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Anomalous random walks and diffusions: From fractals to random media

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Abstract. We present results concerning the behavior of random walks and diffusions on disordered media. Examples treated include fractals and various models of random graphs, such as percolation clusters, trees generated by branching processes, Erdős-Rényi random graphs and uniform spanning trees. As a consequence of the inhomogeneity of the underlying spaces, we observe anomalous behavior of the corresponding random walks and diffusions. In this regard, our main interests are in estimating the long time behavior of the heat kernel and in obtaining a scaling limit of the random walk. We will overview the research in these areas chronologically, and describe how the techniques have developed from those introduced for exactly self-similar fractals to the more robust arguments required for random graphs.

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1. Introduction

Since the mid-sixties, mathematical physicists have investigated anomalous behavior of random walks and diffusions on disordered media (see for example [17]). The random walk on a percolation cluster – the so-called 'ant in the labyrinth' ([24]) – is one of the central examples. Recall that the bond percolation model on the lattice $\mathbb{Z}^d$, $d \geq 2$, is defined as follows: each nearest neighbor bond is open with probability $p \in [0,1]$ and closed otherwise, independently of all the others. It is well-known that this model exhibits a phase transition, whereby if $\theta(p) := P_p(|\mathcal{C}(0)| = +\infty)$, where $\mathcal{C}(0)$ is the open cluster containing 0, then there exists $p_c = p_c(\mathbb{Z}^d) \in (0,1)$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$. For $p > p_c$, there exists a unique open infinite cluster upon which the long time behavior of the simple random walk is similar to that of the simple random walk on $\mathbb{Z}^d$ (see Section 4.1). For the simple random walk on the critical percolation cluster, however, in 1982 Alexander and Orbach [1] made a striking conjecture about how there might be quite different behavior. (To make the problem mathematically precise, one has to consider the critical percolation cluster conditioned to be infinite, as we discuss in

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Section 4.2.) Let $Y = \{Y_n^\omega\}_{n\in\mathbb{N}}$ be the simple random walk on the cluster (i.e. Y_n^ω is in one of the adjacent neighbors of Y_{n-1}^ω with equal probabilities), and $p_n^\omega(x,y)$ be its heat kernel (transition density); see (3.3) for precise definition. Here and in the following, the suffix ω stands for the randomness of the media. Define

$$d_s := -2 \lim_{n\to\infty} \frac{\log p_{2n}^\omega(x,x)}{\log n} \tag{1.1}$$

as the spectral dimension of the cluster if the limit exists. (To be precise, the original definition of d_s was the 'density of states', which gives the asymptotic growth of the eigenvalue counting function.) One formulation of the Alexander-Orbach conjecture is that $d_s = 4/3$ for all $d \geq 2$. Clearly, this expresses anomalous behavior for the random walk, since $d_s = d$ for simple random walk on $\mathbb{Z}^d$. These works stimulated a lot of interest from mathematical physicists in exact fractals as well (see for example [41]).

Mathematical progress on these problems started to be made in the late eighties. In 1986, Kesten wrote two beautiful papers ([31, 32]) in which he constructed an 'incipient infinite cluster' for critical percolation on $\mathbb{Z}^2$ and showed that the random walk on this was anomalous (in the latter work, he also considered random walks on critical models of trees); these were the first significant mathematically rigorous works in this area. Kesten's work and mathematical physicists' work mentioned above triggered intensive research on diffusions on fractals, which are "ideal" disordered media. As part of this, Brownian motion was constructed on typical fractals, such as the Sierpinski gasket, and properties of these processes were obtained (see Section 2). These included detailed heat kernel estimates of the so-called sub-Gaussian form, meaning that the heat kernel is bounded from above and below by

$$c_1 t^{-d_s/2} \exp\Big(- c_2 \Big(\frac{d(x,y)^{d_w}}{t}\Big)^{1/(d_w-1)}\Big)$$

with different pairs of constants (c_1, c_2) for the upper and lower bounds. Here $d_w > 2$ is a constant and $d(\cdot,\cdot)$ is a geodesic distance on the fractal.

While diffusions on fractals had been extensively studied by 2000 and continue to be actively studied, the turn of the century saw increasing moves being made to analyze "fractal-like spaces" instead of working only on ideal fractals. The key issue here is whether the sub-Gaussian estimates mentioned above are stable under perturbations of spaces and operators. (Note that when $d_s = d$ and $d_w = 2$, the corresponding estimates are Gaussian estimates, and such a perturbation theory was extensively developed in the nineties.) In this direction, several functional inequalities have been shown to be equivalent to the sub-Gaussian estimates, some of which are stable under perturbations, meaning that the stability problem has been affirmatively resolved (see Section 3).

It turns out that such a stability theory is useful even for the analysis on random media, including percolation clusters as Kesten considered. Indeed, some functional inequalities have been modified and applied to random walks on various models of disordered media, especially on percolation clusters (see Section 4). Specifically, the Alexander-Orbach conjecture has been affirmatively solved for

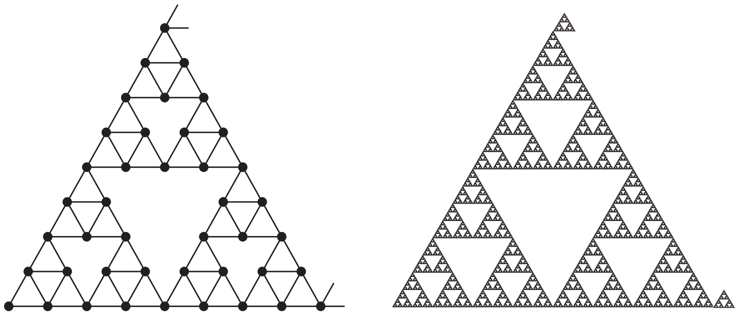

Figure 1. Sierpinski gasket graph V_0 and Sierpinski gasket $\hat{K}$

high dimensions (Theorem 4.4). For some models, scaling limits of random walks have also been established (see Section 4.1 and Section 5); these include supercritical percolation clusters, critical branching processes conditioned to be large, the Erdős-Rényi random graph in the critical window, and the 2-dimensional uniform spanning tree.

The aim of this paper is to give a overview of the stream of research introduced above. It is a very restricted survey and the references are far from complete. Due to space restriction, for papers which are very important but for which details are not discussed in this paper, names of authors and years of publication are mentioned but without inclusion in the list of references. We apologize to the authors of relevant papers which are not cited here. Readers can find more detailed information in the following books/surveys [5, 7, 17, 19, 23, 25, 27, 29, 33, 34, 36, 38, 39, 42, 44, 45].

Notation. We write $f \asymp g$ if there exist constants $c_1, c_2 > 0$ such that $c_1 g(x) \le f(x) \le c_2 g(x)$ for all x, and $f \sim g$ if $\lim_{|x|\to\infty} f(x)/g(x) = 1$.

2. Anomalous heat transfer on fractals

Let $a = (0,0), b = (1,0), c = (1/2, \sqrt{3}/2)$, and set

$$F_1(x) = (x-a)/2 + a,\ \ F_2(x) = (x-b)/2 + b \ \text{ and } \ F_3(x) = (x-c)/2 + c.$$

Then, there exists unique non-void compact set such that $K = \cup_{i=1}^3 F_i(K)$; we call K the 2-dimensional Sierpinski gasket. Define the unbounded Sierpinski gasket as $\hat{K} = \cup_{n=0}^\infty 2^n K$.

We first explain the construction of Brownian motion on $\hat{K}$. Let

$$V_0 = \bigcup_{m=0}^{\infty} 2^m \Big(\bigcup_{i_1,\cdots,i_m=1}^{3} F_{i_1} \circ \cdots \circ F_{i_m}(\{a,b,c\}) \Big), \quad V_m = 2^{-m} V_0.$$

The closure of $\cup_{m\ge 0} V_m$ is $\hat{K}$. Let $\{X(i)\}_{i\ge 0}$ be the simple random walk on V_0. That is, it is a random walk such that $X(i+1)$ is in one of the adjacent neighbors of $X(i)$ in V_0 (i.e. points in the same triangles with length 1 as those $X(i)$ belongs to) with equal probabilities. Let $X_m(i) := 2^{-m}X(i)$ be the simple random walk on V_m. Since X_m moves distance 2^{-m} per unit time, $X_m(i) \to 0$ as $m \to \infty$ for fixed i. So, we must speed up the random walks in order to obtain a non-trivial limit. It is plausible to choose the time scale as the average time for the random walk on V_{m+1} starting from a point in V_m to reach one of the neighboring points in V_m. By the self-similarity and symmetry of $\hat{K}$, this average time is independent of m and it is equal to the average time for X_1 starting from a to arrive at either b or c. A simple calculation deduces that the value is 5. Let $Y_t^{(m)} := X_m([5^m t])$. Then, it can be proved that $\{Y^{(m)}\}$ converges to a non-trivial diffusion on $\hat{K}$ as $m \to \infty$, which is called Brownian motion on $\hat{K}$. (One can construct Brownian motion on K similarly.) Brownian motion on the gasket was first constructed by Goldstein (1987) and Kusuoka (1987) independently. Characterization of Brownian motion is also known; any self-similar diffusion process on $\hat{K}$ whose law is invariant under local translations and reflections on each small triangle is a constant time change of this diffusion ([16]).

The corresponding Laplacian Δ is defined as follows:

$$\Delta f(x) = \lim_{m\to\infty} 5^m \Big(\sum_{x_i:\, x \overset{m}{\sim} x_i} f(x_i) - 4f(x) \Big), \quad x \in \cup_{m\ge 0} V_m \setminus \{0\},$$

for f in a suitable function space, where $x \overset{m}{\sim} y$ means that x and y are adjacent in V_m. Note that the standard approximation for the Laplacian on $\mathbb{R}$ is $\Delta f(x) = \lim_{m\to\infty} 2^{2m}(f(x+2^{-m}) + f(x-2^{-m}) - 2f(x))$ for $f \in C^2(\mathbb{R})$. Set $d_w = \log 5/\log 2$ so that $5 = 2^{d_w}$. Naively, we can say that the Laplacian on the gasket is a "differential operator of order d_w". (One way of stating this rigorously is that the domain of the corresponding Dirichlet form on the gasket is a Besov space of order $d_w/2$ (Jonsson (1996), Grigor'yan-Hu-Lau (2003)).) Kigami (1989) was the first to construct the Laplacian on the gasket directly. It turns out that the theory of Dirichlet forms ([23]) is well-applicable to this area, and diffusions (self-adjoint operators) on fractals have been constructed through Dirichlet forms systematically. Fukushima-Shima (1992) is one of the first who applied the Dirichlet form theory to fractals.

On $\mathbb{R}^d$, we can define $\hat{K}$ similarly from the family of $(d+1)$-th contraction maps with contraction rate $1/2$. (For $d = 1$, $\hat{K} = [0, \infty)$.) The Hausdorff dimension of the d-dimensional gasket is $d_f = \log(d+1)/\log 2$. The time scaling is $d+3$ and $d_w = \log(d+3)/\log 2$.

In order to understand the asymptotic properties of the process, it is very important and useful to obtain detailed heat kernel estimates. Let $\{B(t)\}_{t\ge 0}$ be Brownian motion on the gasket and define

$$P_t f(x) = E^x[f(B(t))] = \int_{\hat{K}} p_t(x,y) f(y) \mu(dy),$$

where μ is the normalized Hausdorff measure on $\hat{K}$. $\{P_t\}_{t\ge 0}$ is the semigroup and $p_t(\cdot,\cdot)$ is the heat kernel (transition density) for Brownian motion on $\hat{K}$. $p_t(\cdot,\cdot)$

is a fundamental solution of the heat equation for the Laplacian. For the case of Brownian motion on $\mathbb{R}^d$, $p_t(x,y)$ is the Gauss-kernel $\frac{1}{(2\pi t)^{d/2}}\exp(-|x-y|^2/(2t))$.

Let $d(x,y)$ be the shortest distance between x and y in $\hat{K}$. The following sub-Gaussian heat kernel estimates are obtained by Barlow-Perkins [16].

Theorem 2.1. *$p_t(x,y)$ obeys the following estimates for $t>0, x,y\in\hat{K}$:*

$$\begin{aligned} c_1 t^{-d_f/d_w}\exp\Big(-c_2\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{1/(d_w-1)}\Big) &\le p_t(x,y) \\ \le c_3 t^{-d_f/d_w}\exp\Big(-c_4\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{1/(d_w-1)}\Big)&. \end{aligned} \tag{2.1}$$

The simple random walk on V_0 also obeys (2.1) for $d(x,y)\le t\in\mathbb{N}$ (Jones (1996)).

From the probabilistic viewpoint, d_w is the order of the diffusion speed of particles and it is called the walk dimension. Indeed, by integrating (2.1), we have $c_5 t^{1/d_w}\le E^x[d(x,B(t))]\le c_6 t^{1/d_w}$. As $d_w>2$, the behavior of the process is anomalous (for a long time, it diffuses slower than Brownian motion on $\mathbb{R}^d$, so the behavior is sub-diffusive). This diffusion does not have finite quadratic variation, so it is not a semi-martingale ([16]). Its martingale dimension is 1 (Kusuoka (1989), Hino (2008)). Set $d_s/2=d_f/d_w$. This d_s, which is the same exponent as in (1.1), gives the asymptotic growth of the eigenvalue counting function for the Laplacian on K, and it is called the spectral dimension. Spectral properties of the Laplacian have been extensively studied (Fukushima-Shima (1992), Kigami-Lapidus (1993), Barlow-Kigami (1997), Teplyaev (1998), etc.). Unlike the Euclidean case, Brownian motion and the Laplacian on the gasket exhibit oscillations in their asymptotics; in the asymptotics of the eigenvalue counting function (Barlow-Kigami (1997)), in the on-diagonal heat kernel asymptotics (Grabner-Woess (1997), Kajino (2013)), and in Schilder's large-deviation principle (Ben Arous-Kumagai (2000)).

(2.1) is a very useful estimate. Various properties of Brownian motion such as laws of the iterated logarithm can be deduced from this estimate. It also implies nice regularity properties of caloric functions $u(t,x)$ (i.e. solutions of the heat equation $\frac{\partial u}{\partial t}=\Delta u$). For $S,R\in(0,\infty), x_0\in\hat{K}$, set

$$Q_-=(S+R^{d_w},S+2R^{d_w})\times B(x_0,R),\ \ Q_+=(S+3R^{d_w},S+4R^{d_w})\times B(x_0,R).$$

The parabolic Harnack inequalities compare the values of caloric functions on Q_- and Q_+ uniformly. They imply uniform Hölder continuity of the caloric functions.

Theorem 2.2 (Generalized parabolic Harnack inequalities and Hölder continuity). *There exist $c_1,c_2,\theta>0$ such that, for any $S,R\in(0,\infty), x_0\in\hat{K}$, if u is a non-negative caloric function on $(S,S+4R^{d_w})\times B(x_0,2R)$, then the following hold:*

$$\sup_{(t,x)\in Q_-} u(t,x)\le c_1\inf_{(t,x)\in Q_+} u(t,x), \tag{$PHI(d_w)$}$$

$$|u(s,x)-u(s',x')|\le c_2\left(\frac{|s-s'|^{1/d_w}+d(x,x')}{R}\right)^{\theta}\|u\|_\infty, \tag{2.2}$$

for any $(s,x),(s',x')\in(S+R^{d_w},S+4R^{d_w})\times B(x_0,R)$.

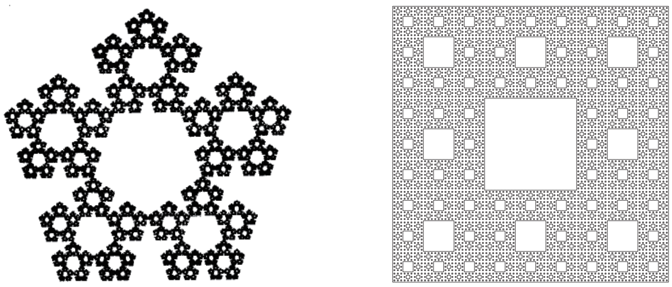

Figure 2. Penta-kun and Sierpinski carpet

In fact, (2.1) and (PHI(d_w)) are equivalent under a suitable volume growth condition as we will see in the next section. (PHI(d_w)) implies various regularity properties of harmonic functions such as the elliptic Harnack inequalities and the Liouville property (i.e. if u is a non-negative harmonic function on $\hat{K}$, then u is a constant function).

For more general fractals such as nested fractals introduced by Lindstrøm (1990) and Sierpinski carpets (see Figure 2, the left figure is an example of nested fractals), Brownian motion is constructed and it is known that the heat kernels obey the sub-Gaussian estimates (2.1) (Barlow-Bass (1989, 1999), Lindstrøm (1990), Kumagai (1993), Fitzsimmons-Hambly-Kumagai (1994)). Characterization of Brownian motion on the fractals are also known (Metz (1996), Sabot (1997), Barlow-Bass-Kumagai-Teplyaev (2010)).

Open problem I: The existing construction of Brownian motion on the carpet requires detailed uniform control of harmonic functions (such as uniform Harnack inequalities) for the approximating processes; see for example [7]. Construct Brownian motion on the carpet without such detailed information.

We refer to [5, 7, 33, 34, 38, 44] for details on diffusions/analysis on fractals.

3. Stability of parabolic Harnack inequalities and sub-Gaussian heat kernel estimates

Since fractals are "ideal" objects in that they have exact self-similarity, it is natural to ask if the inequalities (2.1) and (PHI(d_w)) are stable under perturbations of the state space and the operator.

Let us first briefly overview the history for the case of $d_w = 2$. For any divergence operator $\mathcal{L} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ on $\mathbb{R}^d$ satisfying a uniform elliptic condition, Aronson (1967) proved (2.1) with $d_f = d$ and $d_w = 2$. Later in the last century, there are outstanding results from the field of global analysis on manifolds. Let Δ be the Laplace-Beltrami operator on a complete Riemannian manifold M with the Riemannian metric $d(\cdot,\cdot)$ and with the Riemannian measure μ. Li-Yau

(1986) proved the remarkable fact that if M has non-negative Ricci curvature, then the heat kernel $p_t(x,y)$ satisfies

$$c_1\Phi(x, c_2 d(x,y), t) \le p_t(x,y) \le c_3\Phi(x, c_4 d(x,y), t), \tag{3.1}$$

where $\Phi(x,r,t) = \mu(B(x,t^{1/2}))^{-1}\exp(-r^2/t)$. A few years later, Grigor'yan (1991) and Saloff-Coste (1992) refined the result and proved, in conjunction with the results by Fabes-Stroock (1986) and Kusuoka-Stroock (1987), that (3.1) is equivalent to a volume doubling condition (VD) plus Poincaré inequalities (PI(2)) –see Definition 3.1 and 3.3 for definitions in the graph setting. Their results were later extended to the framework of Dirichlet forms by Sturm (1996) and graphs by Delmotte (1999). Detailed heat kernel estimates are strongly related to the control of harmonic functions. The origin of ideas and techniques used in this field goes back to De Giorgi (1957), Nash (1958), Moser (1961,1964) and there are many other significant works in this area. See for example [25, 42] and the references therein. Summarizing, the following equivalence holds:

$$(3.1) \Leftrightarrow (\mathrm{VD}) + (\mathrm{PI}(2)) \Leftrightarrow (\mathrm{PHI}(2)). \tag{3.2}$$

Since (VD) and (PI(2)) are stable under some perturbations, we see that (3.1) and (PHI(2)) are also stable under the perturbations.

We will discuss the extension of (3.2) to the $d_w > 2$ case. Though such a generalization has also been established under a metric measure space with a local regular Dirichlet form, for simplicity, we will restrict our attention to the graph setting. We first set up notation and definitions.

3.1. Setting. Let G be a countably infinite set, and E a subset of $\{\{x,y\} \in G\times G : x \ne y\}$. We write $x \sim y$ if $\{x,y\} \in E$. A graph is a pair (G,E) and the graph distance $d(x,y)$ for $x,y \in G$ is the length of the shortest path from x to y (we set $d(x,x) = 0$). We assume the graph is connected (i.e. $d(x,y) < \infty$ for all $x,y \in G$) and locally finite (i.e. $|\{y \in G : \{x,y\} \in E\}| < \infty$ for all $x \in G$). For $x \in G$ and $r \ge 0$, denote $B(x,r) = \{y \in G : d(x,y) \le r\}$.

Now assume that the graph G is endowed with a weight (conductance) μ_{xy}, which is a symmetric nonnegative function on $G \times G$ such that $\mu_{xy} > 0$ if and only if $x \sim y$. We call the pair (G,μ) a weighted graph. We can regard it as an electrical network. We define a quadratic form on (G,μ) as follows. Set

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{\substack{x,y\in G\\ x\sim y}}(f(x)-f(y))(g(x)-g(y))\mu_{xy} \qquad \text{for all} \quad f,g \in \mathbb{R}^G.$$

For each $x \in G$, let $\mu_x = \sum_{y\in G}\mu_{xy}$ and for each $A \subset G$, set $\mu(A) = \sum_{x\in A}\mu_x$. μ is a measure on G. Let $\{Y_n\}_{n\ge 0}$ be the discrete time Markov chain whose transition probabilities are given by

$$P(Y_{n+1} = y | Y_n = x) = \frac{\mu_{xy}}{\mu_x} =: P(x,y) \qquad \text{for all} \quad x,y \in G.$$

Y is is called a simple random walk when $\mu_{xy} = 1$ whenever $x \sim y$. The heat kernel of $\{Y_n\}_{n\geq 0}$ can be written as

$$p_n(x,y) := P^x(Y_n = y)/\mu_y \qquad \text{for all} \quad x, y \in G, \tag{3.3}$$

where we set $P^x(\cdot) := P(\cdot|Y_0 = x)$. Clearly, $p_n(x,y) = p_n(y,x)$. We sometimes consider a continuous time Markov chain $\{Y_t\}_{t\geq 0}$ with respect to μ which is defined as follows: each particle stays at a point, say x for (independent) exponential time with parameter 1, and then jumps to another point, say y with probability $P(x,y)$. The heat kernel for the continuous time Markov chain can be expressed as follows.

$$p_t(x,y) = P^x(Y_t = y)/\mu_y = \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}p_n(x,y) \qquad \text{for all} \quad x, y \in G.$$

The discrete Laplacian corresponding to $\{Y_t\}_{t\geq 0}$ is

$$\mathcal{L}f(x) = \sum_{\substack{y\in G\\ y\sim x}} P(x,y)f(y) - f(x) = \frac{1}{\mu_x}\sum_{\substack{y\in G\\ y\sim x}}\Big(f(y) - f(x)\Big)\mu_{xy}.$$

In this section, we assume the following condition on the weighted graph.

Definition 3.1. Let (G,μ) be a weighted graph.
(i) We say (G,μ) has controlled weights if there exists $p_0 > 0$ such that

$$P(x,y) = \mu_{xy}/\mu_x \geq p_0 \qquad \text{for all} \quad x \sim y \in G.$$

(ii) We say (G,μ) satisfies a volume doubling condition (VD) if there exists $c_1 > 1$ such that

$$\mu(B(x,2R)) \leq c_1\mu(B(x,R)) \qquad \text{for all} \quad x \in G, R \geq 1. \tag{3.4}$$

3.2. Stability.

We first introduce two types of perturbations.

Definition 3.2. Let $(G_1,\mu_1), (G_2,\mu_2)$ be weighted graphs with controlled weights.
(i) We say (G_2,μ_2) is a bounded perturbation of (G_1,μ_1) if $G_1 = G_2$ and there exist $c_1, c_2 > 0$ such that $c_1(\mu_1)_{xy} \leq (\mu_2)_{xy} \leq c_2(\mu_1)_{xy}$ for all $x \sim y$.
(ii) A map $T : G_1 \to G_2$ is called a rough isometry if there exist positive constants $c_1, \cdots, c_4 > 0$ such that the following holds for all $x, y \in G_1$ and $y' \in G_2$.

$$c_1^{-1}d_1(x,y) - c_2 \leq d_2(T(x),T(y)) \leq c_1 d_1(x,y) + c_2$$
$$d_2(T(G_1),y') \leq c_3, \quad c_4^{-1}(\mu_1)_x \leq (\mu_2)_{T(x)} \leq c_4(\mu_1)_x.$$

where $d_i(\cdot,\cdot)$ is the the graph distance of (G_i,μ_i), for $i = 1,2$. $(G_1,\mu_1), (G_2,\mu_2)$ are said to be rough isometric if there is a rough isometry between them.

The notion of rough isometry was first introduced by Kanai (1985). Note that rough isometry corresponds to (coarse) quasi-isometry in the field of geometric group theory, which was introduced by Gromov (1981).

We now define some (functional) inequalities.

Definition 3.3. Let (G,μ) be a weighted graph with controlled weights and let $\beta > 1$.
(i) We say (G,μ) satisfies sub-Gaussian heat kernel estimates (HK(β)) if there exist $c_1, \cdots, c_4 > 0$ such that for $x, y \in G, n \geq d(x,y) \vee 1$, the following holds:

$$p_n(x,y) \leq \frac{c_1}{\mu(B(x,n^{1/\beta}))} \exp\Big(-c_2 \Big(\frac{d(x,y)^\beta}{n}\Big)^{1/(\beta-1)}\Big),$$
$$p_n(x,y) + p_{n+1}(x,y) \geq \frac{c_3}{\mu(B(x,n^{1/\beta}))} \exp\Big(-c_4 \Big(\frac{d(x,y)^\beta}{n}\Big)^{1/(\beta-1)}\Big).$$

(ii) We say (G,μ) satisfies (PI(β)), a scaled Poincaré inequality with exponent β, if there exists a constant $c_1 > 0$ such that for any ball $B_R := B(x_0, R) \subset G$ with $x_0 \in G$, $R \geq 1$ and $f : B_R \to \mathbb{R}$,

$$\sum_{x\in B_R} (f(x) - \bar{f}_{B_R})^2 \mu_x \leq c_1 R^\beta \sum_{x\in B_R} \Gamma(f,f)(x).$$

Here $\bar{f}_{B_R} := \mu(B_R)^{-1} \sum_{y\in B_R} f(y)\mu_y$, and $\Gamma(f,f)(x) := \sum_{y\sim x}(f(x)-f(y))^2 \mu_{xy}$.
(iii) We say (G,μ) satisfies (CSA(β)), a cut-off Sobolev inequality in annuli with exponent β, if there exist a constant $c_1 > 0$ such that for every $x_0 \in G, R, r \geq 1$, there exists a cut-off function φ satisfying the following properties:
(a) $\varphi(x) = 1$ if $x \in B_R$, $\varphi(x) = 0$ if $x \in B^c_{R+r}$.
(b) Let $U = B_{R+r} \setminus B_R$. For any $f : U \to \mathbb{R}$,

$$\sum_{x\in U} f(x)^2 \Gamma(\varphi,\varphi)(x) \leq c_1 \Big(\sum_{x\in U} \varphi(x)^2 \Gamma(f,f)(x) + r^{-\beta} \sum_{x\in U} f(x)^2 \mu_x \Big).$$

Theorem 3.4 ([2, 8, 9]). *Let (G,μ) be a weighted graph with controlled weights. Then,*

$$\text{(VD)} + \text{(PI}(\beta)) + \text{(CSA}(\beta)) \Leftrightarrow \text{(PHI}(\beta)) \Leftrightarrow \text{(HK}(\beta)). \tag{3.5}$$

Here and in the following, (PHI(β)) means the discrete version of (PHI(d_w)) in Theorem 2.2 with $d_w = \beta$.

Remark 3.5. (i) There are various other equivalent conditions to (3.5); see [26, 45] and references therein.
(ii) When one of (thus all) the above conditions holds, then it turns out that $\beta \geq 2$.
(iii) (CSA(2)) always holds in the graph context. (Take $\varphi(x) = 1 \wedge r^{-1} d(x, B^c_{R+r})$.) Thus Theorem 3.4 is an extension of (3.2) to the cases of $\beta > 2$ for graphs.
(iv) The main theorem in [2] is the equivalence of the upper bound of (HK(β)) and (CSA(β)) plus the Faber-Krahn inequality with exponent β. The results are stated on metric measure spaces.

For the $\beta = 2$ case, there is a well-known method called Moser's iteration to deduce the Harnack inequality in (3.2). In order for the method to work, it is necessary that the correct order can be deduced using linear cut-off functions. If

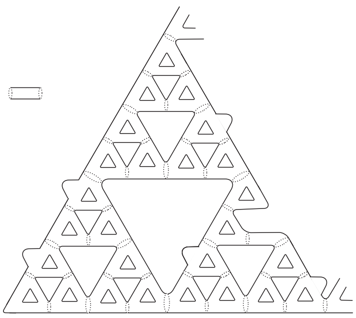

Figure 3. Fractal-like manifold

we adopt similar arguments using the Lipschitz cut-off functions for the $\beta > 2$ case, then the estimates obtained are not sharp enough to establish the Harnack inequality. Roughly speaking, (CSA(β)) guarantees the existence of nice cut-off functions φ that satisfy $\mathcal{E}(\varphi,\varphi) \le c_1 R^{-\beta}\mu(B_R)$. (Note that the order of the energy for the Lipschitz continuous cut-off function is $R^{-2}\mu(B_R)$.) The idea of the proof of the Harnack inequality when $\beta > 2$ is to apply Moser's iteration for weighted measures $\nu_x := \mu_x + R^\beta \Gamma(\varphi,\varphi)(x)$ using (CSA(β)).

Clearly, (VD), (PI(β)) and (CSA(β)) are stable under bounded perturbations. Further, it can be proved that they are stable under rough isometry (Hambly-Kumagai (2004)). We thus obtain the stability of (PHI(β)) and (HK(β)).

As mentioned above, Theorem 3.4 holds in the framework of metric measure spaces with local regular Dirichlet forms (especially Riemannian manifolds). It also holds when the walk dimension β is different for short times and long times. Figure 3 is a 2-dimensional Riemannian manifold whose global structure is like that of the gasket. This can be constructed from the left of Figure 1 by changing each bond to a cylinder and putting projections and dents locally. The diffusion corresponding to the Dirichlet form moves on the surfaces of the cylinders. Using the generalization of Theorem 3.4, one can show that any divergence operator $\mathcal{L} = \sum_{i,j=1}^2 \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ on the manifold which satisfies the uniform elliptic condition obeys (PHI(2)) for $R \le 1$ and (PHI($\log 5/\log 2$)) for $R \ge 1$.

3.3. Strongly recurrent case. The problem with Theorem 3.4 is that it is in general very difficult to check (CSA(β)). Under a stronger volume growth condition, a simpler equivalent condition is known.

For each $x \neq y \in G$, define the effective resistance between them by

$$R_{\mathrm{eff}}(x,y)^{-1} = \inf\left\{\mathcal{E}(f,f) : f(x) = 1, f(y) = 0, f \in \mathbb{R}^G\right\}. \tag{3.6}$$

We define $R_{\mathrm{eff}}(x,x) = 0$ for $x \in G$.

Definition 3.6. (i) We say (G,μ) satisfies the volume growth condition (VG(β_-)) if there exist $K > 1, c_1 > 0$ with $\log c_1/\log K < \beta$ such that

$$\mu(B(x,KR)) \le c_1 \mu(B(x,R)) \quad \text{for all} \quad x \in G, R \ge 1.$$

(ii) We say (G,μ) satisfies (RE(β)), the effective resistance bounds with exponent β, if there exist $c_1, c_2 > 0$ such that

$$\frac{c_1 d(x,y)^\beta}{\mu(B(x,d(x,y)))} \le R_{\text{eff}}(x,y) \le \frac{c_2 d(x,y)^\beta}{\mu(B(x,d(x,y)))} \quad \text{for all} \quad x,y \in G.$$

Theorem 3.7. ([10]) *Let* (G,μ) *be a weighted graph with controlled weights and assume* (VG(β_-))*. Then,*

$$(\mathrm{RE}(\beta)) \Leftrightarrow (\mathrm{PHI}(\beta)) \Leftrightarrow (\mathrm{HK}(\beta)).$$

Under the above conditions, the Markov chain is strongly recurrent in the sense that there exists $p_1 > 0$ such that $P^x(\sigma_{\{y\}} < \sigma_{B(x,2r)^c}) \ge p_1$ for all $x \in G$, $r \ge 1$ and $y \in B(x,r)$, where $\sigma_A = \min\{n \ge 0 : Y_n \in A\}$. Theorem 3.7 is also generalized to the framework of metric measure spaces (Kigami ([34]), Kumagai (2004)).

One can refine the proof of this theorem to a statement which is applicable for random media as we discuss in the next section.

Open problem II: Provide a simpler equivalent condition to (HK(β)) that is applicable to a general graph.

4. Random walk on percolation clusters

From now on, we will discuss random walk on random media. We will consider a random weighted graph $(\mathcal{G}(\omega), \mu(\omega))$ for $\omega \in \Omega$. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space that governs randomness of the weighted graph. Note that we no longer have controlled weights and we cannot expect (VD) in general, so the arguments given in previous sections are not applicable directly. We are interested in the long time behavior of the corresponding Markov chain $\{Y_t^\omega\}_{t\ge 0}$ at the quenched level (i.e. $\mathbb{P}$-a.s. level); we are especially interested in the following two questions:

(Q1) Long time heat kernel estimates for $p_t^\omega(\cdot,\cdot)$.

(Q2) Scaling limit of $\{Y_t^\omega\}_{t\ge 0}$.

(Recall that the suffix ω stands for the randomness of the media.) The prototypical example is random walk on percolation clusters on $\mathbb{Z}^d$, $d \ge 2$.

4.1. Supercritical case. We first consider the supercritical case. In this case, $\{\mu_e : e \in E_d\}$ are Bernoulli random variables; $\mathbb{P}(\mu_e = 1) = p, \mathbb{P}(\mu_e = 0) = 1 - p$ where $p > p_c(\mathbb{Z}^d)$ – see Section 1 for the definition of $p_c(\mathbb{Z}^d)$. We know that there exists a unique infinite connected component of edges with conductance 1, which we denote by $\mathcal{G}(\omega)$. We will condition on the event $\{0 \in \mathcal{G}(\omega)\}$ and define $\mathbb{P}_0(\cdot) := \mathbb{P}(\cdot | 0 \in \mathcal{G})$.

As for (Q1), the following heat kernel estimates are proved in [6].

Theorem 4.1. *There exist constants* $\eta, c_1, \cdots, c_6 > 0$ *and a family of random variables* $\{U_x\}_{x\in\mathbb{Z}^d}$ *with* $\mathbb{P}(U_x \ge n) \le c_1 \exp(-c_2 n^\eta)$ *such that the following holds* $\mathbb{P}_0$*-a.s. for* $t \ge U_x \vee |x-y|$*:*

$$c_3 t^{-d/2} \exp(-c_4 |x-y|^2/t) \le p_t^\omega(x,y) \le c_5 t^{-d/2} \exp(-c_6 |x-y|^2/t). \tag{4.1}$$

The proof uses (3.2) in spirit. A ball $B(x,r)$ is said to be "good" if the volume is comparable to r^d and (PI(2)) holds for the ball. It is proved that a ball is good with high probability and the Borel-Cantelli lemma is used to establish some quenched estimates. Part of the proof of (3.2) is used to establish some heat kernel estimates on good balls.

As for (Q2), it turns out that the quenched invariance principle holds, namely $\varepsilon Y^{\omega}_{t/\varepsilon^2}$ converges as $\varepsilon \to 0$ to Brownian motion on $\mathbb{R}^d$ (with covariance $\sigma^2 I$, $\sigma > 0$) $\mathbb{P}_0$-a.e. ω. This was first proved in [43] for $d \geq 4$ and later extended to all $d \geq 2$ in [18, 40]. The proof for $d \geq 3$ uses Theorem 4.1.

Theorem 4.2. *$\mathbb{P}_0$-a.s., $\varepsilon Y_{t/\varepsilon^2}$ converges (under P^0_ω) in law to Brownian motion on $\mathbb{R}^d$ with covariance $\sigma^2 I$ where $\sigma > 0$ is a non-random constant.*

Furthermore, the quenched local limit theorem holds for this model ([12]).

Let us emphasize that percolation provides one of the natural degenerate models in the sense that uniform ellipticity does not hold, and it is a highly non-trivial fact that the scaling limit is Brownian motion with probability one. For the random conductance model discussed below, when $\mathbb{E}\mu_e < \infty$, a weak form of convergence was already proved in the 1980s that the convergence holds in law under $\mathbb{P}_0 \times P^0_\omega$; a milestone by Kipnis-Varadhan (1986). (See also De Masi-Ferrari-Goldstein-Wick (1989) and Kozlov (1985).) This is sometimes referred to as the annealed (or averaged) invariance principle. It took about two decades to improve the annealed invariance principle to the quenched one.

Remark 4.3. More generally, (Q1) and (Q2) have been extensively studied on the random conductance model. Let $\{\mu_e : e \in E_d\}$ be stationary ergodic that takes non-negative values, and assume $\mathbb{P}(\mu_e > 0) > p_c(\mathbb{Z}^d)$. Then there exists a unique infinite connected component of edges with positive conductance, which we denote by $\mathcal{G}(\omega)$. The random weighted graph $(\mathcal{G}, \mu)$ is the random conductance model. For the i.i.d. case, although there are examples where the heat kernel behaves anomalously (Berger-Biskup-Hoffman-Kozma (2008)), it is proved that quenched invariance principle as in Theorem 4.2 holds; further, $\sigma > 0$ is non-random if $\mathbb{E}\mu_e < \infty$ and $\sigma = 0$ (i.e. the limiting process does not move) if $\mathbb{E}\mu_e = \infty$ (Biskup-Prescott (2007), Mathieu-Piatnitski (2007), Barlow-Deuschel (2010), Andres-Barlow-Deuschel-Hambly (2013)). When $\mathbb{P}(\mu_e \geq u) \sim u^{-\alpha}$as $u \to \infty$ for $\alpha \in (0,1)$, a special case of $\mathbb{E}\mu_e = \infty$, a suitably rescaled Markov chain converges to an anomalous process. It converges to the Fractional-Kinetics (FK) process when $d \geq 2$, where the corresponding heat kernel obeys a fractional time heat equation, and to the Fontes-Isopi-Newman (FIN) diffusion when $d = 1$ (Barlow-Černý (2011), Černý (2011)). See [19, 36] for details. For general ergodic media with $\mathbb{P}(0 < \mu_e < \infty) = 1$, Andres-Deuschel-Slowik ([3]) has proved the quenched invariance principle under some integrability condition of the media. They use Moser's iteration instead of the heat kernel estimates. See Procaccia-Rosenthal-Sapozhnikov (2013) for the quenched invariance principle on a class of degenerate ergodic media such as random interlacements.

4.2. Critical case. We next consider random walk on percolation clusters at criticality. If $d = 2$ or $d \geq 19$ (or $d > 6$ for spread-out models mentioned below) it is known that $\theta(p_c) = 0$, i.e. there is no infinite open cluster $\mathbb{P}$-a.s.; see for example [27]. (Fitzner-van der Hofstad (2014) extends $d \geq 19$ to $d \geq 15$.) It is conjectured that this holds for $d \geq 2$. However, when $p = p_c$, in any box of side n there exist with high probability open clusters of diameter of order n. In order to study mesoscopic properties of these large finite clusters, we will regard them as subsets of an infinite cluster $\mathcal{G}$, called the incipient infinite cluster (IIC for short) and analyze the IIC. This IIC $\mathcal{G} = \mathcal{G}(\omega)$ is our random graph.

The IIC was constructed when $d = 2$ in [31], by taking the limit as $N \to \infty$ of the cluster $\mathcal{C}(0)$ conditioned to intersect the boundary of a box of side N centered at the origin. For large d, a construction of the IIC in $\mathbb{Z}^d$ is given in van der Hofstad-Járai (2004), using the lace expansion. (The results are believed to hold for any $d > 6$.) They also prove the existence and some properties of the IIC for all $d > 6$ for spread-out models: these include the case when there is a bond between x and y with probability pL^{-d} whenever y is in a cube side L with center x, and the parameter L is large enough. The IIC measure can be written as follows:

$$\mathbb{P}_{\mathrm{IIC}}(F) = \lim_{d(0,x)\to\infty} \mathbb{P}_{p_c}(F|0 \leftrightarrow x) \qquad \text{for all} \quad F : \text{cylindrical event}, \tag{4.2}$$

where $\{0 \leftrightarrow x\}$ is the event that 0 and x are in the same open cluster. In the following, we will write $\mathcal{G} = \mathcal{G}_d(\omega)$ for the IIC in $\mathbb{Z}^d$. It is believed that the global properties of $\mathcal{G}$ are the same for all $d > d_c$, both for nearest neighbor and spread-out models, where d_c is the *critical dimension* which is 6 for the percolation model.

Let $Y = \{Y_n^\omega\}_{n\in\mathbb{N}}$ be simple random walk on $\mathcal{G}$, and $p_n^\omega(x, y)$ be its heat kernel. The Alexander-Orbach conjecture mentioned in the introduction can be stated as follows: for any $d \geq 2$, $d_s(\mathcal{G}) = 4/3$, $\mathbb{P}_{\mathrm{IIC}}$-a.e., where d_s was defined in (1.1).

The Alexander-Orbach conjecture turns out to be true on a high dimensional percolation cluster ([35]) as we state in the following.

Theorem 4.4. *There exists $\alpha > 0$ such that the following holds when $d > 6$ for the spread-out model ($d \geq 19$ for the nearest neighbor model): For $\mathbb{P}_{\mathrm{IIC}}$-a.e. $\omega \in \Omega$ and $x \in \mathcal{G}(\omega)$, there exist $N_x(\omega), R_x(\omega) \in \mathbb{N}$ such that*

$$(\log n)^{-\alpha} n^{-\frac{2}{3}} \leq p_{2n}^\omega(x, x) \leq (\log n)^{\alpha} n^{-\frac{2}{3}} \qquad \textit{for all} \quad n \geq N_x(\omega), \tag{4.3}$$

$$(\log R)^{-\alpha} R^3 \leq E_\omega^x \tau_{B(0,R)} \leq (\log R)^{\alpha} R^3 \qquad \textit{for all} \quad R \geq R_x(\omega), \tag{4.4}$$

where $\tau_A := \min\{n \geq 0 : Y_n \notin A\}$.

In the next subsection, we will briefly discuss how this was proved.

4.2.1. Heat kernel estimates on random media. As we mentioned in the end of the last section, Theorem 3.7 (especially its proof) turns out to be useful even for random walk on random media. Below we give a general theorem.

Let $(\mathcal{G}(\omega), \omega \in \Omega)$ be a random graph on $(\Omega, \mathcal{F}, \mathbb{P})$; for $\mathbb{P}$-a.e. ω, we assume that $\mathcal{G}(\omega)$ is a connected locally finite graph that contains a distinguished point

$0 \in \mathcal{G}(\omega)$. For each ω, we put conductance 1 for each bond and let $\{Y_n^\omega\}$ be the simple random walk on $\mathcal{G}$. Let $B(0,R)$ be the ball of radius R centered at 0 with respect to the graph distance $d(\cdot,\cdot)$. For $D, \lambda \geq 1$, we say $B(0,R)$ in $\mathcal{G}$ is λ-good if

$$\frac{R^D}{\lambda} \leq \mu(B(0,R)) \leq \lambda R^D, \quad \frac{R}{\lambda} \leq R_{\text{eff}}(0, B(0,R)^c). \tag{4.5}$$

Here $R_{\text{eff}}(\cdot,\cdot)$ is the effective resistance defined in (3.6). The following are the general estimates in [13, 37].

Theorem 4.5. *If there exist $R_0, \lambda_0 \geq 1$ and $q_0 > 0$ such that*

$$\mathbb{P}(\{\omega : B(0,R) \text{ is } \lambda\text{-good }\}) \geq 1 - \lambda^{-q_0} \qquad \text{for all} \quad R \geq R_0, \lambda \geq \lambda_0, \tag{4.6}$$

then there exists $c > 0$ such that the following holds:
For $\mathbb{P}$-a.e. $\omega \in \Omega$ and $x \in \mathcal{G}(\omega)$, there exist $N_x(\omega), R_x(\omega) \in \mathbb{N}$ such that

$$(\log n)^{-c} n^{-\frac{D}{D+1}} \leq p_{2n}^\omega(x,x) \leq (\log n)^c n^{-\frac{D}{D+1}} \qquad \text{for all} \;\; n \geq N_x(\omega), \tag{4.7}$$

$$(\log R)^{-c} R^{D+1} \leq E_\omega^x \tau_{B(0,R)} \leq (\log R)^c R^{D+1} \qquad \text{for all} \;\; R \geq R_x(\omega). \tag{4.8}$$

In particular, $d_s(\mathcal{G}(\omega)) = \frac{2D}{D+1}$, $\mathbb{P}$–a.s. ω, and the random walk is recurrent.

Furthermore, if (4.6) holds with $\exp(-c_1\lambda^{q_0})$ instead of λ^{-q_0}, then (4.7) and (4.8) hold with $(\log\log \cdot)^{\pm c}$ instead of $(\log \cdot)^{\pm c}$.

In the above statement, the volume growth is of order R^D and the resistance growth is linear. In [37], a general version is given where both growths are controlled by increasing functions with $c_1(R/r)^{\beta_1} \leq f(R)/f(r) \leq c_2(R/r)^{\beta_2}$ for $0 < r < R$, where $0 < \beta_1 \leq \beta_2$ are constants. For this general version, we need to add an extra condition $R_{\text{eff}}(0,z) \leq \lambda f(d(0,z))$ for all $z \in B(0,R)$ in (4.5). Note that this extra condition is always true for the linear case.

Open problem III: Provide a simpler sufficient condition for the heat kernel and exit time estimates for $d_s \geq 2$.

4.2.2. Applying Theorem 4.5 to concrete models. In [35], the condition (4.6) is proved using the control of the two-point function that can be obtained using the lace expansion. Write $x \leftrightarrow y$ if x and y are connected by open edges.

Proposition 4.6. *For the critical bond percolation, assume that the following holds:*

$$c_1|x|^{2-d} \leq \mathbb{P}_{p_c}(0 \leftrightarrow x) \leq c_2|x|^{2-d} \qquad \text{for all} \quad x \in \mathcal{G}(\omega). \tag{4.9}$$

Then (4.6) in Theorem 4.5 holds for $\mathbb{P}_{\text{IIC}}$ with $D = 2$.

When d is high enough, (4.9) is proved using the lace expansion (Hara-van der Hofstad-Slade (2003) for $d > 6$ for the spread-out model, Hara (2008) for $d \geq 19$ for the nearest neighbor model), which implies Theorem 4.4.

There are other models where anomalous behavior of random walk has been proved by verifying (4.6) in Theorem 4.5. We list up some of them. For (i)-(iii), $D = 2$ and $d_s = 4/3$. For (i), (4.6) holds with $\exp(-c_1\lambda^{q_0})$ instead of λ^{-q_0}.

(i) IIC for critical percolation on regular trees ([14]).

(ii) IIC for spread out oriented percolation for $d \geq 6$ ([13]).

(iii) Invasion percolation on a regular tree ([4]).

(iv) IIC for α-stable Galton-Watson trees conditioned to survive forever (Croydon-Kumagai (2008)): $D = \alpha/(\alpha-1)$ and $d_s = 2\alpha/(2\alpha-1)$.

(v) 2-dimensional uniform spanning trees ([15]): $D = 8/5$ and $d_s = 16/13$ – See Section 5.2 for details.

[28] partly generalized the results in [35], and proved the Alexander-Orbach conjecture for the IIC in high dimensions, both for long-range and finite-range percolation.

For the model (i), we have much more detailed estimates ([14]).

Theorem 4.7. *The heat kernel of simple random walk on the IIC for critical percolation on the regular tree obeys the following estimates.*
(i) *(4.3) and (4.4) hold with* $(\log\log\cdot)^{\pm\alpha}$ *instead of* $(\log\cdot)^{\pm\alpha}$.
(ii) *It holds that for* $\mathbb{P}_{\rm IIC}$*-a.e.* ω

$$\liminf_{n\to\infty}(\log\log n)^{1/6}n^{2/3}p_{2n}^{\omega}(0,0) \leq 2.$$

(iii) *The annealed heat kernel* $\mathbb{E}_{\rm IIC}[p_{2n}^{\cdot}(x,y)|x,y\in\mathcal{G}]$ *obeys the sub-Gaussian estimates* (2.1) *with* $d_f = 2, d_w = 3$ *for* $n \geq d(x,y)\vee 1$.

As we have seen above, the quenched estimates have oscillation of $\log\log$ order whereas the annealed estimates do not. Detailed off-diagonal heat kernel estimates (which hold with high probability) are also obtained in [14, Theorem 4.9, 4.10].

4.2.3. Below critical dimensions. For low dimensions, there are only a few rigorous results.

One of the most attractive models is the IIC for 2-dimensional critical percolation. In [32], Kesten proves sub-diffusive behavior of simple random walk on the IIC for 2-dimensional critical percolation cluster (also shows the existence of the IIC in [31]). Namely, let $\{Y_n^{\omega}\}_{n\geq 0}$ be a simple random walk on the IIC, then there exists $\epsilon > 0$ such that the $\mathbb{P}_{\rm IIC}$-distribution of $n^{-\frac{1}{2}+\epsilon}d(0,Y_n)$ is tight. A quenched version of Kesten's result is established both for the IIC and the invasion percolation cluster (Damron-Hanson-Sosoe (2013)). For bond percolation on $\mathbb{Z}^d$, the critical dimension is 6. The Alexander-Orbach conjecture is considered to be false for $d \leq 5$ and some numerical simulations (cf. [17], [29, Section 7.4]) support this. It is a challenging problem to prove this rigorously, especially for $d = 2$.

It is proved in [30] that the effective resistance between the origin and generation n of the incipient infinite oriented branching random walk in $d < 6$ is $O(n^{1-\gamma})$ for some $\gamma > 0$. It is interesting to see that, while the critical dimension of the model is 4, asymptotic behavior of the random walk changes already at $d = 5$. The precise resistance exponent (even its existence) is not known.

Other low dimensional random media for which heat kernel/exit time estimates have been studied include the uniform infinite planar triangulation (Benjamini-Curien (2013); see also Gurel-Gurevich and Nachmias (2013)), the critical percolation cluster for the diamond lattice (Hambly-Kumagai (2010)), and the non-intersecting two-sided random walk trace on $\mathbb{Z}^2$ and $\mathbb{Z}^3$ (Shiraishi (2014+)). See [36, Section 7.4] for details.

Open problems IV: (i) Prove the existence of d_s and d_w for lower dimensional models. Disprove (or prove) the Alexander-Orbach conjecture for the models.
(ii) Compute resistance for random media when the resistance growth is not linear.

Remark 4.8. Heat kernel estimates and scaling limits have been considered for random walks on the long-range percolation model and its variants. See [20, 21] and references therein.

5. Scaling limits of random walks on random media

In this section, we discuss (Q2) (i.e. question about scaling limits of random walks) for random media. It is proved by Croydon (2008) that the distribution of the rescaled simple random walk on critical finite variance Galton-Watson tree converges to Brownian motion on the Aldous tree (see Croydon (2010) for the infinite variance case). Below, we give two more examples.

5.1. Erdős-Rényi random graph in critical window. Let $V_N := \{1, 2, \cdots, N\}$. The Erdős-Rényi random graph is a percolation on the complete graph with vertices in V_N, namely each bond $\{i, j\}$, $i, j \in V_N$ is open with probability $p \in [0, 1]$ and closed otherwise, independently of all the others. Denote its largest connected component by $\mathcal{C}^N$. It is known that this model exhibits a phase transition around $p \sim c/N$ in that the following holds with high probability (Erdős-Rényi (1960)):

$$c < 1 \Rightarrow |\mathcal{C}^N| = O(\log N), \;\; c > 1 \Rightarrow |\mathcal{C}^N| \asymp N, \;\; c = 1 \Rightarrow |\mathcal{C}^N| \asymp N^{2/3}.$$

We will consider finer scaling (the so-called critical window), namely we will take $p = 1/N + \lambda N^{-4/3}$ for fixed $\lambda \in \mathbb{R}$. In this window, the size of the i-th largest connected component is of order $N^{2/3}$ for each $i \in \mathbb{N}$. The following results hold for each i-th largest connected component; for simplicity, we state them for the $\mathcal{C}^N$.

There exists a random compact metric space $\mathcal{M} = \mathcal{M}_\lambda$ such that the following holds in the Gromov-Hausdorff sense

$$N^{-1/3}\mathcal{C}^N \xrightarrow{d} \mathcal{M},$$

where $\mathcal{C}^N$ is considered as a rooted metric space (Addario-Berry, Broutin and Goldschmidt (2012); see also Aldous (1997)). The concrete construction of $\mathcal{M}$ is also known. Let $\{Y_m^{\mathcal{C}^N}\}_{m\geq 0}$ be the simple random walk on $\mathcal{C}^N$. Then the following holds.

Theorem 5.1 ([22]). (i) *There exist Brownian motion* $\{B_t^{\mathcal{M}}\}_{t\ge 0}$ *on* $\mathcal{M}$ *such that*

$$\{N^{-1/3}Y^{\mathcal{C}^N}_{[Nt]}\}_{t\ge 0} \xrightarrow{d} \{B_t^{\mathcal{M}}\}_{t\ge 0}, \qquad \mathbb{P}-a.s.$$

(ii) *There exist a jointly continuous heat kernel* $p_t^{\mathcal{M}}(\cdot,\cdot)$ *of Brownian motion and* $\theta, T_0, c_1, \cdots, c_4 > 0$ *such that for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$,

$$p_t^{\mathcal{M}}(x,y) \le c_1 t^{-\frac{d_f}{d_w}}\ell(t^{-1})^{\theta}\exp\left\{-c_2\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\ell\left(\frac{d(x,y)}{t}\right)^{-\theta}\right\} \tag{5.1}$$

$$p_t^{\mathcal{M}}(x,y) \ge c_3 t^{-\frac{d_f}{d_w}}\ell(t^{-1})^{-\theta}\exp\left\{-c_4\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\ell\left(\frac{d(x,y)}{t}\right)^{\theta}\right\} \tag{5.2}$$

for all $x,y\in\mathcal{M}, t\le T_0$ *with* $\ell(x) := 1\vee\log x$ *and* $d_f = 2, d_w = 3$.

It is known that the L^p-mixing time of the simple random walk on $\mathcal{C}^N$ converges in $\mathbb{P}$-distribution to that of Brownian motion on $\mathcal{M}$ (Croydon-Hambly-Kumagai (2012); see also Nachmias-Peres (2008)).

5.2. 2-dimensional uniform spanning tree. Let $\Lambda_n := [-n,n]^2\cap\mathbb{Z}^2$, which we consider as a graph with edges between lattice neighbors. A spanning tree of Λ_n is a subgraph that connects all the vertices of Λ_n and contains no cycles. Let $\mathcal{U}^{(n)}$ be a spanning tree of Λ_n selected uniformly at random from all possibilities. Pemantle (1991) showed that one could then define a uniform spanning tree (UST) of $\mathbb{Z}^2$, which we denote by $\mathcal{U}$, as the local limit of $\mathcal{U}^{(n)}$ as $n\to\infty$. He also showed that the distribution of $\mathcal{U}$ is independent of the boundary conditions (such as wired, free) on Λ_n. An alternative and very useful construction of $\mathcal{U}$ involves Wilson's algorithm (1996), which can be described as follows. Enumerate $\mathbb{Z}^2$ arbitrarily as $x_0, x_1, \cdots$ and let $\mathcal{U}(0) = \{x_0\}$. For $k\ge 1$, given $\mathcal{U}(k-1)$, run the loop-erased random walk (LERW) from x_k to $\mathcal{U}(k-1)$ and define $\mathcal{U}(k)$ to be the union of the path and $\mathcal{U}(k-1)$. (Here, LERW is a process introduced by Lawler (1980) which is obtained by chronologically erasing loops from the simple random walk.) We then obtain $\mathcal{U} = \cup_{k\ge 0}\mathcal{U}(k)$ – see [39] for more details about the UST.

Now, let M_n be the number of steps of the loop-erasure of a simple random walk on $\mathbb{Z}^2$ from 0 to the circle of radius n. It follows from Lawler (2013) that $E^0M_n \asymp n^{5/4}$ (Note that $\lim_{n\to\infty}\log E^0M_n/\log n = 5/4$ was shown by Kenyon (2000)). Applying this in conjunction with Wilson's algorithm, it has been established that $|B_{\mathcal{U}}(0,R)| \asymp R^{2/(5/4)} = R^{8/5}$ with high probability where $B_{\mathcal{U}}(x,R)$ is the ball with respect to the graph distance. In particular, in [15], the condition of Theorem 4.5 is proved with $D = 8/5$, as mentioned in Section 4.2.2.

In the seminal paper by Schramm (2000), the topological properties of any possible scaling limit of the 2-dimensional UST $\mathcal{U}$ were investigated. (The uniqueness of the scaling limit for a UST in a 2-dimensional domain was established in Lawler-Schramm-Werner (2004).) In [11], the convergence of $\mathcal{U}$ is discussed in terms of the generalized Gromov-Hausdorff-Prohorov topology. It is proved that the law of the UST is tight under rescaling in a space of measured, rooted real trees embedded into Euclidean space. Let $\mathcal{T}$ be the limiting real tree when the lattice spacing is rescaled using the subsequence $\{\delta_i\}_{i\geq 1}$, $\rho_{\mathcal{T}}$ be its root, $\phi_{\mathcal{T}}$ be the random embedding of $\mathcal{T}$ into $\mathbb{R}^2$, and $X^{\mathcal{T}}$ be Brownian motion on $\mathcal{T}$ started from $\rho_{\mathcal{T}}$. Then the following holds, where we write $X^{\mathcal{U}}$ for the simple random walk on $\mathcal{U}$ started from 0.

Theorem 5.2 ([11]). *The annealed law of* $\{(\delta_i X^{\mathcal{U}}_{\delta_i^{-13/4}t} : t \geq 0)\}_{i\geq 1}$ *converges to the annealed law of* $\phi_{\mathcal{T}}(X^{\mathcal{T}})$. *Furthermore, there exists a jointly continuous heat kernel* $p_t^{\mathcal{T}}(\cdot,\cdot)$ *of* $X^{\mathcal{T}}$ *such that, for each* $R > 0$ *and* $\mathbb{P}$*-a.e.* $\omega \in \Omega$, *one can find* $T_0 > 0$ *such that* (5.1) *and* (5.2) *hold for all* $x, y \in B_{\mathcal{T}}(\rho_{\mathcal{T}}, R)$, $t \leq T_0$ *with* $\ell(x) := 1 \vee \log x$ *and* $d_f = 8/5, d_w = d_f + 1 = 13/5$.

Note that the exponent $13/4 = (5/4) \cdot d_w$ above is the walk dimension with respect to the Euclidean distance.

6. Conclusions

We have provided an overview of the stream of research on anomalous random walks and diffusions. Through the detailed study of diffusions on exactly self-similar fractals, it became apparent that Brownian motion on fractals typically obeys sub-Gaussian heat kernel estimates. This motivated the development of stability theory for such anomalous diffusions/random walks which is a generalization of the classical perturbation theory of Gaussian bounds. Then, some of the results in this direction turned out to be useful in analyzing random walks in random media. Although not discussed in this paper, such a stability theory also gives new insights to analysis on metric measure spaces.

There are many interesting random media whose dynamical properties are not yet known. Necessity is the Mother of Invention. We believe that further developments will continue to lead to important interactions between probability, analysis and mathematical physics.

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References

[1] Alexander, S., Orbach, R., Density of states on fractals: "fractons", *J. Physique (Paris) Lett.* **43** (1982), L625–L631.

[2] Andres, S., Barlow, M.T., Energy inequalities for cutoff functions and some applications, *J. reine angew. Math.*, to appear.

[3] Andres, S., Deuschel, J.-D., Slowik, M., Invariance principle for the random conductance model in a degenerate ergodic environment, *Ann. Probab.*, to appear.

[4] Angel, O., Goodman, J., den Hollander, F., Slade, G., Invasion percolation on regular trees, *Ann. Probab.* **36** (2008), 420–466.

[5] Barlow, M.T., *Diffusions on fractals.* Lect. Notes in Math. **1690**, Springer, New York, 1998.

[6] ———, Random walks on supercritical percolation clusters, *Ann. Probab.* **32** (2004), 3024-3084.

[7] ———, *Analysis on the Sierpinski carpet.* Analysis and geometry of metric measure spaces, 27–53, CRM Proc. Lect. Notes **56**, Amer. Math. Soc., Providence, RI, 2013.

[8] Barlow, M.T., Bass, R.F., Stability of parabolic Harnack inequalities, *Trans. Amer. Math. Soc.* **356** (2003), 1501–1533.

[9] Barlow M.T., Bass, R.F., Kumagai, T., Stability of parabolic Harnack inequalities on measure metric spaces, *J. Math. Soc. Japan* **58** (2006), 485–519.

[10] Barlow, M.T., Coulhon, T., Kumagai, T., Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs, *Comm. Pure Appl. Math.* **58** (2005), 1642–1677.

[11] Barlow, M.T., Croydon, D., Kumagai, T., Subsequential scaling limits of simple random walk on the two-dimensional uniform spanning tree, *in preparation.*

[12] Barlow, M.T., Hambly, B.M., Parabolic Harnack inequality and local limit theorem for random walks on percolation clusters, *Electron. J. Probab.* **14** (2009), 1–27.

[13] Barlow, M.T., Járai, A.A., Kumagai, T., Slade, G., Random walk on the incipient infinite cluster for oriented percolation in high dimensions, *Comm. Math. Phys.* **278** (2008), 385–431.

[14] Barlow, M.T., Kumagai, T., Random walk on the incipient infinite cluster on trees, *Illinois J. Math.* **50** (2006), 33–65.

[15] Barlow, M.T., Masson, R., Spectral dimension and random walks on the two dimensional uniform spanning tree, *Comm. Math. Phys.* **305** (2011), 23–57.

[16] Barlow, M.T., Perkins, E.A., Brownian motion on the Sierpiński gasket, *Probab. Theory Relat. Fields* **79** (1988), 543–623.

[17] Ben-Avraham, D., Havlin, S., *Diffusion and reactions in fractals and disordered systems.* Cambridge University Press, Cambridge, 2000.

[18] Berger, N., Biskup, M., Quenched invariance principle for simple random walk on percolation clusters, *Probab. Theory Relat. Fields* **137** (2007), 83–120.

[19] Biskup, M., Recent progress on the random conductance model, *Probability Surveys* **8** (2011), 294–373.

[20] Chen, Z.-Q., Kim, P., Kumagai, T., Discrete approximation of symmetric jump processes on metric measure spaces, *Probab. Theory Relat. Fields* **155** (2013), 703–749.

[21] Crawford, N., Sly, A., Simple random walks on long range percolation clusters
I: heat kernel bounds, *Probab. Theory Relat. Fields* **154** (2012), 753–786.
II: scaling limits, *Ann. Probab.* **41** (2013), 445–502.

[22] Croydon, D.A., Scaling limit for the random walk on the largest connected component of the critical random graph, *Publ. RIMS. Kyoto Univ.* **48** (2012), 279–338.

[23] Fukushima, M., Oshima, Y., Takeda, M., *Dirichlet forms and symmetric Markov processes.* de Gruyter, Berlin, 2011 (2nd Edition).

[24] De Gennes, P.G., La percolation: un concept unificateur, *La Recherche* **7** (1976), 919–927.

[25] Grigor'yan, A., *Heat kernel and analysis on manifolds.* Amer. Math. Soc., Providence, RI; International Press, Boston, MA, 2009.

[26] Grigor'yan, A., Telcs, A., Two-sided estimates of heat kernels on metric measure spaces, *Ann. Probab.* **40** (2012), 1212–1284.

[27] Grimmett, G., *Percolation.* Springer, Berlin, 1999 (2nd Edition).

[28] Heydenreich, M., van der Hofstad, R., Hulshof, T., Random walk on the high-dimensional IIC, *Comm. Math. Phys.*, to appear.

[29] Hughes, B.D., *Random walks and random environments, volume 2: random environments.* Oxford University Press, Oxford, 1996.

[30] Járai, A.A., Nachmias, A., Electrical resistance of the low dimensional critical branching random walk, *ArXiv:1305.1092.*

[31] Kesten, H., The incipient infinite cluster in two-dimensional percolation, *Probab. Theory Relat. Fields* **73** (1986), 369–394.

[32] ———, Subdiffusive behavior of random walk on a random cluster, *Ann. Inst. H. Poincaré Probab. Statist.* **22** (1986), 425–487.

[33] Kigami, J., *Analysis on fractals.* Cambridge Univ. Press, Cambridge, 2001.

[34] ———, Resistance forms, quasisymmetric maps and heat kernel estimates, *Mem. Amer. Math. Soc.* **216** (2012), no. 1015.

[35] Kozma, G., Nachmias, A., The Alexander-Orbach conjecture holds in high dimensions, *Invent. Math.* **178** (2009), 635–654.

[36] Kumagai, T., *Random walks on disordered media and their scaling limits.* Lect. Notes in Math. **2101**, Springer, New York, 2014.

[37] Kumagai, T., Misumi, J., Heat kernel estimates for strongly recurrent random walk on random media, *J. Theoret. Probab.* **21** (2008), 910–935.

[38] Kusuoka, S., *Diffusion processes on nested fractals.* Lect. Notes in Math. **1567**, Springer, New York, 1993.

[39] Lyons, R., Peres, Y., *Probability on trees and networks.* Cambridge University Press, in preparation. Current version available at `http://mypage.iu.edu/~rdlyons/`.

[40] Mathieu, P., Piatnitski, A., Quenched invariance principles for random walks on percolation clusters, *Proc. Roy. Soc. A* **463** (2007), 2287-2307.

[41] Rammal, R., Toulouse, G., Random walks on fractal structures and percolation clusters, *J. Physique Lettres* **44** (1983), L13–L22.

[42] Saloff-Coste, L., *Aspects of Sobolev-type inequalities.* Cambridge Univ. Press, Cambridge, 2002.

[43] Sidoravicius, V., Sznitman, A.-S., Quenched invariance principles for walks on clusters of percolation or among random conductances, *Probab. Theory Relat. Fields* **129** (2004), 219–244.

[44] Strichartz, R.S., *Differential equations on fractals: a tutorial.* Princeton University Press, Princeton, NJ, 2006.

[45] Telcs, A., *The art of random walks.* Lect. Notes in Math. **1885**, Springer, New York, 2006.

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