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Dedicated to Professor Ken-ichi SHINODA

By

Anatol N. KIRILLOV and Hiroshi NARUSE

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Anatol N. Kirillov* and Hiroshi Naruse**

RIMS Kyoto , IPMU* and University of Yamanashi***

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Abstract. We construct double Grothendieck polynomials of classical types which are equivalent to the polynomials defined in [15] and compare with [14].

1. Introduction

Let G be a semisimple Lie group, $B \subset G$ be a Borel subgroup of G, $T \subset B$ be a maximal torus in B, $\mathcal{F} := G/B$ and $W := N_G(T)/T$ be the corresponding flag variety and the Weyl group. Let ℓ be the rank of G.
According to the famous Borel theorem, the cohomology ring $H^*(G/B, \mathbb{Q})$ is isomorphic to the quotient $\mathbb{Q}[x_1, \ldots, x_\ell]/J_\ell$, where $x_i := c_1(L_i) \in H^2(G/B, \mathbb{Q})$, $i = 1, \ldots, \ell$, and $c_1(L_i)$ denotes the first Chern class of the standard line bundle L_i over the flag variety in question, J_ℓ stands for the ideal generated by the fundamental invariants of positive degree associated with the Weyl group W .

To our best knowledge the first systematic and complete treatment of the Schubert Calculus has been done by I.N. Bernstein, I.M. Gelfand and S.I. Gelfand [2] and independently, by M. Demazure [5] in the beginning of 70's of the last century. A Schubert polynomial $\overline{\mathfrak{S}_w}(X_\ell)$, $\ell = rk(G)$, corresponding to an element w of the Weyl group W, by definition is a polynomial which expresses the Poincaré dual class of the homology class of the Schubert variety $X_w := \overline{BwB/B} \subset G/B$ in terms of the Borel generators $x_i, 1 \leq i \leq \ell$, in the cohomology ring of the flag variety $\mathcal{F}$. Therefore by the very definition, a Schubert polynomial $\overline{\mathfrak{S}_w}(X)$ is defined only modulo the ideal J_ℓ.

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Hence it is an interesting problem: does there exist "natural representative" of a Shubert polynomial $\overline{\mathfrak{S}_w}(X_\ell)$ in the ring $\mathbb{Q}[x_1,\dots,x_\ell]$ with "nice" combinatorial, algebraic and geometric properties ?

For the type A_{n-1} flag varieties A. Lascoux and M.-P. Schützenberger constructed a family of double Schubert polynomials $\mathfrak{S}_w(X_{n-1},Y_{n-1})$, $w\in S_n$ with several "nice" properties, when we set $\mathfrak{S}_w(X_{n-1}):=\mathfrak{S}_w(X_{n-1},0)$, such as

1. $\mathfrak{S}_w(X_{n-1})$ is a representative of the Schubert class X_w corresponding to $w\in S_n$, that is $\mathfrak{S}_w(X_{n-1})\equiv\overline{\mathfrak{S}}_w(X_{n-1})(\mod J_{n-1})$,
2. (Compatibility conditions)

$$\partial_i^{(x)}\mathfrak{S}_w(X_{n-1},Y_{n-1})=\begin{cases}\mathfrak{S}_{ws_i}(X_{n-1},Y_{n-1}) & \text{if } l(ws_i)=\ell(w)-1,\\ 0 & \text{otherwise}\end{cases}$$

$$\partial_i^{(y)}\mathfrak{S}_w(X_{n-1},Y_{n-1})=\begin{cases}\mathfrak{S}_{s_iw}(X_{n-1},Y_{n-1}) & \text{if } l(s_iw)=\ell(w)-1,\\ 0 & \text{otherwise}\end{cases}$$

3. $\mathfrak{S}_w(X_{n-1},Y_{n-1})$ has nonnegative integer coefficients,
4. $\mathfrak{S}_w(X_{n-1},Y_{n-1})$ is stable,
5. $\mathfrak{S}_w(X_{n-1},Y_{n-1})$ satisfies the vanishing conditions, that is

$$\mathfrak{S}_w(-v(Y_{n-1}),Y_{n-1})=0,\ \text{unless } w\le v$$

with respect to the Bruhat order $\le$ on the symmetric group S_n,

6. the structural constants for the multiplication of Schubert polynomials $\mathfrak{S}_w(X_{n-1})$, $w\in S_n$, coincide with the triple intersections numbers of Schubert varieties.

A new approach to the theory of type A Schubert polynomials which is based on the study of the type A nil–Coxeter algebras, has been initiated by S. Fomin and R. Stanley. The basic idea of that approach is to consider and study the generating function of all Schubert polynomials simultaneously, namely, to treat the following generating function

$$\mathfrak{S}(X_{n-1})=\sum_{w\in S_n}\mathfrak{S}_w(X_{n-1})u_w,$$

where u_w denotes the standard linear basis in the nil–Coxeter algebra NC_n.

An unexpected and deep result discovered in [9] is that in the algebra $NC_n[x_1,\dots,x_{n-1}]$ the polynomial $\mathfrak{S}_n(X_{n-1})$ is completely factorizable in the product of linear factors. The basic tool to prove the factorizability property is the usage of the Yang–Baxter relation among the elements $h_i(x)=1+xu_i$ in the algebra $NC_n[x,y]$,

namely

$$(1+xu_i)(1+(x+y)u_{i+1})(1+yu_i) = (1+yu_{i+1})(1+(x+y)u_i)(1+xu_{i+1}). \quad (1)$$

The main consequence of the Yang–Baxter relation (1) is that the polynomials $A_k(x) = h_{n-1}(x)h_{n-2}(x)\dots h_k(x)$, commute, namely

$$[A_k(x), A_k(y)] = 0.$$

Now one can prove, [9], [8] that

$$\mathfrak{S}(X_{n-1}) = \sum_{w\in S_n} \mathfrak{S}_w(X_{n-1})u_w = A_1(x_1)A_2(x_2)\dots A_{n-1}(x_{n-1}).$$

This approach can be applied to a construction of type A double Schubert polynomials, Grothendieck and double Grothendieck polynomials, which originally had been introduced by A. Lascoux and M.-P. Schützenberger.

Construction of "good" representatives for the Schubert polynomials corresponding to the flag varieties of classical types B, C, D was initiated by S. Billey and M.Haiman [3] and independently by S. Fomin and A. N. Kirillov, [7]. In [7] the authors extended an algebro-combinatorial approach to a definition and study of the type A Schubert and Grothendieck polynomials to the case of those of types B and C. But it also works for type D as well. The key tool in a construction of the aforementioned polynomials is a unitary exponential solution to the quantum Yang–Baxter equations ([22]) with values in the NiCoxeter algebras of types B, C, D correspondingly. The exponential solution to the quantum Yang–Baxter equation associated with nilCoxeter algebra $NC(R)$, $R := A_{n-1}, B_n, C_n, D_n$, allows to construct a family of elements $R_i(x) \in NC(R)[x]$, $i = 1, \dots, rk(R)$ such that

$$R_i(x)R_i(y) = R_i(y)R_i(x), i = 1, \dots, rk(R).$$

The elements $R_i(x_1), \dots, R_i(x_\ell), i = 1, \dots, \ell := rk(R)$, are building blocks in the construction of the generating function for all Schubert polynomials corresponding to the flag variety associated with the root system R.

Now in order to ensure the coherency conditions one needs to specify the action of simple transpositions of the corresponding Weyl group on the ring of polynomials $\mathbb{Q}[x_1, \dots, x_\ell]$. In [7] and [15] the authors have chosen the *standard* action of the Weyl group on the cohomology ring of the corresponding flag variety G/B. Namely,

$$s_0(x_1) = -x_1, s_0(x_i) = x_i, \text{ if } i \geq 1, (\textit{types } B, C),$$
$$s_{\hat{1}} = -s_1, s_{\hat{1}}(x_i) = x_i \text{ if } i \geq 2, (\textit{type } D).$$

Based on these (= standard !) choice of the action of the simple transpositions, the divided difference operators (resp. *isobaric* ones) are defined uniquely. It is easy to see [7] that for root systems of types B, C, D it is impossible to find "good" representatives for the Schubert classes which satisfy the properties $2, 3, 6$ listed above. Nevertheless

in [7] the authors introduce the so called Schubert polynomials of the second kind with nice combinatorial properties including those $3, 4, 5, 6$, and therefore suitable for computation of the triple intersection numbers for Schubert polynomials of classical type, the main Problem of the Schubert Calculus, see [7] for details.

As for a construction of certain representatives the double Schubert and β-Grothendieck polynomials of classical types B, C, D, the author of [15] has used the following observation: if the a family of polynomials $S_w(X), w \in W$ has "wanted properties" with respect to variables X, the the polynomials

$$S_w(X,Y) := \sum_{\substack{u,v\in W, uv=w \\ \ell(uv)=\ell(u)+\ell(v)}} S_{u^{-1}}(Y)S_v(X)$$

also will have "wanted properties" with respect to the set of variables X and Y. Based on this observation and using the Schubert and Grothendieck polynomials introduced in [6], the author of [15] has introduced a family of polynomials depending on two sets of variable X and Y having nice combinatorial properties including among others, that $3, 4, 5, 6$ listed above. One example of such polynomials is the triple β-Grothendieck polynomial $\mathfrak{G}^W_w(X, Y, Z)$, $w \in W$, where W stands for the Weyl group of classical type $B, C,$ or D. Indeed, let W be of type B, C, D, one can start with the W-type Schubert/Grothendieck expression of the *second kind* $S^W(Z, X) := \sqrt{H^W(Z)}\mathfrak{G}^A(X)$, have been introduced for the Schubert polynomials in [3] for Schubert polynomials of types W, [7] for $W = B, C$ types Schubert polynomials, [16] for Schubert/Grothendieck case. According to *an observation* mentioned above, a "good" candidate for the <u>double</u> Schubert/Grothendieck expression of type W is

$$S^W(Y, T, Z, X) := S^W(-Y, -T)^{-1}S^W(Z, X) = \mathfrak{G}^A(-Y)^{-1}\sqrt{H^W(T)H^W(Z)}\mathfrak{G}^A(X).$$

To deduce this equality we have used the following facts:

$$\Big[H^W(T), H^W(Z)\Big] = 0, H^W(-T)^{-1} = H^W(T).$$

Finally, one can restrict the generating function $S^W(Y, T, Z, X)$ on the diagonal $T = Z$ and come to the following expression for the generating function of a double Schubert/Grothendieck polynomials of type W

$$S^W(Y, Z, X) = \mathfrak{G}^A(-Y)^{-1}H^W(Z)\mathfrak{G}^A(X).$$

Another algero-geometric interpretation of the generating function $S^W(Y, Z, X)$ has been obtained in [12].

Advantage of the algebro-combinatorial approach is, for example, a possibility to define, among others, a plactic versions of polynomials $\mathfrak{G}^W_w(X, Y)$, $\mathfrak{S}^W_w(X, Y)$ and their generalizations, see [16] for the case of root systems of type A

In [3] the authors used non-standard action of Weyl group on the ring of supersymmetric functions of infinite number of variables $\Gamma = (\mathbb{Z}[x_1, x_2, \ldots])^{SS}$ and define another family of Schubert polynomials.

In [12] the second author *et al.* studied the double Schubert polynomials of type B, C, D using localization map of equivariant cohomology. For K-theory there is analogous map and the image has the so called Goresky-Kottwitz-MacPherson property [10]. As mentioned for the case of Grassmannians in [14], the Schubert classes can be characterized by recurrence relations.(c.f. §6.)

2. Definitions and Notations

In this paper $W = W(X)$ is a Weyl group of type $X = A, B, C, D$. I^X is the set of simple reflections in $W(X)$. We index the simple reflections by the same notation as in [12] §3.2. In particular , for type B and C, s_0 corresponds to the left most node of the Dynkin diagram with the relation $(s_0 s_1)^4 = 1$ and $(s_0 s_i)^2 = 1$ for $i \geq 2$. For type D, $s_{\hat{1}} := s_0 s_1 s_0$ and we consider $W(D)$ as the subgroup of $W(B)$ generated by $s_{\hat{1}}, s_1, \ldots$.

Following [7], we prepare some notations. Let β be an indeterminate. We define operations $\oplus$ and $\ominus$ as follows.

$$x \oplus y := x + y + \beta xy, x \ominus y := (x - y)/(1 + \beta y).$$

We also use the convention that

$$\bar{x} := \ominus x = -\frac{x}{1 + \beta x}.$$

Then we have $x \oplus \bar{x} = 0$. For a Weyl group W with the set S of Coxeter generators, we define Id-Coxeter algebra as follows.

Definition 1. (Id-Coxeter algebra)

Id-Coxeter algebra $Id_\beta(W)$ for W is a $\mathbb{Z}[\beta]$ algebra with generators u_i for each $s_i \in S$ and relations as follows.

$$u_i^2 = \beta u_i,$$

$$\underbrace{u_i u_j u_i \cdots}_{m_{i,j}\text{termes}} = \underbrace{u_j u_i u_j \cdots}_{m_{i,j}\text{termes}} \quad \text{if } m_{i,j} \text{ is the order of } s_i s_j.$$

For each $s_i \in I^X$,we define divided-difference operator $\pi_i^{(a)}$ and $\psi_i^{(a)}$ with respect to the variables $a = (a_1, a_2, ...)$ as follows. Assume that $R \supset \mathbb{Z}[\beta]$ is a ring with a group action of $W(X)$. We define the action of $W(X)$ on $R[a, \bar{a}] := R[a_1, a_2, ..., \bar{a}_1, \bar{a}_2, ...]$ as follows.

Definition 2. The action of $s_i \in I^X$ on the variables $a_1, a_2, \ldots, \bar{a}_1, \bar{a}_2, \ldots$

- If $i \geq 1$, $s_i(a_i) = a_{i+1}, s_i(a_{i+1}) = a_i$, $s_i(\bar{a}_i) = \bar{a}_{i+1}, s_i(\bar{a}_{i+1}) = \bar{a}_i$, and $s_i(a_k) = a_k, s_i(\bar{a}_k) = \bar{a}_k$ for $k \neq i, i+1$.
- $s_0(a_1) = \bar{a}_1$, $s_0(\bar{a}_1) = a_1$, and $s_0(a_k) = a_k$, $s_0(\bar{a}_k) = \bar{a}_k$ for $k > 1$.
- $s_{\hat{1}}(a_1) = \bar{a}_2, s_{\hat{1}}(a_2) = \bar{a}_1, s_{\hat{1}}(\bar{a}_1) = a_2, s_{\hat{1}}(\bar{a}_2) = a_1$, and $s_{\hat{1}}(a_k) = a_k, s_{\hat{1}}(\bar{a}_k) = \bar{a}_k$ for $k > 2$.

We write the induced action on $R[a, \bar{a}]$ by $s_i^{(a)}$. Divided difference operators $\pi_i^{(a)}$ and $\psi_i^{(a)}$ are defined as follows. For $f \in R[a, \bar{a}] = R[a_1, a_2, ..., \bar{a}_1, \bar{a}_2, ...]$,

$$\pi_i^{(a)}(f) := \frac{f - (1 + \beta\alpha_i(a))s_i^{(a)}(f)}{\alpha_i(a)} \text{ and } \psi_i^{(a)} := \pi_i^{(a)} + \beta,$$

where $\alpha_i(a)$ is the element in $\mathbb{Z}[\beta][a, \bar{a}]$ corresponding to the root α_i, i.e. $\alpha_i(a) = a_i \oplus \bar{a}_{i+1}$ for $i = 1, 2, ...$, $\alpha_0^B(a) = \bar{a}_1$, $\alpha_0^C(a) = \bar{a}_1 \oplus \bar{a}_1$ and $\alpha_{\hat{1}}(a) = \bar{a}_1 \oplus \bar{a}_2$.

(Formally we can think as $\alpha_i(a) = \dfrac{e^{\beta\alpha_i} - 1}{\beta}$. c.f. [6])

PROPOSITION 1. *We have the following relations of operators.*
$\pi_i^2 = -\beta\pi_i$, $\psi_i^2 = \beta\psi_i$ *for all* $s_i \in I^X$.
$\underbrace{\pi_i\pi_j\pi_i\cdots}_{m_{i,j}\text{termes}} = \underbrace{\pi_j\pi_i\pi_j\cdots}_{m_{i,j}\text{termes}}$, $\underbrace{\psi_i\psi_j\psi_i\cdots}_{m_{i,j}\text{termes}} = \underbrace{\psi_j\psi_i\psi_j\cdots}_{m_{i,j}\text{termes}}$ *if* $m_{i,j}$ *is the order of* s_is_j.

We can check the relations by direct calculations.

The explicit form of $\psi_i^{(a)}$ is as follows,
$\psi_i^{(a)}(F) = \frac{s_i^{(a)}F - F}{a_{i+1} \ominus a_i}$ for $i \geq 1$,
$\psi_{0,B}^{(a)}(F) = \frac{s_0^{(a)}F - F}{a_1}$, $\psi_{0,C}^{(a)}(F) = \frac{s_0^{(a)}F - F}{a_1 \oplus a_1}$ and $\psi_{\hat{1}}^{(a)}(F) = \frac{s_{\hat{1}}^{(a)}F - F}{a_1 \oplus a_2}$.

Similarly we can define divided difference operators $\pi_i^{(b)}$ and $\psi_i^{(b)}$ corresponding to the variables $b_1, b_2, ...$

3. Basic Properties

Let $h_i(x) := 1 + xu_i$. Then it follows that $h_i(x)h_i(y) = h_i(x \oplus y)$.

LEMMA 1. *(Yang-Baxter relation)*

$$\begin{aligned}
h_i(x)h_j(y) &= h_j(y)h_i(x) && m_{i,j} = 2\\
h_i(x)h_j(x \oplus y)h_i(y) &= h_j(y)h_i(x \oplus y)h_j(x) && m_{i,j} = 3\\
h_i(x)h_j(x \oplus y)h_i(x \oplus y \oplus y)h_j(y) &= h_j(y)h_i(x \oplus y \oplus y)h_j(x \oplus y)h_i(x) && m_{i,j} = 4
\end{aligned}$$

These can be proved by direct calculations.

DEFINITION 3.

$A_i^{(n)}(x) := h_{n-1}(x)h_{n-2}(x)\cdots h_i(x) \ \ (i = 1, 2, ..., n-1)$

$F_n^B(x) := A_1^{(n)}(x)\ h_0(x)\ A_1^{(n)}(\bar{x})^{-1}$

$= h_{n-1}(x)h_{n-2}(x)\cdots h_1(x)h_0(x)h_1(x)\cdots h_{n-2}(x)h_{n-1}(x)$

$F_n^C(x) := A_1^{(n)}(x)\ h_0(x)^2\ A_1^{(n)}(\bar{x})^{-1}$

$= h_{n-1}(x)h_{n-2}(x)\cdots h_1(x)h_0(x)^2h_1(x)\cdots h_{n-2}(x)h_{n-1}(x)$

$F_n^D(x) := A_2^{(n)}(x)\ h_{\hat{1}}(x)h_1(x)\ A_2^{(n)}(\bar{x})^{-1}$

$= h_{n-1}(x)\cdots h_2(x)h_1(x)h_{\hat{1}}(x)h_2(x)\cdots h_{n-1}(x)$

LEMMA 2.

(1) $A_i^{(n)}(x)A_i^{(n)}(y) = A_i^{(n)}(y)A_i^{(n)}(x)$

(2) $F_n^X(x)F_n^X(y) = F_n^X(y)F_n^X(x)$ *for* $X = B, C, D$

(3) $F_n^X(x)F_n^X(\bar{x}) = 1$

Note that from (1) we have $A_i^{(n)}(x)A_i^{(n)}(y)^{-1} = A_i^{(n)}(y)^{-1}A_i^{(n)}(x)$ and $A_i^{(n)}(x)^{-1}A_i^{(n)}(y)^{-1} = A_i^{(n)}(y)^{-1}A_i^{(n)}(x)^{-1}$.

Proof.

(1) For the case $i = n-1$ is trivial. By reverse induction on i, we can assume $i < n-1$ and $A_{i+1}^{(n)}(x)A_{i+1}^{(n)}(y) = A_{i+1}^{(n)}(y)A_{i+1}^{(n)}(x)$. Then

$$\begin{aligned}
A_i^{(n)}(x)A_i^{(n)}(y) &= A_{i+1}^{(n)}(x)h_i(x)A_{i+1}^{(n)}(y)h_i(y)\\
&= A_{i+1}^{(n)}(x)A_{i+1}^{(n)}(y)h_{i+1}(\bar{y})h_i(x)h_{i+1}(y)h_i(y \ominus x)h_i(x)\\
&= A_{i+1}^{(n)}(y)A_{i+1}^{(n)}(x)h_{i+1}(\bar{y})h_{i+1}(y \ominus x)h_i(y)h_{i+1}(x)h_i(x)\\
&= A_{i+1}^{(n)}(y)A_{i+1}^{(n)}(x)h_{i+1}(\bar{x})h_i(y)h_{i+1}(x)h_i(x)\\
&= A_{i+1}^{(n)}(y)A_{i+2}^{(n)}(x)h_i(y)h_{i+1}(x)h_i(x)\\
&= A_{i+1}^{(n)}(y)h_i(y)A_{i+2}^{(n)}(x)h_{i+1}(x)h_i(x)\\
&= A_i^{(n)}(y)A_i^{(n)}(x)
\end{aligned}$$

(2) Using Lemma 3.1 and (1) we can show the equalities as follows. For $X = B$,

$$\begin{aligned}
&F_n^B(x)F_n^B(y)\\
=\ & A_1^{(n)}(x)h_0(x)A_1^{(n)}(\bar{x})^{-1}A_1^{(n)}(y)h_0(y)A_1^{(n)}(\bar{y})^{-1}\\
=\ & A_1^{(n)}(x)h_0(x)A_1^{(n)}(y)A_1^{(n)}(\bar{x})^{-1}h_0(y)A_1^{(n)}(\bar{y})^{-1}\\
=\ & A_1^{(n)}(x)A_1^{(n)}(y)h_1(\bar{y})h_0(x)h_1(y)h_1(x)h_0(y)h_1(\bar{x})A_1^{(n)}(\bar{x})^{-1}A_1^{(n)}(\bar{y})^{-1}\\
=\ & A_1^{(n)}(y)A_1^{(n)}(x)h_1(\bar{y})h_0(x)h_1(y)h_1(x)h_0(y)h_1(\bar{x})A_1^{(n)}(\bar{y})^{-1}A_1^{(n)}(\bar{x})^{-1}\\
=\ & A_1^{(n)}(y)A_2^{(n)}(x)h_1(x \oplus \bar{y})h_0(x)h_1(x \oplus y)h_0(y)h_1(\bar{x} \oplus y)A_2^{(n)}(\bar{y})^{-1}A_1^{(n)}(\bar{x})^{-1}\\
=\ & A_1^{(n)}(y)A_2^{(n)}(x)h_0(y)h_1(x \oplus y)h_0(x)A_2^{(n)}(\bar{y})^{-1}A_1^{(n)}(\bar{x})^{-1}\\
=\ & A_1^{(n)}(y)h_0(y)A_1^{(n)}(x)A_1^{(n)}(\bar{y})^{-1}h_0(x)A_1^{(n)}(\bar{x})^{-1}\\
=\ & A_1^{(n)}(y)h_0(y)A_1^{(n)}(\bar{y})^{-1}A_1^{(n)}(x)h_0(x)A_1^{(n)}(\bar{x})^{-1}\\
=\ & F_n^B(y)F_n^B(x)
\end{aligned}$$

Similar arguments with appropriate modifications will give $X = C, D$ cases.
The essential equalities to be used are
$h_1(x \oplus \bar{y})h_0(x \oplus x)h_1(x \oplus y)h_0(y \oplus y)h_1(\bar{x} \oplus y) = h_0(y \oplus y)h_1(x \oplus y)h_0(x \oplus x)$ and
$h_2(x \oplus \bar{y})h_1(x)h_{\hat{1}}(x)h_2(x \oplus y)h_1(y)h_{\hat{1}}(y)h_2(\bar{x} \oplus y) = h_1(y)h_{\hat{1}}(y)h_2(x \oplus y)h_1(x)h_{\hat{1}}(x)$.
(3) This esentially follows by the relation $h_i(x)h_i(\bar{x}) = 1$.

4. β-super symmetric functions

Definition 4. β-super symmetric function is a symmetric function which satisfies the following property.
$f(t, \bar{t}, x_3, ..., x_n) = f(0, 0, x_3, ..., x_n)$ for every t.

Remark
The β-supersymmetric property is translated to usual supersymmetricity by the change of variables x_i to $\dfrac{e^{\beta x_i} - 1}{\beta}$.

Let $SS_\beta(x_1, \ldots, x_n) := \{f \in \mathbb{Z}[\beta][x_1, ..., x_n] \mid f : \beta\text{-supersymmetric}\}$ and set $SS_\beta(x) := \varprojlim_n SS_\beta(x_1, \ldots, x_n)$.

$SS_\beta(x)$ is the ring of β-supersymmetric functions and we denote it as $\Gamma'_\beta(x)$. If $\beta = 0$ this becomes the the ring of supersymmetric functions Γ'.

4.1. K-theoretic Schur functions $GP_\lambda(x), GQ_\lambda(x)$ In [14] β-supersymmetric functions $GP_\lambda(x), GQ_\lambda(x)$ are defined. Let $b_1, b_2, ...$ be indeterminates, and set $[x|b]^k = (x \oplus b_1) \cdots (x \oplus b_k)$ and $[[x|b]]^k = (x \oplus x)(x \oplus b_1) \cdots (x \oplus b_{k-1})$.
Let SP_n be the set of strict partitions of length at most n. i.e. $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$ such that $r \leq n$.

Definition 5. (Ikeda-Naruse [14]) For a strict partition $\lambda \in SP_n$,

$$GP_\lambda(x_1, \ldots, x_n|b) := \frac{1}{(n-r)!} \sum_{w \in S_n} w\left(\prod_{1 \leq i \leq r}\left([x_i|b]^{\lambda_i} \prod_{i<j \leq n} \frac{x_i \oplus x_j}{x_i \ominus x_j}\right)\right)$$

$$GQ_\lambda(x_1, \ldots, x_n|b) := \frac{1}{(n-r)!} \sum_{w \in S_n} w\left(\prod_{1 \leq i \leq r}\left([[x_i|b]]^{\lambda_i} \prod_{i<j \leq n} \frac{x_i \oplus x_j}{x_i \ominus x_j}\right)\right)$$

where $w \in S_n$ acts $x_1, \ldots, x_n$ as permutation of indices.

We also define
$GP_\lambda(x_1, \ldots, x_n) := GP_\lambda(x_1, \ldots, x_n|0)$, $GQ_\lambda(x_1, \ldots, x_n) := GQ_\lambda(x_1, \ldots, x_n|0)$,

$$GP_\lambda(x) := \varprojlim_n GP_\lambda(x_1, ..., x_n) \text{ and } GQ_\lambda(x) := \varprojlim_n GQ_\lambda(x_1, ..., x_n|b).$$

$$GP_\lambda(x|b) := \lim_{\leftarrow n} GP_\lambda(x_1, ..., x_{2n}|b) \text{ and } GQ_\lambda(x|b) := \lim_{\leftarrow n} GQ_\lambda(x_1, ..., x_n|b).$$

Examples.
$GP_1(x_1, \dots, x_n) = x_1 \oplus x_2 \oplus \cdots \oplus x_n$.
$GQ_1(x_1, \dots, x_n) = (x_1 \oplus x_1) \oplus (x_2 \oplus x_2) \oplus \cdots \oplus (x_n \oplus x_n)$.

LEMMA 3.
(1) $GP_\lambda(x_1, \dots, x_n)$ and $GQ_\lambda(x_1, \dots, x_n)$ are β-supersymmetric functions.
(2) $\{GP_\lambda(x_1, \dots, x_n)\}_{\lambda \in SP_n}$ forms a basis of $SS_\beta(x_1, \dots, x_n)$ over $\mathbb{Z}[\beta]$.
(3) Let $SS^C_\beta(x_1, \dots, x_n)$ be the $\mathbb{Z}[\beta]$-subspace of $SS_\beta(x_1, \dots, x_n)$ spanned by $GQ_\lambda(x_1, \dots, x_n)(\lambda \in SP_n)$. Then $\{GQ_\lambda(x_1, \dots, x_n)\}_{\lambda \in SP_n}$ forms a basis of $SS^C_\beta(x_1, \dots, x_n)$ over $\mathbb{Z}[\beta]$.

Proof.
(1) follows from the definition.
(2) and (3) follows from the fact for corresponding properties for usual Schur P, Q-functions.

REMARK 1. We remark that the definition of β-supersymmetry and the polynomials GP_λ, GQ_λ can be generalized in more general setting such as algebraic cobordism [20]. We are planning to study the details elsewhere. (cf.[21])

LEMMA 4. *([14])*
$GP_\lambda(x|b)$ and $GQ_\lambda(x|b)$ are characterized by (left) divided difference relations and initial conditions. i.e.

$$\pi_i^{(b)} GX_\lambda(x|b) = \begin{cases} GX_{\lambda^{(i)}}(x|b) & \text{if} \quad s_i\lambda < \lambda \\ -\beta\, GX_\lambda(x|b) & \text{if} \quad s_i\lambda \geq \lambda \end{cases}$$

and

$$GX_\emptyset(x|b) = 1$$

where $GB_\lambda(x|b) = GP_\lambda(x|0, b), GC_\lambda(x|b) = GQ_\lambda(x|b), GD_\lambda(x|b) = GP_\lambda(x|b)$.

See [14] Theorem 6.1 and Theorem 7.1.

4.2. Stable symmetric functions $\mathcal{F}^X_w(x_1, ..., x_n)$

DEFINITION 6. For $X = B, C, D$, we define

$$F^X_n(x_1, x_2, \dots, x_n) := \prod_{i=1}^n F^X_n(x_i) \text{ and } F^X_\infty(x) := \lim_{\leftarrow n} F^X_n(x_1, x_2, \dots, x_n).$$

We also define $\mathcal{F}_w^X(x_1, ..., x_n)$ and $\mathcal{F}_w^X(x)$ by the following expression.

$$F_n^X(x_1, x_2, \ldots, x_n) = \sum_{w \in W_n^X} \mathcal{F}_w^X(x_1, ..., x_n) u_w \ , F_\infty^X(x) = \sum_{w \in W^X} \mathcal{F}_w^X(x) u_w$$

LEMMA 5. *For each $w \in W_n^X$, $\mathcal{F}_w^X(x_1, x_2, \ldots, x_n)$ is a β-supersymmetric function.*

Proof.
This follows from Lemma 3.3 (2) and (3).

LEMMA 6. *(0) For $X = B, C, D$, $\mathcal{F}_{w^{-1}}^X(x_1, x_2, \ldots, x_n) = \mathcal{F}_w^X(x_1, x_2, \ldots, x_n)$.*
(1) For $X = B$ or D, $\mathcal{F}_w^X(x_1, x_2, \ldots, x_n)$ can be expanded in $GP_\lambda(x_1, x_2, \ldots, x_n)$ with coefficients in $\mathbb{Z}[\beta]$.
(2) For a (maximal) Grassmannian element $w \in W_n^X$,

$$\mathcal{F}_w^B(x_1, x_2, \ldots, x_n) = GP_{\lambda_B(w)}(x_1, x_2, \ldots, x_n)$$
$$\mathcal{F}_w^C(x_1, x_2, \ldots, x_n) = GQ_{\lambda_C(w)}(x_1, x_2, \ldots, x_n)$$
$$\mathcal{F}_w^D(x_1, x_2, \ldots, x_n) = GP_{\lambda_D(w)}(x_1, x_2, \ldots, x_n)$$

where $\lambda_B(w), \lambda_C(w), \lambda_D(w)$ are strict partitions corresponding to w. cf.[14]

Proof.
(0) This follows from the symmetry of F_n^X.
(1) This follows from Lemma 4.3 (2).
(2) This follows from Proposition 6.

REMARK 2. We state conjecture that the coefficients in the expansion of (1) is positive. This will be a consequence of K-theory analogue of "transition equation" for type B,C,D.(cf. [12])

Example
$\mathcal{F}_{s_0}^B(x_1, \ldots, x_n) = GP_1(x_1, \ldots, x_n)$
$\mathcal{F}_{s_0}^C(x_1, \ldots, x_n) = GQ_1(x_1, \ldots, x_n)$
$\mathcal{F}_{s_{\hat{1}}}^D(x_1, \ldots, x_n) = GP_1(x_1, \ldots, x_n)$

PROPOSITION 2. *(Compatible sequence formula) cf. ([3],[7])*

For $w \in W_n^X$, we have

$$\begin{aligned}
\mathcal{F}_w^B(x_1,...,x_n) &= \sum_{\tilde{a}\in\tilde{R}(w)} \sum_{\tilde{b}\in C^B(\tilde{a})} \beta^{\ell(\tilde{a})-\ell(w)} 2^{|\tilde{b}|-\gamma(\tilde{a},\tilde{b})-o^B(\tilde{a})} x_{\tilde{b}} \\
\mathcal{F}_w^C(x_1,...,x_n) &= \sum_{\tilde{a}\in\tilde{R}(w)} \sum_{\tilde{b}\in C^C(\tilde{a})} \beta^{\ell(\tilde{a})-\ell(w)} 2^{|\tilde{b}|-\gamma(\tilde{a},\tilde{b})} x_{\tilde{b}} \\
\mathcal{F}_w^D(x_1,...,x_n) &= \sum_{\tilde{a}\in\tilde{R}(w)} \sum_{\tilde{b}\in C^D(\tilde{a})} \beta^{\ell(\tilde{a})-\ell(w)} 2^{|\tilde{b}|-\gamma(\tilde{a},\tilde{b})-o^D(\tilde{a})} x_{\tilde{b}},
\end{aligned}$$

where we used the following notations.

$\tilde{R}(w)$ is the set of sequence of indices $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_\ell)$ such that $u_{s_{\tilde{a}_1}} \cdots u_{s_{\tilde{a}_\ell}} = u_w$.

$\ell(\tilde{a})$ is the length ℓ of the sequence $\tilde{a}$.

$C^B(\tilde{a}) = C^C(\tilde{a})$ is the set of compatible sequences $\tilde{b}$ with respect to $\tilde{a}$, i.e. $\tilde{b} = (1 \le \tilde{b}_1 \le \tilde{b}_1 \le \cdots \le \tilde{b}_{\ell(\tilde{a})} \le n)$ such that $\tilde{a}_{i-1} \le \tilde{a}_i \ge \tilde{a}_{i+1} \implies \tilde{b}_{i-1} < \tilde{b}_{i+1}$.

$C^D(\tilde{a})$ is the set of compatible sequence for the flattened word $\tilde{\tilde{a}}$ of $\tilde{a}$ with further properties that if $\tilde{a}_i = \tilde{a}_{i+1} = 1$ or $\tilde{a}_i = \tilde{a}_{i+1} = \hat{1}$ then $\tilde{b}_i < \tilde{b}_{i+1}$. Note that the flattened word $\tilde{\tilde{a}}$ is obtained from $\tilde{a}$ by replacing $\hat{1}$ with 1. cf.[3].

$o^B(\tilde{a})$ is the number of appearance of 0's in $\tilde{a}$.

$o^D(\tilde{a})$ is the total number of appearance of 1 and $\hat{1}$ in $\tilde{a}$.

$|\tilde{b}|$ is the number of distinct $\tilde{b}_i$'s.

$\gamma(\tilde{a},\tilde{b}) := \#|\{i|\tilde{a}_i = \tilde{a}_{i+1} \text{ and } \tilde{b}_i = \tilde{b}_{i+1}\}|$.

$x_{\tilde{b}} := x_{\tilde{b}_1} x_{\tilde{b}_2} \cdots x_{\tilde{b}_\ell}$ for $\tilde{b} = (\tilde{b}_1 \ldots, \tilde{b}_\ell)$.

Proof.

This follows essentially from the expansion of the defining generating function.

Example.

type D, $n = 2$ case $w = [\bar{1}, \bar{2}, 3] = s_1 s_{\hat{1}}$

$\tilde{b} = (1,1)$ is a compatible sequence for $\tilde{a} = (1,\hat{1}), (\hat{1},1)$.

$\tilde{b} = (1,2)$ is a compatible sequence for $\tilde{a} = (1,\hat{1}), (\hat{1},1)$.

$\tilde{b} = (2,2)$ is a compatible sequence for $\tilde{a} = (1,\hat{1}), (\hat{1},1)$.

$\tilde{b} = (1,1,2)$ is a compatible sequence for $\tilde{a} = (1,\hat{1},1), (\hat{1},1,\hat{1}), (1,\hat{1},\hat{1}), (\hat{1},1,1)$.

$\tilde{b} = (1,2,2)$ is a compatible sequence for $\tilde{a} = (1,\hat{1},1), (\hat{1},1,\hat{1}), (1,1,\hat{1}), (\hat{1},\hat{1},1)$.

$\tilde{b} = (1,1,2,2)$ is a compatible sequence for

$$\tilde{a} = (1,\hat{1},1,\hat{1}), (\hat{1},1,\hat{1},1), (\hat{1},1,1,\hat{1}), (1,\hat{1},\hat{1},1).$$

There are no other compatible sequences and the sum of the terms becomes $\mathcal{F}^D_{s_1 s_{\hat{1}}}(x_1,x_2) = x_1^2 + 2x_1x_2 + x_2^2 + 2\beta x_1^2 x_2 + 2\beta x_1 x_2^2 + \beta^2 x_1^2 x_2^2 = (x_1 \oplus x_2)^2$.

5. Main results

First we recall the type A Grothendieck polynomials [7].
We set $G_{A_{n-1}}(a_1, ..., a_{n-1}) := A_1^{(n)}(a_1)A_2^{(n)}(a_2)\cdots A_{n-1}^{(n)}(a_{n-1})$.
Then for $w \in S_n$, we define $\mathcal{G}_w^{A_{n-1}}(a)$ as the coefficient of u_w.

$$G_{A_{n-1}}(a_1, ..., a_{n-1}) = \sum_{w\in S_n} \mathcal{G}_w^{A_{n-1}}(a)u_w.$$

Furthermore, we can consider $G_A(a) := \lim_{\leftarrow n} G_{A_{n-1}}(a_1, ..., a_{n-1})$ and get strongly stable polynomials $\mathcal{G}_w^A(a)$ by

$$G_A(a) = \sum_{w\in S_\infty} \mathcal{G}_w^A(a)u_w.$$

Strongly stable means that if $w \in S_n$ then $\mathcal{G}_w^A(a) = \mathcal{G}_w^{A_{n-1}}(a)$ (which does not depend on n).

5.1. The first definition

Definition 7. We define for $X = B, C$ or D,

$$G_n^X(a, b; x) := G_{A_{n-1}}(\bar{b}_1, ..., \bar{b}_{n-1})^{-1} F_n^X(x) G_{A_{n-1}}(a_1, ..., a_{n-1})$$

and define $\mathcal{G}_{n,w}^X(a, b; x)$ as the coefficient of u_w.

$$G_n^X(a, b; x) = \sum_{w\in W_n^X} \mathcal{G}_{n,w}^X(a, b; x)u_w.$$

In this case $\mathcal{G}_{n,w}^X(a, b; x) \in SS_\beta(x_1, \ldots, x_n)[a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}]$.
Furthermore, we can define $\mathcal{G}_w^X(a, b; x)$ by

$$G_A(\bar{b})^{-1} F_\infty(x) G_A(a) = \sum_{w\in W^X} \mathcal{G}_w^X(a, b; x)u_w.$$

Then $\mathcal{G}_w^X(a, b; x)$ has strong stability (cf. Proposition 5), and when we set $\beta = 0$ this is the double Schubert polynomial defined in [12]. It is clear that if $w \in W_n^X$ then $\mathcal{G}_{n,w}^X(a, b; x) = \mathcal{G}_w^X(a, b; x_1, \ldots, x_n, 0, 0, \ldots)$.

We will write $w \cdot v = z$ (called Demazure product) if $u_w u_v = \beta^{\ell(w)+\ell(v)-\ell(z)} u_z$.

Proposition 3. *For $X = B, C, D$ and $w \in W^X$, we have*

$$\mathcal{G}_w^X(a, b; x) = \sum_{(v_1,u,v_2)\in R(w)} \mathcal{G}_{v_1^{-1}}^A(b)\mathcal{F}_u^X(x)\mathcal{G}_{v_2}^A(a)$$

where $R(w) = \{(v_1, u, v_2) \in S_\infty \times W^X \times S_\infty \mid v_1 \cdot u \cdot v_2 = w\}$.

DEFINITION 8. The action of Weyl group W_n^X on $SS_\beta^{(n)}(x) \otimes \mathbb{Z}[\beta][a, \bar{a}] \otimes \mathbb{Z}[\beta][b, \bar{b}]$ is derived from the action as follows. For $f(x) \in SS_\beta(x)$,

$$s_0^{(a)} f(x) = f(a_1, x),\ s_0^{(b)} f(x) = f(b_1, x),$$

$$s_{\hat{1}}^{(a)} f(x) = f(a_1, a_2, x),\ s_{\hat{1}}^{(b)} f(x) = f(b_1, b_2, x).$$

These actions can be clarified by the change of variables explained in the second definition below (cf.§5.2 Remark 3).

PROPOSITION 4. *We have*
$\pi_i^{(a)} G_n^X(a, b; x) = G_n^X(a, b; x)(u_i - \beta)$ *and* $\pi_i^{(b)} G_n^X(a, b; x) = (u_i - \beta) G_n^X(a, b; x)$.

N.B. These mean that

$$\pi_i^{(a)} \mathcal{G}_w^X(a, b; x) = \begin{cases} \mathcal{G}_{ws_i}^X(a, b; x) & \text{if } l(ws_i) = \ell(w) - 1, \\ -\beta \mathcal{G}_w^X(a, b; x) & \text{otherwise} \end{cases}$$

and

$$\pi_i^{(b)} \mathcal{G}_w^X(a, b; x) = \begin{cases} \mathcal{G}_{s_i w}^X(a, b; x) & \text{if } l(s_i w) = \ell(w) - 1, \\ -\beta \mathcal{G}_w^X(a, b; x) & \text{otherwise} \end{cases}.$$

Proof.

We will prove $\psi_i^{(a)} G_n^X(a, b; x) = G_n^X(a, b; x) u_i$. Recall the explicit formula of ψ_i after the Prop. 2.2.

$G_{A_{n-1}}(\bar{b})^{-1}$ is invariant for the action of $s_i^{(a)}$, $i \in I^X$. For $i > 0$, $\psi_i^{(a)} F_n^X(x) = F_n^X(x)$ and $\psi_i^{(a)} G_{A_{n-1}}(a) = G_{A_{n-1}}(a) u_i$(cf. [6]), therefore $\psi_i^{(a)} F_n^X(x) G_{A_{n-1}}(a) = F_n^X(x) G_{A_{n-1}}(a) u_i$.

$\psi_{0,B}^{(a)}(F_n^B(x) G_{A_{n-1}}(a)) = \frac{F_n^B(x) F^B(a_1) G_{A_{n-1}}(\bar{a}_1, a_2, \ldots, a_{n-1}) - F_n^B(x) G_{A_{n-1}}(a)}{a_1} = F_n^B(x) G_{A_{n-1}}(a) u_0$

$\psi_{0,C}^{(a)}(F_n^C(x) G_{A_{n-1}}(a)) = \frac{F_n^C(x) H^C(a_1) G_{A_{n-1}}(\bar{a}_1, a_2, \ldots, a_{n-1}) - F_n^C(x) G_{A_{n-1}}(a)}{a_1 \oplus a_1} = F_n^C(x) G_{A_{n-1}}(a) u_0$

$\psi_{\hat{1}}^{(a)}(F_n^D(x) G_{A_{n-1}}(a)) = \frac{F_n^D(x) F^D(a_1, a_2) G_{A_{n-1}}(\bar{a}_2, \bar{a}_1, \ldots, a_{n-1}) - F_n^D(x) G_{A_{n-1}}(a)}{a_1 \oplus a_2} = F_n^D(x) G_{A_{n-1}}(a) u_{\hat{1}}$

Similar arguments hold for the action of $\psi_i^{(b)}$.

PROPOSITION 5. *(strong stability)*
$\mathcal{G}_w^X(a,b;x)$ *has strong stability i.e. if* $i_n : W_n^X \to W_{n+1}^X$ *is the natural inclusion, then*

$$\mathcal{G}_{i_n(w)}^X(a,b;x) = \mathcal{G}_w^X(a,b;x).$$

PROPOSITION 6. *(Grassmannian elements) For a Grassmannian element* $w \in W^X$*, we have the following equality.*

$$\mathcal{G}_w^B(a,b;x) = GP_{\lambda_B(w)}(x|0,b)$$
$$\mathcal{G}_w^C(a,b;x) = GQ_{\lambda_C(w)}(x|b)$$
$$\mathcal{G}_w^D(a,b;x) = GP_{\lambda_D(w)}(x|b)$$

where $\lambda_X(w)$ *is the strict partition corresponding to* $w \in W^X$ *(cf. [14]).*

5.2. The second definition As [7], we can use "change of variables" for x_i, $i = 1, 2. \ldots$.

$$F(x_i) = \sqrt{F(\bar{a}_i)F(\bar{b}_i)}$$

to define the double Grothendieck polynomial $\mathcal{G}_w^{X_n}(a,b)$ with two sets of variables a, b.

REMARK 3. As $s_0^{(a)}(\sqrt{F(\bar{a}_1,\bar{a}_2,\ldots)}) = \sqrt{F(a_1,\bar{a}_2,\ldots)}$ and by the supersymmetric property of F, this is $= \sqrt{F(a_1,a_1,\bar{a}_1,\bar{a}_2,\ldots)} = F(a_1)\sqrt{F(\bar{a}_1,\bar{a}_2,\ldots)}$. This explains the action $s_0^{(a)}(F(x)) = F(a_1,x)$ and $s_0^{(b)}(F(x)) = F(b_1,x)$. The action of $s_{\hat{1}}^{(a)}$ and $s_{\hat{1}}^{(b)}$ as well.

DEFINITION 9. Let $X = B, C, D$. For $w \in W_n^X$, we define $G_n^X(a)$ and $G_n^X(a,b)$ as follows.

$$G_n^X(a) := \sqrt{F_n^X(\bar{a}_1,...,\bar{a}_n)}G_{A_{n-1}}(a) \qquad \text{and} \qquad G_n^X(a,b) := G_n^X(\bar{b})^{-1}G_n^X(a).$$

By expanding these in terms of u_w, we can define $\mathcal{G}_{n,w}^X(a)$ and $\mathcal{G}_{n,w}^X(a,b)$ by

$$G_n^X(a) = \sum_{w\in W_n^X} \mathcal{G}_{n,w}^X(a)u_w \qquad \text{and} \qquad G_n^X(a,b) = \sum_{w\in W_n^X} \mathcal{G}_{n,w}^X(a,b)u_w.$$

REMARK 4. This double Grothendieck polynomial $\mathcal{G}_{n,w}^X(a,b)$ is essentially the same as defined in [15]. This has weak stability. i.e. $\mathcal{G}_{n,w}^X = \mathcal{G}_{n+1,w}^X|_{a_{n+1}=b_{n+1}=0}$ for $w \in W_n^X$. But it doesn't have strong stability.

Note that for $w \in W_n^X$, then

$\mathcal{G}_{n,w}^X(a) \in \mathbb{Q}[\beta][[a_1,...,a_n,\bar{a}_1,...,\bar{a}_n]]$ and

$\mathcal{G}^X_{n,w}(a,b) \in \mathbb{Q}[\beta][[a_1,...,a_n,\bar{a}_1,...,\bar{a}_n,b_1,...,b_n,\bar{b}_1,...,\bar{b}_n]]$.

Examples

$\mathcal{G}^B_{2,s_0}(a,b) = \frac{\sqrt{1+(\bar{a}_1\oplus\bar{a}_2\oplus\bar{b}_1\oplus\bar{b}_2)\beta}-1}{\beta} = \frac{\bar{a}_1\oplus\bar{a}_2\oplus\bar{b}_1\oplus\bar{b}_2}{2} - \beta\frac{(\bar{a}_1\oplus\bar{a}_2\oplus\bar{b}_1\oplus\bar{b}_2)^2}{8} + \cdots$

$\mathcal{G}^C_{2,s_0}(a,b) = \bar{a}_1 \oplus \bar{a}_2 \oplus \bar{b}_1 \oplus \bar{b}_2$, $\mathcal{G}^C_{3,s_0}(a,b) = \bar{a}_1 \oplus \bar{a}_2 \oplus \bar{a}_3 \oplus \bar{b}_1 \oplus \bar{b}_2 \oplus \bar{b}_3$.

$\mathcal{G}^D_{3,s_{\hat{1}}}(a,b) = \frac{\sqrt{1+(\bar{a}_1\oplus\bar{a}_2\oplus\bar{a}_3\oplus\bar{b}_1\oplus\bar{b}_2\oplus\bar{b}_3)\beta}-1}{\beta}$

PROPOSITION 7. *The following holds for $X = B, C, D$ and $i \in I_{X_n}$.*

$$\pi_i^{(a)} G^X_n(a,b) = G^X_n(a,b)(u_i - \beta)$$

$$\pi_i^{(b)} G^X_n(a,b) = (u_i - \beta) G^X_n(a,b)$$

Proof.
These are Prop. 5.3 with change of variables.

6. Identification with Schubert class

Let $\mathcal{R}^{(b)}_\beta := \mathbb{Z}[\beta][b_1, \bar{b}_1, b_2, \bar{b}_2, \ldots]$. K-theory Schubert classes are determined by the localization (Prop. 2.10 in [17]). And they are determined uniquely by either "right hand" recurrence ((2.12) in [17]) or "left hand" recurrence (Remark 2.3 in [17]).

(right recurrence) $\mathcal{G}_e = 1$, and $\pi_i^{(a)}\mathcal{G}_w = \mathcal{G}_{ws_i}$ if $ws_i < w$ and $\pi_i^{(a)}\mathcal{G}_w = -\beta\mathcal{G}_w$ if $ws_i > w$.

(left recurrence) $\mathcal{G}_e = 1$, and $\pi_i^{(b)}\mathcal{G}_w = \mathcal{G}_{s_i w}$ if $s_i w < w$ and $\pi_i^{(b)}\mathcal{G}_w = -\beta\mathcal{G}_w$ if $s_i w > w$.

Therefore we can identify the polynomials $\mathcal{G}^X_w(a,b;x)$ defined above as Schubert classes. In particular we have

THEOREM 1. *Assume* $\mathcal{G}^X_u(a,b;x)\mathcal{G}^X_v(a,b;x) = \sum_{w\in W^X} c^{w,X}_{u,v}(\beta)\mathcal{G}^X_w(a,b;x)$, *$c^{w,X}_{u,v}(\beta) \in \mathcal{R}^{(b)}_\beta$. Then $c^{w,X}_{u,v}(\beta)|_{\beta=-1}$ is the generalized Littlewood-Richardson coefficient for equivariant K-theory of type X. (b_i is considered as $1 - e^{t_i}$.)*

REMARK 5. $c^w_{u,v}(0)$ is the generalized Littlewood-Richardson coefficient for equivariant cohomology if we replace b_i to $-t_i$. (cf. [12].)

Example
$\mathcal{G}^C_{s_0}(a,b;x)\mathcal{G}^C_{s_0}(a,b;x) = \mathcal{G}^C_{s_1 s_0}(a,b;x) + \beta\mathcal{G}^C_{s_0 s_1 s_0}(a,b;x)$

7. Adjoint polynomials

The Grothendieck polynomial represents the K-theory Schubert class of the structure sheaf $\mathcal{O}_{X_w}$ of the Schubert variety $X^w = \overline{B_- wB/B} \subset X = G/B$. We can also define the adjoint polynomials $\mathcal{H}^X_{n,w}$, for each $w \in W^X_n$, corresponding to the ideal sheaf $\mathcal{O}_{X^w}(-\partial X^w)$ of boundary ∂X^w in X^w. cf. [11, 18]. The pairing $\langle \cdot\, , \cdot \rangle : K_T(X) \otimes_{R(T)} K_T(X) \to R(T)$ is given by

$$\langle v_1, v_2 \rangle = \chi(X, v_1 \otimes v_2) \qquad \text{where} \qquad \chi(X, \mathcal{F}) = \sum_{p \geq 0} (-1)^p \mathrm{ch}\, H^p(X, \mathcal{F}).$$

We define the relative adjoint polynomial $\mathcal{H}^X_{w,v}$ for $w \leq v$ by $\mathcal{H}^X_{w,v} := \psi^{(a)}_{w^{-1}v}(\mathcal{G}^X_v)$. The adjoint polynomial for $w \in W^X_n$ is $\mathcal{H}^X_{n,w} := \mathcal{H}^X_{w,w^{(n)}_0}$, where $w^{(n)}_0$ is the longest element in W^X_n (cf. [18]). These polynomials are no more stable but have similar properties as Grothendieck polynomials.

Proposition 8. *For $w \in W^X_n$*

$$\begin{aligned}
\mathcal{H}^B_{n,e} &= \textstyle\prod_{1\leq i\leq n-1}(1+\beta a_i)^{n-i} \prod_{1\leq i\leq n-1}(1+\beta b_i)^{n-i} \prod_{1\leq i\leq n}(1+\beta x_i)^{2n-1} \\
\mathcal{H}^C_{n,e} &= \textstyle\prod_{1\leq i\leq n-1}(1+\beta a_i)^{n-i} \prod_{1\leq i\leq n-1}(1+\beta b_i)^{n-i} \prod_{1\leq i\leq n}(1+\beta x_i)^{2n} \\
\mathcal{H}^D_{n,e} &= \textstyle\prod_{1\leq i\leq n-1}(1+\beta a_i)^{n-i} \prod_{1\leq i\leq n-1}(1+\beta b_i)^{n-i} \prod_{1\leq i\leq n}(1+\beta x_i)^{2n-2}
\end{aligned}$$

and

$$\mathcal{H}^X_{n,w} = (-1)^{\ell(w)} \mathcal{H}^{X_n}_e \overline{\mathcal{G}^X_{n,w}}$$

where $\overline{\mathcal{G}^X_{n,w}} = \mathcal{G}^X_{n,w}(\overline{a}, \overline{b}; \overline{x})$.

We can derive these formula using generating functions. Let us define $H^X_n(a, b; x)$ as

$$H^X_n(a, b; x) := \sum_{w \in W^X_n} (-1)^{\ell(w)} \mathcal{H}^X_{n,w}(a, b; x) u_w.$$

Then we get the following formula.

Proposition 9.

$$H^X_n(a, b; x) = \mathcal{H}^X_{n,e} G^X_n(\bar{a}, \bar{b}; \bar{x}).$$

Actually we can show the following property.

Proposition 10. *For $s_i \in I^X_n$ we have*

$$\pi^{(a)}_i H^X_n(a, b; x) = H^X_n(a, b; x)(-u_i)$$

$$\pi_i^{(b)} H_n^X(a,b;x) = (-u_i) H_n^X(a,b;x).$$

PROPOSITION 11. *(Interpolation formula) For* $F \in SS_\beta \otimes_{\mathbb{Z}[\beta]} \mathcal{R}_\beta^{(a)} \otimes_{\mathbb{Z}[\beta]} \mathcal{R}_\beta^{(b)}$,

$$F = \sum_{w \in W^X} (\psi_w^{(a)}(F)|_e)\, \mathcal{G}_w^X(a,b;x)$$

where the summation is infinite in general and $|_e$ *means the localization at* e*, i.e. take substitutions* $a_i = \bar{b}_i$ *and* $x_i = 0$ *for all* i*.*

COROLLARY 1. *The equivariant Littlewood-Richardson coefficient can be written as*

$$c_{u,v}^{w,X}(\beta) = \psi_w^{(a)}(\mathcal{G}_u^X(a,b;x)\mathcal{G}_v^X(a,b;x))|_e.$$

THEOREM 2.

$$\mathcal{G}_w^X(a,b;x) = \sum_{uv=w, u \le w} \mathcal{H}_{u,w}^X(\bar{c},b;0)\mathcal{G}_v^X(a,c;x)$$

There is also similar formula using second version of type B, C, D double Grothendieck polynomials.

8. Pipe dream formula

For type A Schubert/Grothendieck polynomials, it is well known that there is an explicit formula using pipe dream ([1, 6]). We can extend this to type B, C, D cases as follows. For this we recall the type A case formula. We give two descriptions, one in terms of excited Young diagrams [14] , another one in terms of compatible sequences [16].

The type A_{n-1} double Grothendieck polynomials $\mathcal{G}_w^{A_{n-1}}(a,b)$ are defined as follows.

$$G_{A_{n-1}}(\bar{b}_1, ..., \bar{b}_{n-1})^{-1} G_{A_{n-1}}(a_1, ..., a_{n-1}) = \sum_{w \in S_n} \mathcal{G}_w^{A_{n-1}}(a,b) u_w.$$

By using Yang-Baxter relations, the left hand side can be expressed as below

$$\prod_{j=n-1}^{1} \prod_{i=1}^{n-j} h_{i+j-1}(a_i \oplus b_j). \tag{2}$$

Therefore the expansion in terms of u_w produces the pipe dream formula.

1. Excited Young diagrams (EYD for short).
Let $N := n(n-1)/2$ and

$$\Delta_n := (s_{n-1})(s_{n-2}, s_{n-1}) \cdots (s_1, s_2, \cdots, s_{n-1}) = (d_1, d_2, \ldots, d_N).$$

Each term in the expansion of (2) corresponds to a subsequence of Δ_n. Give $w \in S_n$ with $\ell(w) = \ell$, let $Rsub(\Delta_n, w)$ be the set of subsequences of Δ_n each element of which gives a reduced expression of w. i.e.

$$Rsub(\Delta_n, w) := \{(d_{j_1}, d_{j_2}, \ldots, d_{j_\ell}) \mid 1 \le j_1 < j_2 < \cdots < j_\ell \le N, d_{j_1} d_{j_2} \cdots d_{j_\ell} = w\}.$$

We will call $D \in Rsub(\Delta_n, w)$ an extended EYD. For an extended EYD $D = (d_{j_1}, d_{j_2}, \ldots, d_{j_\ell}) \in Rsub(\Delta_n, w)$, we define the set $B(D)$ of backward movable positions by, considering $j_{\ell+1} := N + 1$,

$$B(D) := \{d_j \mid j \le N, \exists p \text{ such that } j_p < j < j_{p+1}, d_{j_1} d_{j_2} \cdots d_{j_p} = (d_{j_1} d_{j_2} \cdots d_{j_p}) \cdot d_j\}.$$

We also define weight $wt(d_k) = a_i \oplus b_j$ if $k = (n-j)(n-j-1)/2 + i$. Then we have the following extended EYD formula.

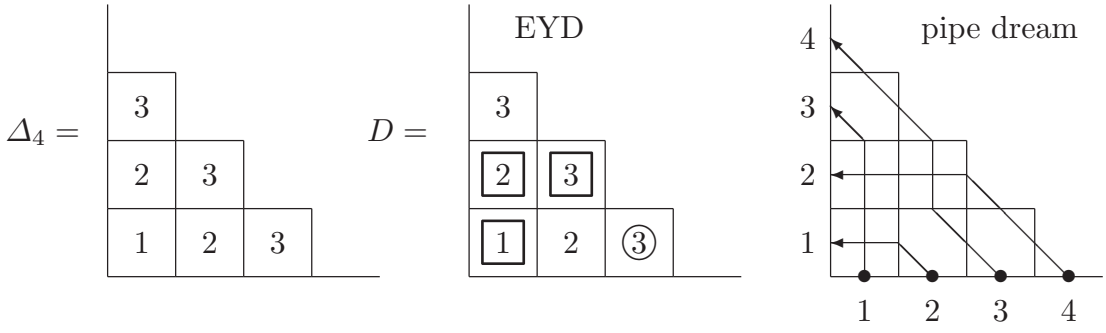
$$\mathcal{G}_w^{A_{n-1}}(a, b) = \sum_{D \in RSub(\Delta_n, w)} Wt(D),$$

where

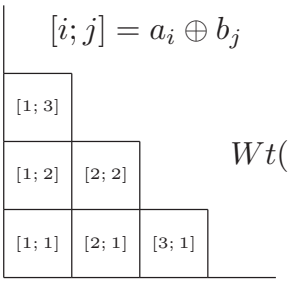
$$Wt(D) = \prod_{\square \in D} wt(\square) \times \prod_{\bigcirc \in B(D)} (1 + \beta wt(\bigcirc)).$$

In the pipe dream diagram two patterns appear. One is which corresponds to the selected box in EYD configuration. The other case we put in the box. Each selected box in D corresponds to a word of the reduced expression of w inside Δ_n. $B(D)$ is the set of backward movable positions (cf. [14]).

Example: type A_3, $w = [3, 1, 4, 2] = s_2 s_3 s_1$.



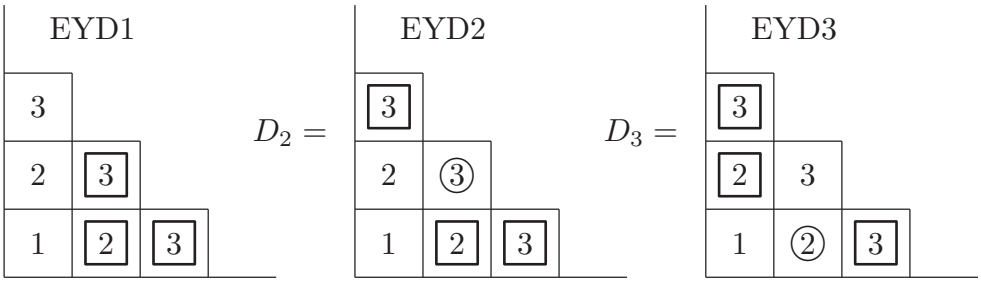
$wt =$



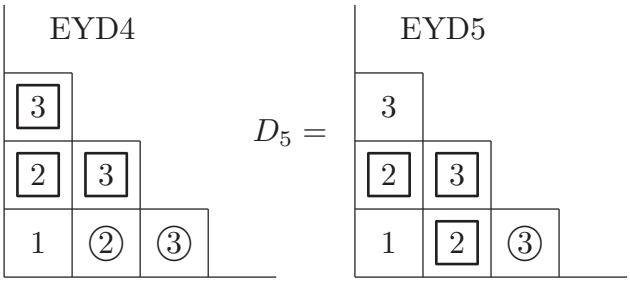
$Wt(D) = (a_1 \oplus b_2)(a_2 \oplus b_2)(a_1 \oplus b_1)(1 + \beta\, a_3 \oplus b_1)$

Example 2 $w = s_2 s_3 s_2 = s_3 s_2 s_3$
One can show that $\mathcal{G}_w^{A_{n-1}}(a,b)|_{a=1,b=0} = 5 + 5\beta + \beta^2$.

$D_1 =$ $D_2 =$ $D_3 =$



$D_4 =$ $D_5 =$



$$
\begin{aligned}
Wt(D_1) &= (a_2 \oplus b_2)(a_2 \oplus b_1)(a_3 \oplus b_1)\\
Wt(D_2) &= (a_1 \oplus b_3)(a_2 \oplus b_1)(a_3 \oplus b_1)(1 + \beta(a_2 \oplus b_2))\\
Wt(D_3) &= (a_1 \oplus b_3)(a_1 \oplus b_2)(a_3 \oplus b_1)(1 + \beta(a_2 \oplus b_1))\\
Wt(D_4) &= (a_1 \oplus b_3)(a_1 \oplus b_2)(a_2 \oplus b_2)(1 + \beta(a_2 \oplus b_1))(1 + \beta(a_3 \oplus b_1))\\
Wt(D_5) &= (a_1 \oplus b_2)(a_2 \oplus b_2)(a_2 \oplus b_1)(1 + \beta(a_3 \oplus b_1))
\end{aligned}
$$

From these data we get
$\mathcal{G}_{s_2 s_3 s_2}^{A_3}(a,b) = Wt(D_1) + Wt(D_2) + Wt(D_3) + Wt(D_4) + Wt(D_5)$.

Actually there is an algorithm to create all the extended EYD diagrams for a given $w \in S_n$. The algorithm is essentially written in [1]. Combinatorics related to extended EYD diagrams (including type B, C, D case) will be discussed elsewhere.

2. Compatible sequence.

Let T be a semistandard tableau and $w(T)$ be the column reading word corresponding to the tableau T. Denote by $R(T)$ (resp. $IR(T)$)the set of words which are plactic (resp. idplactic) equivalent to $w(T)$. Let $\mathbf{a} = (a_1, \cdots, a_n) \in R(T)$, where $n := |T|$ (resp. $\mathbf{a} = (a_1, \cdots, a_m) \in IR(T)$, where $m \geq |T|$).

DEFINITION 10. ([8],[16]) (Compatible sequences $\begin{Bmatrix}\mathbf{a}\\ \mathbf{b}\end{Bmatrix}$)

Given a word $\mathbf{a} \in R(T)$ (esp. $\mathbf{a} \in IR(T)$), denote by $C(\mathbf{a})$ (resp. $IC(\mathbf{a})$) the set of sequences of positive integers, called compatible sequences, $\mathbf{b} := (b_1 \leq b_2 \leq \cdots \leq b_m)$ such that

$$b_i \leq a_i, \text{ and if } a_i \leq a_{i+1}, \text{ then } b_i < b_{i+1}. \tag{3}$$

Finally, define the set $C(T)$ (resp. $IC(T)$) to be the union $\bigcup C(\mathbf{a})$ (resp. the union $\bigcup IC(\mathbf{a})$), where $\mathbf{a}$ runs over all words which are plactic (resp. idplactic) equivalent to the word $w(T)$.

Example Take $T = \begin{matrix}2 & 3\\ 3\end{matrix}$. The corresponding tableau word is $w(T) = 323$. We have $R(T) = \{232, 323\}$ and $IR(T) = R(T) \bigcup \{2323, 3223, 3232, 3233, 3323, 32323, \cdots\}$. Moreover,

$$C(T) = \begin{Bmatrix}\mathbf{a}: & 232 & 323 & 323 & 323 & 323\\ \mathbf{b}: & 122 & 112 & 113 & 123 & 223\end{Bmatrix},$$

$$IC(T) = C(T) \bigcup \begin{Bmatrix}\mathbf{a}: & 2323 & 3223 & 3232 & 3233 & 3323 & 32323\\ \mathbf{b}: & 1223 & 1123 & 1122 & 1123 & 1223 & 11223\end{Bmatrix}.$$

From these data we get single Grothendieck polynomial

$$\mathcal{G}_w^{A_{n-1}}(a) = a_1a_2^2 + a_1^2a_2 + a_1^2a_3 + a_1a_2a_3 + a_2^2a_3 + \beta(2a_1a_2^2a_3 + 2a_1^2a_2a_3 + a_1^2a_2^2) + \beta^2a_1^2a_2^2a_3.$$

For type B_n or C_n case, we can rewrite the generating function of Definition 5.1 as follows.

$$\left(\prod_{j=n-1}^{1}\prod_{i=1}^{n-j} h_{i+j-1}(x_{n-i+1} \oplus b_j)\right)\left(\prod_{i=n}^{1}\prod_{j=n}^{i} h_{j-i}(x_{i,j}^X)\right)\left(\prod_{i=n-1}^{1}\prod_{j=1}^{n-i} h_{i+j-1}(x_i \oplus a_j)\right) \tag{4}$$

where $x_{i,j}^X = x_i \oplus x_j$ if $i \neq j$, $x_{i,i}^B = x_i$ and $x_{i,i}^C = x_i \oplus x_i$.

Comparing this to the type A case, we get the following formula.

PROPOSITION 12. *For $w \in W(A_{n-1}) \subset W(B_n) = W(C_n)$, we have*

$$\mathcal{G}_{n,w}^B(a, b; x) = \mathcal{G}_{n,w}^C(a, b; x) = \mathcal{G}_{1^n \times w}^{A_{2n-1}}(x_1, \ldots, x_n, a_1, \ldots, a_{n-1}, x_1, \ldots, x_n, b_1, \ldots, b_{n-1}).$$

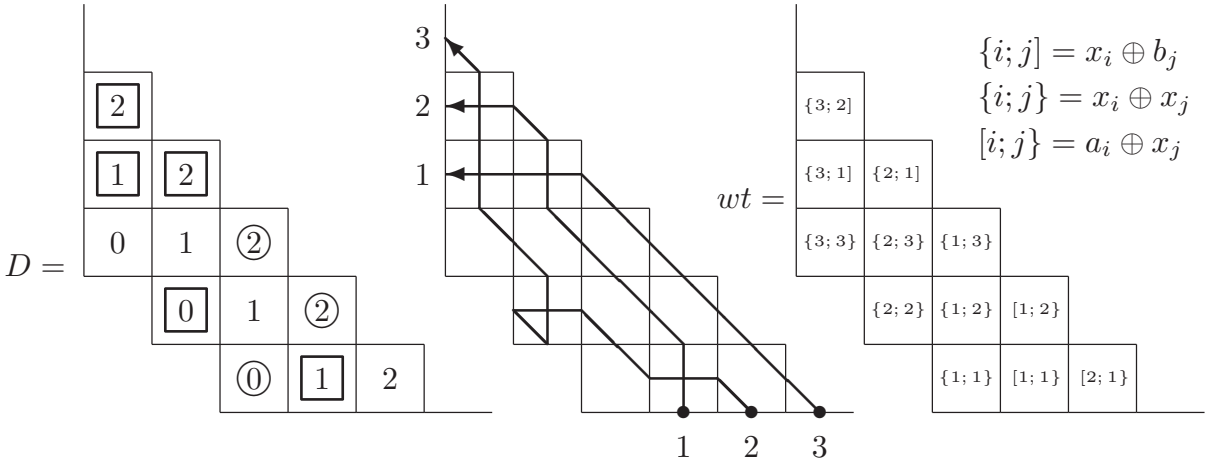
For type D_n case, we assume $n = 2m$ an even integer, for odd $n = 2m-1$ case we can get the formula by just erasing the last variable $x_{2m} = 0$ for $n = 2m$ case.

$$\left(\prod_{i=n}^{2}\prod_{j=i-1}^{1} h_{i+j-1}(x_i \oplus b_j)\right)\left(\prod_{i=n-1}^{1}\prod_{j=n}^{i+1} h_{i,j}(x_i \oplus x_j)\right)\left(\prod_{i=n-1}^{1}\prod_{j=1}^{n-i} h_{i+j-1}(x_i \oplus a_j)\right) \quad (5)$$

where $h_{i,j}(x_i \oplus x_j) := h_{j-i}(x_i \oplus x_j)$ if $j - i \geq 2$, $h_{i,i+1}(x_i \oplus x_{i+1}) := h_{\hat{1}}(x_i \oplus x_{i+1})$ if $i =$odd , and $h_{i,i+1}(x_i \oplus x_{i+1}) := h_1(x_i \oplus x_{i+1})$ if $i =$even.

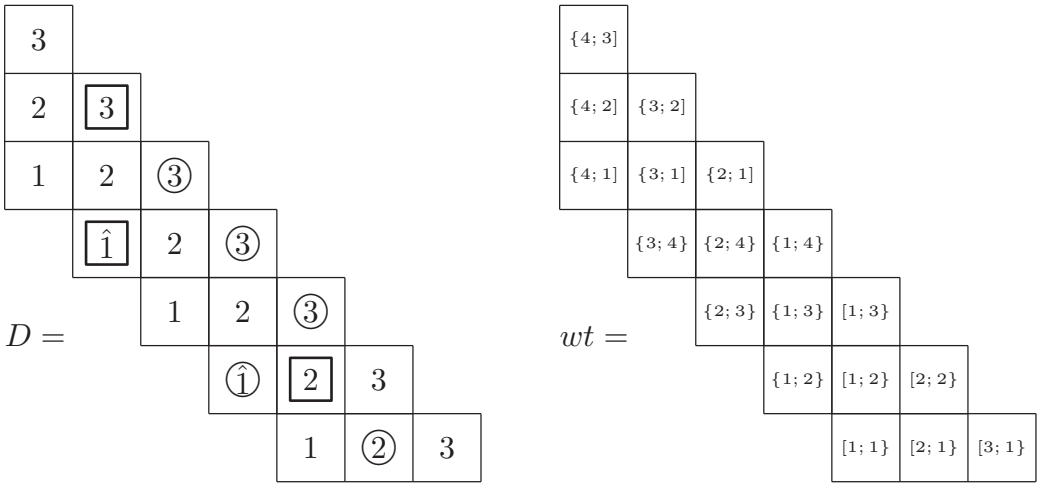
If w is a maximal Grassmannian element of type B_n, C_n or D_n, then the above gives the Excited Young diagram formula of [14] Th 9.2. Therefore the above give a generalization of the EYD formula.

Example
Type C_3, $w = [2, \bar{3}, 1] = s_2 s_1 s_2 s_0 s_1$.



$$Wt(D) = (x_3 \oplus b_2)(x_3 \oplus b_1)(x_2 \oplus b_1)(x_2 \oplus x_2)(x_1 \oplus a_1)$$
$$\times (1 + \beta(x_1 \oplus x_3))(1 + \beta(a_1 \oplus x_2))(1 + \beta(x_1 \oplus x_1))$$

Example
type D_4 , $w = [\bar{2}, 4, \bar{1}, 3] = s_3 s_{\hat{1}} s_2$



$$Wt(D) = (x_3 \oplus b_2)(x_3 \oplus x_4)(a_1 \oplus x_2)$$
$$\times (1+\beta(x_2 \oplus b_1))(1+\beta(x_1 \oplus x_4))(1+\beta(a_1 \oplus x_3))(1+\beta(x_1 \oplus x_2))(1+\beta(a_2 \oplus x_1))$$

Conclusion

In the present paper we compare an algebro-combinatorial [15] and algebro- geometric [14] constructions of the double Schubert/Grothendieck polynomials of types B,C,D, and show that these two approaches give rise to essentially the same polynomial representatives for the Schubert/Grothendieck classes in the cohomology/K-theory rings of the types B,C and D full flag varieties correspondingly. The formulas obtained (4) and (5) lead to combinatorial descriptions of polynomials in questions in terms of either EYD, or compatible sequences, or set-valued tableaux [4].

We expect that after a certain change of Id-Coxeter algebra and replacing $A \oplus B$ in our formulas (4) and (5) by $F(A, B)$, where $F(x, y)$ stands for the universal formal group law, we come to formal power series which have a suitable interpretations in the theory of algebraic cobordism [20] of flag varieties.

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Present Address:
Anatol N. Kirillov
Research Institute for Mathematical Sciences,
RIMS, Kyoto University, Sakyo-ku, 606-8502 Japan
and
The Kavli Institute for Physics and Mathematics of the Universe ,
IPMU, 5-1-5 Kashiwanoha, Kashiwa, 277-8583 Japan.
e-mail: kirillov@kurims.kyoto-u.ac.jp

Hiroshi Naruse
Graduate School of Education, University of Yamanashi,
4-4-37 Takeda, Kofu, Yamanashi, 400-8510 Japan.
e-mail: hnaruse@yamanashi.ac.jp