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Nilpotent Admissible Indigenous Bundles via Cartier Operators in Characteristic Three

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Abstract. — In the present paper, we study the *p-adic Teichmüller theory* in the case where $p = 3$. In particular, we discuss *nilpotent admissible/ordinary* indigenous bundles over a projective smooth curve in characteristic three. The main result of the present paper is a characterization of the *supersingular divisors* of nilpotent admissible/ordinary indigenous bundles in characteristic three by means of various *Cartier operators*. By means of this characterization, we prove that, for every nilpotent *ordinary* indigenous bundle over a projective smooth curve in characteristic three, there exists a connected finite étale covering of the curve on which the indigenous bundle is *not ordinary*. We also prove that every projective smooth curve *of genus two* in characteristic three is *hyperbolically ordinary*. These two applications yield *negative*, *positive partial* answers to *basic questions* in the *p-adic Teichmüller theory*, respectively.

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INTRODUCTION

In the present paper, we study the *p-adic Teichmüller theory* established by S. Mochizuki [cf. [5], [6]] in the case where $p = 3$. In particular, we discuss *nilpotent admissible/ordinary* indigenous bundles over a projective smooth curve in characteristic three. In the Introduction, let p be an *odd* prime number, $g \geq 2$ an integer, S a connected noetherian scheme of characteristic p [i.e., over $\mathbb{F}_p$], and $f\colon X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g over S. Write $f^F\colon X^F \to S$ for the projective smooth curve over S obtained by base-changing $X \to S$ via the absolute Frobenius morphism of S and $\Phi\colon X \to X^F$ for the relative Frobenius morphism over S. We use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

First, let us recall the notion of an *indigenous bundle* and some properties on an indigenous bundle. We shall say that a pair

$$(\pi\colon P \to X, \nabla_P)$$

consisting of a $\mathbb{P}^1$-bundle $\pi\colon P \to X$ over X and a connection ∇_P on P relative to X/S is an *indigenous bundle* over X/S if there exists a [uniquely determined — cf. [5], Chapter I, Proposition 2.4] section [i.e., the *Hodge section*] $\sigma\colon X \to P$ of $\pi\colon P \to X$ such that the Kodaira-Spencer homomorphism $\sigma^*\omega_{P/X} \to \omega_{X/S}$ at σ relative to ∇_P [i.e., the homomorphism obtained by differentiating σ by means of ∇_P] is an isomorphism [cf. [5], Chapter I, Definition 2.2]. The notion of an indigenous bundle was introduced and studied by R. C. Gunning [cf. [2], §2] and enables one to understand the theory of uniformization of [algebraic] Riemann surfaces in a somewhat more algebraic setting.

Let $(\pi\colon P \to X, \nabla_P)$ be an indigenous bundle over X/S. Then the connection ∇_P on P determines a *horizontal* homomorphism [i.e., the *p-curvature*]

$$\mathcal{P}\colon\ \Phi^*\tau_{X^F/S} \ \longrightarrow\ \mathcal{A}d(P) \ \stackrel{\text{def}}{=}\ \pi_*\tau_{P/X}.$$

We shall say that the indigenous bundle $(\pi\colon P \to X, \nabla_P)$ is *nilpotent* (respectively, *admissible*; *dormant*) if the square of $\mathcal{P}$ is 0 (respectively, the zero locus of $\mathcal{P}$ is empty; $\mathcal{P} = 0$) [cf. [5], Chapter II, Definition 2.4 (respectively, [5], Chapter II, Definition 2.4; [6], Chapter II, Definition 1.1)]. Moreover, we shall refer to the composite of the p-curvature $\mathcal{P}$ and the surjection $\mathcal{A}d(P) \twoheadrightarrow \tau_{X/S}$ determined by the Hodge section of $(\pi\colon P \to X, \nabla_P)$ as the *square Hasse invariant* of $(\pi\colon P \to X, \nabla_P)$ [cf. [5], Chapter II, Proposition 2.6, (1)]. Then, by means of this *square Hasse invariant*, one may define the *Frobenius on* $\mathbb{R}^1 f_*\tau_{X/S}$ *induced by* $(\pi\colon P \to X, \nabla_P)$ [cf. the discussion following [5], Chapter II, Lemma 2.11]. We shall say that the indigenous bundle $(\pi\colon P \to X, \nabla_P)$ is *ordinary* if the Frobenius on $\mathbb{R}^1 f_*\tau_{X/S}$ induced by $(\pi\colon P \to X, \nabla_P)$ is an isomorphism [cf. [5], Chapter II, Definition 3.1]. A *nilpotent admissible/ordinary* indigenous bundle plays a central role in the "*classical*" *p-adic Teichmüller theory*, i.e., the *p-adic Teichmüller theory* discussed in [not [6] but] [5].

The first main result of the present paper is the following *uniqueness* of a *dormant* indigenous bundle in characteristic three [cf. Theorem 2.1, Corollary 2.6]:

THEOREM A. — *In the notation introduced at the beginning of the Introduction, suppose that* $p = 3$. *Then there exists a* **unique dormant** *indigenous bundle over* X/S. *In particular, there exists a natural* **bijection** *between*

- $H^0(S, f_*\omega_{X/S}^{\otimes 2}) = H^0(X, \omega_{X/S}^{\otimes 2})$ *and*
- *the set of isomorphism classes of indigenous bundles over* X/S

such that, for $\theta \in H^0(S, f_*\omega_{X/S}^{\otimes 2})$, *the* **dormant locus** *in* S *of the indigenous bundle over* X/S *corresponding to* θ **coincides** *with the zero locus in* S *of* θ.

If an indigenous bundle $(\pi\colon P \to X, \nabla_P)$ over X/S is *nilpotent admissible*, then there exist an invertible sheaf $\mathcal{H}$ on X and a global section χ of $\mathcal{H}$ such that $\mathcal{H}^{\otimes 2} \cong \mathcal{H}om_{\mathcal{O}_X}(\Phi^*\tau_{X^F/S}, \tau_{X/S})$, and, moreover, the square of χ coincides with the square Hasse invariant of $(\pi\colon P \to X, \nabla_P)$ [cf. [5], Chapter II, Proposition 2.6, (3)]. We shall refer to χ as the *Hasse invariant* of $(\pi\colon P \to X, \nabla_P)$ [cf. [5], Chapter II, Proposition 2.6, (3)] and to the zero locus of the Hasse invariant as the *supersingular divisor* of $(\pi\colon P \to X, \nabla_P)$ [cf. [5], Chapter II, Proposition 2.6, (3)]. The *supersingular divisor* is an important invariant of a nilpotent admissible indigenous bundle; for instance, if S is *reduced*, then the isomorphism class of a nilpotent admissible indigenous bundle over X/S is *completely determined* by the *supersingular divisor* [cf. [5], Chapter II, Proposition 2.6, (4)]. The main result of the present paper is a characterization of the *supersingular divisors* of nilpotent admissible/ordinary indigenous bundles in characteristic three by means of various *Cartier operators*.

In order to present the main result of the present paper, let us recall some notions related to the *Cartier operator*. Let $(\mathcal{L}, \tau)$ be a square-trivialized invertible sheaf on X, i.e., a pair consisting of an invertible sheaf $\mathcal{L}$ on X and a trivialization τ of the square of $\mathcal{L}$ [cf. Definition A.3]. Then the [usual] Cartier operator $\Phi_*\omega_{X/S} \to \omega_{X^F/S}$, together with the trivialization τ, determines a homomorphism of $\mathcal{O}_S$-modules

$$C_{(\mathcal{L},\tau)}\colon\ f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})$$

— where we write $\mathcal{L}^F$ for the invertible sheaf on X^F obtained by pulling back $\mathcal{L}$ via the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S. We shall refer to this homomorphism as the *Cartier operator associated to* $(\mathcal{L}, \tau)$ [cf. Definition A.4]. On the other hand, the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S determines a *Frobenius-semi-linear* homomorphism

$$f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}).$$

For a global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$, we shall write u^F for the global section of $\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}$ obtained by forming the image of u via this Frobenius-semi-linear homomorphism. We shall say that a global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *normalized Cartier eigenform* associated to $(\mathcal{L}, \tau)$ if u defines a relative effective Cartier divisor of X/S, and, moreover, $C_{(\mathcal{L},\tau)}(u) = -u^F$ [cf. Definition A.8, (i)].

A part of the main result of the present paper is as follows [cf. Theorem 5.2, (ii)]:

Theorem B. — *In the notation introduced at the beginning of the Introduction, suppose that* $p = 3$. *Let* D *be a relative effective Cartier divisor of* X/S. *Then it holds that* D *is the* **supersingular divisor** *of a* **nilpotent admissible** (*respectively,* **nilpotent ordinary**) *indigenous bundle over* X/S *if and only if* D *is* **of CE-type** (*respectively,* **of CEO-type**) [*cf. Definition* 5.1, (iii)], *i.e., there exist an invertible sheaf* $\mathcal{L}$ *on* X, *a trivialization* τ *of the square of* $\mathcal{L}$, *and a global section* χ *of* $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ *such that*

the following two (*respectively, three*) *conditions* (1), (2) (*respectively,* (1), (2), (3)) *are satisfied:*

(1) *The divisor D is* **étale** *over S and coincides with the zero locus of $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$.*

(2) *The global section $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ is a* **normalized Cartier eigenform** *associated to $(\mathcal{L}, \tau)$.*

(3) *The invertible sheaf $\mathcal{L}$ is* **parabolically ordinary** [*cf. Definition* A.7]*, i.e., the Cartier operator associated to $(\mathcal{L}, \tau)$ is* **injective***, or, equivalently* [*cf. Proposition* A.6]*, one of the following two conditions is satisfied:*

- *$\mathcal{L}$ is* **of relative order one** [*cf. Definition* A.2]*, and, moreover, X is* **parabolically ordinary** [*cf. Definition* A.5*,* (i)]*.*
- *$\mathcal{L}$ is* **of relative order two** [*cf. Definition* A.2]*, and, moreover, the connected finite étale double covering of X which trivializes $\mathcal{L}$* [*determined by τ*] *is* **parabolically new-ordinary** [*cf. Definition* A.5*,* (ii)]*.*

Here, let us recall the following two *basic questions* in the *p-adic Teichmüller theory* discussed in [6], Introduction, §2.1 [cf. [6], Introduction, §2.1, (1), (2)]:

(1) Is every pointed stable curve [of type (g, r), where $2g - 2 + r > 0$] *hyperbolically ordinary*? That is to say, does every pointed stable curve [of type (g, r), where $2g-2+r > 0$] admit, étale locally on S, a *nilpotent ordinary* indigenous bundle?

(2) Let P be a *nilpotent ordinary* indigenous bundle over a pointed stable curve X [of type (g, r), where $2g - 2 + r > 0$] and $Y \to X$ a connected finite [log] étale covering of X. Then is the pull-back of P to Y still *ordinary*?

As a corollary of Theorem B, we obtain the following theorem, which yields a *negative answer* to the above basic question (2) [cf. Corollary 5.4]:

THEOREM C. — *Let X be a projective smooth curve of genus ≥ 2 over an algebraically closed field k of characteristic 3. Then, for every* **nilpotent ordinary** *indigenous bundle P over X/k, there exists a connected finite étale covering $Y \to X$ of X such that the* [*necessarily* **nilpotent admissible**] *indigenous bundle $(Y \to X)^*P$ over Y/k is* **not ordinary**.

In §6, we give, by applying the results obtained in the present paper, a complete list of *nilpotent/nilpotent admissible/nilpotent ordinary* indigenous bundles over a projective smooth curve *of genus two* over an algebraically closed field of characteristic three [cf. Theorem 6.1]. Moreover, we prove the following theorem, which yields a *positive partial answer* to the above basic question (1) [cf. Corollary 6.6, Remark 6.6.1]:

THEOREM D. — *Every projective smooth curve of genus two over a connected noetherian scheme of characteristic three is* **hyperbolically ordinary** [*cf.* [5]*, Chapter II, Definition 3.3*]*.*

1. Construction of a Dormant Indigenous Bundle

In the present §1, we construct a *dormant indigenous bundle* over a projective smooth curve of genus ≥ 2 of characteristic 3 [cf. Proposition 1.1 below]. In the present §1, let $g \geq 2$ be an integer, S a connected noetherian scheme *of characteristic* 3 [i.e., over $\mathbb{F}_3$], and $f\colon X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g over S. Write $f^F\colon X^F \to S$ for the projective smooth curve over S obtained by base-changing $X \to S$ via the absolute Frobenius morphism of S, $\Phi\colon X \to X^F$ for the relative Frobenius morphism over S, $\mathcal{I} \subseteq \mathcal{O}_{X\times_S X}$ for the ideal of $\mathcal{O}_{X\times_S X}$ which defines the diagonal morphism with respect to X/S, and $X_{(n)} \subseteq X \times_S X$ for the closed subscheme of $X \times_S X$ defined by the ideal $\mathcal{I}^{n+1} \subseteq \mathcal{O}_{X\times_S X}$ [where n is a nonnegative integer]. In particular, it follows that $\mathcal{I}/\mathcal{I}^2 = \omega_{X/S}$ (respectively, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = \tau_{X/S}$), where we use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

We shall write

$$\mathcal{B}_\circ \stackrel{\text{def}}{=} \text{Coker}(\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X)$$

for the $\mathcal{O}_{X^F}$-module obtained by forming the cokernel of the natural homomorphism $\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X$ and

$$\mathcal{E}_\circ \stackrel{\text{def}}{=} \Phi^*\mathcal{B}_\circ.$$

Since the homomorphism $\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X$ admits a natural splitting after pulling back via Φ, which thus determines a natural isomorphism of $\mathcal{O}_X$-modules

$$\Phi^*\Phi_*\mathcal{O}_X \stackrel{\sim}{\longrightarrow} \mathcal{O}_X \oplus \mathcal{E}_\circ,$$

and Φ is *finite flat of degree* 3, it follows that $\mathcal{B}_\circ$, hence also $\mathcal{E}_\circ$, is *locally free of rank* 2. We shall write

$$\pi_\circ\colon P_\circ \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{E}_\circ) \longrightarrow X$$

for the $\mathbb{P}^1$*-bundle over* X *associated to* $\mathcal{E}_\circ$.

Next, let us observe that one verifies immediately that the natural morphism

$$X \times_{X^F} X \longrightarrow X \times_S X$$

determines an isomorphism

$$X \times_{X^F} X \stackrel{\sim}{\longrightarrow} X_{(2)}.$$

In particular, the closed immersion $X_{(1)} \hookrightarrow X \times_S X$ determines a closed immersion $X_{(1)} \hookrightarrow X \times_{X^F} X$. Thus, it follows that the $\mathcal{O}_X$-module $\mathcal{E}_\circ$, hence also the $\mathbb{P}^1$-bundle $P_\circ$, on X admits a *natural connection* relative to X/S. We shall write

$$\nabla_{\mathcal{E}_\circ}, \quad \nabla_{P_\circ}$$

for the respective natural connections on $\mathcal{E}_\circ$, $P_\circ$. [So one verifies immediately that the connection $\nabla_{\mathcal{E}_\circ}$ *coincides* with the connection on $\mathcal{E}_\circ = \Phi^*\mathcal{B}_\circ$ determined by the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$.] Moreover, the above isomorphism $X \times_{X^F} X \stackrel{\sim}{\to} X_{(2)}$, together with the cartesian diagram

$$\begin{array}{ccc} X \times_{X^F} X & \xrightarrow{\text{pr}_2} & X \\ {\scriptstyle \text{pr}_1}\downarrow & & \downarrow{\scriptstyle \Phi} \\ X & \xrightarrow[\Phi]{} & X^F, \end{array}$$

determines *isomorphisms* of $\mathcal{O}_X$-modules

$$\Phi^*\Phi_*\mathcal{O}_X \xrightarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X\times_{X^F}X} \xleftarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X_{(2)}},$$

which are *compatible* with the respective natural surjections onto $\mathcal{O}_X$ [arising from the diagonal morphism with respect to X/X^F] from each of these three modules. In particular, by forming the kernels of the respective natural surjections onto $\mathcal{O}_X$, we obtain *isomorphisms* of $\mathcal{O}_X$-modules

$$\mathcal{E}_\circ \xrightarrow{\sim} \mathrm{Ker}(\mathrm{pr}_{1*}\mathcal{O}_{X\times_{X^F}X} \twoheadrightarrow \mathcal{O}_X) \xleftarrow{\sim} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3).$$

We shall write

$$\sigma_\circ\colon X \longrightarrow P_\circ$$

for the section of $\pi_\circ\colon P_\circ \to X$ determined by the composite $\mathcal{E}_\circ \twoheadrightarrow \omega_{X/S}$ of the above isomorphism $\mathcal{E}_\circ \xrightarrow{\sim} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3)$ and the natural surjection $\mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3) \twoheadrightarrow \mathcal{I}/\mathcal{I}^2 = \omega_{X/S}$. Then one verifies easily that the *Kodaira-Spencer homomorphism* $\sigma_\circ^*\omega_{P_\circ/X} \to \omega_{X/S}$ at $\sigma_\circ$ relative to $\nabla_{P_\circ}$ [i.e., the homomorphism obtained by differentiating $\sigma_\circ$ by means of $\nabla_{P_\circ}$] is an *isomorphism*. Thus, it follows immediately from our construction that the following proposition holds:

PROPOSITION 1.1. — *The pair* $(\pi_\circ\colon P_\circ \to X, \nabla_{P_\circ})$ *is an* **indigenous bundle** [*cf.* [5], *Chapter I, Definition 2.2*] *over* X/S *whose* **Hodge section** [*cf.* [5], *Chapter I, Proposition 2.4*] *is given by* $\sigma_\circ$. *Moreover, the indigenous bundle* $(\pi_\circ\colon P_\circ \to X, \nabla_{P_\circ})$ *is* **dormant** [*cf.* [6], *Chapter II, Definition 1.1*].

In the remainder of the present §1, let us consider the invertible sheaves

$$\det(\mathcal{E}_\circ), \quad \det(\mathcal{B}_\circ), \quad \det(\Phi_*\omega_{X/S}).$$

Write $\mathcal{M} \overset{\mathrm{def}}{=} \mathcal{H}om_{\mathcal{O}_{X^F}}(\det(\mathcal{B}_\circ), \omega_{X^F/S})$. First, let us observe that since the $\mathcal{O}_X$-module $\mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3) \cong \mathcal{E}_\circ = \Phi^*\mathcal{B}_\circ$ fits into an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \longrightarrow \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3) \longrightarrow \omega_{X/S} \longrightarrow 0,$$

it follows that

$$\det(\mathcal{E}_\circ) \cong \omega_{X/S}^{\otimes 3},$$

hence also

$$\Phi^*\mathcal{M} \cong \mathcal{O}_X.$$

Next, let us recall from the discussion preceding [8], Théorème 4.1.1, that the map

$$\begin{array}{ccc} \Phi_*\mathcal{O}_X \times \Phi_*\mathcal{O}_X & \longrightarrow & \omega_{X^F/S} \\ (f, g) & \mapsto & c(f \cdot \Phi_* d(g)) \end{array}$$

— where we write $d\colon \mathcal{O}_X \to \omega_{X/S}$ for the exterior differentiation operator and $c\colon \Phi_*\omega_{X/S} \to \omega_{X^F/S}$ for the *Cartier operator* — determines an *isomorphism* of $\mathcal{O}_{X^F}$-modules

$$\mathcal{B}_\circ \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X^F}}(\mathcal{B}_\circ, \omega_{X^F/S}),$$

which thus implies that

$$\mathcal{M}^{\otimes 2} \cong \mathcal{O}_{X^F}.$$

Thus, we obtain:

Lemma 1.2. — *It holds that*

$$\det(\mathcal{E}_\circ) \ \cong \ \omega_{X/S}^{\otimes 3}, \quad \det(\mathcal{B}_\circ) \ \cong \ \omega_{X^F/S}, \quad \det(\Phi_*\omega_{X/S}) \ \cong \ \omega_{X^F/S}^{\otimes 2}.$$

Proof. — The first "$\cong$" has already been verified. Since the homomorphism between the relative Jacobian varieties of X^F/S, X/S induced by Φ is *finite flat of degree* 3^g, it follows from the fact that $\Phi^*\mathcal{M} \cong \mathcal{O}_X$, $\mathcal{M}^{\otimes 2} \cong \mathcal{O}_{X^F}$ verified above that $\mathcal{M}$ lies in $(f^F)^*\mathrm{Pic}(S)$. Thus, again by the fact that $\Phi^*\mathcal{M} \cong \mathcal{O}_X$, the second "$\cong$" follows. The third "$\cong$" follows from the second "$\cong$", together with the *well-known* exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \ \longrightarrow \ \mathcal{O}_{X^F} \ \longrightarrow \ \Phi_*\mathcal{O}_X \ \stackrel{\Phi_* d}{\longrightarrow} \ \Phi_*\omega_{X/S} \ \stackrel{c}{\longrightarrow} \ \omega_{X^F/S} \ \longrightarrow \ 0$$

[cf., e.g., [4], Theorem 7.2]. □

2. The Dormant Trivialization of the Schwarz Torsor

In the present §2, we maintain the notation of the preceding §1. In particular, we have a projective smooth curve $f\colon X \to S$ and a *dormant indigenous bundle* $(\pi_\circ\colon P_\circ \to X, \nabla_{P_\circ})$ over X/S [cf. Proposition 1.1]. In the present §2, we prove the following theorem:

Theorem 2.1. — *Every* **dormant** *indigenous bundle over* X/S *is* **isomorphic** *to the dormant indigenous bundle* $(\pi_\circ\colon P_\circ \to X, \nabla_{P_\circ})$ *of Proposition* 1.1.

Remark 2.1.1. — It follows from [10], Corollary 5.4, that if $p > 2g-2$, then the number of isomorphism classes of *dormant indigenous bundles* over a "*sufficiently general*" projective smooth curve of genus g over an algebraically closed field of characteristic p is equal to

$$\frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{i=1}^{p-1} \frac{1}{\sin^{2g-2}(\frac{\pi \cdot i}{p})} \ = \ \frac{(-p)^{g-1}}{2} \cdot \sum_{\zeta^p=1,\, \zeta \neq 1} \frac{\zeta^{g-1}}{(\zeta-1)^{2g-2}}.$$

On the other hand, one verifies easily that the above quantity in the case where $p = 3$ is always *equal to* 1. Thus, it follows from Theorem 2.1 that the formula of [10], Corollary 5.4, is *valid* for $p = 3$ *without the assumption that* $p > 2g - 2$.

To verify Theorem 2.1, let us first recall some facts on the *p-adic Teichmüller theory* [cf. [5], [6]]. Write

$$\mathcal{M}_g$$

for the moduli stack of projective smooth curves of genus g of characteristic 3 and

$$\mathcal{N}_g[\infty]$$

for the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with *dormant indigenous bundles*. Then the natural (1-)morphism

$$\mathcal{N}_g[\infty] \ \longrightarrow \ \mathcal{M}_g$$

is *finite* and *faithfully flat*; moreover, there exists a *dense open* substack of $\mathcal{M}_g$ on which this (1-)morphism is *étale* [cf. the final portion of [6], Chapter II, Theorem 2.8]. Moreover,

as is well-known, there exists a *dense open* substack of $\mathcal{M}_g$ on which the associated relative Jacobian variety is *ordinary*. Thus, to complete the verification of Theorem 2.1, we may assume without loss of generality, by considering a geometric point on the intersection of these two *dense open* substacks [i.e., a geometric point on the — necessarily *dense open* — substack of $\mathcal{M}_g$ on which the above (1-)morphism is *étale*, and, moreover, the associated relative Jacobian variety is *ordinary*], that

- S is the spectrum of an algebraically closed field [of characteristic 3], and that
- the Jacobian variety of X is *ordinary* [i.e., that X is *parabolically ordinary* — cf. Definition A.5, (i)].

To complete the verification of Theorem 2.1, let

$$(\pi\colon P \to X, \nabla_P)$$

be a *dormant indigenous bundle* over X/S. Let us first observe that it follows from [5], Chapter I, Proposition 2.5, that there exists an isomorphism over X

$$P \ \cong \ P_\circ \ = \ \mathbb{P}(\mathcal{E}_\circ).$$

By means of such an isomorphism, let us identify P with $P_\circ = \mathbb{P}(\mathcal{E}_\circ)$. Next, let us observe that since $(\pi\colon P \to X, \nabla_P)$ is *dormant*, one verifies immediately [cf. also [4], Theorem 5.1] that, by considering *horizontal* local sections of $\pi\colon P \to X$ with respect to ∇_P, we obtain a $\mathbb{P}^1$-bundle

$$\pi_Q\colon \ Q \ \longrightarrow \ X^F$$

over X^F and an isomorphism $P \cong \Phi^*Q$ over X relative to which a local section of π is *horizontal* [with respect to ∇_P] if and only if the local section arises from a local section of π_Q.

Next, let us observe that since S is the spectrum of an algebraically closed field, there exist invertible sheaves $\mathcal{L}_Q$, $\mathcal{L}_{X^F}$ on Q, X^F such that $\mathcal{L}_Q^{\otimes 2} \cong \tau_{Q/X^F}$, $\mathcal{L}_{X^F}^{\otimes 2} \cong \omega_{X^F/S}$, respectively. Thus, one verifies easily that Q is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(\pi_{Q*}\mathcal{L}_Q)$, hence also the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{L}_{X^F} \otimes_{\mathcal{O}_{X^F}} \pi_{Q*}\mathcal{L}_Q)$, over X^F. In particular, since $P = P_\circ$ is isomorphic to Φ^*Q, it follows that there exists an invertible sheaf $\mathcal{M}_X$ on X such that $\mathcal{E}_\circ \otimes_{\mathcal{O}_X} \mathcal{M}_X \cong \Phi^*(\mathcal{L}_{X^F} \otimes_{\mathcal{O}_{X^F}} \pi_{Q*}\mathcal{L}_Q)$. Next, let us observe that, by considering the following *well-known* exact sequence of $\mathcal{O}_Q$-modules

$$0 \ \longrightarrow \ \omega_{Q/X^F} \ \longrightarrow \ \mathcal{H}om_{\mathcal{O}_Q}(\mathcal{L}_Q, \pi_Q^*\pi_{Q*}\mathcal{L}_Q) \ \longrightarrow \ \mathcal{O}_Q \ \longrightarrow \ 0,$$

we obtain that $\pi_Q^* \det(\pi_{Q*}\mathcal{L}_Q) \cong \mathcal{O}_Q$, which thus implies that $\det(\pi_{Q*}\mathcal{L}_Q) \cong \mathcal{O}_{X^F}$. Thus, by considering the determinant of $\mathcal{E}_\circ \otimes_{\mathcal{O}_X} \mathcal{M}_X \cong \Phi^*(\mathcal{L}_{X^F} \otimes_{\mathcal{O}_{X^F}} \pi_{Q*}\mathcal{L}_Q)$, we obtain from Lemma 1.2 that $\mathcal{M}_X^{\otimes 2} \cong \mathcal{O}_X$. Thus, in summary, we obtain:

LEMMA 2.2. — *There exists a locally free coherent $\mathcal{O}_{X^F}$-module $\mathcal{B}_Q$ of rank 2 on X^F which satisfies the following conditions:*

(0) *There exists an isomorphism of Q with the $\mathbb{P}^1$-bundle associated to $\mathcal{B}_Q$ over X^F; moreover, the connection ∇_P coincides with the connection on $P \cong \Phi^*Q$ induced by the connection on $\Phi^*\mathcal{B}_Q$ determined by the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$.*

(1) *There exists a(n) [not necessarily horizontal] isomorphism $\mathcal{E}_\circ \cong \Phi^*\mathcal{B}_Q$ of $\mathcal{O}_X$-modules.*

(2) $\det(\mathcal{B}_Q) \cong \omega_{X^F/S}$.

PROOF. — Since $\mathcal{M}_X^{\otimes 2} \cong \mathcal{O}_X$, S is the spectrum of an algebraically closed field, and the homomorphism between the Jacobian varieties of X^F, X induced by Φ is *finite flat of degree* 3^g, one verifies easily that there exists an invertible sheaf $\mathcal{M}_{X^F}$ on X^F such that $\mathcal{M}_{X^F}^{\otimes 2} \cong \mathcal{O}_{X^F}$ and $\Phi^*\mathcal{M}_{X^F} \cong \mathcal{M}_X$. Then it follows immediately from the above discussion that the $\mathcal{O}_{X^F}$-module

$$\mathcal{B}_Q \stackrel{\text{def}}{=} \mathcal{H}om_{\mathcal{O}_{X^F}}(\mathcal{M}_{X^F}, \mathcal{L}_{X^F} \otimes_{\mathcal{O}_{X^F}} \pi_{Q*}\mathcal{L}_Q)$$

satisfies the conditions in the statement of Lemma 2.2. □

Next, let us observe that the surjection

$$\Phi^*\mathcal{B}_Q \cong \mathcal{E}_\circ \cong \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3) \twoheadrightarrow \omega_{X/S}$$

[cf. Lemma 2.2, (1)] determines a homomorphism of $\mathcal{O}_{X^F}$-modules

$$\alpha\colon \mathcal{B}_Q \longrightarrow \Phi_*\omega_{X/S},$$

hence, by pulling back via Φ, also a homomorphism of $\mathcal{O}_X$-modules

$$\Phi^*\alpha\colon \Phi^*\mathcal{B}_Q \longrightarrow \Phi^*\Phi_*\omega_{X/S}$$

which is *horizontal* [with respect to the connections determined by the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$].

LEMMA 2.3. — *The homomorphism $\Phi^*\alpha$, hence also the homomorphism α, is a* **locally split injection**.

PROOF. — Let us first recall that we have a cartesian diagram

$$\begin{array}{ccc} X \times_{X^F} X \cong X_{(2)} & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \Phi} \\ X & \xrightarrow[\Phi]{} & X^F \end{array}$$

[cf. the discussion preceding Proposition 1.1]. Thus, the closed subschemes $X = X_{(0)} \subseteq X_{(1)} \subseteq X_{(2)}$ determine a filtration

$$\{0\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 = \Phi^*\Phi_*\omega_{X/S}$$

such that, for $i \in \{1, 2, 3\}$,

$$\mathcal{F}_i/\mathcal{F}_{i-1} \cong \omega_{X/S}^{\otimes 4-i}.$$

Moreover, one verifies immediately that the connection on $\Phi^*\Phi_*\omega_{X/S}$ determines an *isomorphism* of invertible sheaves on X

$$\mathcal{F}_2/\mathcal{F}_1 \xrightarrow{\sim} (\mathcal{F}_3/\mathcal{F}_2) \otimes_{\mathcal{O}_X} \omega_{X/S}.$$

Thus, since

$$\det(\Phi^*\mathcal{B}_Q) \cong \det(\mathcal{F}_3/\mathcal{F}_1) \cong \omega_{X/S}^{\otimes 3}$$

[cf. Lemma 2.2, (2)], to complete the verification of Lemma 2.3, it suffices to verify that the composite

$$\Phi^*\mathcal{B}_Q \stackrel{\Phi^*\alpha}{\to} \Phi^*\Phi_*\omega_{X/S} = \mathcal{F}_3 \twoheadrightarrow \mathcal{F}_3/\mathcal{F}_1$$

is *injective*.

Next, let us observe that it follows from the definition of $\Phi^*\alpha$ that the composite of $\Phi^*\alpha$ and the natural homomorphism $\Phi^*\Phi_*\omega_{X/S} \to \omega_{X/S}$ [i.e., the surjection $\Phi^*\Phi_*\omega_{X/S} = \mathcal{F}_3 \twoheadrightarrow \mathcal{F}_3/\mathcal{F}_2 \cong \omega_{X/S}$] determines an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \longrightarrow \Phi^*\mathcal{B}_Q \quad (\cong \mathcal{E}_\circ) \longrightarrow \omega_{X/S} \longrightarrow 0,$$

which gives rise to the *Hodge section* $\sigma_\circ$ of $(\pi_\circ\colon P_\circ \to X, \nabla_{P_\circ})$ [cf. Proposition 1.1], hence also of the indigenous bundle $(\pi\colon P \to X, \nabla_P)$ [cf. [5], Chapter I, Proposition 2.4]. Thus, to complete the verification of Lemma 2.3, it suffices to verify the *injectivity* of the homomorphism

$$\omega_{X/S}^{\otimes 2} \longrightarrow \mathcal{F}_2/\mathcal{F}_1$$

induced by the composite $\Phi^*\mathcal{B}_Q \to \mathcal{F}_3/\mathcal{F}_1$ under consideration, or, equivalently, the *injectivity* of the composite

$$\omega_{X/S}^{\otimes 2} \longrightarrow \mathcal{F}_2/\mathcal{F}_1 \xrightarrow{\sim} (\mathcal{F}_3/\mathcal{F}_2) \otimes_{\mathcal{O}_X} \omega_{X/S}$$

with the isomorphism discussed above. On the other hand, since $\Phi^*\alpha$ is *horizontal*, one verifies immediately that this composite *coincides* with the composite

$$\omega_{X/S}^{\otimes 2} \hookrightarrow \Phi^*\mathcal{B}_Q \to \Phi^*\mathcal{B}_Q \otimes_{\mathcal{O}_X} \omega_{X/S} \twoheadrightarrow \omega_{X/S} \otimes_{\mathcal{O}_X} \omega_{X/S} \cong (\mathcal{F}_3/\mathcal{F}_2) \otimes_{\mathcal{O}_X} \omega_{X/S}$$

— where the second arrow is the connection on $\Phi^*\mathcal{B}_Q$. In particular, if the composite $\omega_{X/S}^{\otimes 2} \to (\mathcal{F}_3/\mathcal{F}_2) \otimes_{\mathcal{O}_X} \omega_{X/S}$ under consideration is *not injective*, then it follows that the invertible subsheaf $\omega_{X/S}^{\otimes 2} \subseteq \Phi^*\mathcal{B}_Q$ is *preserved* by the connection on $\Phi^*\mathcal{B}_Q$, which thus implies [cf. Lemma 2.2, (0)] that the *Hodge section* $\sigma_\circ$ of the indigenous bundle $(\pi\colon P \to X, \nabla_P)$ is *horizontal* with respect to ∇_P; thus, we obtain a *contradiction*. This completes the proof of Lemma 2.3. □

We conclude from Lemma 2.3 that the cokernel $\mathrm{Coker}(\alpha)$ of α is an *invertible sheaf* on X^F which is isomorphic to

$$\mathcal{H}om_{\mathcal{O}_{X^F}}(\det(\mathcal{B}_Q), \det(\Phi_*\omega_{X/S})).$$

In particular, it follows from Lemma 1.2; Lemma 2.2, (2), that we have an exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{B}_Q \xrightarrow{\alpha} \Phi_*\omega_{X/S} \longrightarrow \omega_{X^F/S} \longrightarrow 0.$$

Let us prove Theorem 2.1.

PROOF OF THEOREM 2.1. — Let us recall the *well-known* exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{B}_\circ \longrightarrow \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0$$

— cf. the exact sequence of $\mathcal{O}_{X^F}$-modules which appears in the proof of Lemma 1.2. If $\mathcal{B}_\circ = \mathrm{Im}(\alpha)$, then it follows from Lemma 2.2, (0), that the dormant indigenous bundle $(\pi\colon P \to X, \nabla_P)$ is *isomorphic* to the dormant indigenous bundle $(\pi_\circ\colon P_\circ \to X, \nabla_{P_\circ})$ of Proposition 1.1. Thus, assume that $\mathcal{B}_\circ \neq \mathrm{Im}(\alpha)$. Then since $\det(\mathcal{B}_\circ) \cong \det(\mathcal{B}_Q) \cong \omega_{X^F/S}$ [cf. Lemma 1.2; Lemma 2.2, (2)], it holds that $\mathcal{B}_\circ \not\subseteq \mathrm{Im}(\alpha)$, which thus implies that the composite

$$\mathcal{B}_\circ = \mathrm{Ker}(c) \hookrightarrow \Phi_*\omega_{X/S} \twoheadrightarrow \Phi_*\omega_{X/S}/\mathrm{Im}(\alpha) \cong \omega_{X^F/S}$$

is *nonzero*. In particular, since we have an isomorphism $\mathcal{B}_\circ \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X^F}}(\mathcal{B}_\circ, \omega_{X^F/S})$ of $\mathcal{O}_{X^F}$-modules [cf. the discussion preceding Lemma 1.2], we conclude that the $\mathcal{O}_S$-module $f^F_*\mathcal{B}_\circ$ admits a *nonzero section*. Thus, it follows, in light of the exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \longrightarrow \mathcal{B}_\circ \longrightarrow 0,$$

that the Jacobian variety of X is *not ordinary* — in *contradiction* to our assumption that X is *parabolically ordinary*. This completes the proof of Theorem 2.1. □

It follows from Theorem 2.1 [together with the discussion following Remark 2.1.1] that the natural (1-)morphism

$$\mathcal{N}_g[\infty] \longrightarrow \mathcal{M}_g$$

is an *isomorphism*, hence also *étale*. Thus, by the final portion of [10], Theorem 3.3, we obtain:

Corollary 2.4. — *Every dormant indigenous bundle over X/S is* **dormant ordinary** [*cf.* [10]*, Definition 3.2*].

We shall write

$$\mathcal{C}_g \longrightarrow \mathcal{M}_g$$

for the universal curve over $\mathcal{M}_g$ and

$$\mathcal{S}_g \longrightarrow \mathcal{M}_g$$

for the *Schwarz torsor* over $\mathcal{M}_g$ [cf. [6], Introduction, §0.4], i.e., the torsor over the locally free coherent $\mathcal{O}_{\mathcal{M}_g}$-module of rank $3g-3$

$$(\mathcal{C}_g \to \mathcal{M}_g)_*\omega^{\otimes 2}_{\mathcal{C}_g/\mathcal{M}_g}$$

obtained by forming the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with *indigenous bundles* [cf. also [5], Chapter I, Corollary 2.9]. By considering the composite of the above natural isomorphism $\mathcal{M}_g \xleftarrow{\sim} \mathcal{N}_g[\infty]$ and the natural closed immersion $\mathcal{N}_g[\infty] \hookrightarrow \mathcal{S}_g$ of stacks, we obtain a trivialization

$$\mathcal{M}_g \longrightarrow \mathcal{S}_g$$

of the Schwarz torsor.

Definition 2.5. — We shall refer to this trivialization $\mathcal{M}_g \to \mathcal{S}_g$ of the Schwarz torsor as the *dormant trivialization*.

By the dormant trivialization of Definition 2.5, we obtain an isomorphism of $\mathcal{S}_g$ with the geometric vector bundle over $\mathcal{M}_g$ associated to $(\mathcal{C}_g \to \mathcal{M}_g)_*\omega^{\otimes 2}_{\mathcal{C}_g/\mathcal{M}_g}$. Thus:

COROLLARY 2.6. — *There exists a natural* **bijection** *between the following two sets:*

- $\Gamma(S, f_*\omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2})$.
- *The set of isomorphism classes of indigenous bundles over* X/S.

For $\theta \in \Gamma(S, f_*\omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2})$, *the indigenous bundle over* X/S *corresponding to* θ *is given as follows: Let us recall the pair* $(\mathcal{E}_\circ, \nabla_{\mathcal{E}_\circ})$ *and the exact sequence of* $\mathcal{O}_X$*-modules*

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \longrightarrow \mathcal{E}_\circ \longrightarrow \omega_{X/S} \longrightarrow 0$$

discussed in §1. Write $\phi^\theta \colon \mathcal{E}_\circ \to \mathcal{E}_\circ \otimes_{\mathcal{O}_X} \omega_{X/S}$ *for the homomorphism of* $\mathcal{O}_X$*-modules obtained by forming the composite*

$$\mathcal{E}_\circ \twoheadrightarrow \omega_{X/S} \xrightarrow{\theta} \omega_{X/S}^{\otimes 3} = \omega_{X/S}^{\otimes 2} \otimes_{\mathcal{O}_X} \omega_{X/S} \hookrightarrow \mathcal{E}_\circ \otimes_{\mathcal{O}_X} \omega_{X/S}.$$

We shall write

$$\nabla_{P_\circ}^\theta$$

for the connection on $P_\circ$ *determined by the connection*

$$\nabla_{\mathcal{E}_\circ}^\theta \stackrel{\text{def}}{=} \nabla_{\mathcal{E}_\circ} + \phi^\theta$$

on $\mathcal{E}_\circ$. *Then the indigenous bundle over* X/S *corresponding to* θ *is given by*

$$P_\theta \stackrel{\text{def}}{=} (\pi_\circ \colon P_\circ \to X, \nabla_{P_\circ}^\theta).$$

Moreover, for $\theta \in \Gamma(S, f_*\omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2})$, *the* **dormant locus** *in* S *of* P_θ [*i.e., the maximal closed subscheme* $F \subseteq S$ *of* S *such that the restriction of* P_θ *to* $X \times_S F$ *is* **dormant**] **coincides** *with the zero locus in* S *of* θ [*i.e., the maximal closed subscheme* $F \subseteq S$ *of* S *such that the restriction of* θ *to* $X \times_S F$ *is identically zero*].

3. LOCAL CRITERIA

In the present §3, we prove *local criteria* for some properties on indigenous bundles [cf. Proposition 3.1; Proposition 3.8, (ii), below]. We maintain the notation introduced at the beginning of §1.

Let

$$\theta \in \Gamma(X, \omega_{X/S}^{\otimes 2})$$

be a global section of $\omega_{X/S}^{\otimes 2}$. Thus, it follows from Corollary 2.6 that we obtain a connection

$$\nabla_{P_\circ}^\theta$$

on the $\mathbb{P}^1$-bundle $P_\circ$ such that the pair

$$P_\theta \stackrel{\text{def}}{=} (\pi_\circ \colon P_\circ \to X, \nabla_{P_\circ}^\theta)$$

forms an *indigenous bundle* over X/S.

Let $x \in X$ be a point of X and $t_x = t \in \mathcal{O}_X$ a local parameter of X/S at x. Write $\phi_x = \phi \in \mathcal{O}_X$ for the local function on X at x which fits into the equality

$$\theta = \phi \cdot dt \otimes dt.$$

Then one verifies immediately that the local sections

$$e_1 \stackrel{\text{def}}{=} 1 \otimes t - t \otimes 1, \quad e_2 \stackrel{\text{def}}{=} e_1^2 \in \mathrm{pr}_{1*}\mathcal{O}_{X \times_{X^F} X} \xleftarrow{\sim} \Phi^*\Phi_*\mathcal{O}_X$$

[cf. the discussion preceding Proposition 1.1] are contained in the submodules

$$\mathrm{Ker}(\mathrm{pr}_{1*}\mathcal{O}_{X\times_{X^F}X} \twoheadrightarrow \mathcal{O}_X) \xleftarrow{\sim} \mathcal{E}_\circ,$$

and that, in the natural exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \longrightarrow \mathcal{E}_\circ \longrightarrow \omega_{X/S} \longrightarrow 0,$$

the local section e_2 determines a *local trivialization* of the invertible sheaf $\omega_{X/S}^{\otimes 2}$, and the local section e_1 determines a *local splitting* of the surjection $\mathcal{E}_\circ \twoheadrightarrow \omega_{X/S}$; in particular, $\{e_1, e_2\}$ forms a *local basis* of $\mathcal{E}_\circ$.

Next, let us observe that it follows immediately from the definition of $\nabla_{\mathcal{E}_\circ}$ that

$$\nabla_{\mathcal{E}_\circ}(e_1, e_2) \;=\; (e_1, e_2) \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes dt.$$

Thus, one verifies immediately from the definition of $\nabla^\theta_{\mathcal{E}_\circ}$ [cf. Corollary 2.6] that

$$\nabla^\theta_{\mathcal{E}_\circ}(e_1, e_2) \;=\; (e_1, e_2) \cdot \begin{pmatrix} 0 & 1 \\ \phi & 0 \end{pmatrix} \otimes dt.$$

In particular, it follows that the *p-curvature* $\mathcal{P}^\theta$ of the connection $\nabla^\theta_{\mathcal{E}_\circ}$ [cf., e.g., the discussion preceding [4], Theorem 5.1] is given by

$$\begin{array}{rccc} \mathcal{P}^\theta : & \Phi^*\tau_{X^F/S} & \longrightarrow & \mathcal{A}d_{\mathcal{O}_X}(\mathcal{E}_\circ) \\ & \Phi^{-1}\delta_{t^F} & \mapsto & \left((e_1, e_2) \mapsto (e_1, e_2) \cdot \begin{pmatrix} -\phi' & \phi \\ \phi^2 + \phi'' & \phi' \end{pmatrix}\right) \end{array}$$

— where we write $t^F \in \mathcal{O}_{X^F}$ for the local parameter of X^F/S determined by the local parameter $t \in \mathcal{O}_X$, δ_{t^F} (respectively, δ_t) for the local trivialization of $\tau_{X^F/S}$ (respectively, $\tau_{X/S}$) which maps dt^F (respectively, dt) to 1, ∂_t for the local derivation corresponding to δ_t, "$(-)'$" for "$\partial_t(-)$" [i.e., "$(-)'$" is the "derivative of $(-)$ with respect to t"], and

$$\mathcal{A}d_{\mathcal{O}_X}(\mathcal{E}_\circ) \;\subseteq\; \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_\circ)$$

for the submodule of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_\circ)$ consisting of *trace zero* endomorphisms of locally free coherent $\mathcal{O}_X$-module $\mathcal{E}_\circ$. This local computation [cf. Remark 3.1.1 below] leads us to the following *local criteria* for some properties on indigenous bundles:

Proposition 3.1. — *The following hold:*

(i) *The indigenous bundle P_θ is* **nilpotent** [*cf.* [5], *Chapter II, Definition 2.4*] *if and only if, for every point $x \in X$, the equality*

$$(\phi'_x)^2 + \phi_x \cdot \phi''_x + \phi_x^3 \;=\; 0$$

holds.

(ii) *Suppose that S is the spectrum of an algebraically closed field* [*of characteristic* 3]. *Then the indigenous bundle P_θ is* **admissible** [*cf.* [5], *Chapter II, Definition 2.4*] *if and only if, for every closed point $x \in X$, it holds that*

$$\mathrm{ord}_x(\phi_x) \;\leq\; 2.$$

PROOF. — Assertion (i) follows from the definition, together with the above local computation. To verify assertion (ii), let us observe that

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

forms a local basis of the locally free coherent $\mathcal{O}_X$-module $\mathcal{A}d_{\mathcal{O}_X}(\mathcal{E}_\circ)$. Thus, assertion (ii) follows immediately from the definition, together with the above local computation. □

REMARK 3.1.1. — We note that since $\det(\mathcal{E}_\circ) \cong \omega_{X/S}^{\otimes 3} \not\cong \mathcal{O}_X$ [cf. Lemma 1.2], the pair $(\mathcal{E}_\circ, \nabla_{\mathcal{E}_\circ})$, as well as the pair $(\mathcal{E}_\circ, \nabla_{\mathcal{E}_\circ}^\theta)$ [cf. Corollary 2.6], is *not an indigenous vector bundle* [cf. [5], Chapter I, Definition 2.2; also the discussion preceding [5], Chapter I, Definition 2.2]. One verifies easily from the fact that $\det(\mathcal{B}_\circ) \cong \omega_{X^F/S}$ [cf. Lemma 1.2] that if $\mathcal{L}$ is an invertible sheaf on X^F such that $\mathcal{L}^{\otimes 2} \cong \tau_{X^F/S}$ [note that since 2 is *invertible* on S, such an invertible sheaf always exists after étale localizing S], then an *indigenous vector bundle* whose projectivization is isomorphic to $(\pi_\circ \colon P_\circ \to X, \nabla_{P_\circ})$ is given by tensoring $(\mathcal{E}_\circ, \nabla_{\mathcal{E}_\circ})$ with the invertible sheaf $\Phi^*\mathcal{L}$ equipped with the connection determined by the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$. On the other hand, one also verifies easily that the operation of taking tensor product with a *dormant invertible sheaf* [i.e., an invertible sheaf equipped with a connection whose p-curvature is identically zero] does *not affect* the *local computation of the p-curvature as above* [as well as the *validity* of the following results].

REMARK 3.1.2. — If $\mathcal{P}^\theta = 0$, then it follows from the above local computation that $\phi = 0$, hence also $\theta = 0$. By means of this observation, one can give an *alternative proof* of Theorem 2.1.

Next, let us observe that the natural exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \longrightarrow \mathcal{E}_\circ \longrightarrow \omega_{X/S} \longrightarrow 0$$

determines a homomorphism of $\mathcal{O}_X$-modules

$$\mathcal{A}d_{\mathcal{O}_X}(\mathcal{E}_\circ) \hookrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_\circ) \to \mathcal{H}om_{\mathcal{O}_X}(\omega_{X/S}^{\otimes 2}, \omega_{X/S}) \cong \tau_{X^F/S};$$

moreover, the *square Hasse invariant* [cf. [5], Chapter II, Proposition 2.6, (1)] of the indigenous bundle P_θ is defined as the composite of the p-curvature $\mathcal{P}^\theta$ and this homomorphism. Thus, by the above local computation, we obtain:

PROPOSITION 3.2. — *The* **square Hasse invariant** *of the indigenous bundle P_θ is, up to multiplication by a global section of $\mathcal{O}_S^\times$, given by*

$$\theta \in \Gamma(X, \omega_{X/S}^{\otimes 2}) \cong \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\Phi^*\tau_{X^F/S}, \tau_{X/S})).$$

In particular, if, moreover, the indigenous bundle P_θ is **admissible**, *then the* **double supersingular divisor** [*cf.* [5], *Chapter II, Proposition 2.6,* (2)] *of P_θ* **coincides** *with the zero locus of θ.*

In particular, we obtain the following two corollaries:

Corollary 3.3. — *Suppose that the indigenous bundle P_θ is* **nilpotent** *and* **admissible**. *Then the* **supersingular divisor** [*cf.* [5], *Chapter II, Proposition 2.6,* (3)] *of P_θ is* **finite étale** *over S.*

Proof. — Since [it follows from the definition that] the supersingular divisor of P_θ is *finite flat* over S [cf. also [5], Chapter II, Proposition 2.6, (2)], to complete the verification of Corollary 3.3, it suffices to verify the *unramifiedness*. Thus, we may assume without loss of generality that S is the spectrum of an algebraically closed field [of characteristic 3]. Then the *unramifiedness* follows from Proposition 3.1, (ii); Proposition 3.2, together with the definition of the *supersingular divisor*. □

Corollary 3.4. — *Suppose that S is* **reduced**. *Then the isomorphism class of* **nilpotent** *indigenous bundle over X/S is* **completely determined** *by the zero locus of the square Hasse invariant.*

Proof. — First, let us observe that since S is *reduced*, it follows from [6], Chapter I, Proposition 1.5, that, to verify Corollary 3.4, we may assume without loss of generality that S is the spectrum of an algebraically closed field k [of characteristic 3]. Next, let us observe that one verifies easily that if ϕ is *nonzero* and *satisfies* the equality "$(\phi')^2 + \phi \cdot \phi'' + \phi^3 = 0$" of Proposition 3.1, (i), then, for every $c \in k \setminus \{0, 1\}$, $c \cdot \phi$ does *not satisfy* the equality "$(\phi')^2 + \phi \cdot \phi'' + \phi^3 = 0$" of Proposition 3.1, (i). Thus, Corollary 3.4 follows from Proposition 3.1, (i); Proposition 3.2, together with Corollary 2.6. □

Remark 3.4.1. — Observe that Corollary 3.4 is a *generalization* of [5], Chapter II, Proposition 2.6, (4), in the case where $p = 3$.

Next, let us observe that it follows from the equality of Proposition 3.1, (i), that the following lemma holds:

Lemma 3.5. — *Suppose that S is the spectrum of an algebraically closed field* [*of characteristic* 3]*, and that the indigenous bundle P_θ is* **nilpotent**. *Then, for every closed point $x \in X$, it holds that* $\mathrm{ord}_x(\phi_x) \notin 3\mathbb{Z} + 1$.

Proof. — Assume that $n \stackrel{\text{def}}{=} \mathrm{ord}_x(\phi_x) \in 3\mathbb{Z} + 1$ for some closed point $x \in X$. Write

$$\phi_x \ = \ \sum_{i=0}^{\infty} a_i t_x^i$$

by regarding ϕ_x as an element of the completion $\mathcal{O}_{X,x}^{\wedge}$. Then, by considering the coefficient of the "t_x^{2n-2}" of the left-hand side of the equality "$(\phi')^2+\phi\cdot\phi''+\phi^3 = 0$" of Proposition 3.1, (i), we obtain that $a_n = 0$. Thus, we obtain a *contradiction*. □

By Lemma 3.5, we obtain:

COROLLARY 3.6. — *Suppose that $g = 2$. If a* **nilpotent** *indigenous bundle over X/S is* **active** [*cf.* [6], *Chapter II, Definition 1.1*], *then it is* **admissible**.

PROOF. — Let us first observe that it follows from the definition of the *admissibility* that, to verify Corollary 3.6, we may assume without loss of generality that S is the spectrum of an algebraically closed field k [of characteristic 3]. On the other hand, in this case, since $\deg(\omega_{X/S}^{\otimes 2}) = 4$, it follows immediately from Proposition 3.1, (ii), together with Lemma 3.5, that every *nilpotent* and *active* indigenous bundle over X/S is *admissible*. □

We shall write

$$\mathcal{N}_g$$

for the moduli stack of smooth *nilcurves* [cf. the discussion preceding [6], Introduction, Theorem 0.1] of genus g of characteristic 3, i.e., the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with *nilpotent indigenous bundles*. Note that it follows from [5], Chapter II, Theorem 2.3 [cf. also the discussion following [5], Chapter II, Definition 2.4], that the natural (1-)morphism

$$\mathcal{N}_g \longrightarrow \mathcal{M}_g$$

is *finite flat of degree* 3^{3g-3}.

COROLLARY 3.7. — *Suppose that $g = 2$. Then the open substack of $\mathcal{N}_2$*

$$\mathcal{N}_2 \setminus \mathcal{N}_2[\infty]$$

is **smooth** *over $\mathbb{F}_3$.*

PROOF. — This follows from Corollary 3.6, together with [5], Chapter II, Corollary 2.16. □

PROPOSITION 3.8. — *Suppose that S is the spectrum of an algebraically closed field k [of characteristic 3], and that the indigenous bundle P_θ is* **nilpotent**. *Then the following hold:*

(i) *We shall write*

$$T_\theta$$

for the relative tangent space of $\mathcal{N}_g/\mathcal{M}_g$ at the k-valued point of $\mathcal{N}_g$ corresponding to P_θ. Then T_θ is naturally isomorphic to the subspace of $\Gamma(X, \omega_{X/S}^{\otimes 2})$ consisting of global sections η of $\omega_{X/S}^{\otimes 2}$ such that if, for some closed point $x \in X$, we write

$$\eta \ = \ \psi_x \cdot dt_x \otimes dt_x,$$

then it holds that

$$(\phi_x \cdot \psi_x)'' \ = \ 0.$$

(ii) *It holds that the indigenous bundle P_θ is* **ordinary** [*cf.* [5], *Chapter II, Definition 3.1*] *if and only if the following condition is satisfied: For every nonzero global section η of $\omega_{X/S}^{\otimes 2}$, if, for some closed point $x \in X$, we write*

$$\eta \ = \ \psi_x \cdot dt_x \otimes dt_x,$$

then it holds that

$$(\phi_x \cdot \psi_x)'' \neq 0.$$

Proof. — Assertion (ii) follows immediately from assertion (i). Thus, to complete the verification of Proposition 3.8, it suffices to verify assertion (i). Write $A \stackrel{\text{def}}{=} k[\epsilon]/(\epsilon^2)$, where ϵ is an indeterminate. Then it follows from Proposition 3.1, (i), that T_θ is naturally isomorphic to the subspace of $\Gamma(X, \omega_{X/S}^{\otimes 2})$ consisting of global sections η of $\omega_{X/S}^{\otimes 2}$ such that if, for some closed point $x \in X$, we write

$$\eta = \psi_x \cdot dt_x \otimes dt_x,$$

then the equality

$$((\phi + \epsilon\psi)')^2 + (\phi + \epsilon\psi) \cdot (\phi + \epsilon\psi)'' + (\phi + \epsilon\psi)^3 = 0$$

— where write $\psi \stackrel{\text{def}}{=} \psi_x$ — in $A \otimes_k \Gamma(X, \omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2}) \oplus \epsilon \cdot \Gamma(X, \omega_{X/S}^{\otimes 2})$ holds. On the other hand, again by Proposition 3.1, (i), one verifies easily that it holds that this equality holds if and only if the equality

$$\phi'' \cdot \psi + \phi \cdot \psi'' - \phi' \cdot \psi' \ (= (\phi \cdot \psi)'') = 0$$

holds. This completes the proof of assertion (i). □

Remark 3.8.1. — Proposition 3.8, (ii), also follows immediately from Proposition 3.2; Lemma A.9, (i) [in the case where we take the pair "$(\mathcal{L}, \tau)$" of Lemma A.9, (i), to be the pair consisting of $\mathcal{O}_X$ and the natural identification $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{O}_X$ — cf. Remark A.4.1], together with [5], Chapter II, Proposition 2.12.

Thus, we obtain:

Corollary 3.9. — *Suppose that S is the spectrum of an algebraically closed field k [of characteristic 3], and that the indigenous bundle P_θ is* **nilpotent**. *Then the following conditions are equivalent:*

(1) *The indigenous bundle P_θ is* **dormant**.

(2) *The vector space T_θ over k of Proposition* 3.8, (i), *is* **of dimension $3g - 3$**.

Proof. — If P_θ is *dormant*, then $\theta = 0$ [cf. Corollary 2.6]. Thus, the implication (1) $\Rightarrow$ (2) follows from Proposition 3.8, (i). On the other hand, if condition (2) is satisfied, then it follows from Proposition 3.8, (i) [in the case where we take the "η" of Proposition 3.8, (i), to be θ], that $(\phi^2)'' = 0$. Thus, since $0 = (\phi^2)'' = -(\phi')^2 - \phi \cdot \phi'' = \phi^3$ [cf. Proposition 3.1, (i)], we conclude that $\phi = 0$, hence also $\theta = 0$, i.e., that condition (1) is satisfied [cf. Corollary 2.6]. This completes the proof of Corollary 3.9. □

4. Indigenous Bundles Arising from Squares

In the present §4, we discuss some properties on an indigenous bundle which arises from the square of a "*twisted*" differential form, i.e., the square of a global section of a "*square root*" of the square of the relative cotangent sheaf [cf. Proposition 4.1, Proposition 4.2, Proposition 4.4 below]. In the present §4, we maintain the notation introduced at the beginning of §1.

Let

$$\mathcal{L} \ = \ (\mathcal{L}, \tau \colon \mathcal{L}^{\otimes 2} \stackrel{\sim}{\to} \mathcal{O}_X)$$

be a *square-trivialized invertible sheaf* on X [cf. Definition A.3] and

$$\chi \ \in \ \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$$

a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$. Let us recall [cf. the discussion following Definition A.3] that we have isomorphisms of invertible sheaves

$$\begin{array}{ccccc} \mathcal{L} & \stackrel{\sim}{\longrightarrow} & \mathcal{L}^{\otimes 3} & \stackrel{\sim}{\longrightarrow} & \Phi^* \mathcal{L}^F \\ \tau(l \otimes l) \cdot l & \mapsto & l \otimes l \otimes l & \mapsto & \Phi^{-1} l^F \end{array}$$

— where we write $\mathcal{L}^F$ for the invertible sheaf on X^F obtained by pulling back $\mathcal{L}$ via the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S, l is a local section of $\mathcal{L}$, and l^F is the local section of $\mathcal{L}^F$ determined by l.

Let $x \in X$ be a point of X, $t_x = t \in \mathcal{O}_X$ a local parameter of X/S at x, and $l_x = l \in \mathcal{L}$ a local trivialization of $\mathcal{L}$ at x. Then the global trivialization τ and the local trivialization $l_x = l$ determine a local unit

$$\delta_x \ = \ \delta \ \stackrel{\text{def}}{=} \ \tau(l \otimes l) \ \in \ \mathcal{O}_X^{\times}$$

at x. Moreover, the global section χ determines a local function $\phi_x = \phi \in \mathcal{O}_X$ on X at x which fits into the equality

$$\chi \ = \ \phi \cdot l \otimes dt$$

at x.

Next, let us observe that the trivialization τ determines an isomorphism

$$\tau \colon \ \Gamma(X, (\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2}) \ \stackrel{\sim}{\longrightarrow} \ \Gamma(X, \omega_{X/S}^{\otimes 2}).$$

Thus, by considering the image via this isomorphism of the *square*

$$\theta \ \stackrel{\text{def}}{=} \ \chi \otimes \chi \ \in \ \Gamma(X, (\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2})$$

of χ, we obtain a global section

$$\tau(\theta) \ \in \ \Gamma(X, \omega_{X/S}^{\otimes 2})$$

of $\omega_{X/S}^{\otimes 2}$. On the other hand, it follows from Corollary 2.6 that this global section $\tau(\theta)$ gives rise to an *indigenous bundle* over X/S

$$P_{\tau(\theta)} \ \stackrel{\text{def}}{=} \ (\pi_\circ \colon P_\circ \to X, \nabla_{P_\circ}^{\tau(\theta)}).$$

Proposition 4.1. — *Suppose that χ defines a relative effective Cartier divisor of X/S. Then the following conditions are equivalent:*

(1) *The indigenous bundle $P_{\tau(\theta)}$ is* **nilpotent** *and* **active**.

(2) *The global section* $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ *is a* **normalized Cartier eigenform** *associated to* $\mathcal{L} = (\mathcal{L}, \tau)$ [*cf. Definition* A.8, (i)].

Proof. — Let us first observe that it follows from the definitions of $\tau(\theta)$ that $\tau(\theta)$ fits into the equality

$$\tau(\theta) \ = \ \phi^2 \cdot \delta \cdot dt \otimes dt$$

at x. Thus, it follows from Proposition 3.1, (i), that it holds that $P_{\tau(\theta)}$ is *nilpotent* if and only if, for every point $x \in X$,

$$\begin{aligned} &((\phi^2 \cdot \delta)')^2 + (\phi^2 \cdot \delta) \cdot (\phi^2 \cdot \delta)'' + (\phi^2 \cdot \delta)^3 \\ = \ &\phi^4 \cdot (\delta')^2 - \phi^3 \cdot \phi' \cdot \delta \cdot \delta' - \phi^3 \cdot \phi'' \cdot \delta^2 + \phi^4 \cdot \delta \cdot \delta'' + \phi^6 \cdot \delta^3 \\ = \ &\phi^3 \cdot \delta^3 \cdot (-(\phi \cdot \delta^{-1})'' + \phi^3) \end{aligned}$$

is equal to zero. In particular, Proposition 4.1 follows from Lemma A.9, (ii), together with Corollary 2.6. □

Proposition 4.2. — *Suppose that the indigenous bundle* $P_{\tau(\theta)}$ *is* **nilpotent** *and* **active**. *Then the following conditions are equivalent:*

(1) *The indigenous bundle* $P_{\tau(\theta)}$ *is* **nilpotent** *and* **admissible**.

(2) *The zero locus of the global section* $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ *is* **finite étale** *over* S.

Proof. — Since [one verifies immediately that] the locus [in S] on which condition (1) (respectively, (2)) is satisfied is *open*, to complete the verification of Proposition 4.2, we may assume without loss of generality that S is the spectrum of an algebraically closed field [of characteristic 3]. Then the equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 3.1, (ii), together with the definition of $\tau(\theta)$. □

Remark 4.2.1.

(i) Note that condition (2) of Proposition 4.1 does *not imply* condition (2) of Proposition 4.2. Such a counter-example is as follows: Let k be an algebraically closed field of characteristic 3. Let us consider the following polynomial:

$$f(t) \ = \ t^{12} + t^{10} + 1 \ \in \ k[t].$$

Then one verifies easily that $f(t)$ does *not have* any multiple root, which thus implies that the equation

$$s^2 \ = \ f(t)$$

determines a *hyperelliptic* projective smooth curve C of genus five over k.

Write $\omega \in \Gamma(C, \omega_{C/k})$ for the global section of $\omega_{C/k}$ whose restriction to the open subscheme of X on which f is invertible is of the form

$$\frac{\alpha \cdot t^4}{s} dt$$

— where $\alpha \in k$ satisfies that $\alpha^2 = 2$. Then one verifies easily from Lemma A.9, (ii), that ω is a *normalized Cartier eigenform* associated to $\mathcal{O}_C$ [equipped with the natural

identification $\mathcal{O}_C \otimes_{\mathcal{O}_C} \mathcal{O}_C = \mathcal{O}_C$]. On the other hand, it is immediate that if we write $c \in C$ for the closed point corresponding to $(t, s) = (0, 1)$, then $\mathrm{ord}_c(\omega) = 4$.

(ii) It follows from Corollary 3.6 that a *nilpotent active* indigenous bundle over a projective smooth curve *of genus two* in characteristic three is *admissible*. On the other hand, it follows from the discussion of (i), together with Proposition 4.1 and Proposition 4.2, that there exists a *nilpotent active* indigenous bundle over a projective smooth curve in characteristic three which is *not admissible*.

PROPOSITION 4.3. — *Suppose that S is the spectrum of an algebraically closed field k [of characteristic 3], and that the indigenous bundle $P_{\tau(\theta)}$ is* **nilpotent** *and* **admissible**. *Write*

$$T_{\tau(\theta)}$$

for the relative tangent space of $\mathcal{N}_g/\mathcal{M}_g$ at the k-valued point of $\mathcal{N}_g$ corresponding to $P_{\tau(\theta)}$. Thus, it follows from Proposition 3.8, (i), *that $T_{\tau(\theta)}$ may be regarded as a subspace of $\Gamma(X, \omega_{X/S}^{\otimes 2})$:*

$$T_{\tau(\theta)} \subseteq \Gamma(X, \omega_{X/S}^{\otimes 2}).$$

Then the map

$$\begin{array}{ccc} \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \otimes_k \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) & \longrightarrow & \Gamma(X, \omega_{X/S}^{\otimes 2}) \\ \alpha \otimes \beta & \mapsto & \tau(\alpha \otimes \beta) \end{array}$$

induces an **isomorphism** *of vector spaces over k*

$$\begin{array}{ccc} \mathrm{Ker}(C_{\mathcal{L}}) & \xrightarrow{\sim} & T_{\tau(\theta)} \\ \sigma & \mapsto & \tau(\sigma \otimes \chi) \end{array}$$

— where we write $C_{\mathcal{L}}$ for the Cartier operator associated to $\mathcal{L} = (\mathcal{L}, \tau)$ [cf. Definition A.4].

PROOF. — Let us first observe that [one verifies easily that] the homomorphism of vector spaces over k

$$\begin{array}{cccc} \Xi\colon & \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) & \longrightarrow & \Gamma(X, \omega_{X/S}^{\otimes 2}) \\ & \alpha & \mapsto & \tau(\alpha \otimes \chi) \end{array}$$

is *injective*. Thus, to verify Proposition 4.3, it suffices to verify the following two assertions:

(a) $\Xi(\mathrm{Ker}(C_{\mathcal{L}})) \subseteq T_{\tau(\theta)}$.

(b) The resulting [cf. (a)] homomorphism $\Xi\colon \mathrm{Ker}(C_{\mathcal{L}}) \to T_{\tau(\theta)}$ is *surjective*.

Next, let us recall from the proof of Proposition 4.1 that $\tau(\theta)$ fits into the equality

$$\tau(\theta) = \phi^2 \cdot \delta \cdot dt \otimes dt$$

at x. Thus, it follows from Proposition 3.8, (i), that the subspace $T_{\tau(\theta)} \subseteq \Gamma(X, \omega_{X/S}^{\otimes 2})$ consists of global sections η of $\omega_{X/S}^{\otimes 2}$ such that if, for some closed point $x \in X$, we write

$$\eta = \psi \cdot dt \otimes dt,$$

then it holds that

$$(\phi^2 \cdot \delta \cdot \psi)'' = 0.$$

Now we verify the assertion (a). Let $\sigma \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ be such that $C_{\mathcal{L}}(\sigma) = 0$. Write

$$\sigma \ = \ \mu \cdot l \otimes dt$$

at x. Since $\sigma \in \mathrm{Ker}(C_{\mathcal{L}})$, it holds that $(\mu \cdot \delta^{-1})'' = 0$ [cf. Lemma A.9, (i)]. Thus, since

$$\tau(\sigma \otimes \chi) \ = \ \phi \cdot \mu \cdot \delta \cdot dt \otimes dt$$

at x, and

$$(\phi^3 \cdot \mu \cdot \delta^2)'' \ = \ (\phi^3 \cdot \delta^3)'' \cdot (\mu \cdot \delta^{-1}) - (\phi^3 \cdot \delta^3)' \cdot (\mu \cdot \delta^{-1})' + (\phi^3 \cdot \delta^3) \cdot (\mu \cdot \delta^{-1})'' \ = \ 0,$$

we conclude that $\Xi(\sigma) \in T_{\tau(\theta)}$. This completes the proof of the assertion (a).

Next, we verify the assertion (b). Let η be a global section of $\omega_{X/S}^{\otimes 2}$ which belongs to $T_{\tau(\theta)}$. Write

$$\eta \ = \ \psi \cdot dt \otimes dt$$

at x. Then since

$$0 \ = \ (\phi^2 \cdot \delta \cdot \psi)''$$

$$= \ (-(\phi')^2 - \phi \cdot \phi'') \cdot \delta \cdot \psi + \phi^2 \cdot \delta'' \cdot \psi + \phi^2 \cdot \delta \cdot \psi'' + \phi \cdot \phi' \cdot \delta' \cdot \psi - \phi^2 \cdot \delta' \cdot \psi' + \phi \cdot \phi' \cdot \delta \cdot \psi',$$

and ϕ is *of order* ≤ 1 [at x] by Proposition 4.2, it holds that $\mathrm{ord}_x(\phi) \geq 1$ implies $\mathrm{ord}_x(\psi) \geq 1$. Thus, it follows that $V(\chi) = V(\chi)_{\mathrm{red}} \subseteq V(\eta)_{\mathrm{red}} \subseteq V(\eta)$, where we write "$V(-)$" for the zero locus of "$(-)$", i.e., that $\eta \in \Gamma(X, \omega_{X/S}^{\otimes 2}(-V(\chi))) \subseteq \Gamma(X, \omega_{X/S}^{\otimes 2})$. Now let us observe that since $(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2} \cong \omega_{X/S}^{\otimes 2}$, which thus implies that $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S} \cong \omega_{X/S}^{\otimes 2}(-V(\chi))$, we have an isomorphism

$$\begin{array}{ccc} \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) & \stackrel{\sim}{\longrightarrow} & \Gamma(X, \omega_{X/S}^{\otimes 2}(-V(\chi))) \\ \sigma & \mapsto & \tau(\sigma \otimes \chi). \end{array}$$

Thus, we conclude that there exists a global section σ of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ such that $\eta = \tau(\sigma \otimes \chi)$. Write

$$\sigma \ = \ \mu \cdot l \otimes dt$$

at x, which thus implies that

$$\psi \ = \ \mu \cdot \phi \cdot \delta$$

at x. Then since

$$0 \ = \ (\phi^2 \cdot \delta \cdot \psi)'' \ = \ (\mu \cdot \phi^3 \cdot \delta^2)''$$

$$= \ (\phi^3 \cdot \delta^3)'' \cdot (\mu \cdot \delta^{-1}) - (\phi^3 \cdot \delta^3)' \cdot (\mu \cdot \delta^{-1})' + (\phi^3 \cdot \delta^3) \cdot (\mu \cdot \delta^{-1})''$$

$$= \ \phi^3 \cdot \delta^3 \cdot (\mu \cdot \delta^{-1})'',$$

it holds that $(\mu \cdot \delta^{-1})'' = 0$, i.e., that $\sigma \in \mathrm{Ker}(C_{\mathcal{L}})$ [cf. Lemma A.9, (i)]. This completes the proof of the assertion (b), hence also of Proposition 4.3. $\square$

Proposition 4.4. — *Suppose that the indigenous bundle $P_{\tau(\theta)}$ is* **nilpotent** *and* **admissible**. *Then the following conditions are equivalent:*

(1) *The indigenous bundle $P_{\tau(\theta)}$ is* **nilpotent** *and* **ordinary**.

(2) *The invertible sheaf $\mathcal{L}$ is* **parabolically ordinary** [*cf. Definition* A.7].

PROOF. — Since [one verifies immediately that] the locus [in S] on which condition (1) (respectively, (2)) is satisfied is *open*, to complete the verification of Proposition 4.4, we may assume without loss of generality that S is the spectrum of an algebraically closed field [of characteristic 3]. Then Proposition 4.4 follows from Proposition 4.3. □

5. Nilpotent Admissible Indigenous Bundles via Cartier Operators

In the present §5, we prove the main result of the present paper [cf. Theorem 5.2 below], as well as some corollaries to the main result. In the present §5, we maintain the notation introduced at the beginning of §1.

DEFINITION 5.1.

(i) We shall say that a pair

$$(\mathcal{L}, \chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}))$$

consisting of an invertible sheaf $\mathcal{L}$ on X and a global section χ of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is *of CE-type* [where "CE" stands for "Cartier Eigenform"] if

- $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$,
- $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ is a *Cartier eigenform* associated to $\mathcal{L}$ [cf. Definition A.8, (ii)], and
- the zero locus of χ is *étale* over S.

(ii) We shall say that a pair $(\mathcal{L}, \chi)$ of CE-type is *of CEO-type* [where "CEO" stands for "Cartier Eigenform and Ordinary"] if $\mathcal{L}$ is parabolically ordinary.

(iii) We shall say that a relative effective Cartier divisor D of X/S is *of CE-type* (respectively, *of CEO-type*) if there exists a pair $(\mathcal{L}, \chi)$ of CE-type (respectively, of CEO-type) such that D coincides with the zero locus of χ.

The main result of the present paper is as follows:

THEOREM 5.2. — *Let $g \geq 2$ be an integer, S a connected noetherian scheme of characteristic 3 [i.e., over $\mathbb{F}_3$], and $f : X \to S$ a projective smooth curve of genus g. Write $\omega_{X/S}$ for the relative cotangent bundle of X/S. Then the following hold:*

(i) *Let P be a* **nilpotent admissible** *indigenous bundle over X/S. Write $\mathcal{L}_P$ for the* **Hasse defect** *of P [cf. Definition* B.2*] and $\chi_P \in \Gamma(X, \mathcal{L}_P \otimes_{\mathcal{O}_X} \omega_{X/S})$ for the* **Hasse invariant** *of P [cf.* [5]*, Chapter II, Proposition 2.6,* (3)*; also the final portion of Proposition* B.4*]. Then the pair*

$$(\mathcal{L}_P, \chi_P)$$

is **of CE-type** *[cf. Definition* 5.1*,* (i)*]. Moreover, it holds that P is* **nilpotent ordinary** *if and only if the pair $(\mathcal{L}_P, \chi_P)$ is* **of CEO-type** *[cf. Definition* 5.1*,* (ii)*].*

(ii) *Let D be a relative effective Cartier divisor of X/S. Then it holds that D is the* **supersingular divisor** *[cf.* [5]*, Chapter II, Proposition 2.6,* (3)*] of a* **nilpotent**

admissible (*respectively,* **nilpotent ordinary**) *indigenous bundle over X/S if and only if D is* **of CE-type** (*respectively,* **of CEO-type**) [*cf. Definition* 5.1*,* (iii)].

(iii) *Suppose that S is* **reduced***. Then, by considering the* **supersingular divisors***, we have a* **bijection** *between the following two sets:*

- *The set of isomorphism classes of* **nilpotent admissible** (*respectively,* **nilpotent ordinary**) *indigenous bundles over X/S.*
- *The set of relative effective Cartier divisors of X/S* **of CE-type** (*respectively,* **of CEO-type**).

Proof. — First, we verify the first assertion of assertion (i). Let us first observe that it follows from the final portion of Proposition B.3 that $\mathcal{L}_P^{\otimes 2} \cong \mathcal{O}_X$. Moreover, it follows from Corollary 3.3 that the zero locus of χ_P is *finite étale* over S. Thus, to complete the verification of the first assertion of assertion (i), it suffices to verify that there exists a trivialization $\tau\colon \mathcal{L}_P^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ such that χ_P is a *normalized Cartier eigenform* associated to $(\mathcal{L}_P, \tau)$.

Let us write $\theta_P \in \Gamma(X, \omega_{X/S}^{\otimes 2})$ for the global section of $\omega_{X/S}^{\otimes 2}$ corresponding, via the bijection of Corollary 2.6, to the indigenous bundle P. Fix a trivialization $\tau\colon \mathcal{L}_P^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ of $\mathcal{L}_P^{\otimes 2}$ and write θ for the image via the isomorphism $\Gamma(X, (\mathcal{L}_P \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2}) \xrightarrow{\sim} \Gamma(X, \omega_{X/S}^{\otimes 2})$ induced by τ of the square $\chi_P \otimes \chi_P \in \Gamma(X, (\mathcal{L}_P \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2})$ of χ_P. Then it follows from Proposition 3.2 that there exists a global unit $u \in \Gamma(S, \mathcal{O}_S^\times)$ such that $\theta_P = u \cdot \theta$. Thus, we may assume without loss of generality, by replacing τ by $u^{-1} \cdot \tau$, that $\theta_P = \theta$. In particular, it follows from Proposition 4.1 that χ_P is a *normalized Cartier eigenform* associated to $(\mathcal{L}_P, \tau)$. This completes the proof of the first assertion of assertion (i). Moreover, the final assertion of assertion (i) follows from the first assertion of assertion (i), together with Proposition 4.4 [cf. also the equality "$\theta_P = \theta$" in the proof of the first assertion of assertion (i)]. This completes the proof of assertion (i).

Next, we verify assertion (ii). The *necessity* follows from assertion (i). To verify the *sufficiency*, let D be a relative effective Cartier divisor of X/S *of CE-type* (respectively, *of CEO-type*). Thus, it follows from the definition that there exists a pair $(\mathcal{L}, \chi)$ *of CE-type* (respectively, *of CEO-type*) such that D is defined by χ. Now since $(\mathcal{L}, \chi)$ is *of CE-type*, the zero locus of χ is *étale* over S, and there exists a trivialization $\tau\colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ of $\mathcal{L}^{\otimes 2}$ such that χ is a *normalized Cartier eigenform* associated to $(\mathcal{L}, \tau)$. Thus, it follows from Proposition 4.1 and Proposition 4.2 that the indigenous bundle P over X/S corresponding, via the bijection of Corollary 2.6, to the image via the isomorphism $\Gamma(X, (\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2}) \xrightarrow{\sim} \Gamma(X, \omega_{X/S}^{\otimes 2})$ induced by τ of the square $\chi \otimes \chi \in \Gamma(X, (\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2})$ of χ is *nilpotent* and *admissible*. Moreover, it follows from Proposition 4.4 that if $(\mathcal{L}, \chi)$ is *of CEO-type*, then the indigenous bundle P is *ordinary*. Write χ_P for the *Hasse invariant* of P. Then it follows from Proposition 3.2 that the zero locus of χ_P, i.e., the supersingular divisor of P, *coincides* with the zero locus of χ, i.e., D. This completes the proof of the *sufficiency*, hence also of assertion (ii).

The *injectivity* of the map of assertion (iii) follows from Corollary 3.4 [cf. also [5], Chapter II, Proposition 2.6, (4)]. The *surjectivity* of the map of assertion (iii) follows from assertion (ii). This completes the proof of Theorem 5.2. □

COROLLARY 5.3. — *Let X be a projective smooth curve of genus $g \geq 2$ over an algebraically closed field k of characteristic 3 and $\mathcal{L}$ an invertible sheaf on X such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$. Then the following hold:*

(i) *Suppose that $\mathcal{L} \cong \mathcal{O}_X$ (respectively, $\mathcal{L} \ncong \mathcal{O}_X$). Then the number of isomorphism classes of* **nilpotent admissible** *indigenous bundles over X/k whose Hasse defects are isomorphic to $\mathcal{L}$ is*

$$\leq \sharp\mathbb{P}^{g-1}(\mathbb{F}_3) = \frac{3^g - 1}{3 - 1} \quad \Big(\textit{respectively,} \quad \leq \sharp\mathbb{P}^{g-2}(\mathbb{F}_3) = \frac{3^{g-1} - 1}{3 - 1}\Big).$$

(ii) *The number of isomorphism classes of* **nilpotent admissible** *indigenous bundles over X/k is*

$$\leq \sharp\mathbb{P}^{g-1}(\mathbb{F}_3) + (\sharp((\mathbb{Z}/2\mathbb{Z})^{\oplus 2g}) - 1) \cdot \sharp\mathbb{P}^{g-2}(\mathbb{F}_3) = \frac{1}{2}\Big((3^g - 1) + (2^{2g} - 1)(3^{g-1} - 1)\Big).$$

PROOF. — First, we verify assertion (i). Fix a trivialization $\tau\colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$. Let us first observe that it follows from Theorem 5.2, (iii) [cf. also Remark A.8.1], that, to verify assertion (i), it suffices to verify that the number of subspaces of $V_{\mathcal{L}} \stackrel{\text{def}}{=} \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ *of dimension* 1 which are *preserved* and *not annihilated* by the Cartier operator $V_{\mathcal{L}} \to \Gamma(X^F, \mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})) \xleftarrow{\sim} V_{\mathcal{L}}$ associated to $(\mathcal{L}, \tau)$ is

$$\leq \sharp\mathbb{P}^{g-1}(\mathbb{F}_3) = \frac{3^g - 1}{3 - 1} \quad \Big(\text{respectively,} \quad \leq \sharp\mathbb{P}^{g-2}(\mathbb{F}_3) = \frac{3^{g-1} - 1}{3 - 1}\Big)$$

if $\mathcal{L} \cong \mathcal{O}_X$ (respectively, $\mathcal{L} \ncong \mathcal{O}_X$). To this end, let us observe that one verifies easily from the *Riemann-Roch theorem* that the vector space $V_{\mathcal{L}}$ over k is *of dimension* g (respectively, $g - 1$) if $\mathcal{L} \cong \mathcal{O}_X$ (respectively, $\mathcal{L} \ncong \mathcal{O}_X$). Thus, assertion (i) follows immediately from elementary linear algebra [cf. also [7], p.143, Corollary]. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i). This completes the proof of Corollary 5.3. □

COROLLARY 5.4. — *Let X be a projective smooth curve of genus ≥ 2 over an algebraically closed field k of characteristic 3. Then, for every* **nilpotent ordinary** *indigenous bundle P over X/k, there exists a connected finite étale covering $Y \to X$ of X such that the* [*necessarily* **nilpotent admissible**] *indigenous bundle $(Y \to X)^*P$ over Y/k is* **not ordinary**.

PROOF. — Write $\mathcal{L}_P$ for the *Hasse defect* of P and $Y_1 \to X$ for the connected finite étale covering of X which trivializes $\mathcal{L}_P$. [So if $\mathcal{L}_P \cong \mathcal{O}_X$ (respectively, $\mathcal{L}_P \ncong \mathcal{O}_X$), then $Y_1 \to X$ is *of degree* 1 (respectively, 2).] Next, let $Y_2 \to X$ be a connected finite étale covering of X such that Y_2 is *not parabolically ordinary* [cf., e.g., [9], Théorème 2] and $Y \to X$ a connected finite étale covering of X which dominates $Y_1 \to X$ and $Y_2 \to X$.

Now let us observe that since $Y \to X$ *factors* through $Y_1 \to X$, one verifies immediately that the Hasse defect of the indigenous bundle $(Y \to X)^*P$ over Y/k is *trivial*. Thus, it follows from the final portion of Theorem 5.2, (i), that the indigenous bundle $(Y \to X)^*P$ over Y/k is *ordinary* if and only if Y is *parabolically ordinary*. On the other hand, since $Y \to X$ *factors* through $Y_2 \to X$, and Y_2 is *not parabolically ordinary*, it holds that Y

is *not parabolically ordinary*, which thus implies that the indigenous bundle $(Y \to X)^*P$ over Y/k is *not ordinary*. □

Remark 5.4.1. — Corollary 5.4 yields a *negative answer* to the basic question (2) of [6], Introduction, §2.1.

Finally, we discuss the various *moduli stacks* related to the main result of the present paper. We shall apply the notational conventions for the various stacks established in the Appendix C [in the case where we take the "p" of the Appendix C to be 3]. The following corollary follows immediately from the final portion of Theorem 5.2, (i) [cf. also Proposition C.5]:

Corollary 5.5. — *We have a* **cartesian** *diagram of stacks*

$$\begin{array}{ccc} \mathcal{N}_g^{\mathrm{ord}} & \longrightarrow & \mathcal{J}_g[2]^{\text{pb-ord}} \\ \downarrow & & \downarrow \\ \mathcal{N}_g^{\mathrm{adm}} & \longrightarrow & \mathcal{J}_g[2] \end{array}$$

— where the vertical arrows are the natural open immersions of stacks [*cf. Definition* C.4]*, and the lower horizontal arrow is the* **Hasse defect morphism** [*cf. Definition* C.1]*.*

Next, for a nonnegative integer r, write

$$\mathcal{M}_{g,[r]}$$

for the moduli stack of *hyperbolic curves of type* (g, r) of characteristic 3, i.e., the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with relative effective *étale* Cartier divisors of relative degree r.

Definition 5.6. — It follows from Corollary 3.3, together with Proposition B.3, that the supersingular divisor of the universal nilpotent admissible indigenous bundle over $\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{N}_g^{\mathrm{adm}} \to \mathcal{N}_g^{\mathrm{adm}}$ determines a (1-)morphism over $\mathcal{M}_g$

$$\mathcal{N}_g^{\mathrm{adm}} \longrightarrow \mathcal{M}_{g,[2g-2]}.$$

We shall refer to this (1-)morphism as the *Hasse morphism*.

The following corollary follows immediately from Theorem 5.2, (ii):

Corollary 5.7. — *Let (X, D) be a hyperbolic curve of type $(g, 2g-2)$ over a connected noetherian scheme S of characteristic* 3*. Then the following conditions are equivalent:*

(1) *The classifying* (1-)*morphism $S \to \mathcal{M}_{g,[2g-2]}$ of (X, D)* **factors** *through the* **Hasse morphism** $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{M}_{g,[2g-2]}$.

(2) *The relative effective Cartier divisor D of X/S is* **of CE-type**.

6. The Case of Genus Two

In the present §6, we give, by applying the results obtained in the present paper, a complete list of *nilpotent/nilpotent admissible/nilpotent ordinary* indigenous bundles over a projective smooth curve *of genus two* over an algebraically closed field of characteristic three [cf. Theorem 6.1 below]. Moreover, we also prove that every projective smooth curve of genus two over a connected noetherian scheme of characteristic three is *hyperbolically ordinary* [cf. Corollary 6.6 below]. In the present §6, we maintain the notation introduced at the beginning of §1. Suppose, moreover, that $g = 2$ [i.e., that X is *of genus* 2], and that S is the spectrum of an algebraically closed field k [of characteristic 3].

Since X is of genus 2, X admits a *uniquely determined hyperelliptic involution* ι, which determines a *double* covering

$$\xi\colon\ X\ \longrightarrow\ Q$$

— where we write Q for the [scheme-theoretic] quotient of X by the action of ι. [Thus, Q is isomorphic to the projective line $\mathbb{P}^1_k$ over k.] We shall write

$$\mathrm{WP}\ \subseteq\ X$$

for the ramification locus of ξ, i.e., the zero locus of the global section of the invertible sheaf on X [of degree 6]

$$\mathcal{H}om_{\mathcal{O}_X}(\xi^*\omega_{Q/S}, \omega_{X/S})$$

determined by ξ. [Thus, $\mathrm{WP} \subseteq X$ is the set of *Weierstrass points* of X.] Then, as is well-known, the closed subscheme $\mathrm{WP} \subseteq X$ is *reduced*; moreover, we have a bijection [of finite sets of cardinality 15] between

- the set of subsets of WP *of cardinality* 2

and

- the set of isomorphism classes of invertible sheaves on X *of order* 2

given by mapping $D = \{x_1, x_2\} \subseteq \mathrm{WP}$ to $\mathcal{L}_D \stackrel{\text{def}}{=} \mathcal{O}_X(x_1 - x_2)$. Finally, one verifies immediately that, for a subset $D \subseteq \mathrm{WP}$ of cardinality 2, if we write

$$\xi_D\colon\ X_D\ \longrightarrow\ X$$

for the connected finite étale double covering which trivializes $\mathcal{L}_D$ and

$$E_D$$

for the elliptic curve over k obtained by considering the double covering of Q ($\cong \mathbb{P}^1_k$) whose branch locus coincides with $\xi(\mathrm{WP} \setminus D)$, then the "*new part*" of ξ_D [i.e., the abelian variety obtained by forming the quotient of the Jacobian variety of X_D by the image — via the homomorphism induced by ξ_D — of the Jacobian variety of X] is *isogenous* to E_D.

By this observation, together with the results obtained in the present paper, we give the following complete list of *nilpotent/nilpotent admissible/nilpotent ordinary* indigenous bundles over X/S:

Theorem 6.1. — *The following hold:*

(i) *Every* **nilpotent active** [*i.e.,* **nilpotent nondormant**] *indigenous bundle over* X/S *is* **admissible**.

(ii) *Let* $D \subseteq \mathrm{WP}$ *be a subset* **of cardinality 2** *and* $\theta_D \in \Gamma(X, \omega_{X/S}^{\otimes 2})$ *a* [*uniquely determined, up to multiplication by an element of* $k^\times$] *global section of* $\omega_{X/S}^{\otimes 2}$ *such that the zero locus of* θ_D *coincides with* $2D$ [*if we naturally regard* D *as a reduced divisor of degree* 2]*, and, moreover, the elliptic curve* E_D *is* **ordinary**. *Then a* [*uniquely determined — cf. Proposition* 3.2*, Corollary* 3.4] $k^\times$*-multiple of* θ_D *corresponds, via the bijection of Corollary* 2.6*, to a* **nilpotent** [*necessarily* **admissible** — *cf.* (i)] *indigenous bundle over* X/S.

(iii) *Let* $\omega_{\mathrm{CE}} \in \Gamma(X, \omega_{X/S})$ *be a* **Cartier eigenform** *associated to* $\mathcal{O}_X$. *Then a* [*uniquely determined — cf. Proposition* 3.2*, Corollary* 3.4] $k^\times$*-multiple of* $\omega_{\mathrm{CE}} \otimes \omega_{\mathrm{CE}} \in \Gamma(X, \omega_{X/S}^{\otimes 2})$ *corresponds, via the bijection Corollary* 2.6*, to a* **nilpotent** [*necessarily* **admissible** — *cf.* (i)] *indigenous bundle over* X/S.

(iv) *Every* **nilpotent active** [*i.e.,* **admissible** — *cf.* (i)] *indigenous bundle over* X/S *is obtained as the result of either* (ii) *or* (iii).

(v) *It holds that a* **nilpotent** *indigenous bundle over* X/S *is* **ordinary** *if and only if one of the following two conditions is satisfied:*

(1) *The indigenous bundle is obtained as the result of* (ii).

(2) *The indigenous bundle is obtained as the result of* (iii)*, and, moreover,* X *is* **parabolically ordinary**.

Proof. — Assertion (i) follows from Corollary 3.6. Next, we verify assertion (ii). Let $D \subseteq \mathrm{WP}$ be as in assertion (ii). Then it is immediate that $\mathcal{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S} \cong \mathcal{O}_X(D)$ [if we naturally regard D as a reduced divisor of degree 2]. In particular, it follows that $\Gamma(X, \mathcal{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S})$ is *of dimension* 1, which thus implies that the zero locus of every nonzero global section of $\mathcal{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S}$ *coincides* with [the reduced closed subscheme of X whose underlying subset is] D. Moreover, since E_D is *ordinary*, it follows from Proposition A.6 [cf. also Remark A.8.1] that every nonzero global section of $\mathcal{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *Cartier eigenform* associated to $\mathcal{L}_D$. Thus, it follows immediately from Proposition 4.1 that assertion (ii) holds. This completes the proof of assertion (ii).

Assertion (iii) follows from Proposition 4.1. Next, we verify assertion (iv). Let P be a *nilpotent admissible* indigenous bundle over X/S. If the Hasse defect of P is *trivial*, then it follows from Theorem 5.2, (iii), that P is obtained as the result of (iii). If the Hasse defect of P is *nontrivial*, then it follows from Theorem 5.2, (iii), together with Proposition A.6, that P is obtained as the result of (ii) [cf. also the proof of assertion (ii)]. This completes the proof of assertion (iv). Assertion (v) follows from the final portion of Theorem 5.2, (i), together with assertion (iv). This completes the proof of Theorem 6.1. □

Remark 6.1.1. — It follows immediately from Proposition 3.2, together with the various definitions involved, that the *Hasse invariants* and the *supersingular divisors* of *nilpotent admissible* indigenous bundles over X/S are given as follows:

• Write P for the *nilpotent admissible* indigenous bundle over X/S obtained as the result of Theorem 6.1, (ii), with respect to a subset $D \subseteq \mathrm{WP}$ as in Theorem 6.1, (ii). Then the *supersingular divisor* of P is [the reduced closed subscheme of X whose underlying

subset is] D. Next, let us observe that $\mathcal{O}_X(D)^{\otimes 2} \cong \omega_{X/S}^{\otimes 2}$, and, moreover, the vector space $\Gamma(X, \mathcal{O}_X(D))$ over k is *of dimension* 1. Let s be a nonzero global section of $\mathcal{O}_X(D)$. Then the *Hasse invariant* of the indigenous bundle P is a $k^\times$-multiple of

$$s \ \in \Gamma(X, \mathcal{O}_X(D)).$$

• Write P for the *nilpotent admissible* indigenous bundle over X/S obtained as the result of Theorem 6.1, (iii), with respect to a global section $\omega_{\mathrm{CE}} \in \Gamma(X, \omega_{X/S})$ as in Theorem 6.1, (iii). Then the *Hasse invariant* of P is a $k^\times$-multiple of

$$\omega_{\mathrm{CE}} \ \in \Gamma(X, \omega_{X/S}).$$

The *supersingular divisor* of this indigenous bundle is the zero locus of ω_{CE}.

REMARK 6.1.2. — One verifies immediately that an indigenous bundle [implicitly] discussed in [1], §11, is a nilpotent admissible indigenous bundle obtained as the result of Theorem 6.1, (iii) [cf. the discussion in Remark 6.1.1 concerning *supersingular divisors*; also condition (2) of Theorem 6.1, (v), and the equivalence (a) $\Leftrightarrow$ (b) of [1], Theorem 2.8, (3)].

The following corollary follows immediately from Theorem 2.1 and Theorem 6.1 [cf. also Remark 6.1.1], together with elementary linear algebra [cf. also [7], p.143, Corollary]:

COROLLARY 6.2. — *Write* n_{WP} *for the number of subsets* D *of* WP *of cardinality* 2 *such that the elliptic curve* E_D *is* **ordinary**. *Write, moreover,* γ_X $(\in \{0, 1, 2\})$ *for the p-rank of the Jacobian variety of* X. *Then the following hold:*

(i) *The number of isomorphism classes of* **nilpotent** *indigenous bundles over* X/S *is given by*

$$1 + n_{\mathrm{WP}} + \sharp\mathbb{P}^{\gamma_X - 1}(\mathbb{F}_3)$$

— *where we write* $\sharp\mathbb{P}^{-1}(\mathbb{F}_3) \stackrel{\mathrm{def}}{=} 0$.

(ii) *The number of isomorphism classes of* **nilpotent admissible** *indigenous bundles over* X/S *is given by*

$$n_{\mathrm{WP}} + \sharp\mathbb{P}^{\gamma_X - 1}(\mathbb{F}_3)$$

— *where we write* $\sharp\mathbb{P}^{-1}(\mathbb{F}_3) \stackrel{\mathrm{def}}{=} 0$.

(iii) *If* X *is* **parabolically ordinary** (*respectively,* **not parabolically ordinary**) [*i.e.,* $\gamma_X = 2$ (*respectively,* $\gamma_X \neq 2$)], *then the number of isomorphism classes of* **nilpotent ordinary** *indigenous bundles over* X/S *is given by*

$$n_{\mathrm{WP}} + 4 \quad (\textit{respectively, } n_{\mathrm{WP}}).$$

Next, let us recall the following *well-known* lemma on the p-rank of the Jacobian variety of a projective smooth curve of genus ≤ 2 over an algebraically closed field of characteristic three. The following well-known lemma follows immediately from, for instance, the characterization of the *Cartier operator* [cf., e.g., [4], Theorem 7.2], together with a *well-known* explicit description of the global differential forms on a *hyperelliptic* projective smooth curve:

Lemma 6.3. — *The following hold:*

(i) *Suppose that X admits a dense open subscheme which is isomorphic to the affine scheme over k*

$$\mathrm{Spec}\big(k[s,t]/(s^2 - f(t))\big)$$

— where s and t are indeterminates, and

$$f(t) \ = \ t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \ \in \ k[t].$$

Then it holds that X is **parabolically ordinary** *if and only if $a_1 \neq a_2 \cdot a_4$.*

(ii) *If an elliptic curve E over k admits a dense open subscheme which is isomorphic to the affine scheme over k*

$$\mathrm{Spec}\big(k[s,t]/(s^2 - f(t))\big)$$

— where s and t are indeterminates, and

$$f(t) \ = \ t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \ \in \ k[t]$$

$$(\textit{respectively}, \quad f(t) \ = \ t^3 + a_2 t^2 + a_1 t + a_0 \ \in \ k[t])$$

— then it holds that E is **ordinary** *if and only if $a_2 \neq 0$.*

Corollary 6.4. — *There exists a* **dense open** *substack of $\mathcal{M}_2$ such that every projective smooth curve parametrized by a geometric point on this open substack admits* **exactly 19** *isomorphism classes of* **nilpotent ordinary** *indigenous bundles.*

Proof. — It follows from Theorem 6.1 [cf. also Corollary 6.2, (iii)] that, to verify Corollary 6.4, it suffices to verify the following assertion: There exists a *dense open* substack $U_{\mathrm{ord}} \subseteq \mathcal{M}_2$ (respectively, $U_{\mathrm{WP}} \subseteq \mathcal{M}_2$) of $\mathcal{M}_2$ such that every projective smooth curve parametrized by a geometric point on U_{ord} (respectively, U_{WP}) is *parabolically ordinary* (respectively, satisfies the condition that, for every subset $D \subseteq \mathrm{WP}$ of *cardinality* 2, the elliptic curve E_D is *ordinary*). On the other hand, the existence of "U_{ord}" (respectively, "U_{WP}") as above follows immediately from Lemma 6.3, (i) (respectively, Lemma 6.3, (ii)), together with a straightforward calculation. This completes the proof of Corollary 6.4. □

Remark 6.4.1. — Let us observe that the number "19" in the statement of Corollary 6.4 *coincides* with the result of the formula of [6], Chapter V, Corollary 1.3, (3), i.e., the formula

$$n_{2,0}^{\mathrm{ord}} \ = \ \frac{p}{3}(2p^2 + 1).$$

Finally, we prove the *existence* of a *nilpotent ordinary* indigenous bundle over X/S:

Proposition 6.5. — *The following hold:*

(i) *There exist* **at least 12** *isomorphism classes of* **nilpotent ordinary** *indigenous bundles whose Hasse defects are* **nontrivial** *over X/S.*

(ii) *There exist* **at least 13** *isomorphism classes of* **nilpotent ordinary** *indigenous bundles over X/S.*

PROOF. — Let us identify Q with $\mathbb{P}^1_k$ by an isomorphism over k. Moreover, let us naturally identify the set of closed points of $\mathbb{P}^1_k = Q$ with the set $k \cup \{\infty\}$.

First, I claim that the following assertion holds:

> Claim 6.5.A: If the number "n_{WP}" defined in the statement of Corollary 6.2 is ≤ 12, then, by considering a suitable automorphism of $\mathbb{P}^1_k = Q$, one may take the subset "WP" of $\mathbb{P}^1_k = Q$ to be
>
> $$\{0, 1, \infty, -1, \alpha, -\alpha\} \subseteq k \cup \{\infty\}$$
>
> — where $\alpha \in k$ satisfies that $\alpha^2 = 2$ — i.e., X admits a dense open subscheme which is isomorphic to the affine scheme over k
>
> $$\mathrm{Spec}\big(k[s,t]/(s^2 - f(t))\big)$$
>
> — where s and t are indeterminates, and
>
> $$f(t) = x(x-1)(x+1)(x-\alpha)(x+\alpha) = x^5 + 2x \in k[t].$$

Indeed, suppose that there exist 3 distinct subsets D_1, D_2, $D_3 \subseteq \mathrm{WP}$ of *cardinality* 2 such that the elliptic curves E_{D_1}, E_{D_2}, and E_{D_3} are *not ordinary*.

First, we consider the case where $\mathrm{WP} = D_1 \cup D_2 \cup D_3$. Then let us observe that we may assume without loss of generality, by considering a suitable automorphism of $\mathbb{P}^1_k = Q$, that $\xi(D_1) = \{0, \infty\}$ and $1 \in \xi(D_2)$. Then since E_{D_2} and E_{D_3} are *not ordinary*, it follows from Lemma 6.3, (ii), that there exists an element $\alpha \in k \setminus \{0, 1, -1\}$ such that

$$\xi(D_2) = \{1, -1\}, \quad \xi(D_3) = \{\alpha, -\alpha\}.$$

On the other hand, since

$$\alpha \cdot (-\alpha) + \alpha \cdot 1 + \alpha \cdot (-1) + (-\alpha) \cdot 1 + (-\alpha) \cdot (-1) + 1 \cdot (-1) = -\alpha^2 - 1,$$

and E_{D_1} is *not ordinary*, it follows from Lemma 6.3, (ii), that $\alpha^2 = 2$. Thus, one may take the subset "WP" to be as in Claim 6.5.A.

Next, we consider the case where $\mathrm{WP} \neq D_1 \cup D_2 \cup D_3$. Then let us observe that we may assume without loss of generality, by considering a suitable automorphism of $\mathbb{P}^1_k = Q$, that $\infty \in \xi(\mathrm{WP} \setminus (D_1 \cup D_2 \cup D_3))$, that $0 \in \xi(\mathrm{WP} \setminus (D_1 \cup D_2))$, and that $1 \in \xi(D_1 \setminus (D_1 \cap D_2))$. Then since E_{D_1} and E_{D_2} are *not ordinary*, it follows from Lemma 6.3, (ii), that there exists an element $\alpha \in k \setminus \{0, 1, -1\}$ such that

$$\xi(\mathrm{WP} \setminus D_1) = \{0, \infty, \alpha, -\alpha\}, \quad \xi(\mathrm{WP} \setminus D_2) = \{0, \infty, 1, -1\},$$

which thus implies that

$$\xi(\mathrm{WP}) = \{0, 1, -1, \alpha, -\alpha, \infty\}.$$

Thus, since $\infty \notin \xi(D_3)$, one verifies easily from Lemma 6.3, (ii), that E_{D_3} is *ordinary* — in *contradiction* to our assumption that E_{D_3} is *not ordinary*. This completes the proof of Claim 6.5.A.

Now we verify assertion (i). Suppose that the number "n_{WP}" defined in the statement of Corollary 6.2 is ≤ 12. Then it follows from Claim 6.5.A that one may take the subset "WP" to be as in Claim 6.5.A. In particular, it follows from Lemma 6.3, (ii), together with a straightforward calculation, that the number "n_{WP}" defined in the statement of Corollary 6.2 is *equal to* 12. Thus, assertion (i) follows from Theorem 6.1. This completes the proof of assertion (i).

Next, we verify assertion (ii). Assume that the set of isomorphism classes of *nilpotent ordinary* indigenous bundles over X/S is *of cardinality* ≤ 12. Then it follows from Corollary 6.2, (iii), that the number "n_{WP}" defined in the statement of Corollary 6.2 is ≤ 12. Thus, it follows from Claim 6.5.A that one may take the subset "WP" to be as in Claim 6.5.A. Then it follows from Lemma 6.3, (ii), together with a straightforward calculation, that the number "n_{WP}" defined in the statement of Corollary 6.2 is *equal to* 12. Moreover, since [it follows from Lemma 6.3, (i), that] X is *parabolically ordinary*, it follows from Corollary 6.2, (iii), that X/S admits *exactly* $16\ (= 12+4) > 12$ isomorphism classes of *nilpotent ordinary* indigenous bundles — in *contradiction* to our assumption that the set of isomorphism classes of *nilpotent ordinary* indigenous bundles over X/S is *of cardinality* ≤ 12. This completes the proof of assertion (ii). □

It follows from Proposition 6.5, (ii), together with [5], Chapter II, Proposition 3.4, that the following corollary holds:

Corollary 6.6. — *Every projective smooth curve of genus* 2 *over a connected noetherian scheme of characteristic* 3 *is* **hyperbolically ordinary** [*cf.* [5]*, Chapter II, Definition 3.3*].

Remark 6.6.1. — Corollary 6.6 yields a *positive partial answer* to the basic question (1) of [6], Introduction, §2.1. By Corollary 6.6, we conclude that the image of the natural (1-)morphism discussed in the basic question (1) of [6], Introduction, §2.1, in the case where $(g, r, p) = (2, 0, 3)$ *contains* the open substack $(\mathcal{M}_{2,0})_{\mathbb{F}_3} \subseteq (\overline{\mathcal{M}}_{2,0})_{\mathbb{F}_3}$.

Remark 6.6.2. — Corollary 6.6 also yields an example of a *projective* smooth curve of positive characteristic which is *not parabolically ordinary* but *hyperbolically ordinary* [cf. also Lemma 6.3, (i)].

Appendix A. Cartier Operator Associated to a Square-trivialized Invertible Sheaf

In the Appendix A, let us recall the *Cartier operator* associated to a *square-trivialized* invertible sheaf on a projective smooth curve in positive characteristic. It seems to the author that the content of the Appendix A is *well-known*; however, since a suitable literature could not be found, the author has decided to discuss it in the Appendix A.

In the Appendix A, let p be an *odd* prime number, $g \geq 2$ an integer, S a connected noetherian scheme of characteristic p [i.e., over $\mathbb{F}_p$], and $f\colon X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g. Write $f^F\colon X^F \to S$ for the projective smooth curve obtained by base-changing f via the absolute Frobenius morphism of S and $\Phi\colon X \to X^F$ for the relative Frobenius morphism over S. We shall use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

PROPOSITION A.1. — *Let $\mathcal{L}$ be an invertible sheaf on X such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$. Then the following conditions are equivalent:*

(1) *The restriction of $\mathcal{L}$ to every fiber of f is* **of order two** [*i.e., is* **nontrivial**].

(2) *There exists a point $s \in S$ of S such that the restriction of $\mathcal{L}$ to the fiber of f at s is* **of order two** [*i.e., is* **nontrivial**].

(3) *The invertible sheaf $\mathcal{L}$ does* **not arise** *from an invertible sheaf on S.*

(4) *The image of the classifying morphism of $\mathcal{L}$ [from S to the relative Jacobian variety of X/S] does* **not intersect** *the image of the identity section.*

PROOF. — The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are immediate. The implication (3) $\Rightarrow$ (4) follows immediately from our assumption that S is *connected*, together with the [well-known] fact that the endomorphism of the relative Jacobian variety of X/S obtained by multiplication by 2 is *finite étale*. This completes the proof of Proposition A.1. □

DEFINITION A.2. — Let $\mathcal{L}$ be an invertible sheaf on X such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$. Then we shall say that $\mathcal{L}$ is *of relative order two* (respectively, *one*) if $\mathcal{L}$ satisfies (respectively, does not satisfy) the four conditions in the statement of Proposition A.1.

DEFINITION A.3. — We shall refer to a pair

$$(\mathcal{L}, \tau\colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X)$$

consisting of an invertible sheaf $\mathcal{L}$ on X and a global trivialization τ of the square $\mathcal{L}^{\otimes 2}$ of $\mathcal{L}$ as a *square-trivialized invertible sheaf* on X.

Let

$$\mathcal{L} \ = \ (\mathcal{L}, \tau\colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X)$$

be a *square-trivialized invertible sheaf* on X. Thus, the trivialization τ determines isomorphisms of invertible sheaves on X

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\sim} & \mathcal{L}^{\otimes p} & \xrightarrow{\sim} & \Phi^*\mathcal{L}^F \\ \tau(l \otimes l)^{\frac{p-1}{2}} \cdot l & \mapsto & l^{\otimes p} & \mapsto & \Phi^{-1} l^F \end{array}$$

— where we write $\mathcal{L}^F$ for the invertible sheaf on X^F obtained by pulling back $\mathcal{L}$ via the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S, l is a local section of $\mathcal{L}$, and l^F is the local section of $\mathcal{L}^F$ determined by l.

Let us recall [cf., e.g., [4], Theorem 7.2] that we have an exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0$$

— where we write d for the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$, and c for the Cartier operator. We shall write

$$\mathcal{B}_\circ \stackrel{\text{def}}{=} \text{Coker}(\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X) = \text{Ker}(c\colon \Phi_*\omega_{X/S} \to \omega_{X^F/S})$$

for the *locally free coherent $\mathcal{O}_{X^F}$-module of rank $p-1$* obtained by forming the cokernel of the natural homomorphism $\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X$, or, alternatively, the kernel of the Cartier

operator $c\colon \Phi_*\omega_{X/S} \to \omega_{X^F/S}$. Then, by tensoring with $\mathcal{L}^F$ and applying the above isomorphism $\mathcal{L} \xrightarrow{\sim} \Phi^*\mathcal{L}^F$ determined by τ, we obtain an exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}_\circ \longrightarrow \Phi_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow \mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S} \longrightarrow 0,$$

which thus determines an exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}_\circ) \longrightarrow f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}).$$

Definition A.4. — We shall write

$$C_{\mathcal{L}}$$

for the third arrow of the above exact sequence of $\mathcal{O}_S$-modules and refer to

$$C_{\mathcal{L}}\colon\ f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})$$

as the *Cartier operator* associated to $\mathcal{L} = (\mathcal{L}, \tau)$.

Remark A.4.1. — If we take the pair "$(\mathcal{L}, \tau)$" to be the pair consisting of $\mathcal{O}_X$ and the natural identification $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{O}_X$, then the Cartier operator $f_*\omega_{X/S} \to f_*^F\omega_{X^F/S}$ associated to $(\mathcal{L}, \tau)$ *coincides* with the [homomorphism induced by the] *usual* Cartier operator.

Remark A.4.2. — One verifies easily that since the formation of

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0$$

commutes with arbitrary change of base "$S' \to S$", the formation of

$$0 \longrightarrow f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}_\circ) \longrightarrow f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \xrightarrow{C_{\mathcal{L}}} f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})$$

commutes with arbitrary change of base "$S' \to S$".

If [the underlying invertible sheaf of] $\mathcal{L}$ is *of relative order two*, then we shall write

$$\xi_{\mathcal{L}}\colon\ X_{\mathcal{L}} \longrightarrow X$$

for the connected finite étale *double* covering of X over S which *trivializes* the invertible sheaf $\mathcal{L}$ [determined by τ] and

$$\xi_{\mathcal{L}}^F\colon\ X_{\mathcal{L}}^F \longrightarrow X^F$$

for the connected finite étale covering of X^F over S obtained by base-changing $\xi_{\mathcal{L}}$ via the absolute Frobenius morphism of S. Thus, a trivialization of $\xi_{\mathcal{L}}^*\mathcal{L}$ determines respective isomorphisms of $\mathcal{O}_X$-, $\mathcal{O}_{X^F}$-modules

$$(\xi_{\mathcal{L}})_*\mathcal{O}_{X_{\mathcal{L}}} \cong \mathcal{O}_X \oplus \mathcal{L}, \quad (\xi_{\mathcal{L}}^F)_*\mathcal{O}_{X_{\mathcal{L}}^F} \cong \mathcal{O}_{X^F} \oplus \mathcal{L}^F.$$

Moreover, one verifies immediately that the natural homomorphism of $\mathcal{O}_S$-modules

$$\mathbb{R}^1(f^F \circ \xi_{\mathcal{L}}^F)_*\mathcal{O}_{X_{\mathcal{L}}^F} \longrightarrow \mathbb{R}^1(f \circ \xi_{\mathcal{L}})_*\mathcal{O}_{X_{\mathcal{L}}}$$

determined by the relative Frobenius morphism $X_{\mathcal{L}} \to X_{\mathcal{L}}^F$ over S is *decomposed* into the direct sum of the natural homomorphisms of $\mathcal{O}_S$-modules

$$\mathbb{R}^1 f_*^F\mathcal{O}_{X^F} \longrightarrow \mathbb{R}^1 f_*\mathcal{O}_X, \quad \mathbb{R}^1 f_*^F\mathcal{L}^F \longrightarrow \mathbb{R}^1 f_*\Phi^*\mathcal{L}^F \xleftarrow{\sim} \mathbb{R}^1 f_*\mathcal{L}$$

[cf. the isomorphism given in the discussion following Definition A.3].

DEFINITION A.5.

(i) We shall say that $f\colon X \to S$ is *parabolically ordinary* [cf. the discussion following [5], Chapter II, Definition 3.3] if the Jacobian variety of every fiber of f is ordinary.

(ii) Let $Y \to S$ be a projective smooth curve over S and $\xi\colon Y \to X$ a finite étale covering over S. Then we shall say that $\xi\colon Y \to X$ is *parabolically new-ordinary* if, for every point $s \in S$ of S, the "new part" of ξ at s [i.e., the abelian variety over s obtained by forming the quotient of the Jacobian variety of $Y \times_S s$ by the image — via the homomorphism induced by ξ — of the Jacobian variety of $X \times_S s$] is ordinary.

Thus, we obtain:

PROPOSITION A.6. — *It holds that the Cartier operator* $C_{\mathcal{L}}\colon f_*(\mathcal{L}\otimes_{\mathcal{O}_X}\omega_{X/S}) \to f^F_*(\mathcal{L}^F\otimes_{\mathcal{O}_{X^F}}\omega_{X^F/S})$ *associated to* $\mathcal{L} = (\mathcal{L}, \tau)$ *is* **injective** *if and only if one of the following conditions is satisfied:*

(1) $\mathcal{L}$ *is* **of relative order one**, *and* X/S *is* **parabolically ordinary**.

(2) $\mathcal{L}$ *is* **of relative order two**, *and the connected finite étale double covering* $\xi_{\mathcal{L}}\colon X_{\mathcal{L}} \to X$ *is* **parabolically new-ordinary**.

PROOF. — Let us first observe that it follows from Remark A.4.2 that, to verify Proposition A.6, we may assume without loss of generality that S is the spectrum of an algebraically closed field. Next, let us recall from the discussion preceding Definition A.4 that we have an isomorphism of $\mathcal{O}_S$-modules

$$f^F_*(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}_\circ) \xrightarrow{\sim} \mathrm{Ker}(C_{\mathcal{L}}).$$

In particular, the exact sequence of $\mathcal{O}_{X^F}$-modules in the discussion preceding Definition A.4

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0,$$

together with the isomorphism $\mathcal{L} \xrightarrow{\sim} \Phi^*\mathcal{L}^F$ given in the discussion following Definition A.3, determines an exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \mathrm{Ker}(C_{\mathcal{L}}) \longrightarrow \mathbb{R}^1 f^F_*\mathcal{L}^F \longrightarrow \mathbb{R}^1 f_*\mathcal{L}.$$

Thus, it follows from the discussion preceding Definition A.5 that Proposition A.6 holds. This completes the proof of Proposition A.6. □

DEFINITION A.7. — Let $\mathcal{M}$ be an invertible sheaf on X such that $\mathcal{M}^{\otimes 2} \cong \mathcal{O}_X$. Then we shall say that $\mathcal{M}$ is *parabolically ordinary* if, for some [or, equivalently, every] trivialization $\tau_{\mathcal{M}}\colon \mathcal{M}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$, the square-trivialized invertible sheaf $(\mathcal{M}, \tau_{\mathcal{M}})$ satisfies either (1) or (2) in the statement of Proposition A.6 [i.e., the Cartier operator associated to $(\mathcal{M}, \tau_{\mathcal{M}})$ is injective — cf. Proposition A.6].

Next, let us observe that the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S determines a *Frobenius-semi-linear* homomorphism

$$f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}).$$

For a global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$, we shall write u^F for the global section of $\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}$ obtained by forming the image of u via this Frobenius-semi-linear homomorphism.

Definition A.8.

(i) We shall say that a global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *normalized Cartier eigenform* associated to $\mathcal{L} = (\mathcal{L}, \tau)$ if u defines a relative effective Cartier divisor of X/S, and, moreover, $C_{\mathcal{L}}(u) = -u^F$.

(ii) Let $\mathcal{M}$ be an invertible sheaf on X such that $\mathcal{M}^{\otimes 2} \cong \mathcal{O}_X$. Then we shall say that a global section of $\mathcal{M} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *Cartier eigenform* associated to $\mathcal{M}$ if the global section is a normalized Cartier eigenform associated to the square-trivialized invertible sheaf obtained by equipping $\mathcal{M}$ with a trivialization of the square of $\mathcal{M}$.

Remark A.8.1. — One verifies immediately that if S is the spectrum of an algebraically closed field k [of characteristic p], then it holds that a global section $u \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ is a *Cartier eigenform* associated to [the underlying invertible sheaf of] $\mathcal{L}$ if and only if $u \neq 0$, and, moreover, $C_{\mathcal{L}}(u)$ is a $k^\times$-multiple of u^F; moreover, in this case, the subset of $k^\times$ consisting of $c \in k^\times$ such that cu is a *normalized Cartier eigenform* associated to $\mathcal{L} = (\mathcal{L}, \tau)$ forms an $\mathbb{F}_p^\times$*-torsor*, which thus implies that this subset is *of cardinality* $p - 1$.

Remark A.8.2. — If we take the pair "$(\mathcal{L}, \tau)$" to be the pair consisting of $\mathcal{O}_X$ and the natural identification $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{O}_X$ [i.e., if we are in the situation of Remark A.4.1], then the property of being a [*normalized*] *Cartier eigenform* is closely related to the property of being *locally logarithmic* [cf., e.g., [3], Théorème 2.1.17].

Finally, we consider a local criterion for a *normalized Cartier eigenform*. Let $x \in X$ be a point of X, $t_x = t \in \mathcal{O}_X$ a local parameter of X/S at x, $l_x = l \in \mathcal{L}$ a local trivialization of $\mathcal{L}$ at x, and

$$\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$$

a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$. Then the global trivialization τ and the local trivialization $l_x = l$ determine a local unit

$$\delta_x = \delta \stackrel{\text{def}}{=} \tau(l \otimes l) \in \mathcal{O}_X^\times$$

at x. Moreover, the global section χ determines a local function $\phi_x = \phi \in \mathcal{O}_X$ on X at x which fits into the equality

$$\chi = \phi \cdot l \otimes dt$$

at x. Then it follows immediately from the characterization of the *Cartier operator* [cf., e.g., [4], Theorem 7.2; also the discussion given in [3], §2.1 — especially, the equality (2.1.13) in [3], §2.1], together with the explicit description of the isomorphism $\mathcal{L} \xrightarrow{\sim} \Phi^* \mathcal{L}^F$ given in the discussion following Definition A.3, that the following lemma holds:

LEMMA A.9. — *Write ∂_{t_x} for the local derivation corresponding to the local trivialization of $\tau_{X/S}$ which maps dt_x to 1 [i.e., "$\partial_{t_x}(-)$" is the "derivative of $(-)$ with respect to t_x"]. Then the following hold:*

(i) *It holds that the global section χ of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is* **annihilated** *by the Cartier operator $C_{\mathcal{L}}$ associated to $\mathcal{L} = (\mathcal{L}, \tau)$ if and only if, for every point $x \in X$, the equality*

$$\overbrace{(\partial_{t_x} \circ \cdots \circ \partial_{t_x})}^{p-1}(\delta_x^{-\frac{p-1}{2}} \cdot \phi_x) \ = \ 0$$

holds.

(ii) *It holds that the global section χ of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a* **normalized Cartier eigenform** *associated to $\mathcal{L} = (\mathcal{L}, \tau)$ if and only if, for every point $x \in X$, the local function ϕ_x is* **not a zero-divisor**, *and, moreover, the equality*

$$\phi_x^p \ = \ \overbrace{(\partial_{t_x} \circ \cdots \circ \partial_{t_x})}^{p-1}(\delta_x^{-\frac{p-1}{2}} \cdot \phi_x)$$

holds.

APPENDIX B. THE HASSE BUNDLE OF A NILPOTENT ADMISSIBLE INDIGENOUS BUNDLE

In the Appendix B, we discuss the *Hasse bundle* of a *nilpotent admissible* indigenous bundle. In the Appendix B, let p be an *odd* prime number, $g \geq 2$ an integer, S a connected noetherian scheme of characteristic p [i.e., over $\mathbb{F}_p$], and $f\colon X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g. Write $f^F\colon X^F \to S$ for the projective smooth curve obtained by base-changing f via the absolute Frobenius morphism of S and $\Phi\colon X \to X^F$ for the relative Frobenius morphism over S. We shall use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

Let

$$P \ = \ (\pi\colon P \to X, \nabla_P)$$

be a *nilpotent admissible* [cf. [5], Chapter II, Definition 2.4] indigenous bundle [cf. [5], Chapter I, Definition 2.2] over X/S. Write

$$\sigma_{\mathrm{Hdg}}\colon \ X \ \longrightarrow \ P$$

for the *Hodge section* of P [cf. [5], Chapter I, Proposition 2.4] and

$$\mathcal{I}_{\mathrm{Hdg}} \ \subseteq \ \mathcal{O}_P$$

for the ideal of $\mathcal{O}_P$ which defines the section σ_{Hdg}. Thus, it follows from the definition of an indigenous bundle that the Kodaira-Spencer homomorphism at σ_{Hdg} relative to ∇_P [i.e., the homomorphism obtained by differentiating σ_{Hdg} by means of ∇_P]

$$\sigma_{\mathrm{Hdg}}^* \omega_{P/X} \ \longrightarrow \ \omega_{X/S}$$

is an *isomorphism*.

Proposition B.1. — *There exists a* **unique section** $\sigma_{\mathrm{con}}\colon X \to P$ *of* $\pi\colon P \to X$ *which satisfies the following conditions:*

(1) *The section* σ_{con} *is* **horizontal** *with respect to* ∇_P. *In particular, the connection* ∇_P *induces a connection on the invertible sheaf* $\sigma_{\mathrm{con}}^*\omega_{P/X}$ *on* X.

(2) *There exists a* **horizontal** *isomorphism* $\sigma_{\mathrm{con}}^*\omega_{P/X} \cong \Phi^*\tau_{X^F/S}$, *where we regard* $\Phi^*\tau_{X^F/S}$ *as an invertible sheaf equipped with a connection by equipping* $\Phi^*\tau_{X^F/S}$ *with the connection arising from the exterior differentiation operator* $\mathcal{O}_X \to \omega_{X/S}$.

We shall refer to this section σ_{con} *as the* **conjugate section** *of the indigenous bundle* P.

Proof. — This follows from the second paragraph of [the statement of] [5], Chapter II, Proposition 2.5. □

Write

$$\sigma_{\mathrm{con}}\colon\ X\ \longrightarrow\ P$$

for the *conjugate section* of P and

$$\mathcal{I}_{\mathrm{con}}\ \subseteq\ \mathcal{O}_P$$

for the ideal of $\mathcal{O}_P$ which defines the section σ_{con}.

Definition B.2. — We shall refer to the invertible sheaf on X

$$\sigma_{\mathrm{Hdg}}^*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P)\ =\ \mathcal{H}om_{\mathcal{O}_X}(\sigma_{\mathrm{Hdg}}^*\mathcal{I}_{\mathrm{con}}, \mathcal{O}_X)$$

obtained by pulling back $\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P)$ via σ_{Hdg} as the *Hasse bundle* of P. We shall refer to the invertible sheaf on X

$$\sigma_{\mathrm{Hdg}}^*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P) \otimes_{\mathcal{O}_X} \tau_{X/S}^{\otimes\frac{p-1}{2}}\ =\ \mathcal{H}om_{\mathcal{O}_X}(\sigma_{\mathrm{Hdg}}^*\mathcal{I}_{\mathrm{con}}, \tau_{X/S}^{\otimes\frac{p-1}{2}})$$

obtained by tensoring the Hasse bundle with $\tau_{X/S}^{\otimes\frac{p-1}{2}}$ as the *Hasse defect* of P.

Write

$$\mathcal{H}_P\ \stackrel{\mathrm{def}}{=}\ \sigma_{\mathrm{Hdg}}^*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P)$$

for the *Hasse bundle* of P. Then let us observe that since $\pi\colon P \to X$ is *of genus zero*, and the invertible sheaf on P

$$\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}})$$

is *of relative degree* 0 over X, it follows immediately that the natural homomorphisms

$$\pi_*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}})\ \longrightarrow\ \sigma_{\mathrm{Hdg}}^*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}})\ \cong\ \mathcal{H}_P \otimes_{\mathcal{O}_X} \omega_{X/S},$$

$$\pi_*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}})\ \longrightarrow\ \sigma_{\mathrm{con}}^*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}})\ \cong\ \sigma_{\mathrm{con}}^*\mathcal{I}_{\mathrm{Hdg}} \otimes_{\mathcal{O}_X} \Phi^*\omega_{X^F/S}$$

[cf. the discussion preceding Proposition B.1; Proposition B.1, (2)] are *isomorphisms*. Thus, by means of the natural identification $\sigma_{\mathrm{Hdg}}^*\mathcal{I}_{\mathrm{con}} = \sigma_{\mathrm{con}}^*\mathcal{I}_{\mathrm{Hdg}}$, we obtain:

Proposition B.3. — *There exist isomorphisms of invertible sheaves on* X

$$\mathcal{H}_P^{\otimes 2}\ \cong\ \mathcal{H}om_{\mathcal{O}_X}(\Phi^*\tau_{X^F/S}, \tau_{X/S})\ \cong\ \omega_{X/S}^{\otimes p-1}.$$

In particular, the **square of the Hasse defect** *is* **trivial**.

Moreover, we obtain:

PROPOSITION B.4. — *The global section of*

$$\mathcal{H}om_{\mathcal{O}_X}(\Phi^*\tau_{X^F/S}, \tau_{X/S})$$

obtained, relative to the isomorphism of Proposition B.3*, by forming the square of the global section of $\mathcal{H}_P$ determined by the natural inclusion $\mathcal{I}_{\mathrm{con}} \hookrightarrow \mathcal{O}_P$ coincides, up to multiplication by a global section of $\mathcal{O}_S^\times$, with the* **square Hasse invariant** *of P [cf.* [5]*, Chapter II, Proposition 2.6,* (1)*]. In particular, the global section of $\mathcal{H}_P$ determined by the natural inclusion $\mathcal{I}_{\mathrm{con}} \hookrightarrow \mathcal{O}_P$ coincides, up to multiplication by a global section of $\mathcal{O}_S^\times$, with the* **Hasse invariant** *of P [cf.* [5]*, Chapter II, Proposition 2.6,* (3)*].*

PROOF. — This follows from the discussion in the proof of [5], Chapter II, Proposition 2.6, (3). □

APPENDIX C. VARIOUS MODULI STACKS

In the Appendix C, we consider various moduli stacks related to the notions discussed in the present paper. In the Appendix C, let p be an *odd* prime number and $g \geq 2$ an integer.

We shall write

$$\mathcal{M}_g$$

for the moduli stack of projective smooth curves of genus g of characteristic p;

$$\mathcal{C}_g \longrightarrow \mathcal{M}_g$$

for the universal curve over $\mathcal{M}_g$;

$$\mathcal{J}_g \longrightarrow \mathcal{M}_g$$

for the relative Jacobian variety of $\mathcal{C}_g \to \mathcal{M}_g$;

$$\mathcal{J}_g[n] \subseteq \mathcal{J}_g$$

for the kernel of the endomorphism of $\mathcal{J}_g$ over $\mathcal{M}_g$ obtained by multiplication by n [where n is a nonnegative integer]. Moreover, we shall write

$$\mathcal{N}_g$$

for the moduli stack of smooth *nilcurves* [cf. the discussion preceding [6], Introduction, Theorem 0.1] of genus g of characteristic p, i.e., the moduli stack of projective smooth curves of genus g of characteristic p equipped with *nilpotent indigenous bundles*;

$$\mathcal{N}_g^{\mathrm{adm}} \subseteq \mathcal{N}_g$$

for the moduli stack of projective smooth curves of genus g of characteristic p equipped with *nilpotent admissible indigenous bundles*;

$$\mathcal{N}_g^{\mathrm{ord}} \subseteq \mathcal{N}_g^{\mathrm{adm}}$$

for the moduli stack of projective smooth curves of genus g of characteristic p equipped with *nilpotent ordinary indigenous bundles*.

Definition C.1. — It follows from the final portion of Proposition B.3 that the *Hasse defect* of the universal nilpotent admissible indigenous bundle over $\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{N}_g^{\mathrm{adm}} \to \mathcal{N}_g^{\mathrm{adm}}$ determines a (1-)morphism over $\mathcal{M}_g$

$$\mathcal{N}_g^{\mathrm{adm}} \longrightarrow \mathcal{J}_g[2].$$

We shall refer to this (1-)morphism as the *Hasse defect morphism*.

Proposition C.2. — *The following three open substacks of* $\mathcal{N}_g^{\mathrm{adm}}$ **coincide***:*

(1) *The open substack* $\mathcal{N}_g^{\mathrm{ord}} \subseteq \mathcal{N}_g^{\mathrm{adm}}$.

(2) *The* **étale locus** *of the natural* (1-)*morphism* $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{M}_g$.

(3) *The* **étale locus** *of the Hasse defect morphism* $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{J}_g[2]$.

Proof. — The assertion that the open substack given in (1) *coincides* with the open substack given in (2) follows from the definition. On the other hand, since the (1-)morphism $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{M}_g$ is *flat* [cf. [5], Chapter II, Theorem 2.3], the assertion that the open substack given in (2) *coincides* with the open substack given in (3) follows from the well-known fact that the natural (1-)morphism $\mathcal{J}_g[2] \to \mathcal{M}_g$ is a *finite étale surjection*. □

Now let us observe that since, as is well-known, $\mathcal{J}_g[2]$ is *finite étale* over $\mathcal{M}_g$, the identity section of $\mathcal{J}_g \to \mathcal{M}_g$ determines an isomorphism of stacks over $\mathcal{M}_g$

$$\mathcal{M}_g \sqcup (\mathcal{J}_g[2] \setminus \mathcal{J}_g[1]) \xrightarrow{\sim} \mathcal{J}_g[2].$$

Thus, by considering the Hasse defect morphism, we obtain:

Proposition C.3. — *Let* $U \to \mathcal{N}_g^{\mathrm{adm}}$ *be a scheme over* $\mathcal{N}_g^{\mathrm{adm}}$*. Suppose that there exist two geometric points* s_1, s_2 *of* U *such that the Hasse defect of the nilpotent admissible indigenous bundle corresponding to* s_1 (*respectively,* s_2) *is* **of relative order one** (*respectively,* **two**)*. Then* U *is* **not connected**.

Next, we shall write

$$\mathcal{R}_g$$

for the moduli stack of "*nontrivial*" smooth *Prym curves* of genus g of characteristic p, i.e., the moduli stack of projective smooth curves of genus g of characteristic p equipped with square-trivialized invertible sheaves whose underlying invertible sheaves are of relative order two;

$$\underline{\mathcal{R}}_g \stackrel{\mathrm{def}}{=} \mathcal{J}_g[2] \setminus \mathcal{J}_g[1].$$

Thus, we have a natural (1-)morphism $\mathcal{R}_g \to \underline{\mathcal{R}}_g$ over $\mathcal{M}_g$. For a nonnegative integer d, write

$$\mathcal{A}_d$$

for the moduli stack of principally polarized abelian varieties of dimension d of characteristic p and

$$\mathcal{A}_d^{\mathrm{ord}} \subseteq \mathcal{A}_d$$

for the moduli stack of principally polarized *ordinary* abelian varieties of dimension d of characteristic p.

DEFINITION C.4. — Since $\mathcal{M}_g \sqcup \underline{\mathcal{R}}_g$ is naturally isomorphic to $\mathcal{J}_g[2]$ over $\mathcal{M}_g$, the inverse image of $\mathcal{A}_g^{\mathrm{ord}} \subseteq \mathcal{A}_g$ via the *Torelli morphism* $\mathcal{M}_g \to \mathcal{A}_g$ and the image in $\underline{\mathcal{R}}_g$ of the inverse image of $\mathcal{A}_{g-1}^{\mathrm{ord}} \subseteq \mathcal{A}_{g-1}$ via the *Prym morphism* $\mathcal{R}_g \to \mathcal{A}_{g-1}$ determine an open substack of $\mathcal{J}_g[2]$. We shall write

$$\mathcal{J}_g[2]^{\text{pb-ord}} \subseteq \mathcal{J}_g[2]$$

for this open substack.

Thus, it follows immediately from the various definitions involved that the following proposition holds:

PROPOSITION C.5. — *In the notation introduced at the beginning of the Appendix B, let P be a* **nilpotent admissible** *indigenous bundle over X/S. Then the following conditions are equivalent:*

(1) *The image of the composite*

$$S \longrightarrow \mathcal{N}_g^{\mathrm{ord}} \longrightarrow \mathcal{J}_g[2]$$

of the classifying (1-)*morphism $S \to \mathcal{N}_g^{\mathrm{ord}}$ of P and the Hasse defect morphism is* **contained** *in the open substack $\mathcal{J}_g[2]^{\text{pb-ord}} \subseteq \mathcal{J}_g[2]$.*

(2) *The Hasse defect of P is* **parabolically ordinary**.

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