Notes on Schubert, Grothendieck and Key Polynomials

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Abstract. We introduce common generalization of (double) Schubert, Grothendieck, Demazure, dual and stable Grothendieck polynomials, and Di Francesco–Zinn-Justin polynomials. Our approach is based on the study of algebraic and combinatorial properties of the reduced rectangular plactic algebra and associated Cauchy kernels.

Key words: plactic monoid and reduced plactic algebras; nilCoxeter and idCoxeter algebras; Schubert, β-Grothendieck, key and (double) key-Grothendieck, and Di Francesco–Zinn-Justin polynomials; Cauchy's type kernels and symmetric, totally symmetric plane partitions, and alternating sign matrices; Noncrossing Dyck paths and (rectangular) Schubert polynomials; multi-parameter deformations of Genocchi numbners of the first and the second types; Gandhi–Dumont polynomials and (staircase) Schubert polynomials; double affine nilCoxeter algebras

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To the memory of Alexander Grothendieck (1928–2014)

Contents

Extended abstract

We introduce certain finite-dimensional algebras denoted by $\mathcal{PC}_n$ and $\mathcal{PF}_{n,m}$ which are certain quotients of the plactic algebra $\mathcal{P}_n$, which had been introduced by A. Lascoux and M.-P. Schützenberger [46]. We show that $\dim(\mathcal{PF}_{n,k})$ is equal to the number of symmetric plane partitions fitting inside the box $n \times k \times k$, $\dim(\mathcal{PC}_n)$ is equal to the number of alternating sign matrices of size $n \times n$, moreover,

$$\dim(\mathcal{PF}_{n,n}) = \mathrm{TSPP}(n+1) \times \mathrm{TSSCPP}(n),$$
$$\dim(\mathcal{PF}_{n,n+1}) = \mathrm{TSPP}(n+1) \times \mathrm{TSSCPP}(n+1),$$
$$\dim(\mathcal{PF}_{n+2,n}) = \dim(\mathcal{PF}_{n,n+1}), \qquad \dim(\mathcal{PF}_{n+3,n}) = \frac{1}{2}\dim(\mathcal{PF}_{n+1,n+1}),$$

and study decomposition of the Cauchy kernels corresponding to the algebras $\mathcal{PC}_n$ and $\mathcal{PF}_{n,m}$; as well as introduce polynomials which are common generalizations of the (double) Schubert, β-Grothendieck, Demazure (known also as *key polynomials*), (plactic) key-Grothendieck, (plactic) Stanley and stable β-Grothendieck polynomials. Using a family of the Hecke type divided difference operators we introduce polynomials which are common generalizations of the Schubert, β-Grothendieck, dual β-Grothendieck, β-Demazure–Grothendieck, and Di Francesco–Zinn-Justin polynomials. We also introduce and study some properties of the double affine nilCoxeter algebras and related polynomials, put forward a q-deformed version of the Knuth relations and plactic algebra.

1 Introduction

The Grothendieck polynomials had been introduced by A. Lascoux and M.-P. Schützenberger in [48] and studied in detail in [40]. There are two equivalent versions of the Grothendieck polynomials depending on a choice of a basis in the Grothendieck ring $K^\star(\mathcal{F}l_n)$ of the complete flag variety $\mathcal{F}l_n$. The basis $\{\exp(\xi_1), \ldots, \exp(\xi_n)\}$ in $K^*(\mathcal{F}l_n)$ is one choice, and another choice is the basis $\{1 - \exp(-\xi_j),\ 1 \le j \le n\}$, where $\{\xi_j,\ 1 \le j \le n\}$ denote the Chern classes of the tautological linear bundles L_j over the flag variety $\mathcal{F}l_n$. In the present paper we use the basis in a deformed Grothendieck ring $K^{*,\beta}(\mathcal{F}_n)$ of the flag variety $\mathcal{F}l_n$ generated by the set of elements $\{x_i = x_i^{(\beta)} = 1 - \exp(\beta\xi_i),\ i = 1, \ldots, n\}$. This basis has been introduced and used for construction of the β-Grothendieck polynomials in [17, 19].

A basis in the classical Grothendieck ring of the flag variety in question corresponds to the choice $\beta = -1$. For arbitrary β the ring generated by the elements $\{x_i^{(\beta)},\ 1 \le i \le n\}$ has been identified with the Grothendieck ring corresponding to the generalized cohomology theory associated with the multiplicative formal group law $F(x, y) = x + y + \beta xy$, see [23]. The Grothendieck polynomials corresponding to the classical K-theory ring $K^\star(\mathcal{F}l_n)$, i.e., the case $\beta = -1$, had been studied in depth by A. Lascoux and M.-P. Schützenberger in [49]. The β-Grothendieck polynomials has been studied in [17, 18, 23].

The *plactic monoid* over a finite totally ordered set $\mathbb{A} = \{a < b < c < \cdots < d\}$ is the quotient of the free monoid generated by elements from $\mathbb{A}$ subject to the elementary Knuth transformations [29]

$$bca = bac \quad \& \quad acb = cab, \qquad \text{and} \qquad bab = bba \quad \& \quad aba = baa, \tag{1.1}$$

for any triple $\{a < b < c\} \subset \mathbb{A}$.

To our knowledge, the concept of "plactic monoid" has its origins in a paper by C. Schensted [64], concerning the study of the longest increasing subsequence of a permutation, and a paper by D. Knuth [29], concerning the study of combinatorial and algebraic properties of the Robinson–Schensted correspondence[1].

As far as we know, this monoid and the (unital) algebra $\mathcal{P}(\mathbb{A})$ corresponding to that monoid[2], had been introduced, studied and used by M.-P. Schützenberger, see [65, Section 5], to give the first complete proof of the famous *Littlewood–Richardson rule* in the theory of symmetric functions. A bit later this monoid, was named the "monoïde plaxique" and studied in depth by A. Lascoux and M.-P. Schützenberger [46]. The algebra corresponding to the plactic monoid is commonly known as *plactic algebra*. One of the basic properties of the plactic algebra [65] is that it contains the distinguish commutative subalgebra which is generated by noncommutative elementary (quasi-symmetric) polynomials[3]

$$e_k(\mathbb{A}_n) = \sum_{i_1 > i_2 > \cdots > i_k} a_{i_1} a_{i_2} \cdots a_{i_k}, \qquad k = 1, \ldots, n,$$

see, e.g., [65, Corollary 5.9] and [16].

We refer the reader to nice written overview [44] of the basic properties and applications of the plactic monoid in combinatorics.

It is easy to see that the plactic relations for two letters $a < b$, namely,

$$aba = baa, \qquad bab = bba,$$

imply the commutativity of noncommutative elementary polynomials in two variables. In other words, the plactic relations for two letters imply that

$$ba(a+b) = (a+b)ba, \qquad a < b.$$

It has been proved in [16] that these relations together with the Knuth relations (1.1) for three letters $a < b < c$, imply the commutativity of noncommutative elementary quasi-symmetric polynomials for any number of variables.

In the present paper we prove that in fact the commutativity of noncommutative elementary quasi-symmetric polynomials for $n = 2$ and $n = 3$ implies the commutativity of that polynomials for all n, see Theorem 2.23[4].

One of the main objectives of the present paper is to study combinatorial properties of the generalized plactic Cauchy kernel

$$\mathcal{C}(\mathfrak{P}_n, U) = \prod_{i=1}^{n-1} \left\{ \prod_{j=n-1}^{i} (1 + p_{i,j-i+1} u_j) \right\},$$

where $\mathfrak{P}_n$ stands for the set of parameters $\{p_{ij},\ 2 \le i+j \le n+1,\ i > 1,\ j > 1\}$, and $U := U_n$ stands for a certain noncommutative algebra we are interested in, see Section 5.

[1] See, e.g., wiki/Robinson–Schensted correspondence.

[2] If $\mathbb{A} = \{1 < 2 < \cdots < n\}$, the elements of the algebra $\mathcal{P}(\mathbb{A})$ can be identified with semistandard Young tableaux. It was discovered by D. Knuth [29] that modulo *Knuth equivalence* the equivalence classes of semistandard Young tableaux form an algebra, and he has named this algebra by *tableaux algebra*. It is easily seen that the tableaux algebra introduced by D. Knuth is isomorphic to the algebra introduced by M.-P. Schützenberger [65].

[3] See, e.g., [36] for definition of noncommutative quasi-symmetric functions and polynomials.

[4] Let us stress that conditions necessary and sufficient to assure the commutativity of noncommutative elementary polynomials for the number of variables equals $n = 2$ and $n = 3$ turn out to be weaker then that listed in [16].

We also want to bring to the attention of the reader on some interesting combinatorial properties of *rectangular* Cauchy kernels

$$\mathcal{F}(\mathfrak{P}_{n,m},U)=\prod_{i=1}^{n-1}\left\{\prod_{j=m-1}^{1}(1+p_{i,\overline{i-j+1}^{(m)}}u_j)\right\},$$

where $\mathfrak{P}_{n,m}=\{p_{ij}\}_{\substack{1\le i\le n\\1\le j\le m}}$; see Definition 6.6 for the meaning of symbol $\overline{a}^{(m)}$.

We treat these kernels in the (reduced) plactic algebras $\mathcal{PC}_n$ and $\mathcal{PF}_{n,m}$ correspondingly. The algebras $\mathcal{PC}_n$ and $\mathcal{PF}_{n,m}$ are finite-dimensional and have bases parameterized by certain Young tableaux described in Sections 5.1 and 6 correspondingly. Decomposition of the rectangular Cauchy kernel with respect to the basis in the algebra $\mathcal{PF}_{n,m}$ mentioned above, gives rise to a set of polynomials which are common generalizations of the (double) Schubert. β-Grothendieck, Demazure and Stanley polynomials. To be more precise, the polynomials listed above correspond to certain quotients of the plactic algebra $\mathcal{PF}_{n,m}$ and appropriate specializations of parameters $\{p_{ij}\}$ involved in our definition of polynomials $U_\alpha(\{p_{ij}\})$, see Section 6.

As it was pointed out in the beginning of Introduction, the Knuth (or plactic) relations (1.1) have been discovered in [29] in the course of the study of algebraic and combinatorial properties of the Robinson–Schensted correspondence. Motivated by the study of basic properties of a *quantum* version of the tropical/geometric Robinson–Schensted–Knuth correspondence – work in progress, but see [3, 26, 28, 59, 60] for definition and basic properties of the tropical/geometric RSK, – the author of the present paper came to a discovery that certain deformations of the Knuth relations *preserve* the Hilbert series (resp. the Hilbert polynomials) of the plactic algebras $\mathcal{P}_n$ and $\mathcal{F}_n$ (resp. the algebras $\mathcal{PC}_n$ and $\mathcal{PF}_n$).

More precisely, let $\{q_2,\dots,q_n\}$ be a set of (mutually commuting) parameters, and $\mathbb{U}_n:=\{u_1,\dots,u_n\}$ be a set of generators of the free associative algebra over $\mathbb{Q}$ of rank n. Let $Y,Z\subset[1,n]$ be subsets such that $Y\cup Z=[1,n]$ and $Y\cap Z=\varnothing$. Let us set $p(a)=0$, if $a\in Y$ and $p(a)=1$, if $a\in Z$. Define q-deformed super Knuth relations among the generators $u_1,\dots,u_n$ as follows:

$$\begin{aligned}\mathrm{SPL}_q\colon\ &(-1)^{p(i)p(k)}q_ku_ju_iu_k=u_ju_ku_i, \qquad i<j\le k,\\ &(-1)^{p(i)p(k)}q_ku_iu_ku_j=u_ku_iu_j, \qquad i\le j<k.\end{aligned}$$

We define

- q-deformed superplactic algebra $\mathcal{SQP}_n$ to be the quotient of the free associative algebra $\mathbb{Q}\langle u_1,\dots,u_n\rangle$ by the two-sided ideal generated by the set of $\mathbf{q}$-deformed Knuth relations ($\mathrm{SPL}_{\mathbf{q}}$),
- reduced q-deformed superplactic algebras $\mathcal{SQPC}_n$ and $\mathcal{SQPF}_{n,m}$ to be the quotient of the algebra $\mathcal{SQP}_n$ by the two-sided ideals described in Definitions 5.19 and 6.7 correspondingly.

We state

Conjecture 1.1. *The algebra $\mathcal{SQP}_n$ and the algebras $\mathcal{SQPC}_n$ and $\mathcal{SQPF}_{n,m}$, are flat deformations of the algebras $\mathcal{P}_n$, $\mathcal{PC}_n$ and $\mathcal{PF}_{n,m}$ correspondingly.*

In fact one can consider more general deformation of the Knuth relations, for example take a set of parameters $\mathbf{Q}:=\{q_{ik},\ 1\le i<k\le n\}$ and impose on the set of generators $\{u_1,\dots,u_n\}$ the following relations

$$q_{ik}u_ju_iu_k=u_ju_ku_i, \qquad i<j\le k, \qquad q_{ik}u_iu_ku_j=u_ku_iu_j, \qquad i\le j<k.$$

However we don't know how to describe a set of conditions on parameters $\mathbf{Q}$ which imply the flatness of the corresponding quotient algebra(s), as well as we don't know an interpretation and dimension of the algebras $\mathcal{SQPC}_n$ and $\mathcal{SQPF}_{n,m}$ for a "generic" values of parameters $\mathbf{Q}$. We *expect* the dimension of algebras $\mathcal{SQPC}_n$ and $\mathcal{SQPF}_{n,m}$ each depends piece-wise polynomially on a set of parameters $\{q_{ij} \in \mathbb{Z}_{\geq 0},\, 1 \leq i < j \leq n\}$, and pose a problem to describe its polynomiality chambers.

We also mention and leave for a separate publication(s), the case of algebras and polynomials associated with *superplactic* monoid [38, 56], which corresponds to the relations SPL_q with $q_i = 1, \forall\, i$. Finally we point out an interesting and important paper [55] wherein the case $Z = \varnothing$, and the all deformation parameters are equal to each other, has been independently introduced and studied in depth.

Let us repeat that the important property of plactic algebras $\mathcal{P}_n$ is that the noncommutative elementary polynomials

$$e_k(u_1,\ldots,n_{n-1}) := \sum_{n-1\geq a_1\geq a_2\geq a_k\geq 1} u_{a_1}\cdots u_{a_k}, \qquad k=1,\ldots,n-1,$$

generate a commutative subalgebra inside of the plactic algebra $\mathcal{P}_n$, see, e.g., [16, 46]. Therefore all our finite-dimensional algebras introduced in the present paper, have a distinguish finite-dimensional commutative subalgebra. We have in mined to describe these algebras explicitly in a separate publication.

In Section 2 we state and prove necessary and sufficient conditions in order the elementary noncommutative polynomials form a mutually commuting family. Surprisingly enough to check the commutativity of noncommutative elementary polynomials for any n, it's enough to check these conditions only for $n = 2, 3$. However a combinatorial meaning of a generalization of the Lascoux-Schützenberger plactic algebra $\mathcal{P}_n$ obtained in this way, is still missing.

The plactic algebra $\mathcal{PF}_{n,m}$ introduced in Section 6, has a monomial basis parametrized by the set of Young tableaux of shape $\lambda \subset (n^m)$ filled by the numbers from the set $\{1,\ldots,m\}$. In the case $n = m$ it is well-known [20, 35, 58], that this number is equal to the number of symmetric plane partitions fitting inside the cube $n \times n \times n$. Surprisingly enough this number admits a factorization in the product of the number of totally symmetric plane partitions (TSPP) by the number of totally symmetric self-complementary plane partitions (TSSCPP) fit inside the same cube. A similar phenomenon happens if $|m - n| \leq 2$, see Section 6. More precisely, we add to the well-known equalities

$$\#|\mathcal{B}_{1,n}| = 2^n, \qquad \#|\mathcal{B}_{2,n}| = \binom{2n+1}{n}, \qquad \#|\mathcal{B}_{3,n}| = 2^n\,\mathrm{Cat}_{n+1} \qquad [67, A003645],$$
$$\#|\mathcal{B}_{4,n}| = \frac{1}{2}\,\mathrm{Cat}_{n+1}\,\mathrm{Cat}_{n+2} \qquad [67, A000356],$$
$$\#|\mathcal{B}_{n,5}| = \frac{\binom{n+5}{5}\binom{n+7}{7}\binom{n+9}{9}}{\binom{n+2}{2}\binom{n+4}{4}} \qquad [67, A133348],$$

the following relations

$$\#|\mathcal{B}_{n,n}| = \mathrm{TSPP}(n+1)\times\mathrm{ASM}(n), \qquad \#|\mathcal{B}_{n,n+1}| = \mathrm{TSPP}(n+1)\times\mathrm{ASM}(n+1),$$
$$\#|\mathcal{B}_{n+2,n}| = \#|\mathcal{B}_{n,n+1}|, \qquad \#|\mathcal{B}_{n+3,n}| = \frac{1}{2}\#|\mathcal{B}_{n+1,n+1}|,$$
$$\#|\mathrm{PP}(n)| = \#|\mathrm{TSSCPP}(n)| \times \#|\mathrm{ASMHT}(2n)| = \#|\mathrm{CSSCPP}(2n)| \times \#|\mathrm{CSPP}(n)|,$$
$$\#|\mathrm{CSPP}(2n)| = \#|\mathrm{TSPP}(2n)| \times \#|\mathrm{CSTCPP}(2n)|,$$
$$\#|\mathrm{CSPP}(2n+1)| = 2^{2n}\#|\mathrm{TSPP}(2n+1)| \times \#|\mathrm{TSPP}(2n)|,$$

where $\mathrm{PP}(n)$ stands for the set of plane partitions fit in a cub of size $n \times n \times n$; $\mathrm{AMSHT}(2n)$ denotes the set of alternating sign matrices of size $2n \times 2n$ invariant under a half-turn and $\mathrm{CSSPP}(2n)$ denotes the set of cyclically symmetric self-complementary plane partitions fitting inside a cub of size $2n \times 2n \times 2n$, see, e.g., [6]; $\mathrm{CSTCPP}(n)$ stands for the set of cyclically symmetric transpose complementary plane partitions fitting inside a cub of size $2n \times 2n \times 2n$, see, e.g., [67, *A*051255]. See Section 6 for the definition of the sets $\mathcal{B}_{n,m}$ and examples. In Exercise 6.3 we state some (new) divisibility properties of the numbers $\#|B_{n+4,n}|$.

It is well-known that $\mathrm{ASMHT}(2n) = \mathrm{ASM}(n) \times \mathrm{CSPP}(n)$, where $\mathrm{CSPP}(n)$ denotes the number of cyclically symmetric plane partitions fitting inside n-cube, and $\mathrm{CSSCPP}(2n) = \mathrm{ASM}(n)^2$, see, e.g., [6, 37] and [67, *A*006366].

Problem 1.2.

- *Construct bijection between the set of plane partitions fit inside n-cube and the set of (ordered) triples $(\pi_1, \pi_2, \wp)$, where (π_1, π_2) is a pair of* $\mathrm{TSSCPP}(n)$ *and $\wp$ is a cyclically symmetric plane partition fitting inside n-cube.*
- *Describe the involution $\kappa \colon \mathrm{PP}(n) \longrightarrow \mathrm{PP}(n)$ which is induced by the involution $(\pi_1, \pi_2, \wp) \longrightarrow (\pi_2, \pi_1, \wp)$ on the set* $\mathrm{TSSCPP}(n) \times \mathrm{TSSCP}(n) \times \mathrm{CSPP}(n)$*, and its fixed points. Clearly one has $\#|\mathrm{Fix}(\kappa)| = \mathrm{ASMHT}(2n)$.*
- *Characterize pairs of plane partitions $(\Pi_1, \Pi_2) \in \mathrm{PP}(n) \times \mathrm{PP}(n)$ such that*

$$(a) \quad \wp(\Pi_1) = \wp(\Pi_2); \qquad (b) \quad (\pi_1(\Pi_1), \pi_2(\Pi_1)) = (\pi_1(\Pi_2), \pi_2(\Pi_2)).$$

These relations have straightforward proofs based on the explicit product formulas for the numbers

$$\#|\mathrm{SPP}(n)| = \prod_{1 \le i \le j \le k} \frac{n+i+j+k-1}{i+j+k-1},$$

$$\#|\mathrm{TSPP}(n)| = \prod_{i=1}^{n} \prod_{j=i}^{n} \prod_{k=j}^{n} \frac{i+j+k-1}{i+j+k-2},$$

$$\#|\mathrm{PP}(n)| = \prod_{i=0}^{n} \prod_{j=1}^{n-1} \frac{3n-i-j}{2n-i-j} = \prod_{i=1}^{n} \frac{\binom{2n+i}{n}}{\binom{n+i}{n}},$$

but bijective proofs of these identities are an open problem.

It follows from [43, 46] that the dimension of the (reduced) plactic algebra $\mathcal{PC}_n$ is equal to the number of alternating sign matrices of size $n \times n$ (note that $\mathrm{ASM}(n) = \mathrm{TSSCPP}(n)$). Therefore the key-Grothendieck polynomials can be obtained from U-polynomials (see Section 6, Theorem 6.12) after the specialization $p_{ij} = 0$, if $i + j > n + 1$.

In Section 4 following [27] we introduce and study a family of polynomials which are a common generalization of the Schubert, β-Grothendieck, dual β-Grothendieck, β-Demazure, β-key-Grothendieck, Bott–Samelson and q-Demazure polynomials, Whittaker functions (see [7] and Lemma 4.18) and Di Francesco–Zinn-Justin polynomials (see Section 4). Namely, for any permutation $w \in \mathbb{S}_n$ and composition $\zeta \subset \delta_n := (n-1, n-2, \dots, 2, 1)$, we introduce polynomials

$$\mathcal{KN}_w^{(\beta,\alpha,\gamma,h)}(X_n) = h^{\ell(w)} T_{s_{i_1}} \cdots T_{s_{i_\ell}}\big(x^{\delta_n}\big),$$

$$\mathrm{KD}_\zeta^{(\beta,\alpha,\gamma,h)}(X_n) = h^{\ell(v_\zeta)} T_{s_{i_1}} \cdots T_{s_{i_\ell}}\big(x^{\zeta^+}\big),$$

where

$$T_i := T_i^{(\beta,\alpha,\gamma,h)} = -\alpha + ((\alpha+\beta+\gamma)x_i + \gamma x_{i+1} + h$$

$$+ h^{-1}(\alpha+\gamma)(\beta+\gamma)x_i x_{i+1})\partial_{i,i+1}, \qquad i=1,\dots,n-1,$$

denote a collection of divided difference operators which satisfy the Coxeter and Hecke relations

$$T_iT_jT_i = T_jT_iT_j, \qquad \text{if} \quad |i-j|=1; \qquad T_iT_j = T_jT_i, \qquad \text{if} \quad |i-j| \geq 2,$$
$$T_i^2 = (\beta-\alpha)T_i + \beta\alpha, \qquad i=1,\dots,n-1;$$

by definition for any permutation $w \in \mathbb{S}_n$ we set

$$T_w := T_{s_{i_1}} \cdots T_{s_{i_\ell}},$$

for any reduced decomposition $w = s_{i_1}\cdots s_{i_\ell}$ of a permutation in question; ζ^+ denotes a unique partition obtained from ζ by ordering its parts, and $v_\zeta \in \mathbb{S}_n$ denotes the minimal length permutation such that $v_\zeta(\zeta) = \zeta^+$.

Assume that $h=1$.[5] If $\alpha=\gamma=0$, these polynomials coincide with the β-Grothendieck polynomials [17], if $\beta=\alpha=1$, $\gamma=0$ these polynomials coincide with the Di Francesco–Zinn-Justin polynomials [12], if $\beta=\gamma=0$, these polynomials coincide with dual α-Grothendieck polynomials $\mathcal{H}_w^{(\alpha)}(X_n)$,[6] where by definition we set $X_n := (x_1,\dots,x_n)$.

Conjecture 1.3. *For any permutation $w \in \mathbb{S}_n$ and any composition $\zeta \subset \delta_n$, polynomials $\mathcal{KN}_w^{(\beta,\alpha,\gamma,h)}(X_n)$ and $\mathrm{KD}_\zeta^{(\alpha,\beta,\gamma,h)}(X_n)$ have nonnegative coefficients, i.e.,*

$$\mathcal{KN}_w^{(\beta,\alpha,\gamma,h)}(X_n) \in \mathbb{N}[\alpha,\beta,\gamma,h][X_n], \qquad \mathrm{KD}_\zeta^{(\beta,\alpha,\gamma,h)}(X_n) \in \mathbb{N}[\alpha,\beta,\gamma,h][X_n].$$

We *expect* that these polynomials have some geometrical meaning to be discovered.

More generally we study divided difference type operators of the form

$$T_{ij} := T_{ij}^{(a,b,c,h,e)} = a + (bx_i + cx_j + h + ex_ix_j)\partial_{ij},$$

depending on parameters a, b, c, h, e and satisfying the $2D$-Coxeter relations

$$T_{ij}T_{jk}T_{ij} = T_{jk}T_{ij}T_{jk}, \qquad 1 \leq i < j < k \leq n,$$
$$T_{ij}T_{kl} = T_{kl}T_{ij}, \qquad \text{if} \quad \{i,j\} \cap \{k,l\} = \varnothing.$$

We find that the necessary and sufficient condition which ensure the validity of the $2D$-Coxeter relations is the following relation among the parameters[7]:

$$(a+b)(a-c) + he = 0.$$

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[5] Clearly that if $h \neq 0$, then after rescaling parameters α, β and γ one can assume that $h=1$. However, see, e.g., [27, Section 5], the parameter h plays important role in the study of different specializations of the variables x_i, $1 \leq i \leq n-1$ and parameters α, β and γ.

[6] To avoid the reader's confusion, let us explain that in our paper we use the letter α either as the lower index to denote a composition, or as the upper index to denote a parameter which appears in certain polynomials treated in our paper. For example $\mathcal{H}_\alpha^{(\alpha)}(X_n)$ denotes the dual α-Grothendieck polynomial corresponding to a composition α. Note that the α-Grothendieck polynomial $\mathfrak{G}_w^{(\alpha)}(X_n)$ can be obtained from the polynomial $\mathfrak{G}_w^{(\beta)}(X_n)$ by replacing β by α.

[7] In other words, the divided difference operators $\{T_{ij} := T_{ij}^{(a,b,c,h,e)}\}$ which obey the $2D$-Coxeter relations, have the following form:

$$T_{ij}^{(a,b,c,h)} = \left(1 + \frac{(a+b)}{h}x_i\right)\left(1 + \frac{c-a}{h}x_j\right)\partial_{ij} + a\sigma_{ij}, \qquad \text{if} \quad h \neq 0, \quad a,\ b,\ c \text{ arbitrary.}$$

If $h=0$, then either $T_{ij} = ((c-a)x_j + ex_ix_j)\partial_{ij} + a\sigma_{ij}$, or $T_{ij} = (a+b)x_i + ex_ix_j)\partial_{ij} + a\sigma_{ij}$, where σ_{ij} stands for the exchange operator: $\sigma_{ij}(F(z_i,z_j)) = F(z_j,z_i)$.

Therefore, if the above relation between parameters a,b,c,h,e holds, then for any permutation $w \in \mathbb{S}_n$ the operator

$$T_w := T_w^{(a,b,c,h,e)} = T_{i_1}^{(a,b,c,h,e)} \cdots T_{i_\ell}^{(a,b,c,h,e)},$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is any reduced decomposition of w, is *well-defined*. Hence under the same assumption on parameters, for any permutation $w \in \mathbb{S}_n$ one can attach the well-defined polynomial

$$G_w^{(a,b,c,h,e)}(X,Y) := T_w^{(x)\,(a,b,c,h,e)}\Bigg(\prod_{\substack{i\ge 1, j\ge 1\\ i+j\le n+1}} (x_i+y_j)\Bigg),$$

and in much the same fashion to define polynomials

$$D_\alpha^{(a,b,c,h,e)}(X,Y) := T_{w_\alpha}^{(x)\,(a,b,c,h,e)}\big(x^{\alpha^+}\big)$$

for any composition α such that $\alpha_i \le n-i$, $\forall\, i$. We have used the notation $T^{(x)}{}_w^{(a,b,c,h,e)}$ to point out that this operator acts only on the variables $X=(x_1,\ldots,x_n)$; for any composition $\alpha \in \mathbb{Z}_{\ge 0}^n$, α^+ denotes a unique partition obtained from α by reordering its parts in (weakly) decreasing order, and w_α denotes a unique minimal length permutation in the symmetric group $\mathbb{S}_n$ such that $w_\alpha(\alpha) = \alpha^+$.

In the present paper we are interested in to list a conditions on parameters $A := \{a,b,c,h,e\}$ with the constraint

$$(a+b)(a-c) + he = 0,$$

which ensure that the above polynomials $G_w^{(a,b,c,h,e)}(X)$ and $D_\alpha^{(a,b,c,h,e)}(X)$ or their specialization $x_i = 1$, $\forall\, i$, have nonnegative coefficients. We state the following conjectures:

- $\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(X_n) \in \mathbb{N}[\alpha,\beta,\gamma][X_n]$,
- $G_w^{(-b,a+b+c,c,1,(b+c)(a+c)}(X_n) \in \mathbb{N}[a,b,c][X_n]$,
- $G_w^{(-b,a+b+c,c+d,1,(b+c+d)(a+c)}(x_i = 1, \forall\, i) \in \mathbb{N}[a,b,c,d]$, where a, b, c, d are free parameters.

In the present paper we treat the case

$$A = (-\beta, \beta+\alpha+\gamma, \gamma, 1, (\alpha+\gamma)(\beta+\gamma)). \tag{1.2}$$

As it was pointed above, in this case polynomials $G_w^A(X)$ are common generalization of Schubert, β-Grothendieck and dual β-Grothendieck, and Di Francesco–Zinn-Justin polynomials. We expect a certain interpretation of the polynomials G_w^A for general β, α and γ.

As it was pointed out earlier, one of the basic properties of the plactic monoid $\mathcal{P}_n$ is that the noncommutative elementary symmetric polynomials $\{e_k(u_1,\ldots,u_{n-1})\}_{1\le k\le n-1}$ generate a commutative subalgebra in the plactic algebra in question. One can reformulate this statement as follows. Consider the generating function

$$A_i(x) := \prod_{a=n-1}^{i} (1+xu_a) = \sum_{a=0}^{i} e_a(u_{n-1},\ldots,u_i)x^{i-a},$$

where we set $e_0(U) = 1$. Then the commutativity property of noncommutative elementary symmetric polynomials is equivalent to the following commutativity relation in the plactic as well as in the generic plactic, algebras $\mathcal{P}_n$ and $\mathfrak{P}_n$ [16], and Theorem 2.23,

$$A_i(x)A_i(y) = A_i(y)A_i(x), \qquad 1 \le i \le n-1.$$

Now let us consider the Cauchy kernel

$$\mathcal{C}(\mathfrak{P}_n, U) = A_1(z_1) \cdots A_{n-1}(z_{n-1}),$$

where we assume that the pairwise commuting variables $z_1, \ldots, z_{n-1}$ commute with the all generators of the algebras $\mathcal{P}_n$ and $\mathfrak{P}_n$. In what follows we consider the natural completion $\widehat{\mathfrak{P}}_n$ of the plactic algebra $\mathfrak{P}_n$ to allow consider elements of the form $(1+xu_i)^{-1}$. Elements of this form exist in any Hecke type quotient of the plactic algebra $\widetilde{\mathfrak{P}}_n$. Having in mind this assumption, let us compute the action of divided difference operators $\partial^z_{i,i+1}$ on the Cauchy kernel. In the computation below, the commutativity property of the elements $A_i(x)$ and $A_i(y)$ plays the key role. Let us start computation of $\partial^z_{i,i+1}(\mathcal{C}(\mathfrak{P}_n, U)) = \partial^z_{i,i+1}(A_1(z_1) \cdots A_{n-1}(z_{n-1}))$. First of all write $A_{i+1}(z_{i+1}) = A_i(z_{i+1})(1+z_{i+1}u_i)^{-1}$. According to the basic property of the elements $A_i(x)$, one sees that the expression $A_i(z_i)A_i(z_{i+1})$ is symmetric with respect to z_i and z_{i+1}, and hence is invariant under the action of divided difference operator $\partial^z_{i,i+1}$. Therefore,

$$\begin{aligned}\partial^z_{i,i+1}(\mathcal{C}(\mathfrak{P}_n, U)) &= A_1(z_1) \cdots A_i(z_i)A_i(z_{i+1})\partial^z_{i,i+1}\big((1+z_{i+1}u_i)^{-1}\big) \\ &\quad \times A_{i+2}(z_{i+2}) \cdots A_{n-1}(z_{n-1}).\end{aligned}$$

It is clearly seen that $\partial^z_{i,i+1}((1+z_{i+1}u_i)^{-1}) = (1+z_iu_i)^{-1}(1+z_{i+1}u_i)^{-1}u_i$. Therefore,

$$\partial^z_{i,i+1}(\mathcal{C}(\mathfrak{P}_n, U)) = A_1(z_1) \cdots A_i(z_i)A_{i+1}(z_{i+1})(1+z_iu_i)^{-1}u_iA_{i+2}(z_{i+2}) \cdots A_{n-1}(z_{n-1}).$$

It is easy to see that if one adds Hecke's type relations on the generators

$$u_i^2 = (a+b)u_i + ab, \qquad i = 1, \ldots, n-1,$$

then

$$(1+zu_i)^{-1}u_i = \frac{u_i - zab}{(1+bz)(1-az)}.$$

Therefore in the quotient of the plactic algebra $\mathfrak{P}_n$ by the Hecke type relations listed above and by the "locality" relations

$$u_iu_j = u_ju_i, \qquad \text{if} \quad |i-j| \geq 2,$$

one obtains

$$(-b + (1+z_ib))\partial^z_{i,i+1}\left(A_1(z_1) \cdots A_{n-1}(z_{n-1})\right) = \left(A_1(z_1) \cdots A_{n-1}(z_{n-1})\right)\left(\frac{e_i - b}{1 - az_i}\right).$$

Finally, if $a = 0$, then the above identity takes the following form

$$\partial^z_{i,i+1}\left((1+z_{i+1}b)A_1(z_1) \cdots A_{n-1}(z_{n-1})\right) = \left(A_1(z_1) \cdots A_{n-1}(z_{n-1})\right)(e_i - b).$$

In other words the above identity is equivalent to the statement [19] that in the idCoxeter algebra $\mathcal{IC}_n$ the Cauchy kernel $\mathcal{C}(\mathfrak{P}_n, U)$ is the generating function for the b-Grothendieck polynomials. Moreover, each (generalized) double b-Grothendieck polynomial is a positive linear combination of the key-Grothendieck polynomials.

A proof of this statement is a corollary of the more general statement which will be frequently used throughout the present paper, namely, if an equivalence relation $\approx_2$ is a refinement of that $\approx_1$, that is if assumption $a \approx_2 b \Longrightarrow a \approx_1 b$ holds $\forall\, a,\, b$, then each equivalence class w.r.t. relation $\approx_1$ is disjoint union of the equivalence classes w.r.t. relation $\approx_2$.

In the special case $b = -1$ and $P_{ij} = x_i + y_j$ if $2 \le i + j \le n + 1$, $p_{ij} = 0$, if $i + j > n + 1$, this result had been stated in [39].

As a possible mean to define *affine versions* of polynomials treated in the present paper, we introduce the *double affine nilCoxeter algebra of type A* and give construction of a generic family of Hecke's type elements[8] we will be put to use in the present paper.

In Section 5.1 we suggest a common generalization of some combinatorial formulas from [10] and [21]. Namely, we give explicit formula

$$\prod_{\substack{1\le i\le k,\ 1\le j\le n \\ j-i\le n-k}} \frac{N-i-j+1}{i+j-1} \prod_{\substack{1\le i\ \le\ k,\ 1\le j\le n \\ j-i\ >\ n-k}} \frac{N+i+j-1}{i+j-1} \tag{1.3}$$

for the number of k-tuples of noncrossing Dyck paths connecting the points $(0,0)$ and $(N, N - n - k)$. Interpretations of the number (1.3) as the number of certain k-triangulations of a convex $(N + 1)$-gon, or that of certain alternating sign matrices of size $N \times N$, are interesting tasks.

In the case $N = n + k$ we recover the [10, RHS of formula (2)]. In the case $k = 2$ our formula (1.3) is equivalent to that obtained in [21]. Our proof that the number (1.3) counts certain k-tuples of noncrossing Dyck paths is based on the study of combinatorial properties of the so-called column multi-Schur functions $s^*_\lambda(X_n)$ introduced in Theorem 5.6, cf. [58, 72]. In particular we show that for rectangular partition $\lambda = (n)^k$ the polynomial $s^*_\lambda(X_{n+k})$ is essentially coincide with the Schubert polynomial corresponding to the Richardson permutation $1^k \times w_0^{(n)}$. We introduce also a multivariable deformation of the numbers ASM(n), namely,

$$ASM(X_{n-1};t) := \sum_{\lambda\subset\delta_n} s^*_\lambda(X)\ t^{|\lambda|}. \tag{1.4}$$

Finally, in Section 5.1 we give combinatorial interpretations of rectangular and staircase components of the refined TSSCP vector [12] in terms of k-fans of noncrossing Dyck paths in rectangular case and Gadhi–Dumont polynomials and Genocchi numbers in staircase case.

As Appendix we include several examples of polynomials studied in the present paper to illustrate results obtained in these notes. We also include an expository text concerning the *MacNeille completion* of a poset to draw attention of the reader to this subject. It is an examination of the MacNeille completion of the poset associated with the (strong) Bruhat order on the symmetric group, that was one of the main streams of the study in the present paper. Namely, our concern was the challenge how to attach to each edge e of the MacNeille completion $\mathcal{MN}(\mathbb{S}_n)$ of the Bruhat order poset on the symmetric group $\mathbb{S}_n$ an operator ∂_e acting on the ring of polynomials $\mathbb{Z}[X_n]$, such that $\partial_e(K_{h(e)}) = K_{t(e)}$ together with compatibility conditions among the set of operators $\{\partial_e\}_{e\in\mathcal{MN}(\mathbb{S}_n)}$, that is for any two vertices of $\mathcal{MN}(\mathbb{S}_n)$, say α and β, and a path $p_{\alpha,\beta}$ in the MacNeille completion which connects these vertices, the naturally defined operator $\partial_{p_{\alpha,\beta}}$ depends only on the vertices α and β taken, and doesn't depend on a path $p_{\alpha,\beta}$ selected. As far as I know, this problem is still open.

2 Plactic, nilplactic and idplactic algebras

Definition 2.1 ([46])**.** The *plactic algebra* $\mathcal{P}_n$ is an (unital) associative algebra over $\mathbb{Z}$ generated by elements $\{u_1, \dots, u_{n-1}\}$ subject to the set of relations

$$\begin{aligned} &\text{(PL1)} \quad u_j u_i u_k = u_j u_k u_i, \qquad u_i u_k u_j = u_k u_i u_j, \qquad \text{if} \quad i < j < k, \\ &\text{(PL2)} \quad u_i u_j u_i = u_j u_i u_i, \qquad u_j u_i u_j = u_j u_j u_i, \qquad \text{if} \quad i < j. \end{aligned}$$

[8] Remind that by the name *a family of Hecke's type elements* we mean a set of elements $\{e_1, \dots, e_n\}$ such that $e_i^2 = Ae_i + B$, A, B are parameters (Hecke type relations), $e_i e_j = e_j e_i$, if $|i - j| \ge 2$, $e_i e_j e_i = e_j e_i e_j$, if $|i - j| = 1$ (Coxeter relations).

Proposition 2.2 ([46]). *Tableau words*[9] *in the alphabet* $U = \{u_1, \ldots, u_{n-1}\}$ *form a basis in the plactic algebra* $\mathcal{P}_n$.

In other words, each plactic class contain a unique tableau word. In particular,

$$\operatorname{Hilb}(\mathcal{P}_{n+1}, t) = (1-t)^{-n}\big(1-t^2\big)^{-\binom{n}{2}}.$$

Remark 2.3. There exists another algebra over $\mathbb{Z}$ which has the same Hilbert series as that of the plactic algebra $\mathcal{P}_n$. Namely, define algebra $\mathcal{L}_n$ to be an associative algebra over $\mathbb{Z}$ generated by the elements $\{e_1, e_2, \ldots, e_{n-1}\}$, subject to the set of relations

$$(e_i, (e_j, e_k)) := e_i e_j e_k - e_j e_i e_k - e_j e_k e_i + e_k e_j e_i = 0,$$

for all1 $\le i, j, k \le n-1$, $j < k$. Observe that the number of defining relations in the algebra $\mathcal{L}_n$ is equal to $2\binom{n}{3}$. Note that elements $e_1 + e_2$ and $e_2 e_1$ do not commute in the algebra $\mathcal{L}_3$, but do commute if are considered as elements in the plactic algebra $\mathcal{P}_3$. See Example 5.30(C) for some details.

Exercise 2.4.

- Show that the dimension of the degree k homogeneous component $\mathcal{L}_n^{(k)}$ of the algebra $\mathcal{L}_n$ is equal to the number semistandard Young tableaux of the size k filled by the numbers from the set $\{1, 2, \ldots, n-1\}$.
- Let us set $e_{ij} := (e_i, e_j) := e_i e_j - e_j e_i$, $i < j$. Show that the elements $\{e_{ij}\}_{1 \le i < j \le n-1}$ generate the *center* of the algebra $\mathcal{L}_n$.

Definition 2.5.

(a) The *local plactic algebra* $\mathcal{LP}_n$, see, e.g., [16], is an associative algebra over $\mathbb{Z}$ generated by elements $\{u_1, \ldots, u_{n-1}\}$ subject to the set of relations

$$\begin{aligned} &u_i u_j = u_j u_i, \qquad \text{if} \quad |i-j| \ge 2, \\ &u_j u_i^2 = u_i u_j u_i, \qquad u_j^2 u_i = u_j u_i u_j, \qquad \text{if} \quad |i-j| = 1. \end{aligned}$$

One can show (A.K.) that

$$\operatorname{Hilb}(\mathcal{LP}_n, t) = \prod_{j=1}^{n} \left(\frac{1}{1-t^j}\right)^{n+1-j}.$$

(b) The *affine local plactic algebra* $\widehat{\mathcal{PL}}_n$, see [34], is an associative algebra over $\mathbb{Q}$ generated by the elements $\{e_0, \ldots, e_{n-1}\}$ subject to the set of relations listed in item (a), where all indices are understood modulo n.

[9] For the reader convenience we recall a definition of a *tableau word*. Let T be a (regular shape) semistandard Young tableau. The tableau word $w(T)$ associated with T is the *reading word* of T is the sequence of entries of T obtained by concatenating the columns of T bottom to top consecutively starting from the first column. For example, take

$$T = \begin{matrix} 1 & 2 & 3 & 3 \\ 2 & 3 & 4 & \\ 3 & 4 & & \\ 5 & & & \end{matrix}$$

The corresponding *tableau word* is $w(T) = 5321432433$. By definition, a tableau word is the tableau word corresponding to some (regular shape) semistandard Young tableau. It is well-known [53] that the number of tableau subwords contained in the staircase word $I_0^{(n)} := \underbrace{u_{n-1}u_{n-2}\cdots u_2 u_1}\underbrace{u_{n-1}u_{n-2}\cdots u_2}\cdots\underbrace{u_{n-1}u_{u-2}}\underbrace{u_{n-1}}$ is equal to the number of alternating sign matrices $\mathrm{ASM}(n)$.

(c) Let q be a parameter, the *affine quantum local plactic algebra* $\widehat{\mathcal{PL}}_n^{(q)}$ is an associative algebra over $\mathbb{Q}[q]$ generated by the elements $\{e_0, \dots, e_{n-1}\}$ subject to the set of relations

(i) $e_i e_j = e_j e_i$, if $|i-j| > 1$,

(ii) $(1+q)e_i e_j e_i = q e_i^2 e_j + e_j e_i^2$, $(1+q)e_j e_i e_j = q e_i e_j^2 + e_j^2 e_i$, if $|i-j| = 1$, where all indices appearing in the relation (ii) are understood modulo n.

It was observed in [33] that the relations listed in (ii) are the Serre relations of $U_q^{\geq}\big(\hat{\mathfrak{gl}}(n)\big)$ rewritten[10] in generators $k_i E_i$. Some interesting properties and applications of the local and affine local plactic algebras one can find in [33, 34, 36]. It seems an interesting problem to investigate properties and applications of the affine *pseudoplactic* algebra which is the quotient of the affine local plactic algebra $\widehat{\mathcal{LP}}_n$ by the two-sided ideal generated by the set of elements

$$\{(e_j, (e_i, e_k)),\ 0 \leq i < j < k \leq n-1\}.$$

Definition 2.6 (nil Temperley–Lieb algebra)**.** Denote by $\mathcal{TL}_n^{(0)}$ the quotient of the local plactic algebra $\mathcal{LP}_n$ by the two-sided ideal generated by the elements $\{u_1^2, \dots, u_{n-1}^2\}$.

It is well-known that $\dim \mathcal{TL}_n = C_n$, the n-th Catalan number. One also has

$$\mathrm{Hilb}(\mathcal{TL}_4^{(0)}, t) = (1,3,5,4,1), \qquad \mathrm{Hilb}(\mathcal{TL}_5^{(0)}, t) = (1,4,9,12,10,4,2),$$
$$\mathrm{Hilb}(\mathcal{TL}_6, t) = (1,5,14,25,31,26,16,9,4,1).$$

Proposition 2.7. *The Hilbert polynomial* $\mathrm{Hilb}(\mathcal{TL}_n, t)$ *is equal to the generating function for the number or* 321*-avoiding permutations of the set* $\{1,2,\dots,n\}$ *having the inversion number equals to* k*, see* [67, *A*140717]*, for other combinatorial interpretations of the polynomials* $\mathrm{Hilb}(\mathcal{TL}_n, t)$.

Exercise 2.8.

- Show that $\deg_t \mathrm{Hilb}(\mathcal{TL}_n, t) = \left[\frac{n^2}{4}\right]$.
- Show that

$$\lim_{k\to\infty} t^{k^2}\, \mathrm{Hilb}\left(\mathcal{TL}_{2k}, t^{-1}\right) = \prod_{n\geq 1} \frac{\left(1-t^{4n-2}\right)}{\left(1-t^{2n-1}\right)^4\left(1-t^{4n}\right)} = \sum_{n\geq 0} a_n t^n,$$

where a_n is equal to the number of 2-colored generalized Frobenius partitions, see, e.g., [67, *A*051136] and the literature quoted therein[11].

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[10] For the reader convenient, we recall, see, e.g., [33] the definition of algebra $U_q^{\geq}(\hat{\mathfrak{gl}}(n))$. Namely, this algebra is an unital associative algebra over $\mathbb{Q}(q^{\pm 1})$ generated by the elements $\{k_i^{\pm 1}, E_i\}_{i=0,\dots,n-1}$, subject to the set of relations

(R1) $k_i k_j = k_j k_i$, $k_i k_i^{-1} = k_i^{-1} = 1$, $\forall\, i,\, j$,

(R2) $k_i E_j = q^{\delta_{i,j} - \delta_{i,j+1}} E_j k_i$,

(R3) (Serre relations)

$$E_i^2 E_{i+1} - \left(q+q^{-1}\right) E_i E_{i+1} E_i + E_{i+1} E_i^2 = 0, \qquad E_{i+1}^2 E_i - \left(q+q^{-1}\right) E_{i+1} E_i E_{i+1} + E_i E_{i+1}^2 = 0,$$

where all indices are understood modulo n, e.g., $e_n = e_0$.

It is clearly seen that after rewriting the Serre relations (R3) in terms of the elements $\{e_i := k_i E_i\}_{0\leq i\leq n-1}$, one comes to the relations (ii) which have been listed in (c).

[11] The second equality in the above formula is due to G. Andrews [1]. The second formula for the generating function of the numbers $\{a_n\}_{n\geq 0}$ displayed in [67, *A*051136], either contains misprints or counts something else.

- Show that

$$\lim_{k\to\infty} t^{k(k+1)} \operatorname{Hilb}\left(\mathcal{TL}_{2k+1}, t^{-1}\right) = 2\prod_{n\geq 1} \frac{(1+t^n)\left(1+t^{2n}\right)^2}{1-t^n} = 2\sum_{n\geq 0} b_n t^n.$$

See [67, $A201078$] for more details concerning relations of this exercise with the Ramanujan theta functions[12].

We denote by $\mathcal{TH}_n^{(\beta)}$ the quotient of the local plactic algebra $\mathcal{LP}_n$ by the two-sided ideal generated by the elements $\left\{u_i^2-\beta u_i,\, i=1,\ldots,n-1\right\}$.

Definition 2.9. The modified *plactic algebra* $\mathcal{MP}_n$ is an associative algebra over $\mathbb{Z}$ generated by $\{u_1,\ldots,u_{n-1}\}$ subject to the set of relations (PL1) and that

$$u_j u_j u_i = u_j u_i u_i, \qquad \text{and} \qquad u_i u_j u_i = u_j u_i u_j, \qquad \text{if} \quad 1\leq i<j\leq n-1.$$

Definition 2.10. The *nilplactic algebra* $\mathcal{NP}_n$ is an associative algebra over $\mathbb{Q}$ generated by $\{u_1,\ldots,u_{n-1}\}$, subject to set of relations

$$u_i^2=0, \qquad u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1},$$

the set of relations (PL1), and that $u_i u_j u_i = u_j u_i u_j$, if $|i-j|\geq 2$.

Proposition 2.11 ([52])**.** *Each nilplactic class not containing zero, contains one and only one tableau word.*

Proposition 2.12. *The nilplactic algebra $\mathcal{NP}_n$ has finite dimension, its Hilbert polynomial* $\operatorname{Hilb}(\mathcal{NP}_n,t)$ *has degree $\binom{n}{2}$, and* $\dim_{\mathbb{Q}}(\mathcal{NP}_n)_{\binom{n}{2}}=1$*.*

Example 2.13.

$$\begin{aligned}
&\operatorname{Hilb}(\mathcal{NP}_3,t)=(1,2,2,1), \qquad \operatorname{Hilb}(\mathcal{NP}_4,t)=(1,3,6,6,5,3,1),\\
&\operatorname{Hilb}(\mathcal{NP}_5,t)=(1,4,12,19,26,26,22,15,9,4,1), \qquad \dim_{\mathbb{Q}}(\mathcal{NP}_5)=139,\\
&\operatorname{Hilb}(\mathcal{NP}_6,t)=(1,5,20,44,84,119,147,152,140,224,81,52,29,14,5,1),\\
&\dim_{\mathbb{Q}}(\mathcal{NP}_6)=1008.
\end{aligned}$$

Definition 2.14. The *idplactic algebra* $\mathcal{IP}_n := \mathcal{IP}_n^{(\beta)}$ is an associative algebra over $\mathbb{Q}[\beta]$ generated by elements $\{u_1,\ldots,u_{n-1}\}$ subject to the set of relations

$$u_i^2=\beta u_i, \qquad u_i u_j u_i = u_j u_i u_j, \qquad i<j, \tag{2.1}$$

and the set of relations (PL1).

In other words,the idplactic algebra $\mathcal{IP}_n^{(\beta)}$ is the quotient of the modified plactic algebra $\mathcal{MP}_n$ by the two-sided ideal generated by the elements $\left\{u_i^2-\beta u_i,\, 1\leq i\leq n\right\}$.

Proposition 2.15. *Each idplactic class not containing zero, contains a unique tableau word associated with a row strict semistandard Young tableau*[13]*.*

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[12] See, e.g., `http://en.wikipedia.org/wiki/Ramanujan_theta_function`.

[13] Recall that a row strict semistandard Young tableau, say T, is a tableau such that the numbers in each row and each column of T are strictly increasing. For example,

$$\begin{matrix} 1 & 2 & 3 & 5 & 6\\ 2 & 3 & 6 & 8 & \\ 4 & 5 & 7 & & \end{matrix}$$

For each word w denote by $\mathrm{rl}(w)$ the length of a unique tableau word of minimal length which is idplactic equivalent to w.

Example 2.16. Consider words in the alphabet $\{a < b < c < d\}$. Then

$$\mathrm{rl}(dbadc) = 4 = \mathrm{rl}(cadbd), \qquad \mathrm{rl}(dbadbc) = 5 = \mathrm{rl}(cbadbd).$$

Indeed,

$$dbadc \sim dbdac \sim dbdca \sim ddbca \sim dbac,$$
$$dbadbc \sim dbabdc \sim dabadc \sim adbdac \sim abdbca \sim abbdca \sim dbabc.$$

Note that according to our definition, tableau words $w = 31$, $w = 13$ and $w = 313$ belong to different idplactic classes.

Proposition 2.17. *The idplactic algebra $\mathcal{IP}_n^{(\beta)}$ has finite dimension, and its Hilbert polynomial has degree $\binom{n}{2}$.*

Example 2.18.

$$\mathrm{Hilb}(\mathcal{IP}_3, t) = (1,2,2,1), \qquad \mathrm{Hilb}(\mathcal{IP}_4, t) = (1,3,6,7,5,3,1), \qquad \dim(\mathcal{IP}_4) = 26,$$
$$\mathrm{Hilb}(\mathcal{IP}_5, t) = (1,4,12,22,30,32,24,15,9,4,1), \qquad \dim(\mathcal{IP}_5) = 154,$$
$$\mathrm{Hilb}(\mathcal{IP}_6, t) = (1,5,20,50,100,156,188,193,173,126,84,52,29,14,5,1),$$
$$\dim(\mathcal{IP}_6) = 1197, \qquad \dim(\mathcal{IP}_7) = 9401.$$

Note that for a given n some words corresponding to strict semistandard Young tableaux of size between 5 and $\binom{n-1}{2}$ are idplactic equivalent to zero. For example,

$$\begin{matrix} 1 & 2 & 4 \\ 3 & 4 & \end{matrix} \sim 31424 \sim 31242 \sim 13242 \sim 13422 \sim 0,$$

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 4 & \\ 4 & & \end{matrix} \sim 421423 \sim 421243 \sim 412143 \sim 142413 \sim 124213 \sim 122431 \sim 0.$$

It seems an interesting *problem* to count the number of all row strict semistandard Young tableaux contained in the staircase $(n, n-1, \ldots, 2, 1)$ and bounded by n, as well as count the number the number of such tableaux which are idplactic equivalent to zero. For $n = 2$ these numbers are $(6,6)$, for $n = 3$ these numbers are $(26,26)$, for $n = 4$ these numbers are $(160,154)$ and for $n = 5$ they are $(1427,1197)$.

Definition 2.19. The *idplactic Temperly–Lieb algebra $\mathcal{PTL}_n^{(\beta)}$* is define to be the quotient of the idplactic algebra $\mathcal{IP}_n^{(\beta)}$ by the two-sided ideal generated by the elements

$$\{u_i u_j u_i, \ \forall\, i \neq j\}.$$

For example,

$$\mathrm{Hilb}\left(\mathcal{PTL}_4^{(0)}, t\right) = (1,3,6,4,1)_t, \qquad \mathrm{Hilb}\left(\mathcal{PTL}_5^{(0)}, t\right) = (1,4,12,16,14,4,2)_t,$$
$$\mathrm{Hilb}\left(\mathcal{PTL}_6^{(0)}, t\right) = (1,5,20,40,60,46,32,10,4,1)_t,$$
$$\mathrm{Hilb}\left(\mathcal{PTL}_7^{(0)}, t\right) = (1,6,30,80,170,216,238,152,96,44,14,4,2)_t.$$

One can show that

$$\deg_t \operatorname{Hilb}\left(\mathcal{PTL}_n^{(0)}, t\right) = \left[\frac{n^2}{4}\right],$$

and

$$\operatorname{Coeff}_{t^{\max}} \operatorname{Hilb}(\mathcal{PTL}_n, t) = \begin{cases} 1, & \text{if } n \text{ is even}, \\ 2, & \text{if } n \text{ is odd}. \end{cases}$$

Definition 2.20. The *nilCoxeter algebra* $\mathcal{NC}_n$ is defined to be the quotient of the nilplactic algebra $\mathcal{NP}_n$ by the two-sided ideal generated by elements $\{u_i u_j - u_j u_i,\ |i-j| \geq 2\}$.

Clearly the nilCoxeter algebra $\mathcal{NC}_n$ is a quotient of the modified plactic algebra $\mathcal{MP}_n$ by the two-sided ideal generated by the elements $\{u_i u_j - u_j u_i,\ |i-j| \geq 2\}$.

Definition 2.21. The *idCoxeter algebra* $\mathcal{IC}_n^{(\beta)}$ is defined to be the quotient of the idplactic algebra $\mathcal{IP}_n^{(\beta)}$ by the two-sided ideal generated by the elements $\{u_i u_j - u_j u_i,\ |i-j| \geq 2\}$.

It is well-known, see, e.g., [43], that the algebra $\mathcal{NC}_n$ and $\mathcal{IC}_n^{(\beta)}$ has dimension $n!$, and the elements $\{u_w := u_{i_1} \cdots u_{i_\ell}\}$, where $w = s_{i_1} \cdots s_{i_\ell}$ is any reduced decomposition of $w \in \mathbb{S}_n$, form a *basis* in the nilCoxeter and idCoxeter algebras $\mathcal{NC}_n$ and $\mathcal{IC}_n^{(\beta)}$.

Remark 2.22. There is a common generalization of the algebras defined above which is due to S. Fomin and C. Greene [16]. Namely, define generalized plactic algebra $\widetilde{\mathcal{P}}_n$ to be an associative algebra generated by elements $u_1, \ldots, u_{n-1}$, subject to the relations (PL2) and relations

$$u_j u_i (u_i + u_j) = (u_i + u_j) u_j u_i, \qquad i < j. \tag{2.2}$$

The relation (2.2) can be written also in the form

$$u_j (u_i u_j - u_j u_i) = (u_i u_j - u_j u_i) u_i, \qquad i < j.$$

Theorem 2.23 ([16])**.** *For each pair of numbers $1 \leq i < j \leq n$ define*

$$A_{i,j}(x) = \prod_{k=j}^{i} (1 + x u_k).$$

Then the elements $A_{i,j}(x)$ and $A_{i,j}(y)$ commute in the generalized plactic algebra $\widetilde{\mathcal{P}}_n$.

Corollary 2.24. *Let $1 \leq i < j \leq n$ be a pair of numbers. Noncommutative elementary polynomials $e_a^{ij} := \sum_{j \geq i_1 \geq \cdots \geq i_k \geq i} u_{i_1} \cdots u_{i_a}$, $i \leq a \leq j$, generate a commutative subalgebra $\mathcal{C}_{i,j}$ of rank $j-i+1$ in the plactic algebra $\mathcal{P}_n$.*

Moreover, the algebra $\mathcal{C}_{1,n}$ is a maximal commutative subalgebra of $\mathcal{P}_n$.

To establish Theorem 2.23, we are going to prove more general result. To start with, let us define *generic plactic algebra* $\mathfrak{P}_n$.

Definition 2.25. The *generic plactic algebra* $\mathfrak{P}_n$ is an associative algebra over $\mathbb{Z}$ generated by $\{e_1, \ldots, e_{n-1}\}$ subject to the set of relations

$$e_j (e_i, e_j) = (e_i, e_j) e_i, \qquad \text{if} \quad i < j, \tag{2.3}$$

$$(e_j, (e_i, e_k)) = 0, \qquad \text{if} \quad i < j < k, \tag{2.4}$$

$$(e_j, e_k)(e_i, e_k) = 0, \qquad \text{if} \quad i < j < k. \tag{2.5}$$

Hereinafter we shell use the notation $(a, b) := [a, b] := ab - ba$.

Clearly seen that relations (2.3)–(2.5) are consequence of the plactic relations (PL1) and (PL2), but not vice versa.

Theorem 2.26. *Define*

$$A_n(x) = \prod_{k=j}^{1} (1 + xe_k).$$

Then the elements $A_n(x)$ and $A_n(y)$ commute in the generic plactic algebra $\mathfrak{P}_n$. Moreover the elements $A_n(x)$ and $A_n(y)$ commute if and only if the generators $\{e_1, \dots, e_{n-1}\}$ satisfy the relations (2.3)*–*(2.5)*.*

Proof. For $n = 2, 3$ the statement of Theorem 2.26 is obvious. Now assume that the statement of Theorem 2.26 is true in the algebra $\mathfrak{P}_n$. We have to prove that the commutator $[A_{n+1}(x), A_{n+1}(y)]$ is equal to zero. First of all, $A_{n+1}(x) = (1 + xe_n)A_n(x)$. Therefore

$$[A_{n+1}(x), A_{n+1}(y)] = (1 + xe_n)[A_n(x), 1 + ye_n]A_n(y) - [A_n(y), 1 + xe_n]A_n(x).$$

Using the standard identity $[ab, c] = a[b, c] + [a, c]b$, one finds that

$$\frac{1}{xy}[A_n(x), 1 + ye_n] = \sum_{i=}^{n-1} \prod_{a=n-1}^{i+1} (1 + xe_a)(e_i, e_n) \prod_{a=i-1}^{1} (1 + xe_a).$$

Using relations (2.3) we can move the commutator (e_i, e_n) to the left, since $i < a < n$, till we meet the term $(1 + xe_n)$. Using relations (2.4) we see that $(1 + xe_n)(e_i, n) = (e_i, n)(1 + xe_i)$. Therefore we come to the following relation

$$\frac{1}{xy}[A_n(x), 1 + ye_n]$$
$$= \sum_{i=n-1}^{1} (e_i, e_n) \left((1 + xe_i) \prod_{\substack{a=n-1 \\ a \neq i}}^{1} (1 + xe_a)A_n(y) - (1 + ye_i) \prod_{\substack{a=n-1 \\ a \neq i}}^{1} (1 + ye_a)A_n(x) \right).$$

Finally let us observe that according to the relation (2.5),

$$(e_i, e_n)\big((1 + xe_i)(1 + xe_{n-1}) - (1 + xe_{n-1})(1 + xe_i)\big) = x^2(e_i, e_n)(e_i, e_{n-1}) = 0.$$

Indeed,

$$(e_i, e_n)(e_i, e_{n-1}) = (e_i, e_n)e_ie_{n-1} - (e_i, e_n)e_{n-1}e_i = e_ne_{n-1}(e_i, e_n) - e_{n-1}e_n(e_i, e_n) = 0.$$

Therefore

$$\frac{1}{xy}[A_n(x), 1 + ye_n] = \left(\sum_{i=n-1}^{1} (e_i, e_n) \right) [A_n(x), A_n(y)] = 0$$

according to the induction assumption.

Finally, if $i < j$, then $(e_i + e_j, e_je_i) = 0 \Longleftrightarrow$ (2.3); if $i < j < k$, and the relations 2.3 hold, then $(e_i + e_j + e_k, e_je_i + e_ke_j + e_ke_i) = 0 \Longleftrightarrow$ (2.4); if $i < j < k$, and relations (2.3) and (2.4) hold, then $(e_i + e_j + e_k, e_ke_je_i) = 0 \Longleftrightarrow$ (2.5); the relations $(e_je_i + e_ke_j + e_ke_i, e_ke_je_i) = 0$ are consequences of the above ones. ∎

Let T be a semistandard tableau and $w(T)$ be the column reading word corresponding to the tableau T. Denote by $R(T)$ (resp. $IR(T)$) the set of words which are plactic (resp. idplactic) equivalent to $w(T)$. Let $\mathbf{a} = (a_1, \dots, a_n) \in R(T)$, where $n := |T|$ (resp. $\mathbf{a} = (a_1, \dots, a_m) \in IR(T)$, where $m \geq |T|$).

Definition 2.27 (compatible sequences $\mathbf{b}$)**.** Given a word $\mathbf{a} \in R(T)$ (resp. $\mathbf{a} \in IR(T)$), denote by $C(\mathbf{a})$ (resp. $IC(\mathbf{a})$) the set of sequences of positive integers, called compatible sequences, $\mathbf{b} := (b_1 \leq b_2 \leq \cdots \leq b_m)$ such that

$$b_i \leq a_i, \qquad \text{and if} \quad a_i \leq a_{i+1}, \qquad \text{then} \quad b_i < b_{i+1}.$$

Finally, define the set $C(T)$ (resp. $IC(T)$) to be the union $\bigcup C(\mathbf{a})$ (resp. the union $\bigcup IC(\mathbf{a})$), where $\mathbf{a}$ runs over all words which are plactic (resp. idplactic) equivalent to the word $w(T)$.

Example 2.28. Take $T = \begin{matrix} 2 & 3 \\ 3 & \end{matrix}$. The corresponding tableau word is $w(T) = 323$. We have $R(T) = \{232, 323\}$ and $IR(T) = R(T) \bigcup \{2323, 3223, 3232, 3233, 3323, 32323, \dots\}$. Moreover,

$$C(T) = \left\{ \begin{matrix} \mathbf{a}: & 232 & 323 & 323 & 323 & 323 \\ \mathbf{b}: & 122 & 112 & 113 & 123 & 223 \end{matrix} \right\},$$

$$IC(T) = C(T) \bigcup \left\{ \begin{matrix} \mathbf{a}: & 2323 & 3223 & 3232 & 3233 & 3323 & 32323 \\ \mathbf{b}: & 1223 & 1123 & 1122 & 1123 & 1223 & 11223 \end{matrix} \right\}.$$

Let $\mathfrak{P} := \mathfrak{P}_n := \{p_{i,j},\ i \geq 1,\ j \geq 1,\ 2 \leq i + j \leq n + 1\}$ be the set of (mutually commuting) variables.

Definition 2.29.

(1) Let T be a semistandard tableau, and $n := |T|$. Define the double key polynomial $\mathcal{K}_T(\mathfrak{P})$ corresponding to the tableau T to be

$$\mathcal{K}_T(\mathfrak{P}) = \sum_{\mathbf{b} \in C(T)} \prod_{i=1}^{n} p_{b_i, a_i - b_i + 1}.$$

(2) Let T be a semistandard tableau, and $n := |T|$. Define the double key Grothendieck polynomial $\mathcal{GK}_T(\mathfrak{P})$ corresponding to the tableau T to be

$$\mathcal{GK}_T(\mathfrak{P}) = \sum_{\mathbf{b} \in IC(T)} \prod_{i=1}^{m} p_{b_i, a_i - b_i + 1}.$$

In the case when $p_{i,j} = x_i + y_j$, $\forall\, i, j$, where $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ denote two sets of variables, we will write $\mathcal{K}_T(X, Y)$, $\mathcal{GK}_T(X, Y), \dots$, instead of $\mathcal{K}_T(\mathfrak{P})$, $\mathcal{GK}_T(\mathfrak{P}), \dots$.

Definition 2.30. Let T be a semistandard tableau, denote by $\alpha(T) = (\alpha_1, \dots, \alpha_n)$ the exponent of the *smallest* monomial in the set $\left\{ x^{\mathbf{b}} := \prod_{i=1}^{m} x_i^{b_i},\ \mathbf{b} \in C(T) \right\}$ with respect to the lexicographic order.

We will call the composition $\alpha(T)$ to be the bottom code of tableau T.

3 Divided difference operators

In this subsection we remind some basic properties of divided difference operators will be put to use in subsequent sections. For more details, see [57].

Let f be a function of the variables x and y (and possibly other variables), and $\eta \neq 0$ be a parameter. Define the *divided difference operator* $\partial_{xy}(\eta)$ as follows

$$\partial_{xy}(\eta) f(x,y) = \frac{f(x,y) - f(\eta^{-1}y, \eta x)}{x - \eta^{-1}y}.$$

Equivalently, $(x - \eta^{-1}y)\partial_{xy}(\eta) = 1 - s_{xy}^{\eta}$, where the operator s_{xy}^{η} acts on the variables $(x, y, \dots)$ according to the rule: s_{xy}^{η} transforms the pair (x,y) to $(\eta^{-1}y, \eta x)$, and fixes all other variables. We set by definition, $s_{yx}^{\eta} := s_{xy}^{\eta^{-1}}$.

The operator $\partial_{xy}(\eta)$ takes polynomials to polynomials and has degree -1. The case $\eta = 1$ corresponds to the *Newton divided difference operator* $\partial_{xy} := \partial_{xy}(1)$.

Lemma 3.1.

(0) $s_{xy}^{\eta} s_{xz}^{\eta\xi} = s_{yz}^{\xi} s_{xy}^{\eta}$, $s_{xy}^{\eta} s_{xz}^{\eta\xi} s_{yz}^{\xi} = s_{yz}^{\xi} s_{xz}^{\eta\xi} s_{xy}^{\eta}$,

(1) $\partial_{yx}(\eta) = -\eta \partial_{xy}(\eta^{-1})$, $s_{xy}^{\eta} \partial_{yz}(\xi) = \eta^{-1} \partial_{xz}(\eta\xi) s_{xy}^{\eta}$,

(2) $\partial_{xy}(\eta)^2 = 0$,

(3) (*three term relation*) $\partial_{xy}(\eta)\partial_{yz}(\xi) = \eta^{-1}\partial_{xz}(\eta\xi)\partial_{xy}(\eta) + \partial_{yz}(\xi)\partial_{xz}(\eta\xi)$.

(4) (*twisted Leibniz rule*) $\partial_{xy}(\eta)(fg) = \partial_{xy}(\eta)(f)g + s_{xy}^{\eta}(f)\partial_{xy}(\eta)(g)$,

(5) (*crossing relations, cf.* [19, formula (4.6)])

- $x\partial_{xy}(\eta) = \eta^{-1}\partial_{xy}(\eta)y + 1, y\partial_{xy}(\eta) = \eta\partial_{xy}(\eta)x - \eta$,
- $\partial_{xy}(\eta)y\partial_{yz}(\xi) = \partial_{xz}(\eta\xi)x\partial_{xy}(\eta) + \xi^{-1}\partial_{yz}(\xi)z\partial_{xz}(\eta\xi)$,

(6) $\partial_{xy}\partial_{xz}\partial_{yz}\partial_{xz} = 0$.

Let $x_1, \dots, x_n$ be independent variables, and let $P_n := \mathbb{Q}[x_1, \dots, x_n]$. For each $i < j$ put $\partial_{ij} := \partial_{x_i x_j}(1)$ and $\partial_{ji} = -\partial_{ij}$. From Lemma 3.1 we have

$$\partial_{ij}^2 = 0, \qquad \partial_{ij}\partial_{jk} + \partial_{ki}\partial_{ij} + \partial_{jk}\partial_{ki} = 0,$$
$$\partial_{ij}x_j\partial_{jk} + \partial_{ki}x_i\partial_{ij} + \partial_{jk}x_k\partial_{ki}, \qquad \text{if} \quad i,\ j,\ k \text{ are distinct.}$$

It is interesting to consider also an *additive or affine* analog $\partial_{xy}[k]$ of the divided difference operators $\partial_{xy}(\eta)$, namely,

$$\partial_{xy}[k](f(x,y)) = \frac{f(x,y) - f(y-k, x+k)}{x - y + k}.$$

One has $\partial_{yx}[k] = -\partial_{xy}[-k]$, and $\partial_{xy}[p]\partial_{yz}[q] = \partial_{xz}[p+q]\partial_{xy}[p] + \partial_{yz}[q]\partial_{xz}[p+q]$.

A short historical comments in order. As far as I know, the divided difference operator had been invented by I. Newton in/around the year 1687, see, e.g., [43, Ref. [142]]. Since that time the literature concerning divided differences, a plethora of its generalizations and applications in different fields of mathematics and physics, grows exponentially and essentially is immense. To the best of our knowledge, the first systematic use of the isobaric divided difference operators, namely $\pi_i(f) = \partial_i(x_i f)$, and $\overline{\pi}_i := \pi_i - 1$, goes back to papers by M. Demazure concerning the study of desingularization of Schubert varieties and computation of characters of certain modules which is nowadays called *Demazure modules*, see, e.g., [11] and the literature quoted therein; the first systematic use and applications of divided difference operators to the study of cohomology rings of partial flag varieties G/P and description of Schubert classes inside the former, goes back to a paper by I.N. Berstein, I.M. Gelfand and S.I. Gelfand [4]; it is A. Lascoux and M.-P. Schützenberger who had discovered *a polynomial representative* of a Schubert class in the cohomology ring of the type A complete flag varieties, and developed a rich and beautiful combinatorial theory of these polynomials, see [47, 49, 50, 52, 51] for acquainting the reader with basic ideas and results concerning the Lascoux–Schützenber Schubert polynomials.

4 Schubert, Grothendieck and key polynomials

Let $w \in \mathbb{S}_n$ be a permutation, $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be two sets of variables. Denote by $w_0 \in \mathbb{S}_n$ the longest permutation, and by $\delta_n = (n-1, n-2, \ldots, 1)$ the staircase partition. For each partition λ define

$$R_\lambda(X,Y) := \prod_{(i,j)\in\lambda} (x_i + y_j). \tag{4.1}$$

For $i = 1, \ldots, n-1$, let $s_i = (i, i+1) \in \mathbb{S}_n$ denote the simple transposition that interchanges i and $i+1$ and fixes all other elements of the set $\{1, \ldots, n\}$. If $\alpha = (\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n)$ is a composition, we will write

$$s_i\alpha = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_n).$$

Definition 4.1 (cf. [43] and the literature quoted therein)**.**

- For each permutation $w \in \mathbb{S}_n$ *the double Schubert polynomial* $\mathfrak{S}_w(X,Y)$ is defined to be

$$\partial^{(x)}_{w^{-1}w_0}(R_{\delta_n}(X,Y)).$$

- Let α be a composition. *The key polynomials* $K[\alpha](X)$ are defined recursively as follows: if α is a partition, then $K[\alpha](X) = x^\alpha$; otherwise, if α and i are such that $\alpha_i < \alpha_{i+1}$, then

$$K[s_i(\alpha)](X) = \partial_i\big(x_i K[\alpha](X)\big).$$

- *The reduced key polynomials* $\widehat{K}[\alpha](X)$ are defined recursively as follows: if α is a partition, then $\widehat{K}[\alpha](X) = K[\alpha](X) = x^\alpha$; otherwise, if α and i are such that $\alpha_i < \alpha_{i+1}$, then

$$\widehat{K}[s_i(\alpha)](X) = x_{i+1}\partial_i\big(\widehat{K}[\alpha](X)\big).$$

- For each permutation $w \in \mathbb{S}_n$ the *double β-Grothendieck polynomial* $\mathcal{G}^\beta_w(X,Y)$ is defined recursively as follows: if $w = w_0$ is the longest element, then $\mathcal{G}_{w_0}(X,Y) = R_{\delta_n}(X,Y)$; if w and i are such that $w_i > w_{i+1}$, i.e., $l(ws_i) = l(w) - 1$, then

$$\mathcal{G}^\beta_{ws_i}(X,Y) = \partial^{(x)}_i\big((1+\beta x_{i+1})\mathcal{G}^\beta_w(X,Y)\big).$$

- For each permutation $w \in \mathbb{S}_n$ the *double dual β-Grothendieck polynomial* $\mathcal{H}^\beta_w(X,Y)$ is defined recursively as follows: if $w = w_0$ is the longest element, then $\mathcal{H}_{w_0}(X,Y) = R_{\delta_n}(X,Y)$; if w and i are such that $w_i > w_{i+1}$, i.e., $l(ws_i) = l(w) - 1$, then

$$\mathcal{H}^\beta_{ws_i}(X,Y) = (1+\beta x_i)\partial^{(x)}_i\big(\mathcal{H}^\beta_w(X,Y)\big).$$

- *The key β-Grothendieck polynomials* $\mathrm{KG}[\alpha](X;\beta)$ are defined recursively as follows[14]: if α is a partition, then $\mathrm{KG}[\alpha](X;\beta) = x^{\alpha}$; otherwise, if α and i are such that $\alpha_i < \alpha_{i+1}$, then
$$\mathrm{KG}[s_i(\alpha)](X;\beta) = \partial_i\big((x_i + \beta x_i x_{i+1})\,\mathrm{KG}[\alpha](X;\beta)\big).$$
- *The reduced key β-Grothendieck polynomials* $\widehat{\mathrm{KG}}[\alpha](X;\beta)$ are defined recursively as follows: if α is a partition, then $\widehat{\mathrm{KG}}[\alpha](X;\beta) = x^{\alpha}$; otherwise, if α and i are such that $\alpha_i < \alpha_{i+1}$, then
$$\widehat{\mathrm{KG}}[s_i(\alpha)](X;\beta) = (x_{i+1} + \beta x_i x_{i+1})\partial_i\big(\widehat{\mathrm{KG}}[\alpha](X;\beta)\big).$$

For brevity, we will write $\mathrm{KG}[\alpha](X)$ and $\widehat{\mathrm{KG}}[\alpha](X)$ instead of $\mathrm{KG}[\alpha](X;\beta)$ and $\widehat{\mathrm{KG}}[\alpha](X;\beta)$.

Remark 4.2. We can also introduce polynomials $\mathcal{Z}_w$, which are defined recursively as follows: if $w = w_0$ is the longest element, then $\mathcal{Z}_{w_0}(X) = x^{\delta_n}$; if w and i are such that $w_i > w_{i+1}$, i.e., $l(ws_i) = l(w) - 1$, then
$$\mathcal{Z}_{ws_i}(X) = \partial_i\big((x_{i+1} + x_i x_{i+1})\mathcal{Z}_w(X)\big).$$
However, one can show that
$$\mathcal{Z}_w(x_1, \dots, x_n) = (x_1 \cdots x_n)^{n-1}\mathcal{G}_{w_0 w w_0}\big(x_n^{-1}, \dots, x_1^{-1}\big).$$

Theorem 4.3. *The polynomials $\mathfrak{S}_w(X,Y)$, $K[\alpha](X)$, $\widehat{K}[\alpha](X)$, $\mathcal{G}_w(X,Y)$, $\mathcal{H}_w(X,Y)$, $\mathrm{KG}[\alpha](X)$ and $\widehat{\mathrm{KG}}[\alpha](X)$ have nonnegative integer coefficients.*

The key step in a proof of Theorem 4.3 is an observation that for a given n the all algebras involved in the definition of the polynomials listed in that theorem, happen to be a suitable quotients of the reduced plactic algebra $\mathcal{P}_n$, and can be extracted from the Cauchy kernel associated with the algebra $\mathcal{P}_n$ (or that $\mathcal{P}_{n,n}$).

We will use notation $\mathfrak{S}_w(X)$, $\mathcal{G}_w(X)$, $\dots$, for polynomials $\mathfrak{S}_w(X,0)$, $\mathcal{G}_w(X,0)$, $\dots$.

Definition 4.4. For each permutation $w \in \mathbb{S}_n$ the *Di Francesco–Zinn-Justin polynomials* $\mathcal{DZ}_w(X)$ are defined recursively as follows: if w is the longest element in $\mathbb{S}_n$, then $\mathcal{DZ}_w(X) = R_{\delta_n}(X,0)$; otherwise, if w and i are such that $w_i > w_{i+1}$, i.e., $l(ws_i) = l(w) - 1$, then
$$\mathcal{DZ}_{ws_i}(X) = \big((1 + x_i)\partial_i^{(x)} + \partial_i^{(x)}(x_{i+1} + x_i x_{i+1})\big)\mathcal{DZ}_w(X).$$

Conjecture 4.5.

[14] In the case $\beta = -1$ divided difference operators $D_i := \partial_i(x_i - x_i x_{i+1})$ [42, formula (6)] had been used by A. Lascoux to describe the transition on Grothendieck polynomials, i.e., stable decomposition of any Grothendieck polynomial corresponding to a permutation $w \in \mathbb{S}_n$. into a sum of Grasmannian ones corresponding to a collection of *Grasmannin* permutations $v_\lambda \in \mathbb{S}_\infty$, see [42] for details. The above mentioned operators D_i had been used in [42] to construct a basis $\Omega_\alpha | \alpha \in \mathbb{Z}_{\geq 0}$ that deforms the basis which is built up from the Demazure (known also as key) polynomials. Therefore polynomials $\mathrm{KG}[\alpha](X;\beta=-1)$ coincide with those introduced by A. Lascoux in [42]. In [63] the authors give a conjectural construction for polynomials Ω_α based on the use of *extended Kohnert moves*, see, e.g., [58, Appendix by N. Bergeron] for definition of the Kohnert moves. We state *conjecture* that
$$J_\alpha^{(\beta)} = \mathrm{KG}[\alpha](X;\beta),$$
where polynomials $J_\alpha^{\beta)}$ are defined in [63] using the K-theoretic versions of the Kohnert moves. For $\beta = -1$ this Conjecture has been stated by C. Ross and A. Yong in [63]. It seems an interesting problem to relate the K-theoretic Kohnert moves with certain moves of first introduced in [17].

(1) *Polynomials* $\mathcal{DZ}_w(X)$ *have nonnegative integer coefficients.*

(2) *For each permutation* $w \in \mathbb{S}_n$ *the polynomial* $\mathcal{DZ}_w(X)$ *is a linear combination of key polynomials* $K[\alpha](X)$ *with nonnegative integer coefficients.*

As for defi nition of the double Di Francesco–Zinn-Justin polynomials $\mathcal{DZ}_w(X,Y)$ they are well defined, but may have negative coefficients.

Let β and α be two parameters, consider divided difference operator

$$T_i := T_i^{\beta,\alpha} = -\beta + ((\beta+\alpha)x_i + 1 + \beta\alpha x_i x_{i+1})\partial_{i,i+1}.$$

Definition 4.6. Let $w \in \mathbb{S}_n$, define *Hecke–Grothendieck* polynomials $\mathcal{KN}_w^{\beta,\alpha}(X_n)$ to be

$$\mathcal{KN}_w^{(\beta,\alpha)}(X_n) := T_w^{\beta,\alpha}\big(x^{\delta_n}\big),$$

where as before $x^{\delta_n} := x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$; if $u \in \mathbb{S}_n$, then set

$$T_u^{\beta,\alpha} := T_{i_1}^{\beta,\alpha}\cdots T_{i_\ell}^{\beta,\alpha},$$

where $w = s_{i_1}\cdots s_{i_\ell}$ is any reduced decomposition of a permutation taken.

More generally, let β, α and γ be parameters, consider divided difference operators

$$\begin{aligned} T_i := T_i^{\beta,\alpha,\gamma} = -\beta + ((\alpha+\beta+\gamma)x_i + \gamma x_{i+1} + 1 \\ + (\beta+\gamma)(\alpha+\gamma)x_i x_{i+1})\partial_{i,i+1}, \qquad i = 1,\dots,n-1. \end{aligned}$$

For a permutation $w \in \mathbb{S}_n$ define polynomials

$$\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(X_n) := T_{i_1}^{\beta,\alpha,\gamma}\cdots T_{i_\ell}^{\beta,\alpha,\gamma}\big(x^{\delta_n}\big),$$

where $w = s_{i_1}\cdots s_{i_\ell}$ is any reduced decomposition of w.

Remark 4.7. A few comments in order.

(a) The divided difference operators $\big\{T_i := T_i^{(\beta,\alpha,\gamma)},\ i = 1,\dots,n-1\big\}$ satisfy the following relations

- (Hecke relations)

$$T_i^2 = (\alpha-\beta)T_i + \alpha\beta,$$

- (Coxeter relations)

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, T_iT_j = T_jT_i, \qquad \text{if} \quad |i-j| \ge 2.$$

Therefore the elements $T_w^{\beta,\alpha,\gamma}$ are well defined for any $w \in \mathbb{S}_n$.

- (Inversion)

$$(1+xT_i)^{-1} = \frac{1+(\alpha-\beta)x - xT_i}{(1-\beta x)(1+\alpha x)}.$$

(b) Polynomials $\mathcal{KN}_w^{(\beta,\alpha,\gamma)}$ constitute a common generalization of

- the β-Grothendieck polynomials , namely, $\mathcal{G}_w^{(\beta)} = \mathcal{KN}_{w_0w^{-1}}^{(\beta,\alpha=0,\gamma=0)}$,
- the Di Francesco–Zinn-Justin polynomials, namely, $\mathcal{DZ}_w = \mathcal{KN}_w^{(\beta=\alpha=1,\gamma=0)}$,

- the dual α-Grothendieck polynomials, namely, $\mathcal{KN}_{w_0 w^{-1}}^{(\beta=0,\alpha,\gamma=0)} = \mathcal{H}_w^{\alpha}(X)$.

Proposition 4.8.

- (*Duality*) *Let* $w \in \mathbb{S}_n$, $\ell = \ell(w)$ *denotes its length, then* $(\alpha\beta \neq 0)$

$$\mathcal{KN}_w^{(\beta,\alpha)}(1) = (\beta\alpha)^{\ell} \mathcal{KN}_{w^{-1}}^{(\alpha^{-1},\beta^{-1})}(1).$$

- (*Stability*) *Let* $w \in \mathbb{S}_n$ *be a permutation and* $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ *be any its reduced decomposition. Assume that* $i_a \leq n-3$, $\forall\, 1 \leq a \leq \ell$, *and define permutation* $\widetilde{w} := s_{i_1+1} s_{i_2+1} \cdots s_{i_\ell+1} \in \mathbb{S}_n$. *Then*

$$\mathcal{KN}_w^{(\beta,\alpha)}(1) = \mathcal{KN}_{\widetilde{w}}^{(\beta,\alpha)}(1).$$

It is well-known that

- the number $\mathcal{KN}_{w_0}^{(\beta=1,\alpha=1)}(1)$ is equal to the degree of the variety of pairs commuting matrices of size $n \times n$ [13, 30].
- the bidegree of the affine homogeneous variety V_w, $w \in \mathbb{S}_n$ [12] is equal to

$$A^{\binom{n}{2}-\ell(w)} B^{\binom{n}{2}+\ell(w)} \mathcal{KN}_w^{(\beta=\alpha=A/B)}(1),$$

see [9, 12, 30] for more details and applications.

Conjecture 4.9.

- *Polynomials* $\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(X)$ *have nonnegative integer coefficients*

$$\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(X) \in \mathbb{N}[\beta,\alpha,\gamma][X_n].$$

- *Polynomials* $\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(x_1 = 1, \forall\, i)$ *have nonnegative integer coefficients*

$$\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(x_i = 1, \forall i) \in \mathbb{N}[\beta,\alpha,\gamma].$$

- *Double polynomials*

$$\mathcal{KN}_w^{(\beta=0,\alpha,\gamma)}(X,Y) = T_w^{\beta=0,\alpha,\gamma}(x) \prod_{\substack{i+j \leq n+1 \\ i \geq 1,\, j \geq 1}} (x_i + y_j)$$

are well defined and have nonnegative integer coefficients[15].

- *Consider permutation* $w = [n, 1, 2, \ldots, n-1] \in \mathbb{S}_n$. *Clearly* $w = s_{n-1} s_{n-2} \cdots s_2 s_1$. *The number* $\mathcal{KN}_w^{(\beta=1,\alpha=1,\gamma=0)}(1)$ *is equal to the number of Schröder paths of semilength* $(n-1)$ *in which the* $(2,0)$*-steps come in* 3 *colors and with no peaks at level* 1, *see* [67, *A*162326] *for further properties of these numbers.*

It is well-known, see, e.g., [67, *A*126216], that the polynomial $\mathcal{KN}_w^{(\beta,\alpha=0)}(1)$ counts the number of *dissections* of a convex $(n+1)$-gon according the number of diagonals involved, where as the polynomial $\mathcal{KN}_w^{(\beta,\alpha)}(1)$ (up to a normalization) is equal to the *bidegree* of certain algebraic varieties introduced and studied by A. Knutson [30].

A few comments in order.

[15]Note that the assumption $\beta = 0$ is necessary.

(a) One can consider more general family of polynomials $\mathcal{KN}_w^{(a,b,c,d)}(X_n)$ by the use of the divided difference operators $T_i^{a,b,c,d} := -b + ((b+d)x_i + cx_{i+1} + 1 + d(b+c)x_i x_{i+1})\partial_{i,i+1}^x$ instead of that $T_i^{\beta,\alpha,\gamma}$. However the polynomials $\mathcal{KN}_w^{(a,b,c,d)}(1) \in \mathbb{Z}[a,b,c,d]$ may have negative coefficients in general. *Conjecturally*, to ensure the positivity of polynomials $\mathcal{KN}_w^{(a,b,c,d)}(X_n)$, it is necessary take $d := a+c+r$. In this case we state Conjecture

$$\mathcal{KN}_w^{(a,b,c,a+c+r)}(X_n) \in \mathbb{N}[a,b,c,r].$$

We state more general Conjecture in Introduction. In the present paper we treat only the case $r=0$, since a combinatorial meaning of polynomials $\mathcal{KN}_w^{(a,b,c,a+c+r)}(1)$ in the case $r \neq 0$ is missed for the author.

(b) If $\gamma \neq 0$, the polynomials $\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(X_n) \in \mathbb{Z}[\alpha,\beta,\gamma][X_n]$ may have negative coefficients in general.

Theorem 4.10. *Let T be a semistandard tableau and $\alpha(T)$ be its bottom code, see Definition 2.27. Then*

$$\mathcal{K}_T(X) = K[\alpha(T)](X), \qquad \mathcal{KG}_T(X) = \mathrm{KG}[\alpha(T)](X).$$

Let $\alpha = (\alpha_1 \le \alpha_2 \le \cdots \le \alpha_r)$ be a composition, define partition $\alpha^+ = (\alpha_r \ge \cdots \ge \alpha_1)$.

Proposition 4.11. *If $\alpha = (\alpha_1 \le \alpha_2 \le \cdots \le \alpha_r)$ is a composition and $n \ge r$, then*

$$K[\alpha](X_n) = s_{\alpha^+}(X_r).$$

For example, $K[0,1,2,\ldots,n-1] = \prod_{1\le i<j\le n}(x_i+x_j)$. Note that $\widehat{K}[0,1,2,\ldots,n-1] = \prod_{i=2}^{n} x_i^{i-1}$.

Proposition 4.12. *If $\alpha = (\alpha_1 \le \alpha_2 \le \cdots \le \alpha_r)$ is a composition and $n \ge r$, then*

$$KG[\alpha](X_n) = \mathcal{G}[\alpha^+](X_r).$$

For example, $\mathrm{KG}[0,1,2,\ldots,n-1] = \prod_{1\le i<j\le n}(x_i+x_j+x_ix_j)$. Note that

$$\widehat{\mathrm{KG}}[0,1,2,\ldots,n-1] = \prod_{i=2}^{n} x_i^{i-1} \prod_{i=1}^{n-1}(1+x_i)^{n-i}.$$

Definition 4.13. Define degenerate affine $2D$ nil-Coxeter algebra $\mathcal{ANC}_n^{(2)}$ to be an associative algebra over $\mathbb{Q}$ generated by the set of elements $\{\{u_{i,j}\}_{1\le i<j\le n}, x_1,\ldots,x_n\}$ subject to the set of relations

- $x_ix_j = x_jx_i$ for all $i \neq j$, $x_iu_{j,k} = u_{j,k}x_i$, if $i \neq j,k$,
- $u_{i,j}u_{k,l} = u_{k,l}u_{i,j}$, if i, j, k, l are pairwise distinct,
- ($2D$-Coxeter relations) $u_{i,j}u_{j,k}u_{i,j} = u_{j,k}u_{i,j}u_{j,k}$, if $1 \le i<j<k \le n$,
- $x_iu_{i,j} = u_{i,j}x_j + 1$, $x_ju_{i,j} = u_{i,j}x_i - 1$.

Now for a set of parameters[16] $A := (a,b,c,h,e)$ define elements

$$T_{ij} := a + (bx_i + cx_j + h + ex_ix_j)u_{i,j}, \qquad i<j.$$

Throughout the present paper we set $T_i := T_{i,i+1}$.

[16] By definition, a *parameter* is assumed to be belongs to the center of the algebra in question.

Lemma 4.14.

(1) $T_{i,j}^2 = (2a+b-c)T_{i,j} - a(a+b-c)$, *if* $a=0$, *then* $T_{ij}^2 = (b-c)T_{ij}$.

(2) *2D-Coxeter relations* $T_{i,j}T_{j,k}T_{i,j} = T_{j,k}T_{i,j}T_{j,k}$ *are valid, if and only if the following relation among parameters* a, b, c, e, h *holds*[17]

$$(a+b)(a-c) + he = 0. \tag{4.2}$$

(3) *Yang–Baxter relations* $T_{i,j}T_{i,k}T_{j.k} = T_{j,k}T_{i,k}T_{i,j}$ *are valid if and only if* $b=c=e=0$, *i.e.,* $T_{ij} = a + du_{ij}$.

(4) $T_{ij}^2 = 1$ *if and only if* $a = \pm 1$, $c = b \pm 2$, $he = (b \pm 1)^2$.

(5) *Assume that parameters* a, b, c, h, e *satisfy the conditions* (4.2) *and that* $bc + 1 = he$. *Then*

$$T_{ij}x_iT_{ij} = (he-bc)x_j + (h + (a+b)(x_i+x_j) + ex_ix_j)T_{ij}.$$

Some special cases

- (*Representation of affine modified Hecke algebra* [71].) *If* $A = (a, -a, c, h, 0)$, *then* $T_{ij}x_iT_{ij} = acx_j + hT_{ij}$, $i<j$,
- *If* $A = (-a, a+b+c, c, 1, (a+c)(b+c)$, *then*

$$T_{ij}x_iT_{ij} = abx_j + (1 + (b+c)(x_i+x_j) + (a+c)(b+c)x_ix_j)T_{ij}.$$

(6) (*Quantum Yang–Baxter relations, or baxterization of Hecke's algebra generators.*) *Assume that parameters* a, b, c, h, e *satisfy the conditions* (4.2) *and that* $\beta := 2a+b-c \neq 0$. *Then* (*cf.* [24, 44] *and the literature quoted therein*) *the elements* $R_{ij}(u,v) := 1 + \frac{\lambda-\mu}{\beta\mu}T_{ij}$ *satisfy the twisted quantum Yang–Baxter relations*

$$R_{ij}(\lambda_i,\mu_j)R_{jk}(\lambda_i,\nu_k)R_{ij}(\mu_j,\nu_k) = R_{jk}(\mu_j,\nu_k)R_{ij}(\lambda_i,\nu_k)R_{jk}(\lambda_i,\mu_j), \qquad i<j<k,$$

where $\{\lambda_i,\mu_i,\nu_i\}_{1\le i\le n}$ *are parameters.*

Corollary 4.15. *If* $(a+b)(a-c)+he=0$, *then for any permutation* $w \in \mathbb{S}_n$ *the element*

$$T_w := T_{i_1}\cdots T_{i_l} \in \mathcal{ANC}_n^{(2)},$$

where $w = s_{i_1}\cdots s_{i_l}$ *is any reduced decomposition of* w, *is well-defined.*

Example 4.16.

- Each of the set of elements

$$s_i^{(h)} = 1 + (x_{i+1} - x_i + h)u_{i,i+1} \qquad \text{and}$$
$$t_i^{(h)} = -1 + (x_i - x_{i+1} + h(1+x_i)(1+x_{i+1}))u_{ij}, \qquad i = 1,\dots,n-1,$$

by itself generate the symmetric group $\mathbb{S}_n$.

[17] The relation (4.1) between parameters a, b, c, e, h defines a *rational* four-dimensional hypersurface. Its open chart $\{eh \neq 0\}$ contains, for example, the following set (cf. [42]): $\{a = p_1p_4 - p_2p_3$, $b = p_2p_3$, $c = p_1p_4$, $e = p_1p_3$, $h = p_2p_4\}$, where (p_1,p_2,p_3,p_4) are arbitrary parameters. However the points $(-b, a+b+c, c, 1, (a+c)(b+c)$, $(a,b,c) \in \mathbb{N}^3\}$ do not belong to this set.

- If one adds the affine elements $s_0^{(h)} := \pi s_{n-1}^{(h)}\pi^{-1}$ and $t_0^{(h)} := \pi t_{n-1}^{(h)}\pi^{-1}$, then each of the set of elements $\{s_j^{(h)},\, j \in \mathbb{Z}/n\mathbb{Z}\}$ and $\{t_j^{(h)},\, j \in \mathbb{Z}/n\mathbb{Z}\}$ by itself generate the affine symmetric group $\mathbb{S}_n^{\rm aff}$, see (4.3) for a definition of the transformation π.
- It seems an interesting problem to classify all rational, trigonometric and elliptic divided difference operators satisfying the Coxeter relations. A general divided difference operator with *polynomial coefficients* had been constructed in [50], see also Lemma 4.14, relation (4.1). One can construct a family of *rational* representations of the symmetric group (as well as its affine extension) by "iterating" the transformations $s_j^{(h)}$, $j \in \mathbb{Z}/n\mathbb{Z}$. For example, take parameters a and b, define *secondary* divided difference operator

  $$\partial_{xy}^{[a,b]} := -1 + (b + y - x)\partial_{xy}^{[a]},$$

  where

  $$\partial_{xy}^{[a]} := \frac{1 - \overline{s}_{xy}^{(a)}}{a - x + y}, \qquad \overline{s}_{xy}^{(a)} := -1 + (a + x - y)\partial_{xy}.$$

  Observe that the set of operators $\{s_i^{[a,b]} := s_{x_i,x_{i+1}}^{[a,b]},\, i \in \mathbb{Z}/n\mathbb{Z}\}$ gives rise to a rational representation of the affine symmetric group $\mathbb{S}_n^{\rm aff}$ on the field of rational functions $\mathbb{Z}[a,b](X_n)$. In the special case $a := A$, $b := A/h$, $h := 1 - \beta/2$ the operators $s_i^{[a,b]}$ coincide with operators Θ_i, $i \in \mathbb{Z}/n\mathbb{Z}$ have been introduced in [31, equation (4.17)].

Let $A = (a,b,c,h,e)$ be a sequence of integers satisfying the conditions (4.1). Denote by ∂_i^A the divided difference operator

$$\partial_i^A = a + (bx_i + cx_{i+1} + h + ex_ix_{i+1})\partial_i, \qquad i = 1,\dots,n-1.$$

It follows from Lemma 4.14 that the operators $\{\partial_i^A\}_{1\le i\le n}$ satisfy the Coxeter relations

$$\partial_i^A\partial_{i+1}^A\partial_i^A = \partial_{i+1}^A\partial_i^A\partial_{i+1}^A, \qquad i = 1,\dots,n-1.$$

Definition 4.17.

(1) Let $w \in \mathbb{S}_n$ be a permutation. Define the generalized Schubert polynomial corresponding to permutation w as follows

$$\mathfrak{S}_w^A(X_n) = \partial_{w^{-1}w_0}^A x^{\delta_n}, \qquad x^{\delta_n} := x_1^{n-1}x_2^{n-2}\cdots x_{n-1},$$

and w_0 denotes the longest element in the symmetric group $\mathbb{S}_n$.

(2) Let α be a composition with at most n parts, denote by $w_\alpha \in \mathbb{S}_n$ the permutation such that $w_\alpha(\alpha) = \alpha^+$. Let us recall that α^+ denotes a unique partition corresponding to composition α.

Lemma 4.18. *Let $w \in \mathbb{S}_n$ be a permutation.*

- *If $A = (0,0,0,1,0)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the Schubert polynomial $\mathfrak{S}_w(X_n)$.*
- *If $A = (-\beta,\beta,0,1,0)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X_n)$ introduced in* [17].
- *If $A = (0,\beta,0,1,0)$ then $\mathfrak{S}_w^A(X_n)$ is equal to the dual β-Grothendieck polynomial $\mathcal{H}_w^{(\beta)}(X_n)$, studied in depth for $\beta = -1$ and in the basis $\{x_i := \exp(\xi_l)\}$ in* [40].
- *If $A = (-1,2,0,1,1)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the Di Francesco–Zinn-Justin polynomials introduced in* [12].
- *If $A = (1,-1,1,h,0)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the h-Schubert polynomials.*

In all cases listed above the polynomials $\mathfrak{S}_w^A(X_n)$ have non-negative integer coefficients.

Define the generalized key or Demazure polynomial corresponding to a composition α as follows

$$K_\alpha^A(X_n) = \partial_{w_\alpha}^A x^{\alpha^+}.$$

- If $A = (1,0,1,0,0)$, then $K_\alpha^A(X_n)$ is equal to key (or Demazure) polynomial corresponding to α.
- If $A = (0,0,1,0,0)$, then $K_\alpha^A(X_n)$ is equal to the reduced key polynomial.
- If $A = (1,0,1,0,\beta)$, then $K_\alpha^A(X_n)$ is equal to the key β-Grothendieck polynomial $\mathrm{KG}_\alpha^{(\beta)}(X_n)$.
- If $A = (0,0,1,0,\beta)$, then $K_\alpha^A(X_n)$ is equal to the reduced key β-Grothendieck polynomials.

In all cases listed above the polynomials $\mathfrak{S}_w^A(X_n)$ have non-negative integer coefficients.

- If $A = (-1, q^{-1}, -1, 0, 0)$ and λ is a partition, then (up to a scalar factor) polynomial $K_\lambda^A(X_n)$ can be identify with a certain *Whittaker function* (of type A), see [7, Theorem A]. Note that operator $T_i^A := -1 + (q^{-1}x_i - x_{i+1})\partial_i$, $1 \le i \le n-1$, satisfy the Coxeter and Hecke relations, namely $(T_i^A)^2 = (q^{-1}-1)T_i^A + q^{-1}$. In [7] the operator T_i^A has been denoted by $\mathfrak{T}_i$.
- Let $w \in \mathbb{S}_n$ be a permutation and $\mathfrak{m} = (i_1, \dots, i_\ell)$ be a reduced word for w, i.e., $w = s_{i_1} \cdots s_{i_\ell}$ and $\ell(w) = \ell$. Denote by $Z_\mathfrak{m}$ the *Bott–Samelson* nonsingular variety corresponding to the reduced word $\mathfrak{m}$. It is well-known that the Bott–Samelson variety $Z_\mathfrak{m}$ is birationally isomorphic to the Schubert variety X_w associated with permutation w, i.e., the Bott–Samelson variety $Z_\mathfrak{m}$ is a desingularization of the Schubert variety X_w. Following [7] define the Bott–Samelson polynomials $Z_\mathfrak{m}(x,\lambda,v)$ as follows

  $$Z_\mathfrak{m}(x,\lambda,v) = \big(1 + T_{i_1}^A\big) \cdots \big(1 + T_{i_\ell}^A\big) x^\lambda,$$

  where $A = (-v,-1,1,0)$. Note that $\big(1+T_i^A\big)^2 = (1+v)\big(1+T_i^A\big)$, and the divided difference operators $1 + T_i^A = 1 + v + (x_{i+1} - vx_i)\partial_i$ do *not* satisfy the Coxeter relations.
- If $A = (-\beta, \beta+\alpha, 0, 1, \beta\alpha)$, then $\mathfrak{S}_w^A(X_n)$ constitutes a common generalization of the β-Grothendieck and the Di Francesco–Zinn-Justin polynomials.
- If $A = (t,-1,t,1,0)$, then the divided difference operators

  $$T_i^A := t + (-x_i + tx_{i+1} + 1)\partial_i, \qquad 1 \le i \le n-1,$$

  their *baxterizations* and the *raising operator*

  $$\phi := (x_n - 1)\pi,$$

  where π denotes the *q^{-1}-shift operator*, namely $\pi(x_1,\dots,x_n) = (x_n/q, x_1, \dots, x_{n-1})$, can be used to *generate* the *interpolation Macdonald polynomials* as well as the *nonsymmetric Macdonald polynomials*, see [45] for details.

In similar fashion, relying on the operator ϕ, operators

$$\begin{aligned} T_i^{\beta,\alpha,\gamma,h} := {} & -\beta + ((\alpha+\beta+\gamma)x_i + \gamma x_{i+1} + h \\ & + h^{-1}(\alpha+\gamma)(\beta+\gamma)x_i x_{i+1})\partial_i, \qquad 1 \le i \le n-1, \end{aligned}$$

and their *baxterization*, one (A.K.) can introduce polynomials $M_\delta^{\beta,\alpha,\gamma,q}(X_n)$, where δ is a *composition*. These polynomials are *common generalizations* of the interpolation Macdonald polynomials $M_\delta(X_n;q,t)$ (the case $\beta = -t$, $\alpha = -1$, $\gamma = t$), as well as the Schubert, β-Grothendieck and its *dual*, Demazure and Di Francesco–Zinn-Justin polynomials, and *conjecturally* their *affine* analogues/versions. Details will appear elsewhere.

Double affine nilCoxeter algebra. Let t, q, a, b, c, h, d be parameters.

Definition 4.19. Define double affine nil-Coxeter algebra DANC_n to be (unital) associative algebra over $\mathbb{Q}\big(q^{\pm 1}, t^{\pm 1}\big)$ with the set of generators $\big\{e_1, \ldots, e_{n-1}, x_1, \ldots, x_n, \pi^{\pm 1}\big\}$ subject to relations

- (nilCoxeter relations)

$$e_i e_j = e_j e_i, \quad \text{if} \quad |i-j| \geq 2, \qquad e_i^2 = 0, \quad \forall\, i, \qquad e_i e_j e_i = e_j e_i e_j, \quad \text{if} \quad |i-j| = 1;$$

- (crossing relations)

$$x_i e_k = e_k x_i, \quad \text{if} \quad k \neq i, i+1, \qquad x_i e_i - e_i x_{i+1} = 1, \qquad e_i x_i - x_{i+1} e_i = 1;$$

- (affine crossing relations)

$$\begin{aligned} &\pi x_i = x_{i+1}\pi, \quad \text{if} \quad i < n, \qquad \pi x_n = q^{-1} x_1 \pi, \\ &\pi e_i = e_{i+1}\pi, \quad \text{if} \quad i < n-1, \qquad \pi^2 e_{n-1} = q e_1 \pi^2. \end{aligned}$$

Now let us introduce elements $e_0 := \pi e_{n-1} \pi^{-1}$ and

$$T_0 := T_0^{a,b,c,h,d} = \pi T_{n-1} \pi^{-1} = a + \big(b x_n + q^{-1} c x_1 + h + q^{-1} d x_1 x_n\big) e_0.$$

It is easy to see that $\pi e_0 = q e_1 \pi$,

$$\pi T_0^{a,b,c,h,d} = T_1^{a,b,c,qh,q^{-1}d} e_1 \pi = T_1^{a,b,c,h,d} + \big((1-q)h + \big(1-q^{-1}\big) d x_1 x_2\big) e_1.$$

Now let us assume that $a = t$, $b = -t$, $d = e = 0$, $c = 1$. Then,

$$\begin{aligned} &T_i = t + (x_{i+1} - t x_i) e_i, \quad i = 1, \ldots, n-1, \qquad T_0 = t + \big(q^{-1} x_1 - t x_n\big) e_0, \\ &T_i^2 = (t-1)T + t, \quad 0 \leq i < n, \qquad T_i x_i T_i = t x_{i+1}, \quad 1 \leq i < n, \qquad T_0 x_n T_0 = t q^{-1} x_1, \\ &T_0 T_1 T_0 = T_1 T_0 T_1, \qquad T_{n-1} T_0 T_{n-1} = T_0 T_{n-1} T_0, \qquad T_0 T_i = T_i T_0, \text{if} \quad 2 \leq i < n-1. \end{aligned}$$

The operators $T_i := T_i^{t,-t,1,0,0}$, $0 \leq i \leq n-1$ have been used in [45] to give an "elementary" construction of nonsymmetric Macdonald polynomials. Indeed, one can realize the operator π as follows:

$$\pi(f) = f(x_n/q, x_1, x_2, \ldots, x_{n-1}), \qquad \text{so that} \quad \pi^{-1}(f) = (x_2, \ldots, x_n, q x_1), \tag{4.3}$$

and introduce the raising operator [45] to be

$$\phi(f(X_n)) = (x_n - 1)\pi(f(X_n)).$$

It is easily seen that $\phi T_i = T_{i+1}\phi$, $i = 0, \ldots, n-2$, and $\phi^2 T_{n-1} = T_1 \phi^2$. It has been established in [45] how to use the operators ϕ, $T_1, \ldots, T_{n-1}$ to to give formulas for the *interpolation Macdonald polynomials*. Using operators $\phi, T_i^{(a,b,c,h,d)}, i = 1, \ldots, n-1$ instead of ϕ, $T_1, \ldots, T_{n-1}$, $1 \leq i \leq n-1$, one get a 4-parameter generalization of the interpolation Macdonald polynomials, as well as the nonsymmetric Macdonald polynomials.

It follows from the nilCoxeter relations listed above, that the Dunkl–Cherednik elements, cf. [8],

$$Y_i := \left(\prod_{a=i-1}^{1} T_a^{-1}\right) \pi \left(\prod_{a=n-1}^{i+1} T_a\right), \qquad i = 1, \ldots, n,$$

where $T_i = T_i^{t,-t,1,0,0}$, generate a commutative subalgebra in the double affine nilCoxeter algebra DANC_n. Note that the algebra $DANC_n$ contains lot of other interesting commutative subalgebras, see, e.g., [24].

It seems interesting to give an interpretation of polynomials generated by the set of operators $T_i^{t,-t,1,h,e}$, $i = 0, \dots, n-1$ in a way similar to that given in [45]. We expect that these polynomials provide an affine version of polynomials $\mathcal{KN}_w^{(-t,-1,1,1,0)}(X)$, $w \in \mathbb{S}_n \subset \mathbb{S}_n^{\mathrm{aff}}$, see Remark 4.7.

Note that for any affine permutation $v \in \mathbb{S}_n^{\mathrm{aff}}$, the operator

$$T_v^{(a,b,c,h,d)} = T_{i_1}^{(a,b,c,h,d)} \cdots T_{i_\ell}^{(a,b,c,h,d)},$$

where $v = s_{i_1} \cdots s_{i_\ell}$ is any reduced decomposition of v, is *well-defined* up to the sign ± 1. It seems an interesting *problem* to investigate properties of polynomials $L_v[\alpha](X_n)$, where $v \in \mathbb{S}_n^{\mathrm{aff}}$ and $\alpha \in \mathbb{Z}_{\geq 0}^n$, and find its algebro-geometric interpretations.

5 Cauchy kernel

Let $u_1, u_2, \dots, u_{n-1}$ be a set of generators of the free algebra $\mathcal{F}_{n-1}$, which are assumed also to be commute with the all variables $\mathfrak{P}_n := \{p_{i,j},\ 2 \leq i + j \leq n+1,\ i \geq 1, j \geq 1\}$.

Definition 5.1. The Cauchy kernel $\mathcal{C}(\mathfrak{P}_n, U)$ is defined to be as the ordered product

$$\mathcal{C}(\mathfrak{P}_n, U) = \prod_{i=1}^{n-1} \left\{ \prod_{j=n-1}^{i} (1 + p_{i,j-i+1} u_j) \right\}. \tag{5.1}$$

For example,

$$\mathcal{C}(\mathfrak{P}_4, U) = (1 + p_{1,3}u_3)(1 + p_{1,2}u_2)(1 + p_{1,1}u_1)(1 + p_{2,2}u_3)(1 + p_{2,1}u_2)(1 + p_{3,1}u_3).$$

In the case $\{p_{ij} = x_i, \forall\, j\}$ we will write $\mathcal{C}_n(X, U)$ instead of $\mathcal{C}(\mathfrak{P}_n, U)$.

Lemma 5.2.

$$\mathcal{C}(\mathfrak{P}_n, U) = \sum_{(\mathbf{a},\mathbf{b}) \in \mathcal{S}_n} \prod_{j=1}^{p} p_{\{a_j, b_j\}} w(\mathbf{a}, \mathbf{b}), \tag{5.2}$$

where $\mathbf{a} = (a_1, \dots, a_p)$, $\mathbf{b} = (b_1, \dots, b_p)$, $w(\mathbf{a}, \mathbf{b}) = \prod_{j=1}^{p} u_{a_j + b_j - 1}$, *and the sum in* (5.2) *runs over the set*

$$\mathcal{S}_n := \big\{ (\mathbf{a}, \mathbf{b}) \in \mathbb{N}^p \times \mathbb{N}^p \,|\, \mathbf{a} = (a_1 \leq a_2 \leq \cdots \leq a_p), \\ a_i + b_i \leq n, \ \textit{and if } a_i = a_{i+1} \Longrightarrow b_i > b_{i+1} \big\}.$$

We denote by $\mathcal{S}_n^{(0)}$ the set $\{(\mathbf{a}, \mathbf{b}) \in \mathcal{S}_n \,|\, w(\mathbf{a}, \mathbf{b}) \text{ is a tableau word}\}$.

The number of terms in the r.h.s. of (5.1) is equal to $2^{\binom{n}{2}}$, and therefore is equal to the number $\#|\,\mathrm{STY}(\delta_n, \leq n)|$ of semistandard Young tableaux of the staircase shape $\delta_n := (n-1, n-2, \dots, 2, 1)$ filled by the numbers from the set $\{1, 2, \dots, n\}$. It is also easily seen that the all terms appearing in the r.h.s. of (5.2) are different, and thus $\#|\mathcal{S}_n| = \#|\,\mathrm{STY}(\delta_n, \leq n)|$.

We are interested in the decompositions of the Cauchy kernel $\mathcal{C}(\mathfrak{P}_n, U)$ in the algebras $\mathcal{P}_n$, $\mathcal{NP}_n$, $\mathcal{IP}_n$, $\mathcal{NC}_n$ and $\mathcal{IC}_n$.

5.1 Plactic algebra $\mathcal{P}_n$

Let λ be a partition and α be a composition of the same size. Denote by $\widetilde{\mathrm{STY}}(\lambda,\alpha)$ the set of semistandard Young tableaux T of the shape λ and content α which must satisfy the following conditions: for each $k=1,2,\ldots,$ the all numbers k are located in the first k columns of the tableau T. In other words, the all entries $T(i,j)$ of a semistandard tableau $T\in\widetilde{\mathrm{STY}}(\lambda,\alpha)$ have to satisfy the following conditions: $T_{i,j}\ge j$.

For a given (semi-standard) Young tableau T let us denote by $R_i(T)$ the set of numbers placed in the i-th row of T, and denote by $\widetilde{\mathrm{STY}}_0(\lambda,\alpha)$ the subset of the set $\widetilde{\mathrm{STY}}(\lambda,\alpha)$ involving only tableaux T which satisfy the following constrains

$$R_1(T)\supset R_2(T)\supset R_3(T)\supset\cdots.$$

To continue, let us denote by $\mathcal{A}_n$ (respectively by $\mathcal{A}_n^{(0)}$) the union of the sets $\widetilde{\mathrm{STY}}(\lambda,\alpha)$ (resp. that of $\widetilde{\mathrm{STY}}_0(\lambda,\alpha)$) for all partitions λ such that $\lambda_i\le n-i$ for $i=1,2,\ldots,n-1$, and all compositions α, $l(\alpha)\le n-1$. Finally, denote by $\mathcal{A}_n(\lambda)$ (resp. $\mathcal{A}_n^{(0)}(\lambda)$) the subset of $\mathcal{A}_n$ (resp. $\mathcal{A}_n^{(0)}(\lambda)$) consisting of all tableaux of the shape λ.

Lemma 5.3.

- $|\mathcal{A}_n(\delta_n)|=1$, $|\mathcal{A}_n(\delta_{n-1})|=(n-1)!$, $|\mathcal{A}_n((n-1))|=C_{n-1}$ *the* $(n-1)$*-th Catalan number. More generally,*

  $$\left|\mathcal{A}_n\big((1^k)\big)\right|=\binom{n-1}{k},\qquad |\mathcal{A}_n((k))|=\frac{n-k}{n}\binom{n+k-1}{k}=\dim\mathbb{S}_{n+k-1}^{(n-1,k)},$$

  $k=1\ldots,n-1$; *cf.* [67, A009766]; *here* $\dim\mathbb{S}_n^{\lambda}$ *stands for the dimension of the irreducible representation of the symmetric group* $\mathbb{S}_n$ *corresponding to a partition* $\lambda\vdash n$.

- *Let* $k\ge\ell\ge 2$, $n\ge k+2$,*then*

  $$|\mathcal{A}_n((k,\ell))|=\frac{(n-k)(n-\ell+1)(k-\ell+1)(n^2-n-\ell(k+1))}{\ell!\ (k+1)\ k\ (n+k)}\binom{n+k}{k-1}\prod_{i=1}^{\ell-2}(n+i),$$

  $k=1,\ldots,n$. *The case* $k=\ell$ *has been studied in [21] where one can find a combinatorial interpretation of the numbers* $|\mathcal{A}_n((k,k))|$ *for all positive integers* n*; see also [67, A005701, A033276] for more details concerning the cases* $k=2$ *and* $k=3$*. Note that in the case* $k=\ell$ *one has* $n^2-n-k(k+1)=(n+k)(n-k-1)$*, and the above formula can be rewitten as follows* $(k\ge 2)$

  $$|\mathcal{A}_n((k,k))|=\frac{(n-k-1)(n-k)(n-k-1)}{(k+1)k^2(k-1)}\binom{n+k-2}{k-2}\binom{n+k-1}{k-1}.$$

- *(Boundary case: the number* $|\mathcal{A}_N((n^k))|$ *for* $N=n+k$*.)*

  $$|\mathcal{A}_{k+n}((n^k))|=|\mathcal{A}_{k+n}(((n+1)^k))|=\det|Cat_{n+i+j-1}|_{1\le i,j\le k}=\prod_{1\le i\le j\le n}\frac{i+j+2k}{i+j}.$$

  So far as we know, the third equality has been proved for the first time in [10]. The both sides of the third identity have a big variety of combinatorial interpretations such as the number of k-tuples of noncrossing Dyck paths; that of k-tringulations of a convex $(n+k+1)$-gon; that of semistandard Young tableaux with entries from the set $\{1,\ldots,n\}$ having only columns of an even length and bounded by height $2k$, [10]; that of pipe dreams (*or* compatible sequences) associated with the Richardson permutation $1^k \times w_0^{(n)} \in \mathbb{S}_{n+k}$, *etc.*, see e.g., [67, *A*078920] and the literature quoted therein. It seems an interesting task to read off an alternating sign matrx of size $(n+k) \times (n+k)$ from a given k-triangulation of a convex $n+k+1)$-gon.

- *More generally (A.K.),*

$$|\mathcal{A}_N((n^k))| = \prod_{\substack{1\le i\le k,\ 1\le j\le n\\ j-i\le n-k}} \frac{N-i-j+1}{i+j-1} \prod_{\substack{1\le i\ \le\ k,\ 1\le j\le n\\ j-i\ >\ n-k}} \frac{N+i+j-1}{i+j-1}.$$

Moreover, the number $|\mathcal{A}_N((n^k))|$ *also is equal to the number of k-tuples of noncrossing Dyck paths staring from the point* $(0,0)$ *and ending at the point* $(N, N-n-k)$*.*

So far as we know, for the case $k=2$ an equivalent formula for the number of pairs of noncrossing Dyck paths connecting the points $(0,0)$ and $(N, N-n-2)$, has been obtained for the first time in [21].

- *There exists a bijection* $\rho_n \colon \mathcal{A}_n \longrightarrow \mathrm{ASM}(n)$ *such that the image* $\mathrm{Im}\big(\mathcal{A}_n^{(0)}\big)$ *contains the set of* $n \times n$ *permutation matrices.*
- *The number of row strict* (*as well as column strict*) *diagrams* $\lambda \subset \delta_{n+1}$ *is equal to* 2^n*.*

Recall that a row-strict diagram[18] λ is on such that $\lambda_i - \lambda_{i+1} \ge 1$, $\forall\, i$; $\delta_n := (n-1, n-2, \dots, 2, 1)$.

Example 5.4. Take $n=5$ so that $\mathrm{ASM}(5) = 429$ and $\mathrm{Cat}(5) = 42$. One has

$$\begin{aligned}\big|\mathcal{A}_5^{(0)}\big| &= \sum_{\substack{\lambda\subset\delta_4:=(4,3,2,1)\\ \lambda_i-\lambda_{i+1}\ge 1\,\forall\, i}} |\mathcal{A}_5^{(0)}(\lambda)| = \big|\mathcal{A}_5^{(0)}(\varnothing)\big| + \big|\mathcal{A}_5^{(0)}((1))\big| + \big|\mathcal{A}_5^{(0)}((2))\big| + \big|\mathcal{A}_5^{(0)}((3))\big| \\ &\quad + \big|\mathcal{A}_5^{(0)}((2,1))\big| + \big|\mathcal{A}_5^{(0)}((4))\big| + \big|\mathcal{A}_5^{(0)}((3,1))\big| + \big|\mathcal{A}_5^{(0)}((3,2))\big| + \big|\mathcal{A}_5^{(0)}((4,1))\big| \\ &\quad + \big|\mathcal{A}_5^{(0)}((4,2))\big| + \big|\mathcal{A}_5^{(0)}((3,2,1))\big| + \big|\mathcal{A}_5^{(0)}((4,3))\big| + \big|\mathcal{A}_5^{(0)}((4,2,1))\big| \\ &\quad + \big|\mathcal{A}_5^{(0)}((4,3,1))\big| + \big|\mathcal{A}_5^{(0)}((4,3,2))\big| + \big|\mathcal{A}_5^{(0)}((4,3,2,1))\big| \\ &= 1+4+9+14+6+14+16+4+21+14+4+1+9+2+1+1 = 121 \\ &\quad \text{(sum of 16 terms)}, \end{aligned}$$

$$\sum_{k=0}^{4} |\mathcal{A}_5((k))| = 1+4+9+14+14 = 42.$$

We expect that the image $\rho_n\big(\bigcup_{k=0}^{n-1} \mathcal{A}_n((k))\big)$ coincides with the set of $n \times n$ permutation matrices corresponding to either 321-avoiding (or 132-avoiding) permutations.

Now we are going to define a statistic $n(T)$ on the set $\mathcal{A}_n$.

Definition 5.5. Let λ be a partition, α be a composition of the same size. For each tableau $T \in \widetilde{\mathrm{STY}}(\lambda, \alpha) \subset \mathcal{A}_n(\lambda)$ define

$$n(T) = \alpha_n = \#|\{(i,j) \in \lambda | T(i,j) = n\}|.$$

Clearly, $n(T) \le \lambda_1$.

Define polynomials

$$\mathcal{A}_\lambda(t) := \mathcal{A}_\lambda(n;t) = \sum_{T \in \mathcal{A}_n(\lambda)} t^{\lambda_1 - n(T)}.$$

[18] Known also as a strict partition.

It is instructive to display the numbers $\{\mathcal{A}_n(\lambda), \lambda \subset \delta_n\}$ as a vector of the length equals to the n-th Catalan number. For example,

$$\begin{aligned}&\mathcal{A}_4(\varnothing,(1),(2),(1,1),(3),(2,1),(1,1,1),(3,1),(2,2),(2,1,1),(3,2),(3,1,1),(2,2,1),(3,2,1))\\ &\qquad = (1,3,5,3,5,6,1,6,3,2,3,2,1,1).\end{aligned}$$

It is easy to see that the above data, as well as the corresponding data for $n = 5$, coincide with the list of refined totally symmetric self-complementary plane partitions that fit in the box $2n \times 2n \times 2n$ (TSSCPP(n) for short) listed for $n = 1, 2, 3, 4, 5$ in [12, Appendix D].

In fact we have

Theorem 5.6. *The sequence $\{\mathcal{A}_n(\lambda), \lambda \subset \delta_n\}$ coincides with the set of refined* TSSCPP(n) *numbers as defined in* [12]*. More precisely,*

- $|\mathcal{A}_n(\lambda; N)| = \det \big|\binom{N-i}{\lambda'_j-j+i}\big|_{1\le i,j\le \ell(\lambda)}$,
- *we have*

$$\mathcal{A}_\lambda(N;t) := \det\left|\binom{N-i-1}{\lambda'_j-j+i-1} + t\binom{N-i-1}{\lambda'_j-j+i}\right|_{1\le i,j\le\ell(\lambda)},$$

- *let λ be a partition, $|\lambda| = n$, consider the column multi-Schur polynomial and t-deformation thereof*

$$s^*_\lambda(X_N) := \det|e_{\lambda'_j-j+i}(X_{N-i})|_{1\le i,j\le\ell(\lambda)}, \ \ cf.\ [58, Chapter III], [72], \ \ and$$

$$s^*_\lambda(X_N;t) := \det|x_{N-i}\ e_{\lambda'_j-j+i-1}(X_{N-i-1}) + t\ e_{\lambda'_j-j+i}(X_{N-i-1})|_{1\le i,j\le\ell(\lambda)}.$$

  Then, assuming that $\lambda \subset \delta_n$ and $N \ge \lambda_1 + \lambda'_{\ell(\lambda)}$, the polynomial $s^*_\lambda(X_N)$ has nonnegative (integer) coefficients.
- *polynomial $\mathcal{A}_\lambda(t)$ is equal to a t-analog of refined* TSSCPP(n) *numbers $P_n(\lambda'_{n-1}+1, \dots, \lambda'_{n-i}+i, \dots, \lambda'_1+n-1|t)$ introduced by means of recurrence relations in* [12, relation (3.5)].
- *One has*

$$s^*_{((n^k))}(X_{n+k}) = M_{n,k}(X_{n+k})\ \mathfrak{S}_{1^k\times w_0^{(n)}}(x_1^{-1},\dots,x_{k+n}^{-1}),$$

  where $\mathfrak{S}_{1^k\times w_0^{(n)}}(X_{n+k})$ denotes the Schubert polynomial corresponding to the permutation

$$1^k \times w_0^{(n)} = [1,2,\dots,k,n+k,n+k-1,\dots,k+1],$$

  and $M_{n,k}(X_{n+k}) = \prod_{a=1}^k x_a^{-1}\ \prod_{a=1}^{n+k}\ x_a^{\min(n+k-a+1,n)}$.

In particular, $\sum\limits_{\lambda\subset\delta_n} \mathcal{A}_\lambda(t) = \sum\limits_{1\le j\le n-1} A_{n,j}t^{j-1}$, where $A_{n,j}$ stands for the number of alternating sign matrices (ASM$_n$ for short) of size $n \times n$ with a 1 on top of the j-th column.

Corollary 5.7. *([46, 41]) The number of different tableau subwords in the word*

$$w_0 := \prod_{j=1}^{n-1}\left\{\prod_{a=n-1}^{j} a\right\}$$

is equal to the number of alternating sign matrices of size $n \times n$, i.e.,

$$|\mathcal{A}_n| = |\operatorname{TSSCPP}(n)| = |\operatorname{ASM}_n|.$$

It is well-known [6] that

$$A_{n,j} = \binom{n+j-2}{j-1} \frac{(2n-j-1)!}{(n-j)!} \prod_{i=0}^{n-2} \frac{(3i+1)!}{(n+i)!},$$

and the total number A_n of ASM of size $n \times n$ is equal to

$$A_n \equiv A_{n+1,1} = \sum_{j=1}^{n} A_{n,j} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Example 5.8. Take $\lambda = (3^4)$, then
$\mathcal{A}_\lambda(6;t) = 6\ (14,21,15,5)_t$, $\mathcal{A}_\lambda(6;1) = 330$,
$\mathcal{A}_\lambda(7;t) = 33\ (18,40,45,30,10)_t$, $\mathcal{A}_\lambda(7;1) = 4719$,
$\mathcal{A}_\lambda(8;t) = 11\ (225,660,982,889,429)_t$, $\mathcal{A}_\lambda(8;1) = 35035 = (\frac{5\times 7}{8}) \times (7 \times 8 \times 11 \times 13)$.
It is clear that $Coeff_{t^k}\Big(\mathcal{A}_{(n^k)}(N;t)\Big) = \mathcal{A}_{(n^k)}(N-1;1)$.

Let $\lambda \subset \delta_N$ define $\mathcal{A}_\lambda(X_N;t) := s_\lambda^*(X_N)$, and set $\mathcal{A}_\lambda(X_N:t) := s_\lambda^*(x_i = 1, \forall i \in [1,N])$.

Theorem 5.9. *(The case $\lambda = \delta_n := (n-1,n-2,\ldots,2,1)$ One has*

- $$\mathcal{A}_{\delta_n}(n+1;t) = \prod_{j=2}^{n} (1+jt).$$

- *(Gandhi–Dumont polynomials, [14], [67, A036970])*

  $$\mathcal{A}_{\delta_n}(n+2;t) = \sum_{k=2}^{n} B_{n,k} t^k$$

  where

  $$B_{n,k} = \sum_{\{k_j\}} \prod_{j=2}^{n-1} \binom{2j-k_{j-1}}{2j-k_j},$$

  where the sum runs over set of sequences $\{1 \le k_1 < k_2 < \ldots < k_{n-2} < k_{n-1} = 2n-k\}$.

- *In particular,*

  $$\mathcal{A}_{\delta_n}(n+2;0) = G_{2n}, \quad \mathcal{A}_{\delta_n}(n+2;1) = G_{2n+2},$$

  where $G_{2n} = 2(2^{2n}-1)B_{2n}$ and B_{2n} denotes the unsigned Genocchi [19] numbers, and B_{2n} denotes the Bernoulli number, see, e.g., [67, A027642].

- $\mathcal{A}_{\delta_n}(n+2;-1) = (-1)^n$.

- *Let $\mathcal{A}_{\delta_n}(N;t,q)$ denote the principal specialization $x_i := q^{i-1}, i \ge 1$, of the polynomial $s_{\delta_n}^*(X_N;t)$, and write $\mathcal{A}_{\delta_n}(n+2;t,q) = \sum_{k=2}^{n-1} B_{n,k}(q) t^k$. Then $B_{n,k}(q) \in \mathbb{N}[q]$.*

[19] Recall that the unsigned Genocchi numbers are defined through the generting function

$$\frac{2t}{e^t+1} = \sum_{n \ge 1} \frac{G_{2n}}{(2n)!} (-1)^n t^{2n},$$

see, e.g., [14], or
https://en.wikipedia.org/wiki/Genocchi_number

For example, $\mathcal{A}_{\delta_6}(8;t) = (2073, 8146, 12840, 10248, 4200, 720)_t$, $2073 = G_{12}$, $\mathcal{A}_{\delta_6}(8;1) = 38227 = G_{14}$, cf. [67, *A*036970]; $\mathcal{A}_{(2,1)}(5;t,q) = q\begin{bmatrix}4\\2\end{bmatrix}_q + t(q+q^2)^3 + t^2q^5\begin{bmatrix}3\\1\end{bmatrix}_q$. The last example shows that the polynomials $\mathcal{A}_{(\delta_n)}(n+2;t,q)$ give rise to a q deformation of the polynomials $p_{n+2}(t;\delta_n)$ associated with refined $TSSCPP(n)$introduced in [12] which appeared to be coicide (A.K.) with the Gandhi polynomials introduced, e.g., in [14]. However, the polynomials $\mathcal{A}_{(\delta_n)}(n+2;t,q)$ are different from a q-deformation of Gandhi polynomials defined in [22].

It is easy to see that $\mathcal{A}_{\delta_n}(X_{n+2};t) = \mathcal{A}^{(1)}_{\delta_n}(X_n;t) + x_{n+1}\mathcal{A}^{(2)}_{\delta_n}(X_n;t)$ for some polynomials depending on the variables $\{x_1,\ldots,x_n\}$ with nonnegative integer coefficients.

Theorem 5.10. *(The Genocchi numbers of the second kind, [67, A005439])*

- $\mathcal{A}^{(1)}_{\delta_n}(t;x_i=1,\forall i\in[1,n])$ *is a polynomial of the following form*

$$\mathcal{A}^{(1)}_{\delta_n}(t;x_i=1,\forall i\in[1,n]) = tG^{(2)}_{n-2} + \cdots + (n-1)!\ t^{n-1}$$
$$\mathcal{A}^{(1)}_{\delta_n}(t=1;x_i=1,\forall i\in[1,n]) = G^{(2)}_{n-1},$$

  where $G^{(2)}_n$ stands for the n-th Genocchi number of the second kind. It is well known that $G^{(2)}_n = 2^{n-1}G^{(m)}_n$, where $G^{(m)}_n$ denotes the so-called n-th median Genocchi number [67, A000366].

- $\mathcal{A}^{(2)}_{\delta_n}(t;x_i=1,\forall i\in[1,n]) = G_{n-1} + \ldots + L_n\ t^{n-2}$, *where as before, G_n denotes the n-th Genocchi number (of the first kind), [67, A036970]; $L_n := \frac{n-1}{2}\ n!$ denotes the n-th Lah number, see, e.g., [67, 001286].*

Example 5.11.

(1) Take $\lambda=\delta_3$ and $n=5$, then

$$\mathcal{A}_{\delta_3}(X_4;t) = t(x_1+x_2)(x_3^2 + t(x_1x_2 + x_1\ x_3 + x_2\ x_3))+$$
$$x_4(x_1+x_2+x_3)(t\ x_1 + tx_2 + x_3).$$

Therefore, $\mathcal{A}_{\delta_3}(x_i=1,\forall i\in[1,3];t) = 2t + 6t^2 + x_4(3+6t)$.

(2) Take Take $\lambda=\delta_4$ and $n=6$, then

$$\mathcal{A}_{\delta_4}(x_i=1,\forall i\in[1,4];t) = 8\ t(1+3\ t+3\ t^2) + x_5\ (17+46\ t+36\ t^2).$$

(3) Take Take $\lambda=\delta_5$ and $n=7$, then

$$\mathcal{A}_{\delta_5}(x_i=1,\forall i\in[1,5];t) = t(56,192,240,120)_t + 5\ x_6\ (31,100,114,48)_t.$$

(4) Take $\lambda=\delta_6$ and $n=8$, then

$$\mathcal{A}_{\delta_6}(x_i=1,\forall i\in[1,6];t) = t\ (608,2304,3408,2400,720)_t + x_7(2073,7538,10536,6840,1800)_t.$$

(5) (4) Take $\lambda=\delta_7$ and $n=9$, then

$$\mathcal{A}_{\delta_7}(x_i=1,\forall i\in[1,7];t) = t\ (9440,38688,64464,55440,25200,5040)_t+$$
$$x_8(38227,151984,243390,198576,84000,15120)_t.$$

(6) Principal specialization.

$$q^{-10}\ \mathcal{A}_{\delta_4}(x_i=q^{i-1},\forall i\geq 1;q^2\ t) = (1,2,3,4,4,2,1)_q + t\ (1,5,9,12,13,9,4,1)_q+$$
$$t^2(1,5,11,14,14,10,4,1)_q + t^3(1,3,5,6,5,3,1)_q.$$

Therefore, the polynomials $\mathcal{A}_{\delta_n}(X_{n+1};t)$ and $\mathcal{A}^{(1)}_{\delta_n}(X_n;t)$ define multi-parameter deformations of the Genocchi numbers of the first and the second types correspondingly. It is an interesting task to relate these polynomials with those have been studied in [22], if so.

Proofs of the both Teorems 5.9 and 5.10 are besed on the study of recurrence relations among the determinants of the following "almost upper triangular" matrices

$$\det\left| t\binom{n-i-1}{2(j-i)} + \delta_{1,i}\ x_{n-i}\binom{n-1-1}{2(j-i)+1}\right|_{1\le i,j\le n-1}.$$

Details and q-analogs of some formulas stated in Section 5.1 will appear in a separate publication.

Problem 5.12. *Give combinatorial interpretations of polynomials $\mathcal{A}_{p\delta_n}(N;t,q)$ and $\mathcal{A}_{(n^k)}(N;t,q)$ for all $N \ge pn-1$.*

Let as before $\mathrm{STY}(\delta_n \le n) := \mathcal{ST}_n$ denotes the set of all semistandard Young tableaux of the staircase shape $\delta_n = (n-1, n-2, \ldots, 2, 1)$ filled by the numbers from the set $\{1, \ldots, n\}$. Denote by $\mathcal{ST}^{(0)}_n$ the subset of "anti-diagonally" increasing tableaux, i.e.,

$$\mathcal{ST}^{(0)}_n = \{T \in \mathrm{STY}(\delta_n, \le n) \,|\, T_{i,j} \ge T_{i-1,j+1} \text{ for all } 2 \le i \le n-1,\ 1 \le j \le n-2\}.$$

One (A.K.) can construct bijections

$$\iota_n\colon\ \mathcal{S}_n \sim \mathcal{ST}_n, \qquad \zeta_n\colon\ \mathcal{A}_n \sim \mathcal{ST}^{(0)}_n$$

such that $\mathrm{Im}(\iota_n) = \mathrm{Im}(\zeta_n)$.

Proposition 5.13.

$$\sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_n)\\ \rho_n\ge\lambda}} K_{\rho_n,\lambda}\binom{n}{m_0(\lambda),\quad m_1(\lambda),\quad \ldots,\quad m_n(\lambda)} = 2^{\binom{n}{2}},$$

$$\sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_n)\\ \rho_n\ge\lambda}} \binom{n}{m_0(\lambda),\quad m_1(\lambda),\quad \ldots,\quad m_n(\lambda)} = \mathcal{F}_n,$$

where $\mathcal{F}_n$ denotes the number of forests of trees on n labeled nodes; $K_{\rho_n,\lambda}$ denotes the Kostka number, i.e., the number of semistandard Young tableaux of the shape $\rho_n := (n-1, n-2, \ldots, 1)$ and content/weight λ; for any partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0)$ we set $m_i(\lambda) = \{j \,|\, \lambda_j = i\}$.

Let α be a composition, we denote by α^+ the partition obtained from α by reordering of its parts. For example, if $\alpha = (0, 2, 0, 3, 1, 0)$ then $\alpha^+ = (3, 2, 1)$. Note that $\ell(\alpha) = 6$, but $\ell(\alpha^+) = 3$.

Now let α be a composition such that $\rho_n \ge \alpha^+$, $\ell(\alpha) \le n$, that is $\alpha_j = 0$, if $j > \ell(\alpha)$, $|\alpha| = \binom{n}{2}$ and

$$\sum_{k\le j}(\rho_n)_k \ge \sum_{k\le j}(\alpha^+)_k, \qquad \forall\, j.$$

There is a unique semistandard Young tableau $T_n(\alpha)$ of shape ρ_n and content α which corresponds to the maximal configuration of type $(\rho_n;\alpha)$ and has all quantum numbers (riggings) equal to zero. It follows from Proposition 5.13 that $\#\{\alpha \,|\, \ell(\alpha) \le n,\ \rho_n \ge \alpha^+\} = \mathcal{F}_n$. Therefore there is a natural embedding of the set of forests on n labeled nodes to the set of semistandard Young tableaux of shape ρ_n filled by the numbers from the set $[1, \ldots, n]$. We denote by $\mathcal{FT}_n \subset \mathrm{STY}(\rho_n, \le n)$ the subset $\{T_n(\alpha) \,|\, \rho_n \ge \alpha^+,\ \ell(\alpha) \le n\}$. Note that the set

$\mathcal{K}_n := \{\alpha \,|\, \ell(\alpha) = n,\, (\alpha)^+ = \rho_n\}$ contains $n!$ compositions, and under the rigged configuration bijection the elements of the set $\mathcal{K}_n$ correspond to the *key* tableaux [52] of shape ρ_n. See also [2] for connections of the Lascoux–Schützenberger *keys* and ASM.

Let us say a few words about the Kostka numbers $K_{\rho_n,\alpha}$. First of all, it's clear that if $\alpha = (\alpha_1, \alpha_2, \dots)$ is a composition such that $\alpha_1 = n-1$, then $K_{\rho_n,\alpha} = K_{\rho_{n-1},\alpha[1]}$, where we set $\alpha[1] := (\alpha_2, \dots)$.

Now assume that $n = 2k+1$ is an odd integer, and consider partitions $\nu_n := (k^n)$ and $\mu_n := ((k+1)^k, k^k)$. Then

$$K_{\rho_n,\nu_n} = \mathrm{Coeff}_{(x_1x_2\cdots x_n)^k}\left(\prod_{1\le i<j\le n}(x_i+x_j)\right), \qquad \binom{2k}{k}K_{\rho_n,\mu_n} = K_{\rho_n,\nu_n}.$$

It is well-known that the number K_{ρ_n,ν_n} is equal to number of labeled regular tournaments with $n := 2k+1$ nodes, see, e.g., [67, *A*007079].

In the case when $n = 2k$ is an even number, one can show that

$$K_{\rho_n,\nu_n} = K_{\rho_{n-1},\nu_{n-1}}, \qquad K_{\rho_n,\mu_n} = K_{\rho_{n+1},\mu_{n+1}}.$$

Note that the rigged configuration bijection gives rise to an embedding of the set of labeled regular tournaments with $n := 2k+1$ nodes

to the set $\mathrm{STY}(\rho_n, \le n)$, if n is an odd integer, and

to the set $\mathrm{STY}(\rho_{n-1}, \le n-1)$, if n is even integer.

Theorem 5.14.

(1) *In the plactic algebra $\mathcal{P}_n$ the Cauchy kernel has the following decomposition*

$$\mathcal{C}_n(\mathfrak{P}, U) = \sum_{T\in\mathcal{A}_n}\mathcal{K}_T(\mathfrak{P})u_{w(T)}.$$

(2) *Let $T \in \mathcal{A}_n$, and $\alpha(T)$ be its bottom code. Then*

$$\mathcal{K}_T(\mathfrak{P}) - \prod_{(i,j)\in T} p_{\{i,T(i,j)-j+1\}} \ge 0,$$

and equality holds if and only if the bottom code $\alpha(T)$ is a partition.

Note that the number of *different shapes* among the tableaux in the set $\mathcal{A}_n$ is equal to the Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$.

Problem 5.15. *Construct a bijection between the set $\mathcal{A}_n$ and the set of alternating sign matrices* ASM_n.

Example 5.16. For $n = 4$ one has

$$\begin{aligned}\mathcal{C}_4(X,U) = {}& K[0] + K[1]u_1 + K[01]u_2 + K[001]u_3 + K[11](u_{12}+u_{22}) + K[2](u_{21}+u_{31})\\ &+ K[101]u_{13} + K[02]u_{32} + K[011](u_{23}+u_{33}) + K[3]u_{321} + K[12](u_{312}+u_{322})\\ &+ K[21]u_{212} + K[111](u_{123}+u_{133}+u_{233}+u_{223}+u_{333}) + K[021]u_{323}\\ &+ K[201](u_{313}+u_{213}) + K[31]u_{3212} + K[301]u_{3213}\\ &+ K[22](u_{3132}+u_{2132}+u_{3232}) + K[121](u_{3123}+u_{3233}+u_{3223})\\ &+ K[211](u_{2123}+u_{2133}+u_{3133}) + K[32]u_{32132} + K[311](u_{32123}+u_{32133})\\ &+ K[221](u_{21323}+u_{31323}+u_{32323}) + K[321]u_{321323}.\end{aligned}$$

Let $w \in \mathbb{S}_n$ be a permutation with the Lehmer code $\alpha(w)$.

Definition 5.17. Define the *plactic polynomial* $\mathcal{PL}_w(U)$ to be

$$\mathcal{PL}_w(U) = \left\{ \sum_{T \in \mathcal{A}_n,\, \alpha(T)=\alpha(w)} u_{w(T)} \right\}.$$

Comments 5.18. It is easily seen from a definition of the Cauchy kernel that

$$\mathcal{C}_n(X,U) = \sum_{\alpha \subset \delta_n} K[\alpha](X) \mathcal{PL}_{w_0 w_\alpha^{-1}}(U),$$

where w_α denotes a unique permutation in $\mathbb{S}_n$ with the Lehmer code equals α; $K[\alpha](X)$ denotes the key polynomial corresponding to composition $\alpha \subset \delta_n$. The polynomials $\mathcal{PL}_{w_0 w_\alpha^{-1}}$ can be treated as a plactic version of noncommutative Schur and Schubert polynomials introduced and studied in [16, 29, 43, 52, 54].

Now let $X = \{x_1, \ldots, x_n\}$ be a set of mutually commuting variables, and

$$I_0^{(n)} := \{ \underbrace{n-1, n-2, \ldots, 2, 1}_{n-1}, \ldots, \underbrace{n-1, , n-2, \ldots, k+1, k}_{n-k}, \ldots, \underbrace{n-1, n-2}_{2}, n-1 \}$$

be lexicographically maximal reduced expression for the longest element $w_0 \in \mathbb{S}_n$. Let I be a tableau subword of the set $I_0 := I_0^{(n)}$. One can show (A.K.) that under the specialization

$$u_i = \begin{cases} x_i, & \text{if } \ i \in I_0 \backslash I, \\ 1, & \text{if } \ i \in I \end{cases}$$

the polynomial $\mathcal{PL}_{w_0 w_\alpha^{-1}}(U)$ turns into the Schubert polynomial $\mathfrak{S}_{w_\alpha}(X)$. In a similar fashion, consider the decomposition

$$\mathcal{C}_n(X,U) = \sum_{\alpha \subset \delta_n} \mathrm{KG}[\alpha](X; -\beta) \mathcal{PL}_{w_0 w_\alpha^{-1}}(U; \beta).$$

One can show (A.K.) that under the same specialization as has been listed above, the polynomial $\mathcal{PL}_{w_0 w_\alpha^{-1}}(U;\beta)$ turns into the β-Grothendieck polynomial $\mathcal{G}_{w_\alpha}^{\beta}(X)$.

Definition 5.19. Define algebra $\mathcal{PC}_n$ to be the quotient of the plactic algebra $\mathcal{P}_n$ by the two-sided ideal J_n by the set of monomials

$$\{u_{i_1} u_{i_2} \cdots u_{i_n}\}, \qquad 1 \le i_1 \le i_2 \le \cdots \le i_n \le n, \qquad \#\{a | i_a = j\} \le j, \qquad \forall\, j = 1, \ldots, n.$$

Theorem 5.20.

- *The algebra $\mathcal{PC}_n$ has dimension equals to* $\mathrm{ASM}(n)$,
- $\mathrm{Hilb}(\mathcal{PC}_n, q) = \sum\limits_{\lambda \in \delta_{n-1}} |\mathcal{A}_n(\lambda)| q^{|\lambda|}$,
- $\mathrm{Hilb}((\mathcal{PC}_{n+1})^{ab}, q) = \sum\limits_{k=0}^{n} \frac{n-k+1}{n+1} \binom{n+k}{n} q^k$, *cf.* [67, *A*009766].

Definition 5.21. Denote by $\mathcal{PC}_n^{\sharp}$ the quotient of the algebra $\mathcal{PC}_n$ by the two-sided ideal generated by the elements $\{u_i u_j - u_j u_i,\ |i-j| \ge 2\}$.

Proposition 5.22. *Dimension* $\dim \mathcal{PC}_n^{\sharp}$ *of the algebra* $\mathcal{PC}_n^{\sharp}$ *is equal to the number of Dyck paths whose ascent lengths are exactly* $\{1, 2, \ldots, n+1\}$.

See [67, $A107876$, $A107877$] where the first few of these numbers are displayed.

Example 5.23.

$$\begin{aligned}
&\operatorname{Hilb}\left(\mathcal{PC}_5^\sharp, t\right) = (1,4,12,27,48,56,54,38,20,7,1)_t, \qquad \dim \mathcal{PC}_5^\sharp = 268,\\
&\operatorname{Hilb}\left(\mathcal{PC}_6^\sharp, t\right) = (1,5,18,50,116,221,321,398,414,368,275,175,89,35,9,1)_t,\\
&\dim \mathcal{PC}_6^\sharp = 2496, \qquad \dim \mathcal{PC}_7^\sharp = 28612.
\end{aligned}$$

Example 5.24.

$$\begin{aligned}
&\operatorname{Hilb}(\mathcal{PC}_3, q) = (1,2,3,1)_q, \qquad \operatorname{Hilb}(\mathcal{PC}_4, q) = (1,3,8,12,11,6,1)_q,\\
&\operatorname{Hilb}(\mathcal{PC}_5, q) = (1,4,15,35,69,91,98,70,35,10,1)_q,\\
&\operatorname{Hilb}(\mathcal{PC}_6, q) = (1,5,24,74,204,435,783,1144,1379,1346,1037,628,275,85,15,1)_q,\\
&\operatorname{Hilb}(\mathcal{PC}_7, q) = (1,6,35,133,461,1281,3196,6686,12472,19804,27811,33271,34685,\\
&\qquad\qquad 30527,22864,14124,7126,2828,840,175,21,1)_q.
\end{aligned}$$

Problem 5.25. *Denote by* $\mathfrak{A}_n$ *the algebra generated by the curvature of* 2*-forms of the tautological Hermitian linear bundles* ξ_i, $1 \le i \le n$, *over the flag variety* $\mathcal{F}l_n$ [66]. *It is well-known* [62] *that the Hilbert polynomial of the algebra* $\mathfrak{A}_n$ *is equal to*

$$\operatorname{Hilb}(\mathfrak{A}_n, t) = \sum_{F \in \mathcal{F}_n} t^{\operatorname{inv}(F)} = \sum_{F \in \mathcal{F}_n} t^{\operatorname{maj}(F)},$$

where the sum runs over the set $\mathcal{F}_n$ *of forests* F *on the* n *labeled vertices, and* $\operatorname{inv}(F)$ (*resp.* $\operatorname{maj}(F)$) *denotes the inversion index* (*resp. the major index*) *of a forest* F.[20]

Clearly that

$$\dim(\mathfrak{A}_n)_{\binom{n}{2}} = \dim(\mathcal{PC}_n)_{\binom{n}{2}} = \dim(H^\star(\mathcal{F}l_n, \mathbb{Q}))_{\binom{n}{2}} = 1.$$

For example,

$$\begin{aligned}
&\operatorname{Hilb}(\mathcal{PC}_6, t) = (1,5,24,74,204,435,783,1144,1379,1346,1037,628,275,85,15,1)_t,\\
&\operatorname{Hilb}(\mathfrak{A}_6, t) = (1,5,15,35,70,126,204,300,405,490,511,424,245,85,15,1)_t,\\
&\operatorname{Hilb}(H^\star(\mathcal{F}l_n, \mathbb{Q}), t) = (1,5,14,29,49,71,90,101,101,90,71,49,29,14,5,1)_t.
\end{aligned}$$

We expect that $\dim(\mathcal{PC}_n)_{\binom{n}{2}-1} = \binom{n}{2}$ *and* $\dim(\mathcal{PC}_n)_{\binom{n}{2}-2} = \frac{3n+5}{4}\binom{n+2}{3} = s(n+2,2)$, *where* $s(n,k)$ *denotes the Stirling number of the first kind, see, e.g.,* [67, $A000914$].

Problem 5.26.

(1) *Is it true that* $\operatorname{Hilb}(\mathcal{PC}_n, t) - \operatorname{Hilb}(\mathfrak{A}_n, t) \in \mathbb{N}[t]$*? If so, as we expect, does there exist an embedding of sets* $\iota\colon \mathcal{F}(n) \hookrightarrow \mathcal{A}_n$ *such that* $\operatorname{inv}(F) = n(\iota(F))$ *for all* $F \in \mathcal{F}_n$*? See Section* 5.1, *Definition* 5.5, *for definitions of the set* $\mathcal{A}_n$ *and statistics* $n(T)$, $T \in \mathcal{A}_n$.

[20] For the readers convenience we recall definitions of statistics $\operatorname{inv}(F)$ and $\operatorname{maj}(F)$. Given a forest F on n labeled vertices, one can construct a tree T by adding a new vertex (root) connected with the maximal vertices in the connected components of F.

The inversion index $\operatorname{inv}(F)$ is equal to the number of pairs (i,j) such that $1 \le i < j \le n$, and the vertex labeled by j lies on the shortest path in T from the vertex labeled by i to the root.

The major index $\operatorname{maj}(F)$ is equal to $\sum_{x \in \operatorname{Des}(F)} h(x)$; here for any vertex $x \in F$, $h(x)$ is the size of the subtree rooted at x; the descent set $\operatorname{Des}(F)$ of F consists of the vertices $x \in F$ which have the labeling strictly greater than the labeling of its child.

(2) *Define a "natural" bijection* $\kappa\colon \mathrm{STY}(\delta_n, \leq n) \longleftrightarrow 2^{\delta_n}$ *such that the set* $\kappa(\mathrm{MT}(n))$ *admits a "nice" combinatorial description.*

Here $\mathrm{MT}(n)$ denotes the set of (increasing) monotone triangles, namely, a subset of the set $\mathrm{STY}(\delta_n, \leq n)$ consisting of tableaux $\{T = (t_{i,j}) \,|\, i+j \leq n+1,\, i \geq 1, j \geq 1\}$ such that $t_{i,j} \geq t_{i-1,j+1}$, $2 \leq i \leq n$, $1 \leq j < n$, cf. [68]; $\delta_n = (n-1, n-2, \dots, 2, 1)$; $\mathrm{STY}(\delta_n, \leq n)$ denotes the set of semistandard Young tableaux of shape δ_n with entries bounded by n; 2^{δ_n} stands for the set of all subsets of boxes of the staircase diagram δ_n. It is well-known that $\#|\mathrm{STY}(\delta_n, \leq n)| = 2^{\delta_n} = 2^{\binom{n}{2}}$.

Comments 5.27. One can ask a natural question: when do noncommutative elementary polynomials $e_1(\mathbb{A}), \dots, e_n(\mathbb{A})$ form a *q-commuting family*, i.e., $e_i(\mathbb{A})e_j(\mathbb{A}) = qe_j(\mathbb{A})e_i(\mathbb{A})$, $1 \leq i < j \leq n$?

Clearly in the case of two variables one needs to necessitate the following relations

$$e_ie_je_i + e_je_je_i = qe_je_ie_i + qe_je_ie_j, \qquad i<j.$$

Having in mind to construct a q-deformation of the plactic algebra $\mathcal{P}_n$ such that the wanted q-commutativity conditions are fulfilled, one would be forced to add the following relations

$$qe_je_ie_j = e_je_je_i \qquad \text{and} \qquad qe_je_ie_i = e_ie_ie_je_i, \qquad i<j.$$

It is easily seen that these two relations are compatible iff $q^2 = 1$. Indeed,

$$e_j\underline{e_je_ie_j} = qe_j\underline{e_ie_je_i} = q^2e_je_je_ie_i \quad \Longrightarrow \quad q^2 = 1.$$

In the case $q = 1$ one comes to the Knuth relations (PL1) and (PL2). In the case $q = -1$ one comes to the "odd" analogue of the Knuth relations, or "odd" plactic relations (OPL_n), i.e., (OPL_n) :

$$\begin{aligned} u_ju_iu_k &= -u_ju_ku_i, && \text{if} \quad i<j\leq k\leq n, \qquad \text{and}\\ u_iu_ku_j &= -u_ku_iu_j, && \text{if} \quad i\leq j<k\leq n. \end{aligned}$$

Proposition 5.28 (A.K.)**.** *Assume that the elements* $\{u_1, \dots, u_{n-1}\}$ *satisfy the odd plactic relations* (OPL_n)*. Then the noncommutative elementary polynomials* $e_1(U), \dots, e_n(U)$ *are mutually anticommute.*

More generally, let $\mathcal{Q}_{\mathbf{n}} := \{q_{ij}\}_{1\leq i<j\leq n-1}$ be a set of parameters. Define *generalized plactic algebra* $\mathcal{QP}_n$ to be (unital) associative algebra over the ring $\mathbb{Z}[\{q_{ij}^{\pm1}\}_{1\leq i<j\leq n-1}]$ generated by elements $u_1, \dots, u_{n-1}$ subject to the set of relations

$$\begin{aligned} q_{ik}u_ju_iu_k &= u_ju_ku_i, && \text{if} \quad i<j\leq k, \qquad \text{and}\\ q_{ik}u_iu_ku_j &= u_ku_iu_j, && \text{if} \quad i\leq j<k. \end{aligned} \tag{5.3}$$

Proposition 5.29. *Assume that* $q_{ij} := q_j$, $\forall\, 1 \leq i < j$ *be a set of invertible parameters. Then the reduced generalized plactic algebra* $\mathcal{QPC}_n$ *is a free* $\mathbb{Z}[q_2^{\pm1}, \dots, q_{n-1}^{\pm1}]$*-module of rank equals to the number of alternating sign matrices* $\mathrm{ASM}(n)$*. Moreover,*

$$\mathrm{Hilb}(\mathcal{QPC}_n, t) = \mathrm{Hilb}(\mathcal{PC}_n, t), \mathrm{Hilb}(\mathcal{QP}_n, t) = \mathrm{Hilb}(\mathcal{P}_n, t).$$

Recall that *reduced generalized plactic algebra* $\mathcal{QPC}_n$ is the quotient of the generalized plactic algebra by the two-sided ideal J_n introduced in Definition 5.19.

Example 5.30.

(A) Super plactic monoid, [56, 38]. Assume that the set of generators $U := \{u_1, \ldots, u_{n-1}\}$ is divided on two non-crossing subsets, say Y and Z, $Y \cup Z = U, Y \cap Z = \varnothing$. To each element $u \in U$ let us assign the weight $wt(u)$ as follows: $wt(u) = 0$ if $u \in Y$, and $wt(u) = 1$ if $u \in Z$. Finally, define parameters of the generalized plactic algebra $\mathcal{QP}_n$ to be $q_{ij} = (-1)^{wt(u_i)wt(u_j)}$. As a result we led to conclude that the generalized plactic algebra $\mathcal{QP}_n$ in question coincides with the super plactic algebra $\mathcal{PS}(V)$ introduced in [56]. We will denote this algebra by $\mathcal{SP}_{k,l}$, where $k = |Y|$, $l = |Z|$. We refer the reader to papers [56] and [38] for more details about connection of the super plactic algebra and super Young tableaux, and super analogue of the Robinson–Schensted–Knuth correspondence. We are planning to report on some properties of the Cauchy kernel in the (reduced) super plactic algebra elsewhere.

(B) q-analogue of the plactic algebra. Now let $q \neq 0, \pm 1$ be a parameter, and assume that $q_{ij} = q$, $\forall\, 1 \leq i < j \leq n-1$. This case has been treated recently in [55]. We expect that the generalized Knuth relations (5.3) are related with *quantum version* of the tropical/geometric RSK-correspondence (work in progress), and, as expected, with a q-weighted version of the Robinson–Schensted algorithm, presented in [60]. Another interesting *problem* is to understand a meaning of $\mathcal{Q}$-plactic polynomials coming from the decomposition of the (plactic) Cauchy kernels $\mathcal{C}_n$ and $\mathcal{F}_n$ in the reduced generalized plactic algebra $\mathcal{QPC}_n$ (work in progress).

(C) Quantum pseudoplactic algebra $\mathrm{PPL}_n^{(q)}$, [36]. By definition, the quantum pseudoplactic algebra $\mathrm{PPL}_n\,(q)$ is an associative algebra, generated, say over $\mathbb{Q}$, by the set of elements $\{e_1, \ldots, e_{n-1}\}$ subject to the set of defining relations

$$\begin{aligned}
&(a) \quad (1+q)e_ie_je_i - qe_i^2e_j - e_je_i^2 = 0, \qquad (1+q)e_je_ie_j - e_j^2e_j - qe_ie_j^2 = 0, \qquad i<j,\\
&(b) \quad (e_j,(e_i,e_k)) := e_je_ie_k - e_je_ke_i - e_ie_ke_j + e_ke_ie_j = 0, \qquad i<j<k.
\end{aligned}$$

Note that if $q = 1$, then the relations (a) can be written in the form $(e_i,(e_i,e_j)) = 0$ and $(e_2,(e_j,(e_i,e_j)) = 0$ correspondingly. Therefore, $\mathrm{PPL}_3^{(q=1)}$ is the universal enveloping algebra over $\mathbb{Q}$ of the Lie algebra $\mathfrak{sl}_3^+$. The quotient of the algebra $\mathrm{PPL}_n^{q=1}$ by the two-sided ideal generated by the elements $(e_i,(e_j,e_k))$, i, j, k are distinct, is isomorphic to the algebra from Remark 2.3.

Define noncommutative q-elementary polynomials $\Lambda_k(q;X_n)$, cf. [36], as follows

$$\Lambda_k(X;q) := \sum_{n\geq i_1>i_2>\cdots>i_k\geq 1} (x_{i_1},(x_{i_2},(\ldots,(x_{i_{k-1}},x_{i_k})_q)\cdots)_q)_q. \tag{5.4}$$

Proposition 5.31 ([36]). *The noncommutative q-elementary polynomials $\{\Lambda_k(E_{n-1};q)\}_{1\leq k\leq n-1}\}$ are pairwise commute in the algebra* $\mathrm{PPL}_n^{(q)}$.

5.2 Nilplactic algebra $\mathcal{NP}_n$

Let λ be a partition and α be a composition of the same size. Denote by $\widehat{\mathrm{STY}}(\lambda,\alpha)$ the set of columns and rows strict Young tableaux T of the shape λ and content α such that the corresponding tableau word $w(T)$ is reduced, i.e., $l(w(T)) = |T|$.

Denote by $\mathcal{B}_n$ the union of the sets $\widehat{\mathrm{STY}}(\lambda,\alpha)$ for all partitions λ such that $\lambda_i \leq n-i$ for $i = 1,2,\ldots,n-1$, and all compositions α, $\alpha \subset \delta_n$.

For example, $|\mathcal{B}_n| = 1,2,6,25,139,1008,\ldots$, for $n = 1,2,3,4,5,6,\ldots$.

Theorem 5.32.

(1) *In the nilplactic algebra* $\mathcal{NP}_n$ *the Cauchy kernel has the following decomposition*

$$\mathcal{C}_n(\mathfrak{P}, U) = \sum_{T \in \mathcal{B}_n} \mathcal{K}_T(\mathfrak{P}) u_{w(T)}.$$

(2) *Let* $T \in \mathcal{B}_n$ *be a tableau, and assume that its bottom code is a partition. Then*

$$\mathcal{K}_T(\mathfrak{P}) = \prod_{(i,j) \in T} p_{\{i, T(i,j)-j+1\}}.$$

Example 5.33. For $n = 4$ one has

$$\begin{aligned}\mathcal{C}_4(X, U) = {} & K[0] + K[1]u_1 + K[01]u_2 + K[001]u_3 + K[11]u_{12} + K[2](u_{21} + u_{31}) \\ & + K[101]u_{13} + K[02]u_{32} + K[011]u_{23} + K[3]u_{321} + K[12]u_{312} + K[21]u_{212} \\ & + K[111]u_{123} + K[021]u_{323} + K[201]u_{213} + K[31]u_{3212} + K[301]u_{3213} \\ & + K[22]u_{2132} + K[121]u_{3123} + K[211]u_{2123} + K[32]u_{32132} + K[311]u_{32123} \\ & + K[221]u_{21323} + K[321]u_{321323}.\end{aligned}$$

5.3 Idplactic algebra $\mathcal{IP}_n$

Let λ be a partition and α be a composition of the same size. Denote by $\overline{\mathrm{STY}}(\lambda, \alpha)$ the set of columns and rows strict Young tableaux T of the shape λ and content α such that $l(w(T)) = \mathrm{rl}(w(T))$, i.e., the tableau word[21] $w(T)$ is a unique tableau word of minimal length in the idplactic class of $w(T)$, cf. Example 2.16.

Denote by $\mathcal{D}_n$ the union of the sets $\overline{STY}(\lambda, \alpha)$ for all partitions λ such that $\lambda_i \le n - i$ for $i = 1, 2, \ldots, n-1$, and all compositions α, $l(\alpha) \le n - 1$.

For example, $\#|\mathcal{D}_n| = 1, 2, 6, 26, 154, 1197, \ldots$, for $n = 1, 2, 3, 4, 5, 6, \ldots$.

Theorem 5.34.

(2) *In the idplactic algebra* $\mathcal{IP}_n$ *the Cauchy kernel has the following decomposition*

$$\mathcal{C}_n(X, Y, U) = \sum_{T \in \mathcal{D}_n} \mathrm{KG}_T(X, Y) u_{w(T)}.$$

(2) *Let* $T \in \mathcal{D}_n$ *be a tableau, and assume that its bottom code is a partition. Then*

$$\mathrm{KG}_T(X, Y) = \mathcal{K}_T(X, Y) = \prod_{(i,j) \in T} (x_i + y_{T(i,j)-j+1}).$$

Example 5.35. For $n = 4$ one has

$$\begin{aligned}\mathcal{C}_4(X, U) = {} & \mathrm{KG}[0] + \mathrm{KG}[1]u_1 + \mathrm{KG}[01]u_2 + \mathrm{KG}[001]u_3 + \mathrm{KG}[11]u_{12} + \mathrm{KG}[2](u_{21} + u_{31}) \\ & + \mathrm{KG}[101]u_{13} + \mathrm{KG}[02]u_{32} + \mathrm{KG}[011]u_{23} + \mathrm{KG}[3]u_{321} + \mathrm{KG}[12]u_{312} \\ & + \mathrm{KG}[21]u_{212} + \mathrm{KG}[111]u_{123} + \mathrm{KG}[021]u_{323} + \mathrm{KG}[201](u_{313} + u_{213}) \\ & + K[31]u_{3212} + \mathrm{KG}[301]u_{3213} + \mathrm{KG}[22]u_{2132} + \mathrm{KG}[121]u_{3123} \\ & + \mathrm{KG}[211]u_{2123} + \mathrm{KG}[32]u_{32132} + \mathrm{KG}[311]u_{32123} + \mathrm{KG}[221]u_{21323} \\ & + \mathrm{KG}[321]u_{321323}.\end{aligned}$$

Theorem 5.36. *For each composition* α *the key Grothendieck polynomial* $\mathrm{KG}[\alpha](X)$ *is a linear combination of key polynomials* $K[\beta](X)$ *with nonnegative integer coefficients.*

[21] See page 11 for the definition of *tableau word*.

5.4 NilCoxeter algebra $\mathcal{NC}_n$

Theorem 5.37. *In the nilCoxeter algebra $\mathcal{NC}_n$ the Cauchy kernel has the following decomposition*

$$\mathcal{C}_n(X,Y,U) = \sum_{w \in \mathbb{S}_n} \mathfrak{S}_w(X,Y) u_w.$$

Let $w \in \mathbb{S}_n$ be a permutation, denote by $R(w)$ the set of all its reduced decompositions. Since the nilCoxeter algebra $\mathcal{NC}_n$ is the quotient of the nilplactic algebra $\mathcal{NP}_n$, the set $R(w)$ is the union of nilplactic classes of some tableau words $w(T_i)$: $R(w) = \bigcup C(T_i)$. Moreover, $R(w)$ consists of only one nilplactic class if and only if w is a *vexillary* permutation. In general case we see that the set of compatible sequences $CR(w)$ for permutation w is the union of sets $C(T_i)$.

Corollary 5.38. *Let $w \in \mathbb{S}_n$ be a permutation of length l, then*

(1) $\mathfrak{S}_w(X,Y) = \sum\limits_{\mathbf{b} \in CR(w)} x_{b_1} \cdots x_{b_l}$.

(2) *Double Schubert polynomial $\mathfrak{S}_w(X,Y)$ is a linear combination of double key polynomials $\mathcal{K}_T(X,Y)$, $T \in \mathcal{B}_n, w = w(T)$, with nonnegative integer coefficients.*

5.5 IdCoxeter algebras $\mathcal{IC}_n^{\pm}$

Theorem 5.39. *In the IdCoxeter algebra $\mathcal{IC}_n^+$ with $\beta = 1$, the Cauchy kernel has the following decomposition*

$$\mathcal{C}_n(X,Y,U) = \sum_{w \in \mathbb{S}_n} \mathcal{G}_w(X,Y) u_w.$$

Theorem 5.40. *In the IdCoxeter algebra $\mathcal{IC}_n^-$ with $\beta = -1$, one has the following decomposition*

$$\prod_{i=1}^{n-1} \left\{ \prod_{j=n-1}^{i} ((1+x_i)(1+y_{j-i+1}) + (x_i + y_{j-i+1})u_j) \right\} = \sum_{w \in \mathbb{S}_n} \mathcal{H}_w(X,Y) u_w.$$

A few remarks in order.

(a) The (dual) Cauchy identity (5.4) is still valid in the idplactic algebra with constrain $u_i^2 = -\beta u_i$, $i = 1, \ldots, n-1$.

(b) The left hand side of the identity (5.4) can be written in the following form

$$\prod_{\substack{1 \le i,j \le n \\ i+j \le n}} (x_i + y_j) \prod_{i=1}^{n-1} \left\{ \prod_{j=n-1}^{i} \frac{1}{1 - (x_i + y_{j-i+1} + \beta x_i y_{j-i+1}) u_j} \right\}.$$

Indeed, $(1 + \beta x + x u_i)(1 - x u_i) = 1 + \beta x$, since $u_i^2 = -\beta u_i$.

Let $w \in \mathbb{S}_n$ be a permutation, denote by $\mathrm{IR}(w)$ the set of all decompositions in the idCoxeter algebra $\mathcal{IC}_n$ of the element u_w as the product of the generators u_i, $1 \le i \le n-1$, of the algebra $\mathcal{IC}_n$. Since the idCoxeter algebra $\mathcal{IC}_n$ is the quotient of the idplactic algebra $\mathcal{IP}_n$, the set $\mathrm{IR}(w)$ is the union of idplactic classes of some tableau words $w(T_i)$: $\mathrm{IR}(w) = \bigcup \mathrm{IR}(T_i)$. Moreover, the set of compatible sequences $\mathrm{IC}(w)$ for permutation w is the union of sets $\mathrm{IC}(T_i)$.

Corollary 5.41. *Let $w \in \mathbb{S}_n$ be a permutation of length l, then*

(1) $\mathcal{G}_w(X,Y) = \sum\limits_{\mathbf{b} \in IC(w)} \prod\limits_{i=1}^{l} (x_{b_i} + y_{a_i - b_i + 1})$.

(2) *Double Grothendieck polynomial $\mathcal{G}_w(X,Y)$ is a linear combination of double key Grothendieck polynomials $\mathrm{KG}_T(X,Y)$, $T \in \mathcal{B}_n$, $w = w(T)$, with nonnegative integer coefficients.*

6 $\mathcal{F}$-kernel and symmetric plane partitions

Let us fix natural number n and k, and a partition $\lambda \subset (n^k)$. Clearly the number of such partitions is equal to $\binom{n+k}{n}$; note that in the case $n = k$ the number $\binom{2n}{n}$ is equal to the *Catalan number of type* B_n.

Denote by $\mathcal{B}_{n,k}(\lambda)$ the set of semistandard Young tableaux of shape $\lambda \subset (n^k)$ filled by the numbers from the set $\{1, 2, \dots, n\}$. For a tableau $T \in \mathcal{B}_{n,k}$ set as before,

$$n(T) := \operatorname{Card}\{(i, j) \in \lambda \mid T(i, j) = n\},$$

and define polynomial

$$\mathcal{B}_{n,k}(\lambda)(q) := \sum_{T \in \mathcal{B}_{n,k}(\lambda)} q^{\lambda_1 - n(T)}. \tag{6.1}$$

Denote by $\mathcal{B}_{n,k} := \bigcup_{\lambda \subset (n^k)} \mathcal{B}_{n,k}(\lambda)$.

Lemma 6.1 ([20, 35])**.** *The number of elements in the set $\mathcal{B}_{n,k}$ is equal to*

$$\#|\mathcal{B}_{n,k}| = \prod_{1 \le i \le j \le k} \frac{i + j + n - 1}{i + j - 1} = \prod_{0 \le 2a \le k-1} \frac{\binom{n+2k-2a-1}{n}}{\binom{n+2a}{n}}.$$

See also [67, *A*073165] for other combinatorial interpretations of the numbers $\#|\mathcal{B}_{n,k}|$. For example, the number $\#|\mathcal{B}_{n,k}|$ is equal to the number of symmetric plane partitions that fit inside the box $n \times k \times k$. Note that $B_{n,k} = T(n + k, k)$, where the triangle of positive integers $\{T(n + k, k)\}$ can be found in [67, *A*102539].

Proposition 6.2. *One has*

- $\#|\mathcal{B}_{n,n}| := \mathrm{SPP}(n + 1) = \mathrm{TSPP}(n + 1) \times \mathrm{ASM}(n)$,
  $\#|\mathcal{B}_{n,n+1}| = \mathrm{TSPP}(n + 1) \times \mathrm{ASM}(n + 1)$,

where $\mathrm{TSPP}(n)$ *denotes the number of totally symmetric plane partitions fit inside the* $n \times n \times n$*-box, see, e.g.,* [67, *A*005157]*, whereas* $\mathrm{ASM}(n) = \mathrm{TSSCPP}(2n)$ *denotes the of* $n \times n$ *alternating sign matrices, and* $\mathrm{TSSCPP}(2n)$ *denotes the number of totally symmetric self-complimentary plane partitions fit inside the* $2n \times 2n \times 2n$*-box.*

- $\#|\mathcal{B}_{n+2,n}| = \#|\mathcal{B}_{n,n+1}|$.

Note that in the case $n = k$ the number $\mathcal{B}_n := \mathcal{B}_{n,n}$ is equal to the number of symmetric plane portions fitting inside the $n \times n \times n$-box, see [67, *A*049505]. Let us point out that in general it may happen that the number $\#|\mathcal{B}_{n,n+2}|$ is not divisible by any $\mathrm{ASM}(m)$, $m \ge 3$. For example, $\mathcal{B}_{3,5} = 4224 = 2^5 \times 3 \times 11$. On the other hand, it's possible that the number $\#|\mathcal{B}_{n,n+2}|$ is divisible by $\mathrm{ASM}(n+1)$, but does not divisible by $\mathrm{ASM}(n+2)$. For example, $\mathcal{B}_{4,6} = 306735 = 715 \times 429$, but $306735 \nmid 7436 = \mathrm{ASM}(6)$.

Exercise 6.3.

(a) Show that $B_{n+4,n}$ is divisible by

$$\begin{cases} \mathrm{TSPP}(n+2), & \text{if } n \equiv 1 \pmod 2,\ n \ge 3, \\ \mathrm{ASM}(n+2), & \text{if } n \equiv 2 \pmod 8, \\ \mathrm{ASM}(n+1) \text{ and } \mathrm{ASM}(n+2), & \text{if } n \equiv 4 \pmod 8,\ n \ne 4;\ B_{8,4} = \mathrm{ASM}(5)^2, \\ \mathrm{ASM}(n+1) \text{ and } \mathrm{ASM}(n+2), & \text{if } n \equiv 6 \pmod 8, \\ \mathrm{ASM}(n+1), & \text{if } n \equiv 0 \pmod 8,\ n \ge 1. \end{cases}$$

(*b*) Show that $B_{n,n+4}$ is divisible by

$$\begin{cases} \mathrm{ASM}(n+1), & \text{if } \ n \equiv 0 \pmod 2, \\ \mathrm{TSPP}(n+1), & \text{if } \ n \equiv 1 \pmod 2. \end{cases}$$

In all cases listed in Exercise 6.3, it is an open *problem* to give combinatorial interpretations of the corresponding ratios.

Problem 6.4. *Let a is equal to either 0 or 1. Construct bijection between the set $\mathrm{SPP}(n, n+a, n+a)$ of symmetric plane partitions fitting inside the box $n \times n+a \times n+a$ and the set of pairs (P, M) where P is the totally symmetric plane partitions fitting inside the box $n \times n \times n$ and M is an alternating sign matrix of size $n+a \times n+a$.*

Example 6.5. Take $n = 3$. One has $\#|\mathcal{B}_3| = 112 = 16 \times 7$. The number of partitions $\lambda \subset (3^3)$ is equal to 20, namely, the following partitions

$$\{\varnothing, (1), (2), (1,1), (3), (2,1), (1^3), (3,1), (2,2), (2,1^2), (3,2), (3,1^2), (2^2,1), (3^2), \\ (3,2,1), (2^3), (3^2,1), (3,2^2), (3^2,2), (3^3)\},$$

and

$$\mathcal{B}_3(q) := \sum_{\lambda \subset (3^3)} \#|\mathcal{B}_3(\lambda)| q^{|\lambda|} = (1,3,9,19,24,24,19,9,3,1) \\ = (1+q)^3(1+q^2)\big(1+5q^2+q^4\big).$$

Note, however, that

$$\sum_{\lambda \subset (4^4)} \#|\mathcal{B}_4(\lambda)| q^{|\lambda|} = (1,4,16,44,116,204,336,420,490,420,336,204,116,44,16,4,1)$$

is an irreducible polynomial, but its value at $q = 1$ is equal to $2772 = 66 \times 42$.

Let $\mathbf{p} = (p_{i,j})_{1 \le i \le n, 1 \le j \le k}$ be a $n \times k$ matrix of variables.

Definition 6.6. Define the kernel $\mathcal{F}_{n,k}(\mathbf{p}, U)$ as follows

$$\mathcal{F}_{n,k}(\mathbf{p}, U) = \prod_{i=1}^{k-1} \prod_{j=n-1}^{1} \big(1 + p_{i, \overline{j-i+1}^{(n)}} u_j\big),$$

where for a fixed $n \in \mathbb{N}$ and an integer $a \in \mathbb{Z}$, we set

$$\overline{a} = \overline{a}^{(n)} := \begin{cases} a, & \text{if } \ a \ge 1, \\ n+a-1, & \text{if } \ a \le 0. \end{cases}$$

For example,

$$\mathcal{F}_3(\mathbf{p}, U) = (1 + p_{1,2}u_2)(1 + p_{1,1}u_1)(1 + p_{2,1}u_2)(1 + p_{2,2}u_1).$$

In the plactic algebra $\mathcal{FP}_{3,3}$ one has

$$\begin{aligned} \mathcal{F}_{3,3}(\mathbf{p}, U) = 1 &+ (p_{1,1} + p_{2,2})u_1 + (p_{1,2} + p_{2,1})u_2 + p_{1,1}p_{2,1}u_{11} + p_{1,1}p_{2,1}u_{12} \\ &+ (p_{1,2}p_{1,1} + p_{1,2}p_{2,2} + p_{2,1}p_{2,2})u_{21} + p_{1,2}p_{2,1}u_{22} \\ &+ (p_{1,1}p_{1,2}p_{2,2} + p_{1,2}p_{2,2}p_{2,1})u_{212} + (p_{1,1}p_{1,2}p_{2,2} + p_{1,1}p_{2,2}p_{2,1})u_{211} \\ &+ p_{1,1}p_{1,2}p_{2,1}p_{2,2}u_{2121}. \end{aligned}$$

Definition 6.7. Define algebra $\mathcal{PF}_{n,k}$ to be the quotient of the plactic algebra $\mathcal{P}_n$ by the two-sided ideal I_n generated by the set of monomials

$$\{u_{i_1}u_{i_2}\cdots u_{i_k}\}, \qquad 1 \le i_1 \le i_2 \le \cdots \le i_k \le n-1.$$

Theorem 6.8.

$$\mathrm{Hilb}(\mathcal{PF}_{n,k}, q) = \mathcal{B}_{n-1,k-1}(q),$$

In particular,

- *The algebra $\mathcal{PF}_{n,n}$ has dimension equals to the number of symmetric plane partitions* $\mathrm{SPP}(n-1)$*,*

$$\mathrm{Hilb}(\mathcal{PF}_{n,k}, q) = q^{\frac{kn}{2}}\mathfrak{so}_{(\frac{k}{2})^n}\big(\underbrace{q^{\pm 1},\dots,q^{\pm 1}}_{k},1\big),$$

  where $\mathfrak{so}_{(\frac{k}{2})^n}\big(\underbrace{q^{\pm 1},\dots,q^{\pm 1}}_{k},1\big)$ denotes the specialization $x_{2j}=q$, $x_{2j-1}=q^{-1}$, $1 \le j \le k$, of the character $\mathfrak{so}_\lambda\big(x_1,x_1^{-1}\dots,x_k,x_k^{-1},1\big)$ of the odd orthogonal Lie algebra $\mathfrak{so}(2k+1)$ corresponding to the highest weight $\lambda = \big(\underbrace{\tfrac{k}{2},\dots,\tfrac{k}{2}}_{n}\big)$.

- $\deg_q \mathrm{Hilb}(\mathcal{PF}_{n,k}, q) = (n-1)(k-1)$*, and* $\dim(\mathcal{PF}_{n,k})_{(n-1)(k-1)} = 1$*.*
- *The Hilbert polynomial $\mathrm{Hilb}(\mathcal{PF}_{n,k}, q)$ is symmetric and unimodal polynomial in the variable q.*
- $\mathrm{Hilb}((\mathcal{PF}_{n,k})^{ab}, q) = \sum\limits_{j=0}^{k-1}\binom{n+j-2}{n-2}q^j$, $\dim(\mathcal{PF}_{n,k})^{ab} = \binom{n+k-2}{k-1}$.

The key step in proofs of Lemma 6.1 and Theorem 6.8 is based on the following identity

$$\sum_{\lambda\subset(n^k)} s_\lambda(x_1,\dots,x_k) = (x_1\cdots x_k)^{n/2}\mathfrak{so}_{(\frac{k}{2})^n}\big(x_1,x_1^{-1},\dots,x_k,x_k^{-1},1\big),$$

see, e.g., [58, Chapter I, Section 5, Example 19], [35] and the literature quoted therein.

Problem 6.9. *Let $\Gamma := {\Gamma_{n,m}}^{k,\ell} = (n^k, m^\ell)$, $n \ge m$ be a "fat hook". Find generalizations of the identity* (6.1) *and those listed in* [25, p. 71]*, to the case of fat hooks, namely to find "nice" expressions for the following sums*

$$\sum_{\lambda\subset\Gamma} s_\lambda(X_{k+\ell}), \qquad \sum_{\lambda\subset\Gamma} s_\lambda(X_{k+\ell})s_\lambda(Y_{k+\ell}).$$

Find "bosonic" type formulas for these sum at the limit $n\longrightarrow\infty$, $\ell\longrightarrow\infty$, m, k are fixed.

Example 6.10.

$$\mathrm{Hilb}(\mathcal{PF}_{2,3}, q) = (1,3,9,9,9,3,1)_q, \qquad \dim(\mathcal{PF}_{2,4}) = 35 = 5\times 7,$$

$$\dim(\mathcal{PF}_{2,5}) = 126 = 3\times 42, \qquad \dim\mathcal{PF}_{2,n} = \binom{2n+1}{n} = (2n+1)\,\mathrm{Cat}_n$$

(see, e.g., [67, *A*001700]),

$\mathrm{Hilb}(\mathcal{PF}_{3,4}, q) = (1,4,16,44,81,120,140,120,81,44,16,4,1)_q$,

$\dim(\mathcal{PF}_{3,4}) = 672 = 16\times 42$,

$\mathrm{Hilb}(\mathcal{PF}_{5}, q) = (1,4,16,44,116,204,336,420,490,420,336,204,116,44,16,4,1)_q$,

$\dim(\mathcal{PF}_{5,5}) = 2772 = 66\times 42$.

Proposition 6.11.

$$\mathrm{Hilb}(\mathcal{PF}_{2,n},q)=\sum_{k=0}^{2n}\binom{n}{[\frac{k}{2}]}\binom{n}{[\frac{k+1}{2}]}q^k; \qquad \textit{recall} \qquad \dim(\mathcal{PF}_{2,n})=\binom{2n+1}{n}.$$

Therefore, $\mathrm{Hilb}(\mathcal{PF}_{2,n},q)$ *is equal to the generating function for the number of symmetric Dyck paths of semilength* $2n-1$ *according to the number of peaks, see* [67, *A*088855],

$$\dim(\mathcal{PF}_{3,n})=2^n\,\mathrm{Cat}_{n+1}, \qquad \textit{if} \quad n\geq 1.$$

For example,

$$\dim(\mathcal{PF}_{3,6})=27456=64\times 429,$$
$$\mathrm{Hilb}(\mathcal{PF}_{3,6},q)=(1,6,36,146,435,1056,2066,3276,4326,4760,4326,3276,2066,1056,\\ 435,146,36,6,1).$$

Several interesting interpretations of these numbers are given in [67, *A*003645].

Theorem 6.12.

- *Symmetric plane partitions and Catalan numbers:*

$$\#|\mathcal{B}_{4,n}|=\frac{1}{2}\,\mathrm{Cat}_{n+1}\times\mathrm{Cat}_{n+2}\,.$$

- *Symmetric plane partitions and alternating sign matrices:*

$$\#|\mathcal{B}_{n+3,n}|=\frac{1}{2}\,\mathrm{TSPP}(n+1)\times\mathrm{ASM}(n+1)=\frac{1}{2}\#|\mathcal{B}_{n+1,n+1}|.$$

- *Plane partitions and alternating sign matrices invariant under a half-turn:*

$$\#|\,\mathrm{PP}(n)|=\mathrm{ASM}(n)\times\mathrm{ASMHT}(2n),$$

  where $PP(n)$ *denotes the number of plane partitions fitting inside an* $n\times n\times n$ *box, see, e.g.,* [6, 37, 58], [67, *A*008793] *and the literature quoted theirin;* $\mathrm{ASMHT}(2n)$ *denotes the number of alternating sign* $2n\times 2n$*-matrices invariant under a half-turn, see, e.g.,* [6, 37, 61, 68], [67, *A*005138].

- *Plactic decomposition of the* $\mathcal{F}_n$*-kernel:*

$$\mathcal{F}_{n,m}(\mathbf{p},U)=\sum_T u_T U_T(\{p_{ij}\}), \tag{6.2}$$

  where summation runs over the set of semistandard Young tableaux T *of shape* $\lambda\subset(n)^m$ *filled by the numbers from the set* $\{1,\ldots,m\}$.

- $U_T(\{p_{ij}=1,\forall\, i,j\})=\dim V_{\lambda'}^{\mathfrak{gl}(m)}$, *where* λ *denotes the shape of a tableau* T, *and* λ' *denotes the conjugate/transpose of a partition* λ.

Exercise 6.13. It is well-known [21] that that the number $Cat_{n+1}Cat_{n+2}$ counts the number $S_{n+1}^{(4)}$ of standard Young tableaux having $2n+1$ boxes and at most four rows.
Give a bijective proof of the equality $\#|\mathcal{B}_{4,n}|=\frac{1}{2}S_{n+1}^{(4)}$.

A Appendix

A.1 Some explicit formulas for $n=4$ and compositions α such that $\alpha_i \le n-i$ for $i=1,2,\dots$

(1) Schubert and $(-\beta)$-Grothendieck polynomials $\mathcal{G}^-[\alpha] := \mathcal{G}^{-\beta}[\alpha]$ for $n=4$:

$$
\begin{aligned}
&\mathfrak{S}_{1234} = \mathfrak{S}[0] = 1 = \mathcal{G}^-[0], \qquad \mathfrak{S}_{2134} = \mathfrak{S}[1] = x_1 = \mathcal{G}^-[1],\\
&\mathfrak{S}_{1324} = \mathfrak{S}[01] = x_1+x_2 = \mathcal{G}^-[01] + \beta\mathcal{G}^-[11],\\
&\mathfrak{S}_{1243} = \mathfrak{S}[001] = x_1+x_2+x_3 = \mathcal{G}^-[001] + \beta\mathcal{G}^-[011] + \beta^2\mathcal{G}^-[111],\\
&\mathfrak{S}_{3124} = \mathfrak{S}[2] = x_1^2 = \mathcal{G}^-[2], \qquad \mathfrak{S}_{2314} = \mathfrak{S}[11] = x_1x_2 = \mathcal{G}^-[11],\\
&\mathfrak{S}_{2143} = \mathfrak{S}[101] = x_1^2+x_1x_2+x_1x_3 = \mathcal{G}^-[101] + \beta\mathcal{G}^-[201] + \beta^2\mathcal{G}^-[111],\\
&\mathfrak{S}_{1342} = \mathfrak{S}[011] = x_1x_2+x_1x_3+x_2x_3 = \mathcal{G}^-[011] + 2\beta\mathcal{G}^-[111],\\
&\mathfrak{S}_{1423} = \mathfrak{S}[02] = x_1^2+x_1x_2+x_2^2 = \mathcal{G}^-[02] + \beta\mathcal{G}^-[12] + \beta^2\mathcal{G}^-[22],\\
&\mathfrak{S}_{4123} = \mathfrak{S}[3] = x_1^3 = \mathcal{G}^-[3], \qquad \mathfrak{S}_{3214} = \mathfrak{S}[21] = x_1^2x_2 = \mathcal{G}^-[21],\\
&\mathfrak{S}_{2341} = \mathfrak{S}[111] = x_1x_2x_3 = \mathcal{G}^-[111],\\
&\mathfrak{S}_{2413} = \mathfrak{S}[12] = x_1^2x_2+x_1x_2^2 = \mathcal{G}^-[12] + \beta\mathcal{G}^-[22],\\
&\mathfrak{S}_{1432} = \mathfrak{S}[021] = x_1^2x_2+x_1^2x_3+x_1x_2^2+x_2^2x_3+x_1x_2x_3\\
&\qquad\quad = \mathcal{G}^-[021] + 2\beta\mathcal{G}^-[121] + \beta\mathcal{G}^-[22] + \beta^2\mathcal{G}^-[221],\\
&\mathfrak{S}_{3142} = \mathfrak{S}[201] = x_1^2x_2+x_1^2x_3 = \mathcal{G}^-[201] + \beta\mathcal{G}^-[211],\\
&\mathfrak{S}_{4213} = \mathfrak{S}[31] = x_1^3x_2 = \mathcal{G}^-[31], \qquad \mathfrak{S}_{3412} = \mathfrak{S}[22] = x_1^2x_2^2 = \mathcal{G}^-[22],\\
&\mathfrak{S}_{4132} = \mathfrak{S}[301] = x_1^3x_2+x_1^3x_3 = \mathcal{G}^-[301] + \beta\mathcal{G}^-[311],\\
&\mathfrak{S}_{3241} = \mathfrak{S}[211] = x_1^2x_2x_3 = \mathcal{G}^-[211],\\
&\mathfrak{S}_{2431} = \mathfrak{S}[121] = x_1^2x_2x_3+x_1x_2^2x_3 = \mathcal{G}^-[121] + \beta\mathcal{G}^-[221],\\
&\mathfrak{S}_{4312} = \mathfrak{S}[32] = x_1^3x_2^2 = \mathcal{G}^-[32], \qquad \mathfrak{S}_{4231} = \mathfrak{S}[311] = x_1^3x_2x_3 = \mathcal{G}^-[311],\\
&\mathfrak{S}_{3421} = \mathfrak{S}[221] = x_1^2x_2^2x_3 = \mathcal{G}^-[211], \qquad \mathfrak{S}_{4321} = \mathfrak{S}[321] = x_1^3x_2^2x_3 = \mathcal{G}^-[321].
\end{aligned}
$$

Theorem A.1 (cf. [49, Section 5.5])**.** *Each Schubert polynomial is a linear combination of $(-\beta)$-Grothendieck polynomials with nonnegative coefficients from the ring $\mathbb{N}[\beta]$.*

(2) Key and reduced key polynomials:

$$
\begin{aligned}
&K[0] = 1 = \widehat{K}[0], \qquad K[1] = x_1 = \widehat{K}[1], \qquad K[01] = x_1+x_2, \qquad \widehat{K}[01] = x_2,\\
&K[001] = x_1+x_2+x_3, \qquad \widehat{K}[001] = x_3, \qquad K[2] = x_1^2 = \widehat{K}[2],\\
&K[11] = x_1x_2 = \widehat{K}[11], \qquad K[101] = x_1x_2+x_1x_3, \qquad \widehat{K}[101] = x_1x_3,\\
&K[02] = x_1^2+x_1x_2+x_2^2, \qquad \widehat{K}[02] = x_1x_2+x_2^2, \qquad K[011] = x_1x_2+x_1x_3+x_2x_3,\\
&\widehat{K}[011] = x_2x_3, \qquad K[3] = x_1^3 = \widehat{K}[3], \qquad K[21] = x_1^2x_2 = \widehat{K}[21],\\
&K[111] = x_1x_2x_3 = \widehat{K}[111], \qquad K[12] = x_1^2x_2+x_1x_2^2, \qquad \widehat{K}[12] = x_1x_2^2,\\
&K[021] = x_1^2x_2+x_1^2x_3+x_1x_2^2+x_2^2x_3+x_1x_2x_3, \qquad \widehat{K}[021] = x_1x_2x_3+x_2^2x_3,\\
&K[201] = x_1^2x_2+x_1^2x_3, \qquad \widehat{K}[201] = x_1^2x_3, \qquad K[31] = x_1^3x_2 = \widehat{K}[31],\\
&K[22] = x_1^2x_2^2 = \widehat{K}[22], \qquad K[211] = x_1^2x_2x_3 = \widehat{K}[211], \qquad K[301] = x_1^3x_2+x_1^3x_3,\\
&\widehat{K}[301] = x_1^3x_3, \qquad K[121] = x_1^2x_2x_3+x_1x_2^2x_3, \qquad \widehat{K}[121] = x_1x_2^2x_3,\\
&K[32] = x_1^3x_2^2 = \widehat{K}[32], \qquad K[311] = x_1^3x_2x_3 = \widehat{K}[311], \qquad K[221] = x_1^2x_2^2x_3 = \widehat{K}[221],
\end{aligned}
$$

$$K[321] = x_1^3 x_2^2 x_3 = \widehat{K}[321].$$

Note that if $n = 4$, then $\mathfrak{S}[\alpha] = K[\alpha]$ for all $\alpha \subset \delta_4$, except $\alpha = (101)$ in which $\mathfrak{S}[101] = K[2] + K[101]$.

(3) Grothendieck and dual Grothendieck polynomials for $\beta = 1$:

$$\begin{aligned}
&\mathcal{G}_{1234} = \mathcal{G}[0] = 1 = \mathfrak{S}[0], \qquad \mathcal{H}[0] = (1+x_1)^3(1+x_2)^2(1+x_3),\\
&\mathcal{G}_{2134} = \mathcal{G}[1] = x_1 = \mathfrak{S}[1], \qquad \mathcal{H}[1] = (1+x_1)^2(1+x_2)^2(1+x_3)\mathcal{G}[1],\\
&\mathcal{G}_{1324} = \mathcal{G}[01] = x_1 + x_2 + x_1x_2 = \mathfrak{S}[01] + \mathfrak{S}[11],\\
&\mathcal{H}[01] = (1+x_1)^2(1+x_2)(1+x_3)\mathcal{G}[01],\\
&\mathcal{G}_{1243} = \mathcal{G}[001] = x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3\\
&\qquad = \mathfrak{S}[001] + \mathfrak{S}[011] + \mathfrak{S}[111],\\
&\mathcal{H}[001] = (1+x_1)^2(1+x_2)\mathcal{G}[001],\\
&\mathcal{G}_{3124} = \mathcal{G}[2] = x_1^2 = \mathfrak{S}[2], \qquad \mathcal{H}[2] = (1+x_1)(1+x_2)^2(1+x_3)\mathcal{G}[2],\\
&\mathcal{G}_{2314} = \mathcal{G}[11] = x_1x_2 = \mathfrak{S}[11], \qquad \mathcal{H}[11] = (1+x_1)^2(1+x_2)(1+x_3)\mathcal{G}[11],\\
&\mathcal{G}_{2143} = \mathcal{G}[101] = x_1^2 + x_1x_2 + x_1x_3 + x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_1^2x_2x_3\\
&\qquad = \mathfrak{S}[101] + \mathfrak{S}[201] + \mathfrak{S}[111] + \mathfrak{S}[211],\\
&\mathcal{H}[101] = (1+x_1)(1+x_2)\mathcal{G}[101],\\
&\mathcal{G}_{1342} = \mathcal{G}[011] = x_1x_2 + x_1x_3 + x_2x_3 + 2x_1x_2x_3 = \mathfrak{S}[011] + 2\mathfrak{S}[111],\\
&\mathcal{H}[011] = (1+x_1)^2(1+x_2)(x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3),\\
&\mathcal{G}_{1423} = \mathcal{G}[02] = x_1^2 + x_1x_2 + x_2^2 + x_1^2x_2 + x_1x_2^2 = \mathfrak{S}[02] + \mathfrak{S}[12],\\
&\mathcal{H}[02] = (1+x_1)(1+x_3)(x_1^2 + x_1x_2 + x_2^2 + 2x_1^2x_2 + 2x_1x_2^2 + x_1^2x_2^2),\\
&\mathcal{G}_{4123} = \mathcal{G}[3] = x_1^3 = \mathfrak{S}[3], \qquad \mathcal{H}[3] = (1+x_1)^2(1+x_3)\mathcal{G}[3],\\
&\mathcal{G}_{3214} = \mathcal{G}[21] = x_1^2x_2 = \mathfrak{S}[21], \qquad \mathcal{H}[21] = (1+x_1)(1+x_2)(1+x_3)\mathcal{G}[21],\\
&\mathcal{G}_{2341} = \mathcal{G}[111] = x_1x_2x_3 = \mathfrak{S}[111], \qquad \mathcal{H}[111] = (1+x_1)^2(1+x_2)\mathcal{G}[111],\\
&\mathcal{G}_{2413} = \mathcal{G}[12] = x_1^2x_2 + x_1x_2^2 + x_1^2x_2^2 = \mathfrak{S}[12] + \mathfrak{S}[22],\\
&\mathcal{H}[12] = (1+x_1)(1+x_3)\mathcal{G}[12],\\
&\mathcal{G}_{1432} = \mathcal{G}[021] = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_2x_3 + 2x_1x_2x_3(x_1+x_2)\\
&\qquad\qquad + x_1^2x_2^2 + x_1^2x_2^2x_3 = \mathfrak{S}[021] + 2\mathfrak{S}[121] + \mathfrak{S}[22] + \mathfrak{S}[211],\\
&\mathcal{H}[021] = (1+x_1)\mathcal{G}[021],\\
&\mathcal{G}_{3142} = \mathcal{G}[201] = x_1^2x_2 + x_1^2x_3 + x_1^2x_2x_3 = \mathfrak{S}[201] + \mathfrak{S}[211],\\
&\mathcal{H}[201] = (1+x_1)(1+x_2)\mathcal{G}[201],\\
&\mathcal{G}_{4213} = \mathcal{G}[31] = x_1^3x_2 = \mathfrak{S}[31], \qquad \mathcal{H}[31] = (1+x_2)(1+x_3)\mathcal{G}[31],\\
&\mathcal{G}_{3412} = \mathcal{G}[22] = x_1^2x_2^2 = \mathfrak{S}[22], \qquad \mathcal{H}[22] = (1+x_1)(1+x_3)\mathcal{G}[22],\\
&\mathcal{G}_{4132} = \mathcal{G}[301] = x_1^3x_2 + x_1^3x_3 + x_1^3x_2x_3 = \mathfrak{S}[301] + \mathfrak{S}[311],\\
&\mathcal{H}[301] = (1+x_2)\mathcal{G}[301],\\
&\mathcal{G}_{3241} = \mathcal{G}[211] = x_1^2x_2x_3 = \mathfrak{S}[211], \qquad \mathcal{H}[211] = (1+x_2)\mathcal{G}[211],\\
&\mathcal{G}_{2431} = \mathcal{G}[121] = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2^2x_3 = \mathfrak{S}[121] + \mathfrak{S}[221],\\
&\mathcal{H}[121] = (1+x_1)(1+x_2)\mathcal{G}[121],\\
&\mathcal{G}_{4312} = \mathcal{G}[32] = x_1^3x_2^2 = \mathfrak{S}[32], \qquad \mathcal{H}[32] = (1+x_3)\mathcal{G}[32],\\
&\mathcal{G}_{4231} = \mathcal{G}[311] = x_1^3x_2x_3 = \mathfrak{S}[311], \qquad \mathcal{H}[311] = (1+x_2)\mathcal{G}[311],
\end{aligned}$$

$$\mathcal{G}_{3421} = \mathcal{G}[221] = x_1^2x_2^2x_3 = \mathfrak{S}[221], \qquad \mathcal{H}[221] = (1+x_1)\mathcal{G}[221],$$
$$\mathcal{G}_{4321} = \mathcal{G}[321] = x_1^3x_2^2x_3 = \mathfrak{S}[321] = \mathcal{H}[321].$$

Clearly that any β-Grothendieck polynomial is a linear combination of Schubert polynomials with coefficients from the ring $\mathbb{N}[\beta]$.

(4) Key and reduced key Grothendieck polynomials:

$$\mathrm{KG}[0] = 1 = \widehat{\mathrm{KG}}[0], \qquad \mathrm{KG}[1] = x_1 = \widehat{\mathrm{KG}}[1], \qquad \mathrm{KG}[01] = x_1 + x_2 + x_1x_2,$$
$$\widehat{\mathrm{KG}}[01] = x_2 + x_1x_2, \qquad \mathrm{KG}[001] = x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3,$$
$$\widehat{\mathrm{KG}}[001] = x_3 + x_1x_3 + x_2x_3 + x_1x_2x_3, \qquad \mathrm{KG}[2] = x_1^2 = \widehat{\mathrm{KG}}[2],$$
$$\mathrm{KG}[11] = x_1x_2 = \widehat{\mathrm{KG}}[11], \qquad \mathrm{KG}[101] = x_1x_2 + x_1x_3 + x_1x_2x_3,$$
$$\widehat{\mathrm{KG}}[101] = x_1x_3 + x_1x_2x_3, \qquad \mathrm{KG}[02] = x_1^2 + x_1x_2 + x_2^2 + x_1^2x_2 + x_1x_2^2,$$
$$\widehat{\mathrm{KG}}[02] = x_1x_2 + x_2^2 + x_1^2x_2 + x_1x_2^2,$$
$$\mathrm{KG}[011] = x_1x_2 + x_1x_3 + x_2x_3 + 2x_1x_2x_3, \qquad \widehat{\mathrm{KG}}[011] = x_2x_3 + x_1x_2x_3,$$
$$\mathrm{KG}[3] = x_1^3 = \widehat{\mathrm{KG}}[3], \qquad \mathrm{KG}[21] = x_1^2x_2 = \widehat{\mathrm{KG}}[21], \qquad \mathrm{KG}[111] = x_1x_2x_3 = \widehat{\mathrm{KG}}[111],$$
$$\mathrm{KG}[12] = x_1^2x_2 + x_1x_2^2 + x_1^2x_2^2, \qquad \widehat{\mathrm{KG}}[12] = x_1x_2^2 + x_1^2x_2^2,$$
$$\mathrm{KG}[201] = x_1^2x_2 + x_1^2x_3 + x_1^2x_2x_3, \qquad \widehat{\mathrm{KG}}[201] = x_1^2x_3 + x_1^2x_2x_3,$$
$$\mathrm{KG}[021] = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2^2x_3 + 2x_1^2x_2x_3 + 2x_1x_2^2x_3 + x_1^2x_2^2 + x_1^2x_2^2x_3,$$
$$\widehat{\mathrm{KG}}[021] = x_1x_2x_3 + x_2^2x_3 + x_1^2x_2x_3 + 2x_1x_2^2x_3 + x_1^2x_2^2x_3,$$
$$\mathrm{KG}[31] = x_1^3x_2 = \widehat{\mathrm{KG}}[31], \qquad \mathrm{KG}[22] = x_1^2x_2^2 = \widehat{\mathrm{KG}}[22],$$
$$\mathrm{KG}[211] = x_1^2x_2x_3 = \widehat{\mathrm{KG}}[211], \qquad \mathrm{KG}[301] = x_1^3x_2 + x_1^3x_3 + x_1^3x_2x_3,$$
$$\widehat{\mathrm{KG}}[301] = x_1^3x_3 + x_1^3x_2x_3, \qquad \mathrm{KG}[121] = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2^2x_3,$$
$$\widehat{\mathrm{KG}}[121] = x_1x_2^2x_3 + x_1^2x_2^2x_3, \qquad \mathrm{KG}[32] = x_1^3x_2^2 = \widehat{\mathrm{KG}}[32],$$
$$\mathrm{KG}[311] = x_1^3x_2x_3 = \widehat{\mathrm{KG}}[311], \qquad \mathrm{KG}[221] = x_1^2x_2^2x_3 = \widehat{\mathrm{KG}}[221],$$
$$\mathrm{KG}[321] = x_1^3x_2^2x_3 = \widehat{\mathrm{KG}}[321].$$

(5) 42 (deformed) double key polynomials for $n = 4$:

$$\mathcal{K}_{\mathrm{id}} = 1, \qquad \mathcal{K}_1 = p_{1,1}, \qquad \mathcal{K}_2 = p_{1,2} + p_{2,1}, \qquad \mathcal{K}_3 = p_{1,3} + p_{2,2} + p_{3,1},$$
$$\mathcal{K}_{12} = p_{1,1}p_{2,1}, \qquad \mathcal{K}_{21} = p_{1,2}p_{1,1}, \qquad \mathcal{K}_{23} = p_{1,2}p_{2,2} + p_{1,2}p_{3,1} + p_{2,1}p_{3,1},$$
$$\mathcal{K}_{32} = p_{1,3}p_{1,2} + p_{1,3}p_{2,1} + p_{2,2}p_{2,1}, \qquad \mathcal{K}_{13} = p_{1,1}p_{2,2} + p_{1,1}p_{3,1}, \qquad \mathcal{K}_{31} = p_{1,3}p_{1,1},$$
$$\mathcal{K}_{22} = p_{1,2}p_{2,1}, \qquad \mathcal{K}_{33} = p_{1,3}p_{2,2} + p_{1,3}p_{3,1} + p_{2,2}p_{3,1}, \qquad \mathcal{K}_{123} = p_{1,1}p_{2,1}p_{3,1},$$
$$\mathcal{K}_{133} = p_{1,1}p_{2,2}p_{3,1}, \qquad \mathcal{K}_{212} = p_{1,2}p_{1,1}p_{2,1}, \qquad \mathcal{K}_{213} = p_{1,2}p_{1.1}p_{2,2} + p_{1,2}p_{1,1}p_{3,1},$$
$$\mathcal{K}_{223} = p_{1,2}p_{2,1}p_{3,1}, \qquad \mathcal{K}_{233} = p_{1,2}p_{2,2}p_{3,1}, \qquad \mathcal{K}_{321} = p_{1,3}p_{1,2}p_{1,1},$$
$$\mathcal{K}_{312} = p_{1,3}p_{1,1}p_{2,1} + \boldsymbol{q_{13}^{-1}}p_{1,1}p_{2,2}p_{2,1}, \qquad \mathcal{K}_{313} = p_{1,3}p_{1,1}p_{2,2} + p_{1,3}p_{1,1}p_{3,1},$$
$$\mathcal{K}_{322} = p_{1,3}p_{1,2}p_{2,1} + \boldsymbol{q_{23}^{-1}}p_{1,2}p_{2,2}p_{2,1},$$
$$\mathcal{K}_{323} = p_{1,3}p_{1,2}p_{2,2} + p_{1,3}p_{1,2}p_{3,1} + p_{1,3}p_{2,1}p_{3,1} + p_{2,2}p_{2,1}p_{3,1} + \boldsymbol{q_{23}}p_{1,3}p_{2,2}p_{2,1}$$
$$\mathcal{K}_{333} = p_{1,3}p_{2,2}p_{3,1}, \qquad \mathcal{K}_{2123} = p_{1,2}p_{1,1}p_{2,1}p_{3,1}, \qquad \mathcal{K}_{2132} = p_{1,2}p_{1,1}p_{2,2}p_{2,1},$$
$$\mathcal{K}_{2133} = p_{1,2}p_{1,1}p_{2,2}p_{3,1}, \qquad \mathcal{K}_{3123} = p_{1,3}p_{1,1}p_{2,1}p_{3,1} + \boldsymbol{q_{13}^{-1}}p_{1,1}p_{2,2}p_{2,1}p_{3,1},$$
$$\mathcal{K}_{3132} = p_{1,3}p_{1,1}p_{2,2}p_{2,1}, \qquad \mathcal{K}_{3133} = p_{1,3}p_{1,2}p_{2,2}p_{3,1}, \qquad \mathcal{K}_{3212} = p_{1,3}p_{1,2}p_{1,1}p_{2,1},$$
$$\mathcal{K}_{3213} = p_{1,3}p_{1,2}p_{1,1}p_{2,2} + p_{1,3}p_{1,2}p_{1,1}p_{3,1}, \qquad \mathcal{K}_{3223} = p_{1,3}p_{1,2}p_{2,1}p_{3,1} + \boldsymbol{q_{23}^{-1}}p_{1,2}p_{2,2}p_{2,1}p_{3,1},$$

$$\mathcal{K}_{3232} = p_{1,3}p_{1,2}p_{2,2}p_{2,1}, \qquad \mathcal{K}_{3233} = p_{1,3}p_{1,2}p_{2,2}p_{3,1} + \boldsymbol{q_{23}}p_{1,3}p_{2,2}p_{2,1}p_{3,1},$$
$$\mathcal{K}_{21323} = p_{1,2}p_{1,1}p_{2,2}p_{2,1}p_{3,1}, \qquad \mathcal{K}_{31323} = p_{1,3}p_{1,1}p_{2,2}p_{2,1}p_{3,1},$$
$$\mathcal{K}_{32123} = p_{1,3}p_{1,2}p_{1,1}p_{2,1}p_{3,1}, \qquad \mathcal{K}_{32132} = p_{1,3}p_{1,2}p_{1,1}p_{2,2}p_{2,1},$$
$$\mathcal{K}_{32133} = p_{1,3}p_{1,2}p_{1,1}p_{2,2}p_{3,1}, \qquad \mathcal{K}_{32323} = p_{1,3}p_{1,2}p_{2,2}p_{2,1}p_{3,1},$$
$$\mathcal{K}_{321323} = p_{1,3}p_{1,2}p_{1,1}p_{2,2}p_{2,1}p_{3,1}.$$

Theorem A.2 (cf. [39, the case $\beta = -1$])**.** *Each double β-Grothendieck polynomials is a linear combination of double key polynomials with the coefficients from the ring $\mathbb{N}[\beta]$.*

Let us remind that the total number of double key polynomials is equal to the number of alternating sign matrices. We expect that the interrelations between double key polynomials which follow from the structure of the plactic algebra $\mathcal{PC}_n$, see Section 5.1, can be identified with the graph corresponding to the MacNeile completion of the poset associated with the Bruhat order on the symmetric group $\mathbb{S}_n$, see Section A.2 for a definition of the MacNeile completion. It is an interesting problem to describe interrelation graph associated with the (rectangular) key polynomials corresponding to the Cauchy kernel for the algebra $\mathcal{PF}_{n,m}$.

(6) 26 double key Grothendieck polynomials for $n = 4$:

$$\mathcal{GK}_{\mathrm{id}} = 1, \qquad \mathcal{GK}_1 = p_{1,1} = \mathcal{K}_1 \qquad \mathcal{GK}_2 = p_{1,2} + p_{2,1} + p_{1,2}p_{2,1} = \mathcal{K}_2 + \mathcal{K}_{22},$$
$$\mathcal{GK}_3 = p_{1,3} + p_{1,2} + p_{3,1} + p_{1,3}p_{2,2} + p_{1,3}p_{3,1} + p_{2,2}p_{3,1} + p_{1,3}p_{2,2}p_{3,1} = \mathcal{K}_3 + \mathcal{K}_{33} + \mathcal{K}_{333},$$
$$\mathcal{GK}_{12} = p_{1,1}p_{2,1} = \mathcal{K}_{12}, \qquad \mathcal{GK}_{21} = p_{2,1}p_{1,1} = \mathcal{K}_{21},$$
$$\mathcal{GK}_{13} = p_{1,1}p_{2,2} + p_{1,1}p_{3,1} + p_{1,1}p_{2,2}p_{3,1} = \mathcal{K}_{13} + \mathcal{K}_{133}, \qquad \mathcal{GK}_{31} = p_{3,1}p_{1,1} = \mathcal{K}_{31},$$
$$\mathcal{GK}_{23} = p_{1,1}p_{2,2} + p_{1,2}p_{3,1} + p_{2,1}p_{3,1} + p_{1,2}p_{2,1}p_{3,1} + p_{1,2}p_{2,2}p_{3,1} = \mathcal{K}_{23} + \mathcal{K}_{223} + \mathcal{K}_{233},$$
$$\mathcal{GK}_{32} = p_{1,3}p_{1,2} + p_{1,3}p_{2,1} + p_{2,2}p_{2,1} + p_{1,3}p_{1,2}p_{2,1} + p_{1,2}p_{2,1}p_{2,2} = \mathcal{K}_{32} + \mathcal{K}_{322},$$
$$\mathcal{GK}_{123} = p_{1,1}p_{2,1}p_{3,1} = \mathcal{K}_{123}, \qquad \mathcal{GK}_{212} = p_{1,2}p_{1,1}p_{2,1} = \mathcal{K}_{212},$$
$$\mathcal{GK}_{213} = p_{1,2}p_{1,1}p_{2,2} + p_{2,1}p_{1,1}p_{3,1} + p_{1,2}p_{1,1}p_{2,2}p_{3,1} = \mathcal{K}_{213} + \mathcal{K}_{2133},$$
$$\mathcal{GK}_{312} = p_{1,3}p_{1,1}p_{2,1} + p_{1,1}p_{2,2}p_{2,1} + p_{1,2}p_{1,1}p_{2,2}p_{2,1} = \mathcal{K}_{312} + \mathcal{K}_{2132},$$
$$\mathcal{GK}_{313} = p_{1,3}p_{1,1}p_{2,2} + p_{1,3}p_{1,1}p_{3,1} + p_{1,3}p_{1,1}p_{2,2}p_{3,1} = \mathcal{K}_{313} + \mathcal{K}_{3133},$$
$$\mathcal{GK}_{321} = p_{1,1}p_{1,2}p_{1,3} = \mathcal{K}_{123},$$
$$\begin{aligned}\mathcal{GK}_{323} = {} & p_{1,3}p_{1,2}p_{2,2} + p_{1,3}p_{1,2}p_{3,1} + p_{1,3}p_{2,1}p_{3,1} + p_{2,2}p_{2,1}p_{3,1} + p_{1,3}p_{2,2}p_{2,1} \\ & + p_{1,3}p_{1,2}p_{2,1}p_{2,2} + p_{1,2}p_{2,1}p_{2,2}p_{3,1} + p_{1,3}p_{1,2}p_{2,2}p_{3,1} + p_{1,3}p_{1,2}p_{2,1}p_{3,1} \\ & + p_{1,2}p_{2,2}p_{2,1}p_{3,1} + p_{1,3}p_{1,2}p_{2,2}p_{2,1}p_{3,1} = \mathcal{K}_{323} + \mathcal{K}_{3232} + \mathcal{K}_{3233} + \mathcal{K}_{3223} + \mathcal{K}_{32323},\end{aligned}$$
$$\mathcal{GK}_{2123} = p_{1,2}p_{1,1}p_{2,1}p_{3,1} = \mathcal{K}_{2123}, \qquad \mathcal{GK}_{2132} = p_{1,2}p_{1,1}p_{2,2}p_{2,1} = \mathcal{K}_{2132},$$
$$\mathcal{GK}_{3123} = p_{1,3}p_{1,1}p_{2,1}p_{3,1}p_{1,1}p_{2,2}p_{2,1}p_{3,1} + p_{1,3}p_{1,1}p_{2,2}p_{2,1}p_{3,1} = \mathcal{K}_{3123} + \mathcal{K}_{31323},$$
$$\mathcal{GK}_{3212} = p_{1,3}p_{1,2}p_{1,1}p_{2,1} = \mathcal{K}_{3212},$$
$$\mathcal{GK}_{3213} = p_{1,3}p_{1,2}p_{1,1}p_{2,2} + p_{1,3}p_{1,2}p_{1,1}p_{3,1} + p_{1,3}p_{1,2}p_{1,1}p_{2,2}p_{3,1} = \mathcal{K}_{3213} + \mathcal{K}_{32133},$$
$$\mathcal{GK}_{21323} = p_{1,2}p_{1.1}p_{2,2}p_{2,1}p_{3,1} = \mathcal{K}_{21323}, \qquad \mathcal{GK}_{32123} = p_{1,3}p_{1,2}p_{1,1}p_{2,1}p_{31} = \mathcal{K}_{32123},$$
$$\mathcal{GK}_{32132} = p_{1,3}p_{1,2}p_{1,1}p_{2,2}p_{2,1} = \mathcal{K}_{32132}, \qquad \mathcal{GK}_{321323} = p_{3,1}p_{2,1}p_{1,1}p_{2,2}p_{2,1}p_{3,1} = \mathcal{K}_{321323}.$$

(7) 14 double local key polynomials for $n = 4$:

$$\mathcal{LK}_{\mathrm{id}} = 1, \qquad \mathcal{LK}_1 = \mathcal{K}_1, \qquad \mathcal{LK}_2 = \mathcal{K}_2, \qquad \mathcal{LK}_3 = \mathcal{K}_3, \qquad \mathcal{LK}_{12} = \mathcal{K}_{12},$$
$$\mathcal{LK}_{21} = \mathcal{K}_{21} + \mathcal{K}_{212}, \qquad \mathcal{LK}_{13} = \mathcal{K}_{13} + \mathcal{K}_{31} + \mathcal{K}_{313}, \qquad \mathcal{LK}_{23} = \mathcal{K}_{23},$$
$$\mathcal{LK}_{32} = \mathcal{K}_{32} + \mathcal{K}_{323}, \qquad \mathcal{LK}_{123} = \mathcal{K}_{123}, \qquad \mathcal{LK}_{213} = \mathcal{K}_{213} + \mathcal{K}_{2123},$$
$$\mathcal{LK}_{312} = \mathcal{K}_{312} + \mathcal{K}_{3123}, \qquad \mathcal{LK}_{321} = \mathcal{K}_{321} + \mathcal{K}_{3212} + \mathcal{K}_{3213} + \mathcal{K}_{32123} + \mathcal{K}_{32132} + \mathcal{K}_{321323},$$
$$\mathcal{LK}_{2132} = \mathcal{K}_{2132} + \mathcal{K}_{21323}.$$

(8) 35 $(2,3)$-key polynomials:

$$U_{\mathrm{id}} = 1, \quad U_1 = p_{11}+p_{23}, \quad U_2 = p_{12}+p_{21}, \quad U_3 = p_{13}+p_{22}, \quad U_{11} = p_{11}p_{23},$$
$$U_{12} = p_{11}p_{21}, \quad U_{13} = p_{11}p_{22}, \quad U_{21} = p_{12}p_{11}+p_{12}p_{23}+p_{21}p_{23}, \quad U_{23} = p_{12}p_{22},$$
$$U_{22} = p_{12}p_{21}, \quad U_{31} = p_{13}p_{11}+p_{13}p_{23}+p_{22}p_{23}, \quad U_{32} = p_{13}p_{12}+p_{13}p_{21}+p_{22}p_{21},$$
$$U_{33} = p_{13}p_{22}, \quad U_{211} = p_{12}p_{11}p_{23}+p_{11}p_{21}p_{23}, \quad U_{212} = p_{12}p_{11}p_{21}+p_{12}p_{21}p_{23},$$
$$U_{213} = p_{12}p_{11}p_{22}+p_{12}p_{22}p_{23}, \quad U_{311} = p_{13}p_{11}p_{23}+p_{11}p_{22}p_{23},$$
$$U_{312} = p_{13}p_{11}p_{21}+p_{11}p_{22}p_{21}, \quad U_{313} = p_{13}p_{11}p_{22}+p_{13}p_{22}p_{23},$$
$$U_{321} = p_{13}p_{12}p_{11}+p_{13}p_{12}p_{23}+p_{13}p_{21}p_{23}+p_{22}p_{21}p_{23}, \quad U_{322} = p_{13}p_{12}p_{21}+p_{12}p_{22}p_{21},$$
$$U_{323} = p_{13}p_{12}p_{22}+p_{13}p_{22}p_{21}, \quad U_{2121} = p_{12}p_{11}p_{21}p_{23}, \quad U_{2131} = p_{12}p_{11}p_{22}p_{23},$$
$$U_{2132} = p_{12}p_{11}p_{22}p_{23}, \quad U_{3132} = p_{13}p_{11}p_{22}p_{23}, \quad U_{3131} = p_{13}p_{11}p_{22}p_{23},$$
$$U_{3232} = p_{13}p_{21}p_{22}p_{23}, \quad U_{3211} = p_{13}p_{12}p_{11}p_{23}+p_{13}p_{11}p_{21}p_{23}+p_{11}p_{22}p_{21}p_{23},$$
$$U_{3212} = p_{13}p_{12}p_{11}p_{22}+p_{13}p_{12}p_{21}p_{23}+p_{12}p_{21}p_{22}p_{23},$$
$$U_{3213} = p_{13}p_{12}p_{11}p_{21}+p_{13}p_{22}p_{21}p_{23}+p_{12}p_{22}p_{21}p_{23},$$
$$U_{32121} = p_{12}p_{11}p_{22}p_{21}p_{23}+p_{13}p_{12}p_{11}p_{21}p_{23},$$
$$U_{32131} = p_{13}p_{12}p_{11}p_{22}p_{23}+p_{13}p_{11}p_{22}p_{21}p_{23},$$
$$U_{32132} = p_{13}p_{12}p_{11}p_{22}p_{21}+p_{13}p_{12}p_{22}p_{21}p_{23}, \quad U_{321321} = p_{13}p_{12}p_{11}p_{22}p_{21}p_{23}.$$

(9) Polynomials $\mathcal{KN}_w := \mathcal{KN}_w^{(\beta,\alpha)}(1)$ for $n=4$:

$$\mathcal{KN}_{\mathrm{id}} = 1, \quad \mathcal{KN}_1 = \mathcal{KN}_2 = \mathcal{KN}_3 = \beta+1+\alpha\beta,$$
$$\mathcal{KN}_{12} = 1+2\alpha+\alpha^2+3\alpha\beta+3\alpha^2\beta+\alpha\beta^2+2\alpha^2\beta^2, \quad (13),$$
$$\mathcal{KN}_{21} = 2+3\alpha+\alpha^2+\beta+3\alpha\beta+2\alpha^2\beta+\alpha^2\beta^2, \quad (13),$$
$$\mathcal{KN}_{13} = 1+2\alpha+\alpha^2+2\alpha\beta+2\alpha^2\beta+\alpha^2\beta^2 = (1+\alpha+\alpha\beta)^2, \quad (9),$$
$$\mathcal{KN}_{23} = \mathcal{KN}_{12}, \quad \mathcal{KN}_{32} = \mathcal{KN}_{21},$$
$$\begin{aligned}\mathcal{KN}_{132} &= 2+5\alpha+4\alpha^2+\alpha^3+\beta+7\alpha\beta+10\alpha^2\beta+4\alpha^3\beta+2\alpha\beta^2+7\alpha^2\beta^2+5\alpha^3\beta^2+\alpha^2\beta^3\\ &\quad +2\alpha^3\beta^3 = (1+\alpha+\alpha\beta)\big(2+3\alpha+\alpha^2+\beta+4\alpha\beta+3\alpha^2\beta+\alpha\beta^2+2\alpha^2\beta^2\big), \quad (51),\end{aligned}$$
$$\mathcal{KN}_{121} = 1+3\alpha+3\alpha^2+\alpha^3+4\alpha\beta+7\alpha^2\beta+3\alpha^3\beta+\alpha\beta^2+4\alpha^2\beta^2+3\alpha^3\beta^2+\alpha^3\beta^3, \quad (31),$$
$$\begin{aligned}\mathcal{KN}_{321} &= 5+10\alpha+6\alpha^2+\alpha^3+5\beta+14\alpha\beta+12\alpha^2\beta+3\alpha^3\beta+\beta^2+4\alpha\beta^2+6\alpha^2\beta^2\\ &\quad +3\alpha^3\beta^2+\alpha^3\beta^3, \quad (71),\end{aligned}$$
$$\mathcal{KN}_{232} = \mathcal{KN}_{121},$$
$$\begin{aligned}\mathcal{KN}_{123} &= 1+3\alpha+3\alpha^2+\alpha^3+6\alpha\beta+12\alpha^2\beta+6\alpha^3\beta+4\alpha\beta^2+14\alpha^2\beta^2+10\alpha^3\beta^2+\alpha\beta^3\\ &\quad +5\alpha^2\beta^3+5\alpha^3\beta^3 = \beta^3\alpha^3\mathcal{KN}_{321}^{(\beta^{-1},\alpha^{-1})}(1), \quad (71),\end{aligned}$$
$$\mathcal{KN}_{213} = (\alpha\beta)^3\mathcal{KN}_{132}^{(\alpha^{-1},\beta^{-1})},$$
$$\begin{aligned}\mathcal{KN}_{3121} &= 3+10\alpha+12\alpha^2+6\alpha^3+\alpha^4+2\beta+16\alpha\beta+29\alpha^2\beta+19\alpha^3\beta+4\alpha^4\beta+7\alpha\beta^2\\ &\quad +21\alpha^2\beta^2+20\alpha^3\beta^2+6\alpha^4\beta^2+\alpha\beta^3+4\alpha^2\beta^3+7\alpha^3\beta^3+4\alpha^4\beta^3+\alpha^4\beta^4, \quad (173),\end{aligned}$$
$$\mathcal{KN}_{2321} = (\alpha\beta)^4\mathcal{KN}_{3121}^{(\alpha^{-1},\beta^{-1})},$$
$$\begin{aligned}\mathcal{KN}_{1213} &= 1+4\alpha+6\alpha^2+4\alpha^3+\alpha^4+7\alpha\beta+20\alpha^2\beta+19\alpha^3\beta+6\alpha^4\beta+4\alpha\beta^2\\ &\quad +21\alpha^2\beta^2+29\alpha^3\beta^2+12\alpha^4\beta^2+\alpha\beta^3+7\alpha^2\beta^3+16\alpha^3\beta^3+10\alpha^4\beta^3\\ &\quad +2\alpha^3\beta^4+3\alpha^4\beta^4, \quad (173),\end{aligned}$$
$$\mathcal{KN}_{1232} = (\alpha\beta)^4\mathcal{KN}_{1213}^{(\alpha^{-1},\beta^{-1})},$$

$$\begin{aligned}
\mathcal{KN}_{2132} &= 3+9\alpha+10\alpha^2+5\alpha^3+\alpha^4+3\beta+16\alpha\beta+28\alpha^2\beta+20\alpha^3\beta+5\alpha^4\beta+\beta^2\\
&\quad +7\alpha\beta^2+24\alpha^2\beta^2+28\alpha^3\beta^2+10\alpha^4\beta^2+7\alpha^2\beta^3+16\alpha^3\beta^3+9\alpha^4\beta^3+\alpha^2\beta^4\\
&\quad +3\alpha^3\beta^4+3\alpha^4\beta^4, \quad (209),\\
\mathcal{KN}_{21321} &= 3+12\alpha+19\alpha^2+15\alpha^3+6\alpha^4+\alpha^5+3\beta+21\alpha\beta+49\alpha^2\beta+52\alpha^3\beta\\
&\quad +26\alpha^4\beta+5\alpha^5\beta+\beta^2+9\alpha\beta^2+39\alpha^2\beta^2+64\alpha^3\beta^2+43\alpha^4\beta^2+10\alpha^5\beta^2\\
&\quad +10\alpha^2\beta^3+32\alpha^3\beta^3+32\alpha^4\beta^3+10\alpha^5\beta^3+\alpha^2\beta^4+5\alpha^3\beta^4+9\alpha^4\beta^4\\
&\quad +5\alpha^5\beta^4+\alpha^5\beta^5, \quad (483),\\
\mathcal{KN}_{12312} &= 1+5\alpha+10\alpha^2+10\alpha^3+5\alpha^4+\alpha^5+9\alpha\beta+32\alpha^2\beta+43\alpha^3\beta+26\alpha^4\beta\\
&\quad +6\alpha^5\beta+5\alpha\beta^2+32\alpha^2\beta^2+64\alpha^3\beta^2+52\alpha^4\beta^2+15\alpha^5\beta^2+\alpha\beta^3\\
&\quad +10\alpha^2\beta^3+39\alpha^3\beta^3+49\alpha^4\beta^3+19\alpha^5\beta^3+9\alpha^3\beta^4+21\alpha^4\beta^4+12\alpha^5\beta^4+\alpha^3\beta^5\\
&\quad +3\alpha^4\beta^5+3\alpha^5\beta^5, \quad (483),\\
\mathcal{KN}_{12321} &= 2+9\alpha+16\alpha^2+14\alpha^3+6\alpha^4+\alpha^5+\beta+18\alpha\beta+54\alpha^2\beta+64\alpha^3\beta+33\alpha^4\beta\\
&\quad +6\alpha^5\beta+14\alpha\beta^2+65\alpha^2\beta^2+101\alpha^3\beta^2+64\alpha^4\beta^2+14\alpha^5\beta^2+6\alpha\beta^3+33\alpha^2\beta^3\\
&\quad +65\alpha^3\beta^3+54\alpha^4\beta^3+16\alpha^5\beta^3+\alpha\beta^4+6\alpha^2\beta^4+14\alpha^3\beta^4+18\alpha^4\beta^4+9\alpha^5\beta^4\\
&\quad +\alpha^4\beta^5+2\alpha^5\beta^5, \quad (707),\\
\mathcal{KN}_{121321} &= 1+6\alpha+15\alpha^2+20\alpha^3+15\alpha^4+6\alpha^5+\alpha^6+10\alpha\beta+45\alpha^2\beta+81\alpha^3\beta+73\alpha^4\beta\\
&\quad +33\alpha^5\beta+6\alpha^6\beta+5\alpha\beta^2+44\alpha^2\beta^2+116\alpha^3\beta^2+135\alpha^4\beta^2+73\alpha^5\beta^2+15\alpha^6\beta^2\\
&\quad +\alpha\beta^3+15\alpha^2\beta^3+69\alpha^3\beta^3+116\alpha^4\beta^3+81\alpha^5\beta^3+20\alpha^6\beta^3+\alpha^2\beta^4+15\alpha^3\beta^4\\
&\quad +44\alpha^4\beta^4+45\alpha^5\beta^4+15\alpha^6\beta^4+\alpha^3\beta^5+5\alpha^4\beta^5+10\alpha^5\beta^5+6\alpha^6\beta^5+\alpha^6\beta^6\\
&= \beta^6\alpha^6\mathcal{KN}_{121321}^{(\alpha^{-1},\beta^{-1})}(1), \quad (1145).
\end{aligned}$$

(10) Polynomials $\mathcal{KN}_w^{(\beta,\alpha,\gamma)} := \mathcal{KN}_w^{(\beta,\alpha,\gamma)}(1)$ for $n=3$:

$$\begin{aligned}
\mathcal{KN}_{\mathrm{id}}^{(\beta,\alpha,\gamma)} &= 1,\\
\mathcal{KN}_{1}^{(\beta,\alpha,\gamma)} &= \mathcal{KN}_{2}^{(\beta,\alpha,\gamma)} = 1+(\beta+\gamma)(1+\alpha+\gamma), \quad (7),\\
\mathcal{KN}_{12}^{(\beta,\alpha,\gamma)} &= 1+2\alpha+\alpha^2+3\alpha\beta+3\alpha^2\beta+\alpha\beta^2+2\alpha^2\beta^2+5\gamma+8\alpha\gamma+3\alpha^2\gamma+4\beta\gamma\\
&\quad +11\alpha\beta\gamma+4\alpha^2\beta\gamma+\beta^2\gamma+4\alpha\beta^2\gamma+9\gamma^2+10\alpha\gamma^2+2\alpha^2\gamma^2+8\beta\gamma^2+8\alpha\beta\gamma^2\\
&\quad +2\beta^2\gamma^2+7\gamma^3+4\alpha\gamma^3+4\beta\gamma^3+2\gamma^4, \quad (109),\\
\mathcal{KN}_{21}^{(\beta,\alpha,\gamma)} &= 2+3\alpha+\alpha^2+\beta+3\alpha\beta+2\alpha^2\beta+\alpha^2\beta^2+7\gamma+8\alpha\gamma+2\alpha^2\gamma+4\beta\gamma+7\alpha\beta\gamma\\
&\quad +2\alpha^2\beta\gamma+2\alpha\beta^2\gamma+9\gamma^2+7\alpha\gamma^2+\alpha^2\gamma^2+5\beta\gamma^2+4\alpha\beta\gamma^2+\beta^2\gamma^2+5\gamma^3\\
&\quad +2\alpha\gamma^3+2\beta\gamma^3+\gamma^4, \quad (82),\\
\mathcal{KN}_{121}^{(\beta,\alpha,\gamma)} &= 1+3\alpha+3\alpha^2+\alpha^3+4\alpha\beta+7\alpha^2\beta+3\alpha^3\beta+\alpha\beta^2+4\alpha^2\beta^2+3\alpha^3\beta^2+\alpha^3\beta^3\\
&\quad +6\gamma+15\alpha\gamma+12\alpha^2\gamma+3\alpha^3\gamma+3\beta\gamma+20\alpha\beta\gamma+22\alpha^2\beta\gamma+6\alpha^3\beta\gamma+7\alpha\beta^2\gamma\\
&\quad +12\alpha^2\beta^2\gamma+3\alpha^3\beta^2\gamma+3\alpha^2\beta^3\gamma+15\gamma^2+30\alpha\gamma^2+18\alpha^2\gamma^2+3\alpha^3\gamma^2+12\beta\gamma^2\\
&\quad +37\alpha\beta\gamma^2+24\alpha^2\beta\gamma^2+3\alpha^3\beta\gamma^2+3\beta^2\gamma^2+15\alpha\beta^2\gamma^2+9\alpha^2\beta^2\gamma^2+3\alpha\beta^3\gamma^2\\
&\quad +20\gamma^3+30\alpha\gamma^3+12\alpha^2\gamma^3+\alpha^3\gamma^3+18\beta\gamma^3+30\alpha\beta\gamma^3+9\alpha^2\beta\gamma^3+6\beta^2\gamma^3\\
&\quad +9\alpha\beta^2\gamma^3+\beta^3\gamma^3+15\gamma^4+15\alpha\gamma^4+3\alpha^2\gamma^4+12\beta\gamma^4+9\alpha\beta\gamma^4+3\beta^2\gamma^4+6\gamma^5\\
&\quad +3\alpha\gamma^5+3\beta\gamma^5+\gamma^6, \quad (521).
\end{aligned}$$

(11) Few more examples:

$$\begin{aligned}\mathcal{KN}_{4321}^{(\beta,\alpha,\gamma=0)}(1) &= 14+35\alpha+30\alpha^2+10\alpha^3+\alpha^4+21\beta+65\alpha\beta+70\alpha^2\beta+30\alpha^3\beta+4\alpha^4\beta\\ &\quad+9\beta^2+35\alpha\beta^2+50\alpha^2\beta^2+30\alpha^3\beta^2+6\alpha^4\beta^2+\beta^3+5\alpha\beta^3+10\alpha^2\beta^3\\ &\quad+10\alpha^3\beta^3+4\alpha^4\beta^3+\alpha^4\beta^4,\\ \mathcal{KN}_{4321}^{(\beta=1,\alpha=1,\gamma)}(1) &= (441,1984,3754,3882,2385,885,192,22,1)_\gamma,\\ \mathcal{KN}_{54321}^{(\beta=1,\alpha=1,\gamma)} &= (1+\gamma)(2955,13297,25678,27822,18553,7852,2094,336,29,1)_\gamma.\end{aligned}$$

Note that polynomial $L_n(\gamma) := \mathcal{KN}_{n,n-1,\dots,2,1}^{(\beta=1,\alpha=1,\gamma)}(1)$ has degree $2n$ and $L_n(\gamma=-1)=0$.

$$\begin{aligned}h^3\mathcal{KN}_{321}^{(\beta=1,\alpha=1,\gamma,h)} &= (1,3,7,9,7,1)_h+\gamma(6,18,32,32,18,6)_h+\gamma^2(15,42,58,42,15)_h\\ &\quad+\gamma^3(20,48,48,20)_h+\gamma^4(15,27,15)_h+\gamma^5(6,6)_h+\gamma^6h^6,\\ \mathcal{KN}_{121}^{(a=1,b=1,c,r)}(1) &= 31+112c+168c^2+124c^3+44c^4+6c^5\\ &\quad+\big(60+176c+195c^2+93c^3+16c^4\big)r+\big(38+85c+61c^2+14c^3\big)r^2\\ &\quad+\big(8+12c+4c^2\big)r^3.\end{aligned}$$

Problem A.3. *Let $n \geq k \geq 0$ be integers, consider permutation $w_{n,k} := [k,k-1,\dots 1,n,n-1,n-2,\dots,k+1] \in \mathbb{S}_n$. Give combinatorial interpretations of polynomials $L_{n,k}(\alpha,\beta,\gamma) := \mathcal{KN}_{w_{m,k}}^{(\alpha,\beta,\gamma)}(1)$.*

Conjecture A.4. *Set $d := \gamma - 1$.*

- *For any permutation $w \in \mathbb{S}_n$, $\mathcal{KN}_w^{(\beta,\alpha=1,\gamma=d-1)}(1)$ is a polynomial in β and d with non-negative coefficients.*
- *The polynomial $L_n(d)$ has non-negative coefficients, and polynomial $L_n(d)+d^n$ is symmetric and unimodal.*
- *$L_{n,1}(\alpha=1,\beta,d) \in d\mathbb{N}[\beta,d]$.*
- *$L_{n,1}(\alpha,\beta=0,d) \in d^{n-1}(\alpha+d)\mathbb{N}[\alpha,d]$, $L_{n,1}(\alpha=1,\beta=0,d=1) = 2\,\mathrm{Sch}_{n+1}$, $L_{n,1}(\alpha=0,\beta=0,d=2) = 2^n\,\mathrm{Sch}_{n+1}$ (see* [67, *A156017*] *for a combinatorial interpretation of these numbers), where Sch_n denotes the n-th Schröder number, see, e.g.,* [67, *A001003*].
- *$L_{n,1}(\alpha=0,\beta=t-1,\gamma) \in \mathbb{N}[t,\gamma]$, $L_{n,1}(\alpha=0,\beta=-1,\gamma=1)$ is equal to the number of Dyck $(n+1)$-paths (* [67, *A000108*]*) in which each up step (U) not at ground level is colored red (R) or blue (B),* [67, *A064062*].

Note that the number $2\,\mathrm{Sch}_n$ known also as *large Schröder* number, see, e.g., [67, *A*006318]. For example,

$$\begin{aligned}L_{3,1}(\alpha=1,\beta,d) &= d\big(\beta^2+5\beta d+4\beta^2d+5d^2+14\beta d^2+6\beta^2d^2+\beta^3d^2+10d^3+12\beta d^3\\ &\quad+3\beta^2d^3+6d^4+3\beta d^4+d^5\big),\\ L_{7,1}(1,1,d) &= d(1,27,260,1245,3375,5495,5494,3375,1245,260,27,1)_d,\\ L_{7,1}(\alpha,\beta=0,d) &= d^6(\alpha+d)(1+\alpha+d)\big(1+14\alpha+36\alpha^2+14\alpha^3+\alpha^4+14d+72\alpha d\\ &\quad+42\alpha^2d+4\alpha^3d+36d^2+42\alpha d^2+6\alpha^2d^2+14d^3+4\alpha d^3+d^4\big),\\ L_{7,1}(\alpha=0,\beta=t-1,\gamma=1) &= (14589,39446,39607,18068,3627,246,1)_t.\end{aligned}$$

We expect a similar conjecture for polynomials $L_{n,k}(\alpha=1,\beta=1,\gamma)$, $k \geq 1$.

A.2 MacMeille completion of a partially ordered set

Let[22] $(\Sigma, \le)$ be a partially ordered set (poset for short) and $X \subseteq \Sigma$. Define

- The set of upper bounds for X, namely,

$$X^{up} := \{z \in \Sigma \,|\, x \le z, \,\forall\, x \in X\}.$$

- The set of lower bounds for X, namely,

$$X^{lo} := \{z \in \Sigma \,|\, z \le x \,\forall\, x \in X\}.$$

- A poset $(\mathcal{MN}(\Sigma), \le)$, namely,

$$\mathcal{MN}(\Sigma) := \{\mathcal{MN}(X) | X \subseteq \Sigma\},$$

  where $\mathcal{MN}(X) := \left(X^{\mathrm{up}}\right)^{\mathrm{lo}}$. Clearly, $X \subseteq \mathcal{MN}(X)$ and $\mathcal{MN}(\mathcal{MN}(X)) = \mathcal{MN}(X)$.
- A map $\kappa \colon \Sigma \longrightarrow \mathcal{MN}(\Sigma)$, namely, $\kappa(X) = \mathcal{MN}(X)$, $X \subseteq \Sigma$.

Proposition A.5.

- *The map κ is an embedding, that is for $X, Y \subseteq \Sigma$,*

$$X \le Y \qquad \textit{if and only if} \quad \kappa(X) \subseteq \kappa(Y),$$

- *Poset $(\mathcal{MN}(\Sigma), \le)$ is a lattice, called the MacNeille completion of poset $(\Sigma, \le)$.*

Proposition A.6. *Let $(\Sigma, \le)$ be a poset*[23]*. Then there is a poset $(L, \le)$ and a map $\kappa \colon \Sigma \longrightarrow L$ such that*

(1) *κ is an embedding,*

(2) *$(L, \le)$ is a complete lattice*[24]*,*

(3) *for each element $a \in L$ one has*

  (a) $\mathcal{MN}(\{x \in \Sigma \,|\, \kappa(x) \le a\}) = \{x \in \Sigma \,|\, \kappa(x) \le a\}$,
  (b) $a = \bigvee \{\kappa(x) \,|\, x \in \Sigma,\, \kappa(x) \le a\}$.

Moreover, the pair $(\kappa, (L, \le))$ is defined uniquely up to an order preserving isomorphism.

Therefore, the lattice $(L, \le)$, is an order-isomorphic to the MacNeille completion $\mathcal{MN}(\Sigma)$ of a poset Σ.

Problem A.7. *Let Σ be a (finite) graded poset*[25]*, denote by*

$$r_\Sigma(t) := \sum_{a \in \Sigma} t^{r(a)},$$

the rank generating function of a poset Σ. Here $r(a)$ denotes the rank/degree of an element $a \in \Sigma$. Describe polynomial $r_{\mathcal{MN}(\Sigma)}(t)$.

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[22] For the reader convenience we review a definition and basic facts concerning the MacNeille completion of a poset, see for example, notes by E. Turunen, available at `http://math.tut.fi/~eturunen/AppliedLogics007/Mac1.pdf`.

[23] See `https://en.wikipedia.org/wiki/Graded_poset`.

[24] That is every subset of L has a meet and join, see, e.g., [69, p. 249].

[25] See, e.g., [69, p. 244], or `https://en.wikipedia.org/wiki/Graded_poset`.

In the present paper we are interesting in properties of the MacNeille completion of the Bruhat poset $\mathcal{B}_n = \mathcal{B}(\mathbb{S}_n)$ corresponding to the symmetric group $\mathbb{S}_n$. Below we briefly describe a construction of the MacNeille completion $L_n(\mathbb{S}_n) := \mathcal{MN}_n(\mathcal{B}_n)$ following [46] and [70, p. 552, **d**].

Let $w = (w_1 w_2 \dots w_n) \in \mathbb{S}_n$, associate with w a semistandard Young tableaux $T(w)$ of the staircase shape $\delta_n = (n-1, n-2, \dots, 2, 1)$ filled by integer numbers from the set $[1,n] := \{1,2,\dots,n\}$ as follows: the i-th row of of $T(w)$, denoted by $R_i(w)$, consists of the numbers $w_1, \dots, w_{n-i+1}$ in increasing order. Clearly the tableaux $T(w) = [T_{i,j}(w)]_{1 \le i < j \le n-1}$ obtained in such a manner, satisfies the so-called *monotonic* and *flag* conditions, namely,

(1) (monotonic conditions) $T_{1,i} \ge T_{2,i-1} \ge \dots \ge T_{i,1}$, $i = 1, \dots, n-1$,

(2) (flag conditions) $R_1(w) \supset R_2(w) \supset \dots \supset R_{n-1}(w)$.

Denote by $L(\mathbb{S}_n)$ the subset of the set of all Young tableaux $T \in \mathrm{STY}(\delta_n \le n)$ consisting of that T which satisfies the monotonicity conditions (1). The set $L(\mathbb{S}_n)$ has the natural poset structure denoted by "$\ge$", and defined as follows: if $T^{(1)} = [t_{ij}^{(1)}]_{1 \le i < j \le n-1}$ and $T^{(2)} = [t_{ij}^{(2)}]_{1 \le i < j \le n-1}$ belong to the set $L(\mathbb{S}_n)$, then by definition

$$T^{(1)} \ge T^{(2)} \qquad \text{if and only if} \quad t_{ij}^{(1)} \ge t_{ij}^{(2)} \quad \text{for all} \quad 1 \le i < j \le n-1.$$

It is clearly seen that the set $L(\mathbb{S}_n)$ is closed under the following operations

- $\big(\text{meet } T^{(1)} T^{(2)}\big) \bigwedge \big(T^{(1)}, T^{(2)}\big) := T^{(1)} \bigwedge T^{(2)} = \big[\min\big(t_{i,j}^{(1)}, t_{i,j}^{(2)}\big)\big]$,
- $\big(\text{join } T^{(1)} T^{(2)}\big) \bigvee \big((T^{(1)}, T^{(2)}\big) := T^{(1)} \bigvee T^{(2)} = \big[\max\big(t_{i,j}^{(1)}, t_{i,j}^{(2)}\big)\big]$.

Theorem A.8 ([46])**.** *The poset $L(\mathbb{S}_n)$ is a complete distributive lattice with number of vertices equals to the number* $\mathrm{ASM}(n)$ *that is the number of alternating sigh matrices of size $n \times n$. Moreover, the lattice $L(\mathbb{S}_n)$ is order isomorphic to the MacNeille completion of the Bruhat poset $\mathcal{B}_n$.*

Indeed it is not difficult to prove that the set of all monotonic triangles obtained by applying repeatedly operation $\bigvee$ (= meet) to the set $\{T(w),\, w \in \mathbb{S}_n\}$ of triangles corresponding to all elements of the symmetric group $\mathbb{S}_n$, coincides with the set of all monotonic triangles $L(\mathbb{S}_n)$. The natural map $\kappa\colon \mathbb{S}_n \longrightarrow L(\mathbb{S}_n)$ is obviously embedding, and all other conditions of Proposition A.6 are satisfied. Therefore $L(\mathbb{S}_n) = \mathcal{MN}(\mathcal{B}_n$. The fact that the lattice $L(\mathbb{S}_n$ is a distributive one follows from the well-known identities

$$\max(x, \min(y,z)) = \min\big(\max(x,y), \max(x,z)\big), \qquad x,y,z \in \big(\mathbb{R}_{\ge 0}\big)^3.$$

In the lattice $L(\mathbb{S}_n$ this identity can be written in the following forms

$$T^{(1)} \bigvee \big(T^{(2)} \bigwedge T^{(3)}\big) = \big(T^{(1)} \bigwedge T^{(2)}\big) \bigvee \big(T^{(1)} \bigwedge T^{(3)}\big),$$
$$T^{(1)} \bigwedge \big(T^{(2)} \bigvee T^{(3)}\big) = \big(T^{(1)} \bigvee T^{(2)}\big) \bigwedge \big(T^{(1)} \bigvee T^{(3)}\big).$$

Finally the fact that the cardinality of the lattice $L(\mathbb{S}_n)$ is equal to the number $\mathrm{ASM}(n)$ had been proved by A. Lascoux and M.-P. Schützenberger [46].

If $T = [t_{ij}] \in L(\mathbb{S}_n)$, define *rank* of T, denoted by $r(T)$, as follows:

$$r(T) = \sum_{1 \le i < j \le n-1} t_{ij} - \binom{n}{3}.$$

It had been proved by C. Ehresmann [15] that $v \le w$ with respect to the Bruhat order in the symmetric group $\mathbb{S}_n$ if and only if $T_{i,j}(v) \le T_{i,j}(w)$ for all $1 \le i < j \le n-1$.

It follows from an improved tableau criterion for Bruhat order on the symmetric group [5] that[26] the length $\ell(w)$ of a permutation $w \in \mathbb{S}_n$ can be computed as follows

$$\ell(w) = r(T(w)) - \sum_{(i,j)\in I(w)} (j-i-1),$$

where $I(w) := \{(i,j) \mid 1 \le i < j \le n,\ w_i > w_j\}$ denotes the set of *inversions* of permutation w; a detailed proof can be found in [32].

For example, consider permutation $w = [4,6,2,7,5,1,3]$. Then the code $c(w)$ of w is equal to $c(w) = (3,4,1,3,2)$, and w has the length $\ell(w) = 13$. The corresponding Young tableau or monotonic triangle displayed below

$$T(w) = \begin{bmatrix} 1 & 2 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 6 & 7 & \\ 2 & 4 & 6 & 7 & & \\ 2 & 4 & 6 & & & \\ 4 & 6 & & & & \\ 4 & & & & & \end{bmatrix}.$$

Wherefore, $r(T(w)) = |T(w)| - \binom{7}{3} = 94 - 56 = 38$. On the other hand, the inversion set $I(w) = \{(1,3),(1,6),(1,7),(2,3),(2,5),(2,6),(2,7),(3,6),(4,5),(4,6),(4,7),(5,6),(5,7)\}$, hence $\sum_{(i,j)\in I(w)} (j-i-1) = 10+9+2+3+1 = 25$ and $\ell(w) = 38 - 25 = 13$, as it should be.

It is easily seen that the polynomial $r_{\mathcal{MN}_n}(t)$ is symmetric and $\deg(r_{\mathcal{MN}_n}(t)) = \binom{n+1}{3}$, For example,

$$r(\mathcal{MN}_3) = (1,2,1,2,1), \qquad r(\mathcal{MN}_4) = (1,3,3,5,6,\mathbf{6},6,5,3,3,1),$$
$$r(\mathcal{MN}_5) = (1,4,6,10,16,20,27,34,37,40,\mathbf{39},40,37,34,27,20,16,10,6,4,1),$$
$$r(\mathbb{S}_3 \subset \mathcal{MN}_3) = (1,2,0,2,1), \qquad r(\mathbb{S}_4 \subset \mathcal{MN}_4) = (1,3,1,4,2,\mathbf{2},2,4,1,3,1)),$$
$$r(\mathbb{S}_5 \subset \mathcal{MN}_5) = (1,4,3,6,7,6,4,10,6,10,\mathbf{6},10,6,10,4,6,7,6,3,4,1).$$

Conjecture A.9. *The number* $\mathrm{Coeff}_{[\binom{n+1}{3}/2]} r_{\mathcal{MN}_n}(t)$ *is a divisor of the number* $\mathrm{ASM}(n)$.

Acknowledgements

A bit of history. Originally these notes have been designed as a continuation of [17]. The main purpose was to extend the methods developed in [18] to obtain by the use of plactic algebra, a noncommutative generating function for the key (or Demazure) polynomials introduced by A. Lascoux and M.-P. Schützenberger [53]. The results concerning the polynomials introduced in Section 4, *except* the Hecke–Grothendieck polynomials, see Definition 4.6, has been presented in my lecture-courses "Schubert Calculus" and have been delivered at the Graduate School of Mathematical Sciences, the University of Tokyo, November 1995 – April 1996, and at the Graduate School of Mathematics, Nagoya University, October 1998 – April 1999. I want to thank Professor M. Noumi and Professor T. Nakanishi who made these courses possible. Some early versions of the present notes are circulated around the world and now I was asked to put it for the wide audience. I would like to thank Professor M. Ishikawa (Department of Mathematics, Faculty of Education, University of the Ryukyus, Okinawa, Japan) and Professor S. Okada (Graduate School of Mathematics, Nagoya University, Nagoya, Japan) for valuable comments. My special thanks to the referees for very careful reading of a preliminary version of the present paper and many valuable remarks, comments and suggestions.

[26]It has been proved in [5, Corollary 5], that the Ehresmann criterion stated above is equivalent to either the criterion $T^{(1)}_{i,j} \le T^{(2)}_{i,j}$ for all j such that $w_j > w_{j+1}$ and $1 \le i \le j$, or that $T^{(1)}_{i,j} \le T^{(2)}_{i,j}$ for all $j \in \{1,2,\dots,n-1\} \backslash \{k \mid v_k > v_{k+1}\}$ and $1 \le i \le j$.

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