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Abstract

We give an algebra-combinatorial constructions of (noncommutative) generating functions of double Schubert and double β-Grothendieck polynomials corresponding to the full flag varieties associated to the Lie groups of classical types A, B, C and D. Our approach is based on the decomposition of certain " transfer matrices " corresponding to the exponential solution to the quantum Yang–Baxter equations associated with either NiCoxeter or IdCoxeter Coxeter algebras of classical type.

The "triple" β-Grothendieck polynomials $\mathfrak{G}_w^W(X, Y, Z)$ we have introduced, satisfy, among other things, the coherency and (generalized) vanishing conditions. Their generating function has a nice factorization in the algebra $Id_\beta Coxeter(W)$, and as a consequence, the polynomials $\mathfrak{G}_w^W(X, Y, Z)$ admit a combinatorial description in terms of W-type pipe dreams.

1 Introduction

Let G be a Lie group of one of the classical types A_{n-1}, B_n, C_n, D_n. Let B be a Borel subgroup, B^- be the opposite Borel subgroup, $T = B \cap B^-$ be the maximal torus and $W := W(G)$ be the Weyl group. Let $\mathcal{X} = G/B$ be the flag variety of a classical type. The description of the equivariant cohomology ring $H^*_T(G/B, \mathbb{Z})$ of the flag variety G/B is well-known, and can be presented in the form

$$H^*_T(G/B, \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n]/J_n,$$

where J_n is the ideal generated by

$(type\ \mathbf{A_{n-1}})$ $\quad e_i(x_1, \dots, x_n) - e_i(y_1, \dots, y_n),\ \ 1 \le i \le n.$

$(type\ \mathbf{C_n}\ or\ \mathbf{B_n})$ $\quad e_i(x_1^2, \dots, x_n^2) - e_i(y_1^2, \dots, y_n^2),\ 1 \le i \le n.$

$(type \mathbf{D_n})$ $\quad e_i(x_1^2, \dots, x_n^2) - e_i(y_1^2, \dots, y_n^2), \quad 1 \le i \le n-1,$

and $\quad e_n(x_1, \dots, x_n) - e_n(y_1, \dots, y_n).$

There are distinguish elements $[\mathcal{X}_w]_T$ in the cohomology ring $H^*_T(G/B, \mathbb{Z})$, namely, the Poincare dual classes of the homology classes corresponding to the Schubert subvarieties $\mathcal{X}_w = \overline{B\ w\ B/B} \subset \mathcal{X}$. In the cohomology ring $H^*_T(G/B, \mathbb{Z})$ one can write $[\mathcal{X}_w] := \mathcal{X}_w(X_n, Y_n)$ for a certain (homogeneous) polynomial $\mathcal{X}_w(X_n, Y_n)$ of degree $l(w)$ in each set of variables X_n and Y_n. The sets of variables $X_n := (x_1, \dots, x_n)$ and $Y_n := (y_1, \dots, y_n)$ are known as the *Borel generators* of the equivariant cohomology ring $H^*_T(G/B, \mathbb{Z})$; the variables Y_n correspond to generators of the equivariant cohomology ring of a point, $H^*_T(pt) = \mathbb{Z}[Y_n]$, and the set of variables X_n comes from the Chern classes of some linear vector bundles over the flag variety in question. By definition, the equivariant Schubert polynomials, or double Schubert polynomials $\mathcal{X}_w(X_n, Y_n)$, are polynomials which express the equivariant classes $[\mathcal{X}_w]_T$ in terms of Borel generators.

Polynomials $\mathcal{X}_w(X_n, Y_n),\ \ w \in W$, are defined only modulo the ideal J_n. It is known that for any finite dimensional, semisimple Lie group of rank n the set of polynomials $\mathcal{X}_w(X_n, Y_n),\ \ w \in W$, possess and characterized by the following properties (modulo the ideal J_n of relations in the cohomology ring $H^*_T(G/B, \mathbb{Z})$)

(**A**) Polynomials $\mathcal{X}_w(X_n, Y_n)$, $w \in W$, form a $\mathbb{Z}[Y]$-linear basis of the cohomology ring $H^*_T(G/B, \mathbb{Z})$;

(**B**) Coherency conditions) For any simple root α,

$$\partial^{(x)}_\alpha\ \mathcal{X}_w(X_n, Y_n) = \begin{cases} \mathcal{X}_{ws_\alpha}(X_n, Y_n), & l(ws_\alpha) = l(w) + 1, \\ 0 \quad otherwise; \end{cases}$$

$$\partial_\alpha^{(y)}\ \mathcal{X}_w(X_n,Y_n)=\begin{cases}\mathcal{X}_{s_\alpha w}(X_n,Y_n), & l(s_\alpha w)=l(w)+1,\\ 0 & otherwise;\end{cases}$$

(**C**) (Vanishing conditions) Let $w,v\in W$, then

$$\mathcal{X}_w(X_n,-v(X_n))=0,\quad unless\ \ v\le w,$$

where the symbol $\le$ denotes the Bruhat order on the group W;

(**D**) (Normalization condition) $\mathcal{X}_{id}(X_n,Y_n)=1$, where $id\in W$ is the identity element.

Recall that s_α stands for the reflection corresponding to a simple root α; $\partial_\alpha^{(x)}=\frac{1-s_\alpha}{\alpha}$ denotes the corresponding Demazure operator acting on the variables X_n; $v\in W$ acts on X_n via the reflection representation.

In the case of flag varieties corresponding to the classical groups, polynomials $\mathcal{X}_w(X_n,Y_n)$ possess also the so-called *stability property:*

(**E**) Let G_n be a Lie group of one of the classical series, and $\iota: G_n\hookrightarrow G_{n+1}$ be the canonical inclusion corresponding to the Dynkin diagram's embedding. If $w\in G_n$, then

$$\mathcal{X}_w(X_n,Y_n)=\mathcal{X}_{\iota(w)}(X_{n+1},Y_{n+1})_{x_{n+1}=0=y_{n+1}}.$$

In the case of type A_{n-1} flag varieties, A. Lascoux and M.-P. Schüutzenberger have constructed a family of polynomials $\{\mathfrak{S}_w(X_n,Y_n)\in\mathbb{Z}_{\ge 0}[X_n,Y_n], w\in\mathbb{S}_n\}$, called double Schubert polynomials, that satisfies the all properties (**A**)–(**E**) listed above, see e.g. [18],[19] for detail account. It happened that the Lascoux–Schützenberger double Schubert polynomials have non-negative integer coefficients and possess many nice combinatorial and algebraic properties. One of the basic properties of the double Schubert polynomials is the Cauchy type identity, that connects the simple Schubert polynomials $\mathfrak{S}_w(X_n):=\mathfrak{S}_w(X_n,0)$ with the double ones. Namely,

$$\mathfrak{S}_w(X_n,Y_n)=\sum_{u,v}\ \mathfrak{S}_u(X_n)\ \mathfrak{S}_v(Y_n), \tag{1}$$

where the sum runs over all $u,v\in\mathbb{S}_n$ such that $w=v^{-1}\ u$ and $l(w)=l(u)+l(v)$.

Recall that $\mathfrak{S}_w(X_n)\in\mathbb{Z}_{\ge 0}[X_n]$ denotes the Lascoux– Schützenberger Schubert polynomial corresponding to a permutation $w\in\mathbb{S}_n$. The set of Schubert polynomials $\mathfrak{S}_w(X_n), w\in\mathbb{S}_n$ satisfy the Stability and Coherency Conditions (without passing to the quotient modulo the ideal J_n !) and their

images in the cohomology ring $H^*(\mathcal{F}l_n, \mathbb{Z})$ form a basis. Conversely, if W is a Weyl group of a classical type, and one has a family of polynomials $\phi_w(X_n) \in \mathbb{Z}[X_n], w \in W$, which satisfies the conditions $(\mathbf{A}) - -(\mathbf{E})$ (except that $(\mathbf{C})$), then the family of double polynomials

$$\Phi_w(X_n, Y_n) = \sum_{u,v} \phi_u(X_n)\ \phi_v(Y_n), \tag{2}$$

summed over all $u, v \in W$ such that $w = v^{-1}\ u$ and $l(w) = l(u) + l(v)$, *also satisfies* the conditions $(\mathbf{A})$–$(\mathbf{E})$, except, probably, the Vanishing Conditions. This brings up the natural question:

Consider any Weyl group W of a classical type, does there exist a family of polynomials $\phi_w(X_n) \in \mathbb{Z}[X_n], w \in W$ such that the family of double polynomials $\Phi_w(X_n, Y_n),\ w \in W$ defined by (2) satisfies the Vanishing Conditions $(\mathbf{C})$?.

The main goal of the present paper is to show that the answer on this question is *Yes*. Namely, the Schubert polynomials of the second kind introduced originally in [3] for all Weyl groups of classical types, generate the set of double polynomials, called (B, C, D)-double Schubert polynomials of the second kind, that satisfy the Vanishing conditions (without passing to the quotient). As it was observed in [6], the Schubert polynomials of the first kind introduced in [6], are closely related with the Schubert polynomials for classical groups introduced by S. Billey and M. Haiman [3]. Thus, the main result of the present work can be considered as a generalization of the construction of Schubert polynomials for classical groups given in [3] and [6], to the case of double Schubert polynomials. We also give a construction of double Grothendieck polynomials for classical groups. Our main result in the case of Schubert polynomials can be stated as follows.

Let W be a Weyl group of one the types B, C or D. Consider the Nil–Coxeter algebra $Nil(W)$, see Section 2.3, and the corresponding elements $B^W(x) \in \mathbb{Z}[x][Nil(W)]$. Finally, for any $w \in W$ define *double Schubert polynomial* $\mathfrak{S}^W_w(X_n, Y_n)$ via the decomposition

$$(\mathfrak{S}^{A_{n-1}}(-Y_n))^{-1}\ \sqrt{B^W(Y_n)B^W(-X_n)}\ \mathfrak{S}^{A_{n-1}}(X_n) = \sum_{w \in W} \mathfrak{S}^W_w(X_n, Y_n)\ u_w,$$

where $\mathfrak{S}^{A_{n-1}}(X_n)$ denotes the Schubert expression of type A_{n-1}, introduced in [4], and $B^W(X_n) = B^W(x_1) \cdots B^W(x_n)$.

Theorem 1.1 *The set of polynomials* $\mathfrak{S}_w^W(X_n, Y_n) \in \mathbb{Z}[X_n, Y_n]$, $w \in W$ *satisfies the all conditions* (**A**)–(**E**), *and can be taken as a system of representatives for the equivariant Schubert classes* $[\mathcal{X}_w]_T \in H_T^*(G/B, \mathbb{Z})$.

We define double Grothendieck polynomials $\mathfrak{G}_w^W(X_n, Y_n)$ corresponding to a Weyl group W of classical type [1] via the decomposition of the expression

$$(\mathfrak{S}^{A_{n-1}}(\phi_\beta(-Y_n))^{-1}\ \sqrt{\mathcal{B}^W(Y_n)\mathcal{B}^W(X_n)}\ \mathfrak{S}^{A_{n-1}}(\phi_\beta(X_n)) = \sum_{w\in W}\ \mathfrak{G}_w^W(X_n, Y_n)\ u_w$$

in the Id–Coxeter algebra $Id_\beta(W)$. Hereinafter, $\phi_\beta(x) := x/(1 - \frac{\beta}{2}\ x)$ and $\phi_\beta(-X_n) = (\phi_\beta(-x_1), \ldots, \phi_\beta(-x_n))$.

Theorem 1.2 *The double Grothendieck polynomials* $\mathfrak{G}_w^W(X_n, Y_n)$ *corresponding to a Lie group of classical type, satisfy the all conditions* (**A**)–(**E**), *if one replaces the divided difference operators in the Coherency conditions* (**B**), *on the isobaric divided difference operators.*

To study combinatorial properties of the double Schubert polynomials $\mathfrak{S}_w^W(X_n, Y_n)$ of the second kind, as well as to reveal their connections with the polynomials introduced and studied by S. Billey and M. Haiman [3], it is convenient to introduce the set of polynomials $\mathfrak{S}_w^W(X_n, Y_n, Z_m)$ depending on three set of variables via the decomposition in the Nil–Coxeter algebra $Nil(W)$ of the following expression:

$$(\mathfrak{S}^{A_{n-1}}(-Y_n))^{-1}\ B^W(Z_m)\ \mathfrak{S}^{A_{n-1}}(-X_n) = \sum_{w\in W}\ \mathfrak{S}_w^W(X_n, Y_n, Z_m)\ u_w.$$

These polynomials are common generalization of both the Stanley symmetric polynomials $F_w^W(Z_m)$ of type W, coming from the decomposition $B^W(Z_m) = \sum_{w\in W} F_w(Z_m)\ u_w$, and the double Schubert polynomials of the first kind introduced in [6] and Section 3.

In a similar fashion one can define "triple" β-Grothendieck polynomials of the classical type $W = A, B, C, D$:

$$(\mathfrak{G}^{A_{n-1}}(-Y_n))^{-1}\ B^W(Z_m)\ \mathfrak{G}^{A_{n-1}}(-X_n) = \sum_{w\in W}\ \mathfrak{G}_w^W(X_n, Y_n, Z_m)\ u_w,$$

[1] in the case of Weyl groups of type C one has to use the function $\phi_{2\beta}(-X_n)$ instead of that $\phi_\beta(-X_n)$.

where now the left and the right parts are treated in the Id-Coxeter algebra $Id_\beta(W)$.

An algebra-combinatorial approach is used as the basic tool in the present paper gives rise naturally to the study of the generating functions for the double and triple Schubert and β-Grothendieck polynomials are introduced in the present notes, but in more wider class of algebras such as the Hecke and Temperley–Lieb algebras of classical types, and the plactic and reduced plactic algebras of classical type. Recall that the plactic algebra of classical type W is the quotient of the unital free associative algebra over $\mathbb{Q}$ of rang $n := r(W)$ by the two-sided ideal generated by the W-plactic (or W-Knuth–Kraśkiewicz) relations has been described and studied in [17].

Problem 1.1

(**A**) *To extend results concerning the plactic algebra and plactic polynomials of type A_{n-1} obtained in [13] to the case of the plactic algebras corresponding to plactic monoid of type $W := B_n$, C_n and D_n, introduced in [17]. In particular, describe the MacNeille completion $\mathcal{MN}(W)$ of the the Bruhat graph (poset) associated with the Weyl groups of classical type, as well as describe the decomposition of the W-Cauchy kernel in the reduced W-plactic algebra.*

(**B**) *Find a geometric interpretation of plactic Schubert and Grothendieck polynomials of classical types. Does the MacNeille complition $\mathcal{MN}(W)$ can be realized as a convolution algebra of a certain nonsingular algebraic variety ?*

A few words about the history of problems considered in the present paper in order. The algebraic and combinatorial theory of single and double Schubert polynomials of type A was initiated and studied comprehensively by A. Lascoux and M.-P. Schützenberger in the middle of 80's of the last century. We refer the reader to the nice written books [18] and [19] for detailed exposition of this subject. The general description of the cohomology and equivariant cohomology rings, K-theory and equivariant K-theory of generalized flag varieties corresponding to a symmetrizable Kac–Moody group was created by B. Kostant and S. Kumar. Details can be found in the book [15]

A bit of history concerning the present notes. This paper (as well as [13]) is an update version of my notes written for Course " Schubert Calculus" has been delivered at the Graduate School of Mathematical Sciences, University

of Tokyo (1995/96), and at the Graduate School of Mathematics, Nagoya University (1998/99).

Final remark, in [10] the polynomials $\mathfrak{S}_w^W(X,Y,Z)$ have been rediscovered using a geometrical approach, see also [1].

2 Basic definitions

2.1 Weyl groups of classical types

2.1.1 The symmetric group

The symmetric group $\mathbb{S}_n$, $n \geq 1$, is the group of all permutations of the set $[1,n] := \{1,2,\ldots,n\}$. As is customary, we will identify a permutation $w \in \mathbb{S}_n$ with its image, i.e. with the sequence $(w(1),w(2),\ldots,w(n))$. Sometimes we will write w_i instead of $w(i)$, and $w_1 \ldots w_n$ instead of sequence $(w(1),w(2),\ldots,w(n))$.

For $i = 1,\ldots,n-1$ let s_i denote the transposition that interchanges i and $i+1$, and fixes all other numbers in $[1,n]$. It is well-known that the elements $s_1,\ldots,s_{n-1}$ generate the symmetric group $\mathbb{S}_n$ and satisfy the following relations

(1) $s_i^2 = 1$;

(2) $s_i\ s_j = s_j\ s_i$, if $|i-j| \geq 2$;

(3) $s_i\ s_{i+1}\ s_i = s_{i+1}\ s_i\ s_{i+1}$ for $i = 1,\ldots,n-2$.

For a permutation $w \in \mathbb{S}_n$ let's denote by $D(w)$ the *diagram* of the permutation w, see e.g. [18], i.e.

$$(i,j) \in D(w) \Longleftrightarrow i < w^{-1}(j) \ \ and \ \ j < w(i).$$

It is well-know that $l(w) = |D(w)|$, where $l(w)$ denotes the length of a permutation w, i.e. the minimal number of generators whose product is w.

2.1.2 The hyperochtahedral group

The hyperochtahedral group $B_n := W(B_n)$, $n \geq 2$, is the group of symmetries of the n-dimensional cube. As an abstract group it can be given by the set of generators $s_0, s_1, \ldots, s_{n-1}$ satisfying relations

(1) $s_i^2 = 1$, if $i = 0,1,2,\ldots,n-1$;

(2) $s_i\ s_j = s_j\ s_i$, if $|i-j| \geq 2$;

(3) $s_i\ s_{i+1}\ s_i = s_{i+1}\ s_i\ s_{i+1}$, if $i = 1, \ldots, n-2$;
(4) $s_0\ s_1\ s_0\ s_1 = s_1\ s_0\ s_1\ s_0$.

The elements of B_n can be thought of as *signed permutations*: a generator s_i, $i > 0$, interchanges entries in the i'th and $(i+1)'$st positions, and the generator s_0 changes the sign of the first entry. As in any Coxeter group, the *length l(w)* of an element w is the minimal number of generators whose product is w. Such a factorization of minimal length, or the corresponding sequence of indices, is called a *reduced decomposition* of w. As is customary, we will write $\bar{i}$ instead of $-i$. For example, the action of the generator s_0 looks like $s_0(12\ldots n) = \bar{1}2\ldots n$, and $\bar{\bar{a}} = a$. For any sign permutation $w = w_1 w_2 \ldots w_n$ let $\bar{w}$ denotes the sign permutation $\bar{w}_1 \bar{w}_2 \ldots \bar{w}_n$. It is clear that if $w \in \mathbb{S}_n \subset B_n$, then $l(w) + l(\bar{w}) = n^2$.

2.1.3 The group $W(D_n)$

The group $D_n := W(D_n)$ is a subgroup of elements $w \in W(B_n)$ which make an even number of sign changes. The standard generators for this group are s_i, $i = 1, \ldots, n-1$ and $s_{\hat{1}}$, subject to the set of relations

$$\begin{cases} s_i^2 = 1, & if \quad i = \hat{1}, 1, 2, \ldots, n-1; \\ s_i\ s_j = s_j\ s_i, & if \quad |i-j| \geq 2; \\ s_i\ s_{\hat{1}} = s_{\hat{1}}\ s_i, & if \quad i \neq 2; \\ s_i\ s_{i+1}\ s_i = s_{i+1}\ s_i\ s_{i+1}, & if \quad i = 1, \ldots, n-2; \\ s_2\ s_{\hat{1}}\ s_2 = s_{\hat{1}}\ s_2\ s_{\hat{1}}. \end{cases}$$

The elements of D_n can be thought of as *even signed permutations*: a generator s_i, $i > 0$, interchanges variables x_i and x_{i+1}, and the generator $s_{\hat{1}}$ replaces x_1 with $-x_2$ and x_2 with $-x_1$.

2.2 Divided difference and isobaric divided difference operators

The hyperoctahedral group B_n acts on the ring of polynomials $P_n := \mathbb{Q}[x_1, \ldots, x_n]$ in the natural way. Namely, s_i interchanges x_i and x_{i+1}, for $i = 1, \ldots, n-1$, and the special generator s_0 acts by

$$s_0 f(x_1, x_2, \ldots, x_n) = f(-x_1, x_2, \ldots, x_n).$$

The divided difference operator ∂_i for $i = 1, \ldots, n-1$, acts on the ring of polynomials P_n by

$$\partial_i f(x_1, \ldots, x_n) = \frac{f - s_i(f)}{x_i - x_{i+1}},$$

and the B type divided difference operator $\partial_0 := \partial_0^B$ acts on the ring of polynomials P_n by

$$\partial_0 f(x_1, x_2, \ldots, x_n) = \frac{f(x_1, x_2, \ldots, x_n) - f(-x_1, x_2, \ldots, x_n)}{x_1}.$$

We consider also C and D types divided difference operators ∂_0^C and $\partial_{\hat{1}}^D$ which act on the ring of polynomials by $\partial_0^C(f) := 1/2\ \partial_0^B(f)$ and

$$\partial_{\hat{1}}^D f(x_1, x_2, \ldots, x_n) = \frac{f(x_1, x_2, \ldots, x_n) - f(-x_2, -x_1, \ldots, x_n)}{x_1 + x_2}.$$

Finally, we define *isobaric divided difference operators* π_α^G for each simple root α of the corresponding Lie group G of classical type. Namely, let β be a parameter, define $(1 \leq i \leq n-1)$

$$\pi_i^A(f) = \partial_i((1 + \beta\ x_{i+1})f) = \pi_i^C(f), \quad \pi_i^B(f) = \partial_i((1 + \frac{\beta}{2}\ x_{i+1})\ f) = \pi_i^D(f),$$

$$\pi_0^B(f) = \partial_0^B((1 - \frac{\beta}{2}\ x_1)f), \quad \pi_0^C(f) = \partial_0^C((1 - \beta\ x_1)f),$$

$$\pi_{\hat{1}}^D(f) = \partial_{\hat{1}}^D((1 - \frac{\beta}{2}\ x_1)(1 - \frac{\beta}{2}\ x_2)f).$$

Note that $\pi_i^2 = -\beta\pi_i$ for types A and C; $\pi_i^2 = -\frac{\beta}{2}\pi_i$ for types B and D; $\pi_0^2 = -\beta\pi_0$ for type B; $\pi_0^2 = -2\beta\pi_0$ for type C, and $\pi_{\hat{1}}^2 = -\beta\pi_{\hat{1}}$ for type D.

2.3 Nil–Coxeter and Id–Coxeter algebras of classical types

2.3.1 Nil–Coxeter and Id–Coxeter algebras of type A

• Let NC_n denotes the Nil–Coxeter algebra type A_{n-1}. Recall that NC_n is an associative algebra generated over $\mathbb{Z}$ by the set of generators $\{u_1, \ldots, u_{n-1}\}$ subject to the set of relations

(a) $u_i^2 = 0$ for $i = 1, \ldots, n-1$,

(b) $u_i\ u_j = u_j\ u_i$, if $1 \leq i, j \leq n-1$ and $|i-j| \geq 2$,

(c) $u_i\ u_{i+1}\ u_i = u_{i+1}\ u_i\ u_{i+1}$, if $i = 1, \ldots, n-2$.

It is well-known that $dim NC_n = n!$ and the elements $\{u_w, w \in \mathbb{S}_n\}$ form a $\mathbb{Z}$-linear basis in the algebra NC_n, where by definition we set $u_w = u_{a_1} \ldots u_{a_l}$ for any reduced decomposition $w = s_{a_1} \ldots u_{a_l}$ of $w \in \mathbb{S}_n$ chosen.

• Let β be a parameter. The Id–Coxeter algebra of type A, denoted by $Id(A_{n-1}) := Id_\beta(A_{n-1})$, is an associative algebra generated over $\mathbb{Z}[\beta]$ by the set of generators $\{u_1, \ldots, u_{n-1}\}$ subject to the set of relations (b) and (c) from the definition of the algebra NC_n, and the relations $u_i^2 = \beta\ u_i$ for $i = 1, \ldots, n-1$, instead of that (a). It is well-known that the elements $\{u_w, w \in \mathbb{S}_n\}$ form a $\mathbb{Z}[\beta]$-linear basis in the algebra $Id_\beta(A_{n-1})$.

2.3.2 Nil–Coxeter and Id–Coxeter algebras of type B

• Let $Nil(B_n)$ denotes the nil–Coxeter algebra of type B. Recall that $Nil(B_n)$ is an associative algebra generated over $\mathbb{Z}$ by the set of generators $\{u_0, u_1, \ldots, u_{n-1}\}$ subject the set of relations

(a) $u_i^2 = 0$ for $i = 0, 1, \ldots, n-1$,

(b) $u_i\ u_j = u_j\ u_i$, if $1 \leq i, j \leq n-1$ and $|i-j| \geq 2$,

(c) $u_i\ u_{i+1}\ u_i = u_{i+1}\ u_i\ u_{i+1}$, if $i = 1, \ldots, n-2$,

(d) $u_0\ u_1\ u_0\ u_1 = u_1\ u_0\ u_1\ u_0$, $u_0\ u_i = u_i\ u_0$ for $i = 2, \ldots, n-1$.

It is well-known that $dim Nil(B_n) = 2^n\ n!$ and the elements $\{u_w, w \in B_n\}$ form a $\mathbb{Z}$-linear basis in the algebra $Nil(B_n)$, where by definition we set $u_w = u_{a_1} \ldots u_{a_l}$ for any reduced decomposition $w = s_{a_1} \ldots u_{a_l}$ of $w \in W(B_n)$ chosen.

• Let β be a parameter. The Id–Coxeter algebra of type B, denoted by $Id(B_n) := I_\beta(B_n)$, is an associative algebra generated over the ring $\mathbb{Q}[\beta]$ by the set of generators $\{u_0, u_1, \ldots, u_{n-1}\}$ subject to the set of relations (b), (c) and (d) from the definition of the algebra $Nil(B_n)$, and the relations $u_i^2 = \beta\ u_i$ for $i = 0, 1, \ldots, n-1$, instead of that (a). It is well-known that the elements $\{u_w, w \in W(B_n)\}$ form a $\mathbb{Z}[\beta]$-linear basis in the algebra $Id_\beta(B_n)$.

Let $x_1, \ldots, x_n$ be a set of variables which assumed to be commute with all generators $u_0, \ldots, u_{n-1}$. Define deformed addition $x +_\beta\ y = x + y + \beta\ x\ y$, so that $x -_\beta\ y = (x-y)/(1+\beta\ y)$.

Follow [6], define

$$h_i(x) := 1 + x\ u_i, \quad for \quad i = 1, \ldots, n-1, \quad h_0(x) := h_0^B(x) = 1 + x\ u_0.$$

Define also

$$h_0^C(x) = 1 + 2x\ u_0 \quad and \quad h_{\hat{1}}^D(x) = 1 + x\ u_{\hat{1}} := h_{\hat{1}}(x).$$

Lemma 2.1 *(Cf [4],[6]) The elements $h_i(x)$ satisfy the following relations*

(1) $h_i(x)\ h_j(y) = h_j(y)\ h_i(x)$, *if* $1 \le i, j \le n-1$ *and* $|i-j| \ge 2$,

(2) $h_0(x)\ h_i(y) = h_i(y)\ h_0(x)$, *if* $i = 2, \dots, n-1$,

(2) $h_i(x)\ h_i(y) = h_i(x+y)$ *in the algebra* $Nil(B_n)$,

(2a) $h_i(x)\ h_i(y) = h_i(x + y + \beta\ x\ y) = h_i(x +_\beta y)$ *in the algebra* $Id_\beta(B_n)$,

(3) *(Yang–Baxter equation of type A in the algebra $Nil(B_n)$)*

$$h_i(x)\ h_{i+1}(x+y)\ h_i(y) = h_{i+1}(y)\ h_i(x+y)\ h_{i+1}(x), \quad i = 1, \dots, n-2,$$

(3a) *(Yang–Baxter equation of type A in the algebra $Id_\beta(B_n)$)*

$$h_i(x)\ h_{i+1}(x +_\beta y)\ h_i(y) = h_{i+1}(y)\ h_i(x +_\beta y)\ h_{i+1}(x), \quad i = 1, \dots, n-2,$$

(4) *(Yang–Baxter equation of type B in the algebra $Nil(B_n)$)*

$$h_0(y)\ h_1(x+y)\ h_0(x)\ h_1(x-y) = h_1(x-y)\ h_0(x)\ h_1(x+y)\ h_0(y).$$

(4a) *(Yang–Baxter equation of type B in the algebra $Id_\beta(B_n)$)*

$$h_0(y)\ h_1(x +_\beta y)\ h_0(x)\ h_1(x -_\beta y) = h_1(x -_\beta y)\ h_0(x)\ h_1(x +_\beta y)\ h_0(y).$$

Let us introduce in the algebra $Id_\beta(B_n)$ the elements (cf [4], [5]):

$$\mathcal{A}(x) := A_i(\phi_\beta(x)) := A_i^{(n)}(\phi_\beta(x)) = \prod_{a=n-1}^{i} h_a(\phi_\beta(x)),$$

$$\mathcal{B}(x) := B(\phi_\beta(x)) := \prod_{a=n-1}^{1} h_a(\phi_\beta(x))\ h_0(\phi_\beta(x)) \prod_{a=1}^{n-1} h_a(\phi_\beta(x)),$$

$$\mathcal{C}(x) := C(\phi_{2\beta}(x)) := \prod_{a=n-1}^{1} h_a(\phi_{2\beta}(x))\ h_0(\phi_{2\beta}(2x)) \prod_{a=1}^{n-1} h_a(\phi_{2\beta}(x)),$$

where $\phi_\beta(x) := x/(1 - \frac{\beta}{2}x)$.

In the nil–Coxeter algebra $Nil(B_n)$ these elements can be written in the form

$$B_n(x) := A_1^{(n)}(x)\ h_0(x)\ A_1^{(n)}(-x)^{-1},$$
$$C(x) := C_n(x) = A_1^{(n)}(x)\ h_0(2x)\ A_1^{(n)}(-x)^{-1}.$$

Lemma 2.2 *([4], [6]) One has*

(1) $A_i(x)\ A_i(y) = A_i(y)\ A_i(x),\ \ A_i(x) = A_{i+1}(x)\ h_i(x)$.

(2) $B(x)\ B(y) = B(y)\ B(x),\quad C(x)\ C(y) = C(y)\ C(x)$

in the both algebras $Nil(B_n)$ *and* $Id_\beta(B_n)$. *Therefore,*

(2a) $\mathcal{B}(x)\ \mathcal{B}(y) = \mathcal{B}(y)\ \mathcal{B}(x),\quad \mathcal{C}(x)\ \mathcal{C}(y) = \mathcal{C}(y)\ \mathcal{C}(x)$

in the algebra $Id_\beta(B_n)$.

(3) $B(x)\ B(-x) = 1,\quad C(x)\ C(-x) = 1$ *in the Nil–Coxeter algebra* $Nil(B_n)$.

(3a) $\mathcal{B}(x)\ \mathcal{B}(-x) = 1,\quad \mathcal{C}(x)\ \mathcal{C}(-x) = 1$ *in the algebra* $Id_\beta(B_n)$.

Finally, let us consider the following expressions in the algebra $Nil(B_n)$

$$H(Z_m) = B(z_1)\ B(z_2)\cdots B(z_m) = \sum_{w\in W(B_n)} F_w(Z_m)\ u_w, \tag{3}$$

$$\mathbb{H}(t_1, t_2, \ldots, t_m) = \sqrt{B(t_1)\ B(t_2)\cdots B(t_m)}, \tag{4}$$

and those in the algebra $Id_\beta(B_n)$

$$\mathcal{H}(Z_m) = \mathcal{B}(z_1)\ \mathcal{B}(z_2)\cdots \mathcal{B}(z_m) = \sum_{w\in W(B_n)} \mathcal{F}_w(Z_m)\ u_w, \tag{5}$$

$$\mathfrak{H}(t_1, t_2, \ldots, t_m) = \sqrt{\mathcal{B}(t_1)\ \mathcal{B}(t_2)\cdots \mathcal{B}(t_m)}. \tag{6}$$

It follows from Lemma 2.2 that the $H(Z_m)$ and $\mathcal{H}(Z_m)$ as well as $\mathbb{H}(t_1, \cdots, t_m)$ and $\mathfrak{H}(t_1, t_2, \ldots, t_m)$, are symmetric functions of the variables $z_1, \ldots, z_m$ and $t_1, \ldots, t_m$ respectively.

Lemma 2.3

(1) *For any* $w \in W(B_n)$ *the polynomials* $F_w(Z_m)$ *and* $\mathcal{F}_w(Z_m)$ *are supersymmetric functions of the variables* $Z_m = (z_1, \ldots, z_m)$, *i.e.* $F_w(Z_m)$ *and* $\mathcal{F}_w(Z_m)$ *are polynomials of the odd power sums* $p_1(Z_m), p_3(Z_m), \ldots$.

(2) *([3], [16]) For any* $w \in W(B_n)$, *polynomial* $F_w(Z_m)$ *is a linear combination with non-negative integer coefficients of Schur P-functions.*

(3) *([6]) Assume that the variables* $z_1, z_2, \ldots$ *and* $t_1, t_2, \ldots$ *are related by*

$$\frac{p_k(t_1, t_2, \ldots)}{2} = p_k(z_1, z_2, \ldots), \quad k = 1, 3, 5, \ldots. \tag{7}$$

Then $\mathbb{H}(t_1, t_2, \ldots) = H(z_1, z_2, \ldots)$.

Example 2.1 *We display the polynomials* $F_w(Z_m)$ *for* $n = 2, m = 4$.
$F_{id}(Z_4) = 1$,
$F_{u_0}(Z_4) = z_1 + z_2 + z_3 + z_4$,
$F_{u_1}(Z_4) = 2(z_1 + z_2 + z_3 + z_4)$,
$F_{u_{01}}(Z_4) = F_{u_{10}}(Z_4) = (z_1 + z_2 + z_3 + z_4)^2$,
$F_{u_{010}}(Z_4) = z_1\ z_2(z_1+z_2)+(z_1+z_2)(z_3+z_4)(z_1+z_2+z_3+z_4)+z_3\ z_4(z_3+z_4)$,
$F_{u_{101}}(Z_4) = (z_1 + z_2)(z_1^2 + z_1\ z_2 + z_2^2) + z_3\ z_4(z_3^2 + z_3\ z_4 + z_4^2)+$
$2(z_1 + z_2)(z_3 + z_4)(z_1 + z_2 + z_3 + z_4)$,
$F_{u_{0101}}(Z_4) = (z_1 + z_2 + z_3 + z_4)\ F_{u_{010}}(Z_4)$.

Note that $F_{u_{010}}(Z_4) = s_{(2,1)}(Z_4)$.

2.3.3 Nil–Coxeter and Id–Coxeter algebras of type D

• Denote by $Nil(D_n)$ the nil–Coxeter algebra type D. Recall that $Nil(D_n)$ is an associative algebra generated over $\mathbb{Q}$ by the set of elements $\{u_{\hat{1}}, u_1, u_2, \ldots, u_{n-1}\}$ subject to the set of relations

(a) $u_{\hat{1}}^2 = 0, \quad u_i^2 = 0$ for $i = 1, 2, \ldots, n-1$;
(b) $u_i\ u_j = u_j\ u_i, \quad$ if $|i - j| \geq 2$;
(c) $u_i\ u_{\hat{1}} = u_{\hat{1}}\ u_i,$ if $i \neq 2; \quad u_{\hat{1}}\ u_2\ u_{\hat{1}} = u_2\ u_{\hat{1}}\ u_2$,
(d) $u_i\ u_{i+1}\ u_i = u_{i+1}\ u_i\ u_{i+1}$ for $i = 1, 2, \ldots, n-2$.

It is well-known that the elements $\{u_w, w \in W(D_n)\}$ form a $\mathbb{Z}[\beta]$-linear basis in the algebra $Nil(D_n)$.

• Let β be a parameter. The Id–Coxeter algebra $Id(D_n) := Id_\beta(D_n)$ is an associative algebra generated over $\mathbb{Q}[\beta]$ by the set of generators $\{u_{\hat{1}}, u_1, \ldots, u_{n-1}\}$ subject to the set of relations (b), (c) and (d) from the definition of the algebra $Nil(D_n)$, and the relations $u_i^2 = \beta\ u_i$ for $i = \hat{1}, 1, \ldots, n-1$, instead of that (a). It is well-known that the elements $\{u_w, w \in W(D_n)\}$ form a $\mathbb{Z}[\beta]$-linear basis in the algebra $Id_\beta(D_n)$.

Define $\quad D(x) := D_n(x) = h_{n-1}(x) \cdots h_1(x)\ h_{\hat{1}}(x)\ h_2(x) \cdots h_{n-1}(x)$.

Recall that $h_{\hat{1}}(x) = 1 + x\ u_{\hat{1}}$.

First we study properties of the elements $D(x)$ in the Nil–Coxeter algebra $Nil(D_n)$. It is easy to see that

$$D(x) = A_1(x)\ h_{\hat{1}}(x)\ A_2(-x)^{-1} = A_2(x)\ h_{\hat{1}}(x)\ A_1(-x)^{-1}.$$

Lemma 2.4 $\quad D(x)\ D(y) = D(y)\ D(x), \ \ D(x)\ D(-x) = 1.$

Proof. One has $\quad D(x)\ D(y) =$

$A_1(x)\ h_{\hat{1}}(x)\ A_2(-x)^{-1}\ A_2(y)\ h_{\hat{1}}(y)\ A_1(y)^{-1} =$
$A_1(x)\ A_1(y)\ A_1(y)^{-1}\ h_{\hat{1}}(x)\ A_2(y)\ A_2(-x)^{-1}\ h_{\hat{1}}(y)\ A_1(-x)\ A_1(-x)^{-1}\ A_1(-y)^{-1} =$
$A_1(x)\ A_1(y)\ h_1(-y)\ h_2(-y)\ h_{\hat{1}}(x)\ A_3(y)^{-1} A_2(y)\ A_2(-x)^{-1}\ A_3(-x)$
$h_{\hat{1}}(y)\ h_2(-x)\ \ h_1(-x)\ A_1(-x)^{-1}\ A_1(-y)^{-1} =$
$A_1(x)\ A_1(y)\ h_1(-y)\ h_2(-y)\ h_{\hat{1}}(x)\ h_2(x+y)\ h_{\hat{1}}(y)\ h_2(-x)\ h_1(-x)$
$A_1(-x)^{-1}\ A_1(-y)^{-1} = A_1(x)\ A_1(y)\ h_{\hat{1}}(x+y)\ h_1(-x-y)\ A_1(-x)^{-1}\ A_1(-y)^{-1}.$

The final expression is symmetric with respect to x and y, therefore the elements $D(x)$ and $D(y)$ commute to one another.

Note that to deduce the final equality we have used the Yang–Baxter relation

$$h_2(x)\ h_{\hat{1}}(x+y)\ h_2(y) = h_{\hat{1}}(x)\ h_2(x+y)\ h_{\hat{1}}(y).$$

■

Lemma 2.5 *The elements $D(x)$ and $D(y)$ commute in the algebra $Id(D_n)$ and $D(\phi_\beta(x))\ D(\phi_\beta(-x)) = 1$, where $\phi_\beta(x) = x/(1 - \frac{\beta}{2}\ (x)$.*

Proof. It is clear that the element $D_2(x) = h_1(x)\ h_{\hat{1}}(x)$ commutes with $D_2(y)$. The next case is $n = 3$. We have
$D_3(x)\ D_3(y) = h_2(x)\ h_1(x)\ h_{\hat{1}}(x)\ h_2(x +_\beta y)\ h_{\hat{1}}(y)\ h_1(y)\ h_2(y) =$
$h_2(y)\ h_2(x -_\beta y)\ h_1(x)\ h_2(y)\ h_{\hat{1}}(x +_\beta y)\ h_2(x)\ h_1(y)\ h_2(y -_\beta x)\ h_2(x) =$
$h_2(y)\ h_1(y)\ h_{\hat{1}}(x +_\beta y)\ h_1(x)\ h_2(x) = D_3(x)\ D_3(y).$
Now we make use an induction. We have $D_{n+1}(x)\ D_{n+1}(y) =$
$h_n(x)\ h_{n-1}(x)\ D_{n-1}(x)\ h_{n-1}(x)\ h_n(x +_\beta y)\ h_{n-1}(y)\ D_{n-1}(y)\ h_{n-1}(y)\ h_n(y) =$
$h_n(x)\ h_{n-1}(x)\ h_n(y)\ D_{n-1}(x)\ h_{n-1}(x +_\beta y)\ D_{n-1}(y)\ h_n(x)\ h_{n-1}(y)\ h_n(y) =$
$h_n(x)\ h_{n-1}(x)\ h_n(y)\ h_{n-1}([-x]_\beta) D_n(x)\ D_n(y)\ h_{n-1}([-y]_\beta)\ h_n(x)\ h_{n-1}(y)\ h_n(y) =$
$h_n(x)\ h_{n-1}(x)\ h_n(y)\ h_{n-1}([-x]_\beta) D_n(y)\ D_n(x)\ h_{n-1}([-y]_\beta)\ h_n(x)\ h_{n-1}(y)\ h_n(y) =$
$h_n(x)\ \Big(h_{n-1}(x)\ h_n(y)\ h_{n-1}(y -_\beta x)\Big) D_{n-1}(y)\ h_{n-1}(x +_\beta y)\ D_{n-1}(x)$
$\Big(h_{n-1}(x -_\beta y)\ h_n(x)\ h_{n-1}(y)\Big)\ h_n(y) =$
$h_n(y)\ h_{n-1}(y)\ h_n(x)\ D_{n-1}(y)\ h_{n-1}(x +_\beta y)\ D_{n-1}(x)\ h_n(y)\ h_{n-1}(x)\ h_n(x) =$
$D_{n+1}(y)\ D_{n+1}(x).$
The second statement follows from the identity

$$\phi_\beta(x) +_\beta \phi_\beta(-x) = 0.$$

■

Let us set $\mathcal{D}(x) := \mathcal{D}_n(x) = D(\phi_\beta(x))$. It follows from Lemma 2.5 that in the algebra $Id_\beta(D_n)$ the elements $\mathcal{D}(x)$ and $\mathcal{D}(y)$ commute, and $\mathcal{D}(x)\ \mathcal{D}(-x) = 1$.

3 Schubert and Grothendieck polynomials

3.1 Schubert and Grothendieck expressions

Let us recall, cf [4], [5], the definition of certain elements in the Nil–Coxeter and Id–Coxeter algebras which will be used to define Schubert and Grothendieck expressions.

$$A_i(x) := A_i^{(n)}(x) = \prod_{a=n-1}^{i} h_a(x),\ \ B(x) := B_n(x) = \prod_{a=n-1}^{1} h_a(x)\, h_0(x) \prod_{a=1}^{n-1} h_a(x),$$

$$C(x) := C_n(x) = A_1^{(n)}(x)\ h_0(2x)\ A_1^{(n)}(-x)^{-1}.$$

Definition 3.1 *(The case of Nil–Coxeter algebras)*
(A) ([4]) The type A Schubert expression is

$$\mathfrak{S}^{A_{n-1}}(X_n) := A_1^{(n)}(x_1)\ A_2^{(n)}(x_2)\ \cdots A_{n-1}^{(n)}(x_{n-1}).$$

(B) The type B Schubert expression of the first kind is

$$\mathfrak{S}^{B_n}(X_n) := B_n(x_1)\ B_n(x_2)\cdots B_n(x_n)\ \mathfrak{S}^{A_{n-1}}(-X_n).$$

(Ba) The type B Schubert expression of the second kind is

$$\mathfrak{B}(X_n) := \sqrt{B_n(x_1)\ B_n(x_2)\cdots B_n(x_n)}\ \mathfrak{S}^{A_{n-1}}(-X_n),$$

(C) The type C Schubert expression of the first kind is

$$\mathfrak{S}^{C_n}(X_n) := C_n(x_1)\ C_n(x_2)\cdots C_n(x_n)\ \mathfrak{S}^{A_{n-1}}(-X_n).$$

(Ca) The type C Schubert expression of the second kind is

$$\mathfrak{C}(X_n) := \sqrt{C_n(x_1)\ C_n(x_2)\cdots C_n(x_n)}\ \mathfrak{S}^{A_{n-1}}(-X_n),$$

(D) The type D Schubert expression of the first kind is

$$\mathfrak{S}^{D_n}(X_n) := D_n(x_1)\cdots D_n(x_{n-1})\ A_2(-x_1)\ A_3(-x_2)\cdots A_{n-1}(-x_{n-2}),$$

(Da) The type D Schubert expression of the second kind is

$$\mathfrak{D}(X_n) := \sqrt{D_n(x_1)\cdots D_n(x_n)}\ \mathfrak{S}^{A_{n-1}}(-X_n).$$

In a similar fashion one can define $\mathfrak{S}^A(X_\infty)$, $\mathfrak{S}^B(X_\infty)$, $\mathfrak{B}(X_\infty)$, $\mathfrak{S}^C(X_\infty)$, $\mathfrak{C}(X_\infty)$, $\mathfrak{S}^D(X_\infty)$ and $\mathfrak{D}(X_\infty)$.

Definition 3.2 *(The case of Id-Coxeter algebras)*
(*A*) *The type A Grothendieck expression is*

$$\mathfrak{G}^{A_{n-1}}(X_n) := A_1^{(n)}(x_1)\ A_2^{(n)}(x_2)\ \cdots A_{n-1}^{(n)}(x_{n-1}).$$

(*B*) *The type B Grothendieck expression of the first kind is*

$$\mathfrak{G}^{B_n}(X_n) := \mathcal{B}_n(x_1)\ \mathcal{B}_n(x_2)\cdots\mathcal{B}_n(x_n)\ \mathfrak{S}^{A_{n-1}}(\phi_\beta(-X_n)).$$

(*Ba*) *The type B Grothendieck expression of the second kind is*

$$\mathfrak{G}_{B_n}(X_n) := \sqrt{\mathcal{B}_n(x_1)\ \mathcal{B}_n(x_2)\cdots\mathcal{B}_n(x_n)}\ \mathfrak{S}^{A_{n-1}}(\phi_\beta(-X_n)),$$

(*C*) *The type C Grothendieck expression of the first kind is*

$$\mathfrak{G}^{C_n}(X_n) := \mathcal{C}_n(x_1)\ \mathcal{C}_n(x_2)\cdots\mathcal{C}_n(x_n)\ \mathfrak{S}^{A_{n-1}}(\phi_{2\beta}(-X_n)).$$

(*Ca*) *The type C Grothendieck expression of the second kind is*

$$\mathfrak{G}_C(X_n) := \sqrt{\mathcal{C}_n(x_1)\ \mathcal{C}_n(x_2)\cdots\mathcal{C}_n(x_n)}\ \mathfrak{S}^{A_{n-1}}(\phi_{2\beta}(-X_n)),$$

(*D*) *The type D Grothendieck expression of the first kind is* $\mathfrak{G}^{D_n}(X_n) :=$

$$\mathcal{D}_n(x_1)\cdots\mathcal{D}_n(x_{n-1})\ A_2(\phi_\beta(-x_1))\ A_3(\phi_\beta(-x_2))\cdots A_{n-1}(\phi_\beta(-x_{n-2}),$$

(*Da*) *The type D Grothendieck expression of the second kind is*

$$\mathfrak{G}_{D_n}(X_n) := \sqrt{\mathcal{D}_n(x_1)\cdots\mathcal{D}_n(x_n)}\ \mathfrak{S}^{A_{n-1}}(\phi_\beta(-X_n)).$$

Remark 3.1 Since the Nil–Coxeter algebras in question are finite dimensional (in fact, the both Nil–Coxeter and Id–Coxeter algebras do not have elements of degree $> n$), the Schubert and Grothendieck expressions of the second kind are *polynomials*.

Definition 3.3 *([3], [6])*
(*A*) *Schubert polynomials of type A are defined via the decomposition*

$$\mathfrak{S}^{A_{n-1}}(X_n) = \sum_{w\in\mathbb{S}_n} \mathfrak{S}_w(X_n)\ u_w,$$

(B) The type B Schubert polynomials of the first kind are defined via the decomposition

$$\mathfrak{S}^{B_n}(X_n) = \sum_{w \in W(B_n)} \mathfrak{S}^{B_n}_w(X_n)\ u_w,$$

(Ba) The type B Schubert polynomials of the second kind are defined via the decomposition

$$\mathfrak{B}(X_n) = \sum_{w \in W(B_n)} \mathfrak{B}_w(X_n)\ u_w,$$

(C) The type C Schubert polynomials of the first kind are defined via the decomposition

$$\mathfrak{S}^{C_n}(X_n) = \sum_{w \in W(C_n)} \mathfrak{S}^{C_n}_w(X_n)\ u_w,$$

(Ca) The type C Schubert polynomials of the second kind are defined via the decomposition

$$\mathfrak{C}(X_n) = \sum_{w \in W(C_n)} \mathfrak{C}_w(X_n)\ u_w,$$

(D) The type D Schubert polynomials of the first kind are defined via the decomposition

$$\mathfrak{S}^{D_n}(X_n) = \sum_{w \in W(D_n)} \mathfrak{S}^{D_n}_w(X_n)\ u_w,$$

(Da) The type D Schubert polynomials of the second kind are defined via the decomposition

$$\mathfrak{D}(X_n) = \sum_{w \in W(D_n)} \mathfrak{D}_w(X_n)\ u_w.$$

Remark 3.2 *If we replace in the above formulas the Schubert expressions on the Grothendieck expressions, and decomposition in the Nil–Coxeter algebras on that in the corresponding Id-Coxeter algebras, we will come to the definition of Grothendieck polynomials of type B (resp. of types C and D) of the first or second kind.*

Lemma 3.1 *([6]) (Factorization formula, the case of type B)*

$$\mathfrak{S}^{B_n}(X_n) = \mathfrak{S}^{A_{n-1}}(X_n^{op}) \prod_{i=0}^{n-1} \Big(h_0(x_{n-i}) \prod_{j=1}^{n-i-1} h_j(x_{n-i-j} + x_{n-i}) \Big).$$

Proposition 3.1 *(Factorization formula, the case of type D)*

(1) *Assume that* $n = 2k+1 \geq 3$ *is odd, then* $\mathfrak{S}^{B_n}(X_n) = \mathfrak{S}^{A_{n-1}}(X^{op}_{n-1}, x_n)$

$$\prod_{r=k}^{1} h_{\hat{1}}(x_{2r-1} + x_{2r})\Big(\prod_{a=2}^{2r} h_a(x_{2r-a} + x_{2r}) \prod_{a=1}^{2r-1} h_a(x_{2r-1-a} + x_{2r-1}))\Big).$$

(2) *Assume that* $n = 2k \geq 4$ *is even, then* $\mathfrak{S}^{B_n}(X_n) = \mathfrak{S}^{A_{n-1}}(X^{op}_{n-1}, x_n)$

$$\prod_{r=k-1}^{1} h_{\hat{1}}(x_{2r} + x_{2r+1})\Big(\prod_{a=2}^{2r+1} h_a(x_{2r+1-a} + x_{2r}) \prod_{a=1}^{2r} h_a(x_{2r-a} + x_{2r}))\Big)\ h_{\hat{1}}(x_1),$$

where we set $x_0 = 0$, *and* $X^{op}_n := (x_n, \ldots, x_1)$.

Note that the number of terms in each expression is equal to $n(n-1)$, the length of the maximal element in the group $W(D_n)$. In fact, the both products correspond the "maximal" reduced decomposition of the element of maximal length in the Weyl group $W(D_n)$, as well as for the B_n-case stated in Lemma 3.1.

3.2 Double Schubert and Grothendieck polynomials

Definition 3.4

(A) The double Schubert expression $\mathfrak{S}^{A_{n-1}}(X, Y)$ *and double Schubert polynomials* $\mathfrak{S}^{A_{n-1}}_w(X, Y)$, $w \in W(A_{n-1})$, *of type A are defined as follows*

$$\mathfrak{S}^{A_{n-1}}(X, Y) = (\mathfrak{S}^{A_{n-1}}(-Y))^{-1}\ \mathfrak{S}^{A_{n-1}}(X) = \sum_{w \in \mathbb{S}_n} \mathfrak{S}_w(X, Y)\ u_w,$$

(B) The type B double Schubert expression $\mathfrak{S}^{B_n}(X, Y)$ *and type B double Schubert polynomials of the first kind* $\mathfrak{S}^{B_n}_w(X, Y)$, $w \in W(B_n)$, *are defined as follows*

$$\mathfrak{S}^{B_n}(X, Y) = (\mathfrak{S}^{B_n}(-Y))^{-1}\ \mathfrak{S}^{B_n}(X) = \sum_{w \in W(B_n)} \mathfrak{S}^{B_n}_w(X, Y),$$

(Ba) The type B double Schubert expression $\mathfrak{B}(X, Y)$ *and type B double Schubert polynomials of the second kind* $\mathfrak{B}_w(X_n, Y_n)$, $w \in W(B_n)$, *are defined as follows*

$$\mathfrak{B}(X_n, Y_n) = (\mathfrak{B}(-Y_n))^{-1}\ \mathfrak{B}(X_n) = \sum_{w \in W(B_n)} \mathfrak{B}_w(X_n, Y_n).$$

(C) The type C double Schubert expression $\mathfrak{S}^{C_n}(X,Y)$ *and type C double Schubert polynomials of the first kind* $\mathfrak{S}^{C_n}_w(X,Y)$, $w \in W(C_n)$, *are defined as follows*

$$\mathfrak{S}^{C_n}(X,Y) = (\mathfrak{S}^{C_n}(-Y))^{-1}\ \mathfrak{S}^{C_n}(X) = \sum_{w\in W(C_n)}\ \mathfrak{S}^{C_n}_w(X,Y),$$

(Ca) The type C double Schubert expression $\mathfrak{C}(X_n,Y_n)$ *and type C double Schubert polynomials of the second kind* $\mathfrak{C}_w(X_n,Y_n)$, $w \in W(C_n)$, *are defined as follows*

$$\mathfrak{C}(X_n,Y_n) = (\mathfrak{C}(-Y_n))^{-1}\ \mathfrak{C}(X_n) = \sum_{w\in W(C_n)}\ \mathfrak{C}_w(X_n,Y_n).$$

(D) The type D double Schubert expression $\mathfrak{S}^{D_n}(X,Y)$ *and the type D double Schubert polynomials* $\mathfrak{S}^{D_n}_w(X,Y)$, $w \in W(D_n)$, *of the first kind are defined as follows*

$$\mathfrak{S}^{D_n}(X,Y) = (\mathfrak{S}^{D_n}(-Y))^{-1}\ \mathfrak{S}^{D_n}(X) = \sum_{w\in D_n}\ \mathfrak{S}^{D_n}_w(X,Y),$$

(Da) The type D double Schubert expression $\mathfrak{D}(X_n,Y_n)$ *and the type D double Schubert polynomials* $\mathfrak{D}_w(X_n,Y_n)$, $w \in W(D_n)$, *of the second kind are defined as follows*

$$\mathfrak{D}(X_n,Y_n) = (\mathfrak{D}(-Y_n))^{-1}\ \mathfrak{D}(X_n) = \sum_{w\in D_n}\ \mathfrak{D}_w(X_n,Y_n).$$

Definition 3.5 *The double Grothendieck polynomials of the first and second types are defined by replacing in the above formulas the corresponding double Schubert expressions on the corresponding double Grothendieck ones.*

Lemma 3.2 *([4],[5]) The double Schubert expression of the type A has the following decomposition*

$$\mathfrak{S}^{A_{n-1}}(X,Y) = \prod_{i=1}^{n-1}\ \prod_{j=n-i}^{1} h_{i+j-1}(x_i+y_j).$$

Example 3.1 *(B_2 double Schubert polynomials of the first kind)*
$\mathfrak{S}^{B_2}_{id}(X,Y) = 1$,
$\mathfrak{S}^{B_2}_{u_0}(X,Y) = x_1 + x_2 + y_1 + y_2$,
$\mathfrak{S}^{B_2}_{u_1}(X,Y) = x_1 + 2x_2 + y_1 + 2y_2$,
$\mathfrak{S}^{B_2}_{u_{01}}(X,Y) = x_1\ x_2 + x_2^2 + (y_1+y_2)(x_1+2x_2+y+1+y_2)$,
$\mathfrak{S}^{B_2}_{u_{10}}(X,Y) = (x_1+x_2)(x_1+x_2+y_1+2y_2) + y_1\ y_2 + y_2^2$,
$\mathfrak{S}^{B_2}_{u_{010}}(X,Y) = x_1\ x_2(x_1+x_2)+(x_1+x_2)(y_1+y_2)(x_1+x_2+y_1+y_2)+y_1\ y_2(y_1+y_2)$,
$\mathfrak{S}^{B_2}_{u_{101}}(X,Y) = (x_1+x_2)\ x_2^2 + x_2(x_1+x_2)(y_1+2y_2) + (x_1+2x_2)y_2(y_1+y_2) + (y_1+y_2)y_2^2$,
$\mathfrak{S}^{B_2}_{u_{0101}}(X,Y) = (x_2+y_2)\ \mathfrak{S}^{B_2}_{u_{010}}(X,Y)$.

Let us remark that $\mathfrak{S}^{B_2}_{u_{010}}(X,Y) = s_{(2,1)}(X_2,Y_2)$, and $\mathfrak{S}^{B_2}_{u_{0101}}(X,Y) = P_{(2,1)}(X_2,Y_2)$.

Theorem 3.1
(1) *(Factorization formula)* $\mathfrak{S}^{B_n}(X,Y) =$

$$\prod_{i=1}^{n}\Big(\prod_{j=1}^{i-1} h_j(y_{i-j}+y_i)\ \ h_0(y_i)\Big)\ \mathfrak{S}^{A}(X^{op},Y^{op})\ \Big(\prod_{i=n}^{1} h_0(x_i)\ \prod_{j=1}^{i-1} h_j(x_{i-j}+x_i)\Big).$$

(2) *(Specialization formula for B_n double Schubert expression of the second kind)*
$\mathfrak{B}(X_n,-X_n) = (\mathfrak{S}^{A_{n-1}}(X_n))^{-1}\ B(X_n)\ \mathfrak{S}^{A_{n-1}}(-X_n) =$

$$\prod_{j=1}^{n}\ \Big(\prod_{a=j-1}^{1} h_a(x_a+x_j)\ h_0(x_j)\ \prod_{a=1}^{j-1} h_a(x_j-x_a)\Big).$$

Note that the number of terms in the last product is equal to n^2, i.e. to the length of the maximal element in B_n. Recall that by definition $X_n^{op} = (x_n,\ldots,x_1)$.

3.3 Main properties of B_n double Schubert and Grothendieck polynomials

Theorem 3.2 *(The case of double Schubert polynomials)*
(1) *(Action of divided difference operators)*

$$\partial_i^{(x)}\mathfrak{S}^{B_n}(X,Y) = \mathfrak{S}^{B_n}(X,Y)\ u_i, \qquad \partial_i^{(y)}\mathfrak{S}^{B_n}(X,Y) = u_i\ \mathfrak{S}^{B_n}(X,Y)$$

for all $i = 1, \dots, n-1$,

$$\partial_i^{(x)} \mathfrak{B}(X_n, Y_n) = \mathfrak{B}(X_n, Y_n)\ u_i, \qquad \partial_i^{(y)} \mathfrak{B}(X, Y) = u_i\ \mathfrak{B}(X, Y)$$

for all $i = 0, \dots, n-1$;

(2) *(Stability)* *Let* $\iota : B_n \to B_{n+1}$ *be the standard embedding of the group* $W(B_n)$ *to that* $W(B_{n+1})$. *If* $w \in W(B_n)$, *then*

$$\mathfrak{S}^{B_{n+1}}_{\iota(w)}(X_{n+1}, Y_{n+1}) = \mathfrak{S}^{B_n}_w(X_n, Y_n), \qquad \mathfrak{B}_{\iota(w)}(X_{n+1}, Y_{n+1}) = \mathfrak{B}_w(X_n, Y_n);$$

(3) *(Vanishing property of double Schubert polynomials of the second kind corresponding to a Weyl group* W *of classical type)*
Let $w \in W$ *and* $(a_1, \dots, a_l) \in R(w)$ *be a reduced decomposition of the element* w. *Then*

$$\mathfrak{S}_W(X_n, -w(X_n)) = \prod_{r=l}^{1} h_{a_r}(x_{s_{a_1} \cdots s_{a_{r-1}}(a_r)} - x_{s_{a_1} \cdots s_{a_r}(a_r)}). \tag{8}$$

Therefore, if $v, w \in W$, *then*

$$\mathfrak{B}_w(X_n, -v(X_n)) \neq 0,$$

if and only if $v \leq w$ *with respect to the Bruhat order on the group* W.

(3*a*) *Let* $w \in \mathbb{S}_n \subset W(B_n)$, *then* $\mathfrak{B}(X_n, -w(X_n)) =$

$$\mathfrak{S}^{B_n}(X_n, -w(X_n)) = \prod_{(i,j) \in D(w)} h_{n(i,j)}\ (x_{w(i)} - x_j) = \mathfrak{S}^{A_{n-1}}(-X, w(X)),$$

where $D(w)$ *denotes the diagram of a permutation* $w \in \mathbb{S}_n$, *see e.g. [18], p. 8, and* $n(i,j)$ *is the number in the box* $(i,j) \in D(w)$ *according to the standard numbering of the boxes of the diagram* $D(w)$, *and the product is taken according to the reading of the boxes of the diagram* $D(w)$ *column by column, from the bottom to the top, starting from the first column, next is the second and so on.*

(3*b*) *Let* $w = \bar{u} \in W(B_n)$ *be the sign permutation corresponding to a permutation* $u \in \mathbb{S}_n$. *Then* $\mathfrak{B}(X_n, -w(X_n)) =$

$$(\mathfrak{S}^{A_{n-1}}(u(X_n)))^{-1}\ B(X_n)\ \mathfrak{S}^{A_{n-1}}(-X_n) = \prod_{(i,j) \in \overline{D}(u)} h_{n(i,j)}(x_{w(i)} - x_j),$$

where $\overline{D}(u) = [1,n]^2 \setminus D(u)$; *the product is taken according to the reading of the boxes of the set* $\overline{D}(w)$ *column by column, from the bottom to the top, starting from the first column, next is the second and so on; by definition we set* $x_{\bar{i}} = -x_i$.

(4) *If* $w \in W(B_n)$ *is a type B Grassmannian permutation of shape* $\lambda := \lambda(w)$, *then*

$$\mathfrak{B}_w(X_n, Y_n) = P_\lambda(X_{l(\lambda)} \mid Y_{l(\lambda)}),$$

where $P_\lambda(X,Y)$ *denotes the factorial Schur polynomial corresponding to a partition* λ *introduced and studied in [11], [12].*

Proof

(1) By definition, $\partial_0^{(x)}\mathfrak{B}(X_n,Y_n) := x_1^{-1}\ (\mathfrak{B}(X_n,Y_n) - \mathfrak{B}(-x_1, X_n \setminus \{x_1\}, Y_n))=$

$$x_1^{-1}\left(\mathfrak{S}^{A_{n-1}}(-Y)^{-1}\sqrt{B(Y)B(X)}\left(A_1(-x_1) - B(-x_1)\ A_1(x_1)\right)A_2(x_2)\cdots A_{n-1}(x_{n-1})\right).$$

By definition, $B(-x_1)\ A_1(x_1) = A_1(-x_1)\ h_0(-x_1)$, and therefore, $A_1(-x_1) - B(-x_1)\ A_1(x_1) = x_1\ A_1(-x_1)\ u_0$. Thus, $\partial_0^{(x)}\mathfrak{B}(X_n,Y_n) =$

$$x_1^{-1}\left(\mathfrak{S}^{A_{n-1}}(-Y_n)\sqrt{B(Y)B(X)}x_1\ A_1(-x_1)\ u_0\right)A_2(x_2)\cdots A_{n-1}(x_{n-1}) = \mathfrak{B}(X_n,Y_n)\ u_0.$$

Similar reasoning shows that $\partial_0^{(y)}\ \mathfrak{B}(X_n,Y_n) = u_0\ \mathfrak{B}(X_n,Y_n)$.

Because the both functions $B(X)\ B(Y)$ and $\sqrt{B(X)\ B(Y)}$ are symmetric with respect to variables X (resp. Y), the divided difference operators $\partial_i^{(x)}$ (resp. $\partial_i^{(y)}$), $1 \le i \le n-1$, act in fact only on the component $\mathfrak{S}^{A_{n-1}}(X)$ (resp. $(\mathfrak{S}^{A_{n-1}}(-Y))^{-1}$) of the B_n double Schubert expressions either the first or the second types. It is well-known [4], [5], that $\partial_i^{(x)}\ \mathfrak{S}^{A_{n-1}}(X) = \mathfrak{S}^{A_{n-1}}(X)\ u_i$ (resp. $\partial_i^{(y)}\mathfrak{S}^{A_{n-1}}(-Y)^{-1} = u_i\ \mathfrak{S}^{A_{n-1}}(-Y)^{-1}$).

(2) It is clear that $B_{n+1}(x) = h_n(x)\ B_n(x)\ h_n(-x)$, and $A_\bullet^{(n+1)}(x) = h_n(x)\ A_\bullet^{(n)}(x)$. Thus if $w \in B_n \hookrightarrow B_{n+1}$, then one can erase all appearances of the factor $h_n(x)$ in a Schubert expression for $W(B_{n+1})$ to obtain that for the group $W(B_n)$.

■

Remark 3.3 It follows from arguments used in the proof of Theorem 2 from [2], that for any crystallographic Coxeter group W the polynomial defined in the Nil–Coxeter algebra $Nil(W)$ by means of the RHS of (8) is

well-defined, i.e. does not depend on a choice of a reduced decomposition of $w \in W$ taken. In fact, one can show that this statement is valid for any Coxeter group. One of the main results of our paper states that for a Lie group G of classical type, the polynomial in question is equal to the specialization $Y = -w(X)$ of the corresponding double Schubert expression of the second type, see Definition 3.4. We expect that a similar statement can be generalized for any crystallographic Coxeter group. Let us stress that the equality (8) is true as that among *polynomials*, but not only in the corresponding equivariant cohomology ring.

Theorem 3.3 *(The case of double Grothendieck polynomials)*
(1) *(Action of isobaric divided difference operators)*
(I) $\pi_i^A(\mathfrak{G}^{A_{n-1}}(X_n)) = \mathfrak{G}^{A_{n-1}}(X_n)\ (u_i - \beta)$.

$$\pi_{i;x}^{(B)}(\mathfrak{G}^{B_n}(X,Y)) = \mathfrak{G}^{B_n}(X,Y)\ (u_i - \frac{\beta}{2}), \quad \pi_{i;y}^{(B)}(\mathfrak{G}^{B_n}(X,Y)) = (u_i - \frac{\beta}{2})\ \mathfrak{G}^{B_n}(X,Y)$$

for all $i = 1, \ldots, n-1$;

$$\pi_{i;x}^{(B)}(\mathfrak{G}_{B_n}(X_n,Y_n)) = \mathfrak{G}_{B_n}(X_n,Y_n)\ (u_i - \frac{\beta}{2}), \quad \pi_{i;y}^{(B)}(\mathfrak{G}_{B_n}(X,Y)) = (u_i - \frac{\beta}{2})\ \mathfrak{G}_{B_n}(X,Y)$$

for $i =, \ldots, n-1$;
(II)

$$\pi_{0;x}^{(B)}(\mathfrak{G}_{B_n}(X_n,Y_n)) = \mathfrak{G}_{B_n}(X_n,Y_n)\ (u_0 - \beta), \quad \pi_{0;y}^{(B)}(\mathfrak{G}_{B_n}(X,Y)) = (u_0 - \beta)\ \mathfrak{G}_{B_n}(X,Y).$$

$$\pi_{\hat{1};x}^{(D)}(\mathfrak{G}_{D_n}(X_n,Y_n)) = \mathfrak{G}_{D_n}(X_n,Y_n)\ (u_{\hat{1}} - \beta), \quad \pi_{0;y}^{(D)}(\mathfrak{G}_{D_n}(X,Y)) = (u_{\hat{1}} - \beta)\ \mathfrak{G}_{D_n}(X,Y).$$

(2) *(Stability)* *Let* $\iota : B_n \to B_{n+1}$ *be the standard embedding of the group* $W(B_n)$ *to that* $W(B_{n+1})$. *If* $w \in W(B_n)$, *then*

$$\mathfrak{S}_{\iota(w)}^{B_{n+1}}(X_{n+1},Y_{n+1}) = \mathfrak{S}_w^{B_n}(X_n,Y_n), \qquad \mathfrak{B}_{\iota(w)}(X_{n+1},Y_{n+1}) = \mathfrak{B}_w(X_n,Y_n);$$

(3) *(Vanishing property of double Grothendieck polynomials of the second kind corresponding to a Weyl group* W *of classical type)*
Let $w \in W$ *and* $(a_1, \ldots, a_l) \in R(w)$ *be a reduced decomposition of the element* w. *Then*

$$\mathfrak{G}_W(X_n, -w(X_n)) = \prod_{r=l}^{1}\ h_{a_r}(x_{s_{a_1}\cdots s_{a_{r-1}}(a_r)} -_\beta x_{s_{a_1}\cdots s_{a_r}(a_r)}). \qquad (9)$$

Therefore, if $v, w \in W$, then

$$\mathfrak{G}_w(X_n, -v(X_n)) \neq 0,$$

if and only if $v \leq w$ with respect to the Bruhat order on the group W.

Before to start the proof, let's state a simple, but useful identity

$$h_*(\phi_\beta(x))\ h_*(\phi_\beta(-x)) = 1,$$

where $*$ can be equal to $i, 0 \leq i \leq n-1$, or $\hat{1}$.

Proof

By definition, $\pi_i^A(\mathfrak{G}^{A_{n-1}}(X_n)) =$
$\pi_i^A((A_1(x_1)\cdots A_i(x_i)\ A_i(x_{i+1})\ h_i(x_{i+1})^{-1}\ A_{i+2}(x_{i+2})\ldots A_{n-1}(x_{n-1})) =$
$A_1(x_1)\cdots A_i(x_i)\ A_i(x_{i+1})\ \pi_i^A(h_i(x_{i+1})^{-1})\ A_{i+2}(x_{i+2})\ldots A_{n-1}(x_{n-1})$. Now one can compute $\pi_i^A(h_i(x_{i+1})^{-1}) =$

$$\frac{1}{x_i - x_{i+1}}\Big((1+\beta x_{i+1})(1-\frac{x_{i+1}}{1+\beta x_{i+1}}\ u_i) - (1+\beta x_i)(1-\frac{x_i}{1+\beta x_i}\ u_i)\Big) = u_i - \beta.$$

Therefore, $\pi_i^A(\mathfrak{G}^{A_{n-1}}(X_n)) = \mathfrak{G}^{A_{n-1}}(X_n)\ (u_i - \beta)$.
Similarly, one can compute $\pi_i^B\ \Big(h_i(\phi_\beta(x))^{-1}\Big) = \pi_i^B(h_i(\phi_\beta(-x))) = u_i - \frac{\beta}{2}$.
Therefore, $\pi_{i;x}^{(B)}(\mathfrak{G}^{B_n}(X,Y)) = \mathfrak{G}^{B_n}(X,Y)\ (u_i - \frac{\beta}{2})$, The other cases listed in (I) can be proved in a similar manner. QED.

Now let's compute the action of isobaric divided difference operator $\pi_{0;x}^B$.
We have $\pi_{0;x}^{(B)}(\mathfrak{G}_{B_n}(X_n, Y_n)) =$
$\pi_{0;x}^{(B)}\Big((\mathfrak{S}^{A_{n-1}}(\phi_\beta(-Y_n)))^{-1}\ \sqrt{\mathcal{B}(x_1)\cdots\mathcal{B}(x_n)}\ \mathfrak{S}^{A_{n-1}}(\phi_\beta(-X_n))\Big) =$
$(\mathfrak{S}^{A_{n-1}}(\phi_\beta(-Y_n)))^{-1}\ \sqrt{\mathcal{B}(x_2)\cdots\mathcal{B}(x_n)}$
$\Big(\pi_{0;x}^{(B)}(\sqrt{\mathcal{B}(x_1)}\ \mathcal{A}_1(-x_1))\Big)\ \mathcal{A}_2(-x_2)\cdots\mathcal{A}_{n-1}(-x_{n-1})$.

Now one can compute $\pi_{0;x}^{(B)}(\sqrt{\mathcal{B}(x_1)}\ \mathcal{A}_1(-x_1)) =$
$\sqrt{\mathcal{B}(x_1)}\ \mathcal{A}_1(-x_1)\Big((1-\frac{\beta}{2}x_1) - (1+\frac{\beta}{2})\mathcal{A}_1(-x_1)^{-1}\ \mathcal{B}(-x_1)\ \mathcal{A}_1(-x_1)\Big)\ x_1^{-1} =$
$\sqrt{\mathcal{B}(x_1)}\ \mathcal{A}_1(-x_1)\Big((1-\frac{\beta}{2}x_1) - (1+\frac{\beta}{2})\ h_0(\phi_\beta(-x_1))\Big)\ x_1^{-1} =$
$\sqrt{\mathcal{B}(x_1)}\ \mathcal{A}_1(-x_1)\ (u_0 - \beta)$. Therefore,
$\pi_{0;x}^{(B)}(\mathfrak{G}_{B_n}(X_n, Y_n)) = \mathfrak{G}_{B_n}(X_n, Y_n)\ (u_0 - \beta)$. Formula for the action of $\pi_{0;y}^{(B)}$ can be proved in a similar fashion. QED

Finally, let's compute the action of isobaric divided difference operator $\pi^D_{\hat{1};x}$. Like in the cases considered before, using the identity $\mathcal{D}(x)\ \mathcal{D}(-x) = 1$ it is enough to compute $\pi^D_{\hat{1};x}\Big(\sqrt{\mathcal{D}(x_1)\ \mathcal{D}(x_2)}\ \mathcal{A}_1(-x_1)\ \mathcal{A}_2(-x_2)\Big) = \sqrt{\mathcal{D}(x_1)\ \mathcal{D}(x_2)}\ \mathcal{A}_1(-x_1)\ \mathcal{A}_2(-x_2)\ \Big((1-\frac{\beta}{2}x_1)(1-\frac{\beta}{2}x_2) - (1+\frac{\beta}{2}x_1)(1+\frac{\beta}{2}x_2)\ \mathcal{A}_2(-x_2)^{-1}\ \mathcal{A}_1(-x_1)^{-1}\ \mathcal{D}(-x_2)\ \mathcal{D}(-x_1)\mathcal{A}_1(x_2)\ \mathcal{A}_2(x_1)\Big)$.

Now let's compute the product

$$\mathcal{A}_2(-x_2)^{-1}\ \mathcal{A}_1(-x_1)^{-1}\ \mathcal{D}(-x_2)\ \mathcal{D}(-x_1)\mathcal{A}_1(x_2)\ \mathcal{A}_2(x_1).$$

To accomplish this, we will use the following formulas

$$\mathcal{D}(x) = \mathcal{A}_1(x)\ h_{\hat{1}}(\phi_\beta(x))\ \mathcal{A}_2(-x)^{-1} = \mathcal{A}_2(x)\ h_{\hat{1}}(\phi_\beta(x))\ \mathcal{A}_1(-x)^{-1}.$$

Thus, the product in question is equal to
$\mathcal{A}_2(-x_2)^{-1}\ h_{\hat{1}}(\phi_\beta(-x_1))\ \mathcal{A}_2(x_1)^{-1}\ \mathcal{A}_2(-x_2)\ h_{\hat{1}}(\phi_\beta(-x_1))\ \mathcal{A}_2(x_1) =$
$h_2(\phi_\beta(x_2))\ \mathcal{A}_3(-x_2)^{-1}\ h_{\hat{1}}(\phi_\beta(-x_1))\ \mathcal{A}_3(-x_2)\ h_2(\phi_\beta(-x_2) +_\beta \phi_\beta(-x_1))$
$\mathcal{A}_3(x_1)^{-1}\ h_{\hat{1}}(\phi_\beta(-x_2))\ \mathcal{A}_3(x_1)\ h_2(\phi_\beta(x_1)) =$
$h_2(\phi_\beta(x_2))\ h_{\hat{1}}(\phi_\beta(-x_1))\ h_2(\phi_\beta(-x_2)+_\beta\phi_\beta(-x_1))\ h_{\hat{1}}(\phi_\beta(-x_2))\ h_2(\phi_\beta(-x_1)) =$
$h_{\hat{1}}(\phi_\beta(-x_1) +_\beta \phi_\beta(-x_2)$. The final equality follows from the Yang–Baxter relation

$$h_{\hat{1}}(x)\ h_2(x +_\beta y)\ h_{\hat{1}}(y) = h_2(y)\ h_{\hat{1}}(x +_\beta y)\ h_2(x).$$

Substituting the value of the polynomial in question to the above calculations, one can find
$\pi^{(B)}_{i;x}(\mathfrak{G}_{B_n}(X_n, Y_n)) = \mathfrak{G}_{B_n}(X_n, Y_n)\ \Big\{\frac{1}{x_1+c_2}\ \Big((1-\frac{\beta}{2}x_1)((1-\frac{\beta}{2}x_2) -$
$((1+\frac{\beta}{2}x_1)((1+\frac{\beta}{2}x_2)\ h_{\hat{1}}(\phi_\beta(-x_1) +_\beta \phi_\beta(-x_2))\Big)\Big\} = \mathfrak{G}_{B_n}(X_n, Y_n)\ (u_{\hat{1}} - \beta)$.
To deduce the final equality, we have used the following equality

$$\phi_\beta(-x_1) +_\beta \phi_\beta(-x_2) = -\frac{x_1 + x_2}{(1+\frac{\beta}{2}x_1)((1+\frac{\beta}{2}x_2)}.$$

■

3.4 Main properties of D_n-Schubert polynomials

Let set in this Section $\partial_i := \partial^D_i,\ \hat{1}, 1, 2, \ldots, n-1$.

Theorem 3.4

(1) *(Action of divided difference operators)*

$$\partial_i^{(x)}\mathfrak{S}^{D_n}(X,Y) = \mathfrak{S}^{D_n}(X,Y)\ u_i, \qquad \partial_i^{(y)}\mathfrak{S}^{D_n}(X,Y) = u_i\ \mathfrak{S}^{D_n}(X,Y)$$

for all $i = 1, \ldots, n-1$,

$$\partial_i^{(x)}\mathfrak{D}(X_n,Y_n) = \mathfrak{D}(X_n,Y_n)\ u_i, \qquad \partial_i^{(y)}\mathfrak{D}(X_n,Y_n) = u_i\ \mathfrak{D}(X_n,Y_n)$$

for all $i = \hat{1}, 1, 2, \ldots, n-1$;

(2) *(Stability)* *Let* $\iota : D_n \to D_{n+1}$ *be the standard embedding of the group* $W(D_n)$ *to that* $W(D_{n+1})$. *If* $w \in W(D_n)$, *then*

$$\mathfrak{S}^{D_{n+1}}_{\iota(w)}(X_{n+1},Y_{n+1}) = \mathfrak{S}^{D_n}_w(X_n,Y_n), \qquad \mathfrak{D}_{\iota(w)}(X_{n+1},Y_{n+1}) = \mathfrak{D}_w(X_n,Y_n).$$

Proof

By definition $\partial_{\hat{1}}^{(x)}\ \mathfrak{D}(X_n,Y_n) =$
$\frac{1}{x_1+x_2}\Big(\mathfrak{S}^{A_{n-1}}(-Y)^{-1}\sqrt{D(x_n)\cdots D(x_2)\ D(x_1)}\,\mathfrak{S}^{A_{n-1}}(-x_1,-x_2,-x_3,\ldots,-x_n) -$
$\mathfrak{S}^{A_{n-1}}(-Y)^{-1}\sqrt{D(x_n)\cdots D(-x_2)\ D(-x_1)}\ \mathfrak{S}^{A_{n-1}}(x_2,x_1,-x_3,\ldots,-x_n)\Big) =$
$\mathfrak{S}^{A_{n-1}}(-Y)^{-1}\sqrt{D(x_n)\cdots D(x_2)\ D(x_1)}\ A_1(-x_1)\ A_2(-x_2)\Big(1 -$
$A_2(-x_2)^{-1}\ A_1(-x_1)^{-1}\ D(-x_1)\ D(-x_2)\ A_1(x_2)\ A_2(x_1)\Big)\ A_3(-x_3)\cdots A_{n-1}(-x_{n-1})$.

Now let us simplify the expression

$$A_2(-x_2)^{-1}\ A_1(-x_1)^{-1}\ D(-x_1)\ D(-x_2)\ A_1(x_2)\ A_2(x_1)$$

using the the following formula from Lemma 2.2 :

$$D(x)\ D(y) = A_1(x)\ A_1(y)\ h_{\hat{1}}(x+y)\ h_1(-x-y)\ A_1(-x)^{-1}\ A_1(-y)^{-1}.$$

Thus the expression in question is equal to

$$A_2(-x_2)^{-1}\ A_1(-x_2)\ h_{\hat{1}}(-x_1-x_2)\ h_1(x_1+x_2)\ A_1(x_1)^{-1}\ A_2(x_1) = h_{\hat{1}}(-x_1-x_2).$$

Therefore $\partial_{\hat{1}}^{(x)}\ \mathfrak{D}(X_n,Y_n) =$
$\frac{1}{x_1+x_2}\ \mathfrak{S}^{A_{n-1}}(-Y)^{-1}\sqrt{D(x_n)\cdots D(x_2)\ D(x_1)}A_1(-x_1)\ A_2(-x_2)$
$\Big(1 - h_{\hat{1}}(-x_1-x_2)\Big)\ A_3(-x_3)\cdots A_{n-1}(x_{n-1}) = \mathfrak{D}(X_n,Y_n)\ u_{\hat{1}}$.

3.5 Schubert polynomials of the third kind

The type B and type C (double) Schubert expressions of the first kind have nice combinatorial properties, but they are not compatible with the action of divided difference operators ∂_0^B and ∂_0^C respectively.

Definition 3.6 *Let us set* $\widetilde{B_n(x)} = 1 + B_n(x)$, $\widetilde{D_n(x)} = 1 + D_n(x)$,

$$\widetilde{C_n(x)} = (1+C_n(x))/2 = ((A_1^{(n)}(x)+A_1^{(n)}(-x))/2+x\ A_1^{(n)}(x)\ u_0)\ A_1^{(n)}(-x)^{-1}.$$

It is clear that

$$[\widetilde{B_n(x)}, \widetilde{B_n(y)}] = 0, \quad [\widetilde{C_n(x)}, \widetilde{C_n(y)}] = 0, [\widetilde{D_n(x)}, \widetilde{D_n(y)}] = 0.$$

Lemma 3.3 $\partial_0^B \left(\widetilde{B_n(x_1)}\ A_1(-x_1)\right) = \widetilde{B_n(x_1)}\ A_1(-x_1)\ u_0,$

$$\partial_0^C \left(\widetilde{C_n(x_1)}\ A_1(-x_1)\right) = \widetilde{C_n(x_1)}\ A_1(-x_1)\ u_0,$$

$$\partial_{\hat{1}}^D \left(\widetilde{D_n(x_1)}\ \widetilde{D_n(x_2)}\ A_1(-x_1)\ A_2(-x_2)\right) = \widetilde{D_n(x_1)}\ \widetilde{D_n(x_2)}\ A_1(-x_1)\ A_2(-x_2)\ u_{\hat{1}},$$

Proof

By definition, $\partial_0^C \left(\widetilde{C_n(x_1)}\ A_1(-x_1)\right) = \frac{1}{2x}\Big(x\ A(x)\ u_0 + x\ A(-x)\ u_0\Big) = \frac{1}{2}\ \Big(A(x) + A(-x)\Big)\ u_0 = \widetilde{C_n(x_1)}\ A_1(-x_1)\ u_0.$

Definition 3.7 *The C type Schubert expression and Schubert polynomials of the third kind [6] are given by*

$$\mathfrak{c}(X_n) = \widetilde{C_n(x_1)}\ \widetilde{C_n(x_2)} \cdots \widetilde{C_n(x_n)}\ \mathfrak{S}^{A_{n-1}}(-X_n) = \sum_{w \in W(C_n)} \mathfrak{c}_w(X_n)\ u_w.$$

Proposition 3.2 *$\mathfrak{c}_w(X_n) \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$ for all $w \in W(C_n)$.*

Proof

For each $k \geq 1$ let us introduce polynomials

$$\mathcal{B}_k(x_1, \ldots, x_k) = \frac{A_1(x_1) + A_1(-x_1)}{2} \prod_{a=k}^{2} \prod_{b=1}^{k-1} h_b(x_a) + x_1\ A_1(x_1) \prod_{a=k}^{2} \prod_{b=1}^{k-1} h_b(x_a)\ u_0,$$

$$\mathcal{C}(x_1, \cdots, x_k) = B(x_k) \prod_{a=1}^{k-1} \mathcal{B}_k(x_2, \ldots, x_{a+1}).$$

We claim that

$$\mathfrak{c}(X_n) = \prod_{k=2}^{n} \mathcal{C}(x_1, \ldots, x_k).$$

Proof is straightforward and leaves to the reader.

References

[1] D. Anderson and W. Fulton, *Degeneracy Loci, Pfaffians, and Vexillary Signed Permutations in Types B, C, and D*, arXiv:1210.2066.

[2] S. Billey, *Kostant polynomials and the cohomology ring for G/B,* Duke Math. J. **96** (1999), 205–224.

[3] S. Billey and M. Haiman, *Schubert polynomials for the classical groups,* J. Amer. Math. Soc. **8** (1995), no. 2, 443–482.

[4] S.V. Fomin and R. Stanley, *Schubert polynomials and the nil-Coxeter algebra,* Adv. Math. 103 (1994), no. 2, 196–207.

[5] S.V. Fomin and A.N. Kirillov, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials,* Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993). Discrete Math. **153** (1996), no. 1-3, 123–143.

[6] S.V. Fomin and A.N. Kirillov, *Combinatorial B_n-analogues of Schubert polynomials,* Trans. Amer. Math. Soc. **348** (1996), no. 9, 3591–3620.

[7] W. Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas,* Duke Math. J. **65** (1992), no. 3, 381–420.

[8] W. Fulton, *Schubert varieties in flag bundles for the classical groups,* Israel Math. Conf. Proc. **9** (1996), 241-262.

[9] W. Fulton, *Determinantal formulas for orthogonal and symplectic degeneracy loci,* J. Differential Geom **43** (1996), 276-290.

[10] T. Ikeda, L. Mihalcea and H. Narusec, *Double Schubert polynomials for the classical groups*, Advances in Mathematics **226** (2011), 840–886.

[11] V.N. Ivanov, *Interpolation analogues of Schur Q-functions,* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **307** (2004), 99–119; translation in J. Math. Sci. (N. Y.) **131** (2005), no. 2, 5495–5507.

[12] V.N. Ivanov, *A combinatorial formula for factorial Schur Q-functions,* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 256 (1999), 73–94; translation in J. Math. Sci. (New York) **107** (2001), no. 5, 4195–4211.

[13] A. N. Kirillov, *Notes on Schubert, Grothendieck and Key polynomials*, arXiv:1501.07337 .

[14] Kresch A. and Tamvakis H., *Double Schubert polynomials and degeneracy loci for the classical groups,* Ann. Inst. Fourier (Grenoble) **52** (2002), no. 6, 1681–1727.

[15] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory,* Progress in Mathematics, **204** (2002), *Birkhäuser* Boston, Inc., Boston, MA.

[16] Lam Tao Kai, *B and D analogues of stable Schubert polynomials and related insertion algorithms,* Ph. D. thesis, M.I.T., 1994.

[17] C. Lecouvey, *Schensted-Type Correspondences and Plactic Monoids for Types B_n and D_n*, Journal of Algebraic Combinatorics, **18** (2003). pp. 99–133.

[18] I.G. Macdonald, *Notes on Schubert polynomials,* Publication du LACIM, **6** (2001),*Université* du *Québec à Montréal*.

[19] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci,* SMF/AMS Texts and Monographs, **6** (2001), AMS, Providence, RI.

[20] R. Winkel, *Schubert polynomials of types A–D,* Manuscripta Math. **100** (1999), 55–79.