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Notes on Schubert, Grothendieck and Key polynomials

To the memory of Alexander Grothendieck (1928-2014)

By

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Abstract

We introduce

• certain finite dimensional algebras denoted by $\mathcal{PC}_n$ and $\mathcal{PF}_{n,m}$ which are certain quotients of the plactic algebra $\mathcal{P}_n$, introduced by A. Lascoux and M.-P. Schützenberger [20]; we show that

$dim(\mathcal{PF}_{n,k})$ is equal to the number of symmetric plane partitions fit inside the box $n \times k \times k$, $dim(\mathcal{PC}_n)$ is equal to the number of alternating sign matrices of size $n \times n$, moreover,

$dim(\mathcal{PF}_{n,n}) = TSPP(n) \times TSSCPP(n)$, $dim(\mathcal{PF}_{n,n+1}) = TSPP(n) \times TSSCPP(n+1)$, $dim(\mathcal{PF}_{n+2,n}) = dim(\mathcal{PF}_{n,n+1})$, $dim(\mathcal{PF}_{n+3,n}) = \frac{1}{2}\, dim(\mathcal{PF}_{n+1,n+1})$,

study

• decomposition of the Cauchy kernels corresponding to the algebras $\mathcal{PC}_n$ and $\mathcal{PF}_{n,m}$,

introduce

• polynomials which are common generalization of the (double) Schubert, β- Grothendieck, Demazure (known also as *key polynomials*), (plactic) Key-Grothendieck and (plactic) Stanley and stable β- Grothendieck polynomials.

Using a family of the Hecke type divided difference operators we introduce polynomials which are common generalization of the Schubert, β-Grothendieck, dual β-Grothendieck and Di-Francesco–Zin-Justin polynomials.

We also introduce and study some properties of the double affine nilCoxeter algebras and related polynomials.

Keywords Plactic monoid and reduced plactic algebras; nilCoxeter and IdCoxeter algebras; Schubert, β-Grothendieck, Key and (double) Key-Grothendieck polynomials ; Cauchy's type kernels and symmetric, totally symmetric plane partitions, and alternating sign matrices; double affine nilCoxeter algebras.

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1 Introduction

The Grothendieck polynomials had been introduced by A. Lascoux and M.-P. Schútzenberger in [21] and studied in detail in [26]. There are two equivalent versions of the Grothendieck polynomials depending on a choice of a basis in the Grothendieck ring $K^\star(\mathcal{F}l_n)$ of the complete flag variety $\mathcal{F}l_n$. The basis $\{exp(\xi_1), \ldots, exp(\xi_n)\}$ in $K^*(\mathcal{F}l_n)$ is a one choice, and another choice is the basis $\{1 - exp(-\xi_j),\ 1 \leq j \leq n\}$, where $\xi_j,\ 1 \leq j \leq n\}$ denote the Chern classes of the tautological linear bundles L_j over the flag variety $\mathcal{F}l_n$. In the present paper we use the basis in a deformed Grothendieck ring $K^{*,\beta}(\mathcal{F}_n)$ of the flag variety $\mathcal{F}l_n$ generated by the set of elements $\{x_i = x_i^{(\beta)} = 1 - exp(\beta\ \xi_i),\ i = 1, \ldots, n\}$. This basis has been introduced and used for construction of the β-Grothendieck polynomials in [6],[7].

A basis in the classical Grothendieck ring of the flag variety in question corresponds to the choice $\beta = -1$. For arbitrary β the ring generated by the elements $\{x_i^{(\beta)},\ 1 \leq i \leq n\}$ has been identified with the Grothendieck ring corresponding to the generalized cohomology theory associated with the formal group law $F(x, y) = x + y + \beta\ x\ y$, see [12]. The Grothendieck polynomials corresponding to the classical K-theory ring $K^\star(\mathcal{F}l_n)$, i.e. the case $\beta = -1$, had been studied in depth by A. Lascoux and M.-P. Schützenberger in [22]. The β-Grothendieck polynomials has been studied in [6], [8], [12].

The *plactic monoid* over a finite totally ordered set $\mathbb{A} = \{x < y < z < \ldots < t\}$ is the quotient of the free monoid generated by elements from $\mathbb{A}$ subject to the elementary Knuth transformations [16]

$$bca = bac\ \ \&\ acb = cab, \quad and \quad bab = bba\ \&\ aba = baa, \tag{1.1}$$

for any triple $\{a < b < c\} \subset \mathbb{A}$.

To our knowledge, the concept of "plactic monoid" has its origins in a paper by C.Schensted [36], concerning the study of the longest increasing subsequence of a permutation, and a paper by D. Knuth [16], concerning the study of combinatorial and algebraic properties of the Robinson–Schensted correspondence [1].

As far as we know, this monoid and the (unital) algebra $\mathcal{P}(\mathbb{A})$ corresponding to that monoid [2], had been introduced, studied and used in [37], Section 5, to give the first complete proof of the famous *Littlewood–Richardson rule* in the theory of Symmetric functions. A bit later this monoid, was named the "monoïde plaxique" and studied in depth by A. Lascoux and M.-P. Schützenberger [20]. The algebra corresponding to plactic monoid is commonly known as *plactic algebra*. One of the basic properties of the plactic algebra [37] is that it contains the distinguish commutative subalgebra which is generated by noncommutative

[1] See e.g. wiki/Robinson–Schensted _correspondence.

[2] If $\mathbb{A} = \{1 < 2 < \ldots < n\}$, the elements of the algebra $\mathcal{P}(\mathbb{A})$ can be identified with semistandard Young tableaux. It was discovered by D. Knuth [16] that modulo *Knuth equivalence* the equivalence classes of semistandard Young tableaux form an algebra, and named this algebra as *tableaux algebra*. It is easily seen that the tableaux algebra introduced by D. Knuth is isomorphic to the algebra introduced by M.-P. Schützenberger [37].

elementary symmetric polynomials

$$e_k(\mathbb{A}_n) = \sum_{i_1>i_2>\ldots>i_k} a_{i_1}a_{i_2}\cdots a_{i_k}, \quad k=1,\ldots,n, \tag{1.2}$$

see e.g. [37], Corollary 5.9, [5].

We refer the reader to nice written overview [28] of the basic properties and applications of the plactic monoid in Combinatorics.

It is easy to see that the plactic relations for two letters $a<b$, namely,

$$aba = baa, \quad bab = bba,$$

imply the commutativity of noncommutative elementary polynomials in two variables. In other words, the plactic relations for two letters imply that

$$ba(a+b) = (a+b)ba, \ a<b.$$

It has been proved in [5] that these relations together with the Knuth relations (1.1) for three letters $a<b<c$, imply the commutativity of noncommutative elementary symmetric polynomials for any number of variables.

In the present paper we prove that in fact the commutativity of nocommutative elementary symmetric polynomials for $n=2$ *and* $n=3$ implies the commutativity of that polynomials for all n, see Theorem 2.23 [3].

One of the main objectives of the present paper is to study combinatorial properties of the generalized plactic Cauchy kernel

$$\mathcal{C}(\mathfrak{P}_n, U) = \prod_{i=1}^{n-1} \Big\{ \prod_{j=n-1}^{i} (1 + p_{i,j-i+1}\ u_j) \Big\}, \tag{1.3}$$

where $\mathfrak{P}_n$ stands for the set of parameters $\{p_{ij}, \quad 2 \le i+j \le n+1, \ i>1, j>1\}$, and $U := U_n$ stands for a certain noncommutative algebra we are interested in, see Section 5.

We also want to bring to the attention of the reader some interesting combinatorial properties of *rectangular* Cauchy kernels

$$\mathcal{F}(\mathfrak{P}_{n,m}, U) = \prod_{i=1}^{n-1} \Big\{ \prod_{j=m-1}^{1} (1 + p_{i,\overline{i-j+1}^{(m)}}\ u_j) \Big\},$$

where $\mathfrak{P}_{n,m} = \{p_{ij}\}_{\substack{1\le i\le n \\ 1\le j\le m}}$.

We treat these kernels in the (reduced) plactic algebras $\mathcal{PC}_n$ and $\mathcal{PF}_{n,m}$ correspondingly. The algebras $\mathcal{PC}_n$ and $\mathcal{PF}_{n,m}$ are finite dimensional and have bases parameterized by certain Young tableaux described in Section 5.1 and Section 6 correspondingly. Decomposition of

[3] Let us stress that conditions necessary and sufficient to assure the commutativity of noncommutative elementary polynomials for the number of variables equals $n=2$ and $n=3$ turn out to be weaker then whose listed in [5]

the rectangular Cauchy kernel with respect to the basis in the algebra $\mathcal{PF}_{n,m}$ mentioned above, gives rise to a set of polynomials which are common generalizations of the (double) Schubert. β-Grothendieck, Demazure and Stanley polynomials. To be more precise, the polynomials listed above correspond to certain quotients of the plactic algebra $\mathcal{PF}_{n,m}$ and certain specializations of parameters $\{p_{ij}\}$ involved in our definition of polynomials $U_\alpha(\{p_{ij}\})$, see Section 6.

Let us repeat that the important property of plactic algebras is that the noncommutative elementary polynomials

$$e_k(u_1,\dots,n_{n-1}) := \sum_{n-1\geq a_1\geq a_2\geq a_k\geq 1} u_{a_1}\cdots u_{a_k}, \quad k=1,\dots,n-1,$$

generate a commutative subalgebra inside of the plactic algebra $\mathcal{P}_n$, see e.g. [20], [5]. Therefore the all our finite dimensional algebras introduced in the present paper, have a distinguish finite dimensional commutative subalgebra. We have in mined to describe this algebras explicitly in a separate publication.

In Section 2 we state and prove necessary and sufficient conditions in order the elementary noncommutative polynomials form a mutely commuting family. Surprisingly enough to check the commutativity of noncommutative elementary polynomials for any n, it's enough to check these conditions only for $n=2,3$. However a combinatorial meaning of a generalization of the Lascoux-Schützenberger plactic algebra $\mathcal{P}_n$ invented, is still missing.

The plactic algebra $\mathcal{PF}_{n,m}$ introduced in Section 6, has a monomial basis parameterized by the set of rectangular Young tableaux of shape $\lambda\subset(n^m)$ filled by the numbers from the set $\{1,\dots,m\}$. In the case $n=m$ it is well-known [11], [19], [32], that this number is equal to the number of symmetric plane partitions fit inside the cube $n\times n\times n$. Surprisingly enough this number admits a factorization in the product of the number of totally symmetric plane partitions ($TSPP$) by the number of totally symmetric self-complementary plane partitions(TSSCPP) fit inside the same cube. A similar phenomenon happens if $|m-n|\leq 2$, see Section 6. More precisely,we add to the well-known equalities

- $\#|B_{,n}| = 2^n$, $\#|B_{2,n}| = \binom{2n+1}{n}$, $\#|B_{3,n}| = 2^n\ Cat_n$, [39], $A003645$,

$\#|B_{4,n}| = \frac{1}{2}\ Cat_n\ Cat_{n+1}$, [39], $A000356$, $\#|B_{5,n}| = \frac{\binom{n+5}{5}\binom{n+7}{7}\binom{n+9}{9}}{\binom{n+2}{2}\binom{n+4}{4}}$, [39], $A133348$,
the following relations

- $\#|B_{n,n} = TSPP(n)\times ASM(n)$, $\#|B_{n,n+1} = TSPP(n)\times ASM(n+1)$,

- $\#|B_{n+2,n} = \#|B_{n,n+1}$, $\#|B_{n+3,n} = \frac{1}{2}\ \#|B_{n+1,n+1}$.

These relations have strait forward proofs based on the explicit product formulas for the numbers

$$SPP(n) = \prod_{1\leq i\leq j\leq k} \frac{n+i+j+k-1}{i+j+k-1} \ \ and \ \ TSPP(n) = \prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n}\frac{i+j+k-1}{i+j+k-2},$$

but bijective proofs of these identities is an open problem.

It follows from [20], [29] that the dimension of the (reduced) plactic algebra $\mathcal{PC}_n$ is equal to the number of alternating sign matrices of size $n \times n$ ($ASM(n) = TSSCPP(n)$). Therefore the key-Grothendieck polynomials can be obtained from U-polynomials (see Section 6, Theorem 6.9) after the specialization $p_{ij} = 0,\ if\ i + j > n + 1$.

In Section 4 follow [15] we introduce and study a family of polynomials which are common generalization of the Schubert, β-Grothendieck and the Di-Francesco (see Section 4) polynomials. Namely, for any permutation $w \in \mathbb{S}_n$, we introduce polynomial

$$\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(X_n) = T_{s_{i_1}} \cdots T_{s_{i_\ell}}(x^{\delta_n}),$$

where $T_i := T_i^{(\beta,\alpha,\gamma)} =$

$$-\alpha + (\alpha + \beta + \gamma)\ x_i + \gamma\ x_{i+1} + 1 + (\alpha + \gamma)(\beta + \gamma)\ x_i\ x_{i+1})\partial_{i,i+1}, \quad i = 1, \ldots, n-1,$$

denotes a collection of divided difference operators which satisfy the Coxeter and Hecke relations

$$T_i\ T_j\ T_i = T_j\ T_i\ T_j,\ \ if\ \ |i-j| = 1; \quad T_i\ T_j = T_j\ T_i,\ \ if\ \ |i-j| \geq 2,$$

$$T_i^2 = (\beta - \alpha)T_i + \beta\alpha, \quad i = 1, \ldots, n-1,$$

$$T_w := T_{s_{i_1}} \cdots T_{s_{i_\ell}},$$

for any reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$ of a permutation in question.

If $\alpha = \gamma = 0$, these polynomials coincide with the β- Grothendieck polynomials [6], if $\beta = \alpha = 1$, $\gamma = 0$ these polynomials coincide with the Di Francesco–Zin-Justin polynomials [10], if $\beta = \gamma = 0$, these polynomials coincide with dual α- Grothendieck polynomials $\mathcal{H}_w^\alpha(X)$. We expect that polynomials $\mathcal{KN}_\star^{(\beta,\alpha,\gamma)}(X_n)$ have nonnegative coefficients, i.e. $\mathcal{KN}_\star^{(\beta,\alpha,\gamma)}(X_n) \in \mathbb{N}[\alpha, \beta, \gamma][X_n]$ and have some geometrical meaning to be discovered.

More generally we study divided difference type operators of the form

$$T_{ij} := T_{ij}^{(a,b,c,d,h,e)} = a + (b\ x_i + c\ x_j + h + e\ x_i\ x_j)\ \partial_{ij},$$

depending on parameters a, b, c, h, e and satisfying the $2d$-Coxeter relations

$$T_{ij}\ T_{jk}\ T_{ij} = T_{jk}\ T_{ij}\ T_{jk}, \quad 1 \leq i < j < k \leq n.$$

We find that the necessary and sufficient condition which ensure the validity of the $2d$-Coxeter relations is the following relation among the parameters:

(a+b)(a-c)+h e = 0 .

Therefore, if the above relation between parameters a, b, c, d, h, e hold, the the for any permutation $w \in \mathbb{S}_n$ the operator

$$T_w := T_w^{(a,b,c,d,h,e)} = T_{i_1}^{(a,b,c,d,h,e)} \cdots T_{i_\ell}^{(a,b,c,d,h,e)},$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is any reduced decomposition of w, is well-defined. Hence under the same assumption on parameters, for any permutation $w \in \mathbb{S}_n$ one can attach the well-defined polynomial

$$G_w^{(a,b,c,d,h,e)}(X,Y) := T_w^{(x)^{(a,b,c,d,h,e)}}(\prod_{\substack{i\geq 1, j\geq 1 \\ i+j\leq n+1}} (x_i + y_j),$$

and in much the same fashion to define polynomials $D_\alpha^{(a,b,c,d,h,e)}(X,Y)$ for any composition α such that $\alpha_i \leq n-i,\ \forall i$. We have used the notation $T^{(x)}{}_w^{(a,b,c,d,h,e)}$ to point out that this operator acts only on the variables $X = (x_1, \ldots, x_n)$.

In the present paper we are interested in to list conditions on parameters $A := \{a, b, c, d, h, e\}$ with the constraint $(a+b)(a-c) + h\ e = 0$ which ensure that the above polynomials $G_w^{(a,b,c,d,h,e)}(X,Y)$ and $D_\alpha^{(a,b,c,d,h,e)}(X,Y)$ or their specialization $Y = 0$, have nonnegative coefficients. In the present paper we treat the case

$$A = (-\beta, \beta + \alpha + \gamma, \gamma, 1, (\alpha + \gamma)(\beta + \gamma).$$

As it was pointed above, in this case polynomials $G_w^A(X)$ are common generalization of Schubert,β-Grothendieck and dual β-Grothendieck, and Di Francesco–Zin-Justin polynomials. We expect a certain c interpretation of polynomials G_w^A for general β, α *and* γ.

As it was pointed out earlier, one of the basic properties of the plactic monoid $\mathcal{P}_n$ is that the nonocommutative elementary symmetric polynomials $\{e_k(u_1, \ldots, u_{n-1})\}_{1\leq k\leq n-1}$ generate a commutative subalgebra in the plactic algebra in question. One can reformulate this statement as follows. Consider the generating function

$$A_i(x); = \prod_{a=n-1}^{i} (1 + x\ u_a) = \sum_{a=0}^{i} e_a(u_{n-1}, \ldots, u_i)\ x^{i-a},$$

where we set $e_0(U) = 1$. Then the commutativity property of noncommutative elementary symmetric polynomials is equivalent to the following commutativity relation in the plactic as well as in the generic plactic, algebras $\mathcal{P}_n$ and $|mathfrakP_n$, [5] and Theorem 2.23,

$$A_i(x)\ A_i(y) = A_i(y)\ A_i(x), \quad 1 \leq i \leq n-1.$$

Now let us consider the Cauchy kernel

$$\mathcal{C}(\mathfrak{P}_n, U) = A_1(z_1) \cdots A_{n-1}(z_{n-1}),$$

where we assume that the pairwise commuting variables $z_1, \ldots, z_{n-1}$ commute with the all generators of the plactic algebra $\mathcal{P}_n$. In what follows we consider the natural completion $\widehat{\mathfrak{P}}n$ of the plactic algebra $\mathfrak{P}_n$ to allow consider elements of the form $(1+x\ u_i)^{-1}$. This type elements exists in any Hecke type quotient of the plactic algebra. Having in mind this assumption, let us compute the action of divided difference operators $\partial_{i,i+1}^z$ on the Cauchy kernel. In th computation below, the commutativity property of the elements $A_i(x)$ and $A_i(y)$ plays the key role. Let us start computation of $\partial_{i,i+1}^z(\mathcal{P}_n, U) = \partial_{i,i+1}^z(A_1(z_1) \cdots A_{n-1}(z_{n-1}$. First

of all write $A_{i+1} = A_i(z_{i+1})(1 + z_{i+1}\ u_i)^{-1}$. According to the basic property of the elements $A_i(x)$, one sees that the expression $A_i(z_i)\ A_i(z_{i+1})$ is symmetric with respect to z_i and z_{i+1}, that is invariant under the action of divided difference operator $\partial^z_{i,i+1}$ Therefore.

$$\partial^z_{i,i+1}(\mathcal{C}(\mathfrak{P}_n, U)) = A_1(z_1)\cdots A_i(z_i)\ A_i(z_{i+1})\ \partial^z_{i,i+1}((1 + z_{i+1}\ u_i)^{-1})\ A_{i+2}(z_{i+2})\cdots A_{n-1}(z_{n-1}).$$

It is clearly seen that $\partial^z_{i,i+1}((1 + z_{i+1}\ u_i)^{-1}) = (1 + z_i\ u_i)^{-1})(1 + z_{i+1}\ u_i)^{-1}\ u_i$. Therefore

$$\partial^z_{i,i+1}(\mathcal{C}(\mathfrak{P}_n, U)) = A_1(z_1)\cdots A_i(z_i)\ A_{i+1}(z_{i+1})\ (1 + z_i\ u_i)^{-1}\ u_i\ A_{i+2}(z_{i+2})\cdots A_{n-1}(z_{n-1}).$$

It is easy to see that if one adds Hecke's type relations on the generators

$$u_i^2 = (a + b)\ u_i + a\ b, \quad i = 1, \ldots, n-1,$$

then

$$(1 + z\ u_i)^{-1}\ u_i = \frac{u_i - z\ a\ b}{(1 + b\ z)(1 - a\ z)}.$$

Therefore in the quotient of the plactic algebra $\mathcal{P}_n$ by the Hecke type relations listed above and by the "locality" relations

$$u_i\ u_j = u_j\ u_i, \quad if \quad |i - j| \geq 2,$$

one obtains

$$(-b + (1 + z_i\ b))\ \partial^z_{i,i+1}\Big(A_1(z_1)\cdots A_{n-1}(z_{n-1})\Big) = \Big(A_1(z_1)\cdots A_{n-1}(z_{n-1})\Big)\ (\frac{e_i - b}{1 - a\ z_i}).$$

Finally, if $a = 0$, then the above identity takes the following form

$$\partial^z_{i,i+1}\Big((1 + z_{i+1}\ b)\ A_1(z_1)\cdots A_{n-1}(z_{n-1})\Big) = \Big(A_1(z_1)\cdots A_{n-1}(z_{n-1})\Big)\ (e_i - b).$$

In other words the above identity is equivalent to the statement [7] that in the IdCoxeter algebra $\mathcal{IC}_n$ the Cauchy kernel $\mathcal{C}(\mathfrak{P}_n, U)$ is the generating function for the b- Grothendieck polynomials. Moire over, each (generalized) double b-Grothendieck polynomial is a positive linear combination of the key- Grothendieck polynomials. In the special case $b = -1$ and $P_{ij} = x_i + y_j$ if $2 \leq i + j \leq n + 1$, $p_{ij} = 0$, $if\ i + j > n + 1$ this result had been stated in [30].

As a possible mean to define *affine versions* of polynomials treated in the present paper, we introduce the *double affine nilCoxeter algebra of type A* and give construction of a generic family of Hecke's type elements [4] we will be put to use in the present paper.

[4] Remind that by the name *a family of Hecke's type elements* we mean a set of elements $\{e_1, \cdots, c_n\}$ such that

- (Hecke type relations) $e_i^2 = A\ e_i + B$, A, B are parameters,
- (Coxeter relations)
  $c_i\ c_j = c_j\ c_i, \quad if \quad |i - j| \geq 2, \quad e_i\ e_j\ e_i = e_j\ e_i\ e_j, \quad if \quad |i - j| = 1.$

As Appendix we include several examples of polynomials studied in the present paper to illustrate results obtained in these notes. We also include an expository text concerning the *MacNeille completion* of a poset to draw attention of the reader to this subject. It is the MacNeille completion of the poset associated with the (strong) Bruhat order on the symmetric group, that was one of the main streams of the study in the present paper.

A bit of history. Originally these notes have been designed as a continuation of [6]. The main purpose was to extend the methods developed in [8] to obtain by the use of plactic algebra, a noncommutative generating function for the key (or Demazure) polynomials introduced by A. Lascoux and M.-P. Schützenberger [25]. The results concerning the polynomials introduced in Section 4, except the Hecke– Grothendieck polynomials, see Definition 4.6, has been presented in my lecture-courses "Schubert Calculus" have been delivered in the Graduate School of Mathematical Sciences, the University of Tokyo, November 1995 -April 1996, and in the Graduate School of Mathematics, Nagoya University, October 1998 - April 1999. I want to thank Professor M. Noumi and Professor T. Nakanishi who made these courses possible. Some early versions of the present notes are circulated around the world and now I was asked to put it for the wide audience. I would like to thank Professor M. Ishikawa (Department of Mathematics, Faculty of Education, University of the Ryukyus, Okinawa, Japan) and Professor S.Okada (Graduate School of Mathematics, Nagoya University, Nagoya, Japan) for valuable comments.

2 Plactic, nilplactic and idplactic algebras

Definition 2.1 *([20])* *The* **plactic algebra** $\mathcal{P}_n$ *is an (unital) associative algebra over* $\mathbb{Z}$ *generated by elements* $\{u_1, \cdots, u_{n-1}\}$ *subject to the set of relations*

$$(PL1) \qquad u_j\ u_i\ u_k = u_j\ u_k\ u_i, \quad u_i\ u_k\ u_j = u_k\ u_i\ u_j, \ if \quad i < j < k,$$

$$(PL2) \qquad u_i\ u_j\ u_i = u_j\ u_i\ u_i, \quad u_j\ u_i\ u_j = u_j\ u_j\ u_i, \ if \quad i < j.$$

Proposition 2.2 *([20])Tableau words in the alphabet* $U = \{u_1, \cdots, u_{n-1}\}$ *form a basis in the plactic algebra* $\mathcal{P}_n$.

In other words, each plactic class contain a unique tableau word. In particular,

$$Hilb(\mathcal{P}_n, t) = (1-t)^{-n}(1-t^2)^{-\binom{n}{2}}.$$

Remark 2.3 There exists another algebra over $\mathbb{Z}$ which has the same Hilbert series as that of the plactic algebra $\mathcal{P}_n$. Namely, define algebra $\mathcal{L}_n$ to be an associative algebra over $\mathbb{Z}$ generated by the elements $\{e_1, e_2, \ldots, e_{n-1}\}$, subject to the set of relations

$$(e_i, (e_j, e_k)) := e_i\ e_j\ e_k - e_j\ e_i\ e_k - e_j\ e_k\ e_i + e_k\ e_j\ e_i = 0, \quad for \ \ all \ \ 1 \le i, j, k \le n-1, \ \ j < k.$$

Note that the number of defining relations in the algebra $\mathcal{L}_n$ is equal to $2\binom{n}{3}$.
One can show that the dimension of the degree k homogeneous component $\mathcal{L}_n^{(k)}$ of the algebra $\mathcal{L}_n$ is equal to the number semistandard Young tableaux of the size k filled by the numbers from the set $\{1, 2, \ldots, n\}$. ■

Definition 2.4 *The* **local plactic algebra** $\mathcal{LP}_n$ *is an associative algebra over* $\mathbb{Z}$ *generated by elements* $\{u_1, \ldots, u_{n-1}\}$ *subject to the set of relations*

$$u_i\ u_j = u_j\ u_i, \quad if \quad |i-j| \geq 2, \qquad u_j u_i^2 = u_i u_j u_i, \quad u_j^2 u_i = u_j u_i u_j, \quad if \quad j = i+1.$$

One can show (A.K) that

$$Hilb(\mathcal{LP}_n, t) = \prod_{j=1}^{n} \Big(\frac{1}{1-t^j}\Big)^{n+1-j}.$$

Definition 2.5 *(***Nil Temperley-Lieb algebra***)*

Denote by $\mathcal{TL}_n^{(0)}$ *the quotient of the local plactic algebra* $\mathcal{LP}_n$ *by the two-sided ideal generated by the elements* $\{u_1^2, \ldots, u_{n-1}^2\}$.

It is well-known that $dim\mathcal{TL}_n = C_n$, the n-th Catalan number. One also has
$Hilb(\mathcal{TL}_4^{(0)}, t) = (1, 3, 5, 4, 1)$, $Hilb(\mathcal{TL}_5^{(0)}, t) = (1, 4, 9, 12, 10, 4, 2)$,
$Hilb(\mathcal{TL}_6^{(0)}, t) = (1, 5, 14, 25, 31, 26, 16, 9, 4, 1)$.

Proposition 2.6

The Hilbert polynomial $Hilb(\mathcal{TL}_n^{(0)}, t)$ *is equal to the generating function for the number of* 321*-avoiding permutations of the set* $\{1, 2, ..., n\}$ *having inversion number equal to k, see [39],* A140717*, for other combinatorial interpretations of polynomials* $Hilb(\mathcal{TL}_n^{(0)}, t)$.

We denote by $\mathcal{TL}_n^{(\beta)}$ the quotient of the local plactic algebra $\mathcal{LP}_n$ by the two-sided ideal generated by the elements $\{u_1^2 - \beta\ u_1, \ldots, u_{n-1}^2 - \beta\ u_{n-1}\}$.

Definition 2.7 *The* **modified plactic algebra** $\mathcal{MP}_n$ *is an associative algebra over* $\mathbb{Z}$ *generated by* $\{u_1, \ldots, u_{n-1}\}$ *subject to the set of relations* $(PL1)$ *and that*

$$u_j u_j u_i = u_j u_i u_i \ \ and \ \ u_i u_j u_i = u_j u_i u_j, \quad if \quad 1 \leq i < j \leq n-1.$$

Definition 2.8 *The* **(reduced) nilplactic algebra** $\mathcal{NP}_n$ *is an associative algebra over* $\mathbb{Q}$ *generated by* $\{u_1, \cdots, u_{n-1}\}$ *subject to the relations* [5]

$$u_i^2 = 0, \quad u_i\ u_{i+1}\ u_i = u_{i+1}\ u_i\ u_{i+1}, \tag{2.4}$$

[5]Original definition of the nilplactic relations given in [23] involves only relations $(PL1)$ and

$$u_i u_{i+1} u_i \cong u_{i+1} u_i u_{i+1} \quad \& \quad u_i u_i \cong 0, \quad i = 1, \ldots, n-1.$$

It had been shown [24] that the Schensted construction for the plactic congruence extends to the nilplactic case. However as it seen from the following example, as a consequence of relations $(PL1)$ one has

$$(u_1 + u_2 + u_3, u_2 u_1 + u_3 u_1 + u_3 u_2) \equiv u_1 u_3 u_1 - u_3 u_1 u_3 + u_3^2 u_1 - u_3 u_1^2,$$

and therefore noncommutattive elementary symmetric polynomials $e_1(u_1, u_2, u_3)$ and $e_2((u_1, u_2, u_3)$ do not commute modulo the nilplactic congruence defined in [23]. Indeed, $u_1 u_3 u_1 \not\equiv u_3 u_1 u_3$. In order to guarantee the commutativity of all noncommutative elementary polynomials, we add relations

$$x_i\ x_j\ x_i = 0, \quad if \quad |i-j| \geq 2.$$

Cf with definition of *idplactic* relations listed in Definition 2.11.

the set of relations $(PL1)$, *and that* $x_i\ x_j\ x_i = 0$, *if* $|i-j| \geq 2$.

Proposition 2.9 *([23]) Each nilplactic class not containing 0, contains one and only one tableau word.*

Proposition 2.10 *The nilplactic algebra* $\mathcal{NP}_n$ *has finite dimension, its Hilbert polynomial* $Hilb(\mathcal{NP}_n, t)$ *has degree* $\binom{n}{2}$ *and* $\dim(\mathcal{NP}_n)_{\binom{n}{2}} = 1$.

Example 2.11 $Hilb(\mathcal{NP}_3, t) = (1, 2, 2, 1)$, $Hilb(\mathcal{NP}_4, t) = (1, 3, 6, 6, 5, 3, 1)$,
$Hilb(\mathcal{NP}_5, t) = (1, 4, 12, 19, 26, 26, 22, 15, 9, 4, 1)$, $\dim(\mathcal{NP}_5 = 139$,
$Hilb(\mathcal{NP}_6, t) = (1, 5, 20, 44, 84, 119, 147, 152, 140, 114, 81, 52, 29, 14, 5, 1)$, $\dim(\mathcal{NP}_6) = 1008$.

Definition 2.12 *The* **idplactic algebra** $\mathcal{IP}_n^{(\beta)}$ *is an associative algebra over* $\mathbb{Q}(\beta)$ *generated by* $\{u_1, \cdots, u_{n-1}\}$ *subject to the relations*

$$u_i^2 = \beta u_i, \quad u_i\ u_j\ u_i = u_j\ u_i\ u_j, \qquad i < j, \tag{2.5}$$

and the set of relations $(PL1)$.

In other words, the idplactic algebra $\mathcal{IP}_n$ is the quotient of the plactic algebra $\mathcal{P}_n$ by the the two-sided ideal generated by elements $\{u_i^2 - \beta u_i, \quad 1 \leq i \leq n-1\}$.

Proposition 2.13 *Each idlplactic class contains a unique tableau word of the smallest length.*

For each word w denote by $rl(w)$ the length of a unique tableau word of minimal length which is idplactic equivalent to w.

Example 2.14 *Consider words in the alphabet* $\{a < b < c < d\}$. *Then* $rl(dbadc) = 4 = rl(cadbd)$, $rl(dbadbc) = 5 = rl(cbadbd)$. *Indeed,*

$$dbadc \sim dbdac \sim dbdca \sim ddbca \sim dbac,$$

$$dbadbc \sim dbabdc \sim dabadc \sim adbdac \sim abdbca \sim abbdca \sim dbabc.$$

Note that according to our definition, tableau words $w = 31$, $w = 13$ and $w = 313$ belong to different idplactic classes.

Proposition 2.15 *The idplactic algebra* $\mathcal{IP}_n^{(\beta)}$ *has finite dimension, and its Hilbert polynomial has degree* $\binom{n}{2}$.

Example 2.16
$Hilb(\mathcal{IP}_3, t) = (1, 2, 2, 1)$, $Hilb(\mathcal{IP}_4, t) = (1, 3, 6, 7, 5, 3, 1)$, $\dim(\mathcal{IP}_4) = 26$,
$Hilb(\mathcal{IP}_5, t) = (1, 4, 12, 22, 30, 32, 24, 15, 9, 4, 1)$, $\dim(\mathcal{IP}_5) = 154$,
$Hilb(\mathcal{IP}_6, t) = (1, 5, 20, 50, 100, 156, 188, 193, 173, 126, 84, 52, 29, 14, 5, 1)$, $\dim(\mathcal{IP}_6) = 1197$.

Definition 2.17 *The* **idplactic Temperly-Lieb algebra** $\mathcal{PTL}_n^{(\beta)}$ *is define to be the quotient of the idplactic algebra* $\mathcal{IP}_n^{(\beta)}$ *by the two-sided ideal generated by the elements*

$$\{u_i\ u_j\ u_i,\ \ \forall i \neq j\}.$$

For example, $Hilb(\mathcal{PTL}_4^{(0)}, t) = (1, 3, 6, 4, 1)_t$,, $Hilb(\mathcal{PTL}_5^{(0)}, t) = (1, 4, 12, 16, 14, 4, 2)_t$ $Hilb(\mathcal{PTL}_6^{(0)}, t) = (1, 5, 20, 40, 60, 46, 32, 10, 4, 1)_t$, $Hilb(\mathcal{PTL}_7^{(0)}, t) =$ $(1, 6, 30, 80, 170, 216, 238, 152, 96, 44, 14, 4, 2)_t$. One can show that $\deg_t Hilb(\mathcal{PTL}_n^{(0)}, t) = \left[\frac{n^2}{4}\right]$, and $Coeff_{t^{max}} Hilb(\mathcal{PTL}_n, t) = 1$, *if* n *is even, and* $= 2$, *if* n *is odd.*

Definition 2.18 *The* **nilCoxeter algebra** $\mathcal{NC}_n$ *is defined to be the quotient of the nilplactic algebra* $\mathcal{NP}_n$ *by the two-sided ideal generated by elements* $\{u_i\ u_j - u_j\ u_i,\ \ |i-j| \geq 2\}$.

Clearly the nilCoxeter algebra $\mathcal{NC}_n$ is a quotient of the modified plactic algebra $\mathcal{MP}_n$ by the two-sided ideal generated by the elements $\{u_i u_j - u_j u_i,\ \ |i-j| \geq 2\}$.

Definition 2.19 *The* **idCoxeter algebra** $\mathcal{IC}_n^{(\beta)}$ *is defined to be the quotient of the idplactic algebra* $\mathcal{IP}_n^{(\beta)}$ *by the two-sided ideal generated by the elements* $\{u_i\ u_j - u_j\ u_i,\ \ |i-j| \geq 2\}$.

It is well-known that the algebras $\mathcal{NC}_n$ and $\mathcal{IC}_n^{(\beta)}$ have dimension $n!$, and the elements $\{u_w := u_{i_1} \cdots u_{i_\ell}\}$, where $w = s_{i_1} \cdots s_{i_\ell}$ is any reduced decomposition of $w \in \mathbb{S}_n$, form a *basis* in the nilCoxeter and idCoxeter algebras $\mathcal{NC}_n$ and $\mathcal{IC}_n^{(\beta)}$.

Remark 2.20 There is a common generalization of the algebras defined above which is due to S.Fomin and C.Greene [5]. Namely, define generalized plactic algebra $\widetilde{\mathcal{P}}_n$ to be an associative algebra generated by elements $u_1, \cdots, u_{n-1}$, subject to the relations $(PL2)$ and relations

$$u_j u_i (u_i + u_j) = (u_i + u_j) u_j u_i,\ \ i < j. \tag{2.6}$$

The relation (2.5) can be written also in the form

$$u_j (u_i u_j - u_j u_i) = (u_i u_j - u_j u_i) u_i,\ i < j.$$

■

Theorem 2.21 *([5]) For each pair of numbers* $1 \leq i < j \leq n$ *define*

$$A_{i,j}(x) = \prod_{k=j}^{i} (1 + x\ u_k).$$

Then the elements $A_{i,j}(x)$ *and* $A_{i,j}(y)$ *commute in the generalized plactic algebra* $\widetilde{\mathcal{P}}_n$.

Corollary 2.22 *Let* $1 \leq i < j \leq n$ *be a pair of numbers. Noncommutative elementary polynomials* $e_a^{ij} := \sum_{j \geq i_1 \geq \cdots \geq i_k \geq i} u_{i_1} \cdots u_{i_a}$, $i \leq a \leq j$, *generate a commutative subalgebra* $\mathcal{C}_{i,j}$ *of rank* $j - i + 1$ *in the plactic algebra* $\mathcal{P}_n$.

Moreover, the algebra $\mathcal{C}_{1,n}$ *is a maximal commutative subalgebra of* $\mathcal{P}_n$.

To establish Theorem 2.20 , we are going to prove more general result. To start with, let us define *generic plactic algebra* $\mathfrak{P}_n$.

Definition 2.23 *The* **generic plactic algebra** $\mathfrak{P}_n$ *is an associative algebra over* $\mathbb{Z}$ *generated by* $\{e_1, \cdots, e_{n-1}\}$ *subject to the set of relations*

$$e_j(e_i, e_) = (e_i, e_j)e_i, \quad if \quad i < j, \tag{2.7}$$

$$(e_j, (e_i, e_k)) = 0, \quad if \quad i < j < k, \tag{2.8}$$

$$(e_j, e_k)(e_i, e_k) = 0, \quad if \quad i < j < k. \tag{2.9}$$

Clearly seen that relations (2.6)−(2.8) are consequence of the plactic relations $(PL1)$ and $(PL2)$.

Theorem 2.24 *Define*

$$A_n(x) = \prod_{k=j}^{1}(1 + x\ e_k).$$

Then the elements $A_n(x)$ *and* $A_n(y)$ *commute in the generic plactic algebra* $\mathfrak{P}_n$.

Moreover the elements $A_n(x)$ *and* $A_n(y)$ *commute if and only if the generators* $\{e_1, \ldots, e_{n-1}\}$ *satisfy the relations* $(2.6) - (2.8)$.

Proof For $n = 2, 3$ the statement of Theorem 1.22 is obvious. Now assume that the statement of Theorem 1.22 is true in the algebra $\mathfrak{P}_n$. We have to prove that the commutator $[A_{n+1}(x), A_{n+1}(y)]$ is equal to zero. First of all, $A_{n+1}(x) = (1 + xe_n)A_n(x)$. Therefore

$$[A_{n+1}(x), A_{n+1}(y)] = (1 + xe_n)\ [A_n(x), 1 + ye_n]\ A_n(y) - [A_n(y), 1 + xe_n]\ A_n(x).$$

Using the standard identity $[ab, c] = a[b, c] + [a, c]b$, one finds that

$$\frac{1}{xy}\ [A_n(x), 1 + ye_n] = \sum_{i=}^{n-1}\ \prod_{a=n-1}^{i+1}\ (1 + xe_a)\ (e_i, e_n)\ \prod_{a=i-1}^{1}\ (1 + xe_a).$$

Using relations (2.7) we can move the commutator (e_i, e_n) to the left, since $i < a < n$, till we meet the term $(1 + xe_n)$. Using relations (2.6) we see that $(1 + xe_n)(e_i, n) = (e_i, n)(1 + xe_i)$. Therefore we come to the following relation

$$\frac{1}{xy}\ [A_n(x), 1 + ye_n] =$$

$$\sum_{i=n-1}^{1}(e_i, e_n)\Big((1 + xe_i)\prod_{\substack{a=n-1\\a\neq i}}^{1}\ (1 + xe_a)\ A_n(y) - (1 + ye_i)\prod_{\substack{a=n-1\\a\neq i}}^{1}\ (1 + ye_a)\ A_n(x)\Big).$$

Finally let us observe that

$$(e_i, e_n)\Big((1+xe_i)(1+xe_{n-1})-(1+xe_{n-1})(1+xe_i)\Big) = x^2\ (e_i, e_n)(e_i, e_{n-1}) = 0,\ \ according\ to\ (2.8).$$

Indeed, $(e_i, e_n)(e_i, e_{n-1}) = (e_i, e_n)e_ie_{n-1} - (e_i, e_n)e_{n-1}e_i = e_ne_{n-1}(e_i, e_n) - e_{n-1}e_n(e_i, e_n) = 0$. Therefore $\frac{1}{xy}\,[A_n(x), 1 + ye_n] = \Big(\sum_{i=n-1}^{1}(e_i, e_n)\Big)[A_n(x), A_n(y)] = 0$ according to the induction assumption.

Finally, if $i < j$, then $(e_i + e_j, e_je_i) = 0 \Longleftrightarrow (2.6)$,
if $i < j < k$ and the relations (2.6) hold, then $(e_i + e_j + e_k, e_je_i + e_ke_j + e_ke_i) = 0 \Longleftrightarrow (2.7)$,
if $i < j < k$ and relations (2.6) and (2.7) hold, then $(e_i + e_j + e_k, e_ke_je_i) = 0 \Longleftrightarrow (2.8)$; the relations $(e_je_i + e_ke_j + e_ke_i, e_ke_je_i) = 0$ are a consequence of the above ones. ■

Let T be a semistandard tableau and $w(T)$ be the column reading word corresponding to the tableau T. Denote by $R(T)$ (resp. $IR(T)$) the set of words which are plactic (resp. idplactic) equivalent to $w(T)$. Let $\mathbf{a} = (a_1, \cdots, a_n) \in R(T)$, where $n := |T|$ (resp. $\mathbf{a} = (a_1, \cdots, a_m) \in IR(T)$, where $m \geq |T|$).

Definition 2.25 *(Compatible sequences* $\mathbf{b}$*) Given a word* $\mathbf{a} \in R(T)$ *(resp.* $\mathbf{a} \in IR(T)$*), denote by* $C(\mathbf{a})$ *(resp.* $IC(\mathbf{a})$*) the set of sequences of positive integers, called compatible sequences,* $\mathbf{b} := (b_1 \leq b_2 \leq \cdots \leq b_m)$ *such that*

$$b_i \leq a_i, \quad \textit{and if} \quad a_i \leq a_{i+1}, \quad \textit{then} \quad b_i < b_{i+1}. \tag{2.10}$$

Finally, define the set $C(T)$ (resp. $IC(T)$) to be the union $\bigcup\, C(\mathbf{a})$ (resp. the union $\bigcup\, IC(\mathbf{a})$), where $\mathbf{a}$ runs over all words which are plactic (resp. idplactic) equivalent to the word $w(T)$.

Example 2.26 *Take* $T = \begin{matrix} 2 & 3 \\ 3 \end{matrix}$*. The corresponding tableau word is* $w(T) = 323$*. We have* $R(T) = \{232,\ 323\}$ *and* $IR(T) = R(T) \bigcup \{2323,\ 3223,\ 3232,\ 3233,\ 3323,\ 32323, \cdots\}$*. Moreover,*

$$C(T) = \left\{ \begin{matrix} \mathbf{a}: & 232 & 323 & 323 & 323 & 323 \\ \mathbf{b}: & 122 & 112 & 113 & 123 & 223 \end{matrix} \right\},$$

$$IC(T) = C(T) \bigcup \left\{ \begin{matrix} \mathbf{a}: & 2323 & 3223 & 3232 & 3233 & 3323 & 32323 \\ \mathbf{b}: & 1223 & 1123 & 1122 & 1223 & 1123 & 11223 \end{matrix} \right\}.$$

Let $\mathfrak{P} := \mathfrak{P}_n := \{p_{i,j}, i \geq 1, j \geq 1, 2 \leq i + j \leq n + 1\}$ be the set of (mutually commuting) variables.

Definition 2.27 (1) *Let* T *be a semistandard tableau, and* $n := |T|$*. Define the double key polynomial* $\mathcal{K}_T(\mathfrak{P})$ *corresponding to the tableau* T *to be*

$$\mathcal{K}_T(\mathfrak{P}) = \sum_{\mathbf{b} \in C(T)} \prod_{i=1}^{n} p_{b_i, a_i - b_i + 1}. \tag{2.11}$$

(2) *Let* T *be a semistandard tableau, and* $n := |T|$*. Define the double key Grothendieck polynomial* $\mathcal{GK}_T(\mathfrak{P})$ *corresponding to the tableau* T *to be*

$$\mathcal{GK}_T(\mathfrak{P}) = \sum_{\mathbf{b} \in IC(T)} \prod_{i=1}^{m} p_{b_i, a_i - b_i + 1}. \tag{2.12}$$

In the case when $p_{i,j} = x_i + y_j, \quad \forall i, j,$ where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ denote two sets of variables, we will write $\mathcal{K}_T(X,Y), \mathcal{GK}_T(X,Y), \ldots,$ instead of $\mathcal{K}_T(\mathfrak{P}), \mathcal{GK}_T(\mathfrak{P}), \ldots$.

Definition 2.28 *Let T be a semistandard tableau, denote by $\alpha(T) = (\alpha_1, \cdots, \alpha_n)$ the exponent of the smallest monomial in the set $\{x^{\mathbf{b}} := \prod_{i=1}^{m} x_i^{b_i}, \ \mathbf{b} \in C(T)\}$ with respect to the lexicographic order.*

We will call the composition $\alpha(T)$ to be the bottom code of tableau T.

3 Divided difference operators

In this subsection we remind some basic properties of divided difference operators will be put to use in subsequent Sections. For more details, see [34].

Let f be a function of the variables x and y (and possibly other variables), and $\eta \neq 0$ be a parameter. Define the **divided difference operator** $\partial_{xy}(\eta)$ will as follows

$$\partial_{xy}(\eta)\ f(x,y) = \frac{f(x,y) - f(\eta^{-1}\ y, \eta\ x)}{x - \eta^{-1}\ y}.$$

Equivalently, $(x - \eta^{-1}\ y)\ \partial_{xy}(\eta) = 1 - s^{\eta}_{xy}$, where the operator s^{η}_{xy} acts on the variables $(x, y, \ldots)$ according to the rule: s^{η}_{xy} transforms the pair (x,y) to $(\eta^{-1}\ y, \eta\ x)$, and fixes all other variables. We set by definition, $s^{\eta}_{yx} := s^{\eta^{-1}}_{xy}$.

The operator $\partial_{xy}(\eta)$ takes polynomials to polynomials and has degree -1. The case $\eta = 1$ corresponds to the *Newton divided difference operator* $\partial_{xy} := \partial_{xy}(1)$.

Lemma 3.1

(0) $s^{\eta}_{xy}\ s^{\eta\xi}_{xz} = s^{\xi}_{yz}\ s^{\eta}_{xy}, \qquad s^{\eta}_{xy}\ s^{\eta\xi}_{xz}\ s^{\xi}_{yz} = s^{\xi}_{yz}\ s^{\eta\xi}_{xz}\ s^{\eta}_{xy},$

(1) $\partial_{yx}(\eta) = -\eta\ \partial_{xy}(\eta^{-1}), \quad s^{\eta}_{xy}\ \partial_{yz}(\xi) = \eta^{-1}\partial_{xz}(\eta\ \xi)\ s^{\eta}_{xy},$

(2) $\partial_{xy}(\eta)^2 = 0,$

(3) *(Three term relation)*

$\partial_{xy}(\eta)\ \partial_{yz}(\xi) = \eta^{-1}\ \partial_{xz}(\eta\ \xi)\ \partial_{xy}(\eta) + \partial_{yz}(\xi)\ \partial_{xz}(\eta\ \xi).$

(4) *(Twisted Leibniz rule)*

$\partial_{xy}(\eta)\ (fg) = \partial_{xy}(\eta)\ (f)\ g + s^{\eta}_{xy}(f)\ \partial_{xy}(\eta)\ (g),$

(5) *(Crossing relations, cf [7], (4.6))*

- $x\ \partial_{xy}(\eta) = \eta^{-1}\partial_{xy}(\eta)\ y + 1, \quad y\ \partial_{xy}(\eta) = \eta\ \partial_{xy}(\eta)\ x - \eta,$
- $\partial_{xy}(\eta)\ y\ \partial_{yz}(\xi) = \partial_{xz}(\eta\ \xi)\ x\ \partial_{xy}(\eta) + \xi^{-1}\partial_{yz}(\xi)\ z\ \partial_{xz}(\eta\ \xi),$

(6) $\partial_{xy}\ \partial_{xz}\ \partial_{yz}\ \partial_{xz} = 0.$

Let $x_1, \ldots, x_n$ be independent variables, and let $P_n := \mathbb{Q}[x_1, \ldots, x_n]$. For each $i < j$ put $\partial_{ij} := \partial_{x_i\ x_j}(1)$ and $\partial_{ji} = -\partial_{ij}$. From Lemma 2.1 we have

$$\begin{cases} \partial_{ij}^2 = 0, \\ \partial_{ij}\ \partial_{jk} + \partial_{ki}\ \partial_{ij} + \partial_{jk}\ \partial_{ki} = 0, \\ \partial_{ij}\ x_j\ \partial_{jk} + \partial_{ki}\ x_i\ \partial_{ij} + \partial_{jk}\ x_k\ \partial_{ki}, \quad if \quad i,j,k \quad are \quad distinct. \end{cases}$$

It is interesting to consider also an *additive or affine* analog $\partial_{xy}[k]$ of the divided difference operators $\partial_{xy}(\eta)$, namely,

$$\partial_{xy}[k](f(x,y)) = \frac{f(x,y) - f(y-k, x+k)}{x-y+k}.$$

We have $\partial_{yx}[k] = -\partial_{xy}[-k]$, and
$\partial_{xy}[p]\ \partial_{yz}[q] = \partial_{xz}[p+q]\ \partial_{xy}[p] + \partial_{yz}[q]\ \partial_{xz}[p+q]$.

4 Schubert, Grothendieck and Key polynomials

Let $w \in \mathbb{S}_n$ be a permutation, $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n)$ be two sets of variables. Denote by $w_0 \in \mathbb{S}_n$ the longest permutation, and by $\delta := \delta_n = (n-1, n-2, \cdots, 1)$ the staircase partition. For each partition λ define

$$R_\lambda(X,Y) := \prod_{(i,j)\in\lambda} (x_i + y_j).$$

For $i = 1, \cdots, n-1$, let $s_i = (i, i+1) \in \mathbb{S}_n$ denote the simple transposition that interchanges i and $i+1$ and fixes all other elements of the set $\{1, \cdots, n\}$. If $\alpha = (\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_n)$ is a composition, we will write

$$s_i\alpha = (\alpha_1, \cdots, \alpha_{i+1}, \alpha_i, \cdots, \alpha_n)$$

Definition 4.1

- *For each permutation $w \in \mathbb{S}_n$* **the double Schubert polynomial** $\mathfrak{S}_w(X,Y)$ *is defined to be*

$$\partial^{(x)}_{w^{-1}\ w_0}(R_{\delta_n}(X,Y)).$$

Let α be a composition.

- **The key polynomials** $K[\alpha](X)$ *are defined recursively as follows:*
if α is a partition, then $K[\alpha](X) = x^\alpha$;
otherwise, if α and i are such that $\alpha_i < \alpha_{i+1}$, then

$$K[s_i(\alpha)](X) = \partial_i\Big(x_i\ K[\alpha](X)\Big).$$

- **The reduced key polynomials** $\widehat{K}[\alpha](X)$ *are defined recursively as follows:*
if α is a partition, then $\widehat{K}[\alpha](X) = K[\alpha](X) = x^\alpha$;
otherwise, if α and i are such that $\alpha_i < \alpha_{i+1}$, then

$$\widehat{K}[s_i(\alpha)](X) = x_{i+1}\ \partial_i\Big(\widehat{K}[\alpha](X)\Big).$$

- *For each permutation $w \in \mathbb{S}_n$ the* **double β-Grothendieck polynomial** $\mathcal{G}^\beta_w(X,Y)$ *is defined recursively as follows:*

if $w = w_0$ is the longest element, then $\mathcal{G}_{w_0}(X,Y) = R_\delta(X,Y)$;
if w and i are such that $w_i > w_{i+1}$, i.e. $l(ws_i) = l(w) - 1$, then

$$\mathcal{G}^\beta_{ws_i}(X,Y) = \partial_i^{(x)}\Big((1+\beta\ x_{i+1})\ \mathcal{G}^\beta_w(X,Y)\Big).$$

- *For each permutation $w \in \mathbb{S}_n$ the* **double dual β-Grothendieck polynomial** *$\mathcal{H}^\beta_w(X,Y)$ is defined recursively as follows:*
if $w = w_0$ is the longest element, then $\mathcal{H}_{w_0}(X,Y) = R_\delta(X,Y)$;
if w and i are such that $w_i > w_{i+1}$, i.e. $l(ws_i) = l(w) - 1$, then

$$\mathcal{H}^\beta_{ws_i}(X,Y) = (1+\beta\ x_i)\ \partial_i^{(x)}\Big(\mathcal{H}^\beta_w(X,Y)\Big).$$

- **The key β-Grothendieck polynomials** *$KG[\alpha](X;\beta)$ are defined recursively as follows:*
if α is a partition, then $KG[\alpha](X;\beta) = x^\alpha$;
otherwise, if α and i are such that $\alpha_i < \alpha_{i+1}$, then

$$KG[s_i(\alpha)](X;\beta) = \partial_i\Big((x_i + \beta\ x_i\ x_{i+1})\ KG[\alpha](X;\beta)\Big).$$

- **The reduced key β-Grothendieck polynomials** *$\widehat{KG}[\alpha](X;\beta)$ are defined recursively as follows:*
if α is a partition, then $\widehat{KG}[\alpha](X;\beta) = x^\alpha$;
otherwise, if α and i are such that $\alpha_i < \alpha_{i+1}$, then

$$\widehat{KG}[s_i(\alpha)](X;\beta) = (x_{i+1} + \beta\ x_i\ x_{i+1})\ \partial_i\Big(\widehat{KG}[\alpha](X;\beta)\Big).$$

For brevity, we will write $KG[\alpha](X)$ and $\widehat{KG}[\alpha](X)$ instead of $KG[\alpha](X;\beta)$ and $\widehat{KG}[\alpha](X;\beta)$.

Remark 4.2 We can also introduce polynomials $\mathcal{Z}_w$, which are defined recursively as follows:
if $w = w_0$ is the longest element, then $\mathcal{Z}_{w_0}(X) = x^\delta$;
if w and i are such that $w_i > w_{i+1}$, i.e. $l(ws_i) = l(w) - 1$, then

$$\mathcal{Z}_{ws_i}(X) = \partial_i\Big((x_{i+1} + x_i x_{i+1})\ \mathcal{Z}_w(X)\Big).$$

However, one can show that

$$\mathcal{Z}_w(x_1,\cdots,x_n) = (x_1\cdots x_n)^{n-1}\mathcal{G}_{w_0 w w_0}(x_n^{-1},\cdots,x_1^{-1}).$$

Theorem 4.3 *The polynomials $\mathfrak{S}_w(X,Y)$, $K[\alpha](X)$, $\widehat{K}[\alpha](X)$, $\mathcal{G}_w(X,Y)$, $\mathcal{H}_w(X,Y)$, $KG[\alpha](X)$ and $\widehat{KG}[\alpha](X)$ have* **nonnegative** *integer coefficients.*

We will use notation $\mathfrak{S}_w(X)$, $\mathcal{G}_w(X)$, ..., for polynomials $\mathfrak{S}_w(X,0)$, $\mathcal{G}_w(X,0)$, ...

- **Di Francesco–Zin-Justin polynomials**)

Definition 4.4 *For each permutation $w \in \mathbb{S}_n$ the* **Di Francesco-Zinn-Justin polynomials** $\mathcal{DZ}_w(X)$ *are defined recursively as follows:*
if w is the longest element in $\mathbb{S}_n$, then $\mathcal{DZ}_w(X) = R_\delta(X, 0)$;
otherwise, if w and i are such that $w_i > w_{i+1}$, i.e. $l(ws_i) = l(w) - 1$, then

$$\mathcal{DZ}_{ws_i}(X) = \Big((1 + x_i)\partial_i^{(x)} + \partial_i^{(x)}(x_{i+1} + x_i x_{i+1})\Big)\mathcal{DZ}_w(X).$$

Conjecture 4.5
(1) *Polynomials $\mathcal{DZ}_w(X)$ have* **nonnegative** *integer coefficients.*
(2) *For each permutation $w \in \mathbb{S}_n$ the polynomial $\mathcal{DZ}_w(X)$ is a linear combination of key polynomials $K[\alpha](X)$ with* **nonnegative** *integer coefficients.*

As for definition of the double Di Francesco–Zin-Justin polynomials $\mathcal{DZ}_w(X, Y)$ they are well defined, but may have negative coefficients.

- (**Hecke–Grothendieck** polynomials)
  Let β and α be two parameters, consider divided difference operator

$$T_i := T_i^{\beta,\alpha} = -\beta + ((\beta + \alpha) + 1 + \beta\ \alpha\ x_i x_{i+1})\ \partial_{i,i+1}.$$

Definition 4.6
Let $w \in \mathbb{S}_n$, define **Hecke–Grothendieck** *polynomials $\mathcal{KN}_w^{\beta,\alpha}(X_n)$ to be*

$$\mathcal{KN}_w^{(\beta,\alpha)}(X_n) := T^{\beta,\alpha}_{w^{-1}w_0}(x^{\delta_n}),$$

where as before $x^{\delta_n} := x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$; if $u \in \mathbb{S}_n$, then set

$$T_u^{\beta,\alpha} := T_{i_1}^{\beta,\alpha}\cdots T_{i_\ell}^{\beta,\alpha},$$

where $u = s_{i_1}\cdots s_{i_\ell}$ is any reduced decomposition of a permutation taken.

- *More generally, let β, α and γ be parameters, consider divided difference operators*

$$T_i := T_i^{\beta,\alpha,\gamma} = -\beta + ((\alpha+\beta+\gamma)\ x_i + \gamma\ x_{i+1} + 1 + (\beta+\gamma)(\alpha+\gamma)x_i\ x_{i+1})\partial_{i,i+1}, \quad i = 1, \ldots, n-1.$$

For a permutation $w \in \mathbb{S}_n$ define polynomials

$$\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(X_n) := T_{i_1}^{\beta,\alpha,\gamma}\cdots T_{i_\ell}^{\beta,\alpha,\gamma}(x^{\delta_n}),$$

Remark 4.7 *A few comments in order.*
(a) The divided difference operators $\{T_i := T_{i_1}^{(\beta,\alpha,\gamma)},\ i = 1, \cdots, n-1\}$ satisfy the following relations

- *(Hecke relations)*

$$T_i^2 = (\alpha - \beta)\ T_i + \alpha\ \beta,$$

- *(Coxeter relations)*

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i, \quad if \quad |i - j| \geq 2.$$

Therefore the elements $T_w^{\beta,\alpha}$ are well defined for any $w \in \mathbb{S}_n$.

- *(Inversion)*

$$(1 + xT_i)^{-1} = \frac{1 + (\alpha - \beta)x - xT_i}{(1 - \beta\ x)(1 + \alpha\ x)}.$$

(b) Polynomials $\mathcal{KN}_w^{(\beta,\alpha,\gamma)}$ constitute a common generalization of the β-Grothendieck polynomials , namely, $\mathcal{G}_w^{(\beta)} = \mathcal{KN}_w^{(\beta,\alpha=0,\gamma=0)}$, the Di Francesco–Zin-Justin polynomials, namely, $\mathcal{DZ}_w = \mathcal{KN}_w^{(\beta=\alpha=1,\gamma=0)}$, the dual α-Grothendieck polynomials, namely, $\mathcal{KN}_w^{(\beta=0,\alpha,\gamma=0)} = \mathcal{H}_w^{\alpha}(X)$.

Proposition 4.8

- *(Duality) Let $w \in \mathbb{S}_n$, $\ell = \ell(w)$ denotes its length, then ($\alpha\ \beta \neq 0$)*

$$\mathcal{KN}_w^{(\beta,\alpha)}(1) = (\beta\alpha)^{\ell}\ \mathcal{KN}_{w^{-1}}^{(\alpha^{-1},\beta^{-1})}(1).$$

It is well-known that

- *the number $\mathcal{KN}_{w_0}^{(\beta=1,\alpha=1)}(1)$ is equal to the degree of the variety of pairs commuting matrices of size $n \times n$,*
- *the bidegree of the affine homogeneous variety V_w, $w \in \mathbb{S}_n$, [10], is equal to*

$$A^{\binom{n}{2}-\ell(w)}\ B^{\binom{n}{2}+\ell(w)}\ \mathcal{KN}_w^{(\beta=\alpha=A/B)}(1).$$

see [10], [17], [9] for more details and applications.

Conjecture 4.9

- *Polynomials $\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(X)$ have* **nonnegative** *integer coefficients,*

$$\mathcal{KN}_w^{(\beta,\alpha,\gamma)}(X) \in \mathbb{N}[\beta,\alpha,\gamma][X_n]$$

. • Double polyonomials $\mathcal{KN}_w^{(\beta=0,\alpha,\gamma)}(X,Y) = T_w^{\beta=0,\alpha,\gamma}(x)\prod_{\substack{i+j\leq n+1\\ i\geq 1, j\geq 1}}(x_i + y_j)$ are well defined and have **nonnegative** *integer coefficients.*

■

Theorem 4.10 *Let T be a semistandard tableau and $\alpha(T)$ be its bottom code, see Definition 2.27 Then*

$$\mathcal{K}_T(X) = K[\alpha(T)](X), \quad \mathcal{KG}_T(X) = KG[\alpha(T)](X).$$

Let $\alpha = (\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r)$ be a composition, define partition $\alpha^+ = (\alpha_r \geq \cdots \geq \alpha_1)$.

Proposition 4.11 *If $\alpha = (\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r)$ is a composition and $n \geq r$, then*

$$K[\alpha](X_n) = s_{\alpha^+}(X_r).$$

For example, $K[0,1,2,\cdots,n-1] = \prod_{1\leq i<j\leq n}(x_i + x_j)$. Note that $\widehat{K}[0,1,2,\cdots,n-1] = \prod_{i=2}^{n} x_i^{i-1}$.

Proposition 4.12 *If $\alpha = (\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r)$ is a composition and $n \geq r$, then*

$$KG[\alpha](X_n) = \mathcal{G}[\alpha^+](X_r).$$

For example, $KG[0,1,2,\cdots,n-1] = \prod_{1\leq i<j\leq n}(x_i + x_j + x_i x_j)$. Note that $\widehat{KG}[0,1,2,\cdots,n-1] = \prod_{i=2}^{n} x_i^{i-1} \ \prod_{i=1}^{n-1}(1+x_i)^{n-i}$.

Comments 4.1

Definition 4.13 *Define degenerate affine 2d nil-Coxeter algebra $\mathcal{ANC}_n^{(2)}$ to be an associative algebra over $\mathbb{Q}$ generated by the set of elements $\{\{u_{i,j}\}_{1\leq i<j\leq n} \ \text{ and } \ x_1,\ldots,x_n\}$ subject to the set of relations*

- $x_i x_j = x_j x_i$ *for all* $i \neq j$, $x_i \ u_{j,k} = u_{j,k} \ x_i$, *if* $i \neq j,k$,
- $u_{i.j} \ u_{k,l} = u_{k,l} \ u_{i.j}$, *if* i,j,k,l *are pairwise distinct,*
- *(2d-Coxeter relations)* $u_{i,j} \ u_{j,k} \ u_{i,j} = U_{j,k} \ u_{i,j} \ u_{j,k}$, *if* $1 \leq i < j < k \leq n$,
- $x_i \ u_{i.j} = u_{i,j} \ x_j + 1$, $x_j \ u_{i.j} = u_{i,j} \ x_i - 1$.

Now for a set of parameters [6] $A := (a,b,c,h,e)$ define elements

$$T_{ij} := a + (bx_i + cx_j + h + e \ x_i \ x_j) \ u_{i,j} \quad i < j.$$

Lemma 4.14

(1) $T_{i,j}^2 = (2a+b-c) \ T_{i,j} - a(a+b-c)$,
if $a = 0$, *then* $T_{ij}^2 = (b-c) \ T_{ij}$.
(2) *(2d-Coxeter relations) Relations*

$$T_{i,j} \ T_{j,k} \ T_{i,j} = T_{j,k} \ T_{i,j} \ T_{j,k},$$

are valid, if and only if the following relation among parameters a,b,c,e,h holds

$$(a+b)(a-c) + h \ e = 0. \tag{4.13}$$

(3) *(Yang–Baxter relations) Relations*

$$T_{i,j} \ T_{i,k} \ T_{j.k} = T_{j,k} \ T_{i,k} \ T_{i,j}$$

are valid if and only if $b = c = e = 0$, *i.e.* $T_{ij} = a + d \ u_{ij}$.
(4) $T_{ij}^2 = 1$ *if and only if* $a = \pm 1, c = b \pm 2$, $he = (b \pm 1)^2$.
(5) *Assume that parameters a,b,c,h,e satisfy the conditions* (4.13) *and that* $b \ c+1 = h \ e$. *Then*

$$T_{ij} \ x_i \ T_{ij} = (h \ e - b \ c) \ x_j + (h + (a+b)(x_i + x_j) + e \ x_i \ x_j) \ T_{ij}.$$

Some special cases

- *(Representation of affine modified Hecke algebra [42])*
  If $A = (a,-a,c,h,0)$, *then* $T_{ij} \ x_i \ T_{ij} = a \ c \ x_j + h \ T_{ij}$, $i < j$,

[6] By definition, a **parameter** assumed to be belongs to the center of the algebra in question

- *If* $A = (-a, a+b+c, c, 1, (a+c)(b+c)$, *then*

$T_{ij}\ x_i\ T_{ij} = a\ b\ x_j + (1 + (b+c)(x_i + x_j) + (a+c)(b+c)x_i\ x_j)\ T_{ij}$.

(6) *(Quantum Yang–Baxterization) Assume that parameters* a, b, c, h, e *satisfy the conditions* (4.13) *and that* $\beta := 2a + b - c \neq 0$. *Then (cf [28], [13] and the literature quoted therein)*

the elements $R_{ij}(u, v) := 1 + \frac{\lambda - \mu}{\beta\mu}\ T_{ij}$ *satisfy the* twisted *quantum Yang–Baxter relations*

$$R_{ij}(\lambda_i, \mu_j)R_{jk}(\lambda_i, \nu_k)R_{ij}(\mu_j, \nu_k) = R_{jk}(\mu_j, \nu_k)R_{ij}(\lambda_i, \nu_k)R_{jk}(\lambda_i, \mu_j), \quad i < j < k,$$

where $\{\lambda_i, \mu_i, \nu_i\}_{1 \leq i \leq n}$ *are parameters. .*

Corollary 4.15 *If* $(a+b)(a-c) + he = 0$, *then for any permutation* $w \in \mathbb{S}_n$ *the element*

$$T_w := T_{i_1} \cdots T_{i_l} \in \mathcal{ANC}_n^{(2)},$$

where $w = s_{i_1} \cdots s_{i_l}$ *is any reduced decomposition of* w, *is* well-defined.

Example 4.16

Each of the set of elements

$$s_i^{(h)} = 1 + (x_{i+1} - x_i + h)\ u_{i,i+1} \quad \textit{and}$$

$$t_i^{(h)} = -1 + (x_i - x_{i+1} + h(1 + x_i)(1 + x_{i+1})u_{ij}, \quad i = 1, \ldots, n-1,$$

by itself generate the symmetric group $\mathbb{S}_n$.

Comments 4.2 Let $A = (a, b, c, h, e)$ be a sequence of integers satisfying the conditions (4.5). Denote by ∂_i^A the divided difference operator

$$\partial_i^A = a + (b\ x_i + c\ x_{i+1} + h + e\ x_i\ x_{i+1})\ \partial_i, \quad i = 1, \ldots, n-1.$$

It follows from Lemma 4.10 that the operators $\{\partial_i^A\}_{1 \leq i \leq n}$ satisfy the Coxeter relations

$$\partial_i^A\ \partial_{i+1}^A\ \partial_i^A = \partial_{i+1}^A\ \partial_i^A\ \partial_{i+1}^A, \quad i = 1, \ldots, n-1.$$

Definition 4.17

(1) *Let* $w \in \mathbb{S}_n$ *be a permutation. Define the generalized Schubert polynomial corresponding to permutation* w *as follows*

$$\mathfrak{S}_w^A(X_n) = \partial_{w^{-1}\ w_0}^A\ x^{\delta_n}, \quad \textit{where} \quad x^{\delta_n} := x_1^{n-1}\ x_2^{n-2} \cdots x_{n-1},$$

and w_0 *denotes the longest element in the symmetric group* $\mathbb{S}_n$.

(2) *Let* α *be a composition with at most* n *parts, denote by* $w_\alpha \in \mathbb{S}_n$ *the permutation such that* $w_\alpha(\alpha) = \overline{\alpha}$, *where* $\overline{\alpha}$ *denotes a unique partition corresponding to composition* α.

Lemma 4.18 *Let $w \in \mathbb{S}_n$ be a permutation.*

- *If $A = (0, 0, 0, 1, 0)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the Schubert polynomial $\mathfrak{S}_w(X_n)$.*
- *If $A = (-\beta, \beta, 0, 1, 0)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X_n)$ introduced in [6].*
- *If $A = (0, 1, 0, 1, 0)$ then $\mathfrak{S}_w^A(X_n)$ is equal to the dual Grothendieck polynomial.*
- *If $A = (-1, 2, 0, 1, 1)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the Di-Francesco–Zinn-Justin polynomials introduced in [10].*
- *If $A = (1, -1, 1, h, 0)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the h-Schubert polynomials.*

In all cases listed above the polynomials $\mathfrak{S}_w^A(X_n)$ have non-negative integer coefficients. .

Define the generalized key or Demazure polynomial corresponding to a composition α as follows

$$K_\alpha^A(X_n) = \partial_{w_\alpha}^A \, x^{\overline{\alpha}}.$$

- If $A = (1, 0, 1, 0, 0)$, then $K_\alpha^A(X_n)$ is equal to key (or Demazure) polynomial corresponding to α.
- If $A = (0, 0, 1, 0, 0)$, then $K_\alpha^A(X_n)$ is equal to the reduced key polynomial.
- If $A = (1, 0, 1, 0, \beta)$, then $K_\alpha^A(X_n)$ is equal to the key Grothendieck polynomial $KG_\alpha(X_n)$.
- If $A = (0, 0, 1, 0, \beta)$, then $K_\alpha^A(X_n)$ is equal to the reduced key Grothendieck polynomials.

In all cases listed above the polynomials $\mathfrak{S}_w^A(X_n)$ have non-negative integer coefficients. .

- If $A = (-\beta, \beta + \alpha, 0, 1, \beta\alpha)$, then $\mathfrak{S}_w^A(X_n)$ constitutes a common generalization of the Grothendieck and the Di Francesco–Zin-Justin polynomials.

Comments 4.3 *(Double affine NilCoxeter algebra) Let t, q, a, b, c, h, d be parameters.*

Definition 4.19 *Define double affine nil-Coxeter algebra $DANC_n$ to be (unital) associative algebra over $\mathbb{Q}(q^{\pm 1}, t^{\pm 1})$ with the set of generators $\{e_1, \ldots, e_{n-1},\ x_1, \ldots, x_n,\ \pi^{\pm 1}\}$ subject to relations*

- *(NilCoxeter relations)*

$$e_i\ e_j = e_j\ e_i, \quad if \quad |i-j| \geq 2, \quad e_i^2 = 0,\ \forall i, \quad e_i\ e_j\ e_i = e_j\ e_i\ e_j, \quad if \quad |i-j| = 1;$$

- *(Crossing relations)*

$$x_i\ e_k = e_k\ x_i, \quad if \quad k \neq i, i+1, \quad x_i\ e_i - e_i\ x_{i+1} = 1, \quad e_i\ x_i - x_{i+1}\ e_i = 1;$$

- *(Affine crossing relations)*

$$\pi\ x_i = x_{i+1}\ \pi, \quad if \quad i < n, \quad \pi\ x_n = q^{-1}\ x_1\ \pi,$$

$$\pi\ e_i = e_{i+1}\ \pi, \quad if \quad i < n-1, \quad \pi^2\ e_{n-1} = q\ e_1\ \pi^2.$$

Now let us introduce elements $e_0 := \pi\ e_{n-1}\ \pi^{-1}$ *and* $T_0 := T_0^{a,b,c,h,d} = \pi\ T_{n-1}\ \pi^{-1} = a + (b\ x_n + q^{-1}\ c\ x_1 + h + q^{-1}\ d\ x_1\ x_n)\ e_0$. *It is easy to see that*

$$\pi\ e_0 = q\ e_1\ \pi, \quad \pi\ T_0^{a,b,c,h,d} = T_1^{a,b,c,q\ h,q^{-1}d}\ e_1\ \pi = T_1^{a,b,c,h,d} + ((1-q)\ h + (1-q^{-1})\ d\ x_1\ x_2)\ e_1.$$

Now let us assume that $a = t, b = -t, d = e = 0, c = 1$. *Then,*
$T_i = t + (x_{i+1} - t\ x_i)\ e_i$, $i = 1, \ldots, n-1$, $T_0 = t + (q^{-1}\ x_1 - t\ x_n)\ e_0$,
$T_i^2 = (t-1)\ T + t$, $0 \le i < n$, $T_i\ x_i\ T_i = t\ \ x_{i+1}$, $1 \le i < n$, $T_0\ x_n\ T_0 = t\ q^{-1}\ x_1$,
$T_0\ T_1\ T_0 = T_1\ T_0\ T_1$, $T_{n-1}\ T_0\ T_{n-1} = T_0\ T_{n-1}\ T_0$, $T_0\ T_i = T_i\ T_0$, $if\ \ 2 \le i < n-1$.

The operators $T_i := T_i^{t,-t,1,0,0}$, $0 \le i \le n-1$ *have been used in [31] to give an "elementary" construction of nonsymmetric Macdonald polynomials. It follows from the nilCoxeter relations listed above, that the Dunkl–Cherednik elements, cf [3]*

$$Y_i := \Big(\prod_{a=i-1}^{1}\ T_a^{-1}\Big)\ \pi\ \Big(\prod_{a=n-1}^{i+1}\ T_a\Big), \quad i = 1, \ldots, n,$$

where $T_i = T_i^{t,-t,1,0,0}$, *generate a commutative subalgebra in the double affine nilCoxeter algebra* $DANC_n$. *Note that the algebra* $DANC_n$ *contains lot of other interesting commutative subalgebras, see e.g. [13].*

It seems interesting to give an interpretation of polynomials generated by the set of operators $T_i^{t,-t,1,h,e}$, $i = 0, \cdots, n-1$ *in a way similar to that given in [31]. We expect that these polynomials provide an affine version of polynomials* $\mathcal{KN}_w^{(-t,-1,1,1,0)}(X)$, $w \in \mathbb{S}_n \subset \mathbb{S}_n^{aff}$, *see Remark 4.7.*

5 Cauchy kernel

Let $u_1, u_2, \cdots, u_{n-1}$ be a set of generators of the free algebra $\mathcal{F}_{n-1}$, which assumed also to be commute with the all variables $\mathfrak{P}_n := \{p_{i,j},\ 2 \le i + j \le n+1, i \ge 1, j \ge 1\}$..

Definition 5.1 *The Cauchy kernel* $\mathcal{C}(\mathfrak{P}_n, U)$ *is defined to be as the ordered product*

$$\mathcal{C}(\mathfrak{P}_n, U) = \prod_{i=1}^{n-1}\ \Big\{\prod_{j=n-1}^{i}\ (1 + p_{i,j-i+1}\ u_j)\Big\}. \tag{5.14}$$

For example,
$\mathcal{C}(\mathfrak{P}_4, U) = (1 + p_{1,3}\ u_3)(1 + p_{1,2}\ u_2)(1 + p_{1,1}\ u_1)(1 + p_{2,2}\ u_3)(1 + p_{2,1}\ u_2)(1 + p_{3,1}\ u_3)$.

Lemma 5.2

$$\mathcal{C}(\mathfrak{P}_n, U) = \sum_{(\mathbf{a},\mathbf{b}) \in \mathcal{S}_n}\ \prod_{j=1}^{p} p_{\{a_j,b_j\}}\ w(\mathbf{a}, \mathbf{b}), \tag{5.15}$$

where $\mathbf{a} = (a_1, \ldots, a_p)$, $\mathbf{b} = (b_1, \ldots, b_p)$, $w(\mathbf{a}, \mathbf{b}) = \prod_{j=1}^{p}\ u_{a_j+b_j-1}$, *and the sum in* (4.10) *runs over the set* $\mathcal{S}_n :=$

$$\{(\mathbf{a}, \mathbf{b}) \in \mathbb{N}^p \times \mathbb{N}^p \mid \mathbf{a} = (a_1 \le a_2 \le \ldots \le a_p), a_i + b_i \le n, \ \ and\ \ if\ \ a_i = a_{i+1} \Longrightarrow b_i > b_{i+1}\}.$$

We denote by $\mathcal{S}_n^{(0)}$ the set $\{(\mathbf{a},\mathbf{b}) \in \mathcal{S}_n \mid \ w(\mathbf{a},\mathbf{b})$ *is a tableau word*$\}$.

The number of terms in the right hand side of (5.15) is equal to $2^{\binom{n}{2}}$, and therefore is equal to the number $\#|STY(\delta_n, \leq n)|$ of semistandard Young tableaux of the staircase shape $\delta_n := (n-1, n-2, \ldots, 2, 1)$ filled by the numbers from the set $\{1, 2, \ldots, n\}$. It is also easily seen that the all terms appearing in the $RHS(4.10)$ are different, and thus $\#|\mathcal{S}_n| = \#|STY(\delta_n, \leq n)|$.

We are interested in the decompositions of the Cauchy kernel $\mathcal{C}(\mathfrak{P}_n, U)$ in the algebras $\mathcal{P}_n$, $\mathcal{NP}_n$, $\mathcal{IP}_n$, $\mathcal{NC}_n$ and $\mathcal{IC}_n$.

5.1 Plactic algebra $\mathcal{P}_n$

Let λ be a partition and α be a composition of the same size. Denote by $\widetilde{STY}(\lambda, \alpha)$ the set of semistandard Young tableaux T of the shape λ and content α which must satisfy the following conditions:

- for each $k = 1, 2, \cdots$, the all numbers k are located in the first k columns of the tableau T. In other words, the all entries $T(i, j)$ of a semistandard tableau $T \in \widetilde{STY}(\lambda, \alpha)$ have to satisfy the following conditions: $T_{i,j} \leq j$.

For a given (semi-standard) Young tableau T let us denote by $R_i(T)$ the set of numbers placed in the i-th row of T, and denote by $\widetilde{STY}_0(\lambda, \alpha)$ the subset of the set $\widetilde{STY}_0(\lambda, \alpha)$ involving only tableaux T which satisfy the following constrains : $R_1(T) \supset R_2(T) \supset R_3(T) \supset \ldots$.

To continue, let us denote by $\mathcal{A}_n$ (respectively by $\mathcal{A}_n^{(0)}$) the union of the sets $\widetilde{STY}(\lambda, \alpha)$ (resp. that of $\widetilde{STY}_0(\lambda, \alpha)$) for all partitions λ such that $\lambda_i \leq n - i$ for $i = 1, 2, \cdots, n-1$, and all compositions α, $l(\alpha) \leq n - 1$. Finally, denote by $\mathcal{A}_n(\lambda)$ (resp.$\mathcal{A}_n^{(0)}(\lambda)$) the subset of $\mathcal{A}_n$ (resp. $\mathcal{A}_n^{(0)}(\lambda)$) consisting of all tableaux of the shape λ.

Lemma 5.3

- $|\mathcal{A}_n(\delta_n)| = 1$, $|\mathcal{A}_n(\delta_{n-1})| = (n-1)!$, $|\mathcal{A}_n((n-1))| = C_{n-1}$ *the* $n-1$*-th Catalan number. More generally,*

$$|\mathcal{A}_n((1^k))| = \binom{n-1}{k}, \qquad |\mathcal{A}_n((k))| = \frac{n-k}{n}\binom{n+k-1}{k}, \quad k = 0, \ldots, n-1,$$

*cf [39], A*009766*,*

$$|\mathcal{A}_n((k,1))| = \frac{(n-k+1)(n^2+n-k-1)}{(k+1)(n+2)}\binom{n+k}{k-1}, \ k = 1, \ldots, n.$$

- *There exists a bijection* $\rho_n : \mathcal{A}_n \longrightarrow ASM(n)$ *such that the image* $Im\ (\mathcal{A}_n^{(0)})$ *contains the set of* $n \times n$ *permutation matrices.*
- *The number of* **column strict***, as well as* **row strict** *diagrams which are contained inside the staircase diagram* $(n, n-1, \ldots, 2, 1)$ *is equal to* 2^n.

Example 5.4 *Take* $n = 5$ *so that* $ASM(5) = 429$ *and* $Cat(5) = 42$*. One has*

$|\mathcal{A}_5^{(0)}| = |\mathcal{A}_5^{(0)}(\emptyset)| + |\mathcal{A}_5^{(0)}((1))| + |\mathcal{A}_5^{(0)}((2))| + |\mathcal{A}_5^{(0)}((3))| + |\mathcal{A}_5^{(0)}((2,1))| + |\mathcal{A}_5^{(0)}((4))| + |\mathcal{A}_5^{(0)}((3,1))| + |\mathcal{A}_5^{(0)}((3,2))| + |\mathcal{A}_5^{(0)}((4,1))| + |\mathcal{A}_5^{(0)}((4,2))| + |\mathcal{A}_5^{(0)}((3,2,1))| + |\mathcal{A}_5^{(0)}((4,3))| + |\mathcal{A}_5^{(0)}((4,2,1))| + |\mathcal{A}_5^{(0)}((4,3,1))| + |\mathcal{A}_5^{(0)}((4,3,2))| + |\mathcal{A}_5^{(0)}((4,3,2,1))| =$
$1+4+9+14+6+14+16+4+21+14+4+1+9+2+1+1 = 121,$
$\sum_{k=0}^{4} |\mathcal{A}_5((k))| = 1+4+9+14+14 = 42.$

We expect that the image $\rho_n(\bigcup_{k=0}^{n-1} \mathcal{A}_n((k)))$ coincides with the set of $n \times n$ permutation matrices corresponding to either 321-avoiding or 132-avoiding permutations.

■

Now we are going to define a statistic $n(T)$ on the set $\mathcal{A}_n$.

Definition 5.5 *Let λ be a partition, α be a composition of the same size. For each tableau $T \in \widetilde{STY}(\lambda, \alpha) \subset \mathcal{A}_n(\lambda)$ define*

$$n(T) = \alpha_n = Card\{(i,j) \in \lambda \mid T(i,j) = n\}.$$

Clearly, $n(T) \leq \lambda_1$.

Define polynomials

$$\mathcal{A}_\lambda(t) := \sum_{T \in \mathcal{A}_n(\lambda)} t^{\lambda_1 - n(T)}.$$

It is instructive to display the numbers $\{\mathcal{A}_n(\lambda),\ \lambda \subset \delta_n\}$ as a vector of the length equals to the $n-th$ Catalan number. For example,
$\mathcal{A}_4(\emptyset,(1),(2),(1,1),(3),(2,1),(1,1,1),(3,1),(2,2),(2,1,1),(3,2),(3,1,1),(2,2,1),(3,2,1))=$
$(1,3,5,3,5,6,1,6,3,2,3,2,1,1)$.

It is easy to see that the above data, as well as the corresponding data for $n=5$, coincide with the list of refined totally symmetric self-complementary plane partitions that fit in the box $2n \times 2n \times 2n$ $(TSSCPP(n)$ for short$)$ listed for $n = 1,2,3,4,5$ in [10], Appendix D.

In fact we have

Theorem 5.6 *The sequence $\{\mathcal{A}_n(\lambda),\ \lambda \subset \delta_n\}$ coincides with the set of refined $TSSCPP(n)$ numbers as defined in [10]. More precisely,*

- $|\mathcal{A}_n(\lambda)| = det|\binom{n-i}{\lambda'_j - j + i}|_{1 \leq i,j \leq n-1}$,
- *We have*

$$\mathcal{A}_\lambda(t) := det|\binom{n-i-1}{\lambda'_j - j + i - 1} + t\ \binom{n-i-1}{\lambda'_j - j + i}|_{1 \leq i,j \leq n-1},$$

- *Polynomial $\mathcal{A}_\lambda(t)$ is equal to a t-analog of refined $TSSCPP(n)$ numbers $P_n(\lambda'_{n-1}+1, \cdots, \lambda'_{n-i}+i, \cdots, \lambda'_1+n-1 \mid t)$ introduced by means of recurrence relations in [10],* (3.5).

In particular, $\sum_{\lambda \subset \delta_n} \mathcal{A}_\lambda(t) = \sum_{1 \leq j \leq n-1} A_{n,j}\ t^{j-1}$, where $A_{n,j}$ stands for the number of alternating sign matrices $(ASM_n$ for short$)$ of size $n \times n$ with a 1 on top of the j-th column.

Corollary 5.7 *The number of different tableau subwords in the word*

$$w_0 := \prod_{j=1}^{n-1} \Big\{ \prod_{a=n-1}^{j} a \Big\}$$

is equal to the number of alternating sign matrices of size $n \times n$, *i.e.*

$$|\mathcal{A}_n| = |TSSCPP(n)| = |ASM_n|.$$

It is well-known [2] that

$$A_{n,j} = \binom{n+j-2}{j-1} \frac{(2n-j-1)!}{(n-j)!} \prod_{i=0}^{n-2} \frac{(3i+1)!}{(n+i)!},$$

and the total number A_n of ASM of size $n \times n$ is equal to

$$A_n \equiv A_{n+1,1} = \sum_{j=1}^{n} A_{n,j} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Comments 5.1

(1) Let as before $STY(\delta_n \leq n) := \mathcal{ST}_n$ denotes the set of all semistandard Young tableaux of the staircase shape $\delta_n = (n-1, n-2, \ldots, 2, 1)$ filled by the numbers from the set $\{1, \ldots, n\}$. Denote by $\mathcal{ST}_n^{(0)}$ the subset of "anti- diagonally" increasing tableaux, i.e.

$$\mathcal{ST}_n^{(0)} = \{T \in STY(\delta_n, \leq n) \mid \ T_{i,j} \geq T_{i-1,j+1} \ \ for \ \ all \ \ 2 \leq i \leq n-1, \ \ 1 \leq j \leq n-2\}.$$

One (A.K) can construct bijections

$$\iota_n : \mathcal{S}_n \sim \mathcal{ST}_n, \quad \zeta_n : \mathcal{A}_n \sim \mathcal{ST}_n^{(0)}$$

such that $Im(\iota_n) = Im(\zeta_n)$.

(2)

Proposition 5.8

$$\sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_n) \\ \rho_n \geq \lambda}} K_{\rho_n,\lambda} \ \binom{n}{m_0(\lambda), \ m_1(\lambda), \ \ldots, \ m_n(\lambda)} = 2^{\binom{n}{2}}.$$

$$\sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_n) \\ \rho_n \geq \lambda}} \binom{n}{m_0(\lambda), \ m_1(\lambda), \ \ldots, \ m_n(\lambda)} = \mathcal{F}_n,$$

where $\mathcal{F}_n$ *denotes the number of forests of trees on* n *labeled nodes;*

$K_{\rho_n,\lambda}$ *denotes the Kostka number, i.e. the number of semistandard Young tableaux of the shape* $\rho_n := (n-1, n-2, \ldots, 1)$ *and content/weight* λ*;*

for any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0)$ *we set* $m_i(\lambda) = \{j \mid \lambda_j = i\}$.

Let α be a composition, we denote by α^+ the partition obtained from α by reordering of its parts. For example, if $\alpha = (0, 2, 0, 3, 1, 0)$ then $\alpha^+ = (3, 2, 1)$. Note that $\ell(\alpha) = 6$, but $\ell(\alpha^+) = 3$.

Now let α be a composition such that $\rho_n \geq \alpha^+$, $\ell(\alpha) \leq n$, that is $\alpha_j = 0$, $if \;\; j > \ell(\alpha)$, $|\alpha| = \binom{n}{2}$ and

$$\sum_{k \leq j} (\rho_n)_k \geq \sum_{k \leq j} (\alpha^+)_k, \;\; \forall j.$$

There is a unique semistandard Young tableau $T_n(\alpha)$ of shape ρ_n and content α which corresponds to the maximal configuration of type $(\rho_n; \alpha)$ and has all quantum numbers (riggings) equal to zero. It follows from Proposition 4.8 that $\#\{\alpha \mid \ell(\alpha) \leq n, \;\; \rho_n \geq \alpha^+\} = \mathcal{F}_n$. Therefore there is a natural embedding of the set of forests on n labeled nodes to the set of semistandard Young tableaux of shape ρ_n filled by the numbers from the set $[1, \ldots, n]$. We denote by $\mathcal{FT}_n \subset STY(\rho_n, \leq n)$ the subset $\{T_n(\alpha) \mid \rho_n \geq \alpha^+, \ell(\alpha) \leq n\}$. Note that the set $\mathcal{K}_n := \{\alpha \; | \ell(\alpha) = n, \;\; (\alpha)^+ = \rho_n\}$ contains $n!$ compositions, and under the rigged configuration bijection the elements of the set $\mathcal{K}_n$ correspond to the **key** tableaux of shape ρ_n.

Let us say a few words about the Kostka numbers $K_{\rho_n,\alpha}$. First of all, it's clear that if $\alpha = (\alpha_1, \alpha_2, \cdots)$ is a composition such that $\alpha_1 = n - 1$, then $K_{\rho_n,\alpha} = K_{\rho_{n-1},\alpha[1]}$, where we set $\alpha[1] := (\alpha_2, \ldots)$.

Now assume that $n = 2k + 1$ is an odd integer, and consider partitions $\nu_n := (k^n)$ and $\mu_n := ((k + 1)^k, k^k)$. Then

$$K_{\rho_n,\nu_n} = Coeff_{(x_1 x_2 \cdots x_n)^k}\Big(\prod_{1 \leq i < j \leq n} (x_i + x_j)\Big), \quad \binom{2k}{k} K_{\rho_n,\mu_n} = K_{\rho_n,\nu_n}.$$

It is well-known that the number K_{ρ_n,ν_n} is equal to number of labeled regular tournaments with $n := 2k + 1$ nodes, see e.g. [39], $A007079$.

In the case when $n = 2k$ is an even number, one can show that

$$K_{\rho_n,\nu_n} = K_{\rho_{n-1},\nu_{n-1}}, \quad K_{\rho_n,\mu_n} = K_{\rho_{n+1},\mu_{n+1}}.$$

Note that the rigged configuration bijection gives rise to an embedding of the set of labeled regular tournaments with $n := 2k + 1$ nodes to the set $STY(\rho_n, \leq n)$, if n is an odd integer, and to the set $STY(\rho_{n-1}, \leq n - 1)$, if n is even.

■

Theorem 5.9

(1) *In the plactic algebra $\mathcal{P}_n$ the Cauchy kernel has the following decomposition*

$$\mathcal{C}_n(\mathfrak{P}, U) = \sum_{T \in \mathcal{A}_n} \mathcal{K}_T(\mathfrak{P}) \; u_{w(T)}. \tag{5.16}$$

(2) *Let $T \in \mathcal{A}_n$, and $\alpha(T)$ be its bottom code. Then*

$$\mathcal{K}_T(\mathfrak{P}) - \prod_{(i,j) \in T} p_{\{i, T(i,j) - j + 1\}} \geq 0,$$

and equality holds if and only if the bottom code $\alpha(T)$ is a partition.

Note that the number of *different shapes* among the tableaux in the set $\mathcal{A}_n$ is equal to the Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$.

Problem 5.10 *Construct a bijection between the set $\mathcal{A}_n$ and the set of alternating sign matrices ASM_n.*

Example 5.11 *For $n = 4$ one has* $\mathcal{C}_4(X, U) = K[0]+$
$K[1]u_1+$
$K[01]u_2+$
$K[001]u_3+$
$K[11](u_{12} + u_{22})+$
$K[2](u_{21} + u_{31})+$
$K[101]u_{13}+$
$K[02]u_{32}+$
$K[011](u_{23} + u_{33})+$
$K[3]u_{321}+$
$K[12](u_{312} + u_{322})+$
$K[21]u_{212}+$
$K[111](u_{123} + u_{133} + u_{233} + u_{223} + u_{333})+$
$K[021]u_{323}+$
$K[201](u_{313} + u_{213})+$
$K[31]u_{3212}+$
$K[301]u_{3213}+$
$K[22](u_{3132} + u_{2132} + u_{3232})+$
$K[121](u_{3123} + u_{3233} + u_{3223})+$
$K[211](u_{2123} + u_{2133} + u_{3133})+$
$K[32]u_{32132}+$
$K[311](u_{32123} + u_{32133})+$
$K[221](u_{21323} + u_{31323} + u_{32323})+$
$K[321]u_{321323}$.

Let $w \in \mathbb{S}_n$ be a permutation with the Lehmer code $\alpha(w)$.

Definition 5.12 *Define the plactic polynomial $\mathcal{PL}_w(E)$ to be*

$$\mathcal{PL}_w(U) = \Big\{ \sum_{T \in \mathcal{A}_n, \alpha(T)=\alpha(w)} u_{w(T)} \Big\}|_{u_i \longrightarrow u_{n-i}}.$$

Definition 5.13 *Define algebra $\mathcal{PC}_n$ to be the quotient of the plactic algebra $\mathcal{P}_n$ by the two-sided ideal J_n by the set of monomials*

$$\{u_{i_1} u_{i_2} \cdots u_{i_n}\}, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq n, \quad \#\{a \mid i_a = j\} \leq j, \ \forall j = 1, \ldots, n.$$

Theorem 5.14

- *The algebra $\mathcal{PC}_n$ has dimension equals to $ASM(n)$,*
- $Hilb(\mathcal{PC}_n, q) = \sum_{\lambda \in \delta_{n-1}} |\mathcal{A}_\lambda| \; q^{|\lambda|}$,
- $Hilb((\mathcal{PC}_n)^{ab}, q) = \sum_{k=0}^{n} \frac{n-k+1}{n+1} \binom{n+k}{n} q^k$, *cf [39], A*009766.

Definition 5.15 *Denote by $\mathcal{PC}_n^\sharp$ the quotient of the algebra $\mathcal{PC}_n$ by the two-sided ideal generated by the elements* $\{u_i u_j - u_j u_i, \quad |i-j| \geq 2\}$.

Proposition 5.16 *Dimension* $\dim \mathcal{PC}_n^\sharp$ *of the algebra $\mathcal{PC}_n^\sharp$ is equal to the number of Dyck paths whose ascent lengths are exactly* $\{1, 2, \ldots, n+1\}$.

See [39],*A*107877 where the first few of these numbers are displayed.

Example 5.17 $Hilb(\mathcal{PC}_5^\sharp, t) = (1, 4, 12, 27, 48, 56, 54, 38, 20, 7, 1)_t$,
$Hilb(\mathcal{PC}_6^\sharp, t) = (1, 5, 18, 50, 116, 221, 321, 398, 414, 368, 275, 175, 89, 35, 9, 1)_t$,
$\dim \mathcal{PC}_7^\sharp = 28612$.

Example 5.18 $Hilb(\mathcal{PC}_3, q) = (1, 2, 3, 1)_q$, $Hilb(\mathcal{PC}_4, q) = (1, 3, 8, 12, 11, 6, 1)_q$,
$Hilb(\mathcal{PC}_5, q) = (1, 4, 15, 35, 69, 91, 98, 70, 35, 10, 1)_q$,
$Hilb(\mathcal{PC}_6, q) = (1, 5, 24, 74, 204, 435, 783, 1144, 1379, 1346, 1037, 628, 275, 85, 15, 1)_q$,
$Hilb(\mathcal{PC}_7, q) = (1, 6, 35, 133, 461, 1281, 3196, 6686, 12472, 19804, 27811, 33271, 34685, 30527, 22864, 14124, 7126, 2828, 840, 175, 21, 1)_q$.

Problem 5.19 *Denote by $\mathfrak{A}_n$ the algebra generated by the curvature of 2-forms of the tautological Hermitian linear bundles ξ_i, $1 \leq i \leq n$, over the flag variety $\mathcal{F}l_n$, [38]. It is well-known [35] that the Hilbert polynomial of the algebra $\mathfrak{A}_n$ is equal to*

$$Hilb(\mathfrak{A}_n, t) = \sum_{F \in \mathcal{F}(n)} t^{inv(F)} = \sum_{F \in \mathcal{F}(n)} t^{maj(F)},$$

where the sum runs over the set $\mathcal{F}(n)$ of forests F on the n labeled vertices, and $inv(F)$ (resp. $maj(F)$) denotes the inversion index (resp. the major index) of a forest F. [7]

Clearly that

$$\dim(\mathfrak{A}_n)_{\binom{n}{2}} = \dim(\mathcal{PC}_n)_{\binom{n}{2}} = \dim(H^\star(\mathcal{F}l_n, \mathbb{Q})_{\binom{n}{2}} = 1.$$

For example,
$Hilb(\mathcal{PC}_6, t) = (1, 5, 24, 74, 204, 435, 783, 1144, 1379, 1346, 1037, 628, 275, 85, 15, 1)_t$,

[7] For the readers convenience we recall definitions of statistics $inv(F)$ and $maj(F)$. Given a forest F on n labeled vertices, one can construct a tree T by adding a new vertex (root) connected with the maximal vertices in the connected components of F.

The inversion index $inv(F)$ is equal to the number of pairs (i, j) such that $1 \leq i < j \leq n$, and the vertex labeled by j lies on the shortest path in T from the vertex labeled by i to the root.

The major index $maj(F)$ is equal to $\sum_{x \in Des(F)} h(x)$; here for any vertex $x \in F$, $h(x)$ is the size of the subtree rooted at x; the descent set $Des(F)$ of F consists of the vertices $x \in F$ which have the labeling strictly greater than the labeling of its child.

$Hilb(\mathfrak{A}_6,t) = (1,5,15,35,70,126,204,300,405,490,511,424,245,85,15,1)_t$,
$Hilb(H^\star(\mathcal{F}l_n,\mathbb{Q}),t) = (1,5,14,29,49,71,90,101,101,90,71,49,29,14,5,1)_t$.
We **expect** *that* $dim(\mathcal{PC}_n)_{\binom{n}{2}-1} = \binom{n}{2}$ *and* $dim(\mathcal{PC}_n)_{\binom{n}{2}-2} = \frac{3n+5}{4}\binom{n+2}{3} = s(n+2,2)$, *where* $s(n,k)$ *denotes the Stirling number of the first kind, see e.g. [39],* A000914.

Problems

(1) *Is it true that* $Hilb(\mathcal{PC}_n,t) - Hilb(\mathfrak{A}_n,t) \in \mathbb{N}[t]$ *?*
If so, as we **expect**, *does there exist an embedding of sets* $\iota : \mathcal{F}(n) \hookrightarrow \mathcal{A}_n$ *such that* $inv(F) = n(\iota(F))$ *for all* $F \in \mathcal{F}_n$ *?*

See Section 5.1 for definitions of the set $\mathcal{A}_n$ *and statistics* $n(T)$, $T \in \mathcal{A}_n$, *Definition 5.5.*

(2) *It is well-known that* $\#|STY(\delta_n, \leq n)| = 2^{\delta_n} = 2^{\binom{n}{2}}$, *where* $\delta_n = (n-1, n-2, \ldots, 2, 1)$, $STY(\delta_n, \leq n)$ *denotes the set of semistandard Young tableaux of shape* δ_n *with entries bounded by* n,
2^{δ_n} *stands for the set of all subsets of boxes of the staircase diagram* δ_n.

Define a "natural" bijection $\kappa : STY(\delta_n, \leq n) \longleftrightarrow 2^{\delta_n}$ *such that the set* $\kappa(MT(n))$ *admits a "nice" combinatorial description. Here* $MT(n)$ *denotes the set of (increasing) monotone triangules, i.e. the subset of* $STY(\delta_n, \leq n)$ *consisting of tableaux* $T = (t_{i,j})_{\substack{i+j \leq n+1 \\ i \geq 1, j \geq 1}}$ *such that* $t_{i,j} \geq t_{i-1,j+1}$, $2 \leq i \leq n$, $1 \leq j < n$, *cf [40].*

5.2 Nilplactic algebra $\mathcal{NP}_n$

Let λ be a partition and α be a composition of the same size. Denote by $\widehat{STY}(\lambda, \alpha)$ the set of columns and rows strict Young tableaux T of the shape λ and content α such that the corresponding tableau word $w(T)$ is reduced, i.e. $l(w(T)) = |T|$.

Denote by $\mathcal{B}_n$ the union of the sets $\widehat{STY}(\lambda, \alpha)$ for all partitions λ such that $\lambda_i \leq n - i$ for $i = 1, 2, \cdots, n-1$, and all compositions α, $\alpha \subset \delta_n$.

For example, $|\mathcal{B}_n| = 1, 2, 6, 25, 139, 1008, \cdots$, for $n = 1, 2, 3, 4, 5, 6, \cdots$.

Theorem 5.20

(1) *In the nilplactic algebra* $\mathcal{NP}_n$ *the Cauchy kernel has the following decomposition*

$$\mathcal{C}_n(\mathfrak{P}, U) = \sum_{T \in \mathcal{B}_n} \mathcal{K}_T(\mathfrak{P})\ u_{w(T)}. \tag{5.17}$$

(2) *Let* $T \in \mathcal{B}_n$ *be a tableau, and assume that its bottom code is a partition. Then*

$$\mathcal{K}_T(\mathfrak{P}) = \prod_{(i,j) \in T} p_{\{i, T(i,j)-j+1\}}.$$

Example 5.21 *For* $n = 4$ *one has* $\mathcal{C}_4(X, U) =$
$K[0] + K[1]u_1 + K[01]u_2 + K[001]u_3 + K[11]u_{12} + K[2](u_{21} + u_{31}) + K[101]u_{13} +$
$K[02]u_{32} + K[011]u_{23} + K[3]u_{321} + K[12]u_{312} + K[21]u_{212} +$

$K[111]u_{123} + K[021]u_{323} + K[201]u_{213} + K[31]u_{3212} +$
$K[301]u_{3213} + K[22]u_{2132} + K[121]u_{3123} + K[211]u_{2123} + K[32]u_{32132} +$
$K[311]u_{32123} + K[221]u_{21323} + K[321]u_{321323}$.

5.3 Idplactic algebra $\mathcal{IP}_n$

Let λ be a partition and α be a composition of the same size. Denote by $\widetilde{STY}(\lambda, \alpha)$ the set of columns and rows strict Young tableaux T of the shape λ and content α such that $l(w(T)) = rl(w(T))$, i.e. the tableau word $w(T)$ is a unique tableau word of minimal length in the idplactic class of $w(T)$, cf Example 1.9.

Denote by $\mathcal{D}_n$ the union of the sets $\widetilde{STY}(\lambda, \alpha)$ for all partitions λ such that $\lambda_i \leq n - i$ for $i = 1, 2, \cdots, n-1$, and all compositions α, $l(\alpha) \leq n - 1$.

For example, $|\mathcal{D}_n| = 1, 2, 6, 26, 154, 1197, \cdots$, for $n = 1, 2, 3, 4, 5, 6, \cdots$.

Theorem 5.22

In the idplactic algebra $\mathcal{IP}_n$ the Cauchy kernel has the following decomposition

$$\mathcal{C}_n(X, Y, U) = \sum_{T \in \mathcal{D}_n} \mathcal{KG}_T(X, Y)\ u_{w(T)}. \tag{5.18}$$

(2) *Let $T \in \mathcal{D}_n$ be a tableau, and assume that its bottom code is a partition. Then*

$$\mathcal{KG}_T(X, Y) = \mathcal{K}_T(X, Y) = \prod_{(i,j) \in T} (x_i + y_{T(i,j)-j+1}).$$

Example 5.23 *For $n = 4$ one has $\mathcal{C}_4(X, U) =$*
$KG[0]+KG[1]u_1+KG[01]u_2+KG[001]u_3+KG[11]u_{12}+KG[2](u_{21}+u_{31})+KG[101]u_{13}+$
$KG[02]u_{32} + KG[011]u_{23} + KG[3]u_{321} + KG[12]u_{312} + KG[21]u_{212} +$
$KG[111]u_{123} + KG[021]u_{323} + KG[201](u_{313} + u_{213}) + K[31]u_{3212} +$
$KG[301]u_{3213} + KG[22]u_{2132} + KG[121]u_{3123} + KG[211]u_{2123} + KG[32]u_{32132} +$
$KG[311]u_{32123} + KG[221]u_{21323} + KG[321]u_{321323}$.

Theorem 5.24 *For each composition α the key Grothendieck polynomial $KG[\alpha](X)$ is a linear combination of key polynomials $K[\beta](X)$ with* **nonnegative** *integer coefficients.*

5.4 NilCoxeter algebra $\mathcal{NC}_n$

Theorem 5.25 *In the nilCoxeter algebra $\mathcal{NC}_n$ the Cauchy kernel has the following decomposition*

$$\mathcal{C}_n(X, Y, U) = \sum_{w \in \mathbb{S}_n} \mathfrak{S}_w(X, Y)\ u_w. \tag{5.19}$$

Let $w \in \mathbb{S}_n$ be a permutation, denote by $R(w)$ the set of all its reduced decompositions. Since the nilCoxeter algebra $\mathcal{NC}_n$ is the quotient of the nilplactic algebra $\mathcal{NP}_n$, the set $R(w)$ is the union of nilplactic classes of some tableau words $w(T_i)$: $R(w) = \bigcup C(T_i)$. Moreover, $R(w)$ consists of only one nilplactic class if and only if w is a *vexillary* permutation. In general case we see that the set of compatible sequences $CR(w)$ for permutation w is the union of sets $C(T_i)$.

Corollary 5.26 *Let $w \in \mathbb{S}_n$ be a permutation of length l, then*

(1) $\mathfrak{S}_w(X,Y) = \sum_{\mathbf{b}\in CR(w)} x_{b_1}\cdots x_{b_l}$.

(2) *Double Schubert polynomial $\mathfrak{S}_w(X,Y)$ is a linear combination of double key polynomials $\mathcal{K}_T(X,Y)$, $T \in \mathcal{B}_n$, $w = w(T)$, with* **nonnegative** *integer coefficients.*

5.5 IdCoxeter algebras $\mathcal{IC}_n^{\pm}$

Theorem 5.27 *In the IdCoxeter algebra $\mathcal{IC}_n^+$ with $\beta = 1$, the Cauchy kernel has the following decomposition*

$$\mathcal{C}_n(X,Y,U) = \sum_{w\in\mathbb{S}_n} \mathcal{G}_w(X,Y)\ u_w. \tag{5.20}$$

Theorem 5.28 *In the IdCoxeter algebra $\mathcal{IC}_n^-$ with $\beta = -1$, one has the following decomposition*

$$\prod_{i=1}^{n-1}\ \Big\{ \prod_{j=n-1}^{i} ((1+x_i)(1+y_{j-i+1}) + (x_i + y_{j-i+1})\ u_j)\Big\} = \sum_{w\in\mathbb{S}_n} \mathcal{H}_w(X,Y)\ u_w. \tag{5.21}$$

A few remarks in order.

(a) The (dual) Cauchy identity (5.21) is still valid in the idplactic algebra with constrain $u_i^2 = -\beta u_i,\ i = 1,\ldots,n-1$.

(b) The left hand side of the identity (5.21) can be written in the following form

$$\prod_{\substack{1\le i,j\le n\\ i+j\le n}} (x_i + y_j) \prod_{i=1}^{n-1}\ \Big\{ \prod_{j=n-1}^{i} \frac{1}{1-(x_i + y_{j-i+1} + \beta\ x_i\ y_{j-i+1})\ u_j}\Big\}.$$

Indeed, $(1+\beta\ x + x\ u_i)(1 - x\ u_i) = 1 + \beta\ x,$ since $u_i^2 = -\beta\ u_i$.

■

Let $w \in \mathbb{S}_n$ be a permutation, denote by $IR(w)$ the set of all decompositions in the idCoxeter algebra $\mathcal{IC}_n$ of the element u_w as the product of the generators u_i, $1 \le i \le n-1$, of the algebra $\mathcal{IC}_n$. Since the idCoxeter algebra $\mathcal{IC}_n$ is the quotient of the idplactic algebra $\mathcal{IP}_n$, the set $IR(w)$ is the union of idplactic classes of some tableau words $w(T_i)$: $IR(w) = \bigcup IR(T_i)$. Moreover, the set of compatible sequences $IC(w)$ for permutation w is the union of sets $IC(T_i)$.

Corollary 5.29 *Let $w \in \mathbb{S}_n$ be a permutation of length l, then*

(1) $\mathcal{G}_w(X,Y) = \sum_{\mathbf{b}\in IC(w)} \quad \prod_{i=1}^{l}(x_{b_i} + y_{a_i - b_i + 1})$.

(2) *Double Grothendieck polynomial $\mathcal{G}_w(X,Y)$ is a linear combination of double key Grothendieck polynomials $\mathcal{KG}_T(X,Y)$, $T \in \mathcal{B}_n$, $w = w(T)$, with* **nonnegative** *integer coefficients.*

6 $\mathcal{F}$-kernel and symmetric plane partitions

Let us fix natural number n and k, and a partition $\lambda \subset (n^k)$. Clearly the number of such partitions is equal to $\binom{n+k}{n}$; note that in the case $n = k$ the number $\binom{2n}{n}$ is equal to the *Catalan number of type B_n* .

Denote by $\mathcal{B}_{n,k}(\lambda)$ the set of semistandard Young tableaux of shape λ filled by the numbers from the set $\{1, 2, \ldots, n\}$. For a tableau $T \in \mathcal{B}_{n,k}$ set as before,

$$n(T) := Card\ \{(i,j) \in \lambda \ \mid \ T(i,j) = n\},$$

and define polynomial

$$\mathcal{B}_{n,k}(\lambda)(q) := \sum_{T\in\mathcal{B}_{n,k}(\lambda)} q^{\lambda_1 - n(T)}.$$

Denote by $\mathcal{B}_{n,k} := \bigcup_{\lambda\subset(n^k)} \ \mathcal{B}_{n,k}(\lambda)$.

Lemma 6.1 *([11],[19]) The number of elements in the set $\mathcal{B}_{n,k}$ is equal to*

$$\#|\mathcal{B}_{n,k}| = \prod_{1\le i\le j\le k} \frac{i+j+n-1}{i+j-1} = \prod_{\substack{a\ge 0\\ 4a\le 2k-1}} \frac{\binom{n+2k-2a-1}{n+2a}}{\binom{2k-2a-1}{2a}} = \prod_{a\ge 0} \frac{\binom{n+2k+1-2a}{n}}{\binom{n+2([(k-2)/2]-a)}{n}}.$$

See also [39], *A*073165 for other combinatorial interpretations of the numbers $\#|\mathcal{B}_{n,k}|$. For example, the number $\#|\mathcal{B}_{n,k}|$ is equal to the number of symmetric plane partitions fit inside the box $n \times k \times k$.

Proposition 6.2 *One has*

- $\#|\mathcal{B}_{n,n}| := SPP(n) = TSPP(n) \times ASM(n), \quad \#|\mathcal{B}_{n,n+1}| = TSPP(n) \times ASM(n+1),$

*where $TSPP(n)$ denotes the number of totally symmetric plane partitions fit inside the $n \times n \times n$-box, see e.g. [39], A*005157*, whereas $ASM(n) = TSSCPP(2n)$ denotes the of $n \times n$ alternating sign matrices, and $TSSCPP(2n)$ denotes the number of totally symmetric self-complimentary plane partitions fit inside the $2n \times 2n \times 2n$-box.*

- $\#|\mathcal{B}_{n+2,n}| = \#|\mathcal{B}_{n,n+1}|$.

Note that in the case $n = k$ the number $\mathcal{B}_n := \mathcal{B}_{n,n}$ is equal to the number of symmetric plane portions fit inside the $n \times n \times n$-box, see [39], A049505. Let us point to that in general it may happen that the number $\#|\mathcal{B}_{n,n+2}|$ does not divisible by any $ASM(m), m \ge 3$. For example, $\mathcal{B}_{3,5} = 4224 = 2^5 \times 3 \times 11$. On the other hand, it's possible that the number $\#|\mathcal{B}_{n,n+2}|$ is divisible by $ASM(n=1)$, but does not divisible by $ASM(n+2)$. For example, $\mathcal{B}_{4,6} = 306735 = 715 \times 429$, but $306735 \nmid 7436 = ASM(6)$.

Problem 6.3 *Let a is equal to either 0 or 1. Construct bijection between the set $PP(n, n+a, n+a)$ of symmetric plane partitions fit inside the box $n \times n+a \times n+a$ and and the set of pairs (P, M) where P is the totally symmetric plane partitions fit inside the box $n \times n \times n$ and M is an alternating sign matrix of size $n+a \times n+a$.*

Example 6.4 *Take $n = 3$. The number of partitions $\lambda \subset (3^3)$ is equal to 20, namely, namely, the partitions* $\{\emptyset, (1), (2), (1,1), (3), (2,1), (1^3), (3,1), (2,2), (2,1^2), (3,2), (3,1^2), (2^2,1), (3^2), (3,2,1), (2^3), (3^2,1), (3,2^2), (3^2,2), (3^3)\}$, *and*

$$\mathcal{B}_3(q) := \sum_{\lambda \subset (3^3)} \#|\mathcal{B}_3(\lambda)|\ q^{|\lambda|} = (1,3,9,19,24,24,19,9,3,1) = (1+q)^3(1+q^2)(1+5q^2+q^4).$$

Note, however, that

$$\sum_{\lambda \subset (4^4)} \#|\mathcal{B}_4(\lambda)|\ q^{|\lambda|} = (1,4,16,44,116,204,336,420,490,420,336,204,116,44,16,4,1)$$

is an irreducible polynomial, but its value at $q = 1$ is equal to $2772 = 66 \times 42$.

Let $\mathbf{p} = (p_{i,j})_{1 \le i \le n,\ 1 \le j \le k}$ be a $n \times k$ matrix of variables.

Definition 6.5 *Define the kernel $\mathcal{F}_{n,k}(\mathbf{p}, U)$ as follows*

$$\mathcal{F}_{n,k}(\mathbf{p}, U) = \prod_{i=1}^{k-1} \prod_{j=n-1}^{1} (1 + p_{i,\overline{j-i+1}^{(n)}}\ u_j),$$

where for a fixed $n \in \mathbb{N}$ and an integer $a \in \mathbb{Z}$, we set

$$\overline{a} = \overline{a}^{(n)} := \begin{cases} a, & \text{if } a \ge 1, \\ n+a-1 & \text{if } a \le 0. \end{cases}$$

For example, $\mathcal{F}_3(\mathbf{p}, U) = (1 + p_{1,2}\ u_2)(1 + p_{1,1}\ u_1)(1 + p_{2,1}\ u_2)(1 + p_{2,2}\ u_1)$.

In the plactic algebra $\mathcal{FP}_{3,3}$ one has $\mathcal{F}_{3,3}(\mathbf{p}, U) = 1$
$+ (p_{1,1} + p_{2,2})\ u_1$
$+ (p_{1,2} + p_{2,1}) u_2$
$+ p_{1,1}\ p_{2,1}\ u_{11}$
$+ p_{1,1}\ p_{2,1}\ u_{12}$
$+ (p_{1,2}\ p_{1,1} + p_{1,2}\ p_{2,2} + p_{2,1}\ p_{2,2})\ u_{21}$
$+ p_{1,2}\ p_{2,1}\ u_{22}$
$+ (p_{1,1}\ p_{1,2}\ p_{2,2} + p_{1,2}\ p_{2,2}\ p_{2,1})\ u_{212}$
$+ (p_{1,1}\ p_{1,2}\ p_{2,2} + p_{1,1}\ p_{2,2}\ p_{2,1})\ u_{211}$
$+ p_{1,1}\ p_{1,2}\ p_{2,1}\ p_{2,2}\ u_{2121}$.

Definition 6.6 *Define algebra $\mathcal{PF}_{n,k}$ to be the quotient of the plactic algebra $\mathcal{P}_n$ by the two-sided ideal I_n generated by the set of monomials*

$$\{u_{i_1} u_{i_2} \cdots u_{i_k}\}, \quad 1 \le i_1 \le i_2 \le \cdots \le i_k \le n-1.$$

Theorem 6.7

- $Hilb(\mathcal{PF}_{n,k}, q) = \mathcal{B}_{n-1,k-1}(q)$,

In particular,

- *The algebra $\mathcal{PF}_{n,n}$ has dimension equals to the number of symmetric plane partitions $SPP(n-1)$,*

$$\bullet \quad Hilb(\mathcal{PF}_{n,k}, q) = q^{\frac{kn}{2}}\ \mathfrak{so}_{(\frac{k}{2})^n}\ (\underbrace{q^{\pm 1}, \ldots, q^{\pm 1}}_{k}, 1),$$

where $\mathfrak{so}_{(\frac{k}{2})^n}\ (\underbrace{q^{\pm 1}, \ldots, q^{\pm 1}}_{k}, 1)$ denotes the specialization $x_{2j} = q,\ x_{2j-1} = q^{-1},\ 1 \leq j \leq k$, of the character $\mathfrak{so}_\lambda(x_1, x_1^{-1} \ldots, x_k, x_k^{-1}, 1)$ of the odd orthogonal Lie algebra $\mathfrak{so}(2k+1)$ corresponding to the highest weight $\lambda = (\underbrace{\frac{k}{2}, \ldots, \frac{k}{2}}_{n})$.

- $\deg_q Hilb(\mathcal{PF}_{n,k}, q) = (n-1)(k-1)$, *and* $\dim(\mathcal{PF}_{n,k})_{(n-1)(k-1)} = 1$,
- *The Hilbert polynomial $Hilb(\mathcal{PF}_{n,k}, q)$ is symmetric (unimodal ?) polynomial in the variable q,*
- $Hilb((\mathcal{PF}_{n,k})^{ab}, q) = \sum_{j=0}^{k-1} \binom{n+j-2}{n-2} q^j$.

The key step in proofs of Lemma 6.1 and Theorem 6.5 is based on the following identity

$$\sum_{\lambda \subset (n^k)} s_\lambda(x_1, \ldots, x_k) = (x_1 \cdots x_k)^{n/2}\ \mathfrak{so}_{(\frac{k}{2})^n}(x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}, 1), \tag{6.22}$$

see e.g., [32], Ch.I, Sec.5, Ex.19, [19] and the literature quoted therein.

Problem 6.8

Let $\Gamma := {\Gamma_{n,m}}^{k,\ell} = (n^k, m^\ell)$, $n \geq m$ be a "fat hook". Find generalizations of the identity (6.21) *and those listed in [14], p. 71, to the case of fat hooks, namely to find "nice" expressions for the following sums*

$$\bullet \quad \sum_{\lambda \subset \Gamma} s_\lambda(X_{k+\ell}, \quad \sum_{\lambda \subset \Gamma} s_\lambda(X_{k+\ell}\ s_\lambda(Y_{k+\ell},$$

- *Find "bosonic" type formulas for these sum at the limit $n \longrightarrow \infty,\ \ell \longrightarrow \infty$, m, k are fixed.*

Example 6.9 $Hilb(\mathcal{PF}_{3,4}, q) = (1,3,9,9,9,3,1)_q$, $\dim(\mathcal{PF}_{3,4}) = 35 = 5 \times 7$, *and* $\dim(\mathcal{PF}_{3,5}) = 126 = 3 \times 42$
$Hilb(\mathcal{PF}_{3,5}, q) = (1,4,16,44,81,120,140,120,81,44,16,4,1)_q$, $\dim(\mathcal{PF}_{3,5}) = 672 = 16 \times 42$,
$Hilb(\mathcal{PF}_{5}, q) = (1,4,16,44,116,204,336,420,490,420,336,204,116,44,16,4,1)_q$.

Proposition 6.10

$$\bullet \quad Hilb(\mathcal{PF}_{3,n}, q) = \sum_{k=0}^{2n} \binom{n}{\left[\frac{k}{2}\right]} \binom{n}{\left[\frac{k+1}{2}\right]} q^k, \quad \dim(\mathcal{PF}_{3,n}) = \binom{2n-1}{n}.$$

Therefore, $Hilb(\mathcal{PF}_{3,n}, q)$ is equal to the generating function for the number of symmetric Dyck paths of semilength $2n-1$ according to the number of peaks, see [39], A088855.

- $\dim(\mathcal{PF}_{4,n+1}) = 2^n\ Cat_{n+1}$, *if* $n \geq 4$.

For example, $\dim(\mathcal{PF}_{4,7}) = 27456 = 64 \times 429$, $Hilb(\mathcal{PF}_{4,7}, q) =$ $(1, 6, 36, 146, 435, 1056, 2066, 3276, 4326, 4760, 4326, 3276, 2066, 1056, 435, 146, 36, 6, 1)$. Several interesting interpretations of these numbers are given in [39], $A003645$.

Theorem 6.11

- *(Symmetric plane partitions and Catalan numbers)*

$\#|\mathcal{B}_{4,n}| = \frac{1}{2}\ Cat_{n+1} \times Cat_{n+2}$.

- *(Symmetric plane partitions alternating sign matrices)*

$\#|\mathcal{B}_{n+3,n}| = \frac{1}{2}\ \ TSPP(n+1) \times ASM(n+1) = \frac{1}{2}\ \#|\mathcal{B}_{n+1,n+1}|$.

- *(Plactic decomposition of the $\mathcal{F}_n$-kernel)*

$\mathcal{F}_{n,m} = \sum_T u_T\ U_T(\{p_{ij}\})$,
where summation runs over the set of semistandard Young tableaux T of shape $\lambda \subset (n)^m$ filled by the numbers from the set $\{1, \ldots, m\}$.

- $U_T(\{p_{ij} = 1, \forall i, j\}) = \dim V_{\lambda'}^{\mathfrak{gl}(m)}$, *where λ denotes the shape of a tableau T, and λ' denotes the conjugate/transpose of a partition λ.*

7 Appendix

7.1 Some explicit formulas for $n = 4$ and compositions α such that $\alpha_i \leq n - i$ for $i = 1, 2, \cdots$

(1) **Schubert and $(-\beta)$-Grothendieck polynomials $\mathcal{G}^-[\alpha] := \mathcal{G}^{-\beta}[\alpha]$ for $n = 4$**

$\mathfrak{S}_{1234} = \mathfrak{S}[0] = 1 = \mathcal{G}^-[0]$,
$\mathfrak{S}_{2134} = \mathfrak{S}[1] = x_1 = \mathcal{G}^-[1]$,
$\mathfrak{S}_{1324} = \mathfrak{S}[01] = x_1 + x_2 = \mathcal{G}^-[01] + \beta\ \mathcal{G}^-[11]$,
$\mathfrak{S}_{1243} = \mathfrak{S}[001] = x_1 + x_2 + x_3 = \mathcal{G}^-[001] + \beta\ \mathcal{G}^-[011] + \beta^2\ \mathcal{G}^-[111]$,
$\mathfrak{S}_{3124} = \mathfrak{S}[2] = x_1^2 = \mathcal{G}^-[2]$,
$\mathfrak{S}_{2314} = \mathfrak{S}[11] = x_1x_2 = \mathcal{G}^-[11]$,
$\mathfrak{S}_{2143} = \mathfrak{S}[101] = x_1^2 + x_1x_2 + x_1x_3 = \mathcal{G}^-[101] + \beta\ \mathcal{G}^-[201] + \beta^2\ \mathcal{G}^-[111]$,
$\mathfrak{S}_{1342} = \mathfrak{S}[011] = x_1x_2 + x_1x_3 + x_2x_3 = \mathcal{G}^-[011] + 2\beta\ \mathcal{G}^-[111]$,
$\mathfrak{S}_{1423} = \mathfrak{S}[02] = x_1^2 + x_1x_2 + x_2^2 = \mathcal{G}^-[02] + \beta\ \mathcal{G}^-[12] + \beta^2\ \mathcal{G}^-[22]$,
$\mathfrak{S}_{4123} = \mathfrak{S}[3] = x_1^3 = \mathcal{G}^-[3]$,

$\mathfrak{S}_{3214} = \mathfrak{S}[21] = x_1^2 x_2 = \mathcal{G}^-[21]$,
$\mathfrak{S}_{2341} = \mathfrak{S}[111] = x_1 x_2 x_3 = \mathcal{G}^-[111]$,
$\mathfrak{S}_{2413} = \mathfrak{S}[12] = x_1^2 x_2 + x_1 x_2^2 = \mathcal{G}^-[12] + \beta\ \mathcal{G}^-[22]$,
$\mathfrak{S}_{1432} = \mathfrak{S}[021] = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_2 x_3 = \mathcal{G}^-[021] + 2\beta\ \mathcal{G}^-[121] + \beta\ \mathcal{G}^-[22] + \beta^2\ \mathcal{G}^-[221]$,
$\mathfrak{S}_{3142} = \mathfrak{S}[201] = x_1^2 x_2 + x_1^2 x_3 = \mathcal{G}^-[201] + \beta\ \mathcal{G}^-[211]$,
$\mathfrak{S}_{4213} = \mathfrak{S}[31] = x_1^3 x_2 = \mathcal{G}^-[31]$,
$\mathfrak{S}_{3412} = \mathfrak{S}[22] = x_1^2 x_2^2 = \mathcal{G}^-[22]$,
$\mathfrak{S}_{4132} = \mathfrak{S}[301] = x_1^3 x_2 + x_1^3 x_3 = \mathcal{G}^-[301] + \beta\ \mathcal{G}^-[311]$,
$\mathfrak{S}_{3241} = \mathfrak{S}[211] = x_1^2 x_2 x_3 = \mathcal{G}^-[211]$,
$\mathfrak{S}_{2431} = \mathfrak{S}[121] = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 = \mathcal{G}^-[121] + \beta\ \mathcal{G}^-[221]$,
$\mathfrak{S}_{4312} = \mathfrak{S}[32] = x_1^3 x_2^2 = \mathcal{G}^-[32]$,
$\mathfrak{S}_{4231} = \mathfrak{S}[311] = x_1^3 x_2 x_3 = \mathcal{G}^-[311]$,
$\mathfrak{S}_{3421} = \mathfrak{S}[221] = x_1^2 x_2^2 x_3 = \mathcal{G}^-[211]$,
$\mathfrak{S}_{4321} = \mathfrak{S}[321] = x_1^3 x_2^2 x_3 = \mathcal{G}^-[321]$.

Theorem 7.1 *(cf [22], Section 5.5)*
Each Schubert polynomial is a linear combination of $(-\beta)$- Grothendieck polynomials with nonnegative coefficients from the ring $\mathbb{N}[\beta]$.

(2) Key and reduced key polynomials

$K[0] = 1 = \widehat{K}[0]$,
$K[1] = x_1 = \widehat{K}[1]$,
$K[01] = x_1 + x_2,\ \widehat{K}[01] = x_2$,
$K[001] = x_1 + x_2 + x_3,\ \widehat{K}[001] = x_3$,
$K[2] = x_1^2 = \widehat{K}[2]$,
$K[11] = x_1 x_2 = \widehat{K}[11]$,
$K[101] = x_1 x_2 + x_1 x_3,\ \widehat{K}[101] = x_1 x_3$,
$K[02] = x_1^2 + x_1 x_2 + x_2^2,\ \widehat{K}[02] = x_1 x_2 + x_2^2$,
$K[011] = x_1 x_2 + x_1 x_3 + x_2 x_3,\ \widehat{K}[011] = x_2 x_3$,
$K[3] = x_1^3 = \widehat{K}[3]$,
$K[21] = x_1^2 x_2 = \widehat{K}[21]$,
$K[111] = x_1 x_2 x_3 = \widehat{K}[111]$,
$K[12] = x_1^2 x_2 + x_1 x_2^2,\ \widehat{K}[12] = x_1 x_2^2$,
$K[021] = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_2 x_3,\ \widehat{K}[021] = x_1 x_2 x_3 + x_2^2 x_3$,
$K[201] = x_1^2 x_2 + x_1^2 x_3,\ \widehat{K}[201] = x_1^2 x_3$,
$K[31] = x_1^3 x_2 = \widehat{K}[31]$,
$K[22] = x_1^2 x_2^2 = \widehat{K}[22]$,
$K[211] = x_1^2 x_2 x_3 = \widehat{K}[211]$,
$K[301] = x_1^3 x_2 + x_1^3 x_3,\ \widehat{K}[301] = x_1^3 x_3$,
$K[121] = x_1^2 x_2 x_3 + x_1 x_2^2 x_3,\ \widehat{K}[121] = x_1 x_2^2 x_3$,

$K[32] = x_1^3x_2^2 = \widehat{K}[32]$,
$K[311] = x_1^3x_2x_3 = \widehat{K}[311]$,
$K[221] = x_1^2x_2^2x_3 = \widehat{K}[221]$,
$K[321] = x_1^3x_2^2x_3 = \widehat{K}[321]$;

Note that if $n = 4$, then $\mathfrak{S}[\alpha] = K[\alpha]$ for all $\alpha \subset \delta_4$, except $\alpha = (101)$ in which $\mathfrak{S}[101] = K[2] + K[101]$.

(3) **Grothendieck and dual Grothendieck polynomials for $\beta = 1$**

$\mathcal{G}_{1234} = \mathcal{G}[0] = 1 = \mathfrak{S}[0]$,
$\mathcal{H}[0] = (1+x_1)^3(1+x_2)^2(1+x_3)$,
$\mathcal{G}_{2134} = \mathcal{G}[1] = x_1 = \mathfrak{S}[1]$,
$\mathcal{H}[1] = (1+x_1)^2(1+x_2)^2(1+x_3)\mathcal{G}[1]$,
$\mathcal{G}_{1324} = \mathcal{G}[01] = x_1 + x_2 + x_1x_2 = \mathfrak{S}[01] + \mathfrak{S}[11]$,
$\mathcal{H}[01] = (1+x_1)^2(1+x_2)(1+x_3)\mathcal{G}[01]$,
$\mathcal{G}_{1243} = \mathcal{G}[001] = x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3 = \mathfrak{S}[001] + \mathfrak{S}[011] + \mathfrak{S}[111]$,
$\mathcal{H}[001] = (1+x_1)^2(1+x_2)\mathcal{G}[001]$,
$\mathcal{G}_{3124} = \mathcal{G}[2] = x_1^2 = \mathfrak{S}[2]$,
$\mathcal{H}[2] = (1+x_1)(1+x_2)^2(1+x_3)\mathcal{G}[2]$,
$\mathcal{G}_{2314} = \mathcal{G}[11] = x_1x_2 = \mathfrak{S}[11]$,
$\mathcal{H}[11] = (1+x_1)^2(1+x_2)(1+x_3)\mathcal{G}[11]$,
$\mathcal{G}_{2143} = \mathcal{G}[101] = x_1^2 + x_1x_2 + x_1x_3 + x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_1^2x_2x_3 = \mathfrak{S}[101] + \mathfrak{S}[201] + \mathfrak{S}[111] + \mathfrak{S}[211]$,
$\mathcal{H}[101] = (1+x_1)(1+x_2)\mathcal{G}[101]$,
$\mathcal{G}_{1342} = \mathcal{G}[011] = x_1x_2 + x_1x_3 + x_2x_3 + 2\ x_1x_2x_3 = \mathfrak{S}[011] + 2\ \mathfrak{S}[111]$,
$\mathcal{H}[011] = (1+x_1)^2(1+x_2)(x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3)$,
$\mathcal{G}_{1423} = \mathcal{G}[02] = x_1^2 + x_1x_2 + x_2^2 + x_1^2x_2 + x_1x_2^2 = \mathfrak{S}[02] + \mathfrak{S}[12]$,
$\mathcal{H}[02] = (1+x_1)(1+x_3)(x_1^2 + x_1x_2 + x_2^2 + 2\ x_1^2x_2 + 2\ x_1x_2^2 + x_1^2x_2^2)$,
$\mathcal{G}_{4123} = \mathcal{G}[3] = x_1^3 = \mathfrak{S}[3]$,
$\mathcal{H}[3] = (1+x_1)^2(1+x_3)\mathcal{G}[3]$,
$\mathcal{G}_{3214} = \mathcal{G}[21] = x_1^2x_2 = \mathfrak{S}[21]$,
$\mathcal{H}[21] = (1+x_1)(1+x_2)(1+x_3)\mathcal{G}[21]$,
$\mathcal{G}_{2341} = \mathcal{G}[111] = x_1x_2x_3 = \mathfrak{S}[111]$,
$\mathcal{H}[111] = (1+x_1)^2(1+x_2)\mathcal{G}[111]$,
$\mathcal{G}_{2413} = \mathcal{G}[12] = x_1^2x_2 + x_1x_2^2 + x_1^2x_2^2 = \mathfrak{S}[12] + \mathfrak{S}[22]$,
$\mathcal{H}[12] = (1+x_1)(1+x_3)\mathcal{G}[12]$,
$\mathcal{G}_{1432} = \mathcal{G}[021] = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_2x_3 + 2\ x_1x_2x_3(x_1+x_2) + x_1^2x_2^2 + x_1^2x_2^2x_3 = \mathfrak{S}[021] + 2\ \mathfrak{S}[121] + \mathfrak{S}[22] + \mathfrak{S}[211]$,
$\mathcal{H}[021] = (1+x_1)\mathcal{G}[021]$,
$\mathcal{G}_{3142} = \mathcal{G}[201] = x_1^2x_2 + x_1^2x_3 + x_1^2x_2x_3 = \mathfrak{S}[201] + \mathfrak{S}[211]$,
$\mathcal{H}[201] = (1+x_1)(1+x_2)\mathcal{G}[201]$,
$\mathcal{G}_{4213} = \mathcal{G}[31] = x_1^3x_2 = \mathfrak{S}[31]$,

$\mathcal{H}[31] = (1+x_2)(1+x_3)\mathcal{G}[31]$,
$\mathcal{G}_{3412} = \mathcal{G}[22] = x_1^2x_2^2 = \mathfrak{S}[22]$,
$\mathcal{H}[22] = (1+x_1)(1+x_3)\mathcal{G}[22]$,
$\mathcal{G}_{4132} = \mathcal{G}[301] = x_1^3x_2 + x_1^3x_3 + x_1^3x_2x_3 = \mathfrak{S}[301] + \mathfrak{S}[311]$,
$\mathcal{H}[301] = (1+x_2)\mathcal{G}[301]$,
$\mathcal{G}_{3241} = \mathcal{G}[211] = x_1^2x_2x_3 = \mathfrak{S}[211]$,
$\mathcal{H}[211] = (1+x_2)\mathcal{G}[211]$,
$\mathcal{G}_{2431} = \mathcal{G}[121] = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2^2x_3 = \mathfrak{S}[121] + \mathfrak{S}[221]$,
$\mathcal{H}[121] = (1+x_1)(1+x_2)\mathcal{G}[121]$,
$\mathcal{G}_{4312} = \mathcal{G}[32] = x_1^3x_2^2 = \mathfrak{S}[32]$,
$\mathcal{H}[32] = (1+x_3)\mathcal{G}[32]$,
$\mathcal{G}_{4231} = \mathcal{G}[311] = x_1^3x_2x_3 = \mathfrak{S}[311]$,
$\mathcal{H}[311] = (1+x_2)\mathcal{G}[311]$,
$\mathcal{G}_{3421} = \mathcal{G}[221] = x_1^2x_2^2x_3 = \mathfrak{S}[221]$,
$\mathcal{H}[221] = (1+x_1)\mathcal{G}[221]$,
$\mathcal{G}_{4321} = \mathcal{G}[321] = x_1^3x_2^2x_3 = \mathfrak{S}[321] = \mathcal{H}[321]$.

Clearly that any β-Grothendieck polynomial is a linear combination of Schubert polynomials with coefficients from the ring $\mathbb{N}[\beta]$.

(4) **Key and reduced key Grothendieck polynomials**

$KG[0] = 1 = \widehat{KG}[0]$,
$KG[1] = x_1 = \widehat{KG}[1]$,
$KG[01] = x_1 + x_2 + x_1x_2,\ \widehat{KG}[01] = x_2 + x_1x_2$,
$KG[001] = x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3,\ \widehat{KG}[001] = x_3 + x_1x_3 + x_2x_3 + x_1x_2x_3$,
$KG[2] = x_1^2 = \widehat{KG}[2]$,
$KG[11] = x_1x_2 = \widehat{KG}[11]$,
$KG[101] = x_1x_2 + x_1x_3 + x_1x_2x_3,\ \widehat{KG}[101] = x_1x_3 + x_1x_2x_3$,
$KG[02] = x_1^2 + x_1x_2 + x_2^2 + x_1^2x_2 + x_1x_2^2,\ \widehat{KG}[02] = x_1x_2 + x_2^2 + x_1^2x_2 + x_1x_2^2$,
$KG[011] = x_1x_2 + x_1x_3 + x_2x_3 + 2\ x_1x_2x_3,\ \widehat{KG}[011] = x_2x_3 + x_1x_2x_3$,
$KG[3] = x_1^3 = \widehat{KG}[3]$,
$KG[21] = x_1^2x_2 = \widehat{KG}[21]$,
$KG[111] = x_1x_2x_3 = \widehat{KG}[111]$,
$KG[12] = x_1^2x_2 + x_1x_2^2 + x_1^2x_2^2,\ \widehat{KG}[12] = x_1x_2^2 + x_1^2x_2^2$,
$KG[201] = x_1^2x_2 + x_1^2x_3 + x_1^2x_2x_3,\ \widehat{KG}[201] = x_1^2x_3 + x_1^2x_2x_3$,
$KG[021] = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2^2x_3 + 2\ x_1^2x_2x_3 + 2\ x_1x_2^2x_3 + x_1^2x_2^2 + x_1^2x_2^2x_3$,
$\widehat{KG}[021] = x_1x_2x_3 + x_2^2x_3 + x_1^2x_2x_3 + 2\ x_1x_2^2x_3 + x_1^2x_2^2x_3$,
$KG[31] = x_1^3x_2 = \widehat{KG}[31]$,
$KG[22] = x_1^2x_2^2 = \widehat{KG}[22]$,
$KG[211] = x_1^2x_2x_3 = \widehat{KG}[211]$,
$KG[301] = x_1^3x_2 + x_1^3x_3 + x_1^3x_2x_3,\ \widehat{KG}[301] = x_1^3x_3 + x_1^3x_2x_3$,

$KG[121] = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2^2x_3,\ \widehat{KG}[121] = x_1x_2^2x_3 + x_1^2x_2^2x_3,$
$KG[32] = x_1^3x_2^2 = \widehat{KG}[32],$
$KG[311] = x_1^3x_2x_3 = \widehat{KG}[311],$
$KG[221] = x_1^2x_2^2x_3 = \widehat{KG}[221],$
$KG[321] = x_1^3x_2^2x_3 = \widehat{KG}[321],$

(5) 42 **double key polynomials for** $n = 4$

$\mathcal{K}_{id} = 1,$
$\mathcal{K}_1 = p_{1,1},$
$\mathcal{K}_2 = p_{1,2} + p_{2,1},$
$\mathcal{K}_3 = p_{1,3} + p_{2,2} + p_{3,1},$
$\mathcal{K}_{12} = p_{1,1}\ p_{2,1},$
$\mathcal{K}_{21} = p_{1,2}\ p_{1,1},$
$\mathcal{K}_{23} = p_{1,2}\ p_{2,2} + p_{1,2}\ p_{3,1} + p_{2,1}\ p_{3,1},$
$\mathcal{K}_{32} = p_{1,3}\ p_{1,2} + p_{1,3}\ p_{2,1} + p_{2,2}\ p_{2,1},$
$\mathcal{K}_{13} = p_{1,1}\ p_{2,2} + p_{1,1}\ p_{3,1},$
$\mathcal{K}_{31} = p_{1,3}\ p_{1,1},$
$\mathcal{K}_{22} = p_{1,2}\ p_{2,1},$
$\mathcal{K}_{33} = p_{1,3}\ p_{2,2} + p_{1,3}\ p_{3,1} + p_{2,2}\ p_{3,1},$
$\mathcal{K}_{123} = p_{1,1}\ p_{2,1}\ p_{3,1},$
$\mathcal{K}_{133} = p_{1,1}\ p_{2,2}\ p_{3,1},$
$\mathcal{K}_{212} = p_{1,2}\ p_{1,1}\ p_{2,1},$
$\mathcal{K}_{213} = p_{1,2}\ p_{1.1}\ p_{2,2} + p_{1,2}\ p_{1,1}\ p_{3,1},$
$\mathcal{K}_{223} = p_{1,2}\ p_{2,1}\ p_{3,1},$
$\mathcal{K}_{233} = p_{1,2}\ p_{2,2}\ p_{3,1},$
$\mathcal{K}_{321} = p_{1,3}\ p_{1,2}\ p_{1,1},$
$\mathcal{K}_{312} = p_{1,3}\ p_{1,1}\ p_{2,1} + p_{1,1}\ p_{2,2}\ p_{2,1},$
$\mathcal{K}_{313} = p_{1,3}\ p_{1,1}\ p_{2,2} + p_{1,3}\ p_{1,1}\ p_{3,1},$
$\mathcal{K}_{322} = p_{1,3}\ p_{1,2}\ p_{2,1} + p_{1,2}\ p_{2,2}\ p_{2,1},$
$\mathcal{K}_{323} = p_{1,3}\ p_{1,2}\ p_{2,2} + p_{1,3}\ p_{1,2}\ p_{3,1} + p_{1,3}\ p_{2,1}\ p_{3,1} + p_{2,2}\ p_{2,1}\ p_{3,1} + p_{1,3}\ p_{2,2}\ p_{2,1}$
$\mathcal{K}_{333} = p_{1,3}\ p_{2,2}\ p_{3,1},$
$\mathcal{K}_{2123} = p_{1,2}\ p_{1,1}\ p_{2,1}\ p_{3,1},$
$\mathcal{K}_{2132} = p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{2,1},$
$\mathcal{K}_{2133} = p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{3,1},$
$\mathcal{K}_{3123} = p_{1,3}\ p_{1,1}\ p_{2,1}\ p_{3,1} + p_{1,1}\ p_{2,2}\ p_{2,1}\ p_{3,1},$
$\mathcal{K}_{3132} = p_{1,3}\ p_{1,1}\ p_{2,2}\ p_{2,1},$
$\mathcal{K}_{3133} = p_{1,3}\ p_{1,2}\ p_{2,2}\ p_{3,1},$
$\mathcal{K}_{3212} = p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,1},$
$\mathcal{K}_{3213} = p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,2} + p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{3,1},$
$\mathcal{K}_{3223} = p_{1,3}\ p_{1,2}\ p_{2,1}\ p_{3,1} + p_{1,2}\ p_{2,2}\ p_{2,1}\ p_{3,1},$
$\mathcal{K}_{3232} = p_{1,3}\ p_{1,2}\ p_{2,2}\ p_{2,1},$

$\mathcal{K}_{3233} = p_{1,3}\ p_{1,2}\ p_{2,2}\ p_{3,1} + p_{1,3}\ p_{2,2}\ p_{2,1}\ p_{3,1}$,
$\mathcal{K}_{21323} = p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{2,1}\ p_{3,1}$,
$\mathcal{K}_{31323} = p_{1,3}\ p_{1,1}\ p_{2,2}\ p_{2,1}\ p_{3,1}$,
$\mathcal{K}_{32123} = p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,1}\ p_{3,1}$,
$\mathcal{K}_{32132} = p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{2,1}$,
$\mathcal{K}_{32133} = p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{3,1}$,
$\mathcal{K}_{32323} = p_{1,3}\ p_{1,2}\ p_{2,2}\ p_{2,1}\ p_{3,1}$,
$\mathcal{K}_{321323} = p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{2,1}\ p_{3,1}$.

Theorem 7.2 *(cf [30], the case $\beta = -1$)*
Each double β-Grothendieck polynomials is a linear combination of double key polynomials with the coefficients from the ring $\mathbb{N}[\beta]$.

Let us remind that the total number of double key polynomials is equal to the number of alternating sign matrices. We expect that the interrelations between double key polynomials which follow from the structure of the plactic algebra $\mathcal{PC}_n$, see Section 5.1, can be identified with the graph corresponding to the MacNeile completion of the poset associated with the Bruhat order on the symmetric group $\mathbb{S}_n$, see Section 8.2 for a definition of the MacNeile completion. It is an interesting problem to describe interrelation graph associated with the (rectangular) key polynomials corresponding to the Cauchy kernel for the algebra $\mathcal{PF}_{n,m}$.

(6) 26 **double key Grothendieck polynomials for** $n = 4$

$\mathcal{GK}_{id} = 1$,
$\mathcal{GK}_1 = p_{1,1} = \mathcal{K}_1$
$\mathcal{GK}_2 = p_{1,2} + p_{2,1} + p_{1,2}\ p_{2,1} = \mathcal{K}_2 + \mathcal{K}_{22}$,
$\mathcal{GK}_3 = p_{1,3} + p_{1,2} + p_{3,1} + p_{1,3}\ p_{2,2} + p_{1,3}\ p_{3,1} + p_{2,2}\ p_{3,1} + p_{1,3}\ p_{2,2}\ p_{3,1} = \mathcal{K}_3 + \mathcal{K}_{33} + \mathcal{K}_{333}$,
$\mathcal{GK}_{12} = p_{1,1}\ p_{2,1} = \mathcal{K}_{12}$,
$\mathcal{GK}_{21} = p_{2,1}\ p_{1,1} = \mathcal{K}_{21}$,
$\mathcal{GK}_{13} = p_{1,1}\ p_{2,2} + p_{1,1}\ p_{3,1} + p_{1,1}\ p_{2,2}\ p_{3,1} = \mathcal{K}_{13} + \mathcal{K}_{133}$,
$\mathcal{GK}_{31} = p_{3,1}\ p_{1,1} = \mathcal{K}_{31}$,
$\mathcal{GK}_{23} = p_{1,1}\ p_{2,2} + p_{1,2}\ p_{3,1} + p_{2,1}\ p_{3,1} + p_{1,2}\ p_{2,1}\ p_{3,1} + p_{1,2}\ p_{2,2}\ p_{3,1} = \mathcal{K}_{23} + \mathcal{K}_{223} + \mathcal{K}_{233}$,
$\mathcal{GK}_{32} = p_{1,3}\ p_{1,2} + p_{1,3}\ p_{2,1} + p_{2,2}\ p_{2,1} + p_{1,3}\ p_{1,2}\ p_{2,1} + p_{1,2}\ p_{2,1}\ p_{2,2} = \mathcal{K}_{32} + \mathcal{K}_{322}$,
$\mathcal{GK}_{123} = p_{1,1}\ p_{2,1}\ p_{3,1} = \mathcal{K}_{123}$,
$\mathcal{GK}_{212} = p_{1,2}\ p_{1,1}\ p_{2,1} = \mathcal{K}_{212}$,
$\mathcal{GK}_{213} = p_{1,2}\ p_{1,1}\ p_{2,2} + p_{2,1}\ p_{1,1}\ p_{3,1} + p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{3,1} = \mathcal{K}_{213} + \mathcal{K}_{2133}$,
$\mathcal{GK}_{312} = p_{1,3}\ p_{1,1}\ p_{2,1} + p_{1,1}\ p_{2,2}\ p_{2,1} + p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{2,1} = \mathcal{K}_{312} + \mathcal{K}_{2132}$,
$\mathcal{GK}_{313} = p_{1,3}\ p_{1,1}\ p_{2,2} + p_{1,3}\ p_{1,1}\ p_{3,1} + p_{1,3}\ p_{1,1}\ p_{2,2}\ p_{3,1} = \mathcal{K}_{313} + \mathcal{K}_{3133}$,
$\mathcal{GK}_{321} = p_{1,1}\ p_{1,2}\ p_{1,3} = \mathcal{K}_{123}$,
$\mathcal{GK}_{323} = p_{1,3}\ p_{1,2}\ p_{2,2} + p_{1,3}\ p_{1,2}\ p_{3,1} + p_{1,3}\ p_{2,1}\ p_{3,1} + p_{2,2}\ p_{2,1}\ p_{3,1} + p_{1,3}\ p_{2,2}\ p_{2,1} +$
$+ p_{1,3}\ p_{1,2}\ p_{2,1}\ p_{2,2} + p_{1,2}\ p_{2,1}\ p_{2,2}\ p_{3,1} + p_{1,3}\ p_{1,2}\ p_{2,2}\ p_{3,1} + p_{1,3}\ p_{1,2}\ p_{2,1}\ p_{3,1}$
$+ p_{1,2}\ p_{2,2}\ p_{2,1}\ p_{3,1} + p_{1.3}\ p_{1,2}\ p_{2,2}\ p_{2,1}\ p_{3,1} = \mathcal{K}_{323} + \mathcal{K}_{3232} + \mathcal{K}_{3233} + \mathcal{K}_{3223} + \mathcal{K}_{32323}$,
$\mathcal{GK}_{2123} = p_{1,2}\ p_{1,1}\ p_{2,1}\ p_{3,1} = \mathcal{K}_{2123}$,
$\mathcal{GK}_{2132} = p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{2,1} = \mathcal{K}_{2132}$,

$\mathcal{GK}_{3123} = p_{1,3}\ p_{1,1}\ p_{2,1}\ p_{3,1}\ p_{1,1}\ p_{2,2}\ p_{2,1}\ p_{3,1} + p_{1,3}\ p_{1,1}\ p_{2,2}\ p_{2,1}\ p_{3,1} = \mathcal{K}_{3123} + \mathcal{K}_{31323}$,
$\mathcal{GK}_{3212} = p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,1} = \mathcal{K}_{3212}$,
$\mathcal{GK}_{3213} = p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,2} + p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{3,1} + p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{3,1} = \mathcal{K}_{3213} + \mathcal{K}_{32133}$,
$\mathcal{GK}_{21323} = p_{1,2}\ p_{1.1}\ p_{2,2}\ p_{2,1}\ p_{3,1} = \mathcal{K}_{21323}$,
$\mathcal{GK}_{32123} = p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,1}\ p_{31} = \mathcal{K}_{32123}$,
$\mathcal{GK}_{32132} = p_{1,3}\ p_{1,2}\ p_{1,1}\ p_{2,2}\ p_{2,1} = \mathcal{K}_{32132}$,
$\mathcal{GK}_{321323} = p_{3,1}\ p_{2,1}\ p_{1,1}\ p_{2,2}\ p_{2,1}\ p_{3,1} = \mathcal{K}_{321323}$.

(7) 14 double local key polynomials for $n = 4$.

$\mathcal{LK}_{id} = 1$,
$\mathcal{LK}_1 = \mathcal{K}_1$,
$\mathcal{LK}_2 = \mathcal{K}_2$,
$\mathcal{LK}_3 = \mathcal{K}_3$,
$\mathcal{LK}_{12} = \mathcal{K}_{12}$,
$\mathcal{LK}_{21} = \mathcal{K}_{21} + \mathcal{K}_{212}$,
$\mathcal{LK}_{13} = \mathcal{K}_{13} + \mathcal{K}_{31} + \mathcal{K}_{313}$,
$\mathcal{LK}_{23} = \mathcal{K}_{23}$,
$\mathcal{LK}_{32} = \mathcal{K}_{32} + \mathcal{K}_{323}$,
$\mathcal{LK}_{123} = \mathcal{K}_{123}$,
$\mathcal{LK}_{213} = \mathcal{K}_{213} + \mathcal{K}_{2123}$,
$\mathcal{LK}_{312} = \mathcal{K}_{312} + \mathcal{K}_{3123}$,
$\mathcal{LK}_{321} = \mathcal{K}_{321} + \mathcal{K}_{3212} + \mathcal{K}_{3213} + \mathcal{K}_{32123} + \mathcal{K}_{32132} + \mathcal{K}_{321323}$,
$\mathcal{LK}_{2132} = \mathcal{K}_{2132} + \mathcal{K}_{21323}$.

(8) 35 $(3, 2)$-key polynomials.
$U_{id} = 1$,
$U_1 = p_{11} + p_{23}$,
$U_2 = p_{12} + p_{21}$,
$U_3 = p_{13} + p_{22}$,
$U_{11} = p_{11}p_{23}$,
$U_{12} = p_{11}p_{21}$,
$U_{13} = p_{11}p_{22}$,
$U_{21} = p_{12}p_{11} + p_{12}p_{23} + p_{21}p_{23}$,
$U_{23} = p_{12}p_{22}$,
$U_{22} = p_{12}p_{21}$,
$U_{31} = p_{13}p_{11} + p_{13}p_{23} + p_{22}p_{23}$,
$U_{32} = p_{13}p_{12} + p_{13}p_{21} + p_{22}p_{21}$,
$U_{33} = p_{13}p_{22}$,
$U_{211} = p_{12}p_{11}p_{23} + p_{11}p_{21}p_{23}$,
$U_{212} = p_{12}p_{11}p_{21} + p_{12}p_{21}p_{23}$,
$U_{213} = p_{12}p_{11}p_{22} + p_{12}p_{22}p_{23}$,
$U_{311} = p_{13}p_{11}p_{23} + p_{11}p_{22}p_{23}$,

$U_{312} = p_{13}p_{11}p_{21} + p_{11}p_{22}p_{21}$,
$U_{313} = p_{13}p_{11}p_{22} + p_{13}p_{22}p_{23}$,
$U_{321} = p_{13}p_{12}p_{11} + p_{13}p_{12}p_{23} + p_{13}p_{21}p_{23} + p_{22}p_{21}p_{23}$,
$U_{322} = p_{13}p_{12}p_{21} + p_{12}p_{22}p_{21}$,
$U_{323} = p_{13}p_{12}p_{22} + p_{13}p_{22}p_{21}$,
$U_{2121} = p_{12}p_{11}p_{21}p_{23}$,
$U_{2131} = p_{12}p_{11}p_{22}p_{23}$,
$U_{2132} = p_{12}p_{11}p_{22}p_{23}$,
$U_{3132} = p_{13}p_{11}p_{22}p_{23}$,
$U_{3131} = p_{13}p_{11}p_{22}p_{23}$,
$U_{3232} = p_{13}p_{21}p_{22}p_{23}$,
$U_{3211} = p_{13}p_{12}p_{11}p_{23} + p_{13}p_{11}p_{21}p_{23} + p_{11}p_{22}p_{21}p_{23}$,
$U_{3212} = p_{13}p_{12}p_{11}p_{22} + p_{13}p_{12}p_{21}p_{23} + p_{12}p_{21}p_{22}p_{23}$,
$U_{3213} = p_{13}p_{12}p_{11}p_{21} + p_{13}p_{22}p_{21}p_{23} + +p_{12}p_{22}p_{21}p_{23}$,
$U_{32121} = p_{12}p_{11}p_{22}p_{21}p_{23} + p_{13}p_{12}p_{11}p_{21}p_{23}$,
$U_{33131} = p_{13}p_{12}p_{11}p_{22}p_{23} + p_{13}p_{11}p_{22}p_{21}p_{23}$,
$U_{32132} = p_{13}p_{12}p_{11}p_{22}p_{21} + p_{13}p_{12}p_{22}p_{21}p_{23}$,
$U_{321321} = p_{13}p_{12}p_{11}p_{22}p_{21}p_{23}$.

(9) Polynomials $\mathcal{KN}_w := \mathcal{KN}_w^{(\beta,\alpha)}\ (1)$ for $n = 4$.

$\mathcal{KN}_{id} = 1$,
$\mathcal{KN}_1 = \mathcal{KN}_2 = \mathcal{KN}_3 = \beta + 1 + \alpha\ \beta$,
$\mathcal{KN}_{12} = 1 + 2\alpha + \alpha^2 + 3\alpha\ \beta + 3\alpha^2\ \beta + \alpha\ \beta^2 + 2\alpha^2\ \beta^2$, (13)
$\mathcal{KN}_{21} = 2 + 3\alpha + \alpha^2 + \beta + 3\alpha\ \beta + 2\alpha^2\ \beta + \alpha^2\ \beta^2$, (13),
$\mathcal{KN}_{13} = 1 + 2\alpha + \alpha^2 + 2\alpha\ \beta + 2\alpha^2\ \beta + \alpha^2\ \beta^2 = (1 + \alpha + \alpha\ \beta)^2$, (9),
$\mathcal{KN}_{23} = \mathcal{KN}_{12}$,
$\mathcal{KN}_{32} = \mathcal{KN}_{21}$,
$\mathcal{KN}_{132} = 2+5\alpha+4\alpha^2+\alpha^3+\beta+7\alpha\ \beta+10\alpha^2\ \beta+4\alpha^3\ \beta+2\alpha\beta^2+7\alpha^2\ \beta^2+5\alpha^3\ \beta^2+\alpha^2\ \beta^3+2\alpha^3\ \beta^3 = (1 + \alpha + \alpha\ \beta)(2 + 3\alpha + \alpha^2 + \beta + 4\alpha\ \beta + 3\alpha^2\ \beta + \alpha\ \beta^2 + 2\alpha^2\ \beta^2)$, (51),

$\mathcal{KN}_{121} = 1+3\alpha+3\alpha^2+\alpha^3+4\alpha\ \beta+7\alpha^2\ \beta+3\alpha^3\ \beta+\alpha\ \beta^2+4\alpha^2\beta^2+3\alpha^3\ \beta^2+\alpha^3\ \beta^3$, (31),

$\mathcal{KN}_{321} = 5 + 10\alpha + 6\alpha^2 + \alpha^3 + 5\beta + 14\ \alpha\ \beta + 12\alpha^2\ \beta + 3\alpha^3\ \beta + \beta^2 + 4\alpha\ \beta^2 + 6\alpha^2\ \beta^2 + 3\alpha^3\ \beta^2 + \alpha^3\ \beta^3$, (71),
$\mathcal{KN}_{232} = \mathcal{KN}_{121}$,

$\mathcal{KN}_{123} = 1 + 3\alpha + 3\alpha^2 + \alpha^3 + 6\alpha\ \beta + 12\alpha^2\ \beta + 6\alpha^3\ \beta + 4\ \alpha\ \beta^2 + 14\alpha^2\ \beta^2 + 10\alpha^3\ \beta^2 + \alpha\ \beta^3 + 5\alpha^2\ \beta^3 + 5\alpha^3\ \beta^3 = \beta^3\alpha^3\ \mathcal{KN}_{321}^{(\beta^{-1},\alpha^{-1})}(1)$, (71),
$\mathcal{KN}_{213} = \mathcal{KN}_{132}$,

$\mathcal{KN}_{3121} = 3 + 10\alpha + 12\alpha^2 + 6\alpha^3 + \alpha^4 + 2\beta + 16\alpha\ \beta + 29\alpha^2\ \beta + 19\alpha^3\ \beta + 4\alpha^4\ \beta + 7\alpha\ \beta^2 + 21\alpha^2\ \beta^2 + 20\alpha^3\ \beta^2 + 6\alpha^4\ \beta^2 + \alpha\ \beta^3 + 4\alpha^2\ \beta^3 + 7\alpha^3\ \beta^3 + 4\alpha^4\ \beta^3 + \alpha^4\ \beta^4$, (173),

$\mathcal{KN}_{2321} = \mathcal{KN}_{3121}$,

$\mathcal{KN}_{1213} = 1+4\alpha+6\alpha^2+4\alpha^3+\alpha^4+7\alpha\ \beta+20\alpha^2\ \beta+19\alpha^3\ \beta+6\alpha^4\ \beta+4\alpha\ \beta^2+21\alpha^2\ \beta^2+29\alpha^3\ \beta^2+12\alpha^4\ \beta^2+\alpha\ \beta^3+7\alpha^2\ \beta^3+16\ \alpha^3\ \beta^3+10\alpha^4\ \beta^3+2\alpha^3\ \beta^4+3\alpha^4\ \beta^4$, (173),
$\mathcal{KN}_{1232} = \mathcal{KN}_{1213}$,

$\mathcal{KN}_{2132} = 3+9\alpha+10\alpha^2+5\alpha^3+\alpha^4+3\beta+16\alpha\ \beta+28\alpha^2\ \beta+20\alpha^3\ \beta+5\alpha^4\ \beta+\beta^2+7\alpha\ \beta^2+24\alpha^2\ \beta^2+28\alpha^3\ \beta^2+10\alpha^4\ \beta^2+7\alpha^2\ \beta^3+16\alpha^3\ \beta^3+9\alpha^4\ \beta^3+\alpha^2\ \beta^4+3\alpha^3\ \beta^4+3\alpha^4\ \beta^4$, (209),

$\mathcal{KN}_{21321} = 3+12\alpha+19\alpha^2+15\alpha^3+6\alpha^4+\alpha^5+3\beta+21\alpha\ \beta+49\alpha^2\ \beta+52\alpha^3\ \beta+26\alpha^4\ \beta+5\alpha^5\beta+\beta^2+9\alpha\ \beta^2+39\alpha^2\ \beta^2+64\alpha^3\ \beta^2+43\alpha^4\ \beta^2+10\alpha^5\ \beta^2+10\alpha^2\ \beta^3+32\alpha^3\ \beta^3+32\alpha^4\ \beta^3+10\alpha^5\ \beta^3+\alpha^2\ \beta^4+5\alpha^3\ \beta^4+9\alpha^4\ \beta^4+5\alpha^5\ \beta^4+\alpha^5\ \beta^5$, (483),

$\mathcal{KN}_{12312} = 1+5\alpha+10\alpha^2+10\alpha^3+5\alpha^4+\alpha^5+9\alpha\ \beta+32\alpha^2\ \beta+43\alpha^3\ \beta+26\ \alpha^4\ \beta+6\ \alpha^5\ \beta+5\alpha\ \beta^2+32\alpha^2\ \beta^2+64\alpha^3\ \beta^2+52\alpha^4\ \beta^2+15\ \alpha^5\ \beta^2+\alpha\ \beta^3+10\alpha^2\ \beta^3+39\alpha^3\ \beta^3+49\alpha^4\ \beta^3+19\alpha^5\ \beta^3+9\alpha^3\ \beta^4+21\ \alpha^4\ \beta^4+12\alpha^5\ \beta^4+\alpha^3\ \beta^5+3\alpha^4\ \beta^5+3\alpha^5\ \beta^5$, (483),

$\mathcal{KN}_{12321} = 2+9\alpha+16\alpha^2+14\alpha^3+6\alpha^4+\alpha^5+\beta+18\alpha\ \beta+54\alpha^2\ \beta+64\alpha^3\ \beta+33\ \alpha^4\ \beta+6\ \alpha^5\ \beta+14\alpha\ \beta^2+65\alpha^2\ \beta^2+101\alpha^3\ \beta^2+64\alpha^4\ \beta^2+14\ \alpha^5\ \beta^2+6\alpha\ \beta^3+33\alpha^2\ \beta^3+65\alpha^3\ \beta^3+54\alpha^4\ \beta^3+16\alpha^5\ \beta^3+\alpha\ \beta^4+6\alpha^2\ \beta^4+14\alpha^3\ \beta^4+18\alpha^4\ \beta^4+9\alpha^5\ \beta^4+\alpha^4\ \beta^5+2\alpha^5\ \beta^5$, (707),

$\mathcal{KN}_{121321} = 1+6\alpha+15\alpha^2+20\alpha^3+15\alpha^4+6\alpha^5+\alpha^6+10\alpha\ \beta+45\alpha^2\ \beta+81\alpha^3\ \beta+73\alpha^4\ \beta+33\alpha^5\ \beta+6\alpha^6\ \beta+5\alpha\ \beta^2+44\alpha^2\ \beta^2+116\alpha^3\ \beta^2+135\alpha^4\ \beta^2+73\alpha^5\ \beta^2+15\alpha^6\ \beta^2+\alpha\ \beta^3+15\alpha^2\ \beta^3+69\alpha^3\ \beta^3+116\alpha^4\ \beta^3+81\alpha^5\ \beta^3+20\alpha^6\ \beta^3+\alpha^2\ \beta^4+15\alpha^3\ \beta^4+44\alpha^4\ \beta^4+45\alpha^5\ \beta^4+15\alpha^6\ \beta^4+\alpha^3\ \beta^5+5\alpha^4\ \beta^5+10\alpha^5\ \beta^5+6\alpha^6\ \beta^5+\alpha^6\ \beta^6 = \beta^6\alpha^6\ \mathcal{KN}_{121321}^{(\beta^{-1},\alpha^{-1})}(1)$, (1145).

(10) Polynomials $\mathcal{KN}_w^{(\beta,\alpha,\gamma)} := \mathcal{KN}_w^{(\beta,\alpha,\gamma)}(1)$ for $n = 3$

$\mathcal{KN}_{id}^{(\beta,\alpha,\gamma)} = 1$,
$\mathcal{KN}_1^{(\beta,\alpha,\gamma)} = \mathcal{KN}_2^{(\beta,\alpha,\gamma)} = 1+(\beta+\gamma)(1+\alpha+\gamma)$, (7),

$\mathcal{KN}_{21}^{(\beta,\alpha,\gamma)} = 1+2\alpha+\alpha^2+3\alpha\ \beta+3\alpha^2\ \beta+\alpha\ \beta^2+2\alpha^2\ \beta^2+5\gamma+8\alpha\ \gamma+3\alpha^2\ \gamma+4\beta\ \gamma+11\alpha\ \beta\ \gamma+4\alpha^2\ \beta\ \gamma+\beta^2\ \gamma+4\alpha\ \beta^2\ \gamma+9\gamma^2+10\alpha\ \gamma^2+2\alpha^2\ \gamma^2+8\beta\ \gamma^2+8\alpha\ \beta\ \gamma^2+2\beta^2\ \gamma^2+7\gamma^3+4\alpha\ \gamma^3+4\beta\ \gamma^3+2\gamma^4$, (109),

$\mathcal{KN}_{12}^{(\beta,\alpha,\gamma)} = 2+3\alpha+\alpha^2+\beta+3\alpha\ \beta+2\alpha^2\ \beta+\alpha^2\ \beta^2+7\gamma+8\alpha\ \gamma+2\alpha^2\ \gamma+4\beta\ \gamma+7\alpha\ \beta\ \gamma+2\alpha^2\ \beta\ \gamma+2\alpha\ \beta^2\ \gamma+9\ \gamma^2+7\alpha\ \gamma^2+\alpha^2\ \gamma^2+5\beta\ \gamma^2+4\alpha\ \beta\ \gamma^2+\beta^2\ \gamma^2+5\gamma^3+2\alpha\ \gamma^3+2\beta\ \gamma^3+\gamma^4$, (82),

$\mathcal{KN}_{121}^{(\beta,\alpha,\gamma)} = 1+3\alpha+3\alpha^2+\alpha^3+4\alpha\ \beta+7\alpha^2\ \beta+3\alpha^3\ \beta+\alpha\ \beta^2+4\alpha^2\ \beta^2+3\alpha^3\ \beta^2+\alpha^3\ \beta^3+6\gamma+15\alpha\ \gamma+12\alpha^2\ \gamma+3\alpha^3\ \gamma+3\beta\ \gamma+20\alpha\ \beta\ \gamma+22\alpha^2\ \beta\ \gamma+6\alpha^3\ \beta\ \gamma+7\alpha\ \beta^2\ \gamma+12\alpha^2\ \beta^2\ \gamma+3\alpha^3\ \beta^2\ \gamma+3\alpha^2\ \beta^3\ \gamma+15\gamma^2+30\alpha\ \gamma^2+18\alpha^2\ \gamma^2+3\alpha^3\ \gamma^2+12\beta\ \gamma^2+37\alpha\ \beta\ \gamma^2+24\alpha^2\ \beta\ \gamma^2+3\ \alpha^3\beta\ \gamma^2+3\beta^2\ \gamma^2+15\alpha\ \beta^2\ \gamma^2+9\alpha^2\ \beta^2\ \gamma^2+3\alpha\ \beta^3\ \gamma^2+20\gamma^3+30\alpha\ \gamma^3+12\alpha^2\ \gamma^3+\alpha^3\ \gamma^3+18\beta\ \gamma^3+30\alpha\ \beta\ \gamma^3+9\alpha^2\ \beta\ \gamma^3+6\beta^2\ \gamma^3+9\alpha\ \beta^2\ \gamma^3+\beta^3\ \gamma^3+15\gamma^4+15\alpha\ \gamma^4+3\alpha^2\ \gamma^4+12\beta\ \gamma^4+$

$9\alpha\ \beta\ \gamma^4 + 3\beta^2\ \gamma^4 + 6\gamma^5 + 3\alpha\ \gamma^5 + 3\beta\ \gamma^5 + \gamma^6, \qquad (521).$

7.2 MacMeille completion of a partially ordered set [8]

Let $(\Sigma, \leq)$ be a partially ordered set (poset for short) and $X \subseteq \Sigma$. Define

- The set of upper bounds for X, namely,

$$X^{up} := \{z \in \Sigma \mid x \leq z \ \ \forall x \in X\},$$

- The set of lower bounds for X, namely,

$$X^{lo} := \{z \in \Sigma \mid z \leq x \ \ \forall x \in X\},$$

- A poset $(\mathcal{MN}(\Sigma), \leq)$, namely,

$$\mathcal{MN}(\Sigma) := \{\mathcal{MN}(X) \mid X \subseteq \Sigma\}, \quad where \quad \mathcal{MN}(X) := \left(X^{up}\right)^{lo}.$$

Clearly, $X \subseteq \mathcal{MN}(X)$ and $\mathcal{MN}(\mathcal{MN}(X)) = \mathcal{MN}(X)$,

- A map $\kappa : \Sigma \longrightarrow \mathcal{MN}(\Sigma)$, namely, $\kappa(X) = \mathcal{MN}(X), \quad X \subseteq \Sigma.$

Proposition 7.3

- *The map κ is an embedding, that is for $X, Y \subseteq \Sigma$,*

$$X \leq Y \ \ if \ \ and \ \ only \ \ if \ \ \kappa(X) \subseteq \kappa(Y),$$

- *Poset $(\mathcal{MN}(\Sigma), \leq)$ is a* **lattice**, *called the* **MacNeille completion** *of poset $(\Sigma, \leq)$.*

Proposition 7.4 *([43])*

Let $(\Sigma, \leq)$ be a poset. Then there is a poset $(L, \leq)$ and a map $\kappa : \Sigma \longrightarrow L$ such that

(1) *κ is an embedding,*

(2) *$(L, \leq)$ is a complete lattice* [9]*,*

(3) *For each element $a \in L$ one has*

(a) $\mathcal{MN}(\{x \in \Sigma \mid \kappa(x) \leq a\}) = \{x \in \Sigma \mid \kappa(x) \leq a\}$,

(b) $a = \bigvee\{\kappa(x) \mid x \in \Sigma,\ \kappa(x) \leq a\}$.

Moreover, the pair $(\kappa, (L, \leq))$ is defined uniquely up to an order preserving isomorphism.

Therefore, the lattice $(L, \leq)$, is an order isomorphic to the MacNeille completion $\mathcal{MN}(\Sigma)$ of a poset Σ.

[8] For the reader convenience we review a definition and basic facts concerning the MacNeille completion of a poset, see for example, notes by E.Turunen on web-site
math.tut.fi/ eturunen/AppliedLogics007/Mac1.pdf

[9] That is every subset of L has a meet and join, see e.g. [41], v.1, p.249.

Problem 7.5

Let Σ be a (finite) **graded poset** [10] *, denote by*

$$r_{\Sigma}(t) := \sum_{a \in \Sigma} t^{r(a)}$$

the rank generating function of a poset Σ. Here $r(a)$ denotes the rank/degree of an element $a \in \Sigma$.

Describe polynomial $r_{\mathcal{MN}(\Sigma)}(t)$.

In the present paper we are interesting in properties of the MacNeille completion of the Bruhat poset $\mathcal{B}_n = \mathcal{B}(\mathbb{S}_n)$ corresponding to the symmetric group $\mathbb{S}_n$. Below we briefly describe a construction of the MacNeille completion $L_n(\mathbb{S}_n) := \mathcal{MN}_n(\mathcal{B}_n)$ follow [20], and [41], v. 2, p. 552, **d**.

Let $w = (w_1 w_2 \dots w_n) \in \mathbb{S}_n$, associate wit w a semistandard Young tableaux $T(w)$ of the staircase shape $\delta_n = (n-1, n-2, \dots, 2, 1)$ filled by integer numbers from the set $[1, n] := \{1, 2, \dots, n\}$ as follows :

the i-th row of of $T(w)$, denoted by $R_i(w)$, consists of the numbers $w_1, \dots, w_{n-i+1}$ in increasing order. Clearly the tableaux $T(w) = [T_{i,j}(w)]_{1 \le i < j \le n-1}$ obtained in such a manner, satisfies the so-called *monotonic* and *flag* conditions, namely,

(**1**) (monotonic conditions) $T_{1,i} \ge T_{2,i-1} \ge \dots \ge T_{i,1}$, $i = 1, \dots, n-1$,

(**2**) (flag conditions) $R_1(w) \supset R_2(w) \supset \dots \supset R_{n-1}(w)$.

Denote by $L(\mathbb{S}_n)$ the subset of the set of all Young tableaux $T \in STY(\delta_n \le n)$ consisting of that T which satisfies the monotonicity conditions (**1**). The set $L(\mathbb{S}_n)$ has the natural poset structure denoted by $''\ \ge\ ''$, and defined as follows:
if $T^{(1)} = [t^{(1)}_{ij}]_{1 \le i < j \le n-1}$ and $T^{(2)} = [t^{(2)}_{ij}]_{1 \le i < j \le n-1}$ belong to the set $L(\mathbb{S}_n)$, then by definition

$$T^{(1)} \ge T^{(2)} \ \ if \ \ and \ \ only \ \ if \ \ t^{(1)}_{ij} \ge t^{(2)}_{ij} \ \ for \ \ all \ \ 1 \le i < j \le n-1.$$

It is clearly seen that the set $L(\mathbb{S}_n)$ is closed under the following operations

- (meet $T^{(1)}$ $T^{(2)}$) $\bigwedge(T^{(1)}, T^{(2)}) := T^{(1)} \bigwedge T^{(2)} = [min(t^{(1)}_{i,j}, t^{(2)}_{i,j})]$,
- (join $T^{(1)}$ $T^{(2)}$) $\bigvee((T^{(1)}, T^{(2)}) := T^{(1)} \bigvee T^{(2)} = [max(t^{(1)}_{i,j}, t^{(2)}_{i,j})]$.

Theorem 7.6 *([20])*

The poset $L_(\mathbb{S}_n)$ is a complete distributive lattice with number of vertices equals to the number $ASM(n)$ that is the number of alternating sigh matrices of size $n \times n$. Moreover, the lattice $L_(\mathbb{S}_n)$ is order isomorphic to the MacNeille completion of the Bruhat poset $\mathcal{B}_n$.

Indeed it is not difficult to prove that the set of all monotonic triangles obtained by applying repeatedly operation $\bigvee$(=meet) to the set $\{T(w), w \in \mathbb{S}_n$ of triangles corresponding to all elements of the symmetric group $\mathbb{S}_n$, coincides with the set of all monotonic triangles $L(\mathbb{S}_{\ltimes}$.

[10] See e.g. [41], v.1, p. 244, or *en.wikipedia.org/wiki/Graded_poset.*

The natural map $\kappa : \mathbb{S}_n \longrightarrow L(\mathbb{S}_n$ is obviously embedding, and all other conditions of Proposition 7.2 are satisfied. Therefore $L(\mathbb{S}_n) = \mathcal{MN}(\mathcal{B}_n$.the fact that the lattice $L(\mathbb{S}_n$ is a distributive one follows from the well-known identity

$$max(x, min(y, z)) = min(max(x, y), max(x, z)), \quad x, y, z \in_{\geq 0}^{\mathbb{R}})^3.$$

In the lattice $L(\mathbb{S}_n$ this identity can be written in the following forms

$$T^{(1)} \bigvee (T^{(2)} \bigwedge T^{(3)}) = (T^{(1)} \bigwedge T^{(2)}) \bigvee (T^{(1)} \bigwedge T^{(3)}),$$

$$T^{(1)} \bigwedge (T^{(2)} \bigvee T^{(3)}) = (T^{(1)} \bigvee T^{(2)}) \bigwedge (T^{(1)} \bigvee T^{(3)}).$$

Finally the fact that the cardinality of the lattice $L(\mathbb{S}_n)$ is equal to the number $ASM(n)$ had been proved by A. Lascoux and M.-P. Schützenberger [20].

■

If $T = [t_{ij}] \in L(\mathbb{S}_n)$, define *rank* of T, denoted by $r(T)$, as follows:

$$r(T) = \sum_{1 \leq i < j \leq n-1} t_{ij} - \binom{n}{3}$$

.

It had been proved by C. Ehresmann [4] that

- $v \leq w$ with respect to the Bruhat order in the symmetric group $\mathbb{S}_n$ if and only if $T_{i,j}(v) \leq T_{i,j}(w)$ for all $1 \leq i < j \leq n-1$.

It follows from an improved tableau criterion for Bruhat order on the symmetric group [1] that [11]

- The length $\ell(w)$ of a permutation $w \in \mathbb{S}_n$ can be computed as follows

$$\ell(w) = r(T(w)) - \sum_{(i,j) \in I(w)} (j - i - 1),$$

where $I(w) := \{(i, j) \mid 1 \leq i < j \leq n, \ \ w_i > w_j\}$ denotes the set of *inversions* of permutation w; a detailed proof can be found in [18].

For example, consider permutation $w = [4, 6, 2, 7, 5, 1, 3]$. Then the code $c(w)$ of w is equal to $c(w) = (3, 4, 1, 3, 2)$, and w has the length $\ell(w) = 13$. The corresponding Young tableau or monotonic triangle displayed below

$$T(w) = \begin{bmatrix} 1 & 2 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 6 & 7 & \\ 2 & 4 & 6 & 7 & & \\ 2 & 4 & 6 & & & \\ 4 & 6 & & & & \\ 4 & & & & & \end{bmatrix},$$

[11] It has been proved in [1], Corollary 5, that the Ehresmann criterion stated above is equivalent to either the criterion

$$T_{i,j}^{(1)} \leq T_{i,j}^{(2)} \ \ for \ \ all \ \ j \ such \ \ that \ \ w_j > w_{j+1} \ \ and \ \ 1 \leq i \leq j,$$

or that

$$T_{i,j}^{(1)} \leq T_{i,j}^{(2)} \ \ for \ \ all \ \ j \in \{1, 2, \ldots, n-1\} \backslash \{k \mid v_k > v_{k+1}\} \ \ and \ \ 1 \leq i \leq j.$$

Wherefore, $r(T(w)) = |T(w)| - \binom{7}{3} = 94 - 56 = 38$. On the other hand, the inversion set $I(w) = \{(1,3),(1,6),(1,7),(2,3),(2,5),(2,6),(2,7),(3,6),(4,5),(4,6),(4,7),(5,6),(5,7)\}$, hence $\sum_{(i,j)\in I(w)}(j-i-1) = 10+9+2+3+1 = 25$ and $\ell(w) = 38-25 = 13$, as it should be.

■

It is easily seen that the polynomial $r_{\mathcal{MN}_n}(t)$ is symmetric and $deg(r_{\mathcal{MN}_n}(t)) = \binom{n+1}{3}$, For example,

$r(\mathcal{MN}_3) = (1,2,1,2,1)$, $r(\mathcal{MN}_4) = (1,3,3,5,6,\mathbf{6},6,5,3,3,1)$,
$r(\mathcal{MN}_5) = (1,4,6,10,16,20,27,34,37,40,\mathbf{39},40,37,34,27,20,16,10,6,4,1)$;

$r(\mathbb{S}_3 \subset \mathcal{MN}_3) = (1,2,0,2,1)$, $r(\mathbb{S}_4 \subset \mathcal{MN}_4) = (1,3,1,4,2,\mathbf{2},2,4,1,3,1))$,
$r(\mathbb{S}_5 \subset \mathcal{MN}_5) = (1,4,3,6,7,6,4,10,6,10,\mathbf{6},10,6,10,4,6,7,6,3,4,1)$.

Conjecture 7.7

- *The number* $Coef f_{[\binom{n+1}{3}/2]} r_{\mathcal{MN}_n}(t)$ *is a divisor of the number* $ASM(n)$*;*

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