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Indecomposability of Anabelian Profinite Groups

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Abstract. Classically, it is well-known that various **anabelian profinite groups**, i.e., profinite groups which appear in *anabelian geometry*, are **center-free**. In this paper, we study the **indecomposability** — which is also a *group-theoretic property* of profinite groups — of various **anabelian profinite groups**. For instance, we prove that the étale fundamental group of the *configuration space of a hyperbolic curve* over either a *p-adic local field* or a *number field*, as well as the étale fundamental group of an *affine smooth curve* over an *algebraically closed field of positive characteristic*, are **indecomposable**. Finally, we consider the topic of indecomposability in the context of the theory of *combinatorial anabelian geometry* and pose the question: Is the **Grothendieck-Teichmüller group** GT indecomposable? We give an *affirmative answer* to a *pro-l version* of this question.

Contents

Introduction

Classically, it is well-known that various **anabelian profinite groups**, i.e., profinite groups which appear in *anabelian geometry*, are **center-free**. For instance,

- the absolute Galois group of a *sub-p-adic field* [i.e., a field which is isomorphic to a subfield of a finitely generated extension field of $\mathbb{Q}_p$] is *center-free* [cf. [16], Lemma 15.8]
- the étale fundamental group of a *hyperbolic curve* over an *algebraically closed field* is *center-free* [cf., e.g., Proposition 2.4].

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In this paper, we study the **indecomposability** of various *anabelian profinite groups*. The term *indecomposability* is defined as follows [cf. Definition 1.1]:

> We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \cong G_1 \times G_2$, where G_1, G_2 are profinite groups, it follows that either G_1 or G_2 is the trivial group.

For instance, in the case of *absolute Galois groups*, the following fact is known [cf. Theorem 1.2]:

Fact. *Let k be a* **Hilbertian field** *[cf. [FJ], Chapter 12]. Then the absolute Galois group G_k of k is* **indecomposable***.*

In particular, the absolute Galois group of

(i) a *finitely generated extension field* of $\mathbb{Q}$
(ii) a *finitely generated transcendental extension field* of $\mathbb{Q}_p$
(iii) a *finitely generated transcendental extension field* of $\mathbb{F}_p$

is *indecomposable* [cf. Corollary 1.4]. Here, we note that any *p-adic local field* [i.e., a finite extension field of $\mathbb{Q}_p$] is *non-Hilbertian* [cf. Remark 1.3]. But we can prove that for any p-adic local field k, the absolute Galois group G_k of k is also *indecomposable* [cf. Proposition 1.6]. On the other hand, any *finite field* is also *non-Hilbertian* [cf. Remark 1.3], but its absolute Galois group [$\cong \widehat{\mathbb{Z}}$] is clearly *decomposable*!

Now we consider the case of *étale fundamental groups of curves.* For a connected noetherian scheme $(-)$, we shall write

$$\Pi_{(-)}$$

for the étale fundamental group of $(-)$ [for some choice of basepoint]. First, we prove the following theorem [cf. Theorems 2.1, 2.2] which concerns the case where the base field is *algebraically closed.*

Theorem A. *Let k be an* **algebraically closed field***; X a* **smooth curve** *of type (g, r) such that the pair (g, r) satisfies $2g - 2 + r > 0$ (respectively, $(g, r) \neq (0, 0), (1, 0)$) if the characteristic of k is zero (respectively, positive). Then Π_X is* **indecomposable***.*

The characteristic zero case of Theorem A is shown in [22], Proposition 3.2.

Next, we consider the case that the base field is *non-algebraically closed.* Let k be a field of characteristic $p \geq 0$; $l \neq p$ a prime number. Then for the pair (k, l), we consider the following condition:

$(*^l_k)$ For any *finite extension field k'* of k, the *l-adic cyclotomic character* $\chi_{k'} : G_{k'} \to \mathbb{Z}_l^\times$ of k' is *nontrivial.*

We shall say that k is **l-cyclotomically full** if the pair (k, l) satisfies the condition $(*^l_k)$ [cf. Definition 3.2].

Then we prove the following theorem [cf. Theorem 3.4]:

Theorem B. *Let k be a field of characteristic $p \geq 0$ such that G_k is* **center-free** *and* **indecomposable***; X a* **smooth curve** *of type (g, r) over k such that the pair (g, r) satisfies $2g-2+r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive). Suppose that there exists a prime number $l \neq p$ such that k is* **l-cyclotomically full***. Then Π_X is* **indecomposable***.*

Next, in the case of *étale fundamental group of the configuration space of a hyperbolic curve*, we prove the following [cf. Theorem 3.5]:

Theorem C. *Let n be a positive integer; k a field of* **characteristic zero** *such that G_k is* **center-free** *and* **indecomposable***; X a hyperbolic curve over k; X_n the n-th* **configuration space** *associated to X. Suppose that either k is* **algebraically closed***, or* **l-cyclotomically full** *for a prime number l. Then Π_{X_n} is* **indecomposable***.*

For instance, Theorems B and C imply the following corollary [cf. Corollary 3.8]:

Corollary D. *Let n be a positive integer; k a field; X a* **smooth curve** *of type (g, r) over k such that the pair (g, r) satisfies $2g-2+r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive); X_n the n-th* **configuration space** *associated to X. Then the following hold:*

(i) *If k is a* **finitely generated transcendental extension field** *of $\mathbb{F}_p$, then Π_X is* **indecomposable***.*
(ii) *If k is a* **finitely generated extension field** *of either $\mathbb{Q}$ or $\mathbb{Q}_p$, then Π_{X_n} is* **indecomposable***.*

Moreover, Theorem C implies the following *purely geometric result* [cf. Theorem 3.11]:

Theorem E. *Let n be a positive integer; k a field of* **characteristic zero***; X a hyperbolic curve over k; X_n the n-th* **configuration space** *associated to X. Suppose that there exists an* **isomorphism** *of k-schemes*

$$X_n \xrightarrow{\sim} Y \times_k Z$$

— where Y, Z are **k-varieties** *[i.e., schemes that are of finite type, separated, and geometrically integral over k]. Then it follows that either*

$$Y \cong \mathrm{Spec}(k) \quad \textit{or} \quad Z \cong \mathrm{Spec}(k).$$

Finally, we consider the *Grothendieck-Teichmüller group* GT [cf. Definition 5.1]. One fundamental problem in the theory of GT is the issue of whether or not the well-known *injection*

$$G_{\mathbb{Q}} \hookrightarrow \mathrm{GT}$$

is, in fact, *bijective*. On the other hand, from the point of view of the theory of *combinatorial anabelian geometry* [cf., e.g., [20], [10], [11], [12]], it is more natural to consider the issue of whether or not

GT exhibits **analogous behavior / properties** to $G_{\mathbb{Q}}$

[cf. [12], Introduction]. From this point of view, it is natural to pose the question:

Is GT **indecomposable**?

[Note that $G_{\mathbb{Q}}$ is *indecomposable* [cf. the above **Fact**].] In this paper, we give an *affirmative answer* to a *pro-l version* of this question. More precisely, we prove the following result [cf. Theorem 5.4]:

Theorem F. *Let l be a prime number. Then the* **pro-l Grothendieck-Teichmüller group** GT_l *[cf. Definition 5.1] is* **indecomposable**.

The present paper is organized as follows: In §1, we review various properties of *absolute Galois groups*. Also, we prove a [profinite] group-theoretic result [cf. Proposition 1.8] which is needed in §3. In §2, we prove the *indecomposability* of the *geometric fundamental group* of a smooth [hyperbolic] curve [cf. Theorem A]. In §3, by applying the results of §1 and §2, we prove Theorems B, C and Corollary D. Moreover, by combining Theorem C with Lemma 3.10, we conclude Theorem E. In §4, we first give an *alternative proof* [cf. Theorem 4.7] of the indecomposability of the *maximal pro-l quotient of the absolute Galois group of a number field* without using the theory of *Hilbertian fields*. We then proceed to prove the indecomposability of a certain *almost pro-l group* arising from the *configuration space of a hyperbolic curve* over either an *l-adic local field* or *a number field* [cf. Theorem 4.10, (vi)]. Finally, in §5, after reviewing the definitions of GT and GT_l, we verify Theorem F as a consequence of a certain *anabelian result over finite fields* [cf. [7], Remark 6, (iv)].

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0. Notations and Conventions

In this paper, we follow the terminology and conventions of [22], §0, "Topological Groups", "Curves"; [22], Definition 2.1; [21], Definition 1.1, (ii), (iii).

Numbers:

The notation $\mathbb{Q}$ will be used to denote the field of *rational numbers*. The notation $\mathbb{Z} \subseteq \mathbb{Q}$ will be used to denote the set, group, or ring of *rational integers*. The *profinite completion* of the group $\mathbb{Z}$ will be denoted by $\widehat{\mathbb{Z}}$. If p is a *prime number*, then the notation $\mathbb{Q}_p$ (respectively, $\mathbb{Z}_p$) will be used to denote the *p-adic completion* of $\mathbb{Q}$ (respectively, $\mathbb{Z}$). The notation $\mathbb{F}_p$ will be used to denote the *finite field* $\mathbb{Z}/p\mathbb{Z}$.

A finite extension field of $\mathbb{Q}$ (respectively, $\mathbb{Q}_p$) will be referred to as a *number field* (respectively, *p-adic local field*).

Topological groups:

Let G be a Hausdorff topological group, and $H \subseteq G$ a closed subgroup. Let us write

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot h = h \cdot g, \ \forall h \in H\}$$

for the *centralizer* of H in G. Note that $Z_G(H)$ is *always closed* in G. We shall write $Z(G) \stackrel{\text{def}}{=} Z_G(G)$ for the *center* of G.

We shall say that a profinite group G is *elastic* if it holds that every topologically finitely generated closed normal subgroup $N \subseteq H$ of an open subgroup $H \subseteq G$ of G is either trivial or of finite index in G. If G is elastic, but *not* topologically finitely generated, then we shall say that G is *very elastic*.

We shall say that a profinite group G is *slim* if for every open subgroup $H \subseteq G$, the centralizer $Z_G(H)$ is trivial. A profinite group G is slim if and only if every open subgroup of G has *trivial center* [cf. [17], Remark 0.1.3]. Note that every *finite closed normal subgroup* $N \subseteq G$ of a slim profinite group G is *trivial*. [Indeed, this follows by observing that for any normal open subgroup $H \subseteq G$ such that $N \cap H = \{1\}$, consideration of the inclusion $N \hookrightarrow G/H$ reveals that the conjugation action of H on N is *trivial*, i.e., that $N \subseteq Z_G(H) = \{1\}$.]

Let p be a prime number. Then we shall write $G^{(p)}$ for the *maximal pro-p quotient* of a profinite group G, i.e., the inverse limit of the finite quotients of p-power order of G. We shall refer to a quotient $G \twoheadrightarrow Q$ as *almost pro-p-maximal* if, for some normal open subgroup $N \subseteq G$, $\mathrm{Ker}(G \twoheadrightarrow Q)$ coincides with the kernel of the natural surjection from N to the maximal pro-p quotient of N. If G admits an open subgroup which is pro-p, then we shall say that G is *almost pro-p*.

We shall write G^{ab} for the *abelianization* of a profinite group G, i.e., the quotient of G by the closure of the commutator subgroup of G. We shall denote the group of automorphisms of G by $\mathrm{Aut}(G)$. Conjugation by elements of G determines a homomorphism $G \to \mathrm{Aut}(G)$ whose image consists of the *inner automorphisms* of G. We shall denote by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ by the [normal] subgroup consisting of the inner automorphisms. If, moreover, G is *topologically finitely genertaed*, then one verifies easily that the topology of G admits a basis of *characteristic open subgroups*. Any such basis determines a *profinite topology* on the group $\mathrm{Aut}(G)$, $\mathrm{Out}(G)$.

Curves:

Let S be a scheme and X a scheme over S. If (g, r) is a pair of nonnegative integers, then we shall say that $X \to S$ is a *smooth curve of type* (g, r) over S if there exist an S-scheme $\overline{X}$ which is smooth, proper, of relative dimension 1 with geometrically connected fibers of genus g, and a closed subscheme

$D \subseteq \overline{X}$ which is finite étale of degree r over S such that the complement of D in $\overline{X}$ is isomorphic to X over S.

We shall say that X is a *hyperbolic curve* over S if there exists a pair (g, r) of nonnegative integers with $2g - 2 + r > 0$ such that X is a smooth curve of type (g, r) over S. A *tripod* is a hyperbolic curve of type $(0, 3)$.

Let $X \to S$ be a smooth curve of type (g, r). For positive integers $i, j \leq n$ such that $i < j$, write

$$p_{i,j} : P_n \stackrel{\text{def}}{=} X \times_S \ldots \times_S X \to X \times_S X$$

for the projection of the product P_n of n copies of $X \to S$ to the i-th and j-th factors. Then we shall refer to as the *n-th configuration space associated to $X \to S$* the S-scheme

$$X_n \to S$$

which is the open subscheme determined by the complement in P_n of the union of the various inverse images via the $p_{i,j}$ [as (i, j) ranges over the pairs of positive integers $\leq n$ such that $i < j$] of the image of the diagonal embedding $X \hookrightarrow X \times_S X$.

Write E for the set [of cardinality n] of factors of P_n. Let $E' \subseteq E$ be a subset of cardinality n'; $E'' \stackrel{\text{def}}{=} E \setminus E'$; $n'' \stackrel{\text{def}}{=} n - n'$. Then by "*forgetting*" the factors of E that belong to E', we obtain a *natural projection morphism*

$$X_n \to X_{n''}.$$

In this situation, we shall refer to n' as the *length* of this projection morphism. One verifies immediately that a projection $X_n \to X_{n-1}$ of length 1 may be regarded as a *smooth curve of type* $(g, r + n - 1)$ over X_{n-1}.

Fundamental groups:

Let X be a connected noetherian scheme. Then we shall write

$$\Pi_X$$

for the *étale fundamental group* of X [for some choice of basepoint].

For any field k, we shall write

$$G_k$$

for the *absolute Galois group* of k [for some choice of embedding to a separable closure of k]. We note that $G_k \stackrel{\sim}{\to} \Pi_{\mathrm{Spec}(k)}$.

1. Indecomposability of Absolute Galois Groups

In this section, we review various properties of *absolute Galois groups*. Also, we prove a [profinite] group-theoretic result [cf. Proposition 1.8] which is needed in §3.

Definition 1.1. (cf. [22], Definition 3.1) We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \cong G_1 \times G_2$, where G_1, G_2 are profinite groups, it follows that either G_1 or G_2 is the trivial group. We shall say that G is *strongly indecomposable* if every open subgroup of G is indecomposable.

Theorem 1.2. *Let k be a* **Hilbertian field** *[cf. [FJ], Chapter 12]. Then G_k is* **very elastic***,* **slim***, and* **strongly indecomposable***.*

Proof. The *very elasticity* portion of Theorem 1.2 follows from [4], Lemma 16.11.5; [4], Proposition 16.11.6. Note that for any *open subgroup H* of G_k, there exists a *finite separable extension k_H* of k such that $G_{k_H} \xrightarrow{\sim} H$. Here, by [4], Corollary 12.2.3, k_H is also a *Hilbertian field.* Thus, to verify the *slimness* and the *strong indecomposability* portions of Theorem 1.2, it suffices to show that G_k is *center-free* and *indecomposable.* But this center-freeness (respectively, indecomposability) follows from [4], Proposition 16.11.6 (respectively, the *theorem of Haran-Jarden* [cf. [4], Corollary 13.8.4]). $\square$

Remark 1.3. Let k be either a *finite field* or a *p-adic local field.* Then k is always *non-Hilbertian.* Indeed, G_k is *topologically finitely generated* [cf. Proposition 1.6, below; [4], Lemma 16.11.5].

Corollary 1.4. *The following types of fields are* **Hilbertian***:*

(i) **finitely generated extension fields** *of* $\mathbb{Q}$*,*

(ii) **finitely generated transcendental extension fields** *of* $\mathbb{Q}_p$*,*

(iii) **finitely generated transcendental extension fields** *of* $\mathbb{F}_p$*.*

In particular, their absolute Galois groups are **very elastic***,* **slim***, and* **strongly indecomposable***.*

Proof. The first statement follows from [4], Theorem 13.4.2. The last statement follows from the first, together with Theorem 1.2. $\square$

Lemma 1.5. *Let G be a profinite group. If G is* **elastic***,* **slim***, and* **topologically finitely generated***, then G is* **strongly indecomposable***.*

Proof. First, we note that any open subgroup of G is also *elastic*, *slim*, and *topologically finitely generated.* Thus, to verify the assertion, it suffices to show that G is *indecomposable.* Suppose that we have an isomorphism of profinite groups $G \cong G_1 \times G_2$ such that $G_1 \neq \{1\}$. Then since G_1 is a *nontrivial topologically finitely generated closed normal subgroup* of G, [by the *elasticity* of G] G_1 is *of finite index* in G. In particular, G_1 is an *open subgroup* of G. Thus, by the *slimness* of G, we have $G_2 \subseteq Z_G(G_1) = \{1\}$. $\square$

Proposition 1.6. *Let k be a* **p-adic local field**. *Then G_k, as well as any* **almost pro-p-maximal quotient** *$G_k \twoheadrightarrow Q_k$ of G_k, is* **elastic**, **slim**, *and* **topologically finitely generated**. *In particular, G_k and Q_k are* **strongly indecomposable**.

Proof. The assertions follow from Lemma 1.5; [21], Theorem 1.7, (ii); [23], Theorem 7.4.1. ☐

Lemma 1.7. *Let $G_1, \dots, G_n$ be* **profinite groups**, *where $n \geq 1$ is an integer;*

$$\phi : \Pi \stackrel{\text{def}}{=} \prod_{i=1}^{n} G_i \twoheadrightarrow Q$$

a **surjection** *of profinite groups. Then there exist normal closed subgroups $H_i \subseteq G_i$ [for $i = 1, \dots, n$], $N \subseteq Q$ such that $N \subseteq Z(Q)$, and the composite $\Pi \twoheadrightarrow Q/N$ of ϕ with the surjection $Q \twoheadrightarrow Q/N$ induces an* **isomorphism**

$$\overline{\Pi} \stackrel{\text{def}}{=} \prod_{i=1}^{n} \overline{G}_i \xrightarrow{\sim} Q/N$$

— where we write $\overline{G}_i \stackrel{\text{def}}{=} G_i/H_i$. In particular, if Q is **center-free** *and* **indecomposable**, *then we obtain an isomorphism $\overline{G}_i \xrightarrow{\sim} Q$ for some $i \in \{1, \dots, n\}$.*

Proof. This is the content of [22], Proposition 3.3. ☐

Proposition 1.8. *Let*

$$1 \longrightarrow \Delta \longrightarrow \Pi \xrightarrow{\ p\ } G \longrightarrow 1$$

be an exact sequence of profinite groups. Then the following hold:

(i) *Suppose that Δ is* **indecomposable**, *and G is* **center-free** *and* **indecomposable**. *Then if the* **natural outer Galois representation**

$$G \to \mathrm{Out}(\Delta)$$

associated to the above exact sequence is **nontrivial**, *then Π is also* **indecomposable**.

(ii) *Suppose that Δ is* **nontrivial** *and* **center-free**, *and that G is* **nontrivial**. *Then if Π is* **indecomposable**, *then the* **natural outer Galois representation**

$$G \to \mathrm{Out}(\Delta)$$

associated to the above exact sequence is **nontrivial**.

Proof. (i) Suppose that $\Pi = \Pi_1 \times \Pi_2$, where Π_1, Π_2 are *nontrivial* closed normal subgroups of Π. Then since G is *center-free*, it follows from Lemma 1.7 that there exist normal closed subgroups $H_i \subseteq \Pi_i$ [for $i = 1, 2$] such that $\Pi_1/H_1 \times \Pi_2/H_2 \xrightarrow{\sim} G$. In particular, since G is *indecomposable*, we obtain that either $\Pi_1/H_1 = \{1\}$ or $\Pi_2/H_2 = \{1\}$. Without loss of generality, we may assume that $\Pi_1/H_1 = \{1\}$, so $\Pi_1 = H_1$, $\Pi_2/H_2 \xrightarrow{\sim} G$. Thus, we have $\Pi_1 \times H_2 \xrightarrow{\sim} \Delta$.

Now I *claim* that $H_2 \neq \{1\}$. Indeed, suppose that $H_2 = \{1\}$, so $\Pi_1 \xrightarrow{\sim} \Delta$, $\Pi_2 \xrightarrow{\sim} G$. Then the extension determined by the exact sequence that appears in the statement of Proposition 1.8 is isomorphic to the *trivial extension* of G by Δ

$$1 \longrightarrow \Delta \longrightarrow \Delta \times G \longrightarrow G \longrightarrow 1.$$

Thus, the natural outer Galois representation $G \to \mathrm{Out}(\Delta)$ induced by the conjugation action of G on Δ factors through the *trivial morphism* $G \to \mathrm{Out}(\Delta)$. But this contradicts the assumption that the outer representation $G \to \mathrm{Out}(\Delta)$ is *nontrivial.* This completes the proof of the *claim.*

In light of the *claim*, by the *indecomposability* of Δ, we conclude that $\Pi_1 = \{1\}$, a contradiction. This completes the proof that Π is *indecomposable.*

(ii) Suppose that the representation $G \to \mathrm{Out}(\Delta)$ is *trivial.* Note that both Δ and $Z_\Pi(\Delta)$ are *normal closed subgroups* of Π [cf. the discussion entitled "*Topological groups*" in §0]. Moreover, by the triviality of the representation $G \to \mathrm{Out}(\Delta)$, it follows that Π is generated by Δ and $Z_\Pi(\Delta)$. Thus, since Δ is *center-free*, i.e., $\Delta \cap Z_\Pi(\Delta) = Z(\Delta) = \{1\}$, we obtain that $\Pi \cong \Delta \times Z_\Pi(\Delta)$. Here, we note that since $p(Z_\Pi(\Delta)) = G$ is *nontrivial*, we have $Z_\Pi(\Delta) \neq \{1\}$. Therefore, since Δ is *nontrivial*, we conclude that Π is *not indecomposable*, a contradiction. □

2. Indecomposability of Geometric Fundamental Groups of Curves

In this section, we prove the *indecomposability* of the *geometric fundamental group* of a smooth [hyperbolic] curve.

Theorem 2.1. *Let k be an* **algebraically closed field** *of* **characteristic zero***; X a* **hyperbolic curve** *over k. Then Π_X is* **elastic***,* **slim***, and* **topologically finitely generated***. In particular, Π_X is* **strongly indecomposable***.*

Proof. The fact that Π_X is *elastic* (respectively, *slim*; *topologically finitely generated*) follows from [22], Theorem 1.5 (respectively, [22], Proposition 1.4; [26], EXPOSÉ XIII, Corollaire 2.12). In particular, the *strong indecomposability* of Π_X follows from Lemma 1.5 [cf. also [22], Proposition 3.2; [22], Remark 3.2.1]. □

Theorem 2.2. *Let k be an* **algebraically closed field** *of* **characteristic $\boldsymbol{p > 0}$**; *X a* **smooth curve** *of type (g, r) such that the pair (g, r) satisfies $(g, r) \neq (0, 0)$, $(1, 0)$. Then $G \stackrel{\mathrm{def}}{=} \Pi_X$ is* **strongly indecomposable**.

Proof. First, we note that for any *open subgroup* H of G, there exists a *connected finite étale covering* $X_H \to X$ of X, where X_H is also a curve of *type* $\neq (0, 0)$, $(1, 0)$ over k such that $\Pi_{X_H} \stackrel{\sim}{\to} H$. Thus, to verify the assertion, it suffices to show that G is *indecomposable*. Suppose that we have an isomorphism of profinite groups $G \cong G_1 \times G_2$ such that $G_1 \neq \{1\}$, $G_2 \neq \{1\}$. In particular, by the *slimness* of G [cf. Proposition 2.4, below], it follows that G_1, G_2 are *infinite* [cf. §0].

Now I *claim* that

$(*_1)$ there exists an *open subgroup* U of G such that U is [isomorphic to] the *fundamental group* of a curve of *genus* ≥ 2.

Indeed, this fact is elementary and well-known, but we give a short proof here for completeness. First, we consider the case where the genus of X is 0, i.e., the unique *smooth compactification* of X is $\mathbb{P}^1_k$. Here, note that if we identify the *function field* of $\mathbb{P}^1_k$ with $k(t)$, where t is an indeterminate, then for any *Artin-Schreier equation*

$$x^p - x = t^m \quad (m \in \mathbb{Z}_{>0},\ p \nmid m),$$

one computes easily that the *normalization* of $\mathbb{P}^1_k$ in the extension field $k(t)[x]/(x^p - x - t^m)$ of $k(t)$ determines a *finite ramified covering* $\phi_m : C_m \to \mathbb{P}^1_k$ of $\mathbb{P}^1_k$ branched only at ∞, where C_m is a *smooth, proper curve* of genus $\frac{(m-1)(p-1)}{2}$ [cf., e.g., [29], Example 8.16]. Thus, for any curve X of type $(0, r)$, where $r > 0$, by taking m to be sufficiently large, we obtain a *connected finite étale covering* $X' \to X$ of X such that the genus of X' is ≥ 2. Next, we consider the case where the genus of X is 1, i.e., the unique *smooth compactification* of X is an *elliptic curve* E. Note that by applying the *Riemann-Roch Theorem* to E, we obtain a *finite morphism* $E_1 \stackrel{\mathrm{def}}{=} E \setminus \{p\} \to \mathbb{A}^1_k$ over k, where $p \in E \setminus X$ is a *closed point* of E. Next, let us observe that it follows from the genus 0 case, which has already been verified, that there exists a *connected finite étale covering* $C \to \mathbb{A}^1_k$ of $\mathbb{A}^1_k$ such that the genus of C is ≥ 2. Then any *connected component* of $E_1 \times_{\mathbb{A}^1_k} C$ determines a *connected finite étale covering* $C' \to E_1$ of E_1. Moreover, by applying the *Hurwitz formula* to the *compactification* of the *finite morphism* $C' \hookrightarrow E_1 \times_{\mathbb{A}^1_k} C \to C$, it follows that the genus of C' is also ≥ 2. Thus, for any curve X of type $(1, r)$, where $r > 0$, we obtain a *connected finite étale covering* $X' \to X$ of X such that the genus of X' is ≥ 2. This completes the proof of $(*_1)$.

In light of $(*_1)$ and the fact that G_1, G_2 are *infinite*, we may assume, without loss of generality, that G is the fundamental group of a curve of *genus* ≥ 2.

Next, I *claim* that

($*_2$) for every prime number $l \neq p$, there exist *finite quotients* $G_1 \twoheadrightarrow Q_1$, $G_2 \twoheadrightarrow Q_2$ such that l divides the order of Q_1, Q_2.

Indeed, suppose that l does *not* divide the order of any finite quotient of G_1. Now let $N_1 \subsetneq G_1$ be a *proper normal open subgroup* of G_1. Note that by assumption, we have $N_1^{\mathrm{ab}} \otimes \mathbb{Z}_l = \{1\}$. Write $N \stackrel{\mathrm{def}}{=} N_1 \times G_2$. Then since the *conjugation action* of $G/N \cong G_1/N_1 \times \{1\}$ on

$$N^{\mathrm{ab}} \otimes \mathbb{Z}_l \cong (N_1^{\mathrm{ab}} \otimes \mathbb{Z}_l) \times (G_2^{\mathrm{ab}} \otimes \mathbb{Z}_l) \cong \{1\} \times (G_2^{\mathrm{ab}} \otimes \mathbb{Z}_l)$$

is *trivial*, by Proposition 2.4, below, we conclude that $G/N = \{1\}$, a contradiction. This completes the proof of ($*_2$).

In light of the ($*_2$), by replacing G by the *maximal pro-l quotient* of a suitable open subgroup of G for some $l \neq p$, we may assume without loss of generality that G, G_1, G_2 are *pro-l groups*. Then since G is *slim* [cf. Proposition 2.4, below], it follows that G_1, G_2 are *nonabelian* pro-l groups, so $\dim_{\mathbb{F}_l} H^1(G_1, \mathbb{F}_l) \geq 2$, $\dim_{\mathbb{F}_l} H^1(G_2, \mathbb{F}_l) \geq 2$ [cf. [25], Theorem 7.8.1]. In particular, since we have an *inclusion* $H^1(G_1, \mathbb{F}_l) \otimes H^1(G_2, \mathbb{F}_l) \subseteq H^2(G, \mathbb{F}_l)$, we obtain that $\dim_{\mathbb{F}_l} H^2(G, \mathbb{F}_l) \geq 4$. This contradicts the fact that $\dim_{\mathbb{F}_l} H^2(G, \mathbb{F}_l)$ is either 0 or 1. [Indeed, $H^2(G, \mathbb{F}_l)$ is isomorphic to the *second étale cohomology group* $H^2_{\text{ét}}(X, \mathbb{F}_l)$ of X [cf. [19], Proposition 1.1]; the dimension over $\mathbb{F}_l$ of this last cohomology group is either 0 or 1 [cf. [5], Theorem 7.2.9 (ii); Proposition 7.2.10].] Therefore, G is *indecomposable*. $\square$

Remark 2.3. In the situation of Proposition 2.2, if X is an *affine curve*, then Π_X is *never finitely generated*. [In fact, the *maximal pro-p quotient* of Π_X is a *free pro-p group of rank* $|k|$ — cf. [27], Theorem 12.] In particular, we cannot apply Lemma 1.5 to Proposition 2.2.

The following result is well-known [cf., e.g., [28], Proposition 1.11; [22], Propostion 1.4], but we review it briefly for the sake of completeness.

Proposition 2.4. *Let k be an* **algebraically closed field** *of characteristic $p \geq 0$; $l \neq p$ a prime number; X a* **smooth curve** *of type (g, r) over k such that the pair (g, r) satisfies $2g - 2 + r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive). Then for any* **normal open subgroup** *N of $G \stackrel{\mathrm{def}}{=} \Pi_X$ such that the connected finite étale covering $X_N \to X$ corresponding to N has* **genus $\geq$ 2**, *the* **conjugation action** *of G/N on $N^{\mathrm{ab}} \otimes \mathbb{Z}_l$ is* **faithful**. *In particular, Π_X, as well as its maximal pro-l quotient $\Pi_X^{(l)}$, is* **slim**.

Proof. The *faithfulness* portion of Proposition 2.4 follows immediately from the argument given in [3], Lemma 1.14. The *slimness* portion of Proposition 2.4 follows formally from the *faithfulness* portion of Proposition 2.4. $\square$

3. Indecomposability of Various Fundamental Groups

In this section, by applying the results of §1 and §2, we prove the *indecomposability* of various *fundamental groups*. Moreover, by applying an indecomposability result, we prove the "*scheme-theoretic indecomposability*" of the *configuration space of a hyperbolic curve over a field of characteristic zero* [cf. Theorem 3.11].

Lemma 3.1. *Let k be a field; $\overline{k}$ an algebraic closure of k; X a* **quasi-compact**, **geometrically connected** *scheme over k. Then the sequence of schemes $X \times_k \overline{k} \stackrel{\mathrm{pr}_1}{\to} X \to \mathrm{Spec}(k)$ determines an* **exact** *sequence of profinite groups*

$$1 \longrightarrow \Pi_{X \times_k \overline{k}} \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1.$$

Proof. This is the content of [26], EXPOSÉ IX, Théorème 6.1. □

Definition 3.2. Let k be a field of characteristic $p \geq 0$; $l \neq p$ a prime number. Then for the pair (k, l), we consider the following condition:

$(*^l_k)$ For any *finite extension field* k' of k, the *l-adic cyclotomic character* $\chi_{k'} : G_{k'} \to \mathbb{Z}_l^\times$ of k' is *nontrivial*.

We shall say that k is ***l*-cyclotomically full** if the pair (k, l) satisfies the condition $(*^l_k)$.

Lemma 3.3. *In the notation of Definition 3.2, the following hold:*

(i) *k is* ***l*-cyclotomically full** *if and only if for any* **finite extension field** *k' of k, there exists a positive integer n such that k' does not contain a* **primitive l^n-th root of unity**.

(ii) *Let K be an* **extension field** *of k. Then if K is* ***l*-cyclotomically full**, *then the same is true of k. Suppose further that K is a* **finitely generated** *extension field of k. Then if k is* ***l*-cyclotomically full**, *then the same is true of K.*

(iii) *k is* ***l*-cyclotomically full** *if and only if the image of the l-adic cyclotomic character $\chi_k : G_k \to \mathbb{Z}_l^\times$ of k is* **infinite**.

(iv) *Let X be a* **smooth curve** *of type (g, r) over k such that the pair (g, r) satisfies $(g, r) \neq (0, 0)$, $(0, 1)$ (respectively, $(g, r) \neq (0, 0)$) if the characteristic of k is zero (respectively, positive); $\overline{k}$ an algebraic closure of k. Write $X_{\overline{k}} \stackrel{\mathrm{def}}{=} X \times_k \overline{k}$. Suppose, moreover, that k is* ***l*-cyclotomically full**. *Then the image of the* **natural outer Galois representation**

$$\rho_k : G_k \to \mathrm{Out}(\Pi_{X_{\overline{k}}})$$

associated to the exact sequence of profinite groups

$$1 \longrightarrow \Pi_{X_{\overline{k}}} \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1$$

[cf. Lemma 3.1] is **infinite***, hence, in particular,* **nontrivial***. If, moreover, $(g, r) \neq (0, 1)$, then the image of the naturally induced* **pro-l outer Galois representation**

$$\rho_k^{(l)} : G_k \to \mathrm{Out}(\Pi^{(l)}_{X_{\overline{k}}})$$

is **infinite***, hence, in particular,* **nontrivial***.*

(v) *Let l, p be two distinct prime numbers; $k \in \{\mathbb{Q}, \mathbb{Q}_l, \mathbb{Q}_p, \mathbb{F}_p\}$. Suppose that K is a* **finitely generated** *extension field of k. Then K is* **l-cyclotomically full***.*

Proof. Assertion (i) follows immediately from the definitions.

Assertion (ii) follows immediately from (i) and the well-known fact that the *algebraic closure* of k in K is a *finite extension* of k. [In fact, let $E \subseteq K$ be the *algebraic closure* of k in K; $\{x_1, \ldots, x_n\} \subseteq K$ a *transcendence basis* of K/k. Then we obtain that $[E : k] = [E(x_1, \ldots, x_n) : k(x_1, \ldots, x_n)] \leq [K : k(x_1, \ldots, x_n)] < +\infty$.]

We consider assertion (iii). First, let us prove *necessity.* Suppose that the image of χ_k is *finite.* Then the *kernel* H of χ_k is an *open subgroup* of G_k. Thus, there exists a *finite extension* k' of k such that $G_{k'} \xrightarrow{\sim} H$. In particular, the l-adic cyclotomic character $\chi_{k'} : G_{k'} \xrightarrow{\sim} H \hookrightarrow G_k \to \mathbb{Z}_l^\times$ of k' is *trivial* — a contradiction. Next, we prove *sufficiency.* To this end, let k' be a *finite extension field* of k. Write $\chi_{k'} : G_{k'} \to \mathbb{Z}_l^\times$ for the l-adic cyclotomic character of k', H for the *kernel* of χ_k. Then if we identify $G_{k'}$ with an *open subgroup* of G_k, then $G_{k'}/G_{k'} \cap H$ $[\xrightarrow{\sim} \mathrm{Im}(\chi_{k'})]$ corresponds to an *open subgroup* of G_k/H $[\xrightarrow{\sim} \mathrm{Im}(\chi_k)]$. On the other hand, since $\mathrm{Im}(\chi_k)$ is *infinite*, we thus conclude that $\mathrm{Im}(\chi_{k'})$ is also *infinite*, hence, in particular, *nontrivial.* This completes the proof of assertion (iii).

Next, we consider assertion (iv). First, suppose that $(g, r) = (0, 1)$ [so $p > 0$]. Then observe that one verifies immediately — by considering a suitable *Artin-Schreier covering* of X as in the proof of Theorem 2.2 over a suitable *finite extension* of k and applying [8], Lemma 23, (i), (iii) — that the *infiniteness* [hence, in particular, the *nontriviality*] of the image of ρ_k follows from the corresponding infiniteness in the case of $g \geq 1$. Here, we note that, although, in [8], Lemma 23, "Δ" [in the notation of [8], Lemma 23] is assumed to be *topologically finitely generated*, one verifies immediately that this assumption is in fact *unnecessary.* Thus, in the remainder of the proof of assertion (iv), we may assume without loss of generality that $(g, r) \neq (0, 1)$. Next, observe that to verify the *infiniteness* of ρ_k, it suffices to verify the *infiniteness* of $\rho_k^{(l)}$. Moreover, by replacing k by a suitable finite extension of k, it suffices to verify that $\rho_k^{(l)}$ is *nontrivial.* Suppose that $\rho_k^{(l)}$ is *trivial.* First, we assume that $g \geq 1$. Write $J(\overline{X})$ for the *Jacobian variety* of the smooth compactification $\overline{X}$ of X, $T_l(J(\overline{X}))$ for the *l-adic Tate module* of $J(\overline{X})$. Then it follows that the *natural l-adic Galois representation*

$$G_k \to \mathrm{Aut}(T_l(J(\overline{X})))$$

associated to $J(\overline{X})$ is *trivial.* Then since, as is well-known [cf. the *natural isomorphisms* $\bigwedge^{2g} H^1_{\text{ét}}(\overline{X}_{\overline{k}}, \mathbb{Z}_l) \xrightarrow{\sim} H^{2g}_{\text{ét}}(\overline{X}_{\overline{k}}, \mathbb{Z}_l) \xrightarrow{\sim} \mathbb{Z}_l(-g)$ of $\mathbb{Z}_l[G_k]$-modules discussed in [14], Remark 15.5; [13], Theorem 11.1, (a)], the *determinant* of this representation is a positive power of the *l-adic cyclotomic character* of k, we conclude that some positive power of the l-adic cyclotomic character of k is *trivial.* But this contradicts (iii). Next, we assume that $g = 0$ and $r \geq 2$. Then since $r \geq 2$, we may identify $X_{\overline{k}}$ with an *open subscheme* of $\mathbb{A}^1_{\overline{k}} \setminus \{0\}$. Thus, by considering the *maximal pro-l abelian quotient* of $\Pi_{\mathbb{A}^1_{\overline{k}} \setminus \{0\}}$, we conclude that the *$l$-adic cyclotomic character* of k is *trivial* — a contradiction. [Here, we recall that $H^1_{\text{ét}}(\mathbb{A}^1_{\overline{k}} \setminus \{0\}, \mathbb{Z}_l) \cong \mathbb{Z}_l(-1)$.]

Finally, we consider assertion (v). To verify the assertion, it suffices to show that k is *l-cyclotomically full* [cf. (ii)]. Thus, to verify the assertion, it suffices to show that, for any *finite extension field* k' of k, there exists a positive integer n such that k' does not contain a *primitive l^n-th root of unity* [cf. (i)]. But this follows from the well-known fact that for any *finite extension field* k' of k, the *group of roots of unity in* k' is *finite* [cf. [15], Chapter 5; [24], Chapter 2, §4.3, §4.4]. □

Theorem 3.4. *Let k be a field of characteristic $p \geq 0$ such that G_k is* **center-free** *and* **indecomposable***; X a* **smooth curve** *of type (g, r) over k such that the pair (g, r) satisfies $2g-2+r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive). Suppose that there exists a prime number $l \neq p$ such that k is* **l-cyclotomically full***. Then Π_X is* **center-free** *and* **indecomposable***.*

Proof. Let $\overline{k}$ be an algebraic closure of k. Write $X_{\overline{k}} \overset{\text{def}}{=} X \times_k \overline{k}$. Then by Lemma 3.1, we have the following exact sequence of profinite groups

$$1 \longrightarrow \Pi_{X_{\overline{k}}} \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1.$$

In particular, since G_k and $\Pi_{X_{\overline{k}}}$ are *center-free* [cf. Proposition 2.4], it follows that Π_X is also *center-free*. Here, we note that both G_k and $\Pi_{X_{\overline{k}}}$ are *indecomposable* [cf. Theorems 2.1, 2.2]. Thus, since the *natural outer Galois representation*

$$G_k \to \mathrm{Out}(\Pi_{X_{\overline{k}}})$$

associated to the above sequence is *nontrivial* [cf. Lemma 3.3, (iv)], it follows from Proposition 1.8, (i), that Π_X is also *indecomposable.* □

Theorem 3.5. *Let n be a positive integer; k a field of* **characteristic zero** *such that G_k is* **center-free** *and* **indecomposable***; X a hyperbolic curve over k; X_n the n-th* **configuration space** *associated to X. Suppose that either k is* **algebraically closed***, or* **l-cyclotomically full** *for a prime number l. Then Π_{X_n} is* **center-free** *and* **indecomposable***.*

Proof. First, we note that for $n \geq 1$, any *projection morphism* $X_n \to X_{n-1}$ of *length one* determines a natural exact sequence of profinite groups [cf. [22], Proposition 2.2, (i)]

$$1 \longrightarrow \Pi_{(X_n)_{\overline{x}}} \longrightarrow \Pi_{X_n} \longrightarrow \Pi_{X_{n-1}} \longrightarrow 1$$

— where $\overline{x}$ is a *geometric point* of X_{n-1}; we write $X_0 \stackrel{\text{def}}{=} \text{Spec}(k)$; $(X_n)_{\overline{x}}$ denotes the *fiber* of $X_n \to X_{n-1}$ over $\overline{x}$. In particular, by applying induction on n, we conclude from Proposition 2.4 and Theorem 3.4 that Π_{X_n} is *center-free*. Here, we note that $\Pi_{(X_n)_{\overline{x}}}$ and Π_{X_1} are *indecomposable* [cf. Theorems 2.1, 3.4]. Moreover, it is well-known that the *natural outer Galois representation*

$$\Pi_{X_{n-1}} \to \text{Out}(\Pi_{(X_n)_{\overline{x}}})$$

associated to the above exact sequence is *nontrivial*. [In the case where k is an *algebraically closed field*, the above representation is, in fact, *injective* — cf. [2], Theorem 1.] Thus, by induction on n, it follows from Proposition 1.8, (i), that Π_{X_n} is *indecomposable*. □

Corollary 3.6. *Let n be a positive integer; k a* **Hilbertian field** *of characteristic $p \geq 0$; X a* **smooth curve** *of type (g, r) over k such that the pair (g, r) satisfies $2g - 2 + r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive); X_n the n-th* **configuration space** *associated to X. Suppose that there exists a prime number $l \neq p$ such that k is* **l-cyclotomically full**. *Also, if $p > 0$, then we assume further that $n = 1$. Then Π_{X_n} is* **center-free** *and* **indecomposable**.

Proof. These assertions follow immediately from Corollary 1.2 and Theorems 3.4, 3.5. □

Remark 3.7. The *center-freeness* asserted in Theorems 3.4, 3.5 and Corollary 3.6 holds even if one does not assume that k is l-cyclotomically full.

Corollary 3.8. *Let n be a positive integer; k a field; X a* **smooth curve** *of type (g, r) over k such that the pair (g, r) satisfies $2g-2+r > 0$ (respectively, $(g, r) \neq (0, 0)$, $(1, 0)$) if the characteristic of k is zero (respectively, positive); X_n the n-th* **configuration space** *associated to X. Then the following hold:*

(i) *If k is a* **finitely generated transcendental extension field** *of $\mathbb{F}_p$, then Π_X is* **center-free** *and* **indecomposable**.
(ii) *If k is a* **finitely generated extension field** *of either $\mathbb{Q}$ or $\mathbb{Q}_p$, then Π_{X_n} is* **center-free** *and* **indecomposable**.

Proof. First, we note that every field k which appears in Corollary 3.8 is *l-cyclotomically full* for some prime number l [cf. Lemma 3.3, (v)]. Thus, in the case that k is *Hilbertian* [cf. Corollary 1.4] (respectively, *non-Hilbertian*, i.e., *p-adic local*), the assertions follow from Corollary 3.6 (respectively, Proposition 1.6 and Theorem 3.5). □

Definition 3.9. (cf. [9], Definition 2.5) Let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k. Let X be a variety over k [i.e., a scheme that is of finite type, separated, and geometrically integral over k]. Then we shall say that X is of *LFG-type* if, for any normal variety Y over $\overline{k}$ and any morphism $Y \to X \times_k \overline{k}$ over $\overline{k}$ that is not constant, the image of the outer homomorphism $\Pi_Y \to \Pi_{X \times_k \overline{k}}$ is infinite.

Lemma 3.10. *Let n be a positive integer; k a field of* **characteristic zero***; X a hyperbolic curve over k; X_n the n-th* **configuration space** *associated to X. Then X_n is of* **LFG-type**.

Proof. This follows immediately from [9], Proposition 2.7. □

Theorem 3.11. *Let n be a positive integer; k a field of* **characteristic zero***; X a hyperbolic curve over k; X_n the n-th* **configuration space** *associated to X. Suppose that there exists an* **isomorphism** *of k-schemes*

$$X_n \xrightarrow{\sim} Y \times_k Z$$

— where Y, Z are $\boldsymbol{k}$**-varieties** *[cf. Definition 3.9]. Then it follows that either*

$$Y \cong \mathrm{Spec}(k) \quad or \quad Z \cong \mathrm{Spec}(k).$$

Proof. We may assume that k is *algebraically closed*. Then to verify the assertion, it suffices to show that either $\dim(Y) = 0$ or $\dim(Z) = 0$. First, we note that by the *Künneth formula* [cf. [26], EXPOSÉ XIII, Proposition 4.6], there exists an isomorphism of profinite groups

$$\Pi_{X_n} \xrightarrow{\sim} \Pi_Y \times \Pi_Z.$$

Then since Π_{X_n} is *indecomposable* by Theorem 3.5, we may without loss of generality that $\Pi_Y = \{1\}$. Now we fix a k-rational point $z \in Z(k)$ of Z. Then we obtain a *closed immersion* $Y \xrightarrow{\sim} Y \times_k \{z\} \hookrightarrow Y \times_k Z \xrightarrow{\sim} X_n$. Write $Y' \to Y$ for the [*surjective*] morphism obtained by *normalizing* Y. Here, if we assume that $\dim(Y) \geq 1$, then the composite $Y' \to Y \hookrightarrow X_n$ is *nonconstant*. Thus, since X_n is of *LFG-type* by Lemma 3.10, the image of the outer homomorphism $\Pi_{Y'} \to \Pi_{X_n}$ is *infinite* — a *contradiction*. Therefore, we conclude that $\dim(Y) = 0$. □

4. Alternative Proof of the Indecomposability of the Pro-l Absolute Galois Group of a Number Field

In this section, we first give an *alternative proof* [cf. Theorem 4.7] of the indecomposability of the *maximal pro-l quotient of the absolute Galois group of a number field* without using the theory of *Hilbertian fields*. [In fact, this indecomposability is an easy consequence of the *theorem of Haran-Jarden* [cf. [4], Corollary 13.8.4] in the theory of Hilbertian fields.] Finally, we prove the indecomposability of a certain *almost pro-l group* arising from the *configuration space of a hyperbolic curve* over either an *l-adic local field* or *a number field* [cf. Theorem 4.10, (vi)].

Definition 4.1. Let G be a profinite group. We shall say that G is *meta-abelian* if there exists an abelian closed normal subgroup H of G such that the quotient group G/H is also abelian.

Lemma 4.2. *Let G be a* **meta-abelian profinite group**. *Then the following hold:*

(i) *Let H be a* **closed subgroup** *of G. Then H is also* **meta-abelian**.
(ii) *Let H be a closed normal subgroup of G. Then the* **quotient** G/H *is also* **meta-abelian**.
(iii) *Let G_1, G_2 be* **meta-abelian profinite groups**. *Then the* **direct product** $G_1 \times G_2$ *is also* **meta-abelian**.

Proof. These assertions follow immediately from the definitions. □

Theorem 4.3. *Let k be a* **p-adic local field**; *$V_k \subseteq G_k$ the* **ramification group** *of G_k. Then V_k is a* **free pro-p group**, *and the quotient group G_k/V_k [i.e., the Galois group of the maximal tamely ramified extension of k] is* **meta-abelian**. *In particular, for any prime $l \neq p$, the* **maximal pro-l quotient** $G_k^{(l)}$ *of G_k is also* **meta-abelian**.

Proof. The fact that V_k is *free pro-p* (respectively, G_k/V_k is *meta-abelian*) follows from [23], Proposition 7.5.1 (respectively, [the proof of] [23], Theorem 7.5.3). The last statement follows, by applying the *maximal pro-l quotient functor* [which is *right exact*] to the following natural exact sequence of profinite groups

$$1 \longrightarrow V_k \longrightarrow G_k \longrightarrow G_k/V_k \longrightarrow 1,$$

from the fact that $V_k^{(l)} = \{1\}$ and Lemma 4.2, (ii). □

Lemma 4.4. *Let l be a prime number; F a* **nonabelian free pro-l group**. *Then every* **abelian closed normal subgroup** *of F is* **trivial**.

Proof. This is the content of [25], Proposition 8.7.2. □

Lemma 4.5. *Let l be a prime number; G_1 a* **meta-abelian pro-l group***; G_2 a* **free pro-l group***; $\varphi : G_1 \to G_2$ a morphism of profinite groups. Then* $\mathrm{Im}(\varphi)$ *is* **abelian***.*

Proof. Write $G \stackrel{\mathrm{def}}{=} \mathrm{Im}(\varphi)$. First, we note that, by [25], Corollary 7.7.5, G is *free pro-l.* Now suppose that G is *nonabelian.* Here, since G is *meta-abelian* [cf. Lemma 4.2, (ii)], there exists an *abelian closed normal subgroup* H of G such that G/H is also *abelian.* Then by Lemma 4.4, it follows that H is *trivial*, so that $G \stackrel{\sim}{\to} G/H$, a contradiction. Therefore, G is *abelian.* □

Lemma 4.6. *Let l be a prime number; k a* **number field***; $\overline{k}$ an algebraic closure; $G_k \twoheadrightarrow Q_k$ an* **almost pro-l-maximal quotient** *of G_k. Then Q_k is* **slim***.*

Proof. First, we note that, via the same arguments as the arguments applied to prove [18], Proposition 2.1, we conclude the following:

Let k be a *number field.* Then:

(i) The natural surjection $G_k \twoheadrightarrow Q_k$ induces an *isomorphism*

$$H^i(Q_k, \mathbb{F}_l(1)) \stackrel{\sim}{\to} H^i(G_k, \mathbb{F}_l(1))$$

for all integers $i \geq 0$.

(ii) Write $\widetilde{k} \subseteq \overline{k}$ for the extension of k defined by $\mathrm{Ker}(G_k \twoheadrightarrow Q_k)$. Then for any automorphism σ of the field $\widetilde{k}$ that preserves and acts *nontrivially* on $k \subseteq \widetilde{k}$, the automorphism induced by σ of the set of *one-dimensional $\mathbb{F}_l$-subspaces* of the $\mathbb{F}_l$-vector space

$$H^2(Q_k, \mathbb{F}_l(1))$$

is *nontrivial.*

[Here, we remark that, just as in the proof of [18], Proposition 2.1, (ii), assertion (i) is used in the proof of assertion (ii).]

Then by applying assertion (ii), via the same argument as the argument applied to prove [18], Corollary 2.2, we conclude that Q_k is *slim.* □

Theorem 4.7. *Let l be a prime number; k a* **number field***. Then $G_k^{(l)}$ is* **strongly indecomposable***.*

Proof. To verify the assertion, it suffices to show that $G_k^{(l)}$ is *indecomposable*. Suppose that we have an isomorphism of profinite groups $G_k^{(l)} \cong G_1 \times G_2$ such that $G_1 \neq \{1\}$, $G_2 \neq \{1\}$. In particular, since G_1, G_2 are *nontrivial pro-l groups*, it follows that $\dim_{\mathbb{F}_l} H^1(G_1, \mathbb{F}_l) \geq 1$, $\dim_{\mathbb{F}_l} H^1(G_2, \mathbb{F}_l) \geq 1$. Now I *claim* that it holds that

$$\text{either} \quad \dim_{\mathbb{F}_l} H^2(G_1, \mathbb{F}_l) \geq 1 \quad \text{or} \quad \dim_{\mathbb{F}_l} H^2(G_2, \mathbb{F}_l) \geq 1.$$

Indeed, suppose that $\dim_{\mathbb{F}_l} H^2(G_1, \mathbb{F}_l) = \dim_{\mathbb{F}_l} H^2(G_2, \mathbb{F}_l) = 0$. Then by [25], Theorem 7.7.4, it follows that G_1, G_2 are *free pro-l groups*. Now, by Lemma 4.8, below, there exists a *nonarchimedean valuation* v of k such that the residue characteristic p of the *completion* k_v of k at v satisfies $l \mid p-1$. [In particular, there exists a *primitive l-th root of unity* in k_v.] Then we may identify G_{k_v} with a *closed subgroup* of G_k [well-defined up to conjugation in G_k]. Thus, we have a commutative diagram

$$\begin{tikzcd} G_{k_v} \arrow[r, hook, "\iota"] \arrow[d] & G_k \arrow[d] \\ G_{k_v}^{(l)} \arrow[r, hook, "\iota^{(l)}"] & G_k^{(l)} \end{tikzcd}$$

— where the vertical arrows are the natural surjections; the upper horizontal arrow ι is the natural inclusion; the lower horizontal arrow $\iota^{(l)}$ is the morphism obtained by applying the *maximal pro-l quotient functor* to ι. Here, we note that $\iota^{(l)}$ is also *injective* [cf. [23], Theorem 9.4.3]. We shall write $p_i : G_k^{(l)} \cong G_1 \times G_2 \twoheadrightarrow G_i$ $[i = 1, 2]$ for the *i-th projection*. Then since $G_{k_v}^{(l)}$ is a *meta-abelian pro-l group* [cf. Theorem 4.3], by applying Lemma 4.5 to the morphism

$$p_i \circ \iota^{(l)} : G_{k_v}^{(l)} \to G_i,$$

it follows that $p_i(\mathrm{Im}(\iota^{(l)}))$ is *abelian*. Since we have

$$G_{k_v}^{(l)} \xrightarrow{\sim} \mathrm{Im}(\iota^{(l)}) \hookrightarrow p_1(\mathrm{Im}(\iota^{(l)})) \times p_2(\mathrm{Im}(\iota^{(l)})),$$

we thus conclude that $G_{k_v}^{(l)}$ is *abelian*. This contradicts the fact that $G_{k_v}^{(l)}$ is *nonabelian* [cf. [23], Theorem 7.5.3]. This completes the proof of the *claim*.

On the other hand, by the *Künneth formula* in group cohomology, we have an *inclusion*

$$H^3(G_k^{(l)}, \mathbb{F}_l) \supseteq (H^1(G_1, \mathbb{F}_l) \otimes H^2(G_2, \mathbb{F}_l)) \oplus (H^2(G_1, \mathbb{F}_l) \otimes H^1(G_2, \mathbb{F}_l)).$$

Thus, in light of the *claim*, we obtain that $\dim_{\mathbb{F}_l} H^3(G_k^{(l)}, \mathbb{F}_l) \geq 1$.

Now suppose that either k is *totally imaginary* or $l \neq 2$. Then we obtain a contradiction to the well-known fact that the *l-cohomological dimension* of G_k is ≤ 2 [cf. [23], Proposition 8.3.18]. Therefore, we conclude that $G_k^{(l)}$ is *indecomposable* if either k is *totally imaginary* or $l \neq 2$. Note that this implies that $G_k^{(l)}$ is *strongly indecomposable* if either k is *totally imaginary* or $l \neq 2$.

It remains to consider the case where k is *not totally imaginary*, and $l = 2$. Thus, k does not contain a primitive cubic root of unity. Let k' be a *quadratic extension* of k generated by a *primitive cubic root of unity* $\zeta_3 \in k'$.

Thus, the field extension k'/k determines a *open subgroup* $H_{k'} \subseteq G_k^{(l)}$ of $G_k^{(l)}$. On the other hand, since $G_k^{(l)}$ is *slim* [cf. Lemma 4.6], the proof of the strong indecomposability of $G_k^{(l)}$ may be reduced to that of $H_{k'}$ [cf., e.g., the proof of Theorem 2.2]. But this has already been shown. This completes the proof in the case where k is *not totally imaginary*, and $l = 2$. □

Lemma 4.8. *Let n be a positive integer. Then there exist* **infinitely many prime numbers** $p \equiv 1 \pmod{n}$.

Proof. This is a special case of the *Dirichlet's prime number theorem*. In fact, in this case, an elementary proof may be given by applying the theory of *cyclotomic polynomials* [cf., e.g., [6], Chapter 8, Corollary 5.0.1]. □

Corollary 4.9. *Let l be a prime number; k a* **number field**. *Then any* **almost pro-l-maximal quotient** $G_k \twoheadrightarrow Q_k$ *of G_k is* **strongly indecomposable**.

Proof. Since Q_k is *slim* [cf. Lemma 4.6], the *strong indecomposability* of Q_k follows from Theorem 4.7 [cf., e.g., the proof of Theorem 2.2]. □

Theorem 4.10. *Let l be a prime number; n a positive integer; k a field of* **characteristic zero**; *$\overline{k}$ an algebraic closure of k; X a hyperbolic curve over k; X_n the n-th* **configuration space** *associated to X; Δ the* **maximal pro-l quotient** *of $\Pi_{X_n \times_k \overline{k}}$; Π the quotient of Π_{X_n} by the kernel of the natural surjection $\Pi_{X_n \times_k \overline{k}} \twoheadrightarrow \Delta$; $\rho : G_k \to \mathrm{Out}(\Delta)$ the* **natural outer Galois representation** *associated to the lower exact sequence of the following commutative diagram of profinite groups [cf. Lemma 3.1]*

$$\begin{tikzcd}
1 \arrow[r] & \Pi_{X_n \times_k \overline{k}} \arrow[r] \arrow[d, two heads] & \Pi_{X_n} \arrow[r] \arrow[d, two heads] & G_k \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta \arrow[r] & \Pi \arrow[r] & G_k \arrow[r] & 1.
\end{tikzcd}$$

Then the following hold:

(i) Δ *is* **slim**, **topologically finitely generated**, *and* **indecomposable**.

(ii) *The profinite group* $\mathrm{Out}(\Delta)$ *is* **almost pro-l**.

(iii) *Suppose that k is* **l-cyclotomically full**. *Then ρ is* **nontrivial**.

(iv) *Suppose that G_k is* **center-free** *and* **indecomposable**, *and that k is* **l-cyclotomically full**. *Then Π is* **center-free** *and* **indecomposable**.

(v) *Any open normal* **pro-l** *subgroup* N *of* $\mathrm{Out}(\Delta)$ *[cf.* (ii)*] determines an* **almost pro-l-maximal quotient**

$$p_N : G_k \twoheadrightarrow G_k^{(N)}$$

of G_k*, as well as a factorization* $\rho = \rho_N \circ p_N$*, for a uniquely determined morphism*

$$\rho_N : G_k^{(N)} \to \mathrm{Out}(\Delta).$$

(vi) *In the notation of* (v)*, suppose that* $G_k^{(N)}$ *is* **center-free** *and* **indecomposable***, and that* k *is* **l-cyclotomically full**. *Then the profinite group*

$$\Delta \overset{\mathrm{out}}{\rtimes} G_k^{(N)} \overset{\mathrm{def}}{=} \mathrm{Aut}(\Delta) \times_{\mathrm{Out}(\Delta)} G_k^{(N)}$$

is **center-free** *and* **indecomposable**. *In particular, if* k *is an* **l-adic local field** *or a* **number field***, then* $\Delta \overset{\mathrm{out}}{\rtimes} G_k^{(N)}$ *is* **center-free** *and* **indecomposable**.

Proof. First, we consider assertion (i). The fact that Δ is *slim* and *topologically finitely generated* is the content of [22], Proposition 2.2, (ii). Thus, we verify the *indecomposability* of Δ. Write $X_{\overline{k}} \overset{\mathrm{def}}{=} X \times_k \overline{k}$. Let $(X_{\overline{k}})_n$ be the n-th *configuration space* associated to the *hyperbolic curve* $X_{\overline{k}}$. [In particular, we have a natural isomorphism $(X_{\overline{k}})_n \overset{\sim}{\to} X_n \times_k \overline{k}$ of $\overline{k}$-schemes.] Here, we note that, for $n \geq 1$, any *projection morphism* $(X_{\overline{k}})_n \to (X_{\overline{k}})_{n-1}$ of *length one* determines a natural exact sequence of profinite groups [cf. [22], Proposition 2.2, (i)]

$$1 \longrightarrow \Pi^{(l)}_{((X_{\overline{k}})_n)_{\overline{x}}} \longrightarrow \Pi^{(l)}_{(X_{\overline{k}})_n} \longrightarrow \Pi^{(l)}_{(X_{\overline{k}})_{n-1}} \longrightarrow 1$$

— where $\overline{x}$ is a *geometric point* of $(X_{\overline{k}})_{n-1}$; we write $(X_{\overline{k}})_0 \overset{\mathrm{def}}{=} \mathrm{Spec}(\overline{k})$; $((X_{\overline{k}})_n)_{\overline{x}}$ denotes the *fiber* of $(X_{\overline{k}})_n \to (X_{\overline{k}})_{n-1}$ over $\overline{x}$. Also, note that $\Pi^{(l)}_{((X_{\overline{k}})_n)_{\overline{x}}}$ and $\Pi^{(l)}_{(X_{\overline{k}})_1}$ are *indecomposable* [cf. [22], Proposition 3.2]. Moreover, it is well-known that the *natural outer Galois representation*

$$\Pi^{(l)}_{(X_{\overline{k}})_{n-1}} \to \mathrm{Out}(\Pi^{(l)}_{((X_{\overline{k}})_n)_{\overline{x}}})$$

associated to the above exact sequence is *injective* [cf. [2], Remark following the proof of Theorem 1], hence, in particular, *nontrivial.* Thus, by induction on n, it follows from Proposition 1.8, (i), that Δ is *indecomposable.* Assertion (ii) follows from (i) and [1], Corollary 7.

Next, we consider assertion (iii). Suppose that ρ is *trivial.* Then by considering the composites of the *first projections*

$$X_n \to X_{n-1} \to \ldots \to X_1,$$

it follows immediately that the natural outer Galois representation

$$G_k \to \mathrm{Out}(\Pi^{(l)}_{X_{\overline{k}}})$$

is *trivial.* But this contradicts Lemma 3.3, (iv). Thus, ρ is *nontrivial.*

Next, we consider assertion (iv). The *center-freeness* of Π follows immediately from (i) and the assumption that G_k is *center-free.* Moreover, in light of (i) and (iii), by applying Proposition 1.8, (i), to the exact sequence

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow G_k \longrightarrow 1,$$

we conclude that Π is *indecomposable.* Assertion (v) follows immediately from the various definitions involved.

Finally, we consider assertion (vi). Since Δ is *center-free* [cf. (i)], by *pulling back* the natural exact sequence of profinite groups

$$1 \longrightarrow \Delta \longrightarrow \mathrm{Aut}(\Delta) \longrightarrow \mathrm{Out}(\Delta) \longrightarrow 1$$

via ρ_N, we obtain the following exact sequence of profinite groups

$$1 \longrightarrow \Delta \longrightarrow \Delta \overset{\mathrm{out}}{\rtimes} G_k^{(N)} \longrightarrow G_k^{(N)} \longrightarrow 1.$$

In particular, the *center-freeness* of $\Delta \overset{\mathrm{out}}{\rtimes} G_k^{(N)}$ follows immediately from (i) and the assumption that $G_k^{(N)}$ is *center-free.* Moreover, in light of (i) and (iii) [cf. also (v)], by applying Proposition 1.8, (i), we conclude that $\Delta \overset{\mathrm{out}}{\rtimes} G_k^{(N)}$ is *indecomposable.* Finally, in the case where k is an *l-adic local field* or a *number field*, recall that $G_k^{(N)}$ is *center-free* [cf. Proposition 1.6, Lemma 4.6] and *indecomposable* [cf. Proposition 1.6, Corollary 4.9], and that k is *l-cyclotomically full* [cf. Lemma 3.3, (v)]. □

5. Indecomposability of the Pro-l Grothendieck-Teichmüller Group

In this section, we verify the indecomposability of the *pro-l Grothendieck-Teichmüller group* GT_l [cf. Theorem 5.4] as a consequence of a certain *anabelian result over finite fields* [cf. [7], Remark 6, (iv)].

Definition 5.1. (cf. [20], Definition 1.11, (i)) Let l be a prime number; k an algebraically closed field of characteristic zero; X the tripod $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ over k; X_2 the second configurartion space associated to X. Suppose that $\Pi_1 \in \{\Pi_X, \Pi_X^{(l)}\}$. Write

$$\Pi_2 \overset{\mathrm{def}}{=} \begin{cases} \Pi_{X_2}, & \text{if } \Pi_1 = \Pi_X, \\ \Pi_{X_2}^{(l)}, & \text{if } \Pi_1 = \Pi_X^{(l)}. \end{cases}$$

Then for $n = 1$, 2, we shall write

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$$

for the subgroup of $\mathrm{Out}(\Pi_n)$ consisting of *FC-admissible* outomorphisms of Π_n [cf. [20], Definition 1.1, (ii)];

$$\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

for the subgroup of $\mathrm{Out}(\Pi_n)$ consisting of *FC-admissible* outomorphisms of Π_n that *commute with the outer modular symmetries* [cf. [20], Definition 1.1, (vi)];

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_1)^{\Delta+} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$$

for the *image* of $\mathrm{Out}^{\mathrm{FCS}}(\Pi_2)$ via the natural injection $\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ induced by the first projection $X_2 \to X$ [cf. [20], Definition 1.11, (i); [20], Corollary 1.12, (ii); [20], Corollary 4.2, (i)]. We shall refer to

$$\mathrm{GT} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_X)^{\Delta+} \quad (\text{respectively, } \mathrm{GT}_l \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_X^{(l)})^{\Delta+})$$

as the *Grothendieck-Teichmüller group* (respectively, *pro-l Grothendieck-Teichmüller group*).

Remark 5.2. GT as defined in Definition 5.1 coincides with the Grothendieck-Teichmüller group as defined in more classical works [cf. [20], Remark 1.11.1].

The following result is well-known.

Lemma 5.3. *Let l be a prime number. Then* GT*,* GT_l *are* **slim***.*

Proof. The asserted slimness follows immediately from the [*pro-l*] *Grothendieck Conjecture over number fields* [i.e., [16], Theorem A, applied to a *tripod* over a number field] and [11], Lemma 3.5. □

Theorem 5.4. *Let l be a prime number. Then* GT_l *is* **strongly indecomposable***.*

Proof. To verify the assertion, it suffices to show that for any open subgroup $U \subseteq \mathrm{GT}_l$ of GT_l, U is *indecomposable*. Let F be a *finite field* of characteristic $\neq l$. Write Δ for the maximal pro-l quotient of the étale fundamental group of the *tripod* $\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}$ over $\overline{F}$, where $\overline{F}$ is an algebraic closure of F, and

$$\rho : G_F \to \mathrm{Out}(\Delta)$$

for the *pro-l outer Galois representation* associated to $\mathbb{P}^1_F \setminus \{0, 1, \infty\}$. It follows immediately from the various definitions involved that $G \stackrel{\mathrm{def}}{=} \rho(G_F)$ is *contained* in $\mathrm{GT}_l \subseteq \mathrm{Out}(\Delta)$. Thus, by replacing F by a suitable finite extension of F, we may assume without loss of generality that $G \subseteq U$. Moreover, since $\mathrm{Out}(\Delta)$ is *almost pro-l* [cf. [1], Corollary 7], by replacing F by a suitable finite extension of F, we may assume without loss of generality that ρ factors through the *maximal pro-l quotient* $G_F \twoheadrightarrow G_F^{(l)}$ of G_F. Here, note that since G is *infinite* [cf. Lemmas 3.3, (i); 3.3, (iv)], we have $G \cong \mathbb{Z}_l$.

Now suppose that we have an isomorphism of profinite groups $U \cong H_1 \times H_2$. In the following, we shall *identify* U and $H_1 \times H_2$ via this isomorphism. Then I *claim* that it holds that

$$\text{either} \quad G \cap H_1 \neq \{1\} \quad \text{or} \quad G \cap H_2 \neq \{1\}.$$

Indeed, suppose that $G \cap H_1 = \{1\}$ and $G \cap H_2 = \{1\}$. In particular, it follows that, for $i = 1, 2$, the composite

$$G \hookrightarrow U = H_1 \times H_2 \overset{\mathrm{pr}_i}{\twoheadrightarrow} H_i$$

— where pr_i is i-th projection — is *injective*. Thus, if we write $K_i \subseteq H_i$ for the *image* of the above composite, we obtain that $G \overset{\sim}{\to} K_i$ $[\cong \mathbb{Z}_l]$. Here, note that we have *inclusions*

$$G \subseteq K \overset{\mathrm{def}}{=} K_1 \times K_2 \subseteq H_1 \times H_2.$$

Thus, since K $[\cong \mathbb{Z}_l \times \mathbb{Z}_l]$ is *abelian*, we obtain that

$$K \subseteq Z_{\mathrm{GT}_l}(G) \hookrightarrow \mathbb{Z}_l^{\times}$$

— where "$\hookrightarrow$" is induced by the morphism "$\deg_{\mathbb{P}}$" of [7], Definition 3.1, which is *injective* by [7], Remark 6, (iv); [11], Lemma 3.5. In particular, by considering a suitable open subgroup of K, we obtain that $\mathbb{Z}_l \times \mathbb{Z}_l \cong \mathbb{Z}_l$, a contradiction. This completes the proof of the *claim*.

In light of the *claim*, we may assume without loss of generality that

$$G \cap H_1 \neq \{1\}.$$

Then since $G \cap H_1 \subseteq G$ is a *nontrivial* closed subgroup of $G \cong \mathbb{Z}_l$, it follows that $G \cap H_1$ is *open* in G. Thus, by replacing F by a suitable finite extension, we may assume without loss of generality that $G \subseteq H_1$. In particular, we obtain that

$$H_2 \subseteq Z_{\mathrm{GT}_l}(G) \hookrightarrow \mathbb{Z}_l^{\times}$$

— where "$\hookrightarrow$" denotes the arrow "$\hookrightarrow$" in the final display of the proof of the above *claim*. Thus, it follows that H_2 is *abelian*. On the other hand, since H_2 is *center-free* [cf. Lemma 5.3], we obtain that $H_2 = \{1\}$. Therefore, we conclude that U is *indecomposable*, as desired. $\square$

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