts a certain compatibility of the symplectic structures between the moduli spaces involving dormant indigenous bundles. As an application of the result, we construct a Frobenius-constant quantization on the moduli stack of indigenous bundles over ordinary dormant curves.

Facts, as Norwood Hanson says, are theory-laden; they are as theory-laden as we hope our theories are fact-laden. Or in other words, facts are small theories, and true theories are big facts. This does not mean, I must repeat, that right versions can be arrived at casually, or that worlds are built from scratch. We start, on any occasion, with some old version or world that we have on hand and that we are stuck with until we have the determination and skill to remake it into a new one. Some of the felt stubbornness of fact is the grip if habit: our firm foundation is indeed stolid. **Worldmaking begins with one version and ends with another**.

Nelson Goodman, *Ways of Worldmaking* (1978)

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Introduction

In the present paper, we study symplectic geometry concerning indigenous bundles in positive characteristic. Let

$$^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}$$

(cf. § 3.3) be the moduli stack classifying *ordinary dormant curves* of genus $g > 1$ over a field K of characteristic $p > 0$. It is known (cf. Proposition 3.3.1) that $^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}$ may be represented by a smooth, geometrically connected Deligne-Mumford stack over K of dimension $3g-3$. One may construct a canonical symplectic structure (cf. § 4) on the cotangent bundle $T^{\vee \circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}$ of $^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}$ (resp., the moduli stack $\mathcal{S}_{^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,\mathbb{F}_p}}$ (cf. § 2.3) classifying ordinary dormant curves equipped with an indigenous bundle), which we denote by

$$\omega^{\mathrm{can}}_{\circledcirc} \quad (\text{resp., } \omega^{\mathrm{PGL}}_{\circledcirc}).$$

The main result of the present paper is the following:

Theorem A.
If p is sufficiently large, then the natural isomorphism

$$\Psi : T^{\vee \circledcirc}\mathcal{M}^{Zzz...}_{g,\mathbb{F}_p} \xrightarrow{\sim} \mathcal{S}_{^{\circledcirc}\mathcal{M}^{Zzz...}_{g,\mathbb{F}_p}}$$

(cf. § 4.3) is compatible with the respective symplectic structures, i.e.,

$$\Psi^*(\omega^{\mathrm{PGL}_2}_{\circledcirc}) = \omega^{can}_{\circledcirc}.$$

0.1. The notion of an indigenous bundle was originally introduced and studied by Gunning in the context of compact hyperbolic Riemann surfaces (cf. [11]). Here, recall that an indigenous bundle on a fixed compact hyperbolic Riemann surface X is a $\mathbb{P}^1$-bundle on X, together with a connection, that satisfies certain properties. It may be thought of as an *algebraic* object encodes the (*analytic*, i.e., *non-algebraic*) uniformization data for the X. Various equivalent formulations, involving such diverse types of mathematical objects as certain kinds of differential operators or kernel functions, etc., have been studied by many mathematicians. For example, one may construct (in a natural way) from an indigenous bundle on X a specific *projective structure* on the underlying topological space Σ of X. A projective structure on Σ is, by definition, a maximal atlas covered by coordinate charts on Σ such that the transition functions are expressed as Möbius transformations. (In particular, each projective structure uniquely determines a Riemann surface structure on Σ.) This construction gives a bijective correspondence between the set of (isomorphism classes of) indigenous bundles on X and the set of projective structures on Σ defining the Riemann surface X.

0.2. In the following, we shall recall the works of S. Kawai, P. Arés-Gastesi, and I. BIswas (cf. [17]; [1]; [2]) that assert the compatibility of natural symplectic structures on certain moduli spaces concering projective structures, or equivalently, indigenous bundles.

Let Σ be a connected orientable closed surface of genus $g > 1$. Write $\mathcal{T}^{\Sigma}$ for the Teichmüller space associated to Σ, i.e., the quotient space

$$\mathcal{T}^{\Sigma} := \mathrm{Conf}(\Sigma)/\mathrm{Diff}^0(\Sigma),$$

where $\mathrm{Conf}(\Sigma)$ denotes the space of all conformal structures on Σ compatible with the orientation of Σ, and $\mathrm{Diff}^0(\Sigma)$ denotes the group of all diffeomorphisms of Σ homotopic to the identity map of Σ. Also, write

$$\mathcal{S}_{\mathcal{T}^{\Sigma}} := \mathrm{Proj}(\Sigma)/\mathrm{Diff}^0(\Sigma),$$

where $\mathrm{Proj}(\Sigma)$ denotes the space of all projective structures on Σ. It is known that the cotangent bundle $T^{\vee}\mathcal{T}^{\Sigma}$ of $\mathcal{T}^{\Sigma}$ (resp., the quotient space $\mathcal{S}_{\mathcal{T}^{\Sigma}}$) admits a structure of complex manifold of dimension $6g-6$, equipped with a holomorphic symplectic structure $\omega^{\mathrm{can}}_{\mathcal{T}^{\Sigma}}$ (resp., $\omega^{\mathrm{PGL}}_{\mathcal{T}^{\Sigma}}$ (cf. [9]). Consider a C^{∞} section

$$\mathfrak{unif} : \mathcal{T}^{\Sigma} \rightarrow \mathcal{S}_{\mathcal{T}^{\Sigma}}$$

of the natural projection $\mathcal{S}_{\mathcal{T}^{\Sigma}} \rightarrow \mathcal{T}^{\Sigma}$ arising from the uniformization constructed by either Bers, Schottky, or Earle (cf. [3]; [6]; [7]). In light of a natural affine structure on $\mathcal{S}_{\mathcal{T}^{\Sigma}}$, $\mathfrak{unif}$ may be uniquely extended a diffeomorphism

$$\Psi_{\mathfrak{unif}} \colon T^{\vee}\mathcal{T}^{\Sigma} \xrightarrow{\sim} \mathcal{S}^{\Sigma},$$

whose restriction to the zero section $\mathcal{T}^{\Sigma} \rightarrow \mathcal{S}_{\mathcal{T}^{\Sigma}}$ coincides with $\mathfrak{unif}$. It follows from [17], Theorem, [1], Theorem 1.1, and [1], Remark 3.2 that $\Psi_{\mathfrak{unif}}$ is compatible with the respective symplectic structures $\omega^{\mathrm{PGL}}_{\mathcal{S}^{\Sigma}}$, $\omega^{\mathrm{can}}_{\mathcal{T}^{\Sigma}}$ up to a constant factor, i.e.,

$$\Psi^{*}_{\mathfrak{unif}}(\omega^{\mathrm{PGL}}_{\mathcal{S}^{\Sigma}}) = \pi \cdot \omega^{\mathrm{can}}_{\mathcal{T}^{\Sigma}}.$$

0.3. Our aim in the present paper is to address the question whether a similar result holds for hyperbolic (algebraic) curves of *positive characteristic*. Just as in the case of the theory over $\mathbb{C}$, one may define the notion of an indigenous bundle in positive characteristic and their moduli space. Various properties of such objects were firstly discussed in the context of the p-adic Teichmüller theory developed by S. Mochizuki (cf. [19], [20]). One of the key ingredients in the development of this theory is the study of the p-curvature of indigenous bundles in characteristic p. Recall that the p-curvature of a connection may be thought of as the obstruction to the compatibility of p-power structures that appear in certain associated spaces of infinitesimal (i.e., "Lie") symmetries. We say that an indigenous bundle is *dormant* (cf. Definition 3.1.1) if its p-curvature vanishes identically. This condition on an indigenous bundles implies, in particular, the existence of "sufficiently many" horizontal sections locally in the Zariski topology.

In many aspects, dormant indigenous bundles may be thought of as reasonable (algebraic) products which allow us to develop analogous theory of indigenous bundles on Riemann surfaces. As we explained in § 0.2, each compact hyperbolic Riemann surface X of genus $g > 1$ admits a canonical indigenous bundle $\mathcal{P}_X^{\circledast}$ arising from the section $\mathfrak{unif}$. Thus, the moduli space $\mathcal{M}_{g,\mathbb{C}}$ of compact Riemann surfaces of genus g may be thought of as the moduli space classifying such X's equipped with an indigenous bundle satisfying a certain property (i.e., $\mathcal{P}_X^{\circledast}$). From this point of view, it is natural to regard the moduli stack of hyperbolic curves over a field K of positive characteristic equipped with an indigenous bundle satisfying a certain nice property (i.e., being dormant) as a variant of $\mathcal{M}_{g,\mathbb{C}}$. As our primary geometric objects, we would like to deal with pairs $(X, \mathcal{P}^{\circledast})$, called *(ordinary) dormant curves*, consisting of a hyperbolic curve and a dormant indigenous bundle on it (satisfying a certain condition).

In the present paper, as we have already displayed at the beginning of Introduction (i.e., Theorem A), we answer the question asked above affirmatively by considering the moduli stack ${}^{\circledcirc}\mathcal{M}_{g,K}^{\mathrm{Zzz...}}$ of ordinary dormant curve over K.

Finally, as an application of Theorem A, we construct certain additional structures as discussed in [4] or [5] on the moduli spaces under consideration.

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1. Preliminaries

Throughout this section, we fix a commutative ring R over $\mathbb{Z}[\frac{1}{2}]$.

1.1. We shall write (Set) for the category of (small) sets, and (Gpd) for the category of groupoids. For a Deligne-Mumford stack S, we shall write $(Sch)_S$ for the category of schemes over S. If $S = \mathrm{Spec}(R)$, then we shall write $(Sch)_R := (Sch)_{\mathrm{Spec}(R)}$ for simplicity.

1.2. A basic reference for stacks is [18]. Let S be a Deligne-Mumford stack and $\mathcal{F}$ an $\mathcal{O}_S$-module. We shall denote by $\mathcal{F}^\vee$ its dual sheaf, i.e., $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$, and write

$$\mathbb{A}(\mathcal{F}) := \mathcal{S}pec(S(\mathcal{F}^\vee)),$$

where $S(\mathcal{F}^\vee)$ denotes the symmetric algebra on $\mathcal{F}^\vee$ over $\mathcal{O}_S$. If, moreover, X is a Deligne-Mumford stack over S, then we shall write $\Omega_{X/S}$ for the sheaf of 1-differentials of X over S, $\wedge^i\Omega_{X/S}$ ($i = 1, 2, \cdots$) for its i-th exterior power, and $\mathcal{T}_{X/S}$ for the dual sheaf of $\Omega_{X/S}$ (i.e., the sheaf of derivations of X over S).

1.3. Let X be a smooth Deligne-Mumford stack over R. A *symplectic structure* on X (over R) is nondegenerate closed 2-form $\omega \in \Gamma(X, \wedge^2\Omega_{X/R})$. (Here, we say that ω is *nondegenerate* if the morphism $\Omega_{X/R} \to \mathcal{T}_{X/R}$ induced naturally by ω is an isomorphism.)

We shall denote by $T^\vee X$ the total space of the cotangent bundle to X, i.e.,

$$T^\vee X := \mathbb{A}(\Omega_{X/R})$$

(cf. § 1.2), which is a smooth Deligne-Mumford stack over R. It is well-known that there exists a unique 1-form $\lambda \in \Gamma(T^\vee X, \Omega_{T^\vee X/R})$ on $T^\vee X$ satisfying the following condition: If λ_u is the 1-form on X corresponding to a section $u : X \to T^\vee X$ of the natural projection $T^\vee X \to X$, then $u^*(\lambda) = \lambda_u$. Moreover, its exterior derivative

$$\omega_X^{\mathrm{can}} := d\lambda \in \Gamma(T^\vee X, \wedge^2\Omega_{T^\vee X/R})$$

defines a symplectic structure on $T^\vee X$. If $q_1, \cdots, q_n$ are local coordinates in X (where n is the relative dimension of X over R), then the dual coordinates $p_1, \cdots p_n$ in $T^\vee X$ are the coefficients of the decomposition of the 1-form λ into linear combination of the differentials dq_i, i.e., $\lambda = \sum_{i=1}^n p_i dq_i$. Hence, ω_X^{can} may be expressed locally as $\omega_X^{\mathrm{can}} = \sum_{i=1}^n dp_i \wedge dq_i$.

1.4. If S is a Deligne-Mumford stack, then we define a *curve* over S to be a geometrically connected, smooth relative scheme $f : X \to S$ over S of relative dimension 1. For an integer g, we shall say that a proper curve $f : X \to S$ over S is *of genus* g if the direct image $f_*(\Omega_{X/S})$ is a locally free $\mathcal{O}_S$-module of constant rank g.

For an integer $g > 1$, let us write

$$\mathcal{M}_{g,R}$$

for the moduli stack of proper curves of genus g over R and

$$f_{\mathcal{M}_{g,R}} : \mathcal{C}_{g,R} \to \mathcal{M}_{g,R}$$

for the tautological curve over $\mathcal{M}_{g,R}$. It follows from Serre duality that for a proper curve $f : X \to S$, the $\mathcal{O}_S$-module $\mathbb{R}^1 f_*(\Omega_{X/S})$ is isomorphic to $\mathcal{O}_S$; throughout the present paper, we fix a specific choice of an isomorphism

$$\Theta_{\mathcal{M}_{g,\mathbb{Z}[\frac{1}{2}]}} : \mathbb{R}^1 f_{\mathcal{M}_{g,\mathbb{Z}[\frac{1}{2}]}*}(\Omega_{\mathcal{C}_{g,\mathbb{Z}[\frac{1}{2}]}/\mathcal{M}_{g,\mathbb{Z}[\frac{1}{2}]}}) \xrightarrow{\sim} \mathcal{O}_{\mathcal{M}_{g,\mathbb{Z}[\frac{1}{2}]}}$$

of $\mathcal{O}_{\mathcal{M}_{g,\mathbb{Z}[\frac{1}{2}]}}$-modules.

If $u : T \to \mathcal{M}_{g,R}$ is a relative scheme over $\mathcal{M}_{g,R}$, then we shall write

$$f_T : \mathcal{C}_T \to T$$

for the curve over T classified by u (i.e., $\mathcal{C}_T := \mathcal{C}_{g,R} \times_{\mathcal{M}_{g,R}} T$). Also, we shall write

$$\Theta_T : \mathbb{R}^1 f_{T*}(\Omega_{\mathcal{C}_T/T}) \xrightarrow{\sim} \mathcal{O}_T$$

for the pull-back of the isomorphism $\Theta_{\mathcal{M}_{g,\mathbb{Z}[\frac{1}{2}]}}$ via the composite $T \xrightarrow{u} \mathcal{M}_{g,R} \to \mathcal{M}_{g,\mathbb{Z}[\frac{1}{2}]}$. For each vector bundle $\mathcal{E}$ on $\mathcal{C}_T$, Θ_T determines (in the natural way) a unique isomorphism

$$\Theta_{T,\mathcal{E}} : \mathbb{R}^1 f_{T*}(\mathcal{E}^\vee) \xrightarrow{\sim} f_{T*}(\Omega_{\mathcal{C}_T/T} \otimes \mathcal{E})^\vee$$

arising from Serre duality.

1.5. Let S be a Deligne-Mumford stack over R, $f : X \to S$ a smooth relative scheme over S, and G a smooth algebraic group over R with Lie algebra $\mathfrak{g}$. If $\pi : \mathcal{E} \to X$ is a G-torsor over X, then it induces a short exact sequence

$$0 \to \mathrm{ad}(\mathcal{E}) \to \widetilde{\mathcal{T}}_{\mathcal{E}/S} \xrightarrow{\alpha_{\mathcal{E}}} \mathcal{T}_{X/S} \to 0,$$

where $\mathrm{ad}(\mathcal{E}) := \mathcal{E} \times^G \mathfrak{g}$ denotes the adjoint bundle associated to $\mathcal{E}$, and $\widetilde{\mathcal{T}}_{\mathcal{E}/S}$ denotes the subsheaf of G-invariant sections $(\pi_*(\mathcal{T}_{\mathcal{E}/S}))^G$ of $\pi_*(\mathcal{T}_{\mathcal{E}/S})$. An S-*connection* on $\mathcal{E}$ is a split injection $\nabla : \mathcal{T}_{X/S} \to \widetilde{\mathcal{T}}_{\mathcal{E}/S}$ of the above short exact sequence (i.e., $\alpha_{\mathcal{E}} \circ \nabla = \mathrm{id}$). If X is of relative dimension 1 over S, then any such S-connection is necessarily *integrable*, i.e., compatible with the Lie bracket structures on $\mathcal{T}_{X/S}$ and $\widetilde{\mathcal{T}}_{\mathcal{E}/S} = (\pi_*\mathcal{T}_{\mathcal{E}/S})^G$.

Assume that G is a closed subgroup of GL_n for $n \geq 1$. Then the notion of an S-connection defined here may be identified with the usual definition of an S-connection on the associated vector bundle $\mathcal{E} \times^G (\mathcal{O}_X^{\oplus n})$ (cf. [14], Lemma 2.2.3; [16], (1.0)). In this situation, we shall not distinguish between these notions of connections.

For an S-connection ∇ on $\mathcal{E}$, we shall denote by

$$\nabla_{\mathrm{ad}} : \mathrm{ad}(\mathcal{E}) \to \Omega_{X/S} \otimes \mathrm{ad}(\mathcal{E})$$

the S-connection on the adjoint bundle $\mathrm{ad}(\mathcal{E})$ induced by ∇ via a change of structure group $G \to \mathrm{GL}(\mathfrak{g})$.

Let $\mathcal{V}$ be a vector bundle on X and $\nabla : \mathcal{V} \to \Omega_{X/S} \otimes \mathcal{V}$ an S-connection on X/S. Then ∇ may be thought of as a complex of abelian sheaves concentrated at degree 0 and 1. We shall denote this complex by

$$\mathcal{K}^\bullet_\nabla$$

($\mathcal{K}^0_\nabla := \mathcal{V}$, $\mathcal{K}^1_\nabla := \Omega_{X/S} \otimes \mathcal{V}$). Also, an abelian sheaf $\mathcal{E}$ may be thought of as a complex concentrated at degree 0. For $n \in \mathbb{Z}$, we define the complex

$$\mathcal{E}[n]$$

to be $\mathcal{E}$ shifted down by n, so that $\mathcal{E}[n]^{-n} = \mathcal{E}$ and $\mathcal{E}[n]^i = 0$ $(i \neq -n)$.

2. Indigenous bundles

In this section, we recall the notion of an indigenous bundle on a curve and some properties related to this notion. For the definitions and properties discussed in this section, we refer to [19], [20], and [23].

2.1. Fix an integer $g > 1$, a commutative ring R over $\mathbb{Z}[\frac{1}{2}]$, and a relative scheme S over $\mathcal{M}_{g,R}$ (cf. § 1.4) classifying a proper curve $f_S : \mathcal{C}_S \to S$ of genus g . Write G for the projective linear group over R of rank 2 (i.e., $G := \mathrm{PGL}_2$) and B for a Borel subgroup of G. We recall from [8], §4, or [19], Chap. I, § 2, Definition 2.2 the following:

Definition 2.1.1.

(i) Let $\mathcal{P}^\circledast = (\mathcal{P}_B, \nabla)$ be a pair consisting of a B-torsor $\mathcal{P}_B$ over $\mathcal{C}_S$ and a(n) (necessarily integrable) S-connection ∇ on the G-torsor $\mathcal{P}_G := \mathcal{P}_B \times^B G$ induced by $\mathcal{P}_B$. We shall say that $\mathcal{P}^\circledast$ is an *indigenous bundle* on $\mathcal{C}_S/S$ if the composite

$$\overline{\nabla} : \mathcal{T}_{\mathcal{C}_S/S} \xrightarrow{\nabla} \widetilde{\mathcal{T}}_{\mathcal{P}_G/S} \twoheadrightarrow \widetilde{\mathcal{T}}_{\mathcal{P}_G/S}/\widetilde{\iota}(\widetilde{\mathcal{T}}_{\mathcal{P}_B/S}),$$

where $\widetilde{\iota}$ denotes the natural injection $\widetilde{\mathcal{T}}_{\mathcal{P}_B/S} \to \widetilde{\mathcal{T}}_{\mathcal{P}_G/S}$ (cf. § 2.2), is an isomorphism.

(ii) Let $\mathcal{P}^\circledast = (\mathcal{P}_B, \nabla_\mathcal{P})$, $\mathcal{Q}^\circledast = (\mathcal{Q}_B, \nabla_\mathcal{Q})$ be indigenous bundles on $\mathcal{C}_S/S$. An *isomorphism* from $\mathcal{P}^\circledast$ to $\mathcal{Q}^\circledast$ is an isomorphism $\mathcal{P}_B \xrightarrow{\sim} \mathcal{Q}_B$ of B-torsors such that the induced isomorphism $\mathcal{P}_G \xrightarrow{\sim} \mathcal{Q}_G$ of G-torsors is compatible with the respective S-connections.

2.2. One may construct a natural filtration on the adjoint bundle associated to an indigenous bundle. Let $\mathcal{P}^\circledast = (\mathcal{P}_B, \nabla)$ be an indigenous bundle on $\mathcal{C}_S/S$ as follows. Consider the morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(\mathcal{P}_B) & \longrightarrow & \widetilde{\mathcal{T}}_{\mathcal{P}_B/S} & \longrightarrow & \mathcal{T}_{\mathcal{C}_S/S} & \longrightarrow & 0 \\
 & & {\scriptstyle\iota}\downarrow & & {\scriptstyle\widetilde{\iota}}\downarrow & & {\scriptstyle\mathrm{id}}\downarrow & & \\
0 & \longrightarrow & \mathrm{ad}(\mathcal{P}_G) & \longrightarrow & \widetilde{\mathcal{T}}_{\mathcal{P}_G/S} & \longrightarrow & \mathcal{T}_{\mathcal{C}_S/S} & \longrightarrow & 0
\end{array}$$

arising from the extension of the structure group $\mathcal{P}_B \to \mathcal{P}_G$. This diagram yields a natural isomorphism

$$\widetilde{\mathcal{T}}_{\mathcal{P}_G/S}/\widetilde{\iota}(\widetilde{\mathcal{T}}_{\mathcal{P}_B/S}) \xrightarrow{\sim} \mathrm{ad}(\mathcal{P}_G)/\iota(\mathrm{ad}(\mathcal{P}_B)).$$

Denote by

$$\overline{\nabla}^\dagger : \mathcal{T}_{\mathcal{C}_S/S} \xrightarrow{\sim} \mathrm{ad}(\mathcal{P}_G)/\iota(\mathrm{ad}(\mathcal{P}_B))$$

the isomorphism obtained by composing $\overline{\nabla} : \mathcal{T}_{\mathcal{C}_S/S} \xrightarrow{\sim} \widetilde{\mathcal{T}}_{\mathcal{P}_G/S}/\widetilde{\iota}(\widetilde{\mathcal{T}}_{\mathcal{P}_B/S})$ (cf. Definition 2.1.1 (i)) with this isomorphism.

Now we define a 3-step decreasing filtration $\{\mathrm{ad}(\mathcal{P}_G)^i\}_{i=0}^3$ on $\mathrm{ad}(\mathcal{P}_G)$ as follows:

$$\begin{aligned}
\mathrm{ad}(\mathcal{P}_G)^0 &:= \mathrm{ad}(\mathcal{P}_G),\\
\mathrm{ad}(\mathcal{P}_G)^1 &:= \widetilde{\iota}(\mathrm{ad}(\mathcal{P}_B)),\\
\mathrm{ad}(\mathcal{P}_G)^2 &:= \mathrm{Ker}\big(\mathrm{ad}(\mathcal{P}_G)^1 \stackrel{\nabla_{\mathrm{ad}}|_{\mathrm{ad}(\mathcal{P}_G)^1}}{\rightarrow} \Omega_{\mathcal{C}_S/S} \otimes \mathrm{ad}(\mathcal{P}_G)\\
&\qquad\qquad\qquad \twoheadrightarrow \Omega_{\mathcal{C}_S/S} \otimes \mathrm{ad}(\mathcal{P}_G)/\mathrm{ad}(\mathcal{P}_G)^1\big),\\
\mathrm{ad}(\mathcal{P}_G)^3 &:= 0
\end{aligned}$$

(cf. § 1.5 for the definition of ∇_{ad}). It follows from the definition of an indigenous bundle that for $j = 0, 1$,

$$\nabla_{\mathrm{ad}}(\mathrm{ad}(\mathcal{P}_G)^{j+1}) \subseteq \Omega_{\mathcal{C}_S/S} \otimes \mathrm{ad}(\mathcal{P}_G)^j,$$

and the resulting $\mathcal{O}_X$-*linear* morphism

$$\gamma_{j+1} : \mathrm{ad}(\mathcal{P}_G)^{j+1}/\mathrm{ad}(\mathcal{P}_G)^{j+2} \rightarrow \Omega_{\mathcal{C}_S/S} \otimes (\mathrm{ad}(\mathcal{P}_G)^j/\mathrm{ad}(\mathcal{P}_G)^{j+1})$$

induced by ∇_{ad} via taking subquotients of $\mathrm{ad}(\mathcal{P}_G)$ is an isomorphism. By composing γ_j's and the inverse of $\overline{\nabla}^\dagger$, we obtain an isomorphism

$$\gamma_j^\dagger : \mathrm{ad}(\mathcal{P}_G)^j/\mathrm{ad}(\mathcal{P}_G)^{j+1} \stackrel{\sim}{\rightarrow} \Omega_{\mathcal{C}_S/S}^{\otimes(j-1)}$$

$(j = 0, 1, 2)$ of $\mathcal{O}_X$-modules. The composite

$$\Omega_{\mathcal{C}_S/S}^{\otimes 2} \stackrel{\mathrm{id}\otimes(\gamma_2^\dagger)^{-1}}{\rightarrow} \Omega_{\mathcal{C}_S/S} \otimes \mathrm{ad}(\mathcal{P}_G)^2 \hookrightarrow \Omega_{\mathcal{C}_S/S} \otimes \mathrm{ad}(\mathcal{P}_G)$$

$$\big(\text{resp., } \ \mathrm{ad}(\mathcal{P}_G) \twoheadrightarrow \mathrm{ad}(\mathcal{P}_G)/\mathrm{ad}(\mathcal{P}_G)^1 \stackrel{\gamma_0^\dagger}{\rightarrow} \mathcal{T}_{\mathcal{C}_S/S}\big)$$

determines a morphism of complexes

$$\Omega_{\mathcal{C}_S/S}^{\otimes 2}[-1] \rightarrow \mathcal{K}^\bullet_{\nabla_{\mathrm{ad}}} \ \big(\text{resp., } \mathcal{K}^\bullet_{\nabla_{\mathrm{ad}}} \rightarrow \mathcal{T}_{\mathcal{C}_S/S}[0]\big)$$

(cf. § 1.5), and hence, by applying the functor $\mathbb{R}^1 f_{S*}(-)$, a morphism

$$\gamma^\sharp_{\mathcal{P}^\circledast} : f_{S*}(\Omega_{\mathcal{C}_S/S}^{\otimes 2}) \rightarrow \mathbb{R}^1 f_{S*}(\mathcal{K}^\bullet_{\nabla_{\mathrm{ad}}})$$

$$\big(\text{resp., } \gamma^\flat_{\mathcal{P}^\circledast} : \mathbb{R}^1 f_{S*}(\mathcal{K}^\bullet_{\mathcal{P}^\circledast}) \rightarrow \mathbb{R}^1 f_{S*}(\mathcal{T}_{\mathcal{C}_S/S})\big)$$

of $\mathcal{O}_S$-modules. It follows from [19], Chap. I, § 2, Theorem 2.8, that the sequence

$$0 \rightarrow f_{S*}(\Omega_{\mathcal{C}_S/S}^{\otimes 2}) \stackrel{\gamma^\sharp_{\mathcal{P}^\circledast}}{\rightarrow} \mathbb{R}^1 f_{S*}(\mathcal{K}^\bullet_{\nabla_{\mathrm{ad}}}) \stackrel{\gamma^\flat_{\mathcal{P}^\circledast}}{\rightarrow} \mathbb{R}^1 f_{S*}(\mathcal{T}_{\mathcal{C}_S/S}) \rightarrow 0$$

is exact. We note that this sequence is also obtained by taking cohomology of the differentials in the E_1-term of the spectral sequence

$$E_1^{p,q} = \mathbb{R}^q f_{S*}(\mathcal{K}^p_{\nabla_{\mathrm{ad}}}) \Rightarrow \mathbb{R}^{p+q} f_{S*}(\mathcal{K}^\bullet_{\nabla_{\mathrm{ad}}}).$$

2.3. Let us introduce notations for moduli functors classifying the objects discussed above. Denote by

$$\mathcal{S}_{g,R} : (Sch)_{\mathcal{M}_{g,R}} \to (Set)$$

the (Set)-valued functor on $(Sch)_{\mathcal{M}_{g,R}}$ (cf. § 1.1) which, to any $\mathcal{M}_{g,R}$-scheme T, assigns the set of isomorphism classes of indigenous bundles on the curve $f_T : \mathcal{C}_T \to T$ (cf. § 1.4). Evidently, $\mathcal{S}_{g,R}$ may be thought of as a (Gpd)-valued functor on $(Sch)_R$ classifying proper curves over R of genus g equipped with an indigenous bundle on it. By forgetting the datum of an indigenous bundle, we obtain a (1-)morphism

$$\mathcal{S}_{g,R} \to \mathcal{M}_{g,R}.$$

We shall write

$$\mathcal{S}_S := \mathcal{S}_{g,R} \times_{\mathcal{M}_{g,R}} S.$$

2.4. As we shall discuss in the following, $\mathcal{S}_S$ admits a natural affine structure by means of modular interpretation. Let $\mathcal{P}^{\circledast} = (\mathcal{P}_B, \nabla)$ be an indigenous bundle on $\mathcal{C}_S/S$ and $A \in \Gamma(\mathcal{C}_S, \Omega_{\mathcal{C}_S/S}^{\otimes 2})$, i.e., a globally defined section of the projection $\mathbb{A}(f_{S*}(\Omega_{\mathcal{C}_S/S}^{\otimes 2})) \to S$ (cf. § 1.2). By passing to the composite

$$\begin{aligned}
\Omega_{\mathcal{C}_S/S}^{\otimes 2} &\xrightarrow{\mathrm{id}\otimes(\gamma_2^\dagger)^{-1}} \Omega_{\mathcal{C}_S/S} \otimes \mathrm{ad}(\mathcal{P}_G)^2 \\
&\hookrightarrow \Omega_{\mathcal{C}_S/S} \otimes \mathrm{ad}(\mathcal{P}_G) \\
&\xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{\mathcal{C}_S}}(\mathcal{T}_{\mathcal{C}_S/S}, \mathrm{ad}(\mathcal{P}_G)) \\
&\hookrightarrow \mathcal{H}om_{\mathcal{O}_{\mathcal{C}_S}}(\mathcal{T}_{\mathcal{C}_S/S}, \widetilde{\mathcal{T}}_{\mathcal{P}_G/S})
\end{aligned}$$

(cf. § 2.2), one may think of A as an $\mathcal{O}_X$-linear morphism $\mathcal{T}_{\mathcal{C}_S/S} \to \widetilde{\mathcal{T}}_{\mathcal{P}_G/S}$. Hence, the sum $\nabla + A : \mathcal{T}_{\mathcal{C}_S/S} \to \widetilde{\mathcal{T}}_{\mathcal{P}/S}$ makes sense and turn to be an S-connection on $\mathcal{P}_G$ (cf. § 1.5). Moreover, it follows from the definition of an indigenous bundle that the pair

$$\mathcal{P}^{\circledast}_{+A} := (\mathcal{P}_B, \nabla + A)$$

forms an indigenous bundle on C_S/S. The assignment $(\mathcal{P}^{\circledast}, A) \mapsto \mathcal{P}^{\circledast}_{+A}$ is functorial (in the evident sense) with respect to S, and hence, determines an action

$$\mathcal{S}_S \times_S \mathbb{A}(f_{S*}(\Omega_{\mathcal{C}_S/S}^{\otimes 2})) \to \mathcal{S}_S.$$

We recall from [19], Chap. I, § 2, Corollary 2.9, the following:

Proposition 2.4.1.

The functor $\mathcal{S}_S$ may be represented by an $\mathbb{A}(f_{S*}(\Omega_{\mathcal{C}_S/S}^{\otimes 2}))$-torsor over S with respect to the $\mathbb{A}(f_{S*}(\Omega_{\mathcal{C}_S/S}^{\otimes 2}))$-action just discussed. In particular, if, moreover, S is a smooth Deligne-Mumford stack over R, then the functor $\mathcal{S}_S$ may be represented by a smooth Deligne-Mumford stack over R.

2.5. Finally, we recall cohomological expressions of tangent bundles of stacks involved. Write $\overline{v} : S \rightarrow \mathcal{M}_{g,R}$ for the structure morphism of the $\mathcal{M}_{g,R}$-scheme S, and $v : S \rightarrow \mathcal{S}_{g,R}$ for the S-rational point of $\mathcal{S}_{g,R}$ classifying an indigenous bundle $\mathcal{P}^{\circledast}$ on $\mathcal{C}_S/S$. It follows from well-known generalities concerning deformation theory that there exists a canonical isomorphism

$$a_S : \mathbb{R}^1 f_{S*}(\mathcal{T}_{\mathcal{C}_S/S}) \xrightarrow{\sim} \overline{v}^*(\mathcal{T}_{\mathcal{M}_{g,R}/R}).$$

On the other hand, the structure of $\mathbb{A}(f_{S*}(\Omega^{\otimes 2}_{\mathcal{C}_S/S}))$-torsor on $\mathcal{S}_S$ (cf. Proposition 2.4.1) yields a canonical isomorphism

$$c_{\mathcal{P}^{\circledast}} : f_{S*}(\Omega^{\otimes 2}_{\mathcal{C}_S/S}) \xrightarrow{\sim} v^*(\mathcal{T}_{\mathcal{S}_{g,R}/\mathcal{M}_{g,R}}).$$

Proposition 2.5.1.
There exists a canonical isomorphism

$$b_{\mathcal{P}^{\circledast}} : \mathbb{R}^1 f_{S*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}}) \xrightarrow{\sim} v^*(\mathcal{T}_{\mathcal{S}_{g,R}/R})$$

which fits into the following isomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & f_{S*}(\Omega^{\otimes 2}_{\mathcal{C}_S/S}) \arrow[r] \arrow[d, "c_{\mathcal{P}^{\circledast}}"'] & \mathbb{R}^1 f_{S*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}}) \arrow[r] \arrow[d, "b_{\mathcal{P}^{\circledast}}"'] & \mathbb{R}^1 f_{S*}(\mathcal{T}_{\mathcal{C}_S/S}) \arrow[r] \arrow[d, "a_S"'] & 0 \\
0 \arrow[r] & v^*(\mathcal{T}_{\mathcal{S}_{g,R}/\mathcal{M}_{g,R}}) \arrow[r] & v^*(\mathcal{T}_{\mathcal{S}_{g,R}/R}) \arrow[r] & \overline{v}^*(\mathcal{T}_{\mathcal{M}_{g,R}/R}) \arrow[r] & 0,
\end{tikzcd}$$

where the lower sequence denotes the natural exact sequence of tangent bundles, and the upper sequence denotes the short exact sequence discussed in § 2.2.

Proof. The assertion follows from an argument similar to the argument (in the case where the curve in discussion is under certain assumptions, e.g., $S = \mathrm{Spec}(k)$ for an algebraically closed field k) given in [21], § 3. □

3. Dormant Indigenous bundles

In this section, we recall the definition of a dormant indigenous bundle and discuss various moduli functors related to this notion. Throughout this section, let us fix an odd prime p.

3.1. Let g, R, S, G, B be as in § 2. Suppose further that $R = K$ for a field of characteristic p. First, we recall the definition of the p-curvature map. Let $\pi : \mathcal{E} \rightarrow \mathcal{C}_S$ be a G-torsor over $\mathcal{C}_S$ and $\nabla : \mathcal{T}_{\mathcal{C}_S/S} \rightarrow \widetilde{\mathcal{T}}_{\mathcal{E}/S}$ an S-connection on $\mathcal{E}$, i.e., a section of the surjection $\alpha_{\mathcal{E}} : (\pi_* \mathcal{T}_{\mathcal{E}/S})^G =: \widetilde{\mathcal{T}}_{\mathcal{E}/S} \rightarrow \mathcal{T}_{\mathcal{C}_S/S}$ (cf. § 1.5). If ∂ is a derivation corresponding to a local section ∂ of $\mathcal{T}_{\mathcal{C}_S/S}$ (respectively, $\widetilde{\mathcal{T}}_{\mathcal{E}/S} := (\pi_* \mathcal{T}_{\mathcal{E}/S})^G$), then we shall denote by $\partial^{[p]}$ the p-th iterate of ∂, which is also a derivation corresponding to a local section of $\mathcal{T}_{\mathcal{C}_S/S}$ (respectively, $\widetilde{\mathcal{T}}_{\mathcal{E}/S}$).

Since $\alpha_{\mathcal{E}}(\partial^{[p]}) = (\alpha_{\mathcal{E}}(\partial))^{[p]}$ for any local section of $\mathcal{T}_{\mathcal{C}_S/S}$, the image of the p-linear map from $\mathcal{T}_{X/S}$ to $\widetilde{\mathcal{T}}_{\mathcal{E}/S}$ defined by assigning $\partial \mapsto \nabla(\partial^{[p]}) - (\nabla(\partial))^{[p]}$ is contained in $\mathrm{ad}(\mathcal{E})$ $(= \ker(\alpha_{\mathcal{E}}))$. Thus, we obtain an $\mathcal{O}_X$-linear morphism

$$\psi_{(\mathcal{E},\nabla)} : \mathcal{T}_{\mathcal{C}_S/S}^{\otimes p} \to \mathrm{ad}(\mathcal{E})$$

determined by assigning

$$\partial^{\otimes p} \mapsto \nabla(\partial^{[p]}) - (\nabla(\partial))^{[p]}.$$

We shall refer to the morphism $\psi_{(\mathcal{E},\nabla)}$ as the *p-curvature map* of $(\mathcal{E}, \nabla)$.

Definition 3.1.1.

(i) We shall say that an indigenous bundle $\mathcal{P}^{\circledast} = (\mathcal{P}_B, \nabla)$ on $\mathcal{C}_S/S$ is *dormant* if the p-curvature map of $(\mathcal{P}_G, \nabla)$ vanishes identically on $\mathcal{C}_S$.

(ii) Let T be a K-scheme. A *dormant curve* over T of genus g is a pair $X_{/T}^{\mathrm{Zzz...}} := (X/T, \mathcal{P}^{\circledast})$ consisting of a proper curve X over T of genus g and a dormant indigenous bundle $\mathcal{P}^{\circledast}$ on X/T.

(iii) Let T be a K-scheme, and $X_{/T}^{\mathrm{Zzz...}} := (X/T, \mathcal{P}_X^{\circledast} = (\pi_{\mathcal{P}} : \mathcal{P}_B \to X, \nabla_{\mathcal{P}}))$, $Y_{/T}^{\mathrm{Zzz...}} := (Y/T, \mathcal{P}_Y^{\circledast} = (\pi_{\mathcal{Q}} : \mathcal{Q}_B \to Y, \nabla_{\mathcal{Q}}))$ dormant curves over T of genus g. An *isomorphism* from $X_{/T}^{\mathrm{Zzz...}}$ to $Y_{/T}^{\mathrm{Zzz...}}$ is a pair $(h, \widetilde{h})$ consisting of an isomorphism $h : X \xrightarrow{\sim} Y$ of T-schemes and an isomorphism $\widetilde{h} : \mathcal{P}_B \xrightarrow{\sim} \mathcal{Q}_B$ that makes the square

$$\begin{array}{ccc} \mathcal{P}_B & \xrightarrow{\widetilde{h}} & \mathcal{Q}_B \\ {\scriptstyle \pi_{\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{Q}}} \\ X & \xrightarrow{h} & Y \end{array}$$

commute, and is compatible with the respective B-actions and S-connections in the evident sense.

3.2. Next, we shall introduce the notion of the ordinariness for dormant curves. If $\mathcal{P}^{\circledast} = (\mathcal{P}_B, \nabla)$ is an indigenous bundle on $\mathcal{C}_S/S$, then the natural morphism $\mathrm{Ker}(\nabla_{\mathrm{ad}})[0] \to \mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}}$ determines a morphism

$$\gamma^{\natural}_{\mathcal{P}^{\circledast}} : \mathbb{R}^1 f_{S*}(\mathrm{Ker}(\nabla_{\mathrm{ad}})) \to \mathbb{R}^1 f_{S*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}})$$

of $\mathcal{O}_S$-modules. By composing it and $\gamma^{\flat}_{\mathcal{P}^{\circledast}}$ (cf. § 2.2), we obtain a morphism

$$\gamma^{\heartsuit}_{\mathcal{P}^{\circledast}} : \mathbb{R}^1 f_{S*}(\mathrm{Ker}(\nabla_{\mathrm{ad}})) \to \mathbb{R}^1 f_{S*}(\mathcal{T}_{\mathcal{C}_S/S})$$

of $\mathcal{O}_S$-modules. Note that $\gamma^{\heartsuit}_{\mathcal{P}^{\circledast}}$ is also defined as the morphism obtained by applying the functor $\mathbb{R}^1 f_{S*}(-)$ to the natural composite

$$\mathrm{Ker}(\nabla_{\mathrm{ad}}) \hookrightarrow \mathrm{ad}(\mathcal{P}_G) \twoheadrightarrow \mathrm{ad}(\mathcal{P}_G)/\mathrm{ad}(\mathcal{P}_G)^1 \xrightarrow{\gamma_0^{\dagger}} \mathcal{T}_{\mathcal{C}_S/S}.$$

Definition 3.2.1.
We shall say that a dormant curve $\mathcal{C}^{\text{Zzz...}}_{S/S} = (\mathcal{C}_S/S, \mathcal{P}^{\circledast})$ is *ordinary* if $\gamma^{\heartsuit}_{\mathcal{P}^{\circledast}}$ is an isomorphism.

3.3. We shall denote by

$$\mathcal{M}^{\text{Zzz...}}_{g,K} \quad (\text{resp.}, {}^{\circledcirc}\mathcal{M}^{\text{Zzz...}}_{g,K})$$

the (Gpd)-valued functor on $(Sch)_K$ which, to any K-scheme T, assigns the groupoid of dormant curves (resp., ordinary dormant curves) over T of genus g. $\mathcal{M}^{\text{Zzz...}}_{g,K}$ and ${}^{\circledcirc}\mathcal{M}^{\text{Zzz...}}_{g,K}$ may be naturally thought of as functors over $\mathcal{M}_{g,K}$, and there is a natural sequence of functors

$${}^{\circledcirc}\mathcal{M}^{\text{Zzz...}}_{g,K} \to \mathcal{M}^{\text{Zzz...}}_{g,K} \to \mathcal{S}_{g,K}.$$

over $\mathcal{M}_{g,K}$. We quote a result from p-adic Teichmüller theory due to S. Mochizuki concerning these functors.

Proposition 3.3.1.
The functor $\mathcal{M}^{\text{Zzz...}}_{g,K}$ may be represented by a closed substack of $\mathcal{S}_{g,K}$, finite and faithfully flat over $\mathcal{M}_{g,K}$. The functor ${}^{\circledcirc}\mathcal{M}^{\text{Zzz...}}_{g,K}$ may be repressnted by a dense open substack of $\mathcal{M}^{\text{Zzz...}}_{g,K}$ and coincides with the étale locus of $\mathcal{M}^{\text{Zzz...}}_{g,K}$ over $\mathcal{M}_{g,K}$. In particular, $\mathcal{M}^{\text{Zzz...}}_{g,K}$ and ${}^{\circledcirc}\mathcal{M}^{\text{Zzz...}}_{g,K}$ are Deligne-Mumford stacks over K of dimension $3g-3$.

Proof. The assertion follows from [20], Chap. II, § 2.3, Lemma 2.7; [20], Chap. II, § 2.3, Theorem 2.8 (and its proof). ☐

3.4. Finally, by means of cohomological expressions, we shall describe the differential of the morphism $\mathcal{M}^{\text{Zzz...}}_{g,K} \to \mathcal{S}_{g,K}$. Note that Proposition 3.4.1 below implies the fact that ${}^{\circledcirc}\mathcal{M}^{\text{Zzz...}}_{g,K}$ coincides with the étale locus of $\mathcal{M}^{\text{Zzz...}}_{g,K}$ over $\mathcal{M}_{g,K}$, as we have already asserted in Proposition 3.3.1.

Proposition 3.4.1.
Let v, $\mathcal{P}^{\circledast}$ be as in § 2.5. Suppose further that v factors through the morphism $\mathcal{M}^{\text{Zzz...}}_{g,K} \to \mathcal{S}_{g,K}$ (i.e., $\mathcal{P}^{\circledast}$ is dormant), and denote by $\breve{v} : S \to \mathcal{M}^{\text{Zzz...}}_{g,K}$ the resulting S-rational point of $\mathcal{M}^{\text{Zzz...}}_{g,K}$. Then there exists a canonical isomorphism

$$d_{\mathcal{P}^{\circledast}} : \mathbb{R}^1 f_{S*}(\mathrm{Ker}(\nabla_{\mathrm{ad}})) \xrightarrow{\sim} \breve{v}^*(\mathcal{T}_{\mathcal{M}^{\text{Zzz...}}_{g,K}/K})$$

which makes the square

$$\begin{array}{ccc}
\mathbb{R}^1 f_{S*}(\mathrm{Ker}(\nabla_{\mathrm{ad}})) & \xrightarrow{\gamma^{\natural}_{\mathcal{P}^{\circledast}}} & \mathbb{R}^1 f_{S*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}}) \\
{\scriptstyle d_{\mathcal{P}^{\circledast}}}\big\downarrow & & \big\downarrow{\scriptstyle b_{\mathcal{P}^{\circledast}}} \\
\breve{v}^*(\mathcal{T}_{\mathcal{M}^{\text{Zzz...}}_{g,K}/K}) & \longrightarrow & v^*(\mathcal{T}_{\mathcal{S}_{g,K}/K})
\end{array}$$

commute, where the lower horizontal arrow denotes the $\mathcal{O}_S$-linear morphism arising from the morphism $\mathcal{M}^{\text{Zzz...}}_{g,K} \to \mathcal{S}_{g,K}$.

Proof. See the proof of [20], Chap. II, § 2.3, Theorem 2.8. □

4. Compatibility of symplectic structures

In this section, we state the main result of the present paper, i.e., Theorem A (= Theorem 4.3.1). First, we construct a certain symplectic structure on the moduli stack of indigenous bundles.

4.1. Let g, R, S, G, B be as in § 2, and $\mathcal{P}^\circledast = (\mathcal{P}_B, \nabla)$ an indigenous bundle on $f_S : \mathcal{C}_S \to S$. Denote by $\overline{v} : S \to \mathcal{M}_{g,R}$ and $v : S \to \mathcal{S}_{g,R}$ the structure morphism of the $\mathcal{M}_{g,R}$-scheme S and the classifying morphism of $\mathcal{P}^\circledast$ respectively.

Recall that the Killing form on $\mathfrak{sl}_2$ is a nondegenerate symmetric bilinear map $\kappa : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to k$ defined by $\kappa(a,b) = \frac{1}{4} \cdot \mathrm{tr}(\mathrm{ad}(a)\mathrm{ad}(b))$ $(= \mathrm{tr}(ab))$ for a, $b \in \mathfrak{sl}_2$. By executing a change of structure group via κ, we obtain an $\mathcal{O}_S$-linear morphism

$$\kappa_{\mathcal{P}^\circledast} : \mathrm{ad}(\mathcal{P}_G) \otimes \mathrm{ad}(\mathcal{P}_G) \to \mathcal{O}_{\mathcal{C}_S},$$

which induces an isomorphism

$$\kappa^{\triangleright}_{\mathcal{P}^\circledast} : \mathrm{ad}(\mathcal{P}_G) \xrightarrow{\sim} \mathrm{ad}(\mathcal{P}_G)^\vee.$$

Let us write $d_{\mathrm{univ}} : \mathcal{O}_{\mathcal{C}_S} \to \Omega_{\mathcal{C}_S/S}$ for the universal derivation, and $\nabla_{\mathrm{ad}^{\otimes 2}}$ for the S-connection on the tensor product $\mathrm{ad}(\mathcal{P}_G) \otimes \mathrm{ad}(\mathcal{P}_G)$ induced naturally by ∇_{ad}. Then the morphism $\kappa_{\mathcal{P}^\circledast}$ is compatible with the respective S-connections $\nabla_{\mathrm{ad}^{\otimes 2}}$ and d_{univ}. By composing the morphism $\kappa_{\mathcal{P}^\circledast}$ and the cup product in the de Rham cohomology, we obtain a skew-symmetric $\mathcal{O}_S$-bilinear map on $\mathbb{R}^1 f_{S*}(\mathcal{K}^\bullet_{\nabla_{\mathrm{ad}}})$:

$$\begin{aligned}\mathbb{R}^1 f_{S*}(\mathcal{K}^\bullet_{\nabla_{\mathrm{ad}}}) \otimes \mathbb{R}^1 f_{S*}(\mathcal{K}^\bullet_{\nabla_{\mathrm{ad}}}) &\to \mathbb{R}^2 f_{S*}(\mathcal{K}^\bullet_{\nabla_{\mathrm{ad}^{\otimes 2}}}) \\ &\to \mathbb{R}^2 f_{S*}(\mathcal{K}^\bullet_{d_{\mathrm{univ}}}) \\ &\xrightarrow{\sim} \mathbb{R}^1 f_{S*}(\Omega_{\mathcal{C}_S/S}) \\ &\overset{\Theta_S}{\xrightarrow{\sim}} \mathcal{O}_S\end{aligned}$$

(cf. [13], Corollary 5.6 for the third arrow). If $\mathcal{P}^\circledast$ is taken to be the tautological indigenous bundle on $\mathcal{S}_{g,R}$, then the bilinear map just obtained determines, via the isomorphism $b_{\mathcal{P}^\circledast} : \mathbb{R}^1 f_{S*}(\mathcal{K}^\bullet_{\nabla_{\mathrm{ad}}}) \xrightarrow{\sim} v^*(\mathcal{T}_{\mathcal{S}_{g,R}/R})$ (cf. Proposition 2.5.1), a skew-symmetric $\mathcal{O}_{\mathcal{S}_{g,R}}$-bilinear map on $\mathcal{T}_{\mathcal{S}_{g,R}/R}$, equivalently, a 2-form

$$\omega^{\mathrm{PGL}}_{g,R}$$

on the smooth Deligne-Mumford stack $\mathcal{S}_{g,R}$ (cf. Proposition 2.4.1).

Proposition 4.1.1.
$\omega^{\mathrm{PGL}}_{g,R}$ is a symplectic structure on $\mathcal{S}_{g,R}$.

Proof. The nondegeneracy of $\omega^{\mathrm{PGL}}_{g,R}$ follows from, e.g., the explicit description of hypercohomology in terms of the Čech double complex associated to $\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}}$. Indeed, one verifies easily from such an explicit description that the bilinear map $\kappa_{\mathcal{P}^{\circledast}}$ induces naturally a morphism $\kappa^{\blacktriangleright}_{\mathcal{P}^{\circledast}} : \mathbb{R}^1 f_{S*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}}) \rightarrow \mathbb{R}^1 f_{S*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}})^{\vee}$ and a morphism of spectral sequences from "$E^{p,q}_1 = \mathbb{R}^q f_{S*}(\mathcal{K}^p_{\nabla_{\mathrm{ad}}}) \Rightarrow \mathbb{R}^{p+q} f_{S*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}})$" to "$E^{p,q}_1 = \mathbb{R}^{1-p} f_{S*}(\mathcal{K}^{1-q}_{\nabla_{\mathrm{ad}}})^{\vee} \Rightarrow \mathbb{R}^{2-p-q} f_{S*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}})^{\vee}$" which are compatible with $\kappa^{\blacktriangleright}_{\mathcal{P}^{\circledast}}$ in the evident sense. But the constituents $\mathbb{R}^q f_{S*}(\mathcal{K}^p_{\nabla_{\mathrm{ad}}}) \rightarrow \mathbb{R}^{1-p} f_{S*}(\mathcal{K}^{1-q}_{\nabla_{\mathrm{ad}}})^{\vee}$ in this morphism of spectral sequences are, in fact, the composites of $\kappa^{\triangleright}_{\mathcal{P}^{\circledast}}$ and the isomorphisms arising from Serre duality. It thus follows that $\kappa^{\blacktriangleright}_{\mathcal{P}^{\circledast}}$ is an isomorphism, which implies that $\omega^{\mathrm{PGL}}_{g,R}$ is nondegenerate.

Next, we consider the closedness of $\omega^{\mathrm{PGL}}_{g,R}$. Since the closedness of a differential form is preserved under base change via $\mathrm{Spec}(R) \rightarrow \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, it suffices to verify the case where $R = \mathbb{Z}[\frac{1}{2}]$. But, $\wedge^3 \Omega_{\mathcal{S}_{g,\mathbb{Z}[\frac{1}{2}]}/\mathbb{Z}[\frac{1}{2}]}$ is flat over $\mathbb{Z}[\frac{1}{2}]$, so the assertion of the case where $R = \mathbb{Z}[\frac{1}{2}]$ follows from the assertion of the case of $R = \mathbb{C}$.

Let Σ be a connected orientable closed surface of genus g (cf. § 0.2), and $\pi_1(\Sigma)$ the fundamental group of Σ (with respect to a fixed base-point $z \in \Sigma$). The space $\mathrm{Hom}(\pi_1(\Sigma), G)$ of representations $\pi_1(\Sigma) \rightarrow G$ has a canonical G-action obtained by composing representations with inner automorphisms of G; the orbit space is denoted by $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. If we denote by

$$\mathcal{R} \subseteq \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

the space of all irreducible representations, then $\mathcal{R}$ is a complex manifold of dimension $6g - 6$ equipped with a holomorphic symplectic structure $\omega_{\mathcal{R}}$ (cf. [9]; [17]). Write $\mathcal{S}^{\mathrm{an}}_{g,\mathbb{C}}$ for the analytic stack associated with $\mathcal{S}_{g,\mathbb{C}}$, $\mathcal{S}_{\mathcal{T}^{\Sigma}}$ for the space of all projective structures on Σ (cf. Introduction), and

$$t_{\mathcal{S}} : \mathcal{S}_{\mathcal{T}^{\Sigma}} \rightarrow \mathcal{S}^{\mathrm{an}}_{g,\mathbb{C}}$$

for the natural projection. By taking monodromy of indigenous bundles we obtain a local bi-holomorphic map

$$t_{\mathcal{R}} : \mathcal{S}_{\mathcal{T}^{\Sigma}} \rightarrow \mathcal{R}.$$

It follows from the constructions of the relevant moduli spaces and morphisms between them that $t^*_{\mathcal{R}}(\omega_{\mathcal{R}}) = t^*_{\mathcal{S}}(\omega^{\mathrm{PGL}}_{g,\mathbb{C}})$. By [9], Theorem, $\omega_{\mathcal{R}}$ is closed, which thus implies that $\omega^{\mathrm{PGL}}_{g,\mathbb{C}}$ is closed. This completes the proof of Proposition 4.1.1. $\square$

4.2. Suppose further that $\overline{v} : S \to \mathcal{M}_{g,R}$ is étale (hence S is a smooth Deligne-Mumford stack over R). Consider the composite isomorphism

$$\begin{aligned}\Omega_{S/R}\big(= \mathcal{T}^\vee_{S/R}\big) &\xrightarrow{\sim} \overline{v}^*(\mathcal{T}^\vee_{\mathcal{M}_{g,R}/R}) \\ &\xrightarrow{\sim} \overline{v}^*(\mathbb{R}^1 f_{\mathcal{M}_{g,R}*}(\mathcal{T}_{\mathcal{C}_{g,R}/\mathcal{M}_{g,R}})^\vee) \\ &\xrightarrow{\sim} \overline{v}^*(f_{\mathcal{M}_{g,R}*}(\Omega^{\otimes 2}_{\mathcal{C}_{g,R}/\mathcal{M}_{g,R}})) \\ &\xrightarrow{\sim} f_{S*}(\Omega^{\otimes 2}_{\mathcal{C}_S/S}),\end{aligned}$$

where

1. the first isomorphism follows from the étaleness of $\overline{v}$,
2. the second isomorphism arises from $a_{\mathcal{M}_{g,R}} : \mathbb{R}^1 f_{\mathcal{M}_{g,R}*}(\mathcal{T}_{\mathcal{C}_{g,R}/\mathcal{M}_{g,R}}) \xrightarrow{\sim} \mathcal{T}_{\mathcal{M}_{g,R}/R}$ (cf. § 2.5), and
3. the third isomorphism arises from the isomorphism $\Theta_{\mathcal{M}_{g,R},\Omega_{\mathcal{C}_{g,R}/\mathcal{M}_{g,R}}}$ (cf. § 1.4).

By applying this composite isomorphism, we may consider $T^\vee S$ $(= \mathbb{A}(\Omega_{S/R}))$ as *the* trivial $\mathbb{A}(f_{S*}(\Omega^{\otimes 2}_{\mathcal{C}_S/S}))$-torsor over S. Hence, for each global section $\sigma : S \to \mathcal{S}_S$ of the natural projection $\mathcal{S}_S \to S$, there exists a unique isomorphism

$$\Psi_\sigma : T^\vee S \xrightarrow{\sim} \mathcal{S}_S$$

over S that is compatible with the respective $\mathbb{A}(f_{S*}(\Omega^{\otimes 2}_{\mathcal{C}_S/S}))$-actions and whose restriction $\Psi_\sigma|_{0_S}$ to the zero section $0_S : S \to T^\vee S$ coincides with σ. In particular, Ψ_σ induces an isomorphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & 0_S^*(\mathcal{T}_{T^\vee S/S}) \arrow[r] \arrow[d, "\wr"'] & 0_S^*(\mathcal{T}_{T^\vee S/R}) \arrow[r] \arrow[d, "\wr"'] & \mathcal{T}_{S/R} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \sigma^*(\mathcal{T}_{\mathcal{S}_S/S}) \arrow[r] & \sigma^*(\mathcal{T}_{\mathcal{S}_S/R}) \arrow[r] & \mathcal{T}_{S/R} \arrow[r] & 0. \end{tikzcd}$$

Next, write $\overline{v}_{\mathcal{S}} : \mathcal{S}_S \to \mathcal{S}_{g,R}$ for the base change of $\overline{v}$ via the projection $\mathcal{S}_{g,R} \to \mathcal{M}_{g,R}$. Since $\overline{v}_{\mathcal{S}}$ is étale (i.e., the natural morphism $\overline{v}_{\mathcal{S}}^*(\Omega_{\mathcal{S}_{g,R}/R}) \to \Omega_{\mathcal{S}_S/R}$ is an isomorphism), the pull-back

$$\omega_S^{\mathrm{PGL}} := \overline{v}_{\mathcal{S}}^*(\omega_{g,R}^{\mathrm{PGL}})$$

determines a symplectic structure on $\mathcal{S}_S$ (cf. Propotion 4.1.1).

4.3. By Proposition 3.3.1, one may apply the above discussion to the case where the data $(R, S, \sigma : S \to \mathcal{S}_S)$ is taken to be "$(K, {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}, \sigma_{g,K} : {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K} \to \mathcal{S}_{{}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}})$", where K denotes a field of characteristic $p > 2$, $\sigma_{g,K}$ denotes the section arising from the immersion ${}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K} \to \mathcal{S}_{g,K}$. Thus, we obtain an isomorphism

$$\Psi_{\sigma_{g,K}} : T^\vee {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K} \to \mathcal{S}_{{}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}}$$

over ${}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}$ and a symplectic sturcture on $\mathcal{S}_{{}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}}$, which we denote by

$$\omega^{\mathrm{PGL}}_{\circledcirc}.$$

On the other hand, we recall that $T^\vee {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}$ admits a canonical symplectic structure

$$\omega^{\mathrm{can}}_{\circledcirc} := \omega^{\mathrm{can}}_{{}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}}$$

(cf. § 1.3). The main result of the present paper is as follows.

Theorem 4.3.1.
Let g be an integer > 1 and K a field of characteristic $p > 0$. If p is sufficiently large, then the isomorphism $\Psi_{\sigma_{g,K}}$ is compatible with the respective symplectic structures, i.e.,

$$\Psi^*_{\sigma_{g,K}}(\omega^{\mathrm{PGL}}_{\circledcirc}) = \omega^{can}_{\circledcirc}.$$

5. Proof of Theorem A

This section is devoted to the proof of Theorem 4.3.1.

5.1. First we prove the following lemma.

Lemma 5.1.1.
For sufficiently large prime number p, there exists an étale and dominant morphism $u : U \to {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}$ such that the 2-form ω^{PGL}_U is exact.

Proof. First we choose an affine scheme $U_\mathbb{C}$ of finite type over $\mathbb{C}$ and an étale surjective morphism $u_\mathbb{C} : U_\mathbb{C} \to \mathcal{M}_{g,\mathbb{C}}$. By [10], Theorem 1' and [12], Proposition 7.9.1, there exists a canonical isomorphism of de Rham cohomology groups $H^2_{\mathrm{dR}}(\mathcal{S}_{U_\mathbb{C}}) \cong H^2_{\mathrm{dR}}(U_\mathbb{C})$. It follows that the cohomology class $[\omega^{\mathrm{PGL}}_{U_\mathbb{C}}]$ in $H^2_{\mathrm{dR}}(\mathcal{S}_{U_\mathbb{C}})$ representing $\omega^{\mathrm{PGL}}_{U_\mathbb{C}}$ vanishes after possibly replacing $U_\mathbb{C}$ by an affine étale covering of $U_\mathbb{C}$. Thus, we may assume that $\omega^{\mathrm{PGL}}_{U_\mathbb{C}}$ is exact. It follows from a routine argument that one may obtain, from $U_\mathbb{C}$, a scheme U_R over an étale finite $\mathbb{Z}[\frac{1}{m}]$-ring R for some $m \in \mathbb{Z}$, and an étale and dominant morphism $u_R : U_R \to \mathcal{M}_{g,R}$ over R such that $\omega^{\mathrm{PGL}}_{U_R}$ is exact. For sufficiently large prime p, the reduction $U_R \times_\mathbb{Z} \mathbb{F}_p$ is nonempty, so the composite

$$U_R \times_{\mathcal{M}_{g,\mathbb{Z}}} {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K} \overset{u_R \times \mathrm{id}}{\to} \mathcal{M}_{g,R} \times_{\mathcal{M}_{g,\mathbb{Z}}} {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K} \to {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}$$

is étale and dominant, as desired. $\square$

Let U, u be as in Lemma 5.1.1, and denote by

$$\sigma_U : U \to \mathcal{S}_U \quad (\text{resp., } \Psi_U : T^\vee U \to \mathcal{S}_U)$$

the restriction of $\sigma_{g,K}$ (resp., $\Psi_{\sigma_{g,K}}$) to U. Since the natural map

$$\Gamma(T^\vee {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}, \wedge^2\Omega_{T^\vee {}^{\circledcirc}\mathcal{M}^{\mathrm{Zzz...}}_{g,K}/K}) \to \Gamma(T^\vee U, \wedge^2\Omega_{T^\vee U/K})$$

is injective, the proof of Theorem 4.3.1 reduces to proving the equality

$$\Psi_U^*(\omega_U^{\mathrm{PGL}}) = \omega_U^{\mathrm{can}}.$$

5.2. Let $A \in \Gamma(U, \Omega_{U/K}) \cong \Gamma(U, f_{U*}(\Omega_{\mathcal{C}_U/U}^{\otimes 2}))$ (cf. § 4.2), and write

$$\tau_A : T^\vee U \xrightarrow{\sim} T^\vee U$$

for the translation of $T^\vee U$ by A. If we consider $A \in \Gamma(U, \Omega_{U/K})$ as a global section of $\Omega_{T^\vee U/K}$ via the projection $T^\vee U \to U$, then one verifies that $\tau_A^*(\delta) = \delta + A$ for a locally defined 1-form $\delta \in \Omega_{T^\vee U/K}$. Also, for an exact 2-form ω_0 on $T^\vee U$, it satisfies that $\tau_A^*(\omega_0) = \omega_0 + dA$. Indeed, if we express $\omega_0 = d\delta_0$ for a 1-form $\delta_0 \in \Gamma(T^\vee U, \Omega_{T^\vee U/K})$, then

$$\tau_A^*(\omega_0) = \tau_A^*(d\delta_0) = d(\tau_A^*(\delta_0)) = d(\delta_0 + A) = \omega_0 + dA.$$

It follows from Lemma 5.1.1 and the exactness of ω_U^{can} (cf. § 1.3) that

$$\tau_A^*(\Psi_U^* \omega_U^{\mathrm{PGL}}) = \Psi_A^* \omega_U^{\mathrm{PGL}} + dA, \quad \tau_A^* \omega_U^{\mathrm{can}} = \omega_U^{\mathrm{can}} + dA.$$

In particular, $\big(\sigma_U^*(\omega_U^{\mathrm{PGL}}) = \big)\ 0_U^*(\Psi_U^*(\omega_U^{\mathrm{PGL}})) = 0_U^*(\omega_U^{\mathrm{can}})$ if and only if $(\tau_A \circ 0_U)^*(\Psi_U^*(\omega_U^{\mathrm{PGL}})) = (\tau_A \circ 0_U)^*(\omega_U^{\mathrm{can}})$. Here, after possibly replacing U by a scheme which is étale and dominant over U, we may suppose that U is affine and the vector bundle $\Omega_{U/K}$ is globally free. Under this assumption, $\Psi_U^*(\omega_U^{\mathrm{PGL}}) = \omega_U^{\mathrm{can}}$ if and only if $(\tau_{A'} \circ 0_U)^*(\Psi_U^*(\omega_U^{\mathrm{PGL}})) = (\tau_{A'} \circ 0_U)^*(\omega_U^{\mathrm{can}})$ for all $A' \in \Gamma(U, \Omega_{U/K})$. Thus, it suffices to prove the equality

$$\sigma_U^*(\omega_U^{\mathrm{PGL}}) = 0_U^*(\omega_U^{\mathrm{can}}).$$

5.3. To this end, we first consider the right-hand side (i.e., $0_U^*(\omega_U^{\mathrm{can}})$) of the required equality. The zero section $0_U : U \to T^\vee U$ yields a split injection of the natural exact sequence

$$0 \to 0_U^*(\mathcal{T}_{T^\vee U/U}) \to 0_U^*(\mathcal{T}_{T^\vee U/K}) \to \mathcal{T}_{U/K} \to 0,$$

and hence, a decomposition

$$\begin{aligned} 0_U^*(\mathcal{T}_{T^\vee U/K}) &\xrightarrow{\sim} \mathcal{T}_{U/K} \oplus 0_U^*(\mathcal{T}_{T^\vee U/U}) \\ &\xrightarrow{\sim} \mathcal{T}_{U/K} \oplus \Omega_{U/K} \\ &\xrightarrow{\sim} \mathbb{R}^1 f_{U*}(\mathcal{T}_{\mathcal{C}_U/U}) \oplus f_{U*}(\Omega_{\mathcal{C}_U/U}^{\otimes 2}) \end{aligned}$$

(cf. the beginning of § 4.2 for the last isomorphism). The $\mathcal{O}_U$-bilinear map on $0_U^*(\mathcal{T}_{T^\vee U/K})$ determined by ω_U^{can} is given, via this decomposition, by the pairing $\langle -, - \rangle : \mathbb{R}^1 f_{U*}(\mathcal{T}_{\mathcal{C}_U/U}) \times f_{U*}(\Omega_{\mathcal{C}_U/U}^{\otimes 2}) \to \mathcal{O}_U$ arising from Serre duality. That is, this bilinear map may be expressed as assiging

$$((a, b), (a', b')) \mapsto \langle a, b' \rangle - \langle a', b \rangle$$

for local sections a, $a' \in \mathbb{R}^1 f_{U*}(\mathcal{T}_{\mathcal{C}_U/U})$ and b, $b' \in f_{U*}(\Omega_{\mathcal{C}_U/U}^{\otimes 2})$.

5.4. Next, we consider the left-hand side (i.e., $\sigma_U^*(\omega_U^{\mathrm{PGL}})$) of the required equality. The section σ_U $(= \Psi_U \circ 0_U) : U \to \mathcal{S}_U$ yields a split injection of the exact sequence

$$0 \to \sigma_U^*(\mathcal{T}_{\mathcal{S}_U/U}) \to \sigma_U^*(\mathcal{T}_{\mathcal{S}_U/K}) \to \mathcal{T}_{U/K} \to 0.$$

If we denote $\mathcal{C}_{U/U}^{\text{Zzz...}} = (\mathcal{C}_U/U, \mathcal{P}^{\circledast} = (\mathcal{P}_B, \nabla))$ the ordinary dormant curve classified by u (hence $\gamma^{\heartsuit}_{\mathcal{P}^{\circledast}}$ is an isomorphism), then it follows from Proposition 2.5.1 that this split injection corresponds to a split injection of the exact sequence

$$0 \to f_{U*}(\Omega^{\otimes 2}_{\mathcal{C}_U/U}) \overset{\gamma^{\sharp}_{\mathcal{P}^{\circledast}}}{\to} \mathbb{R}^1 f_{U*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}}) \overset{\gamma^{\flat}_{\mathcal{P}^{\circledast}}}{\to} \mathbb{R}^1 f_{U*}(\mathcal{T}_{\mathcal{C}_U/U}) \to 0.$$

Moreover, if we identify $\mathbb{R}^1 f_{U*}(\mathcal{T}_{\mathcal{C}_U/U})$ with $\mathbb{R}^1 f_{U*}(\mathrm{Ker}(\nabla_{\mathrm{ad}}))$ via the isomorphism $\gamma^{\heartsuit}_{\mathcal{P}^{\circledast}}$, then it follows from Proposition 3.4.1 that it also coincides with the split injection determined by $\gamma^{\natural}_{\mathcal{P}^{\circledast}} : \mathbb{R}^1 f_{U*}(\mathrm{Ker}(\nabla_{\mathrm{ad}})) \to \mathbb{R}^1 f_{U*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}})$. Consider the resulting decomposition

$$\mathbb{R}^1 f_{U*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}}) \overset{\sim}{\to} \mathbb{R}^1 f_{U*}(\mathcal{T}_{\mathcal{C}_U/U}) \oplus f_{U*}(\Omega^{\otimes 2}_{\mathcal{C}_U/U}).$$

By the discussion in § 5.3, the proof of Theorem 4.3.1 reduces to the following

Lemma 5.4.1.
The $\mathcal{O}_U$-bilinear map on $\mathbb{R}^1 f_{U*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}})$ corresponding to ω_U^{PGL} (cf. § 4.1) is given, via the decomposition

$$\mathbb{R}^1 f_{U*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}}) \overset{\sim}{\to} \mathbb{R}^1 f_{U*}(\mathcal{T}_{\mathcal{C}_U/U}) \oplus f_{U*}(\Omega^{\otimes 2}_{\mathcal{C}_U/U})$$

just discussed, by the paring $\mathbb{R}^1 f_{U*}(\mathcal{T}_{\mathcal{C}_U/U}) \otimes_{\mathcal{O}_U} f_{U*}(\Omega^{\otimes 2}_{\mathcal{C}_U/U}) \to \mathcal{O}_U$ arising from Serre duality (in the sense of § 5.3).

Proof. The subsheaf $\mathrm{ad}(\mathcal{P}_G)^1 \subseteq \mathrm{ad}(\mathcal{P}_G)$ is isotropic with respect to $\kappa_{\mathcal{P}^{\circledast}} : \mathrm{ad}(\mathcal{P}_G) \times \mathrm{ad}(\mathcal{P}_G) \to \mathcal{O}_{\mathcal{C}_U}$ (cf. § 4.1), so $\kappa_{\mathcal{P}^{\circledast}}$ induces an $\mathcal{O}_{\mathcal{C}_U}$-bilinear morphism $\mathrm{ad}(\mathcal{P}_G)/\mathrm{ad}(\mathcal{P}_G)^1 \times \mathrm{ad}(\mathcal{P}_G)^2 \to \mathcal{O}_{\mathcal{C}_U}$. By passing to the isomorphisms $\gamma_0^{\dagger} : \mathrm{ad}(\mathcal{P}_G)/\mathrm{ad}(\mathcal{P}_G)^1 \overset{\sim}{\to} \mathcal{T}_{\mathcal{C}_U/U}$ and $\gamma_2^{\dagger} : \mathrm{ad}(\mathcal{P}_G)^2 \overset{\sim}{\to} \Omega_{\mathcal{C}_U/U}$ (and by considering the definition of κ), we may identify this bilinear morphism with the natural paring $\mathcal{T}_{\mathcal{C}_U/U} \times \Omega_{\mathcal{C}_U/U} \to \mathcal{O}_{\mathcal{C}_U}$. Thus the assertion follows from the definition of the bilinear map on $\mathbb{R}^1 f_{U*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}})$ and, e.g., the explicit description of $\mathbb{R}^1 f_{U*}(\mathcal{K}^{\bullet}_{\nabla_{\mathrm{ad}}})$ in terms of the Čech double complex. □

6. Application of Theorem A

As an application of Theorem 4.3.1, we construct certain additional structures on $\mathcal{S}_{\circledcirc \mathcal{M}^{\text{Zzz...}}_{g,K}}$.

6.1. Let us fix a field K of characteristic $p > 2$, a smooth Deligne-Mumford stack X over K, and a symplectic structure ω on X. ω corresponds to a non-degenerate pairing $\mathcal{T}_{X/K} \otimes_{\mathcal{O}_X} \mathcal{T}_{X/K} \rightarrow \mathcal{O}_X$ on $\mathcal{T}_{X/K}$ and gives an identification $\mathcal{T}_{X/K} \xrightarrow{\sim} \mathcal{T}_{X/K}^{\vee} = \Omega_{X/K}$. By applying this identification, ω may be thought of as a nondegenerate pairing $\omega^{-1} : \Omega_{X/K} \otimes_{\mathcal{O}_X} \Omega_{X/K} \rightarrow \mathcal{O}_X$. Thus, we obtain a skew-symmetric K-bilinear map

$$\{-,-\}_\omega : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$$

defined by $\{f,g\}_\omega = \omega^{-1}(df, dg)$. One verifies from the closedness of ω that $\{-,-\}_\omega$ is a Poisson bracket.

Definition 6.1.1.
A *restricted structure* on the pair (X, ω) is a map $(-)^{[p]} : \mathcal{O}_X \rightarrow \mathcal{O}_X$ such that the triple $(\mathcal{O}_X, \{-,-\}_\omega, (-)^{[p]})$ forms a sheaf of restricted Poisson algebras over K. (cf. [5], Definition 1.8 for the definition of a restricted Poisson algebra).

Next, we recall the definition of a Frobenius-constant quantization (cf. [4], Definition 3.3; [5], Definition 1.1 and Definition 1.4). In Definition 6.1.2 below, $X^{(1)}$ denotes the Frobenius twist of X over K (i.e., the base-change of the K-scheme X via the absolute Frobenius morphism of K), and $F : X \rightarrow X^{(1)}$ denotes the relative Frobeius morphism of X over K.

Definition 6.1.2.

(i) Consider a pair $(\mathcal{O}_X^\hbar, \tau)$ consisting of
- a Zariski sheaf $\mathcal{O}_X^\hbar$ of flat $k[[\hbar]]$-algebras on X complete with respect to the $\hbar$-adic filtration, and
- an isomorphism $\tau : \mathcal{O}_X^\hbar/\hbar \xrightarrow{\sim} \mathcal{O}_X$ of sheaves of algebras.

We shall say that the pair $(\mathcal{O}_X^\hbar, \tau)$ is a *quantization* of the pair (X, ω) if the commutator in $\mathcal{O}_X^\hbar$ equals $\hbar \cdot \{-,-\}$ mod $\hbar^2 \cdot \mathcal{O}_X^\hbar$

(ii) A *Frobenius-constant quantization* on (X, ω) is a collection of data

$$\mathfrak{O}^\hbar = (\mathcal{O}^\hbar, \tau, s)$$

consisting of a quantization $(\mathcal{O}_\hbar, \tau)$ of (X, ω) and a morphism $s : \mathcal{O}_{X^{(1)}} \rightarrow \mathcal{Z}^\hbar (\subseteq \mathcal{O}_X^\hbar)$ of sheaves of algebras, where $\mathcal{Z}^\hbar$ denotes the center of $\mathcal{O}_X^\hbar$, whose composite with the morphism $\mathcal{Z}^\hbar \rightarrow (\mathcal{O}_X^\hbar \twoheadrightarrow \mathcal{O}_X^\hbar/\hbar \xrightarrow{\tau}) \mathcal{O}_X$ coincides with the morphism $F^* : \mathcal{O}_{X^{(1)}} \hookrightarrow \mathcal{O}_X$.

See [5], the discussion at the end of § 1.2 or Theorem 1.23 for relationships between the notion of a restricted structure and a Frobenius-constant quantization.

Example 6.1.3.
Let S be a smooth Deligne-Mumford stack over K. One may construct naturally a restricted structure and a Frobenius-constant quantization on the cotangent bundle $T^\vee S$ equipped with the symplectic structure ω_S^{can} as follows.

(i) Let us define a map

$$(-)^{[p]} : \mathcal{O}_{T^\vee S} \rightarrow \mathcal{O}_{T^\vee S}$$

of sheave on $T^\vee S$ as follows: if f is a local section lifted from $\mathcal{O}_S$, then $f^{[p]} = 0$, and if ∂ is a local section lifted from $\mathcal{T}_S$, then $\partial^{[p]}$ is the p-th iterate of ∂. Then, by [5], the discussion at the end of § 1.2 and (ii) below, the map $(-)^{[p]}$ forms a restricted structure on $(T^\vee S, \omega_S^{\mathrm{can}})$.

(ii) Suppose that S is affine. The *sheaf of asymptotic differential operators* $D^\hbar(S)$ on S (cf. [4], Example 3.1) is the $\hbar$-completion of the $k[\hbar]$-algebras generated by $\Gamma(S, \mathcal{O}_S)$ and $\Gamma(S, \mathcal{T}_{S/K})$ subject to the following relations:
- $f_1 \cdot f_2 = f_1 f_2$,
- $f_1 \cdot \xi_1 = f_1 \xi_1$,
- $\xi_1 \cdot f_1 - f_1 \cdot \xi_1 = \hbar \xi_1(f_1)$,
- $\xi_1 \cdot \xi_2 - \xi_2 \cdot \xi_2 = \hbar[\xi_1, \xi_2]$.

for sections f_1, $f_2 \in \Gamma(S, \mathcal{O}_S)$ and ξ_1, $\xi_2 \in \Gamma(S, \mathcal{T}_{S/K})$. We have a natural isomorphism $\tau(S) : D^\hbar(S)/\hbar D^\hbar(S) \xrightarrow{\sim} \Gamma(S, \mathcal{O}_{T^\vee S})$. Also, the natural composite $\Gamma(S^{(1)}, \mathcal{O}_{S^{(1)}}) \rightarrow \Gamma(S, \mathcal{O}_S) \rightarrow D^\hbar(S)$ factors through the inclusion $Z^\hbar(S) \rightarrow D^\hbar(S)$, where $Z^\hbar(S)$ denotes the center of $D^\hbar(S)$. We denote by $s(S) : \Gamma(S^{(1)}, \mathcal{O}_{S^{(1)}}) \rightarrow Z^\hbar(S)$ the resulting morphism. By applying a natural noncommutative localization procedure called *Ore localization* (cf. [15]), we obtain from the triple $(D^\hbar, \tau(S), s(S))$ a Frobenius-constant quantization

$$(\mathcal{D}_S^\hbar, \tau, s)$$

on $(T^\vee S, \omega_S^{\mathrm{can}})$ (cf. [4], Proposition 3.5). In general, by gluing the above construction, one may obtains a Frobenius-constant quantization for any smooth Deligne-Mumford stack S.

6.2. By applying the discussion in Example 6.1.3 to the case $S = {}^{\circledcirc}\mathcal{M}_{g,K}^{\mathrm{Zzz...}}$, we obtain a restricted structure as well as a Frobenius-constant quantization on $(T^{\vee \circledcirc}\mathcal{M}_{g,K}^{\mathrm{Zzz...}}, \omega_{\circledcirc}^{\mathrm{can}})$. Such additional structures may be evidently transported via an isomorphism $\Psi_{\sigma_{g,K}}$, that is compatible with the respective symplectic structure (by Theorem 4.3.1), into $(\mathcal{S}_{{}^\circledcirc\mathcal{M}_{g,K}^{\mathrm{Zzz...}}}, \omega_{\circledcirc}^{\mathrm{PGL}})$. Thus, we have the following

Corollary 6.2.1.
If p is sufficiently large, then there exist canonical restricted structure and Frobenius-constant quantization on $(\mathcal{S}_{{}^\circledcirc\mathcal{M}_{g,K}^{\mathrm{Zzz...}}}, \omega_{\circledcirc}^{\mathrm{PGL}})$.

References

[1] P. Arés-Gastesi, I. Biswas, On the symplectic form of the moduli space of projective structures. *J. Symplectic. Geom.* **6**, (2008), pp. 239-246.
[2] P. Arés-Gastesi, I. Biswas, On the symplectic structure over a moduli space of orbifold projective structures. *math.AG/1308.3353*, (2013) .
[3] L. Bers, Fiber spaces over Teichmüller spaces. *Acta Math.* **130** (1973), pp. 89-126.

[4] R. Bezrukavnikov, D. Kaledin, Mckay equivalence for symplectic quotient singularities. *Proc. of the Steklov Inst. of Math.* **246**, (2004), pp. 13-33.
[5] R. Bezrukavnikov, D. Kaledin. Fedosov quantization in positive characteristic. *math.AG/0405575*, (2004).
[6] I. Biswas, Schottky uniformization and the symplectic structure of the cotangent bundle of a Teichmüller space. *J. Geom. Phys.* **35**, (2000), pp. 57-62.
[7] C. J. Earle, Some intrinsic coordinates on Teichmüller space. *Proc. Amer. Math. Soc.* **83** (1981), pp. 527-531.
[8] E. Frenkel, *Langlands Correspondence for Loop Groups.* Cambridge Studies in Advanced Mathematics **103** Cambridge Univ. Press (2007).
[9] W. M. Goldman, The symplectic nature of fundamental group of surfaces. *Adv. Math.* **54** (1984), pp. 200-225.
[10] A. Grothendieck, On the de Rham cohomology of algebraic varieties. *Publication Mathématique IHES* **29**, (1966), pp. 95-103.
[11] R. C. Gunning, Special coordinate covering of Riemann surfaces. *Math. Ann.* **170** (1967), pp. 67-86.
[12] R. Hartshorn, On the de Rham cohomology of algebraic varieties. *Publication Mathématique IHES* **45**, (1971), pp. 1-99.
[13] L. Illusie, Frobenius and Hodge degeneration. *Introduction to Hodge theory. Translated from the 1996 French original by James Lewis and Peters. SMF/AMS Texts and Monographs,* **8**, Amer. Math. Soc., Providence, RI; Société Mathématique de France, Paris, (2002), x+232 pp.
[14] R. Källström, Smooth modules over Lie algebroids I. *math. AG/9808108* (1998).
[15] M. Kapranov, Noncommutative geometry based on commutator expansions. *J. Reine Angew. Math.* **505** (1998), pp. 73-118.
[16] N. M. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Etudes Sci. Publ. Math.* **39** (1970), pp. 175-232.
[17] S. Kawai, The symplectic nature of the space of projective connections on Riemann surfaces. *Math. Ann.* **305** (1996), pp. 161-182.
[18] G. Laumon, L. Moret-Baily, *Champs Algébriques.* Ergebnisse der Mathematik un ihrer Grenzgebiete **39**, Springer-Verlag, (2000).
[19] S. Mochizuki, A theory of ordinary p-adic curves. *Publ. RIMS* **32** (1996), pp. 957-1151.
[20] S. Mochizuki, *Foundations of p-adic Teichmüller theory.* American Mathematical Society, (1999).
[21] B. Osserman, The generalized Vershiebung map for curves of genus 2. *Math. Ann.* **336** (2006), pp. 963-986.
[22] B, Osserman, Mochizuki's crys-stable bundles: a lexicon and applications. *Publ. Res. Inst. Math. Sci.* **43**, (2007), no. 1, 95-119.
[23] Y. Wakabayashi, An explicit formula for the generic number of dormant indigenous bundles. *RIMS Preprint* **1766** (2013).