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Monomorphisms in Categories of Log Schemes

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Abstract. In the present paper, we study *category-theoretic properties* of *monomorphisms* in categories of log schemes. This study allows one to give a *purely category-theoretic reconstruction* of the *log scheme* that gave rise to the category under consideration. We also obtain analogous results for categories of schemes of locally finite type over the ring of rational integers that are equipped with *"archimedean structures"*. Such reconstructions were discussed in two previous papers by the author, but these reconstructions contained some errors, which were pointed out to the author by C. Nakayama and Y. Hoshi. These errors revolve around certain elementary *combinatorial aspects* of *fan decompositions of two-dimensional rational polyhedral cones* — i.e., of the sort that occur in the classical theory of *toric varieties* — and may be repaired by applying the theory developed in the present paper.

Contents:

Introduction

The purpose of the present paper is to study, in some detail, various aspects of the structure of **categories of log schemes** that revolve around the behavior of **monomorphisms** in such categories. This study leads naturally to a *purely category-theoretic reconstruction* of the *log scheme* that gave rise to the category under consideration. Our main result is the following [cf. Theorem 3.8, (iii)].

Theorem A. **(Category-theoretic reconstruction of log schemes)** *For $i = 1, 2$, let $X_i^{\log}$ be a* **locally noetherian fs log scheme** *[cf. the discussion entitled "Log schemes" in §0]. For $i = 1, 2$, we shall write* $\mathrm{Sch}^{\log}(X_i^{\log})$ *for the*

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category of noetherian fs log schemes of finite type over $X_i^{\log}$ *and morphisms of finite type [cf. the discussion at the beginning of* §*1 for more details]. Let*

$$\Phi : \mathrm{Sch}^{\log}(X_1^{\log}) \xrightarrow{\sim} \mathrm{Sch}^{\log}(X_2^{\log})$$

be an [arbitrary!] **equivalence of categories**. *Then there exists a* **unique isomorphism of log schemes**

$$X_1^{\log} \xrightarrow{\sim} X_2^{\log}$$

such that Φ *is isomorphic to the equivalence of categories induced by this isomorphism of log schemes* $X_1^{\log} \xrightarrow{\sim} X_2^{\log}$.

We also obtain analogous results for categories of *locally noetherian fs schemes*

$$\text{“}\mathrm{SCH}^{\log}(-)\text{”}$$

[cf. Theorem 4.6, (iv)], as well as for versions

$$\text{“}\overline{\mathrm{Sch}}^{\log}(-)\text{”}, \ \text{“}\overline{\mathrm{SCH}}^{\log}(-)\text{”}$$

of the categories “$\mathrm{Sch}^{\log}(-)$”, “$\mathrm{SCH}^{\log}(-)$” for schemes of locally finite type over $\mathbb{Z}$ that are equipped with *“archimedean structures”* [cf. Theorem 4.8, (iv)].

The theory exposed in the present paper arose as an attempt to correct *errors*, pointed out to the author by *Chikara Nakayama* and *Yuichiro Hoshi* in June 2013, in the theory of [LgSch], §2. These errors concern the *category-theoretic properties* of *monomorphisms* in categories of log schemes and are discussed in more detail in Example 0.3 and Remark 1.4.1 of the present paper.

At the level of *main results* of the paper [LgSch], these errors in the theory of [LgSch], §2, do not affect the proof of [LgSch], Theorem A, given in [LgSch], §1, but they *do* affect the *proof* — although *not the validity*! — of [LgSch], Theorem B. This result [LgSch], Theorem B, is given a *correct proof* in §3 of the present paper and corresponds precisely to Theorem A [stated above].

At the level of *main results* of papers of the author *subsequent* to [LgSch], the *only place* where the errors in the theory of [LgSch], §2, have an effect is in the portion of the proof of the *main result* of [ArLgSch] [i.e., [ArLgSch], Theorem 3.4] that involves the theory of [ArLgSch], §2. The affected portions of [ArLgSch], §2, are discussed in more detail in the introduction to §4 of the present paper. The main result [ArLgSch], Theorem 3.4, of [ArLgSch] is given a *correct proof* in §4 of the present paper and corresponds precisely to Theorem 4.8, (iv) [quoted above].

One important invariant of the structure of an fs log scheme is the *rank* of the groupification of the fiber of the characteristic sheaf associated to the log structure at a *geometric point* of the underlying scheme of the log scheme [cf. Definition 1.2, (i)]. For instance, when this *rank* is equal to 0 at all geometric points, the log structure of the fs log scheme under consideration is *trivial*. One *central theme* of the theory of the present paper consists of the phenomenon that

> *the theory of category-theoretic properties of* **monomorphisms** *exhibits quite substantive* **qualitative differences**, *depending upon whether or not it holds that the* **ranks** *just referred to are* ≤ 1.

When it holds that these rank are ≤ 1, the fs log scheme under consideration will be referred to in the present paper as **submonic** [cf. Definition 1.2, (i)].

Thus, in some sense, the *simplest "borderline case"* between submonic and non-submonic fs log schemes is the case of a log scheme whose underlying scheme is the *spectrum of a field* whose absolute Galois group acts *trivially* on geometric fibers of the characteristic sheaf associated to the log structure, and for which the *rank* of the groupification of each such geometric fiber of the characteristic sheaf is *equal to* 2. In this case, the log scheme under consideration will be referred to as **log-nodal** [cf. Definition 1.2, (i)].

One important feature of the *category-theoretic properties of* **monomorphisms** in categories of log schemes lies in the observation that

> these category-theoretic properties of monomorphisms take on a particularly **straightforward** and **intuitive** form whenever it holds that the various fs log schemes under consideration are all **submonic**.

This observation is one of the main themes of the theory discussed in §1 of the present paper. Roughly speaking, the errors pointed out by Nakayama and Hoshi in the theory of [LgSch], §2, may be *summarized* as follows:

> the author wrote [LgSch], §2, under the **misunderstanding** that this "straightforward" and "intuitive" approach to category-theoretic properties of monomorphisms holds *even if* the various fs log schemes under consideration are **not necessarily submonic**.

On the other hand, it turns out [cf. the theory of §2 of the present paper] that the various **complications** that occur in the study of the category-theoretic properties of monomorphisms of *arbitrary* non-submonic fs log schemes already appear in the case of **log-nodal** fs log schemes. Moreover, it turns out that

> these complications essentially revolve around various **combinatorial aspects** of **fan decompositions** *of two-dimensional rational polyhedral cones*, i.e., of the sort that occur in the classical theory of *toric varieties*.

These *elementary combinatorial aspects* are reviewed in §0 of the present paper.

The theory developed in the present paper may be summarized as follows. In §1, we introduce basic terminology and discuss various generalities concerning *monomorphisms* in categories of log schemes. In particular, we discuss [cf., especially, Lemma 1.5] how the elementary combinatorics of two-dimensional fan decompositions reviewed in §0 may be *interpreted* in the context of categories of log schemes. In §2, we apply these elementary combinatorics of two-dimensional fan decompositions [cf. Proposition 2.3] to show, in effect, that certain **connectedness** properties of such fan decompositions allow one to give a *category-theoretic characterization of* **submonic** *fs log schemes*. We then proceed to give, in Theorem 2.6, a *category-theoretic reconstruction of the* **scheme structure** *of a submonic fs log scheme*. This reconstruction is quite "straightforward" and "intuitive" and amounts, in essence, to an application of the techniques of [LgSch], §2. In the remainder of §2, we show [cf. Corollary 2.12] that the various complications that arise in the case of arbitrary non-submonic fs log schemes amount, in essence, to the issue of giving a *category-theoretic algorithm* that allows one

> to **distinguish** a **log-nodal** fs log scheme from a nontrivial **log étale localization** of such a log-nodal fs log scheme [i.e., of the sort that arises from a nontrivial two-dimensional fan decomposition].

Such a category-theoretic algorithm is furnished, in effect, by the theory of **seamless partitions** *of orientable log schemes* developed in §3 [cf. Theorem 3.6]. This theory may be regarded as a **translation** into category theory of the **elementary observation** that

> a *nontrivial* two-dimensional fan decomposition may be distinguished from a *trivial* two-dimensional fan decomposition by considering the **"seamless partition"** constituted by the various constituent cones of the fan decomposition.

Finally, in §4, we observe that the theory developed in §1, §2, §3 may be generalized, without any essential complications, to the case of fs log schemes of locally finite type over $\mathbb{Z}$ that are equipped with *archimedean structures* [cf. Theorems 4.3, 4.8]. Such generalizations allow one to avoid the difficulties that arise from applying the *erroneous* portions of [LgSch], §2, in the theory of [ArLgSch], §2, i.e., by, in essence, isolating the [easily resolved] *submonic* aspects of these difficulties from the [more subtle!] *non-submonic* aspects of these difficulties.

Acknowledgements:

This paper owes its existence to the discovery by *Chikara Nakayama* and *Yuichiro Hoshi* of various *errors* [cf. Example 0.3; Remark 1.4.1] in the arguments of [LgSch], §2. The author wishes to express his gratitude to Nakayama and Hoshi for their careful reading of [LgSch].

Section 0: Notations and Conventions

Numbers:

We will denote by $\mathbb{N}$ the set of *natural numbers*, by which we mean the set of integers $n \geq 0$, and by $\mathbb{Z}$ the *ring of rational integers*. By a slight abuse of notation, we shall also use the notation $\mathbb{N}$, $\mathbb{Z}$ to denote the corresponding *monoids*. We shall denote by $\mathbb{Q}_{\geq 0}$ the *additive monoid of nonnegative rational numbers*.

Generalities on monoids:

We shall refer to a finitely generated, saturated [cf. [Kato2], §1.1] monoid that has no nonzero invertible elements as an *fs monoid*. Thus, if P is an fs monoid, then the natural homomorphism of monoids $P \to P^{\mathrm{gp}}$ from P to its *groupification* P^{gp} is *injective*, and P^{gp} is a finitely generated free abelian group. We shall refer to the rank of P^{gp} as the *rank* $\mathrm{rk}(P)$ *of the fs monoid* P.

A homomorphism of monoids $\phi : P \to Q$ between monoids P, Q will be called *positive* if ϕ maps every nonzero element of P to a nonzero element of Q. A nonzero element $a \in P$ of a monoid P will be called a *sum-dominator* if there exists a positive integer n such that $n \cdot a$ may be written as the sum of a finite collection of generators of P. Thus, if $\phi : P \to Q$ is a *nonzero* homomorphism [i.e., a homomorphism that maps any collection of generators of P to a subset of Q that contains at least one nonzero element!] from an arbitrary monoid P to an *fs monoid* Q, and $a \in P$ is a *sum-dominator*, then $\phi(a) \neq 0$. We shall say that a homomorphism of monoids $\phi : P \to Q$ is *sum-dominating* if it maps every nonzero element of P to a sum-dominator of Q. Thus, a *sum-dominating* homomorphism from an arbitrary monoid to a *nonzero fs monoid* is necessarily *positive*.

Let P be an *fs monoid*. Thus, in the terminology of the discussion entitled "Monoids" of [FrdI], §0, P is *sharp*, *integral*, and *saturated*. In particular, it makes sense to speak of the *perfection* P^{pf} of P, as well as of the set of *primes* $\mathrm{Prime}(P)$ of P — cf. the discussion entitled "Monoids" of [FrdI], §0, for more details.

Rank two fs monoids:

Now let us suppose that P is an *fs monoid of rank two*. Then we recall that there exists an *isomorphism of monoids*

$$P^{\mathrm{pf}} \ \stackrel{\sim}{\to}\ \mathbb{Q}_{\geq 0} \ \oplus \ \mathbb{Q}_{\geq 0}$$

[cf. [ExtFam], Proposition 1.7]. In particular, one verifies immediately that the set of primes $\mathrm{Prime}(P) = \mathrm{Prime}(P^{\mathrm{pf}})$ is of *cardinality two*. Write $\mathrm{Prime}(P) = \mathrm{Prime}(P^{\mathrm{pf}}) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Thus, for each $i = 1, 2$, $\mathfrak{p}_i$ may be regarded as a collection of elements of P^{pf}, which generates a submonoid $P^{\mathrm{pf}}_{\mathfrak{p}_i} \subseteq P^{\mathrm{pf}}$. For simplicity, let us write $P_i \stackrel{\mathrm{def}}{=} P^{\mathrm{pf}}_{\mathfrak{p}_i}$. Then one verifies immediately that the *two direct summands* of the codomain of the isomorphism of the above display correspond precisely to P_1, P_2, i.e., we have a *natural isomorphism*

$$P_1 \oplus P_2 \stackrel{\sim}{\to} \ P^{\mathrm{pf}}$$

and noncanonical isomorphisms of abstract monoids $P_1 \cong P_2 \cong \mathbb{Q}_{\geq 0}$. In particular, these two direct summands are *preserved*, up to *possible permutation*, by any automorphism of the monoid P^{pf}. Note that [since the monoid $\mathbb{Q}_{\geq 0}$ has *no nontrivial automorphisms of finite order*] these observations imply that

> any *finite subgroup* of $\mathrm{Aut}(P^{\mathrm{pf}})$ — or, indeed, of $\mathrm{Aut}(P)$ ($\hookrightarrow \mathrm{Aut}(P^{\mathrm{pf}})$) — is *of order* ≤ 2.

Next, let

$$\phi_0 : P \to J_0 \stackrel{\mathrm{def}}{=} \mathbb{N}$$

be a *positive homomorphism* that induces a surjection on groupifications $\phi_0^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow J_0^{\mathrm{gp}} = \mathbb{Z}$. Thus, $\mathrm{Ker}(\phi_0^{\mathrm{gp}}) \cong \mathbb{Z}$. Fix a *nonzero* element $a \in \mathrm{Ker}(\phi_0^{\mathrm{gp}}) \subseteq P^{\mathrm{gp}}$. For $i = 1, 2$, write

$$(P \ \subseteq)\ J_i \ \subseteq \ P^{\mathrm{gp}}$$

for the *saturation* [cf. [LgSch], Lemma 2.5, (ii)] of the submonoid of P^{gp} generated by P and a if $i = 1$ (respectively, $-a$ if $i = 2$) and

$$\phi_i : P \hookrightarrow J_i$$

for the natural inclusion. Thus, $P^{\mathrm{gp}} = J_i^{\mathrm{gp}}$ for $i = 1, 2$. One verifies immediately that, up to a *possible permutation* of the *indices* "1" and "2", the submonoids J_1 and J_2 of P^{gp} are *independent* of the choice of a. Moreover, we observe that it follows immediately from the definition of J_1 and J_2 that

> if $i = 0$ (respectively, $i = 1$, $i = 2$), then a *positive homomorphism* $\phi : P \to \mathbb{N}$ *factors*, via $\phi_i : P \to J_i$, through a *positive homomorphism* $J_i \to \mathbb{N}$ if and only if the homomorphism induced on groupifications $\phi^{\mathrm{gp}} : P^{\mathrm{gp}} \to \mathbb{Z}$ satisfies the condition $\phi^{\mathrm{gp}}(a) = 0$ (respectively, $\phi^{\mathrm{gp}}(a) > 0$; $\phi^{\mathrm{gp}}(a) < 0$).

In this situation, we shall refer to J_1 and J_2 as *bisecting monoids* of P at ϕ_0.

Before proceeding, we observe the following *"continuity property"* of bisecting monoids:

> Suppose that $P^* \subseteq P^{\mathrm{gp}}$ is a *rank two fs monoid* that arises as a *submonoid* of P^{gp} that *contains* P. For $i = 1, 2$, suppose that $\psi_i : P^* \to \mathbb{N}$ is a *homomorphism* whose restriction to P *factors*, via $\phi_i : P \to J_i$, through a *positive homomorphism* $J_i \to \mathbb{N}$. Then $\phi_0 : P \to \mathbb{N}$ *extends* to a *positive homomorphism* $\psi_0 : P^* \to \mathbb{N}$.

Indeed, if ϕ_0 does *not* admit such an extension ψ_0, then it follows that there exist *nonzero* elements $b \in P$, $c \in P^*$ such that $a + b + c = 0$ for some element $a \in \mathrm{Ker}(\phi_0^{\mathrm{gp}}) \subseteq P^{\mathrm{gp}}$. Then it follows from the above discussion of bisecting monoids that, for some $i \in \{1, 2\}$, $\psi_i^{\mathrm{gp}}(a) \geq 0$. Since the restriction of ψ_i to P is a *positive homomorphism*, we thus conclude that $0 = \psi_i^{\mathrm{gp}}(a) + \psi_i^{\mathrm{gp}}(b) + \psi_i^{\mathrm{gp}}(c) > 0 \in \mathbb{N}$, a *contradiction*. This completes the proof of this "continuity property".

Bisecting monoids may be understood more *explicitly* if one passes to *perfections*. Indeed, by restricting our attention to *perfections*, one verifies immediately that we may assume without loss of generality that

$$P^{\mathrm{pf}} = \mathbb{Q}_{\geq 0} \oplus \mathbb{Q}_{\geq 0}, \quad P_1 = \mathbb{Q}_{\geq 0} \oplus 0, \quad P_2 = 0 \oplus \mathbb{Q}_{\geq 0},$$

and that $\phi_0^{\mathrm{pf}} : P^{\mathrm{pf}} \to \mathbb{Q}_{\geq 0}$ is the homomorphism determined by sending $(1, 0)$ and $(0, 1)$ to 1. Then one computes easily that, if one takes $a \stackrel{\mathrm{def}}{=} (1, -1)$, then J_1^{pf} is equal to the perfection of the submonoid of $(P^{\mathrm{pf}})^{\mathrm{gp}} = \mathbb{Q} \oplus \mathbb{Q}$ generated by $(0, 1)$ and $(1, -1)$, while J_2^{pf} is equal to the perfection of the submonoid of $(P^{\mathrm{pf}})^{\mathrm{gp}} = \mathbb{Q} \oplus \mathbb{Q}$ generated by $(1, 0)$ and $(-1, 1)$. Thus, if ϕ^{pf} maps

$$(1, 0) \mapsto a; \quad (0, 1) \mapsto b$$

for $a, b \in \mathbb{Q}_{\geq 0}$, then one verifies immediately that $\phi^{\mathrm{pf}} : P^{\mathrm{pf}} \to \mathbb{Q}_{\geq 0}$ factors, via $\phi_i^{\mathrm{pf}} : P^{\mathrm{pf}} \to J_i^{\mathrm{pf}}$, through a positive homomorphism $J_i^{\mathrm{pf}} \to \mathbb{Q}_{\geq 0}$

> for $i = 0$ (respectively, $i = 1$; $i = 2$) $\iff a = b$ (respectively, $a > b$; $a < b$).

In the present paper, we shall often consider certain *sequences of submonoids* satisfying certain special properties, as in the following examples.

Example 0.1. Submonoids converging from one side. Let P be an *fs monoid of rank two*, ${}^{\infty}P \subseteq P^{\mathrm{gp}}$ a *bisecting monoid* of P at some *positive homomorphism* ${}^{\infty}\phi : P \to \mathbb{N}$. Then there exists an *infinite descending sequence*

$$P \subseteq {}^{\infty}P \subseteq \ldots \subseteq {}^{n}P \subseteq \ldots \subseteq {}^{1}P \subseteq {}^{0}P$$

— where $n \in \mathbb{N}$ — of submonoids of P^{gp} such that *every positive* homomorphism $\phi : {}^{\infty}P \to \mathbb{N}$ factors through a *positive* homomorphism ${}^{n}P \to \mathbb{N}$ for *some* n [which may depend on ϕ], and, moreover, for each $m \in \mathbb{N}$, ${}^{m}P$ is a *bisecting monoid* of P [hence, in particular, an fs monoid of rank two] whose image ${}^{\infty}\phi^{\mathrm{gp}}({}^{m}P)$ via ${}^{\infty}\phi^{\mathrm{gp}} : P^{\mathrm{gp}} \to \mathbb{Z}$ contains both *positive* and *negative* elements. Indeed, by reasoning as in the above discussion, one reduces immediately to the verification — say, in the case where $P^{\mathrm{pf}} = \mathbb{Q}_{\geq 0} \oplus \mathbb{Q}_{\geq 0}$, ${}^{\infty}\phi^{\mathrm{pf}}$ is the homomorphism $P^{\mathrm{pf}} = \mathbb{Q}_{\geq 0} \oplus \mathbb{Q}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ given by $(a, b) \mapsto a+b$, and ${}^{\infty}P$ is the perfection of the submonoid of $\mathbb{Q} \oplus \mathbb{Q}$ generated by $(-1, 1)$ and $(1, 0)$ — of the existence of an infinite descending sequence

$$P^{\mathrm{pf}} \subseteq {}^{\infty}P^{\mathrm{pf}} \subseteq \ldots \subseteq {}^{n}P^{\mathrm{pf}} \subseteq \ldots \subseteq {}^{1}P^{\mathrm{pf}} \subseteq {}^{0}P^{\mathrm{pf}}$$

— where $n \in \mathbb{N}$ — of submonoids of $(P^{\mathrm{pf}})^{\mathrm{gp}}$ such that *every positive* homomorphism $\psi : {}^{\infty}P^{\mathrm{pf}} \to \mathbb{Q}_{\geq 0}$ factors through a *positive* homomorphism ${}^{n}P^{\mathrm{pf}} \to \mathbb{Q}_{\geq 0}$ for *some* n [which may depend on ϕ], and, moreover, for each $m \in \mathbb{N}$, ${}^{m}P^{\mathrm{pf}}$ is the perfection of a finitely generated submonoid of $\mathbb{Q} \oplus \mathbb{Q}$ such that ${}^{m}P \stackrel{\mathrm{def}}{=} {}^{m}P^{\mathrm{pf}} \cap P^{\mathrm{gp}}$ [so ${}^{m}P^{\mathrm{pf}}$ may be identified with the perfection of ${}^{m}P$, as the notation suggests!] is a *bisecting monoid* of P whose image ${}^{\infty}\phi^{\mathrm{gp}}({}^{m}P)$ contains both *positive* and *negative* elements. Such an infinite descending sequence may be obtained, for instance, by taking ${}^{n}P^{\mathrm{pf}}$ to be the perfection of the submonoid of $\mathbb{Q} \oplus \mathbb{Q}$ generated by $(-1, 1 - \frac{1}{n+2})$ and $(1, 0)$.

Example 0.2. Submonoids converging from the center. Let P be an *fs monoid of rank two*. Then there exists an *infinite descending sequence*

$$P \subseteq \ldots \subseteq {}^{n}P \subseteq \ldots \subseteq {}^{1}P \subseteq {}^{0}P$$

— where $n \in \mathbb{N}$ — of submonoids of P^{gp} such that *every positive* homomorphism $\phi : P \to \mathbb{N}$ factors through a *positive* homomorphism ${}^{n}P \to \mathbb{N}$ for *some* n [which may depend on ϕ], and, moreover, for each $m \in \mathbb{N}$, the inclusion $P \hookrightarrow {}^{m}P$ is a *sum-dominating* homomorphism of *fs monoids*. Indeed, by reasoning as in the above discussion, one reduces immediately to the verification, in the case where $P^{\mathrm{pf}} = \mathbb{Q}_{\geq 0} \oplus \mathbb{Q}_{\geq 0}$, of the existence of an infinite descending sequence

$$P^{\mathrm{pf}} \subseteq \ldots \subseteq {}^{n}P^{\mathrm{pf}} \subseteq \ldots \subseteq {}^{1}P^{\mathrm{pf}} \subseteq {}^{0}P^{\mathrm{pf}}$$

— where $n \in \mathbb{N}$ — of perfections of finitely generated submonoids of $(P^{\mathrm{pf}})^{\mathrm{gp}}$ such that *every positive* homomorphism $\psi : P^{\mathrm{pf}} \to \mathbb{Q}_{\geq 0}$ factors through a *positive* homomorphism ${}^{n}P^{\mathrm{pf}} \to \mathbb{Q}_{\geq 0}$ for *some* n [which may depend on ϕ], and, moreover, for each $m \in \mathbb{N}$, the inclusion $P \hookrightarrow {}^{m}P \stackrel{\mathrm{def}}{=} {}^{m}P^{\mathrm{pf}} \cap P^{\mathrm{gp}}$ [so ${}^{m}P^{\mathrm{pf}}$ may be identified with the perfection of ${}^{m}P$, as the notation suggests!] induced by the inclusion $P^{\mathrm{pf}} \hookrightarrow {}^{m}P^{\mathrm{pf}}$ is a *sum-dominating* homomorphism of *fs monoids*. Such an infinite descending sequence may be obtained, for instance, by taking ${}^{n}P^{\mathrm{pf}}$ to be the perfection of the submonoid of $\mathbb{Q} \oplus \mathbb{Q}$ generated by $(1, -\frac{1}{n+2})$ and $(-\frac{1}{n+2}, 1)$. Finally, we observe that this explicit construction shows that the ${}^{n}P$ may be chosen so as to be *preserved* by any *finite group of automorphisms* of P.

Log schemes:

If X is a *scheme*, then we shall write

$$X_{\text{red}} \subseteq X$$

for the closed subscheme determined by equipping the underlying topological space of the scheme X with the reduced induced scheme structure. If X is the underlying scheme of a *log scheme* $X^{\log}$ [cf. [Kato1], §1.2], then we shall write $X^{\log}_{\text{red}}$ for the log scheme determined by restricting the log structure of $X^{\log}$ to $X_{\text{red}} \subseteq X$.

We shall use the terms *log étale* (respectively, *log smooth*) to refer to morphisms between log schemes which are "étale" (respectively, "smooth") in the sense of [Kato1], §3.3.

We use the term *"fs log scheme"* to refer to a *log scheme* which is *fine* [cf. [Kato1], §2.3] and *saturated* [cf. [the evident étale generalization of] [Kato2], §1.5]. We shall refer to a log scheme as *noetherian* (respectively, *locally noetherian*) if its underlying scheme is noetherian (respectively, locally noetherian). We shall say that a morphism of log schemes is *of finite type* if its underlying morphism of schemes is of finite type. We shall say that a morphism of log schemes is an *open immersion* if its underlying morphism of schemes is an open immersion, and, moreover, the log structure on its domain is obtained as the pull-back of the log structure on its codomain. We shall say that a morphism of log schemes is *dominant* if its underlying morphism of schemes is dominant.

We recall from [LgSch], Lemma 2.6, (i), (ii), (iii), that the natural morphism from the underlying scheme of any *fiber product* in the category of *locally noetherian fs log schemes* to the corresponding fiber product of underlying schemes is *finite*. On the other hand, this natural morphism is *not necessarily surjective*! That is to say, the surjectivity asserted [unfortunately, without an *explicit proof*!] in [LgSch], Lemma 2.6, (iii), is *false*. Indeed, the following example constitutes a *counterexample* to this surjectivity.

Example 0.3. Empty fiber products of log schemes. Consider the *fiber product* determined by the *diagram of log schemes*

$$X^{\log} \quad \to \quad Z^{\log} \quad \leftarrow \quad Y^{\log}$$

obtained by equipping the *diagram of schemes*

$$X \;=\; \operatorname{Spec}(k) \quad \to \quad Z \;=\; \operatorname{Spec}(k) \quad \leftarrow \quad Y \;\stackrel{\text{def}}{=}\; \operatorname{Spec}(k)$$

— where k is a field, and the arrows are the identity morphisms — with the log structures determined by the *diagram of monoids*

$$P_X \;\stackrel{\text{def}}{=}\; \langle (1,0); (-1,1) \rangle \quad \supseteq \quad P_Z \;\stackrel{\text{def}}{=}\; \mathbb{N} \oplus \mathbb{N} \quad \subseteq \quad P_Y \;\stackrel{\text{def}}{=}\; \langle (1,-1); (0,1) \rangle$$

— where the notation "$\langle - \rangle$" denotes the submonoid generated by the element(s) in brackets — and the morphisms of monoids $P_X \to k$, $P_Y \to k$, $P_Z \to k$ that map $0 \mapsto 1 \in k$ and all nonzero elements of the domain to $0 \in k$. Then one verifies immediately that this fiber product is, in fact, *empty*, despite the fact that $X \times_Y Z = \mathrm{Spec}(k) \neq \emptyset$.

Finally, we close by reviewing a well-known example of a *non-log smooth* morphism of fs log schemes that will be of use in the exposition of the present paper.

Example 0.4. A non-log smooth morphism of fs log schemes. Let n be an integer ≥ 2; P_Y an fs monoid; $P_Z \subseteq \mathbb{Z} \oplus P_Y$ a submonoid which is an fs monoid such that $\mathbb{N} \oplus P_Y \subseteq P_Z$ and $(\mathbb{Z} \oplus \{0\}) \cap P_Z = \mathbb{N} \oplus \{0\}$; k a field. Write $P_S \stackrel{\mathrm{def}}{=} P_T \stackrel{\mathrm{def}}{=} P_Z$; $Y \stackrel{\mathrm{def}}{=} S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k)$; $Z \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k[t])$, where t is an indeterminate; $T \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$, where ϵ is an indeterminate, whose image in $k[\epsilon]/(\epsilon^2)$ we denote by δ. Thus, we obtain a *commutative diagram of k-schemes*

$$\begin{array}{ccc} S & \longrightarrow & Z \\ \big\downarrow & & \big\downarrow \\ T & \longrightarrow & Y \end{array}$$

by taking the left-hand vertical arrow to be the arrow determined by the assignment $\delta \mapsto 0$ and the upper horizontal arrow to be the arrow determined by the assignment $t \mapsto 0$. Next, write $Y^{\log}$ (respectively, $Z^{\log}$; $S^{\log}$; $T^{\log}$) for the log scheme with underlying scheme Y (respectively, Z; S; T) and the log structure determined by the morphism of monoids $P_Y \to k$ (respectively, $P_Z \to k[t]$; $P_S \to k$; $P_T \to k[\delta]$) that maps $0 \mapsto 1 \in k$ (respectively, $0 \mapsto 1 \in k[t]$ and $(1,0) \mapsto t^n \in k[t]$; $0 \mapsto 1 \in k[t]$ and $(1,0) \mapsto 0 \in k$; $0 \mapsto 1 \in k[t]$ and $(1,0) \mapsto \delta \in k[\delta]$) and all *nonzero* elements $\in P_Y$ (respectively, $\in P_Z \backslash (\mathbb{N} \oplus \{0\})$; $\in P_S \backslash (\mathbb{N} \oplus \{0\})$; $\in P_T \backslash (\mathbb{N} \oplus \{0\})$) to $0 \in k$. Thus, one verifies immediately that the natural inclusion $P_Y \stackrel{\sim}{\to} \{0\} \oplus P_Y \hookrightarrow P_Z = P_S = P_T$, together with the above diagram of k-schemes, determines a *commutative diagram of fs log schemes*

$$\begin{array}{ccc} S^{\log} & \longrightarrow & Z^{\log} \\ \big\downarrow & & \big\downarrow \\ T^{\log} & \longrightarrow & Y^{\log} \end{array}$$

that does *not* admit a *compatible* [in the evident sense] morphism $T^{\log} \to Z^{\log}$. Indeed, one verifies immediately, by considering the images of $(1,0) \in P_Z = P_S = P_T$, that the *existence* of such a morphism $T^{\log} \to Z^{\log}$ would imply the existence of an *n-th root* of δ in $k[\delta]$, a *contradiction*. In particular, we conclude that the morphism $Z^{\log} \to Y^{\log}$ of the above diagram is *not log smooth*.

Section 1: Generalities on Monomorphisms and Minimal Points

In the present §1, we discuss various definitions and generalities related to *monomorphisms* and *"minimal points"* in categories of log schemes.

We suppose that we are in the situation of [LgSch], §2. That is to say, let $X^{\log}$ be a *locally noetherian fs log scheme* [cf. the discussion entitled "Log schemes" in §0]. Then we denote by

$$\mathrm{Sch}^{\log}(X^{\log})$$

the category whose *objects* are *morphisms of log schemes of finite type* $Y^{\log} \to X^{\log}$, where $Y^{\log}$ is a *noetherian fs log scheme*, and whose *morphisms* [from an object $Y_1^{\log} \to X^{\log}$ to an object $Y_2^{\log} \to X^{\log}$] are *morphisms of finite type* $Y_1^{\log} \to Y_2^{\log}$ lying over $X^{\log}$. To simplify the exposition, we shall often refer to the *domain* $Y^{\log}$ of an arrow $Y^{\log} \to X^{\log}$ which is an object of $\mathrm{Sch}^{\log}(X^{\log})$ as an "object of $\mathrm{Sch}^{\log}(X^{\log})$".

Recall the category $\mathrm{Sch}(X)$ of [LgSch], §1, i.e., the category whose *objects* are *morphisms of finite type* $Y \to X$, where Y is a *noetherian scheme*, and whose *morphisms* [from an object $Y_1 \to X$ to an object $Y_2 \to X$] are *morphisms of finite type* $Y_1 \to Y_2$ lying over X. Note that by associating to an object $Y \to X$ of $\mathrm{Sch}(X)$ the object $Y^{\log} \to X^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ obtained by equipping Y with the log structure obtained by pulling back the log structure on $X^{\log}$ via $Y \to X$, we obtain a *natural embedding*

$$\mathrm{Sch}(X) \hookrightarrow \mathrm{Sch}^{\log}(X^{\log})$$

— which thus allows us to regard $\mathrm{Sch}(X)$ as a *full subcategory* of $\mathrm{Sch}^{\log}(X^{\log})$.

Let $Y^{\log}$ be an *fs log scheme*. Then we shall denote its *underlying scheme* (respectively, the *morphism of monoids* that constitutes its *log structure*) by Y (respectively, $\exp_Y : M_Y \to \mathcal{O}_Y$). Thus, we have an *exact sequence of étale sheaves of monoids on* Y

$$0 \to \mathcal{O}_Y^\times \to M_Y \to P_Y \to 0$$

— where the *"characteristic sheaf"* P_Y is defined so as to make the sequence exact. It follows immediately from the fact that $Y^{\log}$ is an fs log scheme that the fibers of P_Y (respectively, the groupification P_Y^{gp} of P_Y) at geometric points of Y are *fs monoids* [cf. the discussion entitled "Generalities on monoids" in §0] (respectively, are finitely generated free abelian groups). In particular, we have *natural injections*

$$P_Y \hookrightarrow P_Y^{\mathrm{gp}}; \quad M_Y \hookrightarrow M_Y^{\mathrm{gp}}$$

— where the superscript "gp" denotes the groupification associated to a sheaf of monoids. In the following, we shall use *similar notation* for objects associated to arbitrary log schemes "$(-)^{\log}$".

In this situation, we shall apply the terminology introduced in [LgSch], §2:

Definition 1.1. In the notation of the above discussion:

(i) If Y is *reduced* (respectively, *one-pointed* — cf. [LgSch], Proposition 1.1), then we shall say that $Y^{\log}$ is *reduced* (respectively, *one-pointed*). If $Y^{\log}$ is *reduced* and *one-pointed*, i.e., Y is equal to the spectrum of a field k, then one may think of P_Y as consisting of a [discrete] monoid equipped with a continuous action of the *absolute Galois group* G_k of k; when this action is *trivial*, we shall say that $Y^{\log}$ is *split* and, by a slight abuse of notation, denote $\Gamma(Y, P_Y)$ by P_Y.

(ii) An object $Y^{\log} \rightarrowtail X^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ will be called *minimal* if it is non-initial and satisfies the property that any monomorphism $Z^{\log} \rightarrowtail Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$, where $Z^{\log}$ is non-initial, is necessarily an isomorphism [cf. [LgSch], Proposition 2.4].

(iii) Suppose that $Y^{\log}$ is a *one-pointed object* of the category $\mathrm{Sch}^{\log}(X^{\log})$. Then a monomorphism $H^{\log} \rightarrowtail Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$ will be called a *hull* for $Y^{\log}$ if every morphism $S^{\log} \to Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$ from a *minimal* object $S^{\log}$ to $Y^{\log}$ *factors* [necessarily uniquely!] though the given monomorphism $H^{\log} \rightarrowtail Y^{\log}$ [cf. [LgSch], Proposition 2.7]. A hull $H^{\log} \rightarrowtail Y^{\log}$ will be called a *minimal hull* if every monomorphism $H_1^{\log} \rightarrowtail H^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$ for which the composite $H_1^{\log} \rightarrowtail H^{\log} \rightarrowtail Y^{\log}$ is a hull is necessarily an isomorphism [cf. [LgSch], Proposition 2.7]. A one-pointed object $H^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ will be called a *minimal hull* if the identity morphism $H^{\log} \to H^{\log}$ is a minimal hull for $H^{\log}$.

(iv) Suppose that $f^{\log} : Z^{\log} \to Y^{\log}$ is a morphism of $\mathrm{Sch}^{\log}(X^{\log})$. Then [cf. [LgSch], Definition 2.11, (i), (ii)]: $f^{\log}$ will be called *log-like* if the underlying morphism of schemes $f : Z \to Y$ is an isomorphism; $f^{\log}$ will be called *scheme-like* if the log structure on $Z^{\log}$ is the pull-back of the log structure on $Y^{\log}$ via the underlying morphism of schemes $f : Z \to Y$ [i.e., in the terminology of many authors, if $f^{\log}$ is *exact*]. Write

$$\mathrm{Sch}^{\log}(X^{\log})|_{\text{sch-lk}} \subseteq \mathrm{Sch}^{\log}(X^{\log})$$

for the *full subcategory* of objects of $\mathrm{Sch}^{\log}(X^{\log})$ determined by *scheme-like* morphisms $Y^{\log} \to X^{\log}$. Thus, one verifies immediately that the natural embedding $\mathrm{Sch}(X) \hookrightarrow \mathrm{Sch}^{\log}(X^{\log})$ discussed above admits a natural factorization as the composite of a *natural equivalence of categories*

$$\mathrm{Sch}(X) \overset{\sim}{\to} \mathrm{Sch}^{\log}(X^{\log})|_{\text{sch-lk}}$$

with the natural inclusion $\mathrm{Sch}^{\log}(X^{\log})|_{\text{sch-lk}} \hookrightarrow \mathrm{Sch}^{\log}(X^{\log})$.

Also, we introduce some *new terminology* as follows:

Definition 1.2. In the notation of the above discussion:

(i) Let $n \in \mathbb{N}$. Then we shall say that $Y^{\log}$ is *of rank* $\leq n$ (respectively, *of rank* n) and write

$$\mathrm{rk}(Y^{\log}) \leq n \text{ (respectively, } \mathrm{rk}(Y^{\log}) = n)$$

if every fiber of P_Y at a geometric point of Y is of rank $\leq n$ (respectively, rank n) [cf. the discussion entitled "Generalities on monoids" in §0]. We shall say that $Y^{\log}$ is *submonic* if it is of rank ≤ 1. If $Y^{\log}$ is locally noetherian, then we define the *submonic dimension* of $Y^{\log}$ to be the supremum

$$\dim^{\mathrm{sm}}(Y^{\log}) \overset{\text{def}}{=} \sup_{Z^{\log} \rightarrowtail Y^{\log}} \dim(Z) \in \mathbb{N} \cup \{-\infty, +\infty\}$$

— where $Z^{\log} \rightarrowtail Y^{\log}$ ranges over the *monomorphisms* of $\mathrm{Sch}^{\log}(Y^{\log})$ such that $Z^{\log}$ is *submonic*, and "$\dim(Z)$" denotes the scheme-theoretic dimension of the underlying locally noetherian scheme Z of $Z^{\log}$. Thus, the submonic dimension is equal to $-\infty$ if and only if it holds that the underlying scheme of every "$Z^{\log}$" that appears in the supremum of the above display is the empty scheme. We shall say that $Y^{\log}$ is *log-nodal* if it is *reduced*, *one-pointed*, *split*, and of *rank two*.

(ii) Suppose that $Y^{\log}$ arises from an object $Y^{\log} \to X^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$. Then a *minimal point* $Z^{\log} \rightarrowtail Y^{\log}$ of $Y^{\log}$ is defined to be a monomorphism $Z^{\log} \rightarrowtail Y^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ such that $Z^{\log}$ is a *minimal* object of $\mathrm{Sch}^{\log}(X^{\log})$. Thus, a minimal point of $Y^{\log}$ may be thought of as an object of $\mathrm{Sch}^{\log}(Y^{\log})$. We shall write

$$\mathrm{MinPt}(Y^{\log})$$

for the set of isomorphism classes [i.e., as objects of $\mathrm{Sch}^{\log}(Y^{\log})$] of minimal points of $Y^{\log}$.

Proposition 1.3. **(Empty and connected underlying schemes)** *Suppose that $Y^{\log}$ is an object of $\mathrm{Sch}^{\log}(X^{\log})$. Then:*

(i) The underlying scheme Y of $Y^{\log}$ is **empty** *if and only if $Y^{\log}$ is an* **initial** *object in the category $\mathrm{Sch}^{\log}(X^{\log})$.*

(ii) The underlying scheme Y of $Y^{\log}$ is **connected** *if and only if the object $Y^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ is* **non-initial** *and, moreover, does* **not** *admit a representation as a* **coproduct** *of two non-initial objects of $\mathrm{Sch}^{\log}(X^{\log})$ whose fiber product over $Y^{\log}$ is initial.*

Proof. Assertions (i) and (ii) follow immediately from the definitions. ◯

Proposition 1.4. **(First properties of monomorphisms)** *Suppose that $f^{\log} : Z^{\log} \to Y^{\log}$ is a morphism of $\mathrm{Sch}^{\log}(X^{\log})$. Thus, the underlying morphism $f : Z \to Y$ of $f^{\log}$ may be regarded as a morphism of $\mathrm{Sch}(X)$. Then:*

(i) The property of being a **monomorphism** *in the category of fs log schemes (respectively, in the category $\mathrm{Sch}^{\log}(X^{\log})$) is* **stable** *under* **base-change** *in the category of fs log schemes (respectively, in the category $\mathrm{Sch}^{\log}(X^{\log})$).*

(ii) Let $M \to N$ be a morphism of **finitely generated**, **saturated monoids** *such that the induced morphism $M^{\mathrm{gp}} \to N^{\mathrm{gp}}$ is* **surjective**. *Then the induced morphism of fs log schemes*

$$\mathrm{Spec}(\mathbb{Z}[N])^{\log} \to \mathrm{Spec}(\mathbb{Z}[M])^{\log}$$

— where we use the superscript "log" to denote the log structures determined by the tautological charts $M \hookrightarrow \mathbb{Z}[M]$, $N \hookrightarrow \mathbb{Z}[N]$ — is a **monomorphism** *in the category of fs log schemes.*

(iii) If $f^{\log}$ is a **monomorphism** *in $\mathrm{Sch}^{\log}(X^{\log})$, then the induced morphism of sheaves of abelian groups $P_Y^{\mathrm{gp}}|_Z \to P_Z^{\mathrm{gp}}$ is* **surjective**.

(iv) Suppose that $Y^{\log}$ is **submonic**, *and that the morphism $P_Y^{\mathrm{gp}}|_Z \to P_Z^{\mathrm{gp}}$ induced by $f^{\log}$ is* **surjective**. *Then $Z^{\log}$ is* **submonic**, *and $f^{\log}$ is* **scheme-like**.

(v) Suppose that $f^{\log}$ is **scheme-like**. *Then $f^{\log}$ is a* **monomorphism** *in $\mathrm{Sch}^{\log}(X^{\log})$ if and only if f is a* **monomorphism** *in $\mathrm{Sch}(X)$.*

(vi) Suppose that $Y^{\log}$ is **submonic***, and that $f^{\log}$ is a* **monomorphism** *in* $\mathrm{Sch}^{\log}(X^{\log})$*. Then the morphism $P_Y^{\mathrm{gp}}|_Z \to P_Z^{\mathrm{gp}}$ induced by $f^{\log}$ is* **surjective***; $Z^{\log}$ is* **submonic***; $f^{\log}$ is* **scheme-like***; and f is a* **monomorphism** *in* $\mathrm{Sch}(X)$*.*

(vii) Suppose that f is a **monomorphism** *in* $\mathrm{Sch}(X)$*, and that the morphism $P_Y^{\mathrm{gp}}|_Z \to P_Z^{\mathrm{gp}}$ induced by $f^{\log}$ is* **surjective***. Then $f^{\log}$ is a* **monomorphism** *in* $\mathrm{Sch}^{\log}(X^{\log})$*.*

Proof. Assertions (i) and (v) follow immediately from the definitions. Next, before proceeding, let us recall that, for instance in the case of the log scheme $Y^{\log}$,

($*_{\mathrm{sys}}$) the sheaf of monoids that defines the log structure of $Y^{\log}$ may be thought of as the restriction to $P_Y \subseteq P_Y^{\mathrm{gp}}$ of a certain *system of line bundles* [i.e., a system of $\mathbb{G}_{\mathrm{m}}$-torsors] parametrized by the sheaf of abelian groups P_Y^{gp}.

Now assertion (ii) follows immediately from ($*_{\mathrm{sys}}$). Assertion (iii) follows from the argument given in the proof of [LgSch], Proposition 2.3 [but cf. Remark 1.4.1 below!]: That is to say, one reduces immediately to the case where Z and Y are equal to $\mathrm{Spec}(k)$ for some field k; then, under the assumption that the asserted *surjectivity* fails to hold, one constructs scheme-like morphisms $W^{\log} \to Z^{\log}$, where $W^{\log}$ is an fs log scheme whose underlying scheme is an *artinian k-algebra*, whose existence *contradicts* the assumption that $f^{\log}$ is a *monomorphism* in $\mathrm{Sch}^{\log}(X^{\log})$. Assertion (iv) follows immediately from the simple and well-understood structure of the monoid $\mathbb{N}$. Assertion (vi) follows formally from assertions (iii), (iv), and (v). Finally, assertion (vii) follows from the definitions, together with the observation ($*_{\mathrm{sys}}$) discussed above. ○

Remark 1.4.1. Suppose that we are in the situation of Proposition 1.4. Then in general,

> *it is* **not** *necessarily the case that the assumption that $f^{\log}$ is a monomorphism in* $\mathrm{Sch}^{\log}(X^{\log})$ *implies that f is a monomorphism in* $\mathrm{Sch}(X)$*.*

That is to say, the corresponding portion of the *necessity* asserted in [LgSch], Proposition 2.3, is *false* as stated. Such an example may be obtained by considering the *monomorphism* constructed in Proposition 1.4, (ii), in the case where the morphism of monoids $M \to N$ is taken to be the morphism

$$M \overset{\mathrm{def}}{=} \mathbb{N} \oplus \mathbb{N} \to N \overset{\mathrm{def}}{=} \mathbb{N} \oplus \mathbb{N}$$

that maps $M \ni (1, 0) \mapsto (1, 1) \in N$ and $M \ni (0, 1) \mapsto (0, 1) \in N$, i.e., in which case the resulting morphism of schemes is a *"blow-up morphism"* that has *fibers of dimension one.*

Lemma 1.5. (Well-known generalities concerning fs monoids and associated log schemes) *Let k be a* **field***; k^{sep} a* **separable closure** *of k; $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(k^{\mathrm{sep}}/k)$; P an* **fs monoid** *[cf. the discussion entitled "Generalities on monoids" in §0] equipped with a continuous action by G_k [i.e., relative to the discrete topology on P]; M^{gp} an extension, in the category of topological abelian* *groups equipped with continuous* G_k*-actions, of* P^{gp} *by* $(k^{\mathrm{sep}})^\times$ *[i.e., the multiplicative group of nonzero elements of* k^{sep}*, equipped with the discrete topology];* $M \stackrel{\mathrm{def}}{=} M^{\mathrm{gp}} \times_{P^{\mathrm{gp}}} P$*. Write* $T^{\log}$ *for the reduced, one-pointed fs log scheme whose underlying scheme is equal to* $T = \mathrm{Spec}(k^{\mathrm{sep}})$*, and whose log structure is given by the homomorphism of monoids* $M \to k^{\mathrm{sep}}$ *that restricts to the natural inclusion* $(k^{\mathrm{sep}})^\times \hookrightarrow k^{\mathrm{sep}}$ *on* $(k^{\mathrm{sep}})^\times \subseteq M$ *and maps non-invertible elements of* M *to* $0 \in k$*. Thus, the associated characteristic sheaf* P_T *is the constant sheaf on* T *determined by* P*; the log scheme* $T^{\log}$ *admits a natural* G_k*-action, which may be regarded as a collection of [pro-]finite étale descent data that gives rise to a reduced, one-pointed fs log scheme* $S^{\log}$ *whose underlying scheme is* $S = \mathrm{Spec}(k)$*. Then:*

(i) The extension of G_k*-modules* $1 \to (k^{\mathrm{sep}})^\times \to M^{\mathrm{gp}} \to P^{\mathrm{gp}} \to 1$ **splits***.*

(ii) Suppose that $\mathrm{rk}(P) \geq 1$*. Then there exists a* **positive** *[cf. the discussion entitled "Generalities on monoids" in §0],* G_k**-equivariant** *[i.e., with respect to the trivial action of* G_k *on* $\mathbb{N}$*] homomorphism* $\phi : P \to \mathbb{N}$ *that induces a* **surjection** *on groupifications* $\phi^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow \mathbb{N}^{\mathrm{gp}}$*. If, moreover,* $\mathrm{rk}(P) \geq 2$*, then there exists a* **positive** *homomorphism* $\psi : P \to \mathbb{N}$ *that induces a* **surjection** *on groupifications* $\psi^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow \mathbb{N}^{\mathrm{gp}}$ *such that* $\mathrm{Ker}(\phi^{\mathrm{gp}}) \neq \mathrm{Ker}(\psi^{\mathrm{gp}})$*.*

(iii) Suppose that $\mathrm{rk}(P) \geq 2$*. Then there exist an* **fs monoid** Q *of* **rank two** *and a* **positive** *homomorphism* $\xi : P \to Q$ *that induces a* **surjection** *on groupifications* $\xi^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow Q^{\mathrm{gp}}$ *and, moreover, satisfies the following property:*

> *Let* $\zeta : Q \to R$ *be a positive homomorphism of fs monoids of rank* ≥ 1 *and* $\sigma \in G_k$ *such that the composite homomorphism* $\zeta \circ \xi \circ \sigma : P \to R$ **factors** *as the composite* $\zeta_\sigma \circ \xi$ *of* $\xi : P \to Q$ *with some positive homomorphism* $\zeta_\sigma : Q \to R$*. Then* σ **stabilizes** *the subquotient* $P^{\mathrm{gp}} \twoheadrightarrow Q^{\mathrm{gp}} \supseteq Q$ *and induces the* **identity** *on* Q*.*

In particular, if $\tau \in G_k$ **stabilizes** *the subquotient* $P^{\mathrm{gp}} \twoheadrightarrow Q^{\mathrm{gp}} \supseteq Q$*, then* τ *induces the* **identity** *on* Q*.*

(iv) Let $\xi : P \to Q$ *be a* **positive** *homomorphism of fs monoids that induces a* **surjection** *on groupifications* $\xi^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow Q^{\mathrm{gp}}$*. Write* Ξ^{sep} *for the subfunctor of the contravariant functor determined by the terminal object [i.e.,* $T^{\log}$*] of* $\mathrm{Sch}^{\log}(T^{\log})$ *that consists of objects* $Z^{\log} \to T^{\log}$ *of* $\mathrm{Sch}^{\log}(T^{\log})$ *such that the composite homomorphism* $P^{\mathrm{gp}} \to \Gamma(T, P_T^{\mathrm{gp}}) \to \Gamma(Z, P_Z^{\mathrm{gp}})$ **induces***, via* ξ*, a homomorphism* $Q \to \Gamma(Z, P_Z)$*; write* $\Xi_+^{\mathrm{sep}} \subseteq \Xi^{\mathrm{sep}}$ *for the subfunctor corresponding to the condition that, for each fiber* $P_{Z,\overline{z}}$ *of* P_Z *at a geometric point* $\overline{z}$ *of* Z*, the resulting homomorphism* $Q \to P_{Z,\overline{z}}$ *is* **positive***. Then* Ξ^{sep} *may be represented by the object of* $\mathrm{Sch}^{\log}(T^{\log})$ *determined by a* **log étale monomorphism**

$$T^{\log}[\xi] \rightarrowtail T^{\log}$$

of $\mathrm{Sch}^{\log}(T^{\log})$*. If, moreover,* Q *coincides with the* **saturation** *of the image of* ξ*, then the following properties hold:* $\Xi_+^{\mathrm{sep}} = \Xi^{\mathrm{sep}}$*; the closed subscheme* $T[\xi]_{\mathrm{red}} \subseteq T[\xi]$ *[cf. the discussion entitled "Log schemes" in §0] of the underlying scheme* $T[\xi]$ *of* $T^{\log}[\xi]$ *is a* **torus** *over* k^{sep} *of* **dimension** $\mathrm{rk}(P) - \mathrm{rk}(Q)$*; the characteristic sheaf* $P_{T^{\log}[\xi]}$ *is isomorphic to the constant sheaf on* $T[\xi]$ *determined by* Q*; if we write*

$M[\xi] \stackrel{\mathrm{def}}{=} M^{\mathrm{gp}} \times_{P^{\mathrm{gp}}} \mathrm{Ker}(\xi^{\mathrm{gp}})$, *then the group of invertible functions on the torus* $T[\xi]_{\mathrm{red}}$ *may be naturally identified with* $M[\xi]$.

(v) Suppose that we are in the situation of (iv). Write $H \subseteq G_k$ *for the open subgroup of elements that* **stabilize** *the subquotient* $P^{\mathrm{gp}} \twoheadrightarrow Q^{\mathrm{gp}} \supseteq Q$ *determined by* ξ*;* $S_H^{\log}$ *for the reduced, one-pointed fs log scheme obtained by descending* $T^{\log}$ *via* $H \subseteq G_k$*;* Ξ *for the subfunctor of the contravariant functor determined by the terminal object [i.e.,* $S_H^{\log}$*] of* $\mathrm{Sch}^{\log}(S_H^{\log})$ *that consists of objects* $Z^{\log} \to S_H^{\log}$ *of* $\mathrm{Sch}^{\log}(S_H^{\log})$ *such that the object* $Z^{\log} \times_{S_H^{\log}} T^{\log} \to T^{\log}$ *of* $\mathrm{Sch}^{\log}(T^{\log})$ *determined by base-changing from* $S_H^{\log}$ *to* $T^{\log}$ *determines an element of* $\Xi^{\mathrm{sep}}(Z^{\log} \times_{S_H^{\log}} T^{\log})$*;* Ξ_+ *for the subfunctor of* Ξ *determined by the subfunctor* Ξ_+^{sep} *of* Ξ^{sep}*. Then* Ξ *may be represented by the object of* $\mathrm{Sch}^{\log}(S_H^{\log})$ *determined by a* **log étale monomorphism**

$$S^{\log}[\xi] \rightarrowtail S_H^{\log}$$

of $\mathrm{Sch}^{\log}(S_H^{\log})$ *which may be obtained, via [pro-]finite étale descent, from the* **natural** H**-action** *on the monomorphism* $T^{\log}[\xi] \rightarrowtail T^{\log}$ *of (iv).*

(vi) Suppose that we are in the situation of (v). Let $S_+^{\log}[\xi] \rightarrowtail S^{\log}[\xi]$ *be a monomorphism of* $\mathrm{Sch}^{\log}(S_H^{\log})$ *that determines an element of* $\Xi_+(-) \subseteq \Xi(-)$*. Then if either* $\mathrm{rk}(Q) = 1$ *or* ξ *is as in (iii), then the composite*

$$S_+^{\log}[\xi] \rightarrowtail S^{\log}[\xi] \rightarrowtail S_H^{\log} \to S^{\log}$$

— i.e., where the second arrow is the monomorphism of the final display of (v); the third arrow is the natural morphism $S_H^{\log} \to S^{\log}$ *— is a* **monomorphism** *in* $\mathrm{Sch}^{\log}(S^{\log})$.

(vii) Suppose that $\mathrm{rk}(P) = 2$*, and that we have been given a* **positive homomorphism** $\phi_0 : P \to J_0 \stackrel{\mathrm{def}}{=} \mathbb{N}$ *that induces a surjection on groupifications* $\phi_0^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow J_0^{\mathrm{gp}} = \mathbb{Z}$*. Then, in the notation of the discussion entitled "Rank two fs monoids" in §0, for* $i = 0, 1, 2$*, let us write* $\phi_i : P \to J_i$ *for the associated positive homomorphism of fs monoids [which is well-defined, up to* **possible permutation** *of the indices "1" and "2"]. For* $i = 0, 1, 2$*, write* Φ_i^{sep} *for the subfunctor of the contravariant functor determined by the terminal object [i.e.,* $T^{\log}$*] of* $\mathrm{Sch}^{\log}(T^{\log})$ *that consists of objects* $Z^{\log} \to T^{\log}$ *of* $\mathrm{Sch}^{\log}(T^{\log})$ *such that, for each fiber* $P_{Z,\overline{z}}$ *of* P_Z *at a geometric point* $\overline{z}$ *of* Z*, the composite homomorphism* $P^{\mathrm{gp}} \to \Gamma(T, P_T^{\mathrm{gp}}) \to \Gamma(Z, P_Z^{\mathrm{gp}}) \to P_{Z,\overline{z}}^{\mathrm{gp}}$ **induces***, via* $\phi_i : P \to J_i$*, a positive homomorphism* $J_i \to P_{Z,\overline{z}}$*. If* $E \subseteq \{0, 1, 2\}$ *is a subset, then write* Φ_E^{sep} *for the subfunctor of the contravariant functor determined by the terminal object [i.e.,* $T^{\log}$*] of* $\mathrm{Sch}^{\log}(T^{\log})$ *that consists of the* **[disjoint!] union** *of the* Φ_i^{sep}*, for* $i \in E$*. Then, for any* $E \subseteq \{0, 1, 2\}$ *such that* $0 \in E$*,* Φ_E^{sep} *may be represented by the object of* $\mathrm{Sch}^{\log}(T^{\log})$ *determined by a* **log étale monomorphism**

$$T^{\log}[\phi_E] \rightarrowtail T^{\log}$$

of $\mathrm{Sch}^{\log}(T^{\log})$ *which satisfies the following properties:* $T^{\log}[\phi_E]$ *is* **connected** *[hence* **nonempty***]. If* $E = \{0\}$*, then* $T^{\log}[\phi_E] \rightarrowtail T^{\log}$ *may be identified with* *the morphism* $T^{\log}[\phi_0] \rightarrowtail T^{\log}$ *of (iv); in particular, in this case, the closed subscheme* $T[\phi_E]_{\mathrm{red}} \subseteq T[\phi_E]$ *of the underlying scheme* $T[\phi_E]$ *of* $T^{\log}[\phi_E]$ *is a* **one-dimensional torus** *over* k^{sep}*. Finally, if* $0 \in E \subseteq E^* \subseteq \{0,1,2\}$*, then the resulting morphism of log schemes* $T^{\log}[\phi_E] \to T^{\log}[\phi_{E^*}]$ *is a* **dominant open immersion** *[cf. the discussion entitled "Log schemes" in §0].*

(viii) Suppose that we are in the situation of (vii). Write $H \subseteq G_k$ *for the open subgroup of elements that* **stabilize** *the subquotient* $P^{\mathrm{gp}} \twoheadrightarrow J_0^{\mathrm{gp}} \supseteq J_0$ *determined by* ϕ_0*;* $S_H^{\log}$ *for the reduced, one-pointed fs log scheme obtained by descending* $T^{\log}$ *via* $H \subseteq G_k$*. Thus,* H *acts naturally on* $\mathrm{Prime}(P)$*, hence also on the set of indices* $\{0,1,2\}$ *[i.e., where we regard the index "0" as being stabilized by the action of* H*]. Let* $E \subseteq \{0,1,2\}$ *be a subset that is* **stabilized** *by this natural action of* H*. Write* Φ_E *for the subfunctor of the contravariant functor determined by the terminal object [i.e.,* $S_H^{\log}$*] of* $\mathrm{Sch}^{\log}(S_H^{\log})$ *that consists of objects* $Z^{\log} \to S_H^{\log}$ *of* $\mathrm{Sch}^{\log}(S_H^{\log})$ *such that the object* $Z^{\log} \times_{S_H^{\log}} T^{\log} \to T^{\log}$ *of* $\mathrm{Sch}^{\log}(T^{\log})$ *determined by base-changing from* $S_H^{\log}$ *to* $T^{\log}$ *determines an element of* $\Phi_E^{\mathrm{sep}}(Z^{\log} \times_{S_H^{\log}} T^{\log})$*. Suppose that* $0 \in E$*. Then* Φ_E *may be represented by the object of* $\mathrm{Sch}^{\log}(S_H^{\log})$ *determined by a* **log étale monomorphism**

$$S^{\log}[\phi_E] \rightarrowtail S_H^{\log}$$

of $\mathrm{Sch}^{\log}(S_H^{\log})$ *which may be obtained, via [pro-]finite étale descent, from the* **natural** H**-action** *on the monomorphism* $T^{\log}[\phi_E] \rightarrowtail T^{\log}$ *of (vii).*

(ix) Suppose that we are in the situation of (viii). Suppose further that $\phi : P \to J_0$ *satisfies the following property:*

> *Let* $\zeta : P \to \mathbb{N}$ *be a positive homomorphism of fs monoids;* $\sigma \in G_k$*;* $i_0, i_1 \in \{0,1\}$*. Suppose that, for* $m \in \{0,1\}$*,* $\zeta \circ \sigma^m : P \to \mathbb{N}$ **factors***, via* $\phi_{i_m} : P \to J_{i_m}$*, through a* **positive** *homomorphism* $J_{i_m} \to \mathbb{N}$*. Then* σ *acts* **trivially** *on* P*.*

Then [one verifies immediately, by taking "ζ*" to be* ϕ_0 *that]* H *fixes the index "1". Moreover, the composite* $S^{\log}[\phi_{\{0,1\}}] \rightarrowtail S_H^{\log} \to S^{\log}$ *of the monomorphism of the final display of (viii) with the natural morphism* $S_H^{\log} \to S^{\log}$ *is a* **monomorphism** *in* $\mathrm{Sch}^{\log}(S^{\log})$*.*

Proof. Since P^{gp} is a *finitely generated free abelian group*, assertion (i) follows immediately from the well-known fact from elementary Galois theory [i.e., Hilbert's "Theorem 90"] that $H^1(G_k, (k^{\mathrm{sep}})^\times) = 0$. Next, we consider assertion (ii). The existence of ϕ follows immediately from [LgSch], Lemma 2.5, (iii), i.e., by considering the [*finite*!] sum of G_k-conjugates of the homomorphism $P \to \mathbb{N}$ whose existence is asserted in [LgSch], Lemma 2.5, (iii); the existence of ψ then follows by applying [LgSch], Lemma 2.5, (iii), to two distinct elements of P that map, via ϕ, to the *same* nonzero element of $\mathbb{N}$. This completes the proof of assertion (ii).

Next, we consider assertion (iii). First, we observe that the final portion of assertion (iii) concerning $\tau \in G_k$ follows immediately from the property in the display of assertion (iii) by taking $\zeta : Q \to R$ to be the identity automorphism of Q. Next, we observe that the homomorphisms ϕ and ψ of assertion (ii) determine a positive homomorphism $(\phi, \psi) : P \to \mathbb{N} \oplus \mathbb{N}$ whose image $I \subseteq \mathbb{N} \oplus \mathbb{N}$ generates a *rank two* subgroup I^{gp} of $\mathbb{N}^{\mathrm{gp}} \oplus \mathbb{N}^{\mathrm{gp}} = \mathbb{Z} \oplus \mathbb{Z}$. Thus, for some positive integer n, it holds that $n \cdot \mathbb{N}^{\mathrm{gp}} \ \oplus \ n \cdot \mathbb{N}^{\mathrm{gp}} \subseteq I^{\mathrm{gp}}$. In particular, we have $n \cdot \mathbb{N} \ \oplus \ n \cdot \mathbb{N} \subseteq J \stackrel{\mathrm{def}}{=} I^{\mathrm{gp}} \ \cap \ (\mathbb{N} \oplus \mathbb{N}) \subseteq \mathbb{N} \oplus \mathbb{N}$; $J^{\mathrm{gp}} = I^{\mathrm{gp}}$ [i.e., since $I^{\mathrm{gp}} \subseteq J^{\mathrm{gp}} \subseteq I^{\mathrm{gp}}$]. One verifies immediately that this implies that the monoid $Q \subseteq \mathbb{N} \oplus \mathbb{N}$ obtained by forming the *saturation* of J in $J^{\mathrm{gp}} = I^{\mathrm{gp}}$ [cf. [LgSch], Lemma 2.5, (ii)] is an *fs monoid of rank two* such that $I \subseteq J \subseteq Q$, $I^{\mathrm{gp}} = J^{\mathrm{gp}} = Q^{\mathrm{gp}}$. Write $\xi : P \to Q$ for the resulting *positive* homomorphism of monoids. Note that ξ induces a *surjection* on groupifications $\xi^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow Q^{\mathrm{gp}}$ $(= J^{\mathrm{gp}} = I^{\mathrm{gp}})$.

Now suppose that $\zeta : Q \to R$ is a positive homomorphism of fs monoids of rank ≥ 1 such that the composite homomorphism $\zeta \circ \xi \circ \sigma : P \to R$ *factors* as the composite $\zeta_\sigma \circ \xi$ of $\xi : P \to Q$ with some positive homomorphism $\zeta_\sigma : Q \to R$. Here, we note that, by applying assertion (ii) in the case where we take "P", "k", and "M^{gp}" to be R, k^{sep}, and $R^{\mathrm{gp}} \times (k^{\mathrm{sep}})^{\times}$, respectively, we may assume without loss of generality that $R = \mathbb{N}$. Also, by replacing R by a suitable submonoid of R, we may assume without loss of generality that ζ, ζ_σ induce *surjections* $\zeta^{\mathrm{gp}}, \zeta_\sigma^{\mathrm{gp}} : Q^{\mathrm{gp}} \twoheadrightarrow R^{\mathrm{gp}} = \mathbb{N}^{\mathrm{gp}} = \mathbb{Z}$. Next, let us observe that, by restricting the first projection $\mathbb{N} \oplus \mathbb{N} \twoheadrightarrow \mathbb{N}$ to $Q \subseteq \mathbb{N} \oplus \mathbb{N}$, one may regard $\phi : P \to \mathbb{N}$ as the composite $\eta \circ \xi$ of $\xi : P \to Q$ with a homomorphism of monoids $\eta : Q \to \mathbb{N}$. Since η *vanishes* on $0 \ \oplus \ n \cdot \mathbb{N} \subseteq J \subseteq Q$, it follows that η is *not positive*, and hence that $\mathrm{Ker}(\eta^{\mathrm{gp}}) \neq \mathrm{Ker}(\zeta^{\mathrm{gp}})$, $\mathrm{Ker}(\eta^{\mathrm{gp}}) \neq \mathrm{Ker}(\zeta_\sigma^{\mathrm{gp}})$. Since ξ^{gp} is *surjective*, we thus conclude that, if we write $\theta \stackrel{\mathrm{def}}{=} \zeta \circ \xi$, $\theta_\sigma \stackrel{\mathrm{def}}{=} \zeta_\sigma \circ \xi$, then $\mathrm{Ker}(\phi^{\mathrm{gp}}) \neq \mathrm{Ker}(\theta^{\mathrm{gp}})$, $\mathrm{Ker}(\phi^{\mathrm{gp}}) \neq \mathrm{Ker}(\theta_\sigma^{\mathrm{gp}})$, and hence that both $\mathrm{Ker}(\phi^{\mathrm{gp}}) \ \cap \ \mathrm{Ker}(\theta^{\mathrm{gp}}) \subseteq P^{\mathrm{gp}}$ and $\mathrm{Ker}(\phi^{\mathrm{gp}}) \ \cap \ \mathrm{Ker}(\theta_\sigma^{\mathrm{gp}}) \subseteq P^{\mathrm{gp}}$ are submodules of rank $\mathrm{rk}(P^{\mathrm{gp}}) - 2$ that *contain* $\mathrm{Ker}(\xi^{\mathrm{gp}})$. Since $\mathrm{Ker}(\xi^{\mathrm{gp}})$ is also a submodule of P^{gp} of rank $\mathrm{rk}(P^{\mathrm{gp}}) - 2$, we thus conclude [since $P^{\mathrm{gp}}/\mathrm{Ker}(\xi^{\mathrm{gp}}) \xrightarrow{\sim} Q^{\mathrm{gp}}$ is *torsion-free*] that $\mathrm{Ker}(\phi^{\mathrm{gp}}) \ \cap \ \mathrm{Ker}(\theta^{\mathrm{gp}}) = \mathrm{Ker}(\phi^{\mathrm{gp}}) \ \cap \ \mathrm{Ker}(\theta_\sigma^{\mathrm{gp}}) = \mathrm{Ker}(\xi^{\mathrm{gp}})$. But, since ϕ is *G_k-equivariant*, this implies that $\mathrm{Ker}(\xi^{\mathrm{gp}})$ is *stabilized* by σ, i.e., that σ induces an automorphism of the quotient $\xi^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow Q^{\mathrm{gp}}$, as well as of the quotient $\eta^{\mathrm{gp}} : Q^{\mathrm{gp}} \twoheadrightarrow \mathbb{N}^{\mathrm{gp}} = \mathbb{Z}$, and maps the quotient $\zeta^{\mathrm{gp}} : Q^{\mathrm{gp}} \twoheadrightarrow \mathbb{N}^{\mathrm{gp}} = \mathbb{Z}$ to the quotient $\zeta_\sigma^{\mathrm{gp}} : Q^{\mathrm{gp}} \twoheadrightarrow \mathbb{N}^{\mathrm{gp}} = \mathbb{Z}$.

Now to complete the proof of assertion (iii), it suffices to verify that σ induces the *identity* on Q^{gp}. Thus, we suppose that σ does *not* induce the identity on Q^{gp}. Then since σ clearly *stabilizes* the *fs monoid of rank two* obtained by forming the *saturation* of the image of $\xi : P \to Q$ in Q [cf. [LgSch], Lemma 2.5, (ii)], it follows [cf. the discussion entitled "Rank two fs monoids" in §0] that σ acts on Q^{gp} as an automorphism of *order* 2, and hence that σ permutes the quotients determined by ζ^{gp}, $\zeta_\sigma^{\mathrm{gp}}$. In particular, σ *stabilizes* the kernel of the homomorphism on groupifications $\zeta_+^{\mathrm{gp}} : Q^{\mathrm{gp}} \to \mathbb{Z}$ determined by the *positive* homomorphism $\zeta_+ : Q \to \mathbb{N}$ obtained by forming the *sum* of ζ, ζ_σ. Since σ acts *nontrivially* on Q^{gp}, this implies that $\mathrm{Ker}(\zeta_+^{\mathrm{gp}}) = \mathrm{Ker}(\eta^{\mathrm{gp}})$. Thus, the *positivity* of ζ_+ *contradicts* the *non-positivity* of η. This completes the proof of assertion (iii).

Next, we observe that assertions (iv), (v), (vii), and (viii) are immediate consequences of the *well-known correspondence* between the theory of log schemes and the classical theory of *toric varieties*. Next, we consider assertion (vi). First of all, given two $S^{\log}$-morphisms $\alpha : Z^{\log} \to S_+^{\log}[\xi]$, $\beta : Z^{\log} \to S_+^{\log}[\xi]$, to verify that $\alpha = \beta$, it suffices to verify that α and β coincide after base-change from k to k^{sep}. Moreover, since the morphism $S_H \to S$ is *finite étale*, and the morphism

$S_+^{\log}[\xi] \rightarrowtail S_H^{\log}$ is already known to be a monomorphism, one verifies immediately that we may assume without loss of generality that $Z^{\log}$ is *reduced* and *one-pointed* — an assumption which reduces the assertion under consideration to an *assertion concerning fs monoids*, i.e., the assertion that if, for some $\sigma \in G_k$, there exist positive homomorphisms of fs monoids $\zeta : Q \to R$ and $\zeta_\sigma : Q \to R$ such that $\zeta \circ \xi \circ \sigma = \zeta_\sigma \circ \xi : P \to R$, then σ *stabilizes* the subquotient $P^{\text{gp}} \twoheadrightarrow Q^{\text{gp}} \supseteq Q$ determined by ξ. But this assertion concerning fs monoids follows immediately, i.e., if one assumes either that $\text{rk}(Q) = 1$ or that ξ satisfies the properties stated in (iii). This completes the proof of assertion (vi). Finally, we observe that assertion (ix) may be verified by a *similar argument* to the argument applied in the proof of assertion (vi). ◯

Proposition 1.6. **(Minimal objects)** *Suppose that $Y^{\log}$ is an object of $\text{Sch}^{\log}(X^{\log})$. Then:*

(i) Suppose that $Y^{\log}$ is a **nonempty** *object of $\text{Sch}^{\log}(X^{\log})$. Then there exists a* **minimal point** *$Z^{\log} \to Y^{\log}$ such that $Z^{\log}$ is* **submonic**, *and, moreover, the underlying morphism of schemes $Z \to Y$ is a* **monomorphism** *in $\text{Sch}(X)$. If, moreover, $Y^{\log}$ is* **not submonic**, *then there exists a* **minimal point** *$W^{\log} \to Y^{\log}$, where $W^{\log}$ is* **submonic**, *that is* **not isomorphic** *[i.e., over $Y^{\log}$] to $Z^{\log} \to Y^{\log}$.*

(ii) $Y^{\log}$ is a **minimal** *object of $\text{Sch}^{\log}(X^{\log})$ if and only if $Y^{\log}$ is* **reduced**, **one-pointed**, *and* **submonic**. *Put another way, $Y^{\log}$ is a* **minimal** *object of $\text{Sch}^{\log}(X^{\log})$ if and only if, for some field k, $Y^{\log}$ is either equal to $\text{Spec}(k)$ equipped with the* **trivial** *log structure or equal to $\text{Spec}(k)$ equipped with the log structure $\mathbb{N} \ni 1 \mapsto 0 \in k$.*

(iii) Suppose that $Y^{\log}$ and $Z^{\log}$ are **minimal** *objects of $\text{Sch}^{\log}(X^{\log})$. If $f^{\log} : Z^{\log} \to Y^{\log}$ is a morphism in $\text{Sch}^{\log}(X^{\log})$, then let us write*

$$\text{MinLg}(f^{\log}) \ \in \ \mathbb{N} \ \cup \ \{+\infty\}$$

for the **"minimal length"** *of $f^{\log}$: that is to say, we set $\text{MinLg}(f^{\log}) \stackrel{\text{def}}{=} 0$ if $f^{\log}$ is an* **isomorphism**; *if $f^{\log}$ is not an isomorphism, then we take $\text{MinLg}(f^{\log})$ to be the* **supremum** *of the set of positive integers n such that $f^{\log}$ admits a factorization*

$$Z_n^{\log} \stackrel{\text{def}}{=} Z^{\log} \ \to \ Z_{n-1}^{\log} \ \to \ \dots \ \to \ Z_1^{\log} \ \to \ Z_0^{\log} \stackrel{\text{def}}{=} Y^{\log}$$

as a composite of morphisms of $\text{Sch}^{\log}(X^{\log})$ which are **not isomorphisms** *such that, for each $i = 1, \dots, n$, $Z_i^{\log}$ is a minimal object of $\text{Sch}^{\log}(X^{\log})$. Then $Y^{\log}$ is of* **rank one** *if and only if $\text{MinLg}(f^{\log})$ is* **finite** *for every morphism $f^{\log}$ in $\text{Sch}^{\log}(X^{\log})$ with codomain equal to $Y^{\log}$ [and domain given by some minimal object].*

Proof. First, we consider assertion (i). Observe that we may assume without loss of generality that $Y^{\log}$ is *reduced* and *one-pointed* [cf. [LgSch], Proposition 1.1, (i)], and hence that the underlying scheme Y of $Y^{\log}$ may be written in the form $\text{Spec}(k_Y)$, for a suitable field k_Y. Next, let us consider the situation discussed in Lemma 1.5, (v), in the case where

- one takes the data that gives rise to "$S^{\log}$" to be the data that arises from $Y^{\log}$ [so "k" corresponds to k_Y];
- if $\mathrm{rk}(Y^{\log}) = 0$, then one takes the positive homomorphism "ξ" to be the identity morphism;
- if $\mathrm{rk}(Y^{\log}) \geq 1$, then one takes the positive homomorphism "ξ" to be the homomorphism "$\phi : P \to \mathbb{N}$" of Lemma 1.5, (ii).

Then one verifies immediately from the description of the torus "$T[\xi]_{\mathrm{red}}$" in Lemma 1.5, (iv), that any *splitting* as in Lemma 1.5, (i) — which, in the terminology of [ExtFam], Definition 1.3, may be regarded as a *"Galois-equivariant clean chart"* — determines a *k-rational point* of $S[\xi]$. In particular, by restricting the log structure of the *submonic* log scheme $S^{\log}[\xi]$ to this k-rational point, we obtain, by Proposition 1.4, (vii); Lemma 1.5, (vi), a *log-like monomorphism* $f^{\log} : Z^{\log} \to Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$, for some *submonic* $Z^{\log}$. Since, by [LgSch], Proposition 2.4, (ii), (iii), $Z^{\log}$ is necessarily *minimal*, we thus conclude that the morphism $f^{\log}$ determines a *minimal point* of $Y^{\log}$, as desired. In a similar vein, if $Y^{\log}$ is *not submonic* [i.e., is of rank $n \geq 2$], then we consider the situation discussed in Lemma 1.5, (v), in the case where one takes the data that gives rise to "$S^{\log}$" to be the data that arises from $Y^{\log}$ [so "k" corresponds to k_Y], and one takes the positive homomorphism "ξ" to be the homomorphism "$\psi : P \to \mathbb{N}$" of Lemma 1.5, (ii). Then a *splitting* as in Lemma 1.5, (i) determines a *rational point* of $S[\xi]$ over a suitable finite separable extension field k_W of k_Y [i.e., corresponding to the subgroup "$H \subseteq G_k$" of Lemma 1.5, (v)]. Now, by restricting the log structure of the *submonic* log scheme $S^{\log}[\xi]$ to this k_W-rational point, we obtain, by Proposition 1.4, (vii); Lemma 1.5, (vi), a *monomorphism* $W^{\log} \to Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$, for some *submonic* $W^{\log}$ whose underlying scheme W is equal to $\mathrm{Spec}(k_W)$, which determines, by [LgSch], Proposition 2.4, (iii), a *minimal point* of $Y^{\log}$ that is *not isomorphic* [i.e., over $Y^{\log}$] to $f^{\log} : Z^{\log} \to Y^{\log}$. This completes the proof of assertion (i).

Next, we consider assertion (ii). The *sufficiency* portion of assertion (ii) follows immediately from [LgSch], Proposition 2.4, (ii), (iii). Thus, to complete the proof of assertion (ii), it suffices to verify the *necessity* portion of assertion (ii). To this end, suppose that $Y^{\log}$ is *minimal*. Then it follows from [LgSch], Proposition 2.4, (i), that $Y^{\log}$ is *reduced* and *one-pointed*, i.e., that $Y = \mathrm{Spec}(k_Y)$, for some field k_Y, and hence, from assertion (i), that there exists a *minimal point* $f^{\log} : Z^{\log} \to Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$, for some *submonic* $Z^{\log}$. If $Y^{\log}$ is *not submonic*, then it follows that $f^{\log}$ is *not* an isomorphism, i.e., in *contradiction* to the assumed *minimality* of $Y^{\log}$. This completes the proof of assertion (ii).

Finally, we consider assertion (iii). First, let us observe that it follows from assertion (ii) that the underlying scheme Y (respectively, Z) of $Y^{\log}$ (respectively, $Z^{\log}$) may be written in the form $\mathrm{Spec}(k_Y)$ (respectively, $\mathrm{Spec}(k_Z)$), for a suitable field k_Y (respectively, k_Z). Then if $Y^{\log}$ is of *rank one*, then the *finiteness* of $\mathrm{MinLg}(f^{\log})$ follows immediately by considering the finiteness of the extension degree $[k_Z : k_Y]$, together with the simple, well-understood structure of the monoid $\mathbb{N}$. On the other hand, if $Y^{\log}$ is of *rank zero*, but $Z^{\log}$ is of *rank one*, then the fact that $\mathrm{MinLg}(f^{\log}) = +\infty$ follows by considering the *infinite descending sequence of submonoids* $\mathbb{N} \supseteq 2 \cdot \mathbb{N} \supseteq \ldots \supseteq 2^n \cdot \mathbb{N} \supseteq \ldots$, for $1 \leq n \in \mathbb{N}$. The completes the proof of assertion (iii). ○

Proposition 1.7. (Monomorphisms from log-nodal objects into non-submonic objects) *Suppose that* $Y^{\log}$ *is a* **non-submonic** *object of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then there exists a* **log-nodal** *object* $Z^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$ *that admits a* **monomorphism** $Z^{\log} \rightarrowtail Y^{\log}$.

Proof. As in the proof of Proposition 1.6, (i), one verifies immediately that we may assume without loss of generality that $Y^{\log}$ is *reduced* and *one-pointed*, i.e., that $Y = \mathrm{Spec}(k_Y)$, for some field k_Y. Now we consider the situation discussed in Lemma 1.5, (v), in the case where one takes the data that gives rise to "$S^{\log}$" to be the data that arises from $Y^{\log}$ [so "k" corresponds to k_Y], and one takes the positive homomorphism "ξ" to be the homomorphism "$\xi : P \to Q$" of Lemma 1.5, (iii). Then one verifies immediately that any *splitting* as in Lemma 1.5, (i), determines a *rational point* of $S[\xi]$ over a suitable finite separable extension field k_Z of k_Y [i.e., corresponding to the subgroup "$H \subseteq G_k$" of Lemma 1.5, (v)] such that the log scheme $Z^{\log}$ obtained by restricting the log structure of the log scheme $S^{\log}[\xi]$ to this k_Z-rational point determines an element of $\Xi_+(-) \subseteq \Xi(-)$. Thus, we obtain, by Proposition 1.4, (vii); Lemma 1.5, (vi), a *monomorphism* $Z^{\log} \to Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$, for some *reduced*, *one-pointed*, *split* [cf. the final portion of Lemma 1.5, (iii)] $Z^{\log}$ of *rank two* [cf. Lemma 1.5, (iii)] whose underlying scheme Z is equal to $\mathrm{Spec}(k_Z)$, as desired. ◯

Proposition 1.8. (Submonic one-pointed log schemes) *Suppose that* $Y^{\log}$ *is an object of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then* $Y^{\log}$ *is* **submonic** *and* **one-pointed** *if and only if* $\mathrm{MinPt}(Y^{\log})$ *is of* **cardinality one**.

Proof. First, we verify *necessity.* Suppose that $Y^{\log}$ is *submonic* and *one-pointed.* Then it follows that $Y^{\log}_{\mathrm{red}}$ [cf. the discussion entitled "Log schemes" in §0] is *reduced*, *one-pointed*, and *submonic*, hence, by Proposition 1.6, (ii), that $Y^{\log}_{\mathrm{red}}$ is *minimal.* Since any morphism from a [necessarily *reduced*, by Proposition 1.6, (ii)!] minimal object of $\mathrm{Sch}^{\log}(X^{\log})$ to $Y^{\log}$ clearly factors uniquely through $Y^{\log}_{\mathrm{red}}$, we thus conclude that $\mathrm{MinPt}(Y^{\log})$ is of *cardinality one*, and that the unique element of $\mathrm{MinPt}(Y^{\log})$ arises from the natural inclusion $Y^{\log}_{\mathrm{red}} \hookrightarrow Y^{\log}$. This completes the proof of *necessity.* Next, we verify *sufficiency.* Suppose that $\mathrm{MinPt}(Y^{\log})$ is of *cardinality one.* Then by applying the initial portion of Proposition 1.6, (i), to the objects "$Z^{\log}$" of $\mathrm{Sch}^{\log}(X^{\log})$ obtained by considering *scheme-like* monomorphisms $Z^{\log} \rightarrowtail Y^{\log}$ that arise from monomorphisms $Z \rightarrowtail Y$ in $\mathrm{Sch}(X)$ for reduced, one-pointed Z [cf. Proposition 1.4, (vii); [LgSch], Proposition 1.1, (iii)], we conclude that $Y^{\log}$ is *one-pointed.* Thus, by applying the final portion of Proposition 1.6, (i), to $Y^{\log}$, we conclude that $Y^{\log}$ is *submonic.* This completes the proof of *sufficiency.* ◯

Before proceeding, we review a well-known consequence of the general theory of fs log schemes.

Lemma 1.9. (Specialization morphisms associated to characteristic sheaves) *Suppose that the underlying scheme* Y *of* $Y^{\log}$ *is the spectrum of a* **strict**

henselian domain A. *Write $\overline{s}$ for the tautological geometric point of Y associated to the unique closed point of Y. Let $\overline{\eta}$ be a geometric point of Y whose image in Y is the unique generic point of Y. In the following, we shall use subscripted "$\overline{s}$'s" and "$\overline{\eta}$'s" to denote the respective fibers at $\overline{s}$, $\overline{\eta}$ of sheaves on the étale site of Y. Then the natural* **"specialization morphism"**

$$P_{Y,\overline{s}} \ \rightarrow \ P_{Y,\overline{\eta}}$$

is **surjective**. *In particular, this specialization morphism is an* **isomorphism** *if and only if* $\mathrm{rk}(P_{Y,\overline{s}}) = \mathrm{rk}(P_{Y,\overline{\eta}})$. *Finally, if* $\mathrm{rk}(P_{Y,\overline{\eta}}) \geq 1$, *and* $a \in P_{Y,\overline{s}}$ *is a* **sum-dominator** *[cf. the discussion entitled "Generalities on monoids" in §0] such that, for elements $a^* \in M_{Y,\overline{s}}$ and $f \in A$, it holds that $a^* \mapsto a$, $a^* \mapsto f$, then $f = 0$.*

Proof. The asserted *surjectivity* follows immediately from the existence, étale locally, of *charts* that give rise to the log structure of $Y^{\log}$. If $\mathrm{rk}(P_{Y,\overline{s}}) = \mathrm{rk}(P_{Y,\overline{\eta}})$, then we thus obtain a *surjection* $P^{\mathrm{gp}}_{Y,\overline{s}} \twoheadrightarrow P^{\mathrm{gp}}_{Y,\overline{\eta}}$ between free abelian groups of the same rank; since such a surjection is necessarily an *isomorphism*, we thus conclude from the inclusion $P_Y \hookrightarrow P^{\mathrm{gp}}_Y$, that the specialization morphism $P_{Y,\overline{s}} \ \rightarrow \ P_{Y,\overline{\eta}}$ is an *isomorphism*, as desired. Finally, we observe that if $\mathrm{rk}(P_{Y,\overline{\eta}}) \geq 1$, and $M_{Y,\overline{s}} \ni a^* \mapsto f \in A$, where a^* lifts a *sum-dominator* $a \in P_{Y,\overline{s}}$, then, in light of the *surjectivity* of the specialization morphism $P_{Y,\overline{s}} \rightarrow P_{Y,\overline{\eta}}$, it follows immediately from the discussion of *sum-dominators* in §0 that a maps to a *nonzero* element $b \in P_{Y,\overline{\eta}}$. On the other hand, if we write K for the quotient field of A, then it follows immediately from the definition of the notion of a *log structure* that the image $f \in A \subseteq K$ of any lifting $b^* \in M_{Y,\overline{\eta}}$ of the element $b \in P_{Y,\overline{\eta}}$ in K is *noninvertible*, hence 0, as desired. This completes the proof of Lemma 1.9. ◯

Proposition 1.10. (Lower bounds on the submonic dimension) *Suppose that $Y^{\log}$ is an object of* $\mathrm{Sch}^{\log}(X^{\log})$, *and that $Z^{\log} \rightarrowtail Y^{\log}$ is a monomorphism of* $\mathrm{Sch}^{\log}(X^{\log})$ *such that, for suitable $n, d \in \mathbb{N}$, the log scheme $Z^{\log}$ is of* **rank n**, *and the underlying scheme Z of $Z^{\log}$ is of* **dimension d**. *Then if $n \geq 1$ (respectively, $n = 0$), then the* **submonic dimensions** $\dim^{\mathrm{sm}}(Y^{\log})$, $\dim^{\mathrm{sm}}(Z^{\log})$ *of $Y^{\log}$, $Z^{\log}$ satisfy the inequality*

$$\dim^{\mathrm{sm}}(Y^{\log}) \geq \dim^{\mathrm{sm}}(Z^{\log}) = d + n - 1$$
$$\textit{(respectively, } \dim^{\mathrm{sm}}(Y^{\log}) \geq \dim^{\mathrm{sm}}(Z^{\log}) = d\textit{)}.$$

Proof. First, let us observe that it follows immediately from the definition of *submonic dimension* [cf. Definition 1.2, (i)] that $\dim^{\mathrm{sm}}(Y^{\log}) \geq \dim^{\mathrm{sm}}(Z^{\log})$. In particular, we may assume without loss of generality that $Z^{\log} = Y^{\log}$. Thus, it follows immediately from Lemma 1.9 that the characteristic sheaf P_Y is *locally constant*. Next, by replacing $Y^{\log}$ by the log scheme determined by a suitable subscheme of Y, one verifies immediately we may assume without loss of generality that the scheme Y is *integral*. Now the case where $n = 0$ is immediate [cf. Proposition 1.4, (vi)], so we may assume without loss of generality that $n \geq 1$. Thus, we may apply the theory reviewed in Lemma 1.5 to the *generic point* of Y. Moreover, one verifies immediately from the fact that P_Y is *locally constant* that the objects [and properties of these objects] discussed in this theory extend to objects [and properties of these objects] over the *entire scheme* Y [i.e., not just the generic point of Y]. In particular, by applying Lemma 1.5, (iv), (v), (vi), where we take the fs monoid "Q" to be $\mathbb{N}$, we conclude that given any *monomorphism* $W^{\log} \rightarrowtail Y^{\log}$, where $W^{\log}$ is a *submonic* object of $\mathrm{Sch}^{\log}(X^{\log})$ whose underlying scheme W is *integral*, there exists a *monomorphism* $V^{\log} \rightarrowtail Y^{\log}$, where $V^{\log}$ is a *submonic* object of $\mathrm{Sch}^{\log}(X^{\log})$ whose underlying scheme V is a *family of* $(n-1)$*-dimensional tori* [cf. Lemma 1.5, (iv)] over Y, such that the monomorphism $W^{\log} \rightarrowtail Y^{\log}$ *factors* as a composite of *monomorphisms* $W^{\log} \rightarrowtail V^{\log} \rightarrowtail Y^{\log}$. In particular, $\dim(W) \leq \dim(V) = d + n - 1$ [cf. Proposition 1.4, (vi)], so we conclude that $\dim^{\mathrm{sm}}(Y^{\log}) = d + n - 1$, as desired. ◯

The following generalities on *log-like* and *scheme-like* morphisms will be of use in the remainder of the present paper.

Proposition 1.11. **(Generalities on log-like and scheme-like morphisms)** *Let* $f^{\log} : Z^{\log} \to Y^{\log}$ *be a morphism of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then:*

(i) Write $U^{\log}$ *for the log scheme whose underlying scheme is equal to the underlying scheme* Z *of* $Z^{\log}$ *and whose log structure is the pull-back of the log structure of* $Y^{\log}$ *via the underlying morphism of schemes* $f : Z = U \to Y$ *associated to* $f^{\log}$. *Then* $U^{\log}$ *may be regarded, in a natural way, as an object of* $\mathrm{Sch}^{\log}(X^{\log})$, *and there exists a* **natural factorization**

$$Z^{\log} \xrightarrow{f_1^{\log}} U^{\log} \xrightarrow{f_2^{\log}} Y^{\log}$$

of $f^{\log}$ *in* $\mathrm{Sch}^{\log}(X^{\log})$, *where* $f_1^{\log}$ *is* **log-like**, *and* $f_2^{\log}$ *is* **scheme-like**.

(ii) The factorization $Z^{\log} \xrightarrow{f_1^{\log}} U^{\log} \xrightarrow{f_2^{\log}} Y^{\log}$ *of (i) may be* **characterized**, *up to a* **unique isomorphism**, *via the following* **universal property**: *The morphism* $f_2^{\log}$ *is* **scheme-like**, *and, moreover, if*

$$Z^{\log} \xrightarrow{h_1^{\log}} V^{\log} \xrightarrow{h_2^{\log}} Y^{\log}$$

is a factorization of $f^{\log}$ *in* $\mathrm{Sch}^{\log}(X^{\log})$ *such that* $h_2^{\log}$ *is* **scheme-like**, *then there exists a* **unique scheme-like** *morphism* $g^{\log} : U^{\log} \to V^{\log}$ *such that* $h_1^{\log} = g^{\log} \circ f_1^{\log}$, $h_2^{\log} \circ g^{\log} = f_2^{\log}$.

(iii) Base-change via the morphism $f_1^{\log} : Z^{\log} \to U^{\log}$ *of (i) determines an* **equivalence of categories**

$$\mathrm{Sch}^{\log}(U^{\log})|_{\text{sch-lk}} \xrightarrow{\sim} \mathrm{Sch}^{\log}(Z^{\log})|_{\text{sch-lk}}$$

[cf. the notational conventions of Definition 1.1, (iv)]. The morphism $f_2^{\log} : U^{\log} \to Y^{\log}$ *of (i) — which may be regarded as an* **object** *of* $\mathrm{Sch}^{\log}(Y^{\log})|_{\text{sch-lk}}$ *— determines an* **equivalence of categories**

$$\mathrm{Sch}^{\log}(U^{\log})|_{\text{sch-lk}} \xrightarrow{\sim} \left\{ \mathrm{Sch}^{\log}(Y^{\log})|_{\text{sch-lk}} \right\}_{f_2^{\log}}$$

of $\mathrm{Sch}^{\log}(U^{\log})$ *with the category* $\left\{\mathrm{Sch}^{\log}(Y^{\log})|_{\text{sch-lk}}\right\}_{f_2^{\log}}$ *of* **objects** *of the category* $\mathrm{Sch}^{\log}(Y^{\log})|_{\text{sch-lk}}$ *equipped with a structure morphism to the object* $f_2^{\log}$ *of* $\mathrm{Sch}^{\log}(Y^{\log})|_{\text{sch-lk}}$ *and* **morphisms** *of the category* $\mathrm{Sch}^{\log}(Y^{\log})|_{\text{sch-lk}}$ *that are compatible with the structure morphisms to the object* $f_2^{\log}$*.*

Proof. Assertions (i), (ii), and (iii) follow immediately from the various definitions involved. ◯

Section 2: The Scheme Structure of Submonic Log Schemes

In the present §2, we give a *category-theoretic reconstruction* of the underlying *scheme structure* of *submonic* objects of the categories of log schemes defined in §1.

We maintain the notation of §1.

Definition 2.1. Let $f^{\log} : Z^{\log} \to Y^{\log}$ be a morphism of $\mathrm{Sch}^{\log}(X^{\log})$. Then we shall say that the morphism $f^{\log}$ is *SLEM* [i.e., a *"submonically log étale monomorphism"*] if $f^{\log}$ is a *monomorphism* in $\mathrm{Sch}^{\log}(X^{\log})$, and, moreover, for any commutative diagram

$$\begin{array}{ccc} V^{\log} & \longrightarrow & Z^{\log} \\ \downarrow & & \downarrow f^{\log} \\ W^{\log} & \longrightarrow & Y^{\log} \end{array}$$

— where $V^{\log}$ and $W^{\log}$ are *one-pointed* and *submonic*, and the left-hand vertical arrow is a *monomorphism* in $\mathrm{Sch}^{\log}(X^{\log})$ — of objects and morphisms in $\mathrm{Sch}^{\log}(X^{\log})$, there exists a unique [*"lifting"*] morphism $W^{\log} \to Z^{\log}$ that renders the two resulting triangles in the above diagram *commutative*.

Proposition 2.2. (SLEM morphisms and open immersions) *Let* $f^{\log} : Z^{\log} \to Y^{\log}$ *be a morphism of* $\mathrm{Sch}^{\log}(X^{\log})$*. Thus, the underlying morphism* $f : Z \to Y$ *of* $f^{\log}$ *may be regarded as a morphism of* $\mathrm{Sch}(X)$*. Then:*

(i) If $f^{\log}$ *is an* **open immersion** *[cf. the discussion entitled "Log schemes" in §0], then* $f^{\log}$ *is* **SLEM**.

(ii) If $Y^{\log}$ *is* **submonic***, and* $f^{\log}$ *is* **SLEM***, then* $f^{\log}$ *is an* **open immersion**.

Proof. First, let us observe recall that any *monomorphism* between *one-pointed objects* in $\mathrm{Sch}(X)$ is necessarily a *closed immersion* between spectra of *artinian rings* [cf., e.g., the proof of [LgSch], Corollary 1.2]. In particular, it follows from Proposition 1.4, (vi), that any monomorphism $V^{\log} \to W^{\log}$ as in Definition 2.1 is necessarily *scheme-like*, and, moreover, that the underlying morphism of schemes associated to any monomorphism $V^{\log} \to W^{\log}$ as in Definition 2.1 is necessarily a *closed immersion* between spectra of *artinian rings*. Thus, it is immediate that if

$f^{\log}$ is an *open immersion*, then $f^{\log}$ is *SLEM*. This completes the proof of assertion (i). Now suppose that $Y^{\log}$ is *submonic*, and $f^{\log}$ is *SLEM*. Thus, it follows from Proposition 1.4, (vi), that $f^{\log}$ is *scheme-like*, and, hence, from Proposition 1.4, (v), that f is a *monomorphism* in $\mathrm{Sch}(X)$. In particular, the existence of unique *liftings* as stipulated in Definition 2.1 implies that f is an *étale monomorphism* in $\mathrm{Sch}(X)$, hence [cf., e.g., [LgSch], Corollary 1.3] an *open immersion*. This completes the proof of assertion (ii). $\bigcirc$

Proposition 2.3. (Connectedness with respect to SLEM localizations)

(i) Let $S^{\log}$ *be a* **connected** *[hence* **nonempty***] object of* $\mathrm{Sch}^{\log}(X^{\log})$ *[cf. Proposition 1.3];* $U^{\log}$, $\{V_i^{\log}\}_{i\in\mathbb{N}}$ *nonempty objects of* $\mathrm{Sch}^{\log}(X^{\log})$*;* $U^{\log} \rightarrowtail S^{\log}$, $\{V_i^{\log} \rightarrowtail S^{\log}\}_{i\in\mathbb{N}}$ **SLEM** *morphisms of* $\mathrm{Sch}^{\log}(X^{\log})$ *such that, for each* $i \in \mathbb{N}$, *the morphism* $V_i^{\log} \rightarrowtail S^{\log}$ *admits a [necessarily unique] factorization* $V_i^{\log} \rightarrowtail V_{i+1}^{\log} \rightarrowtail S^{\log}$ *through the morphism* $V_{i+1}^{\log} \rightarrowtail S^{\log}$, *and, moreover, the fiber product* $U^{\log} \times_{S^{\log}} V_i^{\log}$ *[in* $\mathrm{Sch}^{\log}(X^{\log})$*] is* **empty**. *Then the natural map*

$$\mathrm{MinPt}(U^{\log}) \coprod \left\{ \bigcup_{i\in\mathbb{N}} \mathrm{MinPt}(V_i^{\log}) \right\} \to \mathrm{MinPt}(S^{\log})$$

is **injective**.

(ii) In the situation of (i), suppose further that $S^{\log}$ *is* **submonic**. *Then the natural map of (i) is* **never surjective**.

(iii) Suppose that $S^{\log}$ *is a* **log-nodal** *object of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then, for suitable choices of* $U^{\log} \rightarrowtail S^{\log}$ *and* $\{V_i^{\log} \rightarrowtail S^{\log}\}_{i\in\mathbb{N}}$ *as in (i), the natural map of (i) is* **surjective**.

(iv) Let $T^{\log}$ *be an object of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then* $T^{\log}$ *is* **non-submonic** *if and only if there exist morphisms* $U^{\log} \rightarrowtail S^{\log}$ *and* $\{V_i^{\log} \rightarrowtail S^{\log}\}_{i\in\mathbb{N}}$ *as in (i), together with a monomorphism* $S^{\log} \rightarrowtail T^{\log}$ *in* $\mathrm{Sch}^{\log}(X^{\log})$, *such that the natural map of (i) is* **surjective**.

Proof. First, we consider assertion (i). Let $i \in \mathbb{N}$. Then the injectivity of each of the natural maps $\mathrm{MinPt}(U^{\log}) \to \mathrm{MinPt}(S^{\log})$, $\mathrm{MinPt}(V_i^{\log}) \to \mathrm{MinPt}(S^{\log})$ follows immediately from the the definition of "$\mathrm{MinPt}(-)$". The fact that the images of these two maps are disjoint follows immediately from the the definition of "$\mathrm{MinPt}(-)$", together with the assumption that the fiber product $U^{\log} \times_{S^{\log}} V_i^{\log}$ is *empty*. This completes the proof of assertion (i).

Next, we consider assertion (ii). Since $S^{\log}$ is *submonic*, it follows from Proposition 2.2, (ii), that the morphisms $U^{\log} \rightarrowtail S^{\log}$, $\{V_i^{\log} \rightarrowtail S^{\log}\}_{i\in\mathbb{N}}$ are *open immersions*. Since [the underlying scheme of] $S^{\log}$ is *connected*, it thus follows from the assumption that the objects $U^{\log}$, $\{V_i^{\log}\}_{i\in\mathbb{N}}$ are *nonempty*, whereas the fiber products $\{U^{\log} \times_{S^{\log}} V_i^{\log}\}_{i\in\mathbb{N}}$ are *empty*, that the open subscheme of $S^{\log}$ determined by the union of the images of the morphisms $U^{\log} \rightarrowtail S^{\log}$, $\{V_i^{\log} \rightarrowtail S^{\log}\}_{i\in\mathbb{N}}$ does *not* coincide with $S^{\log}$, and hence [cf. Proposition 1.6, (i)] that the natural map of (i) is *not surjective*. This completes the proof of assertion (ii).

Next, we consider assertion (iii). First, let us observe that, in light of the various assumptions imposed on $S^{\log}$, one verifies immediately that $S^{\log}$ may be regarded as the "$S^{\log}$" that appears in Lemma 1.5, (viii). Here, the *positive homomorphism* $\phi_0 : P \to J_0 = \mathbb{N}$ of Lemma 1.5, (viii), may be taken to be the positive homomorphism "ϕ" of Lemma 1.5, (ii). In particular, we also obtain homomorphisms $\phi_1 : P \to J_1$ and $\phi_2 : P \to J_2$. Now we apply Example 0.1, where we take "P" to be P and "${}^{\infty}P$" to be J_2. This yields an *infinite descending sequence*

$$P \subseteq J_2 \subseteq \ldots \subseteq {}^{i}P \subseteq \ldots \subseteq {}^{1}P \subseteq {}^{0}P$$

— where $i \in \mathbb{N}$ — of submonoids of P^{gp} satisfying various properties as described in Example 0.1. Suppose that, for $i \in \mathbb{N}$, ${}^{i}P$ is obtained as the *bisecting monoid* of P at a *positive homomorphism* ${}^{i}\psi_0 : P \to \mathbb{N}$ that is assigned the *index* "2".

Thus, for $i \in \mathbb{N}$, the *log étale monomorphism*

$$S^{\log}[{}^{i}\psi_{\{0,2\}}] \rightarrowtail S^{\log}$$

of Lemma 1.5, (vii), (viii) [i.e., where we take "ϕ_0" to be ${}^{i}\psi_0$] *factors* through the *log étale monomorphism*

$$S^{\log}[{}^{i+1}\psi_{\{0,2\}}] \rightarrowtail S^{\log}$$

of Lemma 1.5, (vii), (viii) [i.e., where we take "ϕ_0" to be ${}^{i+1}\psi_0$], as well as through the *log étale monomorphism*

$$S^{\log}[\phi_{\{0,2\}}] \rightarrowtail S^{\log}$$

of Lemma 1.5, (vii), (viii) [i.e., where we take "ϕ_0" to be ϕ_0]. In particular, it follows from the fact that $S^{\log}[\phi_{\{0,1\}}] \times_{S^{\log}} S^{\log}[\phi_{\{0,2\}}] = S^{\log}[\phi_{\{0\}}]$ [cf. Lemma 1.5, (vii), (viii)], together with the discussion of Example 0.1, that the fiber product $S^{\log}[\phi_{\{0,1\}}] \times_{S^{\log}} S^{\log}[{}^{i}\psi_{\{0,2\}}]$ is *empty*.

Thus, in summary, if we take $U^{\log} \rightarrowtail S^{\log}$ to be the morphism

$$S^{\log}[\phi_{\{0,1\}}] \rightarrowtail S^{\log}$$

and, for $i \in \mathbb{N}$, $V_i^{\log} \rightarrowtail S^{\log}$ to be the morphism

$$S^{\log}[{}^{i}\psi_{\{0,2\}}] \rightarrowtail S^{\log}$$

discussed above, then we obtain *data as in assertion (i)*. Note, moreover, that it follows immediately from the discussion of Example 0.1 that the natural map of assertion (i) is *surjective*, as desired. This completes the proof of assertion (iii).

Finally, we observe that the *sufficiency* (respectively, *necessity*) portion of assertion (iv) follows formally from assertion (ii) (respectively, (iii)), together with Proposition 1.4, (vi) (respectively, together with Proposition 1.7, applied to $T^{\log}$). This completes the proof of assertion (iv). ◯

Proposition 2.4. (Characterization of scheme-like morphisms between minimal objects) *Let $h^{\log} : T^{\log} \to S^{\log}$ be a morphism between* **minimal** *objects* *of* $\mathrm{Sch}^{\log}(X^{\log})$*. Set* $r \stackrel{\mathrm{def}}{=} \mathrm{rk}(S^{\log}) \in \{0,1\}$ *[cf. Proposition 1.6, (ii)]. Then* $h^{\log}$ *is* **scheme-like** *if and only if there exists a* **connected**, **submonic** *object* $Z^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$ *such that the domain of every minimal point of* $Z^{\log}$ *is of* **rank** $\boldsymbol{r}$*, and, moreover,* $h^{\log}$ *admits a* **factorization**

$$T^{\log} \to Z^{\log} \to S^{\log}$$

as the composite of a **minimal point** $T^{\log} \rightarrowtail Z^{\log}$ *of* $Z^{\log}$ *with a morphism* $Z^{\log} \to S^{\log}$ *that admits a* **section** $S^{\log} \to Z^{\log}$ *[i.e., such that the composite* $S^{\log} \to Z^{\log} \to S^{\log}$ *is the identity morphism].*

Proof. First, we observe that since the underlying morphism of schemes $T \to S$ necessarily arises from [i.e., by applying "Spec(−)" to] a *finite extension of fields*, the asserted *necessity* follows immediately by taking $Z^{\log} \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\mathbb{Z}} \times_{\mathbb{Z}} S^{\log}$ [i.e., the *projective line* over $S^{\log}$]. Here, we note that the fact that "the domain of every minimal point of *this* $Z^{\log}$ is of *rank* r" follows immediately from Proposition 1.4, (vi). Thus, it remains to verify *sufficiency*. First, let us observe that it follows from the manifestly *constructible nature* of the characteristic sheaf P_Z [cf. also Propositions 1.4, (vi); 1.6, (i), (ii)] that the assumption that "the domain of every minimal point of $Z^{\log}$ is of *rank* r" implies that $Z^{\log}$ itself is of *rank* r, and hence [cf. Lemma 1.9] that the characteristic sheaf P_Z is *locally constant*. Since the monoids 0 and $\mathbb{N}$ have *no nontrivial automorphisms*, we thus conclude that the characteristic sheaf P_Z is *constant*, with fibers isomorphic to the monoid 0 (respectively, $\mathbb{N}$) if $r = 0$ (respectively, $r = 1$). The existence of the *section* $S^{\log} \to Z^{\log}$ thus implies that the morphism $Z^{\log} \to S^{\log}$ is *scheme-like*. Since the monomorphism $T^{\log} \rightarrowtail Z^{\log}$ is also *scheme-like* [cf. Proposition 1.4, (vi)], we thus conclude that $h^{\log}$ is *scheme-like*, as desired. This completes the proof of *sufficiency* and hence of Proposition 2.4. ◯

Proposition 2.5. **(Characterization of scheme-like morphisms between submonic objects)** *Let* $f^{\log} : Z^{\log} \to Y^{\log}$ *be a morphism between* **submonic** *objects of* $\mathrm{Sch}^{\log}(X^{\log})$*. Then* $f^{\log}$ *is* **scheme-like** *if and only if, for every* **minimal point** $T^{\log} \rightarrowtail Z^{\log}$ *of* $Z^{\log}$*, there exists a* **minimal point** $S^{\log} \rightarrowtail Y^{\log}$ *of* $Y^{\log}$ *and a* **scheme-like** *morphism* $T^{\log} \to S^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$ *that fit into a commutative diagram*

$$\begin{array}{ccc} T^{\log} & \rightarrowtail & Z^{\log} \\ \downarrow & & \downarrow f^{\log} \\ S^{\log} & \rightarrowtail & Y^{\log} \end{array}$$

of objects of $\mathrm{Sch}^{\log}(X^{\log})$.

Proof. The asserted *necessity* is immediate from the definitions and Propositions 1.4, (vi), (vii); 1.6, (ii). The asserted *sufficiency* follows immediately, in light of the manifestly *constructible nature* of the characteristic sheaves P_Z, P_Y, from the definitions and Propositions 1.4, (vi); 1.6, (i), (ii). ◯

Theorem 2.6. **(Reconstruction of the scheme structure of submonic objects)** *For* $i = 1, 2$*, let* $X_i^{\log}$ *be a* **locally noetherian fs log scheme** *[cf. the* *discussion entitled "Log schemes" in §0]. For $i = 1, 2$, we shall write* $\mathrm{Sch}^{\log}(X_i^{\log})$ *for the category defined at the beginning of §1. Let*

$$\Phi : \mathrm{Sch}^{\log}(X_1^{\log}) \ \stackrel{\sim}{\to}\ \mathrm{Sch}^{\log}(X_2^{\log})$$

be an [arbitrary!] **equivalence of categories***. Then:*

(i) Φ *preserves the following:*
(i-a) **monomorphisms***;*
(i-b) **empty** *objects;*
(i-c) **connected** *objects;*
(i-d) **minimal** *objects;*
(i-e) **minimal points***;*
(i-f) **submonic one-pointed** *objects;*
(i-g) **ranks** *of minimal objects;*
(i-h) **SLEM** *morphisms;*
(i-i) **submonic** *objects;*
(i-j) **scheme-like** *morphisms between* **minimal** *objects;*
(i-k) **scheme-like** *morphisms between* **submonic** *objects;*
(i-l) the **submonic dimension** *of objects.*

(ii) For $i = 1, 2$, let $Y_i^{\log}$ be an object of $\mathrm{Sch}^{\log}(X_i^{\log})$*; write Y_i for the underlying scheme of $Y_i^{\log}$. Suppose further that $\Phi(Y_1^{\log}) = Y_2^{\log}$. Thus, [cf. the portion of (i) concerning (i-i)] $Y_1^{\log}$ is* **submonic** *if only if $Y_2^{\log}$ is. Suppose that $Y_i^{\log}$ is* **submonic** *for $i = 1, 2$. Then Φ induces an* **equivalence of categories**

$$\Big(\mathrm{Sch}(Y_1)\ \stackrel{\sim}{\to}\Big)\quad \mathrm{Sch}^{\log}(Y_1^{\log})|_{\text{sch-lk}} \ \stackrel{\sim}{\to}\ \mathrm{Sch}^{\log}(Y_2^{\log})|_{\text{sch-lk}} \quad \Big(\stackrel{\sim}{\to}\ \mathrm{Sch}(Y_2)\Big)$$

— i.e., where the equivalences in parentheses are the natural equivalences of Definition 1.1, (iv) — that is **functorial** *[in the evident sense!] with respect to $Y_1^{\log}$, $Y_2^{\log}$. Finally, the composite of the equivalences of categories in the above display induces, by applying [LgSch], Theorem 1.7, (ii), an* **isomorphism of schemes**

$$Y_1 \ \stackrel{\sim}{\to}\ Y_2$$

that is **functorial** *[in the evident sense!] with respect to $Y_1^{\log}$, $Y_2^{\log}$.*

Proof. First, we consider assertion (i). The preservation of (i-a) is a matter of general nonsense. The preservation of (i-b) follows from Proposition 1.3, (i). The preservation of (i-c) follows from Proposition 1.3, (ii). The preservation of (i-d) and (i-e) follows immediately from the preservation of (i-a). The preservation of (i-f) follows immediately, in light of Proposition 1.8, from the preservation of (i-e). The preservation of (i-g) follows immediately, in light of Proposition 1.6, (iii), from the preservation of (i-d). The preservation of (i-h) follows immediately from the preservation of (i-a) and (i-f). The preservation of (i-i) follows immediately, in light of Proposition 2.3, (iv), from the preservation of (i-a), (i-b), (i-c), and (i-h). The preservation of (i-j) follows immediately, in light of Proposition 2.4, from the preservation of (i-c), (i-d), (i-e), (i-g), and (i-i). The preservation of (i-k) follows immediately, in light of Proposition 2.5, from the preservation of (i-e), (i-i), and (i-j). This completes the proof of assertion (i), except for the verification of the preservation of (i-l). Assertion (ii) follows immediately [i.e., in the spirit of [LgSch], Corollary 2.15] from the portion of assertion (i) concerning the preservation of (i-k). Here, we note that the *functoriality* of the isomorphism of schemes in the final display in the statement of assertion (ii) follows immediately from the *characterization* given in Proposition 1.11, (ii), of the *factorization* discussed in Proposition 1.11, (i), together with the *natural equivalences of categories* discussed in Proposition 1.11, (iii). Finally, the portion of assertion (i) concerning the preservation of (i-l) follows from the portion of assertion (i) concerning the preservation of (i-a), (i-i), together with the *isomorphisms of schemes* obtained in assertion (ii). ◯

Lemma 2.7. (Characterization of isomorphisms among positive homomorphisms) *Let* $\xi : P \to Q$ *be a* **positive** *homomorphism between* **fs monoids** *such that* $\mathrm{rk}(P) \geq \mathrm{rk}(Q)$*, and, moreover, the following condition is satisfied:*

> *Every positive homomorphism* $\phi : P \to \mathbb{N}$ *admits a* **factorization** $P \to Q \to \mathbb{N}$ *as a composite of* ξ *with a positive homomorphism* $\psi : Q \to \mathbb{N}$.

Then ξ *is an* **isomorphism***.*

Proof. First, let us observe that, by Lemma 1.5, (ii), there exists a *positive homomorphism* $\phi^\dagger : P \to \mathbb{N}$. Next, let us observe that if p is a prime number, then given a surjective homomorphism $\overline{\zeta} : P^{\mathrm{gp}} \twoheadrightarrow \mathbb{F}_p$, there exists a homomorphism $\zeta : P^{\mathrm{gp}} \to \mathbb{Z}$ whose composite with the natural surjection $\mathbb{Z} \twoheadrightarrow \mathbb{F}_p$ is equal to $\overline{\zeta}$. [Indeed, this follows immediately from the fact that P^{gp} is a *finitely generated free abelian group* — cf. the discussion entitled "Generalities on monoids" in §0.] In particular, it follows from the fact that P is a *finitely generated monoid* that, for *sufficiently large* $n \in \mathbb{N}$, the homomorphism $(\zeta + p^n \cdot (\phi^\dagger)^{\mathrm{gp}}) : P^{\mathrm{gp}} \to \mathbb{Z}$ *coincides* with $\overline{\zeta}$ when composed with the natural surjection $\mathbb{Z} \twoheadrightarrow \mathbb{F}_p$ and, moreover, determines a *positive* homomorphism $\phi : P \to \mathbb{N}$. In particular, it follows from the hypotheses imposed on ξ that ϕ admits a *factorization* $P \to Q \to \mathbb{N}$ as a composite of ξ with a positive homomorphism $\psi : Q \to \mathbb{N}$. Since the resulting composite $P^{\mathrm{gp}} \to Q^{\mathrm{gp}} \to \mathbb{Z} \twoheadrightarrow \mathbb{F}_p$ coincides with $\overline{\zeta}$, we thus conclude, by allowing p and $\overline{\zeta}$ to *vary*, that the reduction of the homomorphism of finitely generated free abelian groups $\xi^{\mathrm{gp}} : P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ modulo any prime number is *injective*, and, hence, since $\mathrm{rk}(P) \geq \mathrm{rk}(Q)$, that $\xi^{\mathrm{gp}} : P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is an *isomorphism.* That is to say, P and Q may be regarded as finitely generated saturated monoids within a *single* $\mathbb{Z}$-module $P^{\mathrm{gp}} \xrightarrow{\sim} Q^{\mathrm{gp}}$. In particular, it follows from well-known properties of fs monoids [cf., e.g., [LgSch], Lemma 2.5, (iv)] that the hypotheses imposed on ξ imply that ξ is an *isomorphism*, as desired. ◯

Proposition 2.8. (Characterization of scheme-like morphisms between reduced, one-pointed, non-minimal objects) *Let* $f^{\log} : Z^{\log} \to Y^{\log}$ *be a morphism between* **reduced***,* **one-pointed***,* **non-minimal** *objects of* $\mathrm{Sch}^{\log}(X^{\log})$*. Then* $f^{\log}$ *is* **scheme-like** *if and only if* $\dim^{\mathrm{sm}}(Z^{\log}) \leq \dim^{\mathrm{sm}}(Y^{\log})$*, and, moreover, the following condition is satisfied:*

Let $S^{\log}$ *be a* **minimal** *object of* $\mathrm{Sch}^{\log}(X^{\log})$, $h^{\log} : S^{\log} \to Y^{\log}$ *a morphism of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then there exists a commutative diagram of morphisms of* $\mathrm{Sch}^{\log}(X^{\log})$

$$\begin{array}{ccc} T^{\log} & \to & Z^{\log} \\ \downarrow & & \downarrow f^{\log} \\ S^{\log} & \xrightarrow{h^{\log}} & Y^{\log} \end{array}$$

in which the left-hand vertical arrow $T^{\log} \to S^{\log}$ *is a* **scheme-like** *morphism between* **minimal** *objects of* $\mathrm{Sch}^{\log}(X^{\log})$.

Proof. First of all, we observe that the asserted *necessity* follows immediately from Proposition 1.10, together with the definition of the term "scheme-like". Thus, it suffices to verify the *sufficiency* of the *condition* that appears in the statement of Proposition 2.8. To this end, let us first observe that it follows [cf. Proposition 1.6, (ii)] from the assumption that $Z^{\log}$ and $Y^{\log}$ are *non-minimal* that $\mathrm{rk}(Z^{\log}) \geq 2$, $\mathrm{rk}(Y^{\log}) \geq 2$. Thus, it follows from Proposition 1.10 that $\mathrm{rk}(Z^{\log}) = \dim^{\mathrm{sm}}(Z^{\log}) + 1 \leq \dim^{\mathrm{sm}}(Y^{\log}) + 1 = \mathrm{rk}(Y^{\log})$. Next, let us observe — i.e., by applying Lemma 1.5, (v), as in the proof of Proposition 1.6, (i) — that the *condition* under consideration implies that the restriction to a geometric point of $Z^{\log}$ of the morphism of characteristic sheaves $P_Y|_Z \to P_Z$ induced by $f^{\log}$ satisfies the condition discussed in Lemma 2.7. In particular, we conclude from Lemma 2.7 that this morphism $P_Y|_Z \to P_Z$ is, in fact, an *isomorphism*, and hence that $f^{\log}$ is *scheme-like*, as desired. ◯

Definition 2.9.

(i) Let Z be a scheme. Then we shall refer to a point z of the underlying topological space of Z as a *locally closed point* if z determines a closed point of some open subscheme of Z. Write

$$\mathrm{LCPt}(Z)$$

for the set of locally closed points of Z.

(ii) Let $Z^{\log}$ be an object of $\mathrm{Sch}^{\log}(X^{\log})$. For $i = 1, 2$, let $U_i^{\log}$ be a *minimal object* of $\mathrm{Sch}^{\log}(X^{\log})$ and $f_i^{\log} : U_i^{\log} \to Z^{\log}$ an arrow of $\mathrm{Sch}^{\log}(X^{\log})$. Then we shall say that $f_1^{\log}$ and $f_2^{\log}$ are *point-equivalent* if there exist a morphism $f_W^{\log} : W^{\log} \to Z^{\log}$ and, for each $i = 1, 2$, a morphism $h_i^{\log} : V_i^{\log} \to U_i^{\log}$ between *minimal* objects of $\mathrm{Sch}^{\log}(X^{\log})$ such that $W^{\log}$ is *log-nodal*, and, moreover, for each $i = 1, 2$, the composite morphism $f_i^{\log} \circ h_i^{\log} : V_i^{\log} \to Z^{\log}$ admits a *factorization* $V_i^{\log} \to W^{\log} \to Z^{\log}$ through $f_W^{\log} : W^{\log} \to Z^{\log}$.

(iii) Let $Z^{\log}$ be an object of $\mathrm{Sch}^{\log}(X^{\log})$ whose underlying scheme we denote by Z, $z \in \mathrm{LCPt}(Z^{\log}) \stackrel{\mathrm{def}}{=} \mathrm{LCPt}(Z)$. Then a monomorphism $H^{\log} \rightarrowtail Z^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$ will be called a *point-hull at* z if $H^{\log}$ is *one-pointed*, and, moreover, every morphism $S^{\log} \to Z^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$ from a *minimal* object $S^{\log}$ to $Z^{\log}$ that maps the unique point of the underlying scheme S of $S^{\log}$ to z *factors* [necessarily uniquely!] though the given monomorphism $H^{\log} \rightarrowtail Z^{\log}$. A point-hull

$H^{\log} \rightarrowtail Z^{\log}$ at z will be called a *minimal point-hull at* z if every monomorphism $H_1^{\log} \rightarrowtail H^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$ for which the composite $H_1^{\log} \rightarrowtail H^{\log} \rightarrowtail Z^{\log}$ is a point-hull at z is necessarily an isomorphism. An arrow of $\mathrm{Sch}^{\log}(X^{\log})$ which is a minimal point-hull at some element of $\mathrm{LCPt}(-)$ of the codomain of the arrow will be referred to as a *minimal point-hull*. Thus, if $Z^{\log}$ is *one-pointed*, and one *restricts* one's attention to monomorphisms with *one-pointed* domains, then the notion of a point-hull (respectively, minimal point-hull) at z is identical to the notion of a hull (respectively, minimal hull) [cf. Definition 1.1, (iii)].

Proposition 2.10. (Point-classes and minimal point-hulls) *Let* $Z^{\log}$ *be an object of* $\mathrm{Sch}^{\log}(X^{\log})$*. For* $i = 1, 2$*, let* $U_i^{\log}$ *be a* **minimal** *object of* $\mathrm{Sch}^{\log}(X^{\log})$ *and* $f_i^{\log} : U_i^{\log} \to Z^{\log}$ *be an arrow of* $\mathrm{Sch}^{\log}(X^{\log})$*. For* $i = 1, 2$*, write* Z*,* U_i *for the underlying schemes of* $Z^{\log}$*,* $U_i^{\log}$*, respectively. Then:*

(i) $Z^{\log}$ *is* **one-pointed** *if and only if the set* $\mathrm{LCPt}(Z^{\log}) = \mathrm{LCPt}(Z)$ *is of* **cardinality one***.*

(ii) Let z *be a point of the underlying topological space of* Z*. Then the following conditions are equivalent: (ii-a)* z *is* **locally closed***; (ii-b)* z *appears as the image of a morphism* $U \to Z$ *of* $\mathrm{Sch}(X)$ *for some* **minimal** *object* U *[cf. [LgSch], Proposition 1.1, (ii)] of* $\mathrm{Sch}(X)$*; (ii-c)* z *appears as the image of a morphism* $U^{\log} \to Z^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$ *for some* **minimal** *object* $U^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$*.*

(iii) Write z_i *for the image in* Z *via [the underlying morphism of schemes associated to]* $f_i^{\log}$ *of the unique point of* U_i*. Then the arrows* $f_1^{\log}$ *and* $f_2^{\log}$ *are* **point-equivalent** *if and only if* $z_1 = z_2$*. In particular, the notion of point-equivalence determines an equivalence relation on the collection [i.e., which, strictly speaking, is not necessarily a set!] of arrows in* $\mathrm{Sch}^{\log}(X^{\log})$ *from* **minimal** *objects of* $\mathrm{Sch}^{\log}(X^{\log})$ *to* $Z^{\log}$*. Write*

$$\mathrm{PtCl}(Z^{\log})$$

for the set of equivalence classes of such arrows. We shall refer to an element of $\mathrm{PtCl}(Z^{\log})$ *as a* **point-class** *of* $Z^{\log}$*.*

(iv) If $f^{\log} : U^{\log} \to Z^{\log}$ *is an arrow that determines a point-class of* $Z^{\log}$*, then let us write* $\mathrm{Im}(f^{\log})$ *for the image in* Z *via [the underlying morphism of schemes associated to]* $f^{\log}$ *of the unique point of the underlying scheme* U *of* $U^{\log}$*. Then the assignment* $f^{\log} \mapsto \mathrm{Im}(f^{\log})$ *determines a* **bijection** *of sets*

$$\mathrm{PtCl}(Z^{\log}) \xrightarrow{\sim} \mathrm{LCPt}(Z^{\log}) = \mathrm{LCPt}(Z)$$

that is **functorial** *[in the evident sense] with respect to* $Z^{\log}$*.*

(v) Let $z \in \mathrm{LCPt}(Z)$*. Write* $z^{\log}$ *for the* **reduced***,* **one-pointed object** *of* $\mathrm{Sch}^{\log}(X^{\log})$ *obtained by restricting the log structure of* $Z^{\log}$ *to* z*. Then a monomorphism* $h^{\log} : H^{\log} \rightarrowtail Z^{\log}$ *in* $\mathrm{Sch}^{\log}(X^{\log})$ *is a* **minimal point-hull at** z *if and only if* $h^{\log}$ *induces an* **isomorphism** $H^{\log} \xrightarrow{\sim} z^{\log}$*.*

Proof. First, we observe that assertion (i) follows immediately from the various definitions involved [cf. also [LgSch], Proposition 1.1, (i)]. Next, we consider assertion (ii). First, we recall from [LgSch], Proposition 1.1, (ii), that an object of

$\mathrm{Sch}(X)$ is *minimal* if and only if it is *reduced* and *one-pointed*. Next, we recall from Proposition 1.6, (ii), that a *minimal* object of $\mathrm{Sch}^{\log}(X^{\log})$ is necessarily *reduced* and *one-pointed*. Now the implication (ii-a) $\Longrightarrow$ (ii-b) follows immediately. In a similar vein, the implication (ii-a) $\Longrightarrow$ (ii-c) follows immediately, by applying Proposition 1.6, (i). To verify the implications (ii-b) $\Longrightarrow$ (ii-a), (ii-c) $\Longrightarrow$ (ii-a), it suffices to verify that if U is a *one-pointed* object of $\mathrm{Sch}(X)$, then the image via any morphism $U \to Z$ of $\mathrm{Sch}(X)$ of the unique point of U is a *locally closed* point of Z. Note that, by considering the schematic closure of such a morphism in a suitable affine open of Z, we may assume without loss of generality that U and Z are *affine*, and that the morphism [*of finite type*!] $U \to Z$ has *dense image*. Since this image [which consists of a single point!] is necessarily *constructible*, hence contains a *dense open* subset of the underlying topological space of Z, we thus conclude that we may assume, after replacing Z by a suitable affine open of Z, that the morphism $U \to Z$ is *surjective*, i.e., that Z is *one-pointed*. This completes the proof of assertion (ii).

Next, we consider assertion (iii). Since *minimal* objects of $\mathrm{Sch}^{\log}(X^{\log})$ are necessarily *one-pointed* [cf. Proposition 1.6, (ii)], the *necessity* portion of the asserted equivalence follows immediately from the various definitions involved. Thus, it suffices to verify the *sufficiency* portion of the asserted equivalence. To this end, let us first observe that we may assume without loss of generality that $Z^{\log}$ is *reduced* and *one-pointed*. Also, by *base-changing* to a suitable *finite extension* of the field whose spectrum is Z, we conclude that we may assume without loss of generality that $f_1^{\log}$ and $f_2^{\log}$ are *log-like*, and that $Z^{\log}$ is *split*. Thus, by considering a suitable *splitting* as in Lemma 1.5, (i), one verifies immediately that, to complete the proof of *sufficiency*, it suffices to verify the following assertion concerning *fs monoids*:

> Let P be an *fs monoid*. For $i = 1, 2$, let $\phi_i : P \to \mathbb{N}$ be a *positive* homomorphism of fs monoids. Then there exist an *fs monoid* Q of *rank two* and a *positive* homomorphism $\psi : P \to Q$ of fs monoids such that, for $i = 1, 2$, the homomorphism $2 \cdot \phi_i : P \to \mathbb{N}$ [i.e., the composite of ϕ_i with the positive homomorphism $\mathbb{N} \to \mathbb{N}$ given by multiplication by 2] admits a *factorization* $P \to Q \to \mathbb{N}$ as the composite of ψ with some *positive* homomorphism $\psi_i : Q \to \mathbb{N}$.

This assertion concerning fs monoids may be verified as follows. For $i = 1, 2$, write $N_i^{\mathrm{gp}} \subseteq P^{\mathrm{gp}}$ for the *kernel* of the morphism $\phi_i^{\mathrm{gp}} : P^{\mathrm{gp}} \to \mathbb{Z}$. If $N_1^{\mathrm{gp}} = N_2^{\mathrm{gp}}$, then one verifies immediately that one obtains data as desired by considering the factorization $\mathbb{N} \to \mathbb{N} \oplus \mathbb{N} \to \mathbb{N}$ [i.e., determined by the assignments $\mathbb{N} \ni 1 \mapsto (1, 1) \in \mathbb{N} \oplus \mathbb{N}$ and $\mathbb{N} \oplus \mathbb{N} \ni (a, b) \mapsto a + b \in \mathbb{N}$] of the homomorphism $\mathbb{N} \to \mathbb{N}$ given by multiplication by 2. Thus, we may assume without loss of generality that $N_1^{\mathrm{gp}} \neq N_2^{\mathrm{gp}}$. Write Q for the *saturation* [cf. [LgSch], Lemma 2.5, (ii)] of the image of P in $(P^{\mathrm{gp}}/N_1^{\mathrm{gp}}) \oplus (P^{\mathrm{gp}}/N_2^{\mathrm{gp}})$ $(\cong \mathbb{Z} \oplus \mathbb{Z})$. Thus, we obtain a natural *positive* homomorphism of monoids $\psi : P \to Q$ such that, for $i = 1, 2$, $\phi_i : P \to \mathbb{N}$ admits a *factorization* $P \to Q \to \mathbb{N}$ as the composite of ψ with some *positive* homomorphism $\psi_i : Q \to \mathbb{N}$. Here, we note that the positivity of ψ and ψ_i follows immediately from the positivity of ϕ_i. Also, we observe that the positivity of ψ_i implies that the monoid Q has *no nonzero invertible elements*. We thus conclude that Q is an *fs monoid* of *rank two*, as desired. This completes the proof of assertion (iii). Assertion (iv) follows immediately from assertion (iii), together with the equivalence (ii-a) $\Longleftrightarrow$ (ii-c) of assertion (ii).

Finally, we consider assertion (v). First, we consider the *sufficiency* portion of the asserted equivalence. To verify this sufficiency, it suffices to verify that the natural monomorphism $h_z^{\log} : z^{\log} \rightarrowtail Z^{\log}$ [cf. Proposition 1.4, (vii)] is a *minimal point-hull at z*. The fact that $h_z^{\log}$ is a *point-hull at z* follows immediately from the various definitions involved. Now suppose that $h_1^{\log} : H_1^{\log} \rightarrowtail z^{\log}$ is a monomorphism such that the composite $h_z^{\log} \circ h_1^{\log} : H_1^{\log} \rightarrowtail Z^{\log}$ is a point-hull at z [so both $z^{\log}$ and $H_1^{\log}$ are *one-pointed*]. Then one verifies immediately that, by applying Lemma 1.5, (v), as in the proof of Proposition 1.6, (i), it follows from Proposition 1.4, (iii), and Lemma 2.7 that $h_1^{\log}$ is *scheme-like*, and hence, by Proposition 1.4, (v); [LgSch], Proposition 1.1, (ii), that $h_1^{\log}$ is an *isomorphism*, as desired. Thus, to complete the proof of assertion (v), it suffices to verify the *necessity* portion of the asserted equivalence. First, let us observe that it follows from the existence of the natural monomorphism $H_{\mathrm{red}}^{\log} \rightarrowtail H^{\log}$, together with the definition of the notion of a *minimal point-hull*, that $H^{\log}$ is *reduced* and *one-pointed*. Thus, it follows immediately from Proposition 1.6, (i), that $h^{\log}$ induces a *monomorphism* $H^{\log} \rightarrowtail z^{\log}$. Since we have already verified that $h_z^{\log}$ is a *minimal point-hull at z*, we thus conclude that this monomorphism $H^{\log} \rightarrowtail z^{\log}$ is an *isomorphism*, as desired. This completes the proof of assertion (v). ◯

Proposition 2.11. (Characterization of scheme-like morphisms between arbitrary objects) *Let $f^{\log} : Z^{\log} \to Y^{\log}$ be a morphism between* **arbitrary** *objects of* $\mathrm{Sch}^{\log}(X^{\log})$*. Then $f^{\log}$ is* **scheme-like** *if and only if, for every* **minimal point-hull** $h^{\log} : T^{\log} \to Z^{\log}$*, there exists a commutative diagram of morphisms of* $\mathrm{Sch}^{\log}(X^{\log})$

$$\begin{array}{ccc} T^{\log} & \xrightarrow{h^{\log}} & Z^{\log} \\ \downarrow & & \downarrow f^{\log} \\ S^{\log} & \to & Y^{\log} \end{array}$$

in which the lower horizontal arrow $S^{\log} \to Y^{\log}$ is a **minimal point-hull***, and the left-hand vertical arrow $T^{\log} \to S^{\log}$ is a* **scheme-like** *morphism between* **reduced**, **one-pointed** *objects of* $\mathrm{Sch}^{\log}(X^{\log})$.

Proof. The asserted equivalence follows immediately, in light of the manifestly *constructible nature* of the characteristic sheaves P_Z, P_Y, from Proposition 2.10, (v), together with the definition of the term "scheme-like". ◯

Corollary 2.12. (Conditional reconstruction of the scheme structure of arbitrary objects) *Suppose that we are in the situation of Theorem 2.6, and that Φ satisfies the following condition:*

($*_{\mathrm{nod}}$) *an object of* $\mathrm{Sch}^{\log}(X_1^{\log})$ *is* **log-nodal** *if and only if its image via Φ is a log-nodal object of* $\mathrm{Sch}^{\log}(X_2^{\log})$.

Then:

(i) Φ preserves the following:
(i-a) **point-equivalent** *pairs of arrows;*

(i-b) the set-valued **functor** $\mathrm{LCPt}(-)$ *[up to natural equivalence];*
(i-c) arrows which are **minimal point-hulls***;*
(i-d) **scheme-like** *morphisms between* **arbitrary** *objects.*

(ii) For $i = 1, 2$, let $Y_i^{\log}$ be an object of $\mathrm{Sch}^{\log}(X_i^{\log})$; write Y_i for the underlying scheme of $Y_i^{\log}$. Suppose further that $\Phi(Y_1^{\log}) = Y_2^{\log}$. Then Φ induces an **equivalence of categories**

$$\Big(\mathrm{Sch}(Y_1) \ \stackrel{\sim}{\to}\Big) \quad \mathrm{Sch}^{\log}(Y_1^{\log})|_{\text{sch-lk}} \ \stackrel{\sim}{\to}\ \mathrm{Sch}^{\log}(Y_2^{\log})|_{\text{sch-lk}} \quad \Big(\ \stackrel{\sim}{\to}\ \mathrm{Sch}(Y_2)\Big)$$

— i.e., where the equivalences in parentheses are the natural equivalences of Definition 1.1, (iv) — that is **functorial** *[in the evident sense!] with respect to $Y_1^{\log}$, $Y_2^{\log}$. Finally, the composite of the equivalences of categories in the above display induces, by applying [LgSch], Theorem 1.7, (ii), an* **isomorphism of schemes**

$$Y_1 \ \stackrel{\sim}{\to}\ Y_2$$

that is **functorial** *[in the evident sense!] with respect to $Y_1^{\log}$, $Y_2^{\log}$.*

Proof. First, we consider assertion (i). The preservation of (i-a) follows immediately, in light of the preservation of (i-d) asserted in Theorem 2.6, (i), from the condition $(*_{\mathrm{nod}})$, together with the definition of the term "point-equivalent". The prservation of (i-b) now follows from the preservation of (i-a), together with the *bijection* of Proposition 2.10, (iv). The preservation of (i-c) then follows from the preservation of (i-b) [cf. also the preservation of (i-a), (i-d) asserted in Theorem 2.6, (i)], together with the *equivalence* of Proposition 2.10, (i). The preservation of (i-d) follows, in light of the preservation of (i-c), from Propositions 2.8; 2.10, (v); 2.11 [cf. also the preservation of (i-d), (i-j), (i-l) asserted in Theorem 2.6, (i)]. This completes the proof of assertion (i). Now assertion (ii) follows immediately [i.e., in the spirit of Theorem 2.6, (ii); [LgSch], Corollary 2.15] from the portion of assertion (i) concerning the preservation of (i-d). Here, we note that the *functoriality* of the isomorphism of schemes in the final display in the statement of assertion (ii) follows immediately from the *characterization* given in Proposition 1.11, (ii), of the *factorization* discussed in Proposition 1.11, (i), together with the *natural equivalences of categories* discussed in Proposition 1.11, (iii). ○

Section 3: Seamless Partitions of Orientable Log Schemes

In the present §3, we discuss the notion of a *seamless partition* of an *orientable* log scheme. This notion leads naturally to a *category-theoretic characterization* of *log-nodal* objects, which we apply to eliminate the dependence on the condition "$(*_{\mathrm{nod}})$" in Corollary 2.12.

We maintain the notation of §2.

Definition 3.1.

(i) Suppose that $Y^{\log}$ is a object of $\mathrm{Sch}^{\log}(X^{\log})$. Then we shall say that $Y^{\log}$ is *log-Dedekind* if it satisfies the following conditions:

(i-a) $\dim^{\mathrm{sm}}(Y^{\log}) \leq 1$;
(i-b) if $Z^{\log}$ is a *minimal* object of $\mathrm{Sch}^{\log}(X^{\log})$ such that there exists a morphism $Z^{\log} \to Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$, then $Z^{\log}$ is of *rank one*;
(i-c) if $Z^{\log}$ is a *nonempty submonic* object of $\mathrm{Sch}^{\log}(X^{\log})$, with underlying scheme Z, such that there exists a *SLEM* morphism $Z^{\log} \rightarrowtail Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$, then the closed subscheme $Z_{\mathrm{red}} \subseteq Z$ is *regular* and of *positive dimension*.

If y is a point of the underlying scheme Y of a *log-Dedekind* object $Y^{\log}$, and the fiber of P_Y at some geometric point of Y that maps to y is of *rank two*, then we shall say that y is a *nodal point* of $Y^{\log}$.

(ii) Suppose that $Y^{\log}$ is a *log-Dedekind* object of $\mathrm{Sch}^{\log}(X^{\log})$. For $i = 1, 2$, let $Z_i^{\log}$ be a *connected* [hence *nonempty*], *submonic* object of $\mathrm{Sch}^{\log}(X^{\log})$ and

$$f_i^{\log} : Z_i^{\log} \rightarrowtail Y^{\log}$$

a *SLEM* morphism. We shall say that $f_1^{\log}$ and $f_2^{\log}$ are *submonically equivalent* if the fiber product $Z_{12}^{\log} \stackrel{\mathrm{def}}{=} Z_1^{\log} \times_{Y^{\log}} Z_2^{\log}$ determined by $f_1^{\log}$ and $f_2^{\log}$ is *nonempty*. [Here, we note that, for $i = 1, 2$, the projection $Z_{12}^{\log} \to Z_i^{\log}$, is *SLEM*, hence, by Proposition 2.2, (ii), an *open immersion*, whose image is, by condition (i-c), *dense* whenever it is nonempty.] Thus, one verifies immediately that the notion of *submonic equivalence* determines an *equivalence relation* on the collection [i.e., which, strictly speaking, is not necessarily a set!] of arrows of $\mathrm{Sch}^{\log}(X^{\log})$ which are SLEM morphisms from connected, submonic objects of $\mathrm{Sch}^{\log}(X^{\log})$ to $Y^{\log}$. Write

$$\mathrm{SmCp}(Y^{\log})$$

for the set of equivalence classes of such arrows. We shall refer to an element of $\mathrm{SmCp}(Y^{\log})$ as a *submonic component* of $Y^{\log}$.

(iii) Suppose that $Y^{\log}$ is a *log-Dedekind* object of $\mathrm{Sch}^{\log}(X^{\log})$. If $h^{\log} : H^{\log} \rightarrowtail Y^{\log}$ is a monomorphism of $\mathrm{Sch}^{\log}(X^{\log})$, then we shall write

$$\mathrm{Chn}(h^{\log}) \subseteq \mathrm{SmCp}(Y^{\log})$$

for the subset of submonic components for which there exists a representative arrow $Z^{\log} \rightarrowtail Y^{\log}$ that admits a *factorization* $Z^{\log} \rightarrowtail H^{\log} \rightarrowtail Y^{\log}$ through $h^{\log} : H^{\log} \rightarrowtail Y^{\log}$. If $C \subseteq \mathrm{SmCp}(Y^{\log})$ is a *nonempty* subset, then we shall refer to C as a *chain* if there exists a *SLEM* morphism $h^{\log} : H^{\log} \rightarrowtail Y^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ such that $H^{\log}$ is *connected* [hence *nonempty*!], and $C = \mathrm{Chn}(h^{\log})$. If $C \subseteq \mathrm{SmCp}(Y^{\log})$ is a subset, then we shall refer to C as an $\mathbb{N}$-*chain* if there exists a collection $\{C_i\}_{i \in \mathbb{N}}$ of chains $C_i \subseteq \mathrm{SmCp}(Y^{\log})$ such that $C = \cup_{i \in \mathbb{N}} C_i$, and $C_i \subseteq C_{i+1}$ for all $i \in \mathbb{N}$.

Proposition 3.2. (First properties of log-Dedekind objects) *Suppose that* $Y^{\log}$ *is a* **log-Dedekind** *object of* $\mathrm{Sch}^{\log}(X^{\log})$*. Then:*

(i) $Y^{\log}$ *is of* **rank** $\leq$ **2**.

(ii) The **non-nodal** *points of the underlying scheme* Y *of* $Y^{\log}$ *form an* **open subset** *of the underlying topological space of* Y*. Write* $Y_{\mathrm{sm}} \subseteq Y$ *for the corresponding open subscheme and* $Y^{\log}_{\mathrm{sm}}$ *for the log scheme obtained by restricting the log structure of* $Y^{\log}$ *to* Y_{sm}*. Then the complement of* Y_{sm} *in* Y *is a* **closed subscheme** *of* Y *of* **dimension zero***, and* $Y^{\log}_{\mathrm{sm}}$ *is* **submonic***. We shall refer to* $Y^{\log}_{\mathrm{sm}}$ *as the* **submonic locus** *of* $Y^{\log}$*.*

(iii) Let $Z^{\log}$ *be a* **nonempty submonic** *object of* $\mathrm{Sch}^{\log}(X^{\log})$ *and* $Z^{\log} \rightarrowtail Y^{\log}$ *a* **SLEM** *morphism. Then the closed subscheme* $Z_{\mathrm{red}} \subseteq Z$ *of the underlying scheme* Z *of* $Z^{\log}$ *is* **regular** *and of* **dimension one***, and* $Z^{\log}$ *is of* **rank one***. In particular, [cf. Proposition 2.2, (i)]* $(Y_{\mathrm{sm}})_{\mathrm{red}}$ *is regular and of dimension one, and* $Y^{\log}_{\mathrm{sm}}$ *is of rank one.*

(iv) Let $f^{\log} : Z^{\log} \rightarrowtail Y^{\log}$ *be a* **SLEM** *morphism from a* **connected***,* **submonic** *object* $Z^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$ *to* $Y^{\log}$*. Then* $f^{\log}$ *either admits a* **factorization** $Z^{\log} \rightarrowtail Y^{\log}_{\mathrm{sm}} \rightarrowtail Y^{\log}$ *as the composite of an open immersion* $Z^{\log} \rightarrowtail Y^{\log}_{\mathrm{sm}}$ *with the natural monomorphism* $Y^{\log}_{\mathrm{sm}} \rightarrowtail Y^{\log}$ *or maps the entire underlying scheme* Z *of* $Z^{\log}$ *to some* **nodal point** y *of* $Y^{\log}$*. In the former case, we shall say that* $f^{\log}$ *is* **non-nodal***; in the latter case, we shall say that* $f^{\log}$ *is* **nodal** *and lies over* y*. We shall also apply this terminology "non-nodal"/"nodal" to the element of* $\mathrm{SmCp}(Y^{\log})$ *determined by* $f^{\log}$*.*

(v) Let y *be a* **nodal point** *of* $Y^{\log}$*. Then the subset*

$$\mathrm{SmCp}(Y^{\log})_y \ \subseteq\ \mathrm{SmCp}(Y^{\log})$$

of **nodal** *elements that lie over* y *forms an* $\mathbb{N}$**-chain***. Moreover, every morphism* $H^{\log} \to Y^{\log}$ *in* $\mathrm{Sch}^{\log}(X^{\log})$ *from a* **minimal** *object* $H^{\log}$ *to* $Y^{\log}$ *that maps the unique point of the underlying scheme of* $H^{\log}$ *to* y **factors** *through some representative of an element of* $\mathrm{SmCp}(Y^{\log})_y$*.*

(vi) Every element $\gamma \in \mathrm{SmCp}(Y^{\log})$ *admits a* **"maximal"** *representative arrow* $f^{\log} : Z^{\log} \rightarrowtail Y^{\log}$*, i.e., a representative arrow such that every arrow* $U^{\log} \rightarrowtail Y^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$ *that is submonically equivalent to* $f^{\log}$ *admits a* **factorization**

$$U^{\log} \ \rightarrowtail\ Z^{\log} \ \rightarrowtail\ Y^{\log}$$

as the composite of some **open immersion** $U^{\log} \rightarrowtail Z^{\log}$ *with* $f^{\log}$*. If, moreover,* γ *is* **non-nodal***, then such a maximal representative* $f^{\log} : Z^{\log} \rightarrowtail Y^{\log}$ *arises from an* **isomorphism** *of* $Z^{\log}$ *onto some* **connected component** *of* $Y^{\log}_{\mathrm{sm}}$*.*

Proof. First, let us observe that the inequality $\dim^{\mathrm{sm}}(Y^{\log}) \leq 1$ of Definition 3.1, (i-a), together with the restriction imposed by Definition 3.1, (i-b) [cf. also Proposition 1.6, (i)], imply that the integers "d" and "n" in Proposition 1.10 satisfy the following *conditions*:

$(*_1)$ $n \in \{1, 2\}$;
$(*_2)$ $n = 2 \implies d = 0$;
$(*_3)$ $n = 1 \implies d \leq 1$.

Assertion (i) thus follows from $(*_1)$ [cf. also Lemma 1.9]. Assertion (ii) follows from $(*_1)$, $(*_2)$ [cf. also Lemma 1.9]. Assertion (iii) follows from $(*_1)$, $(*_3)$, together with Definition 3.1, (i-c) [cf. also Proposition 2.2, (i)].

Next, we consider assertion (iv). If y is a *nodal point* of $Y^{\log}$, then write $y^{\log}$ for the log scheme obtained by restricting the log structure of $Y^{\log}$ to the closed subscheme, equipped with the reduced induced scheme structure, of Y determined by y. Write $Z_y^{\log} \stackrel{\mathrm{def}}{=} Z^{\log} \times_{Y^{\log}} y^{\log}$. Thus, the underlying scheme Z_y of $Z_y^{\log}$ may be identified with the scheme-theoretic *fiber* of Z over y. Note that if $Z_y = \emptyset$ for *every nodal point* y of $Y^{\log}$, then $f^{\log}$ admits a *factorization* $Z^{\log} \rightarrowtail Y_{\mathrm{sm}}^{\log} \rightarrowtail Y^{\log}$ as the composite of a *monomorphism* $Z^{\log} \rightarrowtail Y_{\mathrm{sm}}^{\log}$ with the natural monomorphism $Y_{\mathrm{sm}}^{\log} \rightarrowtail Y^{\log}$; moreover, since $f^{\log}$ is *SLEM*, it follows immediately that the morphism $Z^{\log} \rightarrowtail Y_{\mathrm{sm}}^{\log}$ is *SLEM* and hence, by assertion (ii) and Proposition 2.2, (ii), an *open immersion*. Thus, since, by assertion (iii), Z_{red} is *regular* and of *dimension one*, it follows immediately — i.e., by possibly replacing $Z^{\log}$ by the log scheme determined by a suitable dense open subscheme of Z — that, to complete the proof of assertion (iv), it suffices to verify, under the additional assumption that $Z_y^{\log}$ is *connected* [hence *nonempty*] for *some fixed nodal point* y of $Y^{\log}$, that $\dim(Z_y) = 1$. To this end, let us first observe that the natural morphism $Z_y^{\log} \to y^{\log}$ is *SLEM*. Since $Z_y^{\log}$ is *connected* and [by assertion (iii)] of *rank one*, it follows from Lemma 1.5, (v) [where we take "$S^{\log}$" to be $y^{\log}$], that the *monomorphism* $Z_y^{\log} \to y^{\log}$ admits a *factorization* as a composite of *monomorphisms*

$$Z_y^{\log} \rightarrowtail y_Z^{\log} \rightarrowtail y^{\log}$$

— where $y_Z^{\log}$ is, in the notation of Lemma 1.5, (v), a log scheme of *rank one* of the form "$S^{\log}[\xi]$". Since $Z_y^{\log} \rightarrowtail y^{\log}$ is *SLEM*, it follows immediately that $Z_y^{\log} \rightarrowtail y_Z^{\log}$ is *SLEM* and hence, by Proposition 2.2, (ii), an *open immersion*. Since the underlying scheme of $y_Z^{\log}$ is of *dimension one* [cf. Lemma 1.5, (iv), (v)], we thus conclude that $\dim(Z_y) = 1$, as desired. This completes the proof of assertion (iv).

Next, we consider assertion (v). Write k for the *residue field* of Y at y, $S^{\log} \stackrel{\mathrm{def}}{=} Y^{\log} \times_Y \mathrm{Spec}(k)$ [where the morphism implicit in the right-hand factor of the fiber product is the tautological morphism $\mathrm{Spec}(k) \to Y$ associated to y], $\mathcal{O}_{\widehat{Y}}$ for the *complete noetherian local ring* obtained by completing Y along y, $\widehat{Y} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_{\widehat{Y}})$, $\widehat{Y}^{\log} \stackrel{\mathrm{def}}{=} Y^{\log} \times_Y \widehat{Y}$, $\widehat{y}$ for the unique closed point of $\widehat{Y}$. Thus, $S^{\log}$ is a log scheme of the sort that appears in Lemma 1.5, so, in the following discussion, we shall apply the notational conventions introduced at the beginning of Lemma 1.5. Write $\mathcal{O}_{\widehat{Y}^{\mathrm{sep}}}$ for the *completion* of the *strict henselization* of $\mathcal{O}_{\widehat{Y}}$ determined by k^{sep}, $\widehat{Y}^{\mathrm{sep}} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_{\widehat{Y}^{\mathrm{sep}}})$ [so $\widehat{Y}^{\mathrm{sep}}$ is equipped with a natural action by G_k], $(\widehat{Y}^{\mathrm{sep}})^{\log} \stackrel{\mathrm{def}}{=} Y^{\log} \times_Y \widehat{Y}^{\mathrm{sep}}$, $\widehat{y}^{\mathrm{sep}}$ for the *unique closed point* of $\widehat{Y}^{\mathrm{sep}}$.

Next, let us *fix a G_k-equivariant splitting* as in Lemma 1.5, (i). Note that since $H^1(G_k, k^{\mathrm{sep}}) = 0$ [i.e., Hilbert's "Theorem 90"], this G_k-equivariant splitting *lifts* to a *G_k-equivariant chart* $P \to \mathcal{O}_{\widehat{Y}^{\mathrm{sep}}}$ of $(\widehat{Y}^{\mathrm{sep}})^{\log}$. Note that since $Y^{\log}$ is a *log-Dedekind* object of $\mathrm{Sch}^{\log}(X^{\log})$, it follows immediately from assertion (ii) that the *support* of the closed subscheme $\widehat{Y}^{\mathrm{sep}}_* \subseteq \widehat{Y}^{\mathrm{sep}}$ determined by the ideal generated by the image via this *chart* of $P \setminus \{0\}$ is equal to $\{\widehat{y}^{\mathrm{sep}}\}$.

Next, let

$$Q \subseteq P^{\mathrm{gp}}$$

be a *finitely generated, saturated submonoid* such that $P \subseteq Q \neq P^{\mathrm{gp}}$. Write $G_Q \subseteq G_k$ for the open subgroup of elements that *preserve* Q [i.e., relative to the natural action of G_k on P^{gp}]. Moreover, we *assume* further that *one* of the following [mutually exclusive!] conditions holds:

(v-a) $G_Q = G_k$, and, moreover, the natural inclusion $P \subseteq Q$ is a *sum-dominating* homomorphism of *fs monoids* [cf. the discussion entitled "Generalities on monoids" in §0].

(v-b) There exists a *positive homomorphism* $\xi : P \to \mathbb{N}$ which induces a *surjection* on groupifications $\xi^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow \mathbb{Z}$ such that Q coincides with the *saturation* [cf. [LgSch], Lemma 2.5, (ii)] of the submonoid of P^{gp} generated by P and $\mathrm{Ker}(\xi^{\mathrm{gp}})$.

Thus, even when $G_Q \neq G_k$ [which implies that *condition (v-b)* holds], one verifies immediately that the natural inclusion $P \subseteq Q$ is a *sum-dominating* homomorphism. That is to say, the natural inclusion $P \subseteq Q$ is a *sum-dominating* homomorphism, *no matter* which of the two conditions (v-a), (v-b) one assumes.

Next, let us observe that the inclusion $P \hookrightarrow Q$ determines a *log étale monomorphism*

$$Z^{\log}[Q] \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathbb{Z}[Q])^{\log} \to Z^{\log}[P] \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathbb{Z}[P])^{\log}$$

as in Proposition 1.4, (ii) [cf. also [Kato1], Proposition 3.4]. Moreover, the G_k-*equivariant chart* $P \to \mathcal{O}_{\widehat{Y}^{\mathrm{sep}}}$ of $(\widehat{Y}^{\mathrm{sep}})^{\log}$ determines a G_k-*equivariant morphism* $(\widehat{Y}^{\mathrm{sep}})^{\log} \to Z^{\log}[P]$ and hence a *fiber product* [of fs log schemes]

$$(\widehat{Z}^{\mathrm{sep}})^{\log} \stackrel{\mathrm{def}}{=} (\widehat{Y}^{\mathrm{sep}})^{\log} \times_{Z^{\log}[P]} Z^{\log}[Q]$$

equipped with a natural action by G_Q. This natural G_Q-action in turn determines *descent data* for the projection morphism $(\widehat{Z}^{\mathrm{sep}})^{\log} \to (\widehat{Y}^{\mathrm{sep}})^{\log}$, which may be used to descend this projection morphism to a *log étale monomorphism* $\widehat{Z}^{\log} \to \widehat{Y}^{\log}_Q$, where we write $\widehat{Y}_Q \to \widehat{Y}$ for the *finite étale covering* corresponding to the open subgroup $G_Q \subseteq G_k$, $\widehat{Y}^{\log}_Q \stackrel{\mathrm{def}}{=} \widehat{Y}^{\log} \times_{\widehat{Y}} \widehat{Y}_Q$.

Next, let us observe that since $Y^{\log}$ is a *log-Dedekind* object of $\mathrm{Sch}^{\log}(X^{\log})$ [or, equivalently, of $\mathrm{Sch}^{\log}(Y^{\log})$], it follows immediately from assertions (ii) and (iii) that any *minimal* object of $\mathrm{Sch}^{\log}(\widehat{Z}^{\log})$ is of *rank one*. Thus, since the inclusion $P \subseteq Q$ is *sum-dominating*, it follows from the final portion of Lemma 1.9 that any regular function on the underlying scheme $\widehat{Z}^{\mathrm{sep}}$ of $(\widehat{Z}^{\mathrm{sep}})^{\log}$ that arises [i.e., via the various *charts* implicit in the above discussion] from an element $\in P \setminus \{0\}$ necessarily *vanishes* at every point of $\widehat{Z}^{\mathrm{sep}}$, hence [since $\widehat{Z}^{\mathrm{sep}}$ is *noetherian*] is necessarily *nilpotent*. Since, as observed above, the *support* of the closed suscheme $\widehat{Y}^{\mathrm{sep}}_* \subseteq \widehat{Y}^{\mathrm{sep}}$ is equal to $\{\widehat{y}^{\mathrm{sep}}\}$, we thus conclude that the natural morphism $\widehat{Z}^{\mathrm{sep}} \to \widehat{Y}^{\mathrm{sep}}$ *factors* through a closed subscheme of $\widehat{Y}^{\mathrm{sep}}$ whose support is equal to $\{\widehat{y}^{\mathrm{sep}}\}$.

This in turn implies that, if we write $\widehat{Z}$ for the underlying scheme of $\widehat{Z}^{\log}$, then the *composite morphism* $\widehat{Z} \to \widehat{Y}_Q \to \widehat{Y}$ *factors* through a closed subscheme of $\widehat{Y}$ whose support is equal to $\{\widehat{y}\}$.

Next, I *claim* that the *composite morphism*

$$\widehat{Z}^{\log} \ \to \ \widehat{Y}_Q^{\log} \ \to \ \widehat{Y}^{\log}$$

is a *log étale monomorphism*. Indeed, in light of what has already been verified, it suffices to prove, in the case where $G_Q \neq G_k$ [which implies that *condition (v-b)* holds], that this composite morphism is a *monomorphism*. Since the morphism $\widehat{Z}^{\log} \to \widehat{Y}_Q^{\log}$ is already known to be a *monomorphism*, and the morphism $\widehat{Y}_Q^{\log} \to \widehat{Y}^{\log}$ is a *scheme-like* morphism whose underlying morphism of schemes is *finite étale*, one verifies immediately that to complete the proof of the *claim*, it suffices to verify [cf. the argument applied in the proof of Lemma 1.5, (vi); the fact that the *composite morphism* $\widehat{Z} \to \widehat{Y}_Q \to \widehat{Y}$ *factors* through a closed subscheme of $\widehat{Y}$ whose support is equal to $\{\widehat{y}\}$] that the *base-change* of the morphism $\widehat{Z}^{\log} \to \widehat{Y}^{\log}$ via the natural morphism $S^{\log} \to \widehat{Y}^{\log}$ is a *monomorphism*. On the other hand, one verifies immediately that this base-changed morphism $\widehat{Z}^{\log} \times_{\widehat{Y}^{\log}} S^{\log} \to S^{\log}$ may be *identified* with the morphism "$S^{\log}[\xi] \to S^{\log}$" of Lemma 1.5, (vi) [i.e., where the objects "ξ", "H" of Lemma 1.5, (vi), correspond, respectively, to ξ and G_Q in the present discussion; we observe that it follows immediately from *condition (v-b)* that "$\Xi_+ = \Xi$"]. Thus, the fact that this base-changed morphism $\widehat{Z}^{\log} \times_{\widehat{Y}^{\log}} S^{\log} \to S^{\log}$ is a *monomorphism* follows from Lemma 1.5, (vi). This completes the proof of the *claim*.

Thus, in summary, the *composite morphism* $\widehat{Z}^{\log} \to \widehat{Y}^{\log} \to Y^{\log}$ may be regarded as a *log étale monomorphism* of $\mathrm{Sch}^{\log}(Y^{\log})$, or, indeed, of $\mathrm{Sch}^{\log}(X^{\log})$. In the following, we shall use the notation

$$f^{\log} : Z^{\log} \to Y^{\log}$$

to denote this composite morphism. Moreover, one computes easily that, if we write Z for the underlying scheme of $Z^{\log}$, then $Z_{\mathrm{red}} \times_{\widehat{Y}_Q} \widehat{Y}^{\mathrm{sep}}$ may be identified with the reduced closed subscheme of $\mathrm{Spec}(k^{\mathrm{sep}}[Q])$ determined by forming the zero locus of the set of functions $P \setminus \{0\} \subseteq Q$. Thus, if *condition (v-a)* holds, then one verifies immediately, by applying an *isomorphism* $Q^{\mathrm{pf}} \xrightarrow{\sim} \mathbb{Q}_{\geq 0} \oplus \mathbb{Q}_{\geq 0}$ as in the discussion entitled "Rank two fs monoids" in §0 [cf. also Lemma 1.5, (iv)], that $Z_{\mathrm{red}} \times_{\widehat{Y}_Q} \widehat{Y}^{\mathrm{sep}}$ may be regarded as the codomain of a finite surjective morphism whose domain consists of *two copies of the affine line* over k^{sep} *glued together at a single point*, hence, in particular, is *connected*. On the other hand, if *condition (v-b)* holds, then one verifies immediately that $Z_{\mathrm{red}} \times_{\widehat{Y}_Q} \widehat{Y}^{\mathrm{sep}}$ is a *one-dimensional torus* [cf. the situation discussed in Lemma 1.5, (iv)], hence, in particular, is *connected*.

Thus, in summary, the morphism $f^{\log} : Z^{\log} \to Y^{\log}$ is a *log étale monomorphism* with *connected* domain such that the resulting *chain*

$$\mathrm{Chn}(f^{\log}) \ \subseteq \ \mathrm{SmCp}(Y^{\log})$$

is *contained* in $\mathrm{SmCp}(Y^{\log})_y$. Now we consider the monoids constructed in Example 0.2, where we allow $n \in \mathbb{N}$ to *vary*. Then it follows immediately from the discussion of Example 0.2 that given any element $\gamma \in \mathrm{SmCp}(Y^{\log})_y$, it holds that $\gamma \in \mathrm{Chn}(f^{\log})$, if, in the notation of Example 0.2, we take $Q \stackrel{\mathrm{def}}{=} {}^nP$ — a submonoid which, as discussed in Example 0.2, may be constructed in such a way that *condition (v-a)* holds — for n *sufficiently large*.

Finally, let $H^{\log} \to Y^{\log}$ be a morphism in $\mathrm{Sch}^{\log}(X^{\log})$ from a *minimal* object $H^{\log}$ to $Y^{\log}$ that maps the unique point of the underlying scheme H of $H^{\log}$ to y. Thus, if we regard H as the spectrum of a finite subextension of k in the perfection of k^{sep}, then the morphism $H^{\log} \to Y^{\log}$ determines, by considering the induced morphism on *log structures*, a *positive homomorphism* $\xi : P \to \mathbb{N}$ and submonoid $Q \subseteq P^{\mathrm{gp}}$ that satisfy *condition (v-b)*. Moreover, it follows immediately from the construction of $f^{\log}$ that $Z^{\log}$ is *submonic* [so $f^{\log}$ may be regarded as a representative of an element of $\mathrm{SmCp}(Y^{\log})_y$], and that the morphism $H^{\log} \to Y^{\log}$ *factors* through $f^{\log}$. This completes the proof of assertion (v).

Finally, we consider assertion (vi). If γ is *non-nodal*, then assertion (vi) follows immediately from assertions (iii) and (iv). Thus, we may assume without loss of generality that γ is *nodal*. Then assertion (vi) follows immediately by *gluing*, in the notation of Definition 3.1, (ii), the various $Z_i^{\log} \rightarrowtail Y^{\log}$ that constitute an element of $\mathrm{SmCp}(Y^{\log})$ along the *open immersions* $Z_{12}^{\log} \rightarrowtail Z_i^{\log}$. Here, we note that it follows immediately from the fact that the log scheme $y_Z^{\log}$ that appeared in the proof of assertion (iv) is *noetherian* that this gluing process *terminates* after a *finite* number of steps. This completes the proof of assertion (vi). ◯

Definition 3.3. Suppose that $Y^{\log}$ is a *connected*, *non-submonic*, *log-Dedekind* object of $\mathrm{Sch}^{\log}(X^{\log})$. Let $\gamma \in \mathrm{SmCp}(Y^{\log})$. Write

$$\mathrm{Mono}(Y^{\log})$$

for the *full subcategory* of $\mathrm{Sch}^{\log}(Y^{\log})$ determined by the arrows $H^{\log} \to Y^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ which are *monomorphisms* in $\mathrm{Sch}^{\log}(X^{\log})$.

(i) Let $C_1, C_2 \subseteq \mathrm{SmCp}(Y^{\log})$ be *chains*. Then we shall say that the pair of chains $\{C_1, C_2\}$ forms a *partition at* γ if the chains C_1, C_2 satisfy the following conditions:

(i-a) $C_1 \cup C_2 = \mathrm{SmCp}(Y^{\log})$, $C_1 \cap C_2 = \{\gamma\}$;

(i-b) for $i = 1, 2$, the subset $C_i \setminus \{\gamma\} \subseteq \mathrm{SmCp}(Y^{\log})$ is an $\mathbb{N}$-*chain* [hence *nonempty*];

(i-c) the $\mathbb{N}$-chains of (i-b) are *"maximal"* in the sense that every $\mathbb{N}$-chain $C \subseteq \mathrm{SmCp}(Y^{\log})$ such that $\gamma \notin C$ is contained in C_i for some $i \in \{1, 2\}$;

(i-d) if, for $i = 1, 2$, we write Ψ_i for the *subfunctor* of the contravariant functor determined by the terminal object [i.e., $Y^{\log}$] of $\mathrm{Mono}(Y^{\log})$ that consists of objects $h^{\log} : H^{\log} \rightarrowtail Y^{\log}$ of $\mathrm{Mono}(Y^{\log})$ such that *every composite morphism* $H_*^{\log} \rightarrowtail H^{\log} \rightarrowtail Y^{\log}$, where $H_*^{\log} \rightarrowtail H^{\log}$ is a *minimal point* of $H^{\log}$, *factors* through some representative of an element $\in C_i$ ($\subseteq \mathrm{SmCp}(Y^{\log})$), then Ψ_i is *representable* by an object $h_i^{\log} : Y_i^{\log} \rightarrowtail Y^{\log}$ of $\mathrm{Mono}(Y^{\log})$.

We shall say that $Y^{\log}$ is *orientable* if $Y^{\log}$ admits a partition at every element of $\mathrm{SmCp}(Y^{\log})$.

(ii) Let $\{C_1, C_2\}$ be a *partition at* γ. Suppose that $h_1^{\log}$, $h_2^{\log}$ are as in (i-d). Then we shall say that the partition $\{C_1, C_2\}$ is *seamless* if the following condition is satisfied:

> a *monomorphism* $h^{\log} : H^{\log} \rightarrowtail Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$ is an *isomorphism* if and only if, for $i = 1, 2$, the projection $H^{\log} \times_{Y^{\log}} Y_i^{\log} \to Y_i^{\log}$ associated to the fiber product determined by $h^{\log}$ and $h_i^{\log}$ is an *isomorphism*.

We shall say that $Y^{\log}$ is *homogeneous* if $Y^{\log}$ is *orientable*, and, moreover, *no partition* at an element $\in \mathrm{SmCp}(Y^{\log})$ is *seamless*.

Proposition 3.4. (First properties of partitions)

(i) Suppose that $Y^{\log}$ is an **orientable** *object of* $\mathrm{Sch}^{\log}(X^{\log})$. *Let $\{C_1, C_2\}$ be a* **partition** *at an element $\gamma \in \mathrm{SmCp}(Y^{\log})$. Then, up to a possible permutation of the indices "1", "2",* **every partition** *at γ* **coincides** *with $\{C_1, C_2\}$.*

(ii) Suppose that $Y^{\log}$ is an **orientable** *object of* $\mathrm{Sch}^{\log}(X^{\log})$. *Let $\{C_1, C_2\}$ be a* **partition** *at a* **non-nodal** *element $\gamma \in \mathrm{SmCp}(Y^{\log})$; $h_1^{\log} : Y_1^{\log} \rightarrowtail Y^{\log}$, $h_2^{\log} : Y_2^{\log} \rightarrowtail Y^{\log}$ monomorphisms as in Definition 3.3, (i-d). Then, for $i = 1, 2$, $h_i^{\log} : Y_i^{\log} \rightarrowtail Y^{\log}$ is an* **open immersion***, and the fiber product $Y_1^{\log} \times_{Y^{\log}} Y_2^{\log}$ determined by $h_1^{\log}$ and $h_2^{\log}$ is a* **maximal** *representative for γ, i.e., in the sense of Proposition 3.2, (vi). In particular, the partition $\{C_1, C_2\}$ is* **seamless***.*

(iii) Suppose that $Y^{\log}$ is a **homogeneous** *object of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then $Y^{\log}$ is* **one-pointed***, and $Y_{\mathrm{sm}}^{\log}$ is* **empty***.*

(iv) Suppose that $Y^{\log}$ is a **log-nodal** *object of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then $Y^{\log}$ is* **homogeneous***, hence, in particular,* **orientable***. Moreover, relative to the notational conventions introduced in Definition 1.1, (i),* $\mathrm{SmCp}(Y^{\log})$ *may be naturally identified with the set of* **positive** *homomorphisms $\xi : P_Y \to \mathbb{N}$ such that ξ induces a* **surjection** *on groupifications $\xi^{\mathrm{gp}} : P_Y^{\mathrm{gp}} \twoheadrightarrow \mathbb{Z}$.*

(v) Suppose that $Y^{\log}$ is a **reduced***,* **one-pointed***,* **non-split** *object of* **rank two** *of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then $Y^{\log}$ is* **log-Dedekind***, but* **not orientable***. In particular, $Y^{\log}$ is* **not homogeneous***. If, moreover, $Y = \mathrm{Spec}(k_Y)$ for some field k_Y, and k_Z is a finite* **Galois** *extension of k_Y such that $Z^{\log} \stackrel{\mathrm{def}}{=} Y^{\log} \times_{k_Y} k_Z$ is* **log-nodal***, then* $\mathrm{SmCp}(Y^{\log})$ *may be naturally identified with the set of* $\mathrm{Gal}(k_Z/k_Y)$**-orbits** *of the set* $\mathrm{SmCp}(Z^{\log})$ *[i.e., which was described explicitly in (iv)].*

Proof. Assertion (i) follows, by applying *entirely formal set-theoretic considerations*, from Definition 3.3, (i-a), (i-b), (i-c). Next, we consider assertion (ii). If one *restricts* the morphisms $h_i^{\log} : Y_i^{\log} \rightarrowtail Y^{\log}$ to the *open subscheme* $Y_{\mathrm{sm}} \subseteq Y$ [cf.

Proposition 3.2, (ii)], then assertion (ii) follows immediately from Proposition 3.2, (vi). Next, let y be a *nodal point* of $Y^{\log}$. Then, since γ is *non-nodal*, it follows immediately from Propositions 1.6, (i); 3.2, (iv), (v); Definition 3.3, (i-a), (i-c), (i-d), that there exists a $j \in \{1, 2\}$ such that, if $i = j$ (respectively, $i \neq j$), then $\mathrm{SmCp}(Y^{\log})_y \subseteq C_i$ (respectively, $\mathrm{SmCp}(Y^{\log})_y \cap C_i = \emptyset$), and, moreover, the *restriction* of $h_i^{\log}$ to the *formal scheme* obtained by *completing* Y along y is an *isomorphism* (respectively, has *empty domain*). Thus, it follows immediately [cf. Proposition 3.2, (ii)] that there exists a Zariski open neighborhood U of y in Y such that, for $i = 1, 2$, the *restriction* $h_i^{\log}|_U$ of $h_i^{\log}$ to U is *scheme-like*, and, moreover, the underlying morphism of schemes associated to $h_i^{\log}|_U$ is an *étale monomorphism* [cf. Proposition 1.4, (v)], hence, by elementary scheme theory, an *open immersion*, whose image *contains* y if $i = j$. The *seamlessness* of the partition $\{C_1, C_2\}$ thus follows from elementary scheme theory [i.e., an easy case of *"Zariski descent"*]. This completes the proof of assertion (ii).

Next, we consider assertion (iii). First, let us observe that it follows formally from assertion (ii) that *every submonic component* of a *homogeneous* object of $\mathrm{Sch}^{\log}(X^{\log})$ is necessarily *nodal*. It thus follows formally [cf. Proposition 3.2, (vi)] that $Y_{\mathrm{sm}}^{\log}$ is *empty* and hence, by Proposition 3.2, (ii), that Y is of *dimension zero*. Since homogeneous objects of $\mathrm{Sch}^{\log}(X^{\log})$ are, by definition, *connected* [hence *nonempty*], we thus conclude that $Y^{\log}$ is *one-pointed*. This completes the proof of assertion (iii).

Next, we consider assertion (iv). First, let us observe that $Y^{\log}$ satisfies the hypotheses imposed on the log scheme "$S^{\log}$" of Lemma 1.5. Thus, Lemma 1.5, (iv), (v), which we apply in the case where, in the notation of *loc. cit.*, "Q" is of rank one, yields a *log étale monomorphism* "$S^{\log}[\xi] \rightarrowtail S^{\log}$", whose domain is *connected* and *submonic*. In particular, it follows immediately from the *existence* and *functorial interpretation* [cf. Lemma 1.5, (iv), (v)] of such monomorphisms "$S^{\log}[\xi] \rightarrowtail S^{\log}$" that $Y^{\log}$ is *log-Dedekind* [cf. Propositions 1.4, (vi); 2.2, (ii)]. Next, for simplicity, let us write $P \stackrel{\mathrm{def}}{=} P_Y$. Then observe that, since $Y^{\log}$ is *split*, it follows immediately from the various definitions involved that any element $\gamma \in \mathrm{SmCp}(Y^{\log})$ determines — i.e., by considering the morphism induced on *log structures* by a representative of γ [cf. Proposition 1.4, (iii)] — a *positive* homomorphism $\xi_\gamma : P \to \mathbb{N}$ such that ξ_γ induces a *surjection* on groupifications $\xi_\gamma^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow \mathbb{Z}$. Moreover, it follows immediately from Proposition 3.2, (vi), together with the various properties of the monomorphisms "$S^{\log}[\xi] \rightarrowtail S^{\log}$" discussed in Lemma 1.5, (v), that the assignment

$$\gamma \mapsto \xi_\gamma$$

just discussed determines a *natural bijection* between $\mathrm{SmCp}(Y^{\log})$ and the set of *positive* homomorphisms $\xi : P \to \mathbb{N}$ such that ξ induces a *surjection* on groupifications $\xi^{\mathrm{gp}} : P^{\mathrm{gp}} \twoheadrightarrow \mathbb{Z}$. In the following, we shall apply this natural bijection to *identify* these two sets.

Next, let $\gamma \in \mathrm{SmCp}(Y^{\log})$. Write $\phi_0 : P \to J_0 \stackrel{\mathrm{def}}{=} \mathbb{N}$ for the element ξ_γ discussed above. In the notation of the discussion entitled "Rank two fs monoids" in §0, for $i = 1, 2$, let us write $\phi_i : P \to J_i$ for the associated positive homomorphism of fs monoids [which is well-defined, up to *possible permutation* of the indices "1" and "2"] and $C_i \subseteq \mathrm{SmCp}(Y^{\log})$ for the subset of elements $\delta \in \mathrm{SmCp}(Y^{\log})$ such that $\xi_\delta : P \to \mathbb{N}$ *factors* through either ϕ_0 or ϕ_i. Then I *claim* that

$\{C_1, C_2\}$ is a *partition* at γ which is *not seamless.*

Indeed, let us first observe that condition (i-a) of Definition 3.3 follows immediately from the discussion of *bisecting monoids* in §0. Next, let us observe that, if we take the log scheme "$S^{\log}$" in Lemma 1.5 to be $Y^{\log}$, then it follows, by applying Lemma 1.5, (vii), (viii), to ϕ_0, that, for $i = 1, 2$, the *log étale monomorphism* "$S^{\log}[\phi_{\{0,i\}}] \rightarrowtail S^{\log}$" yields an object $h_i^{\log} : Y_i^{\log} \rightarrowtail Y^{\log}$ as in condition (i-d) of Definition 3.3. Next, we verify condition (i-c) of Definition 3.3. To this end, suppose that $C \subseteq \mathrm{SmCp}(Y^{\log}) \setminus \{\gamma\}$ is a *chain* that *intersects* both $C_1 \setminus \{\gamma\}$ and $C_2 \setminus \{\gamma\}$. Then it follows immediately from the *connectedness* assumption in the definition of a *chain* [cf. Definition 3.1, (iii)], together with Proposition 1.4, (iii); Lemma 1.9, that there exists a *rank two fs monoid* P^* that arises as a *submonoid* of P^{gp} that *contains* P and, moreover, for $i = 1, 2$, admits a *homomorphism* $\psi_i : P^* \to \mathbb{N}$ whose restriction to P determines an element of $C_i \setminus \{\gamma\}$. Moreover, it follows immediately from the description given above of $\mathrm{SmCp}(Y^{\log})$ [i.e., by considering suitable *minimal points* — cf. also Proposition 2.2, (ii)] that P^* may be chosen so that any positive homomorphism $P^* \to \mathbb{N}$ that induces a surjection on groupifications determines an element of C. On the other hand, it follows immediately from the *"continuity property"* of bisecting monoids discussed in §0 that ϕ_0 *extends* to a *positive homomorphism* $P^* \to \mathbb{N}$ and hence that $\gamma \in C$, a *contradiction.* This completes the verification of condition (i-c) of Definition 3.3. Next, we observe that condition (i-b) of Definition 3.3 — i.e., the fact that, for $i = 1, 2$, $C_i \setminus \{\gamma\}$ is an $\mathbb{N}$-*chain* — follows immediately by considering the *log étale monomorphisms* "$S^{\log}[\phi_{\{0,i\}}] \rightarrowtail S^{\log}$" that arise by applying Lemma 1.5, (vii), (viii) [for an appropriate choice of the indices "1" and "2"], to a *sequence of bisecting monoids* as in Example 0.1, where we take "$P \subseteq {}^{\infty}P$" to be the inclusion of monoids $P \subseteq J_i$ that appears in the present discussion. This completes the proof of the fact that $\{C_1, C_2\}$ is a *partition* at γ. The fact that this partition is *not seamless* follows immediately from the existence of the *log étale monomorphism* "$S^{\log}[\phi_{\{0,1,2\}}] \rightarrowtail S^{\log}$" that arises by applying Lemma 1.5, (vii), (viii), to ϕ_0. This completes the proof of the *claim.* Now it follows formally that $Y^{\log}$ is *homogeneous.* This completes the proof of assertion (iv).

Finally, we consider assertion (v). First, we observe that the fact that $Y^{\log}$ is *log-Dedekind* follows immediately from assertion (iv), via a routine *étale descent* argument; the description given in the statement of assertion (v) of the set $\mathrm{SmCp}(Y^{\log})$ also follows immediately, in light of the various definitions involved, via a routine *étale descent* argument [cf. also Proposition 3.2, (vi)]. Now let $\delta \in \mathrm{SmCp}(Y^{\log})$ be an element that arises from a $\mathrm{Gal}(k_Z/k_Y)$-*invariant* element $\gamma \in \mathrm{SmCp}(Z^{\log})$. Here, we note that the *existence* of such an element of $\mathrm{SmCp}(Z^{\log})$ follows immediately from the description of $\mathrm{SmCp}(Z^{\log})$ given in assertion (iv), together with Lemma 1.5, (ii), which implies the existence of a suitable *positive homomorphism* $\xi_\gamma : P \stackrel{\mathrm{def}}{=} P_Z \to \mathbb{N}$. Then to complete the proof that $Y^{\log}$ is *not orientable*, it suffices to verify that $Y^{\log}$ does *not* admit a partition at δ. Moreover, to verify that $Y^{\log}$ does *not* admit a partition at δ, it suffices, in light of conditions (i-b), (i-c) of Definition 3.3, to show that $\mathrm{SmCp}(Y^{\log}) \setminus \{\delta\}$ is an $\mathbb{N}$-*chain.*

To this end, we consider the *sequence of bisecting monoids* $\{{}^nP\}_{n \in \mathbb{N}}$ of Example 0.1, where we take "$P \subseteq {}^{\infty}P$" to be one of the two *bisecting monoids* of P at ξ_γ. Thus, the homomorphism "${}^{\infty}\phi$" of Example 0.1 corresponds to ξ_γ in the present discussion. Now let us consider the *log étale monomorphisms*

$$\text{“}S^{\log}[\phi_{\{0,1\}}] \rightarrowtail S^{\log}\text{”}$$

that arise by applying Lemma 1.5, (vii), (viii), (ix), where we take the log scheme “$S^{\log}$” of *loc. cit.* to be $Y^{\log}$, and we take “$\phi_1 : P \to J_1$” to be the inclusion $P \subseteq {}^nP$, for $n \in \mathbb{N}$. Here, we observe that if $\zeta : P \to \mathbb{N}$ and σ are as in the *condition* of the display of Lemma 1.5, (ix), and σ acts *nontrivially* on P, then it follows immediately from the $\mathrm{Gal}(k_Z/k_Y)$*-invariance* of ξ_γ [i.e., “${}^\infty\phi$”] that σ acts *nontrivially* on $\mathrm{Ker}(\xi_\gamma^{\mathrm{gp}})$ ($\cong \mathbb{Z}$), and hence [since both ζ and $\zeta \circ \sigma$ are assumed to factor through J_1 and hence through “${}^\infty P$”] that ζ^{gp} *vanishes* on $\mathrm{Ker}(\xi_\gamma^{\mathrm{gp}})$; but this implies that we may assume without loss of generality that $\zeta = \xi_\gamma$, which in turn implies [cf. Example 0.1] that $\zeta^{\mathrm{gp}}(J_1) = \xi_\gamma^{\mathrm{gp}}(J_1) \subseteq \mathbb{Z}$ contains both *positive* and *negative* elements, in *contradiction* to the assumptions imposed on ζ. That is to say, the *condition* of the display of Lemma 1.5, (ix), is satisfied.

Thus, in summary, we obtain a *collection*

$$\{Z_n^{\log} \rightarrowtail Y^{\log}\}_{n\in\mathbb{N}}$$

of *log étale monomorphisms* with *connected* domains [cf. Lemma 1.5, (vii), (viii)] such that [cf. the discussion of Example 0.1] $\delta \notin \mathrm{Chn}(Z_n^{\log} \rightarrowtail Y^{\log}) \subseteq \mathrm{SmCp}(Y^{\log})$, and, moreover, $\cup_{n\in\mathbb{N}} \mathrm{Chn}(Z_n^{\log} \rightarrowtail Y^{\log}) = \mathrm{SmCp}(Y^{\log}) \setminus \{\delta\}$. This completes the proof of the fact that $\mathrm{SmCp}(Y^{\log}) \setminus \{\delta\}$ is an $\mathbb{N}$-chain and hence of assertion (v). ○

Proposition 3.5. (Characterization of log-nodal objects)

(i) Suppose that $Y^{\log}$ *is* **nonempty** *object of* $\mathrm{Sch}^{\log}(X^{\log})$*. Then* $Y^{\log}$ *is* **one-pointed** *if and only if the following* **condition** *is satisfied:*

> *For* $i = 1, 2$*, let* $U_i^{\log}$ *be a* **minimal object** *of* $\mathrm{Sch}^{\log}(X^{\log})$ *and* $f_i^{\log} : U_i^{\log} \to Y^{\log}$ *an arrow of* $\mathrm{Sch}^{\log}(X^{\log})$*. Then there exist a morphism* $f_W^{\log} : W^{\log} \to Y^{\log}$ *and, for each* $i = 1, 2$*, a morphism* $h_i^{\log} : V_i^{\log} \to U_i^{\log}$ *between* **minimal** *objects of* $\mathrm{Sch}^{\log}(X^{\log})$ *such that* $W^{\log}$ *is* **homogeneous***, and, moreover, for each* $i = 1, 2$*, the composite morphism* $f_i^{\log} \circ h_i^{\log} : V_i^{\log} \to Y^{\log}$ *admits a* **factorization** $V_i^{\log} \to W^{\log} \to Y^{\log}$ *through* $f_W^{\log} : W^{\log} \to Y^{\log}$*.*

(ii) Suppose that $Y^{\log}$ *is an object of* $\mathrm{Sch}^{\log}(X^{\log})$*. Then* $Y^{\log}$ *is* **log-nodal** *if and only if* $Y^{\log}$ *is* **homogeneous***, and the identity morphism* $Y^{\log} \to Y^{\log}$ *is a* **minimal point-hull** *in* $\mathrm{Sch}^{\log}(X^{\log})$*.*

Proof. First, we consider assertion (i). Since, by Proposition 3.4, (iii), *homogeneous* objects are *one-pointed*, one verifies immediately from the *sufficency* portion of Proposition 2.10, (iii), that the *condition* under consideration implies that $\mathrm{PtCl}(Y^{\log})$ is of *cardinality one*, and hence, by Proposition 2.10, (i), (iv), that $Y^{\log}$ is *one-pointed*, as desired. Now suppose that $Y^{\log}$ is *one-pointed*. Then, by Proposition 2.10, (i), (iv), it follows that $\mathrm{PtCl}(Y^{\log})$ is of *cardinality one*. Since, by Proposition 3.4, (iv), *log-nodal* objects are *homogeneous*, we thus conclude from the definition of the notion of *"point-equivalence"* that the *condition* under consideration is satisfied. This completes the proof of assertion (i).

Next, we consider assertion (ii). The *necessity* portion of assertion (ii) follows immediately from Propositions 2.10, (v); 3.4, (iv). The *sufficiency* portion of assertion (ii) follows immediately, in light of the definition of the term *"homogeneous"*, from Propositions 2.10, (v); 3.2, (i); 3.4, (v). This completes the proof of assertion (ii). ◯

Theorem 3.6. (Reconstruction of the scheme structure of arbitrary objects) *For $i = 1, 2$, let $X_i^{\log}$ be a* **locally noetherian fs log scheme** *[cf. the discussion entitled "Log schemes" in §0]. For $i = 1, 2$, we shall write* $\mathrm{Sch}^{\log}(X_i^{\log})$ *for the category defined at the beginning of §1. Let*

$$\Phi : \mathrm{Sch}^{\log}(X_1^{\log}) \stackrel{\sim}{\to} \mathrm{Sch}^{\log}(X_2^{\log})$$

be an [arbitrary!] **equivalence of categories**. *Then:*

(i) Φ preserves the following:
(i-a) **log-Dedekind** *objects;*
(i-b) the set $\mathrm{SmCp}(-)$ *associated to a log-Dedekind object;*
(i-c) the subsets of the set $\mathrm{SmCp}(-)$ *of (i-b) which are* **[$\mathbb{N}$-]chains***;*
(i-d) **partitions** *at elements of the set* $\mathrm{SmCp}(-)$ *of (i-b);*
(i-e) **orientable** *objects;*
(i-f) **homogeneous** *objects;*
(i-g) **one-pointed** *objects;*
(i-h) **point-hulls** *with one-pointed codomains;*
(i-i) **minimal point-hulls** *with one-pointed codomains;*
(i-j) **log-nodal** *objects.*

(ii) For $i = 1, 2$, let $Y_i^{\log}$ be an object of $\mathrm{Sch}^{\log}(X_i^{\log})$*; write Y_i for the underlying scheme of $Y_i^{\log}$. Suppose further that $\Phi(Y_1^{\log}) = Y_2^{\log}$. Then Φ induces an* **equivalence of categories**

$$\left(\mathrm{Sch}(Y_1) \stackrel{\sim}{\to}\right) \quad \mathrm{Sch}^{\log}(Y_1^{\log})|_{\text{sch-lk}} \stackrel{\sim}{\to} \mathrm{Sch}^{\log}(Y_2^{\log})|_{\text{sch-lk}} \quad \left(\stackrel{\sim}{\to} \mathrm{Sch}(Y_2)\right)$$

— i.e., where the equivalences in parentheses are the natural equivalences of Definition 1.1, (iv) — that is **functorial** *[in the evident sense!] with respect to $Y_1^{\log}$, $Y_2^{\log}$. Finally, the composite of the equivalences of categories in the above display induces, by applying [LgSch], Theorem 1.7, (ii), an* **isomorphism of schemes**

$$Y_1 \stackrel{\sim}{\to} Y_2$$

that is **functorial** *[in the evident sense!] with respect to $Y_1^{\log}$, $Y_2^{\log}$.*

Proof. First, we consider assertion (i). The preservation of (i-a) follows immediately from the preservation of (i-b), (i-d), (i-g), (i-h), (i-i), (i-l) asserted in Theorem

2.6, (i), together with the *isomorphisms of schemes* obtained in Theorem 2.6, (ii). The preservation of (i-b) follows immediately from the preservation of (i-b), (i-c), (i-h), (i-i) asserted in Theorem 2.6, (i). The preservation of (i-c) follows immediately, in light of the preservation of (i-b), from the preservation of (i-c), (i-h) asserted in Theorem 2.6, (i). The preservation of (i-d) follows immediately, in light of the preservation of (i-a), (i-b), (i-c), from the preservation of (i-a), (i-c), (i-d), (i-i) asserted in Theorem 2.6, (i). The preservation of (i-e) follows formally from the preservation of (i-b), (i-d). The preservation of (i-f) follows formally from the preservation of (i-b), (i-d), (i-e), together with the preservation of (i-a) asserted in Theorem 2.6, (i). The preservation of (i-g) follows immediately, in light of the preservation of (i-f) and the characterization given in Proposition 3.5, (i), from the preservation of (i-b), (i-d) asserted in Theorem 2.6, (i). The preservation of (i-h), (i-i) follows immediately, in light of the preservation of (i-g), from the preservation of (i-a), (i-d) asserted in Theorem 2.6, (i). The preservation of (i-j) follows immediately from the preservation of (i-f), (i-i), together with the characterization given in Proposition 3.5, (ii). Finally, assertion (ii) follows formally, in light of the portion of assertion (i) concerning the preservation of (i-j), from Corollary 2.12, (ii). ◯

It remains to reconstruct, in a *category-theoretic* fashion, the *log structures* of the various log schemes under consideration. The approach taken in the present paper to achieving this goal *differs* somewhat from the approach taken in [LgSch]. Nevertheless, we begin by introducing notation as in the discussion preceding [LgSch], Lemma 2.16: Write $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[t])$ [where t is an indeterminate] for the *affine line* over $\mathbb{Z}$; $(\mathbb{A}^1_{\mathbb{Z}})^{\log}$ for the affine line $\mathbb{A}^1_{\mathbb{Z}}$ over $\mathbb{Z}$ equipped with the log structure determined by the *divisor* $V(t)$ [i.e., "the origin"]; $\exp_{\mathbb{A}} : \mathbb{A}^{\log}_{\mathbb{Z}} \to \mathbb{A}_{\mathbb{Z}}$ for the natural morphism determined by "forgetting the log structure";

$$\exp_{Y^{\log}} : \mathbb{A}^{\log}_{Y^{\log}} \to \mathbb{A}_{Y^{\log}}$$

for the *"exponentiation morphism"* obtained by base-changing $\exp_{\mathbb{A}}$ via the natural morphism $Y^{\log} \to \mathrm{Spec}(\mathbb{Z})$;

$$\mathbb{A}^{\times}_{Y^{\log}} \hookrightarrow \mathbb{A}_{Y^{\log}}$$

for the *open immersion* determined by the *complement* of the *origin* of $\mathbb{A}_{Y^{\log}}$; $\mathbb{A}^{\times}_{Y}$, $\mathbb{A}_Y$ for the underlying schemes of $\mathbb{A}^{\times}_{Y^{\log}}$, $\mathbb{A}_{Y^{\log}}$;

$$0_Y : Y \to \mathbb{A}_Y, \quad 1_Y : Y \to \mathbb{A}_Y$$

for the sections determined by the assignments $t \mapsto 0$, $t \mapsto 1$. Thus, the map induced by $\exp_{Y^{\log}}$ on $Y^{\log}$-valued points may be *naturally identified* with $\exp_Y : M_Y \to \mathcal{O}_Y$. Moreover, one verifies easily that the morphism $\mathbb{A}_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}} \to \mathbb{A}_{\mathbb{Z}}$ that defines the *multiplication operation* on the *ring scheme* $\mathbb{A}_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ determines a *morphism of log schemes* over $Y^{\log}$

$$\mathbb{A}^{\log}_{Y^{\log}} \times_{Y^{\log}} \mathbb{A}^{\log}_{Y^{\log}} \to \mathbb{A}^{\log}_{Y^{\log}}$$

that induces, i.e., on $Y^{\log}$-valued points, the *monoid operation* on M_Y. In the following, we shall always regard $\mathbb{A}_{Y^{\log}}$ as being equipped with the *"ring log scheme"* structure — i.e., the ring object structure in the category of log schemes — determined by the *ring scheme* structure of $\mathbb{A}_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$.

One verifies immediately that any *automorphism* of the log scheme $\mathbb{A}_{Y^{\log}}$ that lies over the identity automorphism of $Y^{\log}$ and is *compatible* with the ring log scheme structure of $\mathbb{A}_{Y^{\log}}$ is necessarily equal to the *identity automorphism*. Write

$$\mathcal{O}^{\times}_{\mathbb{A}^{\times}_{Y}}, \quad \mathcal{O}_{\mathbb{A}^{\times}_{Y}}, \quad M_{\mathbb{A}^{\times}_{Y}}, \quad P_{\mathbb{A}^{\times}_{Y}}$$

for the sheaves on the étale site of $\mathbb{A}^{\times}_{Y}$ associated to the log structure of $\mathbb{A}^{\times}_{Y^{\log}}$;

$$\mathcal{O}^{\times}_{\mathbb{A}_{Y}}, \quad \mathcal{O}_{\mathbb{A}_{Y}}, \quad M_{\mathbb{A}_{Y}}, \quad P_{\mathbb{A}_{Y}}$$

for the sheaves on the étale site of $\mathbb{A}_{Y}$ associated to the log structure of $\mathbb{A}_{Y^{\log}}$;

$$_{\mathbb{A}}\mathcal{O}^{\times}_{\mathbb{A}^{\times}_{Y}}, \quad _{\mathbb{A}}\mathcal{O}_{\mathbb{A}^{\times}_{Y}}, \quad _{\mathbb{A}}M_{\mathbb{A}^{\times}_{Y}}, \quad _{\mathbb{A}}P_{\mathbb{A}^{\times}_{Y}}$$

for the push-forwards to $\mathbb{A}_{Y}$ of the sheaves [on the étale site of $\mathbb{A}^{\times}_{Y}$] $\mathcal{O}^{\times}_{\mathbb{A}^{\times}_{Y}}$, $\mathcal{O}_{\mathbb{A}^{\times}_{Y}}$, $M_{\mathbb{A}^{\times}_{Y}}$, $P_{\mathbb{A}^{\times}_{Y}}$. Finally, if $Y^{\log}$ is an object of $\mathrm{Sch}^{\log}(X^{\log})$, then we observe that $\exp_{Y^{\log}} : \mathbb{A}^{\log}_{Y^{\log}} \to \mathbb{A}_{Y^{\log}}$ may be regarded, in a natural way, as an arrow between objects of $\mathrm{Sch}^{\log}(X^{\log})$.

Proposition 3.7. (Categories of quasi-exponentiation morphisms) *We maintain the notation of the above discussion. Suppose that $Y^{\log}$ is an object of $\mathrm{Sch}^{\log}(X^{\log})$. Thus, $\mathbb{A}_{Y^{\log}}$ may be regarded, in a natural way, as an object of $\mathrm{Sch}^{\log}(X^{\log})$. Write*

$$\mathrm{QExp}(Y^{\log}) \subseteq \mathrm{Sch}^{\log}(\mathbb{A}_{Y^{\log}})$$

for the full subcategory of $\mathrm{Sch}^{\log}(\mathbb{A}_{Y^{\log}})$ consisting of objects $f^{\log} : Z^{\log} \to \mathbb{A}_{Y^{\log}}$ [i.e., "quasi-exponentiation morphisms"] that satisfy the following conditions:

(a) the morphism $Z^{\log} \to Y^{\log}$ determined by $f^{\log}$ is **log smooth***;*

(b) $f^{\log}$ induces an **isomorphism** *$f : Z \xrightarrow{\sim} \mathbb{A}_{Y}$ between the underlying schemes of $Z^{\log}$, $\mathbb{A}_{Y^{\log}}$;*

(c) the base-change of $f^{\log}$ via the open immersion $\mathbb{A}^{\times}_{Y^{\log}} \hookrightarrow \mathbb{A}_{Y^{\log}}$ is an **isomorphism***;*

(d) if

$$\begin{array}{ccccc} T^{\log} & \longrightarrow & Z^{\log} & \xrightarrow{f^{\log}} & \mathbb{A}_{Y^{\log}} \\ \big\downarrow & & \big\downarrow & & \\ S^{\log} & \to & Y^{\log} & & \end{array}$$

is a commutative diagram of morphisms of $\mathrm{Sch}^{\log}(X^{\log})$ in which the horizontal arrows of the square are **minimal point-hulls***, and the resulting fiber product $T^{\log} \times_{\mathbb{A}_{Y^{\log}}} \mathbb{A}^{\times}_{Y^{\log}}$ is the* **empty** *object of $\mathrm{Sch}^{\log}(X^{\log})$, then*

$$\mathrm{rk}(T^{\log}) = \mathrm{rk}(S^{\log}) + 1$$

— i.e., if $S^{\log}$ is a minimal object of rank zero, then $T^{\log}$ is a minimal object of rank one; if $S^{\log}$ is not a minimal object of rank zero, then $\mathrm{rk}(T^{\log}) - 1 = \dim^{\mathrm{sm}}(T^{\log}) = \dim^{\mathrm{sm}}(S^{\log}) + 1 = \mathrm{rk}(S^{\log})$ [cf. Proposition 1.10].

Thus, $\exp_{Y^{\log}} : \mathbb{A}^{\log}_{Y^{\log}} \to \mathbb{A}_{Y^{\log}}$ *may be regarded as an object of* $\mathrm{QExp}(Y^{\log})$.

(i) Let $f^{\log} : Z^{\log} \to \mathbb{A}_{Y^{\log}}$ *be an object of* $\mathrm{QExp}(Y^{\log})$. *Then the following properties hold:*

(i-a) ${}_{\mathbb{A}}\mathcal{O}^{\times}_{\mathbb{A}^{\times}_{Y}}$ *and* ${}_{\mathbb{A}}M_{\mathbb{A}^{\times}_{Y}}$ *admit natural* **free** $\mathcal{O}^{\times}_{\mathbb{A}_Y}$**-actions***;*

(i-b) the quotient sheaf ${}_{\mathbb{A}}\mathcal{O}^{\times}_{\mathbb{A}^{\times}_{Y}}/\mathcal{O}^{\times}_{\mathbb{A}_Y}$ *is* **supported** *on the image of* 0_Y*;*

(i-c) the pull-back sheaf

$$0^*_Y({}_{\mathbb{A}}\mathcal{O}^{\times}_{\mathbb{A}^{\times}_{Y}}/\mathcal{O}^{\times}_{\mathbb{A}_Y})$$

is **naturally isomorphic** *to the constant sheaf* $\mathbb{Z}_Y$ *with fiber* $\mathbb{Z}$ *on* Y*;*

(i-d) the commutative diagram of **natural inclusions**

$$\begin{array}{ccc} \mathcal{O}^{\times}_{\mathbb{A}_Y} & \longrightarrow & {}_{\mathbb{A}}\mathcal{O}^{\times}_{\mathbb{A}^{\times}_{Y}} \\ \big\downarrow & & \big\downarrow \\ M_{\mathbb{A}_Y} & \to & {}_{\mathbb{A}}M_{\mathbb{A}^{\times}_{Y}} \end{array}$$

of sheaves with free $\mathcal{O}^{\times}_{\mathbb{A}_Y}$*-actions is* **co-cartesian***;*

(i-e) we have a **natural isomorphism**

$$\{{}_{\mathbb{A}}\mathcal{O}^{\times}_{\mathbb{A}^{\times}_{Y}}/\mathcal{O}^{\times}_{\mathbb{A}_Y}\} \times P_{\mathbb{A}_Y} \overset{\sim}{\to} {}_{\mathbb{A}}P_{\mathbb{A}^{\times}_{Y}}$$

[cf. (i-d)] of étale sheaves on $\mathbb{A}_Y$*;*

(i-f) we have a **natural isomorphism**

$$\mathbb{Z}_Y \times 0^*_Y(P_{\mathbb{A}_Y}) \overset{\sim}{\to} 0^*_Y({}_{\mathbb{A}}P_{\mathbb{A}^{\times}_{Y}})$$

[cf. (i-c), (i-e)] of étale sheaves on Y*;*

(i-g) the natural morphism of sheaves with free $\mathcal{O}^{\times}_{Z}$*-actions*

$$M_Z \to {}_{\mathbb{A}}M_{\mathbb{A}^{\times}_{Y}}|_Z$$

[cf. conditions (b), (c)] is **injective***;*

(i-h) the **image** I_Z *of the inclusion*

$$(f^{-1} \circ 0_Y)^* P_Z \hookrightarrow 0^*_Y({}_{\mathbb{A}}P_{\mathbb{A}^{\times}_{Y}}) \overset{\sim}{\leftarrow} \mathbb{Z}_Y \times 0^*_Y(P_{\mathbb{A}_Y})$$

[cf. (i-f), (i-g)] satisfies the following conditions

$$\mathbb{N}_Y \times 0^*_Y(P_{\mathbb{A}_Y}) \subseteq I_Z, \quad (\mathbb{Z}_Y \times \{0\}) \cap I_Z = \mathbb{N}_Y \times \{0\}$$

— where we write $\mathbb{N}_Y \subseteq \mathbb{Z}_Y$ *for the constant subsheaf with fiber* $\mathbb{N}$.

(ii) An object $f^{\log} : Z^{\log} \to \mathbb{A}_{Y^{\log}}$ *of* $\mathrm{QExp}(Y^{\log})$ *is a* **terminal** *object of* $\mathrm{QExp}(Y^{\log})$ *if and only if* $f^{\log}$ *is* **isomorphic** *to the object of* $\mathrm{QExp}(Y^{\log})$ *determined by* $\exp_{Y^{\log}} : \mathbb{A}^{\log}_{Y^{\log}} \to \mathbb{A}_{Y^{\log}}$.

Proof. First, we consider assertion (i). Properties (i-a), (i-b), (i-c), and (i-d) are immediate from the definitions. Property (i-e) follows formally from properties (i-a) and (i-d). Property (i-f) follows formally from properties (i-c) and (i-e). Next, we consider properties (i-g) and (i-h). First, we observe that it follows immediately from conditions (b), (c) that the morphism of property (i-g) is an *isomorphism* away from the image of $f^{-1} \circ 0_Y$. Moreover, since the domain and codomain of the morphism of property (i-g) are equipped with *free* $\mathcal{O}_Z^\times$*-actions*, one verifies immediately, by restricting to a *locally closed point* of Y, that, in our verification of properties (i-g) and (i-h), we may assume without loss of generality that Y is the *spectrum of a field*. Then it follows immediately from condition (d) that, to complete the verification of property (i-g), it suffices to verify that the following property holds:

$(*_{\neq 0})$ if ζ is a geometric point of Z that lies over the image of $f^{-1} \circ 0_Y$, then the morphism of property (i-g) induces a *composite morphism* on *fibers* at ζ

$$M_Z|_\zeta \ \to \ {}_{\mathbb{A}}M_{\mathbb{A}_Y^\times}|_\zeta \ \to \ {}_{\mathbb{A}}\mathcal{O}_{\mathbb{A}_Y^\times}|_\zeta$$

— where the second arrow is the arrow determined by the log structure of $\mathbb{A}_{Y^{\log}}^\times$ — that maps some *noninvertible* element of $M_Z|_\zeta$ to a *nonzero* element of ${}_{\mathbb{A}}\mathcal{O}_{\mathbb{A}_Y^\times}|_\zeta$.

But this property $(*_{\neq 0})$ follows immediately from the observation that if $(*_{\neq 0})$ does *not* hold, then it would follow formally from the theory of *charts* for *log structures* [cf. also Lemma 1.9] that condition (c) *fails to hold*. This completes the proof of property (i-g). Next, let us observe that it follows immediately from the various definitions involved that the property $(*_{\neq 0})$ may also be interpreted as asserting that the *image* I_Z referred to in property (i-h) contains the *subsheaf*

$$n \cdot \mathbb{N}_Y \ \times \ 0_Y^*(P_{\mathbb{A}_Y}) \ \subseteq \ \mathbb{Z}_Y \ \times \ 0_Y^*(P_{\mathbb{A}_Y})$$

for some *nonzero* $n \in \mathbb{N}$, where we may assume, without loss of generality, that n is chosen to be *minimal*. Moreover, it follows immediately from the *saturated* nature of the *log structure* of $Z^{\log}$ that $(\mathbb{Z}_Y \ \times \ \{0\}) \cap I_Z = n \cdot \mathbb{N}_Y \ \times \ \{0\}$ and $I_Z \ \subseteq \ n \cdot \mathbb{Z}_Y \ \times \ 0_Y^*(P_{\mathbb{A}_Y})$. Thus, one verifies immediately — i.e., by considering *counterexamples* to *log smoothness* [i.e., condition (a)] as in Example 0.4 when $n \neq 1$ — that $n = 1$. This completes the proof of property (i-h) and hence of assertion (i). Assertion (ii) now follows immediately from the observation that $f^{\log}$ is *isomorphic* to the object of $\mathrm{QExp}(Y^{\log})$ determined by $\exp_{Y^{\log}} : \mathbb{A}_{Y^{\log}}^{\log} \to \mathbb{A}_{Y^{\log}}$ if and only if, in the situation discussed in property (i-h) of assertion (i), the *image* referred to in property (i-h) [not only contains, but] is in fact *equal* to the *subsheaf*

$$\mathbb{N}_Y \ \times \ 0_Y^*(P_{\mathbb{A}_Y}) \ \subseteq \ \mathbb{Z}_Y \ \times \ 0_Y^*(P_{\mathbb{A}_Y})$$

discussed in property (i-h). ◯

The following result may be regarded as the *culmination* of the theory developed in the present paper and corresponds to Theorem B [or, more precisely, Theorem 2.19, (ii)] of [LgSch], the proof of which [i.e., as given in [LgSch]] is, unfortunately, *incomplete*.

Theorem 3.8. (Reconstruction of the log scheme structure of arbitrary objects) *For* $i = 1, 2$, *let* $X_i^{\log}$ *be a* **locally noetherian fs log scheme** *[cf. the* *discussion entitled "Log schemes" in §0]. For $i = 1, 2$, we shall write* $\mathrm{Sch}^{\log}(X_i^{\log})$ *for the category defined at the beginning of §1. Let*

$$\Phi : \mathrm{Sch}^{\log}(X_1^{\log}) \ \stackrel{\sim}{\to}\ \mathrm{Sch}^{\log}(X_2^{\log})$$

be an [arbitrary!] **equivalence of categories**. *Then:*

(i) Φ preserves the following constructions [i.e., up to, in the case of (i-a), (i-c), a unique isomorphism] associated to an object "$(-)$":

(i-a) the **ring object** $\mathbb{A}_{(-)}$*;*

(i-b) the **full subcategory** $\mathrm{QExp}(\ (-)\ ) \subseteq \mathrm{Sch}^{\log}(\mathbb{A}_{(-)})$*;*

(i-c) the **exponentiation morphism** $\exp_{(-)} : \mathbb{A}^{\log}_{(-)} \to \mathbb{A}_{(-)}$*;*

(i-d) the **monoid object** *structure on the object* $\mathbb{A}^{\log}_{(-)}$ *of (i-c).*

(ii) For $i = 1, 2$, let $Y_i^{\log}$ be an object of $\mathrm{Sch}^{\log}(X_i^{\log})$*; write Y_i for the underlying scheme of $Y_i^{\log}$. Suppose further that $\Phi(Y_1^{\log}) = Y_2^{\log}$. Then Φ induces an* **isomorphism of log schemes**

$$Y_1^{\log} \ \stackrel{\sim}{\to}\ Y_2^{\log}$$

that is **functorial** *[in the evident sense!] with respect to $Y_1^{\log}$, $Y_2^{\log}$ and* **compatible** *with the isomorphism of schemes of Theorem 3.6, (ii).*

(iii) There exists a **unique isomorphism of log schemes**

$$X_1^{\log} \ \stackrel{\sim}{\to}\ X_2^{\log}$$

such that Φ is isomorphic to the equivalence of categories induced by this isomorphism of log schemes $X_1^{\log} \stackrel{\sim}{\to} X_2^{\log}$.

Proof. First, we consider assertion (i). The preservation of (i-a) follows immediately from Theorem 3.6, (ii); [LgSch], Proposition 1.6, (iii). To verify the preservation of (i-b), it suffices to verify the preservation of the *conditions* (a), (b), (c), (d) in the statement of Proposition 3.7. The preservation of *condition* (a) follows immediately, in light of the *functorial* definition of *log smoothness*, from Theorem 3.6, (ii). The preservation of *condition* (b) follows formally from Theorem 3.6, (ii). The preservation of *condition* (c) follows immediately from the preservation of (i-a) [i.e., which has already been verified], together with Theorem 3.6, (ii). The preservation of *condition* (d) follows immediately from the preservation of (i-c) asserted in Corollary 2.12, (i) [which is applicable in light of the preservation of (i-j) asserted in Theorem 3.6, (i)], together with the preservation of (i-b), (i-d), (i-g), (i-l) asserted in Theorem 2.6, (i). The completes the proof of the preservation of (i-b). The preservation of (i-c) follows formally, in light of the *criterion* given in Proposition 3.7, (ii), from the preservation of (i-b). The preservation of (i-d) follows immediately, in light of Theorem 3.6, (ii), from the easily verified [i.e., by *restricting* to $\mathbb{A}^{\times}_{(-)}$ — cf. the *injectivity* discussed in property (i-g) of Proposition 3.7, (i)] fact that the morphism that determines the *monoid object* structure on the object $\mathbb{A}^{\log}_{(-)}$ of (i-c) is *completely determined*, i.e., as a morphism *over* "$(-)$", by the morphism that it induces between the *underlying schemes* of its domain and codomain. This completes the proof of assertion (i).

Since the map induced by the *exponentiation morphism* $\exp_{(-)}$ on $(-)$-valued points may be *naturally identified* with the morphism between sheaves of monoids that defines the log structure of "$(-)$" [cf. the discussion preceding Proposition 3.7], assertion (ii) follows immediately from assertion (i); Theorem 3.6, (ii). Finally, assertion (iii) follows immediately from the existence of the *functorial isomorphisms of log schemes* discussed in assertion (ii), by considering, for $i = 1, 2$, a suitable *ind-object* of $\mathrm{Sch}^{\log}(X_i^{\log})$

$$\{{}^{\alpha_i}Y_i^{\log}\}_{\alpha_i \in A_i}$$

— where the *transition morphisms* [notation for which was omitted for the sake of simplicity!] are assumed to be *open immersions* — that *"represents* $X_i^{\log}$*"* in $\mathrm{Sch}^{\log}(X_i^{\log})$. [Here, we recall that if $X_i^{\log}$ *fails* to be *quasi-compact*, then $X_i^{\log}$ does not determine an object of $\mathrm{Sch}^{\log}(X_i^{\log})$ in the usual sense.] ○

Section 4: Category-theoretic Representation of Archimedean Structures

In the present §4, we explain the relatively minor modifications to the theory developed in the present paper for log schemes that are necessary in order to accommodate categories of log schemes equipped with *archimedean structures* as discussed in [ArLgSch]. At a more concrete level, we observe that

· Theorem 1.1;
· Proposition 2.7;
· Proposition 2.8

of [ArLgSch] depend on the portions of the theory of [LgSch] that [cf. Example 0.3; Remark 1.4.1] are *in error*. Thus, in the present §4, we explain how these results, as well as the *main theorem* of [ArLgSch] [i.e., [ArLgSch], Theorem 3.4], may be repaired by applying the theory developed thus far in the present paper.

We begin by reviewing [and slightly modifying] the notation introduced at the beginning of [ArLgSch], §2. Write

$$\overline{\mathrm{SCH}}$$

for the *category of arithmetic schemes*,

$$\overline{\mathrm{SCH}}^{\log}$$

for the *category of arithmetic log schemes* [cf. [ArLgSch], Definition 2.2, and the following discussion], and

$$\mathrm{SCH} \subseteq \overline{\mathrm{SCH}}; \quad \mathrm{SCH}^{\log} \subseteq \overline{\mathrm{SCH}}^{\log}$$

for the *full subcategories* determined by the *purely nonarchimedean* objects [cf. [ArLgSch], Definition 2.3, (i)]. Let $\overline{X}^{\log}$ be an object of $\overline{\mathrm{SCH}}^{\log}$. Thus, $\overline{X}^{\log}$ determines underlying objects $X^{\log}$, $\overline{X}$, and X of the categories $\mathrm{SCH}^{\log}$, $\overline{\mathrm{SCH}}$, and SCH, respectively. Write

$$\overline{\mathrm{SCH}}^{\log}(\overline{X}^{\log}) \stackrel{\mathrm{def}}{=} (\overline{\mathrm{SCH}}^{\log})_{\overline{X}^{\log}}; \quad \mathrm{SCH}^{\log}(X^{\log}) \stackrel{\mathrm{def}}{=} (\mathrm{SCH}^{\log})_{X^{\log}};$$

$$\overline{\mathrm{SCH}}(\overline{X}) \stackrel{\mathrm{def}}{=} (\overline{\mathrm{SCH}}^{\log})_{\overline{X}}; \quad \mathrm{SCH}(X) \stackrel{\mathrm{def}}{=} (\mathrm{SCH})_X$$

for the respective categories of "objects over the subscripted objects" [cf. the notational conventions introduced in the discussion entitled "Categories" in [ArLgSch], §0] and

$$\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}) \subseteq \overline{\mathrm{SCH}}^{\log}(\overline{X}^{\log}); \quad \mathrm{Sch}^{\log}(X^{\log}) \subseteq \mathrm{SCH}^{\log}(X^{\log});$$

$$\overline{\mathrm{Sch}}(\overline{X}) \subseteq \overline{\mathrm{SCH}}(\overline{X}); \quad \mathrm{Sch}(X) \subseteq \mathrm{SCH}(X)$$

for the *full subcategories* determined by the *noetherian* objects. To simplify the exposition, we shall often refer to the *domain* of an arrow which is an object of any of the categories of the preceding display as an "object" of the category.

Note that the notation just introduced is *consistent* with the notational conventions introduced at the beginning of §1 of the present paper for "$\mathrm{Sch}^{\log}(X^{\log})$" and "$\mathrm{Sch}(X)$". Indeed, if $X^{\log}$ is *any locally noetherian fs log scheme*, then one may define [in a fashion consistent with the notation introduced above!]

$$\mathrm{SCH}^{\log}(X^{\log})$$

to be the category whose *objects* are *morphisms of log schemes of locally finite type* $Y^{\log} \to X^{\log}$, where $Y^{\log}$ is a *locally noetherian fs log scheme*, and whose *morphisms* [from an object $Y_1^{\log} \to X^{\log}$ to an object $Y_2^{\log} \to X^{\log}$] are *morphisms of locally finite type* $Y_1^{\log} \to Y_2^{\log}$ lying over $X^{\log}$. In a similar vein, if X is *any locally noetherian scheme*, then one may define [in a fashion consistent with the notation introduced above!]

$$\mathrm{SCH}(X)$$

to be the category whose *objects* are *morphisms of schemes of locally finite type* $Y \to X$, where Y is a *locally noetherian scheme*, and whose *morphisms* [from an object $Y_1 \to X$ to an object $Y_2 \to X$] are *morphisms of locally finite type* $Y_1 \to Y_2$ lying over X.

Definition 4.1.

(i) We shall apply *similar terminology* to data [i.e., such as collections of objects and collections of morphisms] associated to any of the categories

$$\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}), \quad \overline{\mathrm{SCH}}^{\log}(\overline{X}^{\log}), \quad \mathrm{Sch}^{\log}(X^{\log}), \quad \mathrm{SCH}^{\log}(X^{\log}),$$

$$\overline{\mathrm{Sch}}(\overline{X}), \quad \overline{\mathrm{SCH}}(\overline{X}), \quad \mathrm{Sch}(X), \quad \mathrm{SCH}(X)$$

to the terminology that has already been established earlier in the present paper for "$\mathrm{Sch}^{\log}(X^{\log})$" or in [LgSch], §1, for "$\mathrm{Sch}(X)$" whenever this terminology may be defined in an evidently analogous fashion for the category of the above display under consideration. When it is necessary, in order to avoid confusion, to *specify* the category of the above display with respect to which the terminology is to be understood, we shall *append an appropriate prefix* such as

$$\overline{\mathrm{Sch}}^{\log}\text{-}, \quad \overline{\mathrm{SCH}}^{\log}\text{-}, \quad \mathrm{Sch}^{\log}\text{-}, \quad \mathrm{SCH}^{\log}\text{-} \quad \overline{\mathrm{Sch}}\text{-}, \quad \overline{\mathrm{SCH}}\text{-}, \quad \mathrm{Sch}\text{-}, \quad \mathrm{SCH}\text{-}$$

to the terminology in question. This convention concerning *prefixes* will be applied, in particular, when the terminology is to be understood as being applied to the *underlying object* in one of the categories of the first display that is determined by another of the categories of the first display.

(ii) Let $\overline{\mathcal{C}}^{\log} \in \{\overline{\mathrm{Sch}}^{\log}, \overline{\mathrm{SCH}}^{\log}\}$, $\overline{X}^{\log}$ an *arithmetic log scheme*, $\overline{Y}^{\log}$ an object of $\overline{\mathcal{C}}^{\log}(\overline{X}^{\log})$. Then we shall say that $\overline{Y}^{\log}$ is *submonically nonarchimedean* if it holds that such that every *submonic one-pointed* object $\overline{Z}^{\log}$ of $\overline{\mathcal{C}}^{\log}(\overline{X}^{\log})$ that admits a morphism to $\overline{Y}^{\log}$ is *purely nonarchimedean*.

Theorem 4.2. **(Equivalences of categories of schemes)** *Let* $\mathcal{C} \in \{\mathrm{Sch}, \mathrm{SCH}\}$*. For* $i = 1, 2$*, let* X_i *be a* **locally noetherian scheme***. Then, relative to the notation introduced at the beginning of the present §4, any* **equivalence of categories**

$$\Phi : \mathcal{C}(X_1) \xrightarrow{\sim} \mathcal{C}(X_2)$$

arises from a **unique isomorphism of schemes** $X_1 \xrightarrow{\sim} X_2$*.*

Proof. When $\mathcal{C} = \mathrm{Sch}$, Theorem 4.2 is precisely the content of [LgSch], Theorem 1.7, (ii). When $\mathcal{C} = \mathrm{SCH}$, Theorem 4.2 follows from an entirely similar argument. ○

Theorem 4.3. **(Equivalences of categories of arithmetic schemes)** *Let* $\overline{\mathcal{C}} \in \{\overline{\mathrm{Sch}}, \overline{\mathrm{SCH}}\}$*. For* $i = 1, 2$*, let* $\overline{X}_i$ *be an* **arithmetic scheme** *[cf. [ArLgSch], Definition 2.2, (i)]. Then, relative to the notation introduced at the beginning of the present §4, any* **equivalence of categories**

$$\Phi : \overline{\mathcal{C}}(\overline{X}_1) \xrightarrow{\sim} \overline{\mathcal{C}}(\overline{X}_2)$$

arises from a **unique isomorphism of arithmetic schemes** $\overline{X}_1 \xrightarrow{\sim} \overline{X}_2$*.*

Proof. If $\overline{\mathcal{C}} = \overline{\mathrm{Sch}}$, then set $\mathcal{C} \stackrel{\mathrm{def}}{=} \mathrm{Sch}$; if $\overline{\mathcal{C}} = \overline{\mathrm{SCH}}$, then set $\mathcal{C} \stackrel{\mathrm{def}}{=} \mathrm{SCH}$. Then Theorem 4.3 follows, in effect, by combining the theory of [LgSch], §1, with the *non-logarithmic portion* of the theory developed in [ArLgSch], §2, §3. [That is to say, the *errors* in [ArLgSch] discussed at the beginning of the present §4 concern subtleties that arise from the *log structures* of the log schemes involved and hence have no effect on the non-logarithmic portion of the theory.] Indeed, let $i \in \{1, 2\}$; write X_i for the underlying scheme of $\overline{X}_i$. Then one verifies immediately that the $\overline{\mathcal{C}}$-*minimal* objects of $\overline{\mathcal{C}}(\overline{X}_i)$ are the *purely nonarchimedean* objects that arise from the $\mathcal{C}$-*minimal* objects of $\mathcal{C}(X_i)$. Thus, the *one-pointed* objects of $\overline{\mathcal{C}}(\overline{X}_i)$ are precisely the objects $\overline{Y}$ such that $\mathrm{MinPt}(\overline{Y}) = \mathrm{MinPt}(Y)$ [where we write Y for the object of $\mathcal{C}(X_i)$ determined by the underlying scheme of $\overline{Y}$] is of *cardinality one*. This *characterization* of one-pointed objects of $\overline{\mathcal{C}}(\overline{X}_i)$ allows one to *circumvent* the application of [ArLgSch], Proposition 2.7, in the theory of [ArLgSch], §2. In particular, we obtain a *category-theoretic characterization* of $\overline{\mathcal{C}}$-*minimal point-hulls* as in [ArLgSch], Proposition 2.8, (iii). One thus obtains — i.e., by considering *epimorphisms* as in [ArLgSch], Proposition 2.9 — a *category-theoretic characterization* of the *purely nonarchimedean* one-pointed objects of $\overline{\mathcal{C}}(\overline{X}_i)$ as in [ArLgSch], Corollary 2.10, (i), and of the *purely archimedean* morphisms of $\overline{\mathcal{C}}(\overline{X}_i)$ as in [ArLgSch], Corollary 2.10, (ii). In particular, we obtain a *category-theoretic characterization*, as in [ArLgSch], Corollary 2.11, of the *purely nonarchimedean* objects of $\overline{\mathcal{C}}(\overline{X}_i)$ and hence, by applying Theorem 4.2, a *category-theoretic reconstruction* of the *underlying scheme* of an object of $\overline{\mathcal{C}}(\overline{X}_i)$, as in [ArLgSch], Corollary 2.12. Now, to complete the proof of Theorem 4.3 [cf. the proof of [ArLgSch], Theorem 3.4], it suffices to apply the *"non-logarithmic global compatibility"* established in [ArLgSch], Lemma 3.2. ○

Next, we consider *analogues* of Theorem 2.6 for $\mathrm{SCH}^{\log}$, $\overline{\mathrm{Sch}}^{\log}$, and $\overline{\mathrm{SCH}}^{\log}$.

Theorem 4.4. (Reconstruction of the scheme structure of submonic objects for $\mathrm{SCH}^{\log}$**)** *For $i = 1, 2$, let $X_i^{\log}$ be a* **locally noetherian fs log scheme** *[cf. the discussion entitled "Log schemes" in §0]. We shall apply the notation introduced at the beginning of the present §4. Let*

$$\Phi : \mathrm{SCH}^{\log}(X_1^{\log}) \xrightarrow{\sim} \mathrm{SCH}^{\log}(X_2^{\log})$$

be an [arbitrary!] **equivalence of categories**. *Then:*

(i) Φ preserves the following:
(i-a) **monomorphisms***;*
(i-b) **empty** *objects;*
(i-c) **connected** *objects;*
(i-d) **minimal** *objects;*
(i-e) **minimal points***;*
(i-f) **submonic one-pointed** *objects;*
(i-g) **ranks** *of minimal objects;*
(i-h) **SLEM** *morphisms;*
(i-i) **submonic** *objects;*
(i-j) **scheme-like** *morphisms between* **minimal** *objects;*
(i-k) **scheme-like** *morphisms between* **submonic** *objects;*
(i-l) the **submonic dimension** *of objects.*

(ii) For $i = 1, 2$, let $Y_i^{\log}$ be an object of $\mathrm{SCH}^{\log}(X_i^{\log})$; write Y_i for the underlying scheme of $Y_i^{\log}$. Suppose further that $\Phi(Y_1^{\log}) = Y_2^{\log}$. Thus, [cf. the portion of (i) concerning (i-i)] $Y_1^{\log}$ is **submonic** *if only if $Y_2^{\log}$ is. Suppose that $Y_i^{\log}$ is* **submonic** *for $i = 1, 2$. Then Φ induces an* **equivalence of categories**

$$\Big(\mathrm{SCH}(Y_1) \xrightarrow{\sim}\Big) \quad \mathrm{SCH}^{\log}(Y_1^{\log})|_{\text{sch-lk}} \xrightarrow{\sim} \mathrm{SCH}^{\log}(Y_2^{\log})|_{\text{sch-lk}} \quad \Big(\xrightarrow{\sim} \mathrm{SCH}(Y_2)\Big)$$

— i.e., where the equivalences in parentheses are the evident analogues for SCH, $\mathrm{SCH}^{\log}$ *of the natural equivalences of Definition 1.1, (iv) — that is* **functorial** *[in the evident sense!] with respect to $Y_1^{\log}$, $Y_2^{\log}$. Finally, the composite of the equivalences of categories in the above display induces, by applying Theorem 4.2, an* **isomorphism of schemes**

$$Y_1 \xrightarrow{\sim} Y_2$$

that is **functorial** *[in the evident sense!] with respect to* $Y_1^{\log}$, $Y_2^{\log}$.

Proof. The proof is entirely similar to the proof of Theorem 2.6. ○

Theorem 4.5. **(Reconstruction of the scheme structure of submonic objects for** $\overline{\mathrm{Sch}}^{\log}$**,** $\overline{\mathrm{SCH}}^{\log}$**)** *Let* $\overline{\mathcal{C}}^{\log} \in \{\overline{\mathrm{Sch}}^{\log}, \overline{\mathrm{SCH}}^{\log}\}$. *If* $\overline{\mathcal{C}}^{\log} = \overline{\mathrm{Sch}}^{\log}$, *then set* $\mathcal{C}^{\log} \stackrel{\mathrm{def}}{=} \mathrm{Sch}^{\log}$, $\mathcal{C} \stackrel{\mathrm{def}}{=} \mathrm{Sch}$*; if* $\overline{\mathcal{C}}^{\log} = \overline{\mathrm{SCH}}^{\log}$, *then set* $\mathcal{C}^{\log} \stackrel{\mathrm{def}}{=} \mathrm{SCH}^{\log}$, $\mathcal{C} \stackrel{\mathrm{def}}{=} \mathrm{SCH}$. *For* $i = 1, 2$, *let* $\overline{X}_i^{\log}$ *be an* **arithmetic log scheme** *[cf. [ArLgSch], Definition 2.2, (ii)]. We shall apply the notation introduced at the beginning of the present §4. Let*

$$\Phi : \overline{\mathcal{C}}^{\log}(\overline{X}_1^{\log}) \stackrel{\sim}{\to} \overline{\mathcal{C}}^{\log}(\overline{X}_2^{\log})$$

be an [arbitrary!] **equivalence of categories**. *Then:*

(i) Φ *preserves the following:*
(i-a) **monomorphisms***;*
(i-b) **empty** *objects;*
(i-c) **connected** *objects;*
(i-d) **minimal** *objects;*
(i-e) **minimal points***;*
(i-f) **submonic one-pointed** *objects;*
(i-f$^{\mathrm{non}}$*)* **purely nonarchimedean submonic one-pointed** *objects;*
(i-g) **ranks** *of minimal objects;*
(i-h) $\mathcal{C}^{\log}$**-SLEM** *morphisms;*
(i-i) **submonic** *objects;*
(i-i$^{\mathrm{non}}$*)* **purely nonarchimedean submonic** *objects;*
(i-j) $\mathcal{C}^{\log}$**-scheme-like** *morphisms between* **minimal** *objects;*
(i-k) $\mathcal{C}^{\log}$**-scheme-like** *morphisms between* **submonic** *objects;*
(i-l) the **submonic dimension** *of objects.*

(ii) For $i = 1, 2$, *let* $\overline{Y}_i^{\log}$ *be an object of* $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$*; write* $Y_i^{\log}$ *for the underlying log scheme of* $\overline{Y}_i^{\log}$, Y_i *for the underlying scheme of* $\overline{Y}_i^{\log}$. *Suppose further that* $\Phi(\overline{Y}_1^{\log}) = \overline{Y}_2^{\log}$. *Thus, [cf. the portion of (i) concerning (i-i), (i-i*$^{\mathrm{non}}$*)]* $\overline{Y}_1^{\log}$ *is* **submonic** *if only if* $\overline{Y}_2^{\log}$ *is;* $\overline{Y}_1^{\log}$ *is* **purely nonarchimedean submonic** *if only if* $\overline{Y}_2^{\log}$ *is. Suppose that* $\overline{Y}_i^{\log}$ *is* **submonic** *for* $i = 1, 2$. *Then* Φ *induces an* **equivalence of categories**

$$\Big(\mathcal{C}(Y_1) \stackrel{\sim}{\to} \Big) \quad \mathcal{C}^{\log}(Y_1^{\log})|_{\text{sch-lk}} \stackrel{\sim}{\to} \mathcal{C}^{\log}(Y_2^{\log})|_{\text{sch-lk}} \quad \Big(\stackrel{\sim}{\to} \mathcal{C}(Y_2)\Big)$$

— i.e., where the equivalences in parentheses are the evident analogues for $\mathcal{C}$, $\mathcal{C}^{\log}$ *of the natural equivalences of Definition 1.1, (iv) — that is* **functorial** *[in the evident sense!] with respect to* $\overline{Y}_1^{\log}$, $\overline{Y}_2^{\log}$. *Finally, the composite of the equivalences of categories in the above display induces, by applying Theorem 4.2, an* **isomorphism of schemes**

$$Y_1 \stackrel{\sim}{\to} Y_2$$

that is **functorial** *[in the evident sense!] with respect to* $\overline{Y}_1^{\log}$, $\overline{Y}_2^{\log}$.

Proof. First, we consider assertion (i). The preservation of (i-a), (i-b), (i-c), (i-d), (i-e), (i-f), and (i-g) follows from an entirely similar argument to the argument applied in the proof of the preservation of the corresponding properties in Theorem 2.6, (i). Here, we note that [one verifies immediately that]

> the *minimal* objects of $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$ are precisely the *purely nonarchimedean* objects of $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$ that arise from the *minimal* objects of $\mathcal{C}^{\log}(X_i^{\log})$, where we write $X_i^{\log}$ for the underlying log scheme of $\overline{X}_i^{\log}$.

The preservation of (i-f$^{\mathrm{non}}$) now follows, in light of the preservation of (i-f), from an entirely similar argument — i.e., by considering *epimorphisms* as in [ArLgSch], Proposition 2.9 — to the argument applied to verify the *category-theoretic characterization* of *purely nonarchimedean one-pointed* objects given in [ArLgSch], Corollary 2.10, (i). In light of the preservation of (i-f$^{\mathrm{non}}$), the preservation of (i-h) follows from an entirely similar argument to the argument applied in the proof of the preservation of (i-h) in Theorem 2.6, (i). In light of the preservation of (i-h), the preservation of (i-i) follows from an entirely similar argument to the argument applied in the proof of the preservation of (i-i) in Theorem 2.6, (i). The preservation of (i-i$^{\mathrm{non}}$) now follows from the preservation of (i-f), (i-f$^{\mathrm{non}}$), (i-i), since [one verifies immediately that] the *purely nonarchimedean submonic* objects $\overline{Y}^{\log}$ of $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$ may be characterized as the *submonically nonarchimedean submonic* objects $\overline{Y}^{\log}$ of $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$. In light of the preservation of (i-i), the preservation of (i-j), (i-k) follows from an entirely similar argument to the argument applied in the proof of the preservation of (i-j), (i-k) in Theorem 2.6, (i). This completes the proof of assertion (i), except for the verification of the preservation of (i-l).

Next, we consider assertion (ii). Suppose that $\overline{Y}_i^{\log}$ is *submonic* for $i = 1, 2$. Let $\overline{Z}_i^{\log} \to \overline{Y}_i^{\log}$ be a *purely archimedean* morphism of $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$ such that $\overline{Z}_i^{\log}$ is *purely nonarchimedean submonic*. Here, one verifies immediately that such a morphism $\overline{Z}_i^{\log} \to \overline{Y}_i^{\log}$ *exists*, and, moreover, that $\overline{Z}_i^{\log}$ may be *characterized up to isomorphism* as an object *over* $\overline{Y}_i^{\log}$ by the property that any arrow $\overline{T}^{\log} \to \overline{Y}_i^{\log}$ in $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$ such that $\overline{T}^{\log}$ is *purely nonarchimedean submonic* admits a *unique factorization* $\overline{T}^{\log} \to \overline{Z}_i^{\log} \to \overline{Y}_i^{\log}$. Thus, it follows from the portion of assertion (i) concerning the preservation of (i-i$^{\mathrm{non}}$) that we may assume without loss of generality that $\Phi(\overline{Z}_1^{\log}) = \overline{Z}_2^{\log}$. Moreover, since $\overline{Z}_i^{\log}$ is *purely nonarchimedean*, one verifies immediately from the various definitions involved that the *full subcategory*

$$\overline{\mathcal{C}}^{\log}(\overline{Z}_i^{\log}) \subseteq \overline{\mathcal{C}}^{\log}(\overline{Y}_i^{\log})$$

admits a *natural equivalence of categories* $\mathcal{C}^{\log}(Y_i^{\log}) \overset{\sim}{\to} \overline{\mathcal{C}}^{\log}(\overline{Z}_i^{\log})$ [cf. the statement of [ArLgSch], Corollary 2.12]. Thus, by applying the portion of assertion (i) concerning the preservation of (i-k), one verifies immediately that assertion (ii) follows immediately follows from an entirely similar argument to the argument applied to verify Theorem 2.6, (ii). Finally, the portion of assertion (i) concerning the preservation of (i-l) follows from an entirely similar argument to the argument applied in the proof of the preservation of (i-l) in Theorem 2.6, (i). ◯

Next, we consider the *analogue* of Corollary 2.12 and Theorems 3.6, 3.8 for $\mathrm{SCH}^{\log}$.

Theorem 4.6. **(Reconstruction of the log scheme structure of arbitrary objects for** $\mathrm{SCH}^{\log}$**)** *For* $i = 1, 2$*, let* $X_i^{\log}$ *be a* **locally noetherian fs log scheme** *[cf. the discussion entitled "Log schemes" in §0]. We shall apply the notation introduced at the beginning of the present §4. Let*

$$\Phi : \mathrm{SCH}^{\log}(X_1^{\log}) \xrightarrow{\sim} \mathrm{SCH}^{\log}(X_2^{\log})$$

be an [arbitrary!] **equivalence of categories***. Then:*

(i) Φ *preserves the following:*
(i-a) **log-Dedekind** *objects;*
(i-b) the set $\mathrm{SmCp}(-)$ *associated to a log-Dedekind object;*
(i-c) the subsets of the set $\mathrm{SmCp}(-)$ *of (i-b) which are* [$\mathbb{N}$-]**chains***;*
(i-d) **partitions** *at elements of the set* $\mathrm{SmCp}(-)$ *of (i-b);*
(i-e) **orientable** *objects;*
(i-f) **homogeneous** *objects;*
(i-g) **one-pointed** *objects;*
(i-h) **point-hulls** *with one-pointed codomains;*
(i-i) **minimal point-hulls** *with one-pointed codomains;*
(i-j) **log-nodal** *objects.*

(ii) Φ *preserves the following:*
(ii-a) **point-equivalent** *pairs of arrows;*
(ii-b) the set-valued **functor** $\mathrm{LCPt}(-)$ *[up to natural equivalence];*
(ii-c) arrows which are **minimal point-hulls***;*
(ii-d) **scheme-like** *morphisms between* **arbitrary** *objects.*

(iii) For $i = 1, 2$*, let* $Y_i^{\log}$ *be an object of* $\mathrm{SCH}^{\log}(X_i^{\log})$*; write* Y_i *for the underlying scheme of* $Y_i^{\log}$*. Suppose further that* $\Phi(Y_1^{\log}) = Y_2^{\log}$*. Then* Φ *induces an* **equivalence of categories**

$$\Big(\mathrm{SCH}(Y_1) \xrightarrow{\sim}\Big) \quad \mathrm{SCH}^{\log}(Y_1^{\log})|_{\text{sch-lk}} \xrightarrow{\sim} \mathrm{SCH}^{\log}(Y_2^{\log})|_{\text{sch-lk}} \quad \Big(\xrightarrow{\sim} \mathrm{SCH}(Y_2)\Big)$$

— i.e., where the equivalences in parentheses are the evident analogues for $\mathrm{SCH}^{\log}$ *of the natural equivalences of Definition 1.1, (iv) — that is* **functorial** *[in the evident sense!] with respect to* $Y_1^{\log}$*,* $Y_2^{\log}$*. Finally, the composite of the equivalences of categories in the above display induces, by applying Theorem 4.2, an* **isomorphism of schemes**

$$Y_1 \xrightarrow{\sim} Y_2$$

that is **functorial** *[in the evident sense!] with respect to* $Y_1^{\log}$*,* $Y_2^{\log}$*.*

(iv) There exists a **unique isomorphism of log schemes**

$$X_1^{\log} \xrightarrow{\sim} X_2^{\log}$$

such that Φ *is isomorphic to the equivalence of categories induced by this isomorphism of log schemes* $X_1^{\log} \xrightarrow{\sim} X_2^{\log}$*.*

Proof. In light of Theorem 4.4, the proof of assertion (i) (respectively, assertion (ii)) is entirely similar to the proof of Theorem 3.6, (i) (respectively, Corollary 2.12, (i)). Now assertion (iii) follows from the portion of assertion (ii) concerning the preservation of (ii-d) by applying an entirely similar argument to the argument applied to verify Corollary 2.12, (ii). Finally, it follows immediately from assertion (iii) that Φ preserves objects whose underlying scheme is *noetherian* [i.e., *quasi-compact*], and hence that Φ induces an *equivalence of categories*

$$\mathrm{Sch}^{\log}(X_1^{\log}) \stackrel{\sim}{\to} \mathrm{Sch}^{\log}(X_2^{\log})$$

[i.e., as in Theorem 3.8]. Thus, assertion (iv) follows immediately from Theorem 3.8, (iii). ◯

Finally, we consider *analogues* of Theorems 3.6, 3.8 for $\overline{\mathrm{Sch}}^{\log}$, $\overline{\mathrm{SCH}}^{\log}$. In order to formulate and prove these analogues, it will be necessary to introduce some new terminology [patterned after the terminology introduced in Definition 3.3], as follows.

Definition 4.7. Let $\overline{\mathcal{C}}^{\log} \in \{\overline{\mathrm{Sch}}^{\log}, \overline{\mathrm{SCH}}^{\log}\}$. If $\overline{\mathcal{C}}^{\log} = \overline{\mathrm{Sch}}^{\log}$, then set $\mathcal{C}^{\log} \stackrel{\mathrm{def}}{=} \mathrm{Sch}^{\log}$; if $\overline{\mathcal{C}}^{\log} = \overline{\mathrm{SCH}}^{\log}$, then set $\mathcal{C}^{\log} \stackrel{\mathrm{def}}{=} \mathrm{SCH}^{\log}$. Let $\overline{X}^{\log}$ be an *arithmetic log scheme*. We shall apply the notation introduced at the beginning of the present §4. Let $\overline{Y}^{\log}$ be a *connected*, *non-submonic*, $\mathcal{C}^{\log}$*-log-Dedekind*, *submonically nonarchimedean* object of $\overline{\mathcal{C}}^{\log}(\overline{X}^{\log})$; write $Y^{\log}$ for the underlying log scheme of $\overline{Y}^{\log}$. Let $\gamma \in \mathcal{C}^{\log}$-SmCp$(\overline{Y}^{\log}) \stackrel{\mathrm{def}}{=} \mathrm{SmCp}(Y^{\log})$. Write

$$\mathrm{Mono}(\overline{Y}^{\log})$$

for the *full subcategory* of $\overline{\mathcal{C}}^{\log}(\overline{Y}^{\log})$ determined by the arrows $\overline{H}^{\log} \to \overline{Y}^{\log}$ of $\overline{\mathcal{C}}^{\log}(\overline{X}^{\log})$ which are *monomorphisms* in $\overline{\mathcal{C}}^{\log}(\overline{X}^{\log})$.

(i) Let $C_1, C_2 \subseteq \mathcal{C}^{\log}$-SmCp$(\overline{Y}^{\log})$ be $\mathcal{C}^{\log}$*-chains*. Then we shall say that the pair of $\mathcal{C}^{\log}$-chains $\{C_1, C_2\}$ forms a $\ddot{\mathcal{C}}^{\log}$*-partition at* γ if the $\mathcal{C}^{\log}$-chains C_1, C_2 satisfy the following conditions:

(i-a) $C_1 \cup C_2 = \mathcal{C}^{\log}$-SmCp$(\overline{Y}^{\log})$, $\quad C_1 \cap C_2 = \{\gamma\}$;

(i-b) for $i = 1, 2$, the subset $C_i \setminus \{\gamma\} \subseteq \mathcal{C}^{\log}$-SmCp$(\overline{Y}^{\log})$ is a $\mathcal{C}^{\log}$-$\mathbb{N}$*-chain* [hence *nonempty*];

(i-c) the $\mathcal{C}^{\log}$-$\mathbb{N}$-chains of (i-b) are *"maximal"* in the sense that every $\mathcal{C}^{\log}$-$\mathbb{N}$-chain $C \subseteq \mathcal{C}^{\log}$-SmCp$(Y^{\log})$ such that $\gamma \notin C$ is contained in C_i for some $i \in \{1, 2\}$;

(i-d) if, for $i = 1, 2$, we write Ψ_i for the *subfunctor* of the contravariant functor determined by the terminal object [i.e., $\overline{Y}^{\log}$] of Mono$(\overline{Y}^{\log})$ that consists of objects $\overline{h}^{\log} : \overline{H}^{\log} \rightarrowtail \overline{Y}^{\log}$ of Mono$(\overline{Y}^{\log})$ such that *every composite morphism* $\overline{H}_*^{\log} \rightarrowtail \overline{H}^{\log} \rightarrowtail \overline{Y}^{\log}$, where $\overline{H}_*^{\log} \rightarrowtail \overline{H}^{\log}$ is a *minimal point* of $\overline{H}^{\log}$, determines an underlying morphism in $\mathcal{C}^{\log}(Y^{\log})$ that *factors* through some representative of an element $\in C_i$ $(\subseteq \mathcal{C}^{\log}\text{-SmCp}(\overline{Y}^{\log}))$, then Ψ_i is *representable* by an object $\overline{h}_i^{\log} : \overline{Y}_i^{\log} \rightarrowtail \overline{Y}^{\log}$ of $\mathrm{Mono}(\overline{Y}^{\log})$.

We shall say that $\overline{Y}^{\log}$ is $\ddot{\mathcal{C}}^{\log}$-*orientable* if $\overline{Y}^{\log}$ admits a $\ddot{\mathcal{C}}^{\log}$-partition at every element of $\mathcal{C}^{\log}\text{-SmCp}(\overline{Y}^{\log})$.

(ii) Let $\{C_1, C_2\}$ be a $\ddot{\mathcal{C}}^{\log}$-*partition at* γ. Suppose that $\overline{h}_1^{\log}$, $\overline{h}_2^{\log}$ are as in (i-d). Then we shall say that the $\ddot{\mathcal{C}}^{\log}$-partition $\{C_1, C_2\}$ is $\ddot{\mathcal{C}}^{\log}$-*seamless* if the following condition is satisfied:

a *monomorphism* $\overline{h}^{\log} : \overline{H}^{\log} \rightarrowtail \overline{Y}^{\log}$ in $\overline{\mathcal{C}}^{\log}(\overline{X}^{\log})$ is an *isomorphism* if and only if, for $i = 1, 2$, the projection $\overline{H}^{\log} \times_{\overline{Y}^{\log}} \overline{Y}_i^{\log} \to \overline{Y}_i^{\log}$ associated to the fiber product determined by $\overline{h}^{\log}$ and $\overline{h}_i^{\log}$ is an *isomorphism*.

We shall say that $\overline{Y}^{\log}$ is $\ddot{\mathcal{C}}^{\log}$-*homogeneous* if $\overline{Y}^{\log}$ is $\ddot{\mathcal{C}}^{\log}$-*orientable*, and, moreover, *no* $\ddot{\mathcal{C}}^{\log}$-*partition* at an element $\in \mathcal{C}^{\log}\text{-SmCp}(\overline{Y}^{\log})$ is $\ddot{\mathcal{C}}^{\log}$-*seamless*.

Theorem 4.8. **(Reconstruction of the arithmetic log scheme structure of arbitrary objects for $\overline{\mathrm{Sch}}^{\log}$, $\overline{\mathrm{SCH}}^{\log}$)** *Let $\overline{\mathcal{C}}^{\log} \in \{\overline{\mathrm{Sch}}^{\log}, \overline{\mathrm{SCH}}^{\log}\}$. If $\overline{\mathcal{C}}^{\log} = \overline{\mathrm{Sch}}^{\log}$, then set $\mathcal{C}^{\log} \stackrel{\mathrm{def}}{=} \mathrm{Sch}^{\log}$; if $\overline{\mathcal{C}}^{\log} = \overline{\mathrm{SCH}}^{\log}$, then set $\mathcal{C}^{\log} \stackrel{\mathrm{def}}{=} \mathrm{SCH}^{\log}$. For $i = 1, 2$, let $\overline{X}_i^{\log}$ be an* **arithmetic log scheme** *[cf. [ArLgSch], Definition 2.2, (ii)]. We shall apply the notation introduced at the beginning of the present §4. Let*

$$\Phi : \overline{\mathcal{C}}^{\log}(\overline{X}_1^{\log}) \ \stackrel{\sim}{\to} \ \overline{\mathcal{C}}^{\log}(\overline{X}_2^{\log})$$

be an [arbitrary!] **equivalence of categories***. Then:*

(i) Let $\overline{Y}^{\log}$, $\overline{Z}^{\log}$ be objects of $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$, for some $i \in \{1, 2\}$; write $Y^{\log}$, $Z^{\log}$ for the underlying log schemes of $\overline{Y}^{\log}$, $\overline{Z}^{\log}$, respectively. Suppose further that $\overline{Z}^{\log}$ is **purely nonarchimedean***. Then the following properties hold:*

*(i-a$_Y$) every $\ddot{\mathcal{C}}^{\log}$-***partition** *at an element $\gamma \in \mathcal{C}^{\log}\text{-SmCp}(\overline{Y}^{\log})$ determines a $\mathcal{C}^{\log}$-***partition** *at γ;*

(i-a$_Z$) there is a natural **bijective** *correspondence between $\ddot{\mathcal{C}}^{\log}$-***partitions** *at elements $\in \mathcal{C}^{\log}\text{-SmCp}(\overline{Z}^{\log})$ and $\mathcal{C}^{\log}$-***partitions** *at elements $\in \mathcal{C}^{\log}\text{-SmCp}(\overline{Z}^{\log})$;*

*(i-b$_Y$) if $\overline{Y}^{\log}$ is $\ddot{\mathcal{C}}^{\log}$-***orientable***, then $\overline{Y}^{\log}$ is $\mathcal{C}^{\log}$-***orientable***;*

*(i-b$_Z$) $\overline{Z}^{\log}$ is $\ddot{\mathcal{C}}^{\log}$-***orientable** *if and only if $\overline{Z}^{\log}$ is $\mathcal{C}^{\log}$-***orientable***;*

*(i-c$_Z$) a $\ddot{\mathcal{C}}^{\log}$-partition at an element $\in \mathcal{C}^{\log}\text{-SmCp}(\overline{Z}^{\log})$ is $\ddot{\mathcal{C}}^{\log}$-***seamless** *if and only if it corresponds to a $\mathcal{C}^{\log}$-partition [cf. (i-a$_Z$)] that is $\mathcal{C}^{\log}$-***seamless***;*

*(i-d$_Y$) if $\overline{Y}^{\log}$ is $\ddot{\mathcal{C}}^{\log}$-***homogeneous***, then it is* **one-pointed***, and $Y_{\mathrm{sm}}^{\log}$ is* **empty***;*

(i-d$_Z$) $\overline{Z}^{\log}$ *is* $\ddot{\mathcal{C}}^{\log}$**-homogeneous** *if and only if* $\overline{Z}^{\log}$ *is* $\mathcal{C}^{\log}$**-homogeneous**.

(ii) Φ *preserves the following:*
(ii-a) $\mathcal{C}^{\log}$**-log-Dedekind** *objects;*
(ii-b) the set $\mathcal{C}^{\log}$-SmCp$(-)$ *associated to a* $\mathcal{C}^{\log}$*-log-Dedekind object;*
(ii-c) the subsets of the set $\mathcal{C}^{\log}$-SmCp$(-)$ *of (ii-b) which are* $\mathcal{C}^{\log}$**-[**$\mathbb{N}$**-]chains***;*
(ii-d) $\ddot{\mathcal{C}}^{\log}$**-partitions** *at elements of the set* $\mathcal{C}^{\log}$-SmCp$(-)$ *of (ii-b);*
(ii-e) $\ddot{\mathcal{C}}^{\log}$**-orientable** *objects;*
(ii-f) $\ddot{\mathcal{C}}^{\log}$**-homogeneous** *objects;*
(ii-g) **one-pointed** *objects;*
(ii-h) **point-hulls** *with one-pointed codomains;*
(ii-i) **minimal point-hulls** *with one-pointed codomains.*

(iii) For $i = 1, 2$, *let* $\overline{Y}_i^{\log}$ *be an object of* $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$*; write* $Y_i^{\log}$ *for the underlying log scheme of* $\overline{Y}_i^{\log}$*. Suppose further that* $\Phi(\overline{Y}_1^{\log}) = \overline{Y}_2^{\log}$*. Then* $\overline{Y}_1^{\log}$ *is* **purely nonarchimedean** *if and only if* $\overline{Y}_2^{\log}$ *is. In particular,* Φ *induces an* **equivalence of categories**

$$\mathcal{C}^{\log}(Y_1^{\log}) \xrightarrow{\sim} \mathcal{C}^{\log}(Y_2^{\log})$$

that is **functorial** *[in the evident sense!] with respect to* $\overline{Y}_1^{\log}$, $\overline{Y}_2^{\log}$*. Finally, the equivalence of categories in the above display induces, by applying Theorems 3.8, (iii); 4.6, (iv), an* **isomorphism of log schemes**

$$Y_1^{\log} \xrightarrow{\sim} Y_2^{\log}$$

that is **functorial** *[in the evident sense!] with respect to* $\overline{Y}_1^{\log}$, $\overline{Y}_2^{\log}$*.*

(iv) There exists a **unique isomorphism of arithmetic log schemes**

$$\overline{X}_1^{\log} \xrightarrow{\sim} \overline{X}_2^{\log}$$

such that Φ *is isomorphic to the equivalence of categories induced by this isomorphism of arithmetic log schemes* $\overline{X}_1^{\log} \xrightarrow{\sim} \overline{X}_2^{\log}$*.*

Proof. First, we consider assertion (i). Properties (i-a$_Y$), (i-a$_Z$), (i-b$_Y$), (i-b$_Z$), (i-c$_Z$), and (i-d$_Z$) follow formally from the definitions [cf. also the first display in the proof of Theorem 4.5]. Property (i-d$_Y$) then follows, in light of properties (i-a$_Y$) and (i-b$_Y$) [cf. also the first display in the proof of Theorem 4.5], by applying a similar argument to the argument [i.e., involving Proposition 3.4, (ii)] applied in the proof of Proposition 3.4, (iii). Here, we note that one must apply the assumption [cf. the beginning of Definition 4.7] that any $\ddot{\mathcal{C}}^{\log}$-homomogeneous object is *submonically nonarchimedean* in order to conclude that any $\ddot{\mathcal{C}}^{\log}$-partition that determines a $\mathcal{C}^{\log}$-*seamless* $\mathcal{C}^{\log}$*-partition* as in Proposition 3.4, (ii), is necessarily $\ddot{\mathcal{C}}^{\log}$*-seamless.* That is to say, this assumption that any $\ddot{\mathcal{C}}^{\log}$-homomogeneous object is *submonically nonarchimedean* implies that the *compact subset* that determines the archimedean structure of the object [cf. [ArLgSch], Definition 2.2, (ii)] is *supported* over the *nodal* *points* of the object and hence [cf. the discussion of such compact subsets in the proof of [ArLgSch], Lemma 2.5] that the *discrepancy* between $\ddot{\mathcal{C}}^{\log}$-/$\mathcal{C}^{\log}$-*seamless* $\ddot{\mathcal{C}}^{\log}$-/$\mathcal{C}^{\log}$-*partitions* may — at least in the case of $\mathcal{C}^{\log}$-*seamless* $\mathcal{C}^{\log}$-*partitions* as in Proposition 3.4, (ii) — be *ignored*. This completes the proof of assertion (i).

Next, we observe that, in light of Theorem 4.5, (i), (ii), assertion (ii) follows by applying a similar argument to the argument applied to verify Theorem 3.6, (i). Here, we observe that the preservation of the crucial property of being *submonically nonarchimedean* [cf. the beginning of Definition 4.7] follows formally from the portion of Theorem 4.5, (i), concerning the preservation of (i-f), (i-$\mathrm{f}^{\mathrm{non}}$). Also, we observe, with regard to the preservation of (ii-g), that, by applying

· the property (i-d_Y) of assertion (i) in place of Proposition 3.4, (iii), and
· the property (i-d_Z) of assertion (i), together with the evident analogue for $\mathcal{C}^{\log}$ of Proposition 3.4, (iv), in place of Proposition 3.4, (iv),

one obtains a *suitable analogue for* $\overline{\mathcal{C}}^{\log}$ — i.e., by considering $\ddot{\mathcal{C}}^{\log}$-*homogeneous* objects — of the *characterization of one-pointed objects* given in Proposition 3.5, (i). This completes the proof of assertion (ii).

Next, we consider assertion (iii). First, let us observe that the portion of assertion (ii) concerning the preservation of (ii-g), (ii-i) allows one to *circumvent* the application of [ArLgSch], Propositions 2.7, 2.8, in the theory of [ArLgSch], §2. One thus obtains — i.e., by considering *epimorphisms* as in [ArLgSch], Proposition 2.9 — a *category-theoretic characterization* of the *purely nonarchimedean* one-pointed objects of $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$ as in [ArLgSch], Corollary 2.10, (i), and of the *purely archimedean* morphisms of $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$ as in [ArLgSch], Corollary 2.10, (ii) [cf. also Proposition 1.4, (iii), (v), of the present paper]. In particular, we obtain a *category-theoretic characterization*, as in [ArLgSch], Corollary 2.11, of the *purely nonarchimedean* objects of $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$ and hence, by applying Theorems 3.8, (iii); 4.6, (iv), a *category-theoretic reconstruction* of the *underlying log scheme* of an object of $\overline{\mathcal{C}}^{\log}(\overline{X}_i^{\log})$, as in [ArLgSch], Corollary 2.12. This completes the proof of assertion (iii). Finally, assertion (iv) follows from assertion (iii) [cf. the proof of [ArLgSch], Theorem 3.4], by applying the *"logarithmic global compatibility"* established in [ArLgSch], Lemma 3.3. ○

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