Rigged Configurations and Catalan, Stretched Parabolic Kostka Numbers and Polynomials : Polynomiality, Unimodality and Log-concavity

ANATOL N. KIRILLOV

Research Institute of Mathematical Sciences (RIMS)
Kyoto 606-8502, Japan
URL: http://www.kurims.kyoto-u.ac.jp/~kirillov

and

The Kavli Institute for the Physics and Mathematics of the Universe (IPMU),
5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan

Dedicated to Professor Masatoshi NOUMI on the occasion of his 60th Birthday

Abstract

We will look at the Catalan numbers from the *Rigged Configurations* point of view originated [9] from an combinatorial analysis of the Bethe Ansatz Equations associated with the higher spin anisotropic Heisenberg models . Our strategy is to take a combinatorial interpretation of the Catalan number C_n as the number of standard Young tableaux of rectangular shape (n^2), or equivalently, as the Kostka number $K_{(n^2),1^{2n}}$, as the starting point of our research. We observe that the rectangular (or multidimensional) Catalan numbers $C(m,n)$, introduced and studied by P. MacMahon [21], [30], see also [31], can be identified with the corresponding Kostka numbers $K_{(n^m),1^{mn}}$, and therefore can be treated by the Rigged Configurations technique. Based on this technique we study the stretched Kostka numbers and polynomials, and give a proof of a *strong rationality* of the stretched Kostka polynomials. This result implies a polynomiality property of the stretched Kostka and stretched Littlewood–Richardson coefficients [7], [26], [16].

Another application of the Rigged Configuration technique presented, is a new family of counterexamples to Okounkov's log-concavity conjecture [25].

Finally, we apply Rigged Configurations technique to give a combinatorial proof of the unimodality of the principal specialization of the internal product of Schur functions. In fact we prove a combinatorial (fermionic) formula for generalized q-Gaussian polynomials which is a far generalization of the so-called KOH-identity [24], as well as it manifests the unimodality property of the q-Gaussian polynomials.

2000 Mathematics Subject Classifications: 05E05, 05E10, 05A19.

1 Introduction

The literature devoted to the study of Catalan [1] and Narayana numbers [2] , their different combinatorial interpretations (more than 200 in fact, [29]), numerous generalizations, applications to Combinatorics, Algebraic Geometry, Probability Theory and so on and so forth, are enormous, see [29] and the literature quoted therein. There exists a wide variety of different generalizations of Catalan numbers, such as the Fuss–Catalan numbers [3] and the Schröder numbers [4] , higher *genus* multivariable Catalan numbers [22], higher dimensional Catalan [5] and Narayana numbers [21], [30], and many and varied other generalizations. Each a such generalization comes from a generalization of a certain combinatorial interpretation of Catalan numbers, taken as a starting point for investigation. One a such interpretation of Catalan numbers has been taken as the starting point of the present paper, is the well-known fact that the Catalan number C_n is equal to the number of *standard* Young tableaux of the shape (n^2).

Now let us look at the Catalan numbers from *Rigged Configurations* side. Since $C_n = K_{(n^2),1^{2n}}$ we can apply a *fermionic formula* for Kostka polynomials [8], and come to the following combinatorial expressions for Catalan and Narayana numbers

$$C_n = \sum_{\nu \vdash n} \prod_{j \geq 1} \binom{2n - 2(\sum_{a \leq j} \nu_j) + \nu_j - \nu_{j+1}}{\nu_j - \nu_{j+1}},$$

where the sum runs over all partitions ν of size n;

$$N(n,k) = \sum_{\substack{\nu \vdash n \\ \nu_1 = k}} \prod_{j \geq 1} \binom{2n - 2(\sum_{a \leq j} \nu_j) + \nu_j - \nu_{j+1}}{\nu_j - \nu_{j+1}},$$

where the sum runs over all partitions ν of size n, $\nu_1 = k$.
A q-versions of these formulas one can find, for example, in [12].
Let us illustrate our combinatorial formulas for $n = 6$. There are 11 partitions of size 6. We display below the distribution of contributions to the combinatorial formulae for the Catalan and Narayana numbers presented above, which come from partitions ν of size 6 and $k = 1, 2, \ldots, 6$.
$N(1,6) = 1$,
$N(2,6) = \binom{9}{1} + \binom{5}{4} + 1 = 9 + 5 + 1 = 15$,
$N(3,6) = \binom{8}{6} + \binom{3}{1}\binom{7}{1} + 1 = 28 + 21 + 1 = 50$,
$N(4,6) = \binom{7}{4} + \binom{6}{4} = 35 + 15 = 50$,

---

[1] *en.wikipedia.org/wiki/Catalan_number*

[2] *en.wikipedia.org/wiki/Narayana_number*

[3] *en.wikipedia.org/wiki/Fuss − Catalan_number*

[4] wolfram.com/SchröederNumber.html

[5] We denote the multidimensional Catalan numbers (as well as the set thereof) by $C(m,n)$. It might be well to point out that the set $C(m,n)$ is different from the set of *Fuss-Catalan* paths (or numbers) denoted commonly by $C_n^{(m)}$.

$N(5,6) = \binom{6}{4} = 15, \qquad N(6,6) = 1.$

A few comments in order.

- In [10] we gave a combinatorial interpretation of the shape of first (admissible) configuration $\nu^{(1)}$ corresponding to a given semistandard tableau T in terms of the set of *secondary descent sets* associated with the Young tableau T in question. In the case of *standard* Young tableaux of rectangular shape (n,n) there exist only one admissible configuration ν, $|\nu| = n$, and a combinatorial rule how to describe partition ν stated in [10], can be restated as follows: By the use of classical bijection between the set of standard Young tableaux of shape (n,n) and that of rooted plane trees with n nodes. one can associate to a given tableau $T \in STY((n,n))$ a rooted plane tree $\mathcal{T}$ on n nodes (out of the root). The number of external nodes of a tree $\mathcal{T}$ is equal to $p := p(T) = \#(DES(T))$, where $DES(T)$ denotes the descent set of the tableau in question. Now for any external node b of the tree $\mathcal{T}$ mentioned, denote by π_b a unique path in the tree $\mathcal{T}$ from the node b to the root. Let $\kappa_b(\mathcal{T})$ stands for the number of *edges* in the path π_b.

Lemma 1.1 *Let $T \in STY((n,n)$ be a standard Young tableau of shape (n,n), and $\nu \vdash n$ be a configuration corresponding to T under the Rigged Configuration bijection. Then*

$$\nu_1 = \kappa^{(1)}(\mathcal{T}) := \max(\kappa_1(\mathcal{T}), \ldots, \kappa_p(\mathcal{T})).$$

Now we proceed by induction. Namely, consider the most left node b in the tree $\mathcal{T}$ such that $\kappa_b = \nu_1$. Let $\mathcal{T}_1$ denotes a *forest* of rooted trees associated with the complement $\mathcal{T} \setminus \pi_b$. Let $\mathcal{T}_1 = \mathcal{T}_1^{(1)} \bigcup \mathcal{T}_1^{(2)} \ldots \bigcup \mathcal{T}_1^{(s)}$ be the union of distinct rooted trees making up the forest $\mathcal{T}_1$. Let now b be a node which belongs, say, to a (unique) subtree $\mathcal{T}_1^{(a)}$ of the forest $\mathcal{T}$, denote as before, by $\pi_b^{(1)}$ and $\kappa_b^{(2)}$ a unique path from the node b to the root of the tree $\mathcal{T}_1^{(a)}$, and its number of edges. <u>Then</u>

$$\nu_2 = \kappa^{(2)}(\mathcal{T}) := max(\kappa_b^{(2)}),$$

where maximum is taken over the all nodes of the forest $\mathcal{T}_1$. Now consider forest $\mathcal{T}_2 = \mathcal{T}_1 \setminus \pi_b^{(1)}$ and repeat the above procedure. As a result we obtain a sequence of numbers $\kappa = (\kappa^{(1)}, \ldots, \kappa^{(p)})$ such that

$$\nu^{(1)} = \kappa.$$

It is easy to see that for a given partition $\lambda \vdash n$, $\lambda = (m_1^{a_1}, \ldots, m_k^{a_k})$, $m_1 > m_2 > \ldots > m_k > 0$, $a_i \geq 1$, $\forall i$, the rooted plane tree corresponding to the *maximal* rigged configuration of type λ [11], looks as follows. It is a rooted plane tree $\mathcal{T}_{max}$ with a unique branching point at the root and external nodes $b_1, \ldots, b_k$ such that $\kappa_{b_1}(\mathcal{T}_{max}) = \cdots = \kappa_{b_{a_1}}(\mathcal{T}_{max}) = m_1$, $\kappa_{b_{a_1}+1}(\mathcal{T}_{max}) = \cdots = \kappa_{b_{a_1+a_2}}(\mathcal{T}_{max}) = m_2$, and so on. The rooted plane tree corresponding to the *minimal* rigged configuration, i.e. that with all zero riggings, corresponds to the *mirror image* of the tree $\mathcal{T}_{max}$.

The Rigged Configuration Bijection allows to attach a non-negative integer to each node of the corresponding rooted plane tree, It is an interesting **Problem** to read off these numbers from the associated tree directly.

• q-versions of formulas for Catalan and Narayana numbers displayed above coincide with the Carlitz–Riordan q-analog of Catalan numbers [28] and q-analog of Narayana numbers correspondingly.

• It is well-known that partitions of n with respect to the dominance ordering, form a lattice denoted by L_n. One (A.K) can define an ordering [6] on the set of admissible configurations of type (λ, μ) as well. In the case $\lambda = (n^2)$, $\mu = (1^{2n})$ the poset of admissible configurations of type (λ, μ) is essentially the same as the lattice of partitions L_n. Therefore, to each vertex ν of the lattice L_n one can attach the space of rigged configurations $RC_{\lambda,\mu}(\nu)$ associated with partition ν. Under a certain evolution a configuration (ν, J) evolves and touches the boundary of the set $RC_{\lambda,\mu}(\nu)$. When such is the case, "state" (ν, J) suffers "a phase transition" , executes the wall-crossing, and end up as a newborn state of some space $RC_{\lambda,\nu}(\nu^{'})$. A precise description of this process is the essence of the Rigged Configuration Bijection [14], [15]. It seems an interesting task to write out in full the evolution process going on in the space of triangulations of a convex $(n+2)$-gon under the Rigged Configuration Bijection (*work in progress*).

• It is an open **Problems** to count the number of admissible configurations associated with the multidimensional Catalan numbers $C(m,n)$ for general n and $m \ge 3$, and describe a structure of the corresponding poset on the set of admissible configurations, as well as to trace out a dynamics of riggings in the poset associated, for example, with the set $SYT((n,n))$. If $m = 3$, the set of of admissible configurations consists of pairs of partitions $(\nu^{(1)}, \nu^{(2)})$ such that $\nu^{(2)} \vdash n$ and $\nu^{(1)} \ge \nu^{(2)} \vee \nu^{(2)}$ [7]. One can check that the number of admissible configurations of type $(n^3, 1^{3n})$ is equal to $1, 3, 6, 16, 33, 78,$ for $n = 1, 2, 3, 4, 5, 6$.

• It is well-known that the q-Narayana numbers [8] obey the symmetry property, namely, $N(k,n) = N(n-k+1,n)$. Therefore it implies some non trivial relations among the products of q-binomial coefficients, combinatorial proofs of whose are desirable.

• It is well-known that the Narayana number $N(k,n)$ counts the number of Dyck paths of the semilength n with exactly k peaks. Therefore, the set of rigged configurations $\{\nu\}$ which associated with the Catalan number C_n and have fixed $\nu_1 = k$, is in one-to-one correspondence with the set of the semilength n Dyck paths with exactly k peaks, as well as the number of rooted plane trees with n edges and k ends.

Thus it looks natural to find and study combinatorial properties of the number of standard

[6] This ordering is inherited from an *evolution* of the maximal configuration $\Delta(\lambda, \mu)$ of type (λ, μ), see e.g. [11], [16], under certain transformations [11] on the set of admissible configurations $C(\lambda, \mu)$ which are coming from representation theory of the Lie algebra $\mathfrak{gl}_n$. An evolution of the *maximal rigged configuration* $(\Delta(\lambda, \mu), J = \{P_j^{(k)}(\Delta(\lambda, \mu))\})$ induces a certain poset structure on the set of rigged configurations $R_{\lambda,\mu}$. We expect some connections between certain poset (lattice ?) structures arising on the set of Rigged Configurations of type $(n^2, 1^{2n})$ coming from an analysis of the Bethe Ansatz, and the known poset and lattice structures on a variety of sets counting by Catalan numbers, such as the sets of *binary trees*, *rooted plane trees*, *triangulations of a regular gons* and so forth, see e.g. [29], [23]. It is an interesting and important **Problem** to study poset structures on the set(s) counting by the multidimensional Catalan and Narayana numbers. Details will appear elsewhere.

[7] Recall that for any partitions λ and μ, $\lambda \vee \mu$ denotes partition corresponding to composition $(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots)$.

[8] Recall that the q- Narayana number $N(k,n \mid q) = \frac{1-q}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q$.

Young tableaux of an arbitrary rectangular shape (n^m),that is the Kostka number $K_{(n^m),1^{mn}}$, which are inherent in the classical Catalan and Narayana numbers. For example, one can expect that a multidimensional Catalan number is the sum of multidimensional Narayana ones (this is so !), or expect that a multidimensional Narayana polynomial is the δ-vector of a certain convex lattice polytope, see e.g. [27] for the case of classical Catalan and Narayana numbers [9].

• Combinatorial analysis of the Bethe Ansatz Equations [9], gives rise to a natural interpretation of the Catalan and rectangular Catalan and Narayana numbers in terms of rigged configurations, and pose **Problem** to elaborate combinatorial structures induced by rigged configurations on any chosen combinatorial interpretation of Catalan numbers. For example, how to describe all triangulations of a convex $(n+2)$-gon which are in a "natural" bijection with the set of all rigged configurations (μ, J) corresponding to a given configuration ν of type $((n^2), 1^{2n})$? One can ask similar questions concerning Dyck paths and its multidimensional generalizations [31], and so on.

In Section 5.1 we present an example to illustrate some basic properties of the Rigged Configuration Bijection.

■

In the present paper we are interested in to investigate combinatorics related with the higher dimensional Catalan numbers, had been introduced and studied in depth by P. MacMahon [21]. It is highly possible that the starting point to introduce the higher dimensional Catalan numbers in [21] was an interpretation of classical Catalan numbers as the number of rectangular shape (n^2) standard Young tableaux mentioned above.

Our main objective in the present paper is to look on the multidimensional Catalan numbers $C(m,n) := C(m,n|1)$, defined as the value of the Kostka– Foulkes polynomials $K_{(n^m),(1^{mn})}(q)$ at $q=1$, from the point of view of Rigged Configurations Theory. In other words, we want to study the multidimensional Catalan and Narayana numbers introduced in [21], [30], by means of a fermionic formula for parabolic Kostka polynomials due to the author, e.g. [13], [16]. In particular, we apply the fermionic formula for parabolic Kostka polynomials cited above, to the study a stretched (parabolic) Kostka polynomials $K_{N\lambda,N\{\mathcal{R}\}}(q)$. At this way we obtain the following results.

Theorem 1.2 *(Polynomiality)*

Let λ be partition and $\{\mathcal{R}\}$ be a dominant sequence of rectangular shape partitions. Then

$$\sum_{N \geq 0} K_{N\lambda,N\mathcal{R}}(q)\ t^N = \frac{P_{\lambda,\mathcal{R}}(q,t)}{Q_{\lambda,\mathcal{R}}(q,t)},$$

were a polynomial $P_{\lambda,\mathcal{R}}(q,t)$ is such that $P_{\lambda,\mathcal{R}}(0,0)=1$;
a polynomial $Q_{\lambda,\mathcal{R}}(q,t) = \prod_{s \in S}(1-q^s t)$ for a some set of non-negative integers $S := S(\lambda,\mathcal{R})$, depending on data λ and $\mathcal{R}$.

[9]The multidimensional Catalan and Narayana numbers, as well as the first *expectation*, had been introduced and proved by P.MacMahon [21]. The second *expectation* will be treated in the present paper, Section 3.

Corollary 1.3 *([7], [26], [16])*

Let λ be partition and $\{\mathcal{R}\}$ be a dominant sequence of rectangular shape partitions. Then

- $\mathcal{K}_{\lambda,\mathcal{R}}(N) := K_{N\lambda,N\mathcal{R}}(1)$ *is a polynomial of N with rational coefficients.*
- *(Littlewood–Richardson polynomials, [20],[26],[16])*

Let λ, μ and ν be partitions such that $|\lambda|+|\mu|=|\nu|$.
The Littlewood–Richardson number $c_{\lambda,\mu}^{\nu}(N) := c_{N\lambda,N\mu}^{N\nu}$ is a polynomial of N with rational coefficients.

Problem 1.4 *Compute [10] the degree of polynomial $\mathcal{K}_{\lambda,,\mathcal{R}}(N)$.*

Our next objective is to define a lattice convex polytope $\mathcal{P}(n,m)$ which has the δ-vector [11] equals to the sequence of multidimensional Narayana numbers $\{N(m,n;k|1),\ \ 1 \le k \le (m-1)(n-1)\}$.

As a preliminary step we recall the definition of a Gelfand -Tsetlin polytope.
Let $\lambda = (\lambda_1,\ldots,\lambda_n)$ be partition and $\mu = (\mu_1,\ldots,\mu_n)$ be composition, $|\lambda| = |\mu|$. The Gelfand– Tsetlin polytope of type (λ,μ), denoted by $GT(\lambda,\mu)$, is the convex hull of all points $(x_{ij})_{1\le i\le j\le n} \in \mathbb{R}_{+}^{\binom{n+1}{2}}$ which satisfy the following set of inequalities and equalities

$$x_{i,j+1} \ge x_{ij} \ge x_{i+1,j+1} \ge 0, \quad x_{1j} = \lambda_j, \ \ 1 \le j \le n, \quad \sum_{a=1}^{j} x_{aj} = \sum_{a=1}^{j} \mu_a.$$

It is well-known that the number of integer points in the Gelfand–Tsetlin polytope $GT(\lambda,\mu)$, i.e. points $(x_{ij}) \in GT(\lambda,\mu)$ such that $x_{ij} \in \mathbb{Z}_{\ge 0}$, $\forall 1 \le i \le j \le n$, is equal to the Kostka number $K_{\lambda,\mu}(1)$. Therefore the stretched Kostka number $K_{N\lambda,N\mu}(1)$ counts the number of integer points in the polytope $GT(N\lambda,N\mu) = N \cdot GT(\lambda,\mu)$. So As far as is we know, there is no general criterion to decide where or not the Gelfand–Tsetlin polytope $GT(\lambda,\mu)$ has only integral vertices, but see [4],[7], [1] for particular cases treated.

In the present paper we are interested in the h-vectors of Gelfand–Tsetlin polytopes $GT(n,1^d)$ and that $GT((n^k,1^{kd}),(1^k)^{n+d})$. We expect (cf [1]) that the polytope $GT(n,1^d)$ is an integral one, but we don't know how to describe the set of parameters (n,k,d) such that the polytope $GT((n^k,(1^k)^{n+d})$ is an integral one.

Theorem 1.5

(1) *Let $\lambda := \lambda_{n,d} = (n,1^d)$ and $\mu = \mu_{n,d} := (1^{n+d})$. Then*

[10] It seems that the formulas for the degree of the stretched Kostka polynomials stated in [7], [26]. [16] are valid only for a special choice of λ, μ or $\mathcal{R}$.

[11] By definition the δ-vector of an integral convex polytope $\mathcal{P} \subset \mathbb{R}^{\mathcal{N}}$ of dimension d is equal to

$$\delta(\mathcal{P}) = \sum_{J=0}^{d} \delta_j t^j = (1-t)^{d+1} \sum_{m=0}^{\infty} \iota(\mathcal{P},m)\ t^m,$$

where $\iota(\mathcal{P},m) := \#(m\mathcal{P} \cap \mathbb{Z}^N)$ denotes the number of integer points in the stretched polytope $m\mathcal{P} := \{mx \mid x \in \mathcal{P}\}$, $m \ge 1$, and we set $\iota(\mathcal{P},0) = 1$.

$$\sum_{N \geq 0} K_{N\lambda, N\mu}(q) \ t^N = \frac{C_{d,n-1}(q^{\binom{n}{2}} \ t, q)}{(q^{\binom{n}{2}} \ t; q))_{d(n-1)+1}},$$

where $C_{d,m}(t,q) = \sum_{k=1}^{(d-1)(m-1)} N(d,m,k \mid q) \ t^{k-1}$ *stands for a* (q,t)*–analogue of the rectangular* (d,n)*-Catalan number.*
In particular, the normalized volume of the Gelfand–Tsetlin polytope $GT((n,1^d),1^{n+d})$ *is equal to the d–dimensional Catalan number*

$$C_{d,n}(1,1) := (dn)! \prod_{j=0}^{d-1} \frac{j!}{(n+j)!} = f^{(n^d)} = f^{(d^n)},$$

(2) *Let* $\lambda := \lambda_{n,1,2} = (n^2, 1^2)$ *and* $\mu = ((1,1)^{n+1})$, $n \geq 2$. *Then*

$$\sum_{N \geq 0} K_{N(n,n,1,1),N(1,1)^{n+1}}(1) \ t^N = \frac{P_{2,n}(t)}{(1-t)^{4n-6}},$$

and $P_{2,n}(1) = C_{n-3} \ C_{n-2}$, *product of two Catalan numbers.*
(3) *Let* $\lambda := \lambda_{n,k,d} = (n^k, 1^{kd})$ *and* $\mu = ((1^k)^{n+d})$, $d \geq 1$. *Then*

$$\sum_{N \geq 0} K_{N(n^k,1^{kd}).N(1^k)^{n+d}}(1) \ t^N = \frac{P_{k,d,n}(t)}{Q_{k,d,n}(t)}.$$

Moreover, $P_{k,d,n}(0) = 1$,

$$Q_{k,d,n}(t) = (1-t)^{k^2(d(n-1)-1)+2+(k-1)\delta_{n,2} \ \delta_{d,1}},$$

and the polynomial $P_{k,d,n}(t)$ *is symmetric with respect to variable t;*

$$deg_t(P_{k,k,n}(t)) = (k-1)(k(n-2) + 2(\delta_{n,2} - 1)).$$

One can see from Theorem 1.4, (1), that the degree of the stretched Kostka polynomial $\mathcal{K}_{(n,1),1^{n+1}}(N) := K_{N(n,1),N(1^{n+1})}(1)$ is equal to $n-1$, whereas it follows from Theorem 1.4, (2) that

$$deg_N(\mathcal{K}_{2(n,1),2(1)^{n+1}}(N)) = 4n - 7 > 3 \ deg_N(\mathcal{K}_{(n,1),1^{n+1}}(N)), \quad if \quad N > 4.$$

Therefore one comes to an infinite family of counterexamples to Okounkov log- concavity conjecture for the Littlewood–Richardson coefficients [24].

Corollary 1.6

- *Let* $n \geq 3$. *There exists an integer* $N_0(n)$ *such that*

$$K_{2N(n,1),2N(1)^{n+1}}(1) \quad > \quad \Big(K_{N(n,1),N(1)^{n+1}}(1)\Big)^2 \quad for \quad all \quad N \geq N_0(n); \tag{1.1}$$

- *Let $n \geq 5$ be an integer, choose ϵ, $0 \leq \epsilon < \frac{n-4}{n-1}$. There is an integer $N_0(n;\epsilon)$ such that*

$$K_{2N(n,1),2N(1)^{n+1}}(1) \quad > \quad \Big(K_{N(n,1),N(1)^{n+1}}(1)\Big)^{3+\epsilon} \qquad for \quad all \quad N \geq N_0(n;\epsilon).$$

(3) *Let $n > 1 + \frac{k^2+2}{k^2 d}$. There exist an integer $N_0(n,k,d)$ such that*

$$K_{2N(n^k,1^{kd}),2N(1^k)^{n+d}}(1) \quad > \quad \Big(K_{N(n^k,1^{kd}),N(1^k)^{n+d}}(1)\Big)^{3} \qquad for \quad all \quad N \geq N_0(n,k,d).$$

For example,

- $K_{2N(5,1),2N(1,1)^6}(1) > (K_{N(5,1),N(1^6)}(1))^3$

if and only if $N \geq 49916$.
Let us recall the well-known fact that any parabolic Kostka number $K_{\lambda,\mathcal{R}}(1)$ can be realized as the Littlewood–Richardson coefficient $c_{\lambda,M}^{\Lambda}$ for uniquely defined partitions Λ and M, see Section 3.2 for details

It should be stressed that for $n = 3$ the example (1.1) has been discovered in [3], and independently by the author (unpublished notes [18]). In this case the minimal value of $N_0(3)$ is equal to 23; one can show (A.K.) that $N_0(4) = 8$.

Our next objective of the present paper is to prove the unimodality of the principal specialization $s_\alpha * s_\beta(q,\ldots,q^{N-1})$ of Schur functions [12],[13]. Proofs given in *loc. cit.* is based on an identification of the principal specialization of internal product of Schur functions with a certain parabolic Kostka polynomial.

Theorem 1.7 *(***Principal specialization of the internal product of Schur functions and parabolic Kostka polynomials***)*

Let α,β be partitions such that $|\alpha| = |\beta|$, $\alpha_1 \leq r$ and $\beta_1 \leq k$. For given integer N such that $\alpha_1 + \beta_1 \leq Nr$, consider partition

$$\lambda_N := (rN - \beta'_k, rN - \beta'_{k-1}, \ldots, rN - \beta'_1, \alpha')$$

and a sequence of rectangular shape partitions

$$R_N := (\underbrace{(r^k),\ldots,(r^k)}_{N}).$$

Then [12]

$$K_{\lambda_N,R_N}(q) \doteq s_\alpha * s_\beta(q,\ldots,q^{N-1}). \tag{1.2}$$

[12] Hereinafter we shall use the notation $A(q) \doteq B(q)$ to mean that the ratio $A(q)/B(q)$ is a certain power of q.

Now we state a fermionic formula for polynomials

$$V_{\alpha,\beta}^{N}(q):=s_{\alpha}*s_{\beta}(q,\ldots,q^{N-1}),$$

which is our main tool to give a combinatorial proof of the unimodality of the principal specialization of the Schur functions, and that of the generalized q-Gaussian polynomials $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q$ associated with a partition λ, as a special case.

Theorem 1.8 *Let α and β be two partitions of the same size, and $r:=\ell(\alpha)$ be the length of α. Then*

$$s_{\alpha}*s_{\beta}(q,\ldots,q^{N-1})=\sum_{\{\nu\}}q^{c(\{\nu\})}\prod_{k,j\geq 1}\begin{bmatrix} P_j^{(k)}(\nu)+m_j(\nu^{(k)})+N(k-1)\delta_{j,\beta_1}\theta(r-k) \\ P_j^{(k)}(\nu) \end{bmatrix}_q, \quad (1.3)$$

where the sum runs over the set of admissible configurations $\{\nu\}$ of type $([\alpha,\beta]_N,(\beta_1)^N)$. Here for any partition λ, λ_j denotes its j-th component.

See Section 4, Theorem 4.2 for details concerning notation. An important property which is specific to admissible configurations of type $[\alpha,\beta]_N,\beta_1^n)$, is the following relations

$$2c(\nu)+\sum_{k,j\geq 1}P_j^{(k)}(\nu)\Big[m_j(\nu^{(k)})+N(k-1)\delta_{j,\beta_1}\theta(r-k)\Big]=N|\alpha|,$$

which imply the unimodality of polynomials $V_{\alpha,\beta}^{N}(q)$, and $\begin{bmatrix} N \\ \alpha \end{bmatrix}_q=V_{\alpha,(|\alpha|)}^{N}(q)$. Let us stress that the sum in the $RHS(1.3)$ runs over the set of admissible configurations of type $([\alpha,\beta]_N,(\beta_1)^N)$. Remark, that the $RHS(1.3)$ has a natural generalization to the case $|\alpha|\equiv|\beta|$ $(mod\ N)$, but in this case a representation-theoretical meaning of the $LHS(1.3)$ is unclear to the author.

Acknowledgments

The present paper is an extended and update version of notes have been paved the way to my Colloquium talk at the University of Queensland, Brisbane, Australia, November 2013, and is based on unpublished preprint [18]. I'm very appreciate to Professor Ole Warnaar (University of Queensland) for kind invitation, fruitful discussions and financial support of my visit.

2 Higher dimensional Catalan and Narayana numbers, [21], [16], [30]

2.1 Rectangular Catalan and Narayana polynomials, and MacMahon polytope, [16]

2.1.1 Rectangular Catalan and Narayana numbers and polynomials

Define **rectangular Catalan polynomial**

$$C(n,m|q) = \frac{(q;q)_{nm}}{\displaystyle\prod_{i=1}^{n}\prod_{j=1}^{m}(1-q^{i+j-1})} = [d\ n]_q! \prod_{j=0}^{d-1} \frac{[j]_q!}{[n+j]_q!}, \tag{2.4}$$

where $[n]_q := \frac{1-q^n}{1-q}$ stands for the q-analogue of an integer n, and by definition $[n]_q! := \prod_{j=1}^{n} [j]_q$.

Proposition 2.1

$$q^{m\binom{n}{2}} C(n,m|q) = K_{(n^m),(1^{nm})}(q). \tag{2.5}$$

Thus, $C(n,m|q)$ is a polynomial of degree $nm(n-1)(m-1)/2$ in the variable q with non–negative integer coefficients. Moreover,

$$C(n,2|q) = C(2,n|q) = c_n(q) = \frac{1-q}{1-q^{n+1}} \left[\begin{array}{c} 2n \\ n \end{array}\right]_q$$

coincides with "the most obvious" q–analog of the Catalan numbers, see e.g. [5], p.255, or [28], and [21],

$$C(n,3|q) = \frac{[2]_q\ [3\ n]_q\ !}{[n]_q\ !\ [n+1]_q\ !\ [n+2]_q\ !}.$$

It follows from (2.5) that the rectangular Catalan number $C(n,m|1)$ counts the number of *lattice* words

$$w = a_1 a_2 \cdots a_{nm}$$

of weight (m^n), i.e. lattice words in which each i between 1 and m occurs exactly n times. Let us recall that a word $a_1 \cdots a_p$ in the symbols $1,\ldots,m$ is said to be a *lattice* word, if for $1 \le r \le p$ and $1 \le j \le m-1$, the number of occurrences of the symbol j in $a_1 \cdots a_r$ is not less than the number of occurrences of $j+1$:

$$\#\{i|1 \le i \le r \ \text{ and } \ a_i = j\} \ge \#\{i|1 \le i \le r \ \text{ and } \ a_i = j+1\}. \tag{2.6}$$

For any word $w = a_1 \cdots a_k$, in which each a_i is a positive integer, define the major index

$$\text{maj}(w) = \sum_{i=1}^{k-1} i\chi(a_i > a_{i+1}),$$

and the number of descents

$$\mathrm{des}(w) = \sum_{i=1}^{k-1} \chi(a_i > a_{i+1}).$$

Finally, for any integer k between 0 and $(n-1)(m-1)$, define **rectangular q–Narayana** number

$$N(n,m;k \mid q) = \sum_{w} q^{\mathrm{maj}(w)},$$

where w ranges over all lattice words of weight (m^n) such that $\mathrm{des}(w) = k$.

Equivalently, $N(n,m;k)$ is equal to the number of rectangular standard Young tableaux with n rows and m columns having k descents, i.e. k occurrences of an integer j appearing in a lower row that that $j+1$.

Example 2.2 Take $n = 4$, $m = 3$, then

$$\sum_{k=0}^{6} N(3,4;k \mid 1) t^k = 1 + 22t + 113t^2 + 190t^3 + 113t^4 + 22t^5 + t^6.$$

We summarize the basic known properties of the rectangular Catalan and Narayana numbers in Proposition 2.3 below.

Proposition 2.3 *([21], [30], [13])*

(A) *(***Lattice words and rectangular Catalan numbers***)*
$C(n,m|q) = \sum_{w} q^{\mathrm{maj}(w)}$, *where w ranges over all lattice words of weight (m^n);*

(B) *(***Bosonic formula for multidimensional Narayana numbers***)*

$$N(n,m;k \mid q) = \sum_{a=0}^{k} (-1)^{k-a} \, q^{\binom{k-a}{2}} \left[\begin{matrix} n\ m+1 \\ k-a \end{matrix} \right]_q \prod_{b=0}^{n-1} \frac{[b]! \ [m+a+b]!}{[m+b]! \ [a+b]!}, \qquad (2.7)$$

(C) *(***Summation formula***) Let r be a positive integer, then*

$$\sum_{k=0}^{r} \left[\begin{matrix} n\ m+r-k \\ r-k \end{matrix} \right]_q N(n,m;k \mid q) = \prod_{a=0}^{m-1} \frac{[a]! \ [n+r+a]!}{[n+a]! \ [r+a]!} =$$

$$= \prod_{a=0}^{n-1} \frac{[a]! \ [m+r+a]!}{[m+a]! \ [r+a]!} = \prod_{a=0}^{r-1} \frac{[a]! \ [n+m+a]!}{[n+a]! \ [m+a]!}.$$

(D) *(***Symmetry***)*

$$N(n,m;k \mid q) = q^{nm((n-1)(m-1)/2-k)} N(n,m;(n-1)(m-1)-k \mid q^{-1}) = N(m,n;k \mid q),$$

for any integer k, $0 \le k \le (n-1)(m-1)/2$;

(E) *(*q**-Narayana numbers***)*

$$N(2,n;k \mid q) = q^{k(k+1)} \frac{1-q}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \doteq \dim_q V_{(k,k)}^{\mathfrak{gl}(n-k+1)}, \quad 0 \le k \le n-1,$$

where $V_{(k,k)}^{\mathfrak{gl}(n-k+1)}$ *stands for the irreducible representation of the Lie algebra* $\mathfrak{gl}(n-k+1)$ *corresponding to the two row partition* (k,k)*; recall that for any finite dimensional* $\mathfrak{gl}(N)$*–module* V *the symbol* $\dim_q V$ *denotes its* q*–dimension, i.e. the principal specialization of the character of the module* V*:*

$$\dim_q V = (\mathrm{ch} V)(1,q,\ldots,q^{N-1});$$

(F) $N(n,m;1 \mid 1) = \sum_{j \ge 2} \binom{n}{j} \binom{m}{j} = \binom{n+m}{n} - nm - 1;$

(G) *(***Fermionic formula for** q**–Narayana numbers***, [16])*

$$q^{m\binom{n}{2}} N(n,m;l \mid q) = \sum_{\{\nu\}} q^{c(\nu)} \prod_{k,j \ge 1} \begin{bmatrix} P_j^{(k)}(\nu) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{bmatrix}_q, \tag{2.8}$$

summed over all sequences of partitions $\{\nu\} = \{\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(m-1)}\}$ *such that*

- $|\nu^{(k)}| = (m-k)n$, $1 \le k \le m-1$;
- $(\nu^{(1)})'_1 = (m-1)n - l$, *i.e. the length of the first column of the diagram* $\nu^{(1)}$ *is equal to* $(m-1)n-l$, $l = 0, \ldots, (m-1)(n-1)$;
- $P_j^{(k)}(\nu) := Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) \ge 0$, *for all* $k, j \ge 1$,

where by definition we put $\nu^{(0)} = (1^{nm})$*; for any diagram* λ *the number* $Q_j(\lambda) = \lambda'_1 + \cdots \lambda'_j$ *is equal to the number of cells in the first* j *columns of the diagram* λ*, and* $m_j(\lambda)$ *is equal to the number of parts of* λ *of size* j*;*

- $c(\nu) = \sum_{k,j \ge 1} \binom{(\nu^{(k-1)})'_j - (\nu^{(k)})'_j}{2}$.

Example 2.4 *Consider the case* $m = 3$, $n = 4$. *In this case* $C(3,4 \mid 1) = 462$, *and the sequences of Narayana numbers is* $(1, 22, 113, 190, 113, 22, 1)$. *Let us display below the distribution of Narayana numbers which is coming from the counting the number of admissible rigged configurations of type* $((4^3),(1^{12}))$ *according to the number* $(m-1)n - \ell(\nu^{(1)})$, *where* $\ell(\nu^{(1)})$ *denotes the length of the first configuration* $\nu^{(1)}$*:*
$N(3,4;0 \mid 1) = 1$, $\quad N(3,4;1 \mid 1) = 1 + 21$, $\;N(3,4;2 \mid 1) = 15 + 35 + 63$,
$N(3,4;3 \mid 1) = 140 + 15 + 35$, $\quad N(3,4;4 \mid 1) = 21 + 28 + 63$,
$N(3,4;5 \mid 1) = 6 + 16$, $\quad N(3,4;6 \mid 1) = 1$.

Conjecture 2.5 *If* $1 \le k \le (n-1)(m-1)/2$, *then*

$$N(n,m;k-1 \mid 1) \le N(n,m;k \mid 1),$$

i.e. the sequence of rectangular Narayana numbers $\{N(n,m;k \mid 1)\}_{k=0}^{(n-1)(m-1)}$ *is symmetric and* **unimodal**.

For definition of unimodal polynomials/sequences see e.g. [27], where one may find a big variety of examples of unimodal sequences which frequently appear in Algebra, Combinatorics and Geometry.

2.1.2 Volume of the MacMahon polytope and rectangular Catalan and Narayana numbers

Let $\mathfrak{M}_{mn}$ be the convex polytope in $\mathbb{R}^{nm}$ of all points $\mathbf{x} = (x_{ij})_{1\le i\le n, 1\le j\le m}$ satisfying the following conditions

$$0 \le x_{ij} \le 1, \quad x_{ij} \ge x_{i-1,j}, \quad x_{ij} \ge x_{i,j-1}, \tag{2.9}$$

for all pairs of integers (i,j) such that $1 \le i \le n$, $1 \le j \le m$, and where by definition we set $x_{i0} = 0 = x_{0j}$.

We will call the polytope $\mathfrak{M}_{nm}$ by *MacMahon polytope*. The MacMahon polytope is an integral polytope of dimension nm with $\binom{m+n}{n}$ vertices which correspond to the set of (0,1)–matrices satisfying (2.9).

If k is a positive integer, define $i(\mathfrak{M}_{nm};k)$ to be the number of points $\mathbf{x} \in \mathfrak{M}_{nm}$ such that $k\mathbf{x} \in \mathbb{Z}^{nm}$. Thus, $i(\mathfrak{M}_{nm};k)$ is equal to the number of plane partitions of rectangular shape (n^m) with all parts do not exceed k. By a theorem of MacMahon (see e.g. [19], Chapter I, §5, Example 13)

$$i(\mathfrak{M}_{nm};k) = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{k+i+j-1}{i+j-1}. \tag{2.10}$$

It follows from (2.10) that the Ehrhart polynomial $\mathcal{E}(\mathfrak{M}_{nm};t)$ of the MacMahon polytope $\mathfrak{M}_{nm}$ is completely resolved into linear factors:

$$\mathcal{E}(\mathfrak{M}_{nm};t) = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{t+i+j-1}{i+j-1}.$$

Hence, the normalized volume

$$\widetilde{\mathrm{vol}}(\mathfrak{M}_{nm}) = (nm)!\mathrm{vol}(\mathfrak{M}_{nm})$$

of the MacMahon polytope $\mathfrak{M}_{nm}$ is equal to the rectangular Catalan number $C(n,m|1)$, i.e. the number of standard Young tableaux of rectangular shape (n^m). We refer the reader to [28], Section 4.6, and [6], Chapter IX, for definition and basic properties of the Ehrhart polynomial $\mathcal{E}(\mathcal{P};t)$ of a convex integral polytope $\mathcal{P}$.

Proposition 2.6

$$\sum_{k\ge 0} i(\mathfrak{M}_{nm};k)z^k = \left(\sum_{j=0}^{(n-1)(m-1)} N(n,m;j)z^j\right) / (1-z)^{nm+1}, \tag{2.11}$$

where

$$N(n,m;j) := N(n,m;j|1)$$

denotes the rectangular Narayana number.

Thus, the sequence of Narayana numbers

$$(1 = N(n,m;0), N(n,m;1), \ldots, N(n,m;(n-1)(m-1)) = 1)$$

is the δ*-vector* (see e.g. [28], p. 235) of the MacMahon polytope. In the case $n = 2$ (or $m = 2$) all these results may be found in [28], Chapter 6, Exercise 6.31.
Question. (*Higher associahedron*) Does there exist an $(m-1)(n-1)$-dimensional integral convex (simplicial?) polytope $Q_{n,m}$ which has δ-vector

$$\delta = (\delta_0(Q_{n,m}), \delta_1(Q_{n,m}), \ldots, \delta_{(n-1)(m-1)}(Q_{n,m}))$$

given by the *rectangular Narayana numbers* $N(n,m;k)$:

$$\sum_{i=0}^{(n-1)(m-1)} \delta_i(Q_{n,m}) t^i = C(n,m|t) \ ?$$

We refer the reader to [6], Chapter I, §6 and Chapter III, for definitions and basic properties of the h-vector and δ-vector of a simplicial polytope; see also, R. Stanley (J. Pure and Appl. Algebra **71** (1991), 319-331).

An answer on this question is known if either n or m is equal to 2, see e.g. R. Simion (Adv. in Appl. Math. **18** (1997), 149-180, Example **4** (the Associahedron)).

Definition 2.7 *([21], [30]) Define* **rectangular** *Schröder polynomial*

$$S(n,m|t) := C(n,m|1+t),$$

and put

$$S(n,m|t) = \sum_{k \geq 0}^{(n-1)(m-1)} S(n,m||k) t^k.$$

A combinatorial interpretations of the numbers $S(n,m||k)$ and $S(n,m|1)$ have been done by R. Sulanke [30].

2.1.3 Rectangular Narayana and Catalan numbers, and d dimensional lattice paths, [30]

Let $\mathcal{C}(d,n)$ denote the set of d–dimensional lattice paths using the steps

$$X_1 = (1,0,\cdots,0),\ X_2 = (0,1,\cdots,0), \cdots, X_d = (0,0,\cdots,1),$$

running from $(0,0,\cdots,0)$ to $(n,n,\cdots,n)$, and lying in the region

$$\{(x_1,x_2,\cdots,x_d)\in\mathbb{R}^d_{\geq 0}\mid x_1\leq x_2\leq\cdots\leq x_d\}.$$

For each path $P := p_1p_2\cdots p_{nd}\in\mathcal{C}(d,n)$ define the statistics

$$asc(P) := \#\{j\mid p_jp_{j+1}=X_k\ X_l,\ k<l\}.$$

Definition 2.8 *The n–th d–dimensional MacMahon–Narayana number of level k, $MN(d,n,k)$ counts the paths $P\in\mathcal{C}(d,n)$ with $asc(P)=k$.*

Proposition 2.9 *(Cf [30]) For any $d\geq 2$ and for $0\leq k\leq(d-1)(n-1)$,*

$$MN(d,n,k)=\sum_{j=0}^{k}(-1)^{k-j}\binom{dn+1}{k-j}\prod_{a=0}^{j-1}\frac{a!\ (d+n+a)!}{(d+a)!\ (n+a)!}.$$

Note that the product $\prod_{a=0}^{j-1}\frac{a!\ (d+n+a)!}{(d+a)!\ (n+a)!}$ is equal to the number of plane partitions of the rectangular shape (n^d), all the parts do not exceed j.

Definition 2.10 *For $d\geq 3$ and $n\geq 1$ the n-th d-Narayana polynomial defined to be*

$$N_{d,n}(t)=\sum_{k=0}^{(d-1)(n-1)}MN(d,n,k)\ t^k.$$

Corollary 2.11 *(Recurrence relations, [30]) For any integer $m\geq 0$ one has*

$$\sum_{k=0}^{m}\binom{dn+m-k}{m-k}MN(d,n,k)=\prod_{a=0}^{d-1}\frac{a!\ (n+m+a)!}{(n+a)!\ (m+a)!}.$$

Corollary 2.12 *The MacMahon–Narayana number $MN(d,n,k)$ is equal to the rectangular Narayana number $N(d,n;k)$.*

2.1.4 Gelfand–Tsetlin polytope $GT((n,1^d),(1)^{n+d})$ and rectangular Narayana numbers

Theorem 2.13 *Let $\lambda := \lambda_{n,d}=(n,1^d)$ and $\mu=\mu_{n,d}:=(1^{n+d})$. Then*

$$\sum_{N\geq 0}K_{N\lambda,N\mu}(q)\ t^N=\frac{C_{d,n-1}(q^{\binom{n}{2}}\ t,q)}{(q^{\binom{n}{2}}\ t;q))_{d(n-1)+1}},$$

where $C_{d,m}(t,q)=\sum_{k=0}^{(d-1)(m-1)}N(d,m,k\mid q)\ t^k$ stands for a (q,t)–analog of the rectangular (d,n)-Catalan number.
In particular, the normalized volume of the Gelfand–Tsetlin polytope $GT((n,1^d),1^{n+d})$ is equal to the d–dimensional Catalan number

$$C_{d,n}(1,1):=(dn)!\prod_{j=0}^{d-1}\frac{j!}{(n+j)!}=f^{(n^d)}=f^{(d^n)},$$

where for any partition λ, f^{λ} denotes the number of standard Young tableaux of shape λ.

3 Rigged configurations, stretched Kostka numbers, log-concavity and unimodality

3.1 Stretched Kostka numbers $K_{N(n^k.1^{kd}),N(1^k)^{n+d}}(1)$

Theorem 3.1 (1)

$$\sum_{N\geq 0} K_{N(n,n,1,1)),N((1,1)^{n+1})}(1)\ t^N = \frac{P_{2,n}(t)}{(1-t)^{4n-6}},$$

and $P_{2,n}(1)=C_{n-3}\ C_{n-2}$.

(2) *Let* $d\geq 1$*, then*

$$\sum_{N\geq 0} K_{N(n^k,1^{kd}).N(1^k)^{n+d}}(1)\ t^N = \frac{P_{k,d,n}(t)}{Q_{k,d,n}(t)}.$$

Moreover, $P_{k,d,n}(0)=1$,

$$Q_{k,d,n}(t) = (1-t)^{k^2(d(n-1)-1)+2+(k-1)\delta_{n,2}\ \delta_{d,1})},$$

and the polynomial $P_{k,d,n}(t)$ *is symmetric with respect to variable* t*;*

$$deg_t(P_{k,k,n}(t)) = (k-1)(k(n-2)+2(\delta_{n,2}-1)).$$

■

For example, assume that $d=1$ and set $P_{k,n}(t) := P_{k,1,n}(t)$. Then
$P_{2,3}(t)=1$, $P_{2,4}(t)=(1,0,1), P_{2,5}(t)=(1,1,6,1,1)$, $P_{2,6}(t)=(1,3,21,20,21,3,1)$,
$P_{2,7}(t)=(1,6,56,126,210,126,56,6,1)$,
$P_{2,8}(t)=(1,10,125,500,1310,1652,1310,500,125,10,1)$, $P_{3,3}(t)=(1,-1,1)$,
$P_{3,4}(t)=(1,0,20,20,55,20,20,0,1), P_{3,5}(t)=(1,6,141,931,4816,13916,$
$27531,33391,27531,13916,4816,931,141,6,1), P_{4,1,3}(t)=(1,-3,9,-8,9,-3,1)=P_{4,2,2}(t)$.

It follows from the duality theorem for parabolic Kostka polynomials [16] that

$$K_{N\lambda,N\mu}(1) = K_{(Nn,N^d)',((N)^{n+d})')}(1) = K_{((d+1)^N,1^{N(n-1)}),(1^N)^{n+d})}(1),$$

and

$$K_{((2d+2)^N,2^{N(n-1)}),((2)^N)^{n+d})}(1) = K_{(Nn,Nn,N^{2d}),((N,N)^{n+d})}(1).$$

Now consider the case $d=1$, that is $\lambda=(n,1)$, $\mu=(1^{n+1})$. Then

$$K_{N\lambda,N\mu}(1) = K_{(Nn,N),(N^{n+1})}(1) = \binom{N+n-1}{n-1}.$$

The second equality follows from a more general result [12], [16],

Proposition 3.2 *Let λ be a partition and N be a positive integer. Consider partitions $\lambda_N := (N|\lambda|, \lambda)$ and $\mu_N := (|\lambda|^{N+1}) = (\underbrace{|\lambda|, \ldots, |\lambda|}_{N+1}|)$. Then*

$$K_{\lambda_N,\mu_N}(q) \stackrel{\bullet}{=} \begin{bmatrix} N \\ \lambda \end{bmatrix} = dim_q V_\lambda^{\mathfrak{gl}(N)},$$

where the symbol $P(q) \stackrel{\bullet}{=} R(q)$ means that the ratio $P(q)/R(q)$ is a power of q; the symbol $\begin{bmatrix} N \\ \lambda \end{bmatrix}$ stands for the generalized Gaussian coefficient corresponding to a partition λ, see [19] for example.

3.2 Counterexamples to Okounkov's log-concavity conjecture

On the other hand,

$$K_{2\lambda_N,2\mu_N}(1) = K_{N(n,n,1,1),N(1,1)^{n+1}}(1) = Coeff_{t^N}\left(\frac{P_{2,n}(t)}{(1-t)^{4n-6}}\right).$$

Therefore the number $K_{2\lambda_N,2\mu_N}(1)$ is a **polynomial** of the degree $4\ n-7$ with respect to parameter N. Recall that the number $K_{N\lambda,N\mu}(1) = \binom{N+n-1}{n-1}$ is a polynomial of degree $n-1$ with respect to parameter N. Therefore we come to the following infinite set of examples which violate the log-concavity Conjecture stated by A. Okounkov [25].

Corollary 3.3 *For any integer $n > 4$, there exists a constant $N_0(n)$ such that*

$$K_{2\lambda_N,2\mu_N}(1) > \Big(K_{N\lambda,N\mu}(1)\Big)^3,$$

for all $N > N_0(n)$.

Recall that $\lambda = (n,1),\ \mu = (1^{n+1})$.

Now take $n = 3$. One has [3]

$$K_{N(3,1),N(1^4)}(1) = \binom{N+2}{2}, \quad K_{N(3,3,1,1),N(1,1)^4}(1) = \binom{N+5}{5}.$$

One can check [3] that

$$K_{N(3,3,1,1),N(1,1)^4}(1) > \Big(K_{N(3,1),N(1^3)}(1)\Big)^2.$$

if (and only if) $N \geq 21$.

Indeed,

$$K_{N(3,3,1,1),N(1,1)^4}(1) - \Big(K_{N(3,1),N(1^3)}(1)\Big)^2 = \frac{N^2-18N-43}{20}\binom{n+2}{3}..$$

Now take $n = 4$. One has

$$K_{N(4,1),N(1^5)}(1) = \binom{N+3}{3}, \quad K_{N(4,4,1,1),N(1,1)^5}(1) = \binom{N+9}{9} + \binom{N+7}{9}.$$

One can check that

$$K_{N(4,4,1,1),N(1,1)^5}(1) > \Big(K_{N(4,1),N(1^5)}(1)\Big)^2,$$

if (and only if) $N \geq 8$,
Now take $n = 5$.

Proposition 3.4 *Let $\nu_N := N(5,1)$ and $\eta_N := N(1)^6$. Then*

- $K_{2\nu_N,2\eta_N}(1) > (K_{\nu_N,\eta_N}(1))^2$

if and only if $N \geq 6$,

- $K_{2\nu_N,2\eta_N}(1) > (K_{\nu_N,\eta_N}(1))^3$

if and only if $N \geq 49916$.

Indeed,

$$K_{N(5,1),N(1^6)}(1) = \binom{N+4}{4},$$

$$K_{N(5,5,1,1),N(1,1)^6}(1) = \binom{N+13}{13} + \binom{N+12}{13} + 6\binom{N+11}{13} + \binom{N+10}{13} + \binom{N+9}{13},$$

and $51891840 \times \Big[K_{N(5,5,1,1),N(1,1)^6}(1) - \Big(K_{N(5,1),N(1^6)}(1)\Big)^3\Big] = \binom{N+4}{5} \times$
$(-78631416 - 172503780\ N - 174033932\ N^2 - 101206400\ N^3 - 35852065\ N^4 -$
$7638110\ N^5 - 899548\ N^6 - 44990\ N^7 + N^8)$.

Note, see e.g. [19], that for any set of partitions $\lambda, \mu^{(1)}, \ldots, \mu^{(p)}$ the parabolic Kostka number $K_{\lambda,\mu^{(1)},\ldots,\mu^{(p)}}(1)$ is equal to the Littlewood–Richardson number $c^{\Lambda}_{\lambda,M}$, where partitions $\Lambda \supset M$ are such that $\Lambda \setminus M = \coprod_i \mu^{(i)}$ is a disjoint union of partitions $\mu^{(i)},\ i = 1, \ldots, p$.

4 Internal product of Schur functions

The irreducible characters χ^λ of the symmetric group S_n are indexed in a natural way by partitions λ of n. If $w \in S_n$, then define $\rho(w)$ to be the partition of n whose parts are the cycle lengths of w. For any partition λ of m of length l, define the power–sum symmetric function

$$p_\lambda = p_{\lambda_1} \ldots p_{\lambda_l},$$

where $p_n(x) = \sum x_i^n$. For brevity write $p_w := p_{\rho(w)}$. The Schur functions s_λ and power–sums p_μ are related by a famous result of Frobenius

$$s_\lambda = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda(w) p_w. \tag{4.1}$$

For a pair of partitions α and β, $|\alpha| = |\beta| = n$, let us define the internal product $s_\alpha * s_\beta$ of Schur functions s_α and s_β:

$$s_\alpha * s_\beta = \frac{1}{n!} \sum_{w \in S_n} \chi^\alpha(w) \chi^\beta(w) p_w. \tag{4.2}$$

It is well–known that

$$s_\alpha * s_{(n)} = s_\alpha, \quad s_\alpha * s_{(1^n)} = s_{\alpha'},$$

where α' denotes the conjugate partition to α.

Let α, β, γ be partitions of a natural number $n \geq 1$, consider the following numbers

$$g_{\alpha\beta\gamma} = \frac{1}{n!} \sum_{w \in S_n} \chi^\alpha(w) \chi^\beta(w) \chi^\gamma(w). \tag{4.3}$$

The numbers $g_{\alpha\beta\gamma}$ coincide with the structural constants for multiplication of the characters χ^α of the symmetric group S_n:

$$\chi^\alpha \chi^\beta = \sum_\gamma g_{\alpha\beta\gamma} \chi^\gamma. \tag{4.4}$$

Hence, $g_{\alpha\beta\gamma}$ are non–negative integers. It is clear that

$$s_\alpha * s_\beta = \sum_\gamma g_{\alpha\beta\gamma} s_\gamma. \tag{4.5}$$

4.1 Internal product of Schur functions, principal specialization, fermionic formulas and unimodality

Let $N \geq 2$, consider the principal specialization $x_i = q^i$, $1 \leq i \leq N-1$, and $x_i = 0$, if $i \geq N$, of the internal product of Schur functions s_α and s_β:

$$s_\alpha * s_\beta(q, q^2, \ldots, q^{N-1}) = \frac{1}{n!} \sum_{w \in S_n} \chi^\alpha(w) \chi^\beta(w) \prod_{k \geq 1} \left(\frac{q^k - q^{kN}}{1 - q^k} \right)^{\rho_k(w)}, \tag{4.6}$$

where $\rho_k(w)$ denotes the number of the length k cycles of w.

By a result of R.-K. Brylinski [2], Corollary 5.3, the polynomials

$$s_\alpha * s_\beta(q, \ldots, q^{N-1})$$

admit the following interpretation. Let $P_{n,N}$ denote the variety of n by n complex matrices z such that $z^N = 0$. Denote by

$$R_{n,N} := \mathbb{C}[P_{n,N}]$$

the coordinate ring of polynomial functions on $P_{n,N}$ with values in the field of complex numbers $\mathbb{C}$. This is a graded ring:

$$R_{n,N} = \oplus_{k \geq 0} R_{n,N}^{(k)},$$

where $R^{(k)}_{n,N}$ is a finite dimensional $\mathfrak{g}l(n)$–module with respect to the adjoint action. Let α and β be partitions of common size. Then [2]

$$s_\alpha * s_\beta(q,\ldots,q^{N-1}) = \sum_{k\geq 0} \langle V_{[\alpha,\beta]_n}, P^{(k)}_{n,N}\rangle q^k,$$

as long as $n \geq \max(Nl(\alpha), Nl(\beta), l(\alpha)+l(\beta))$. Here the symbol $\langle \bullet\ ,\bullet\ \rangle$ denotes the scalar product on the ring of symmetric functions such that $\langle s_\lambda\ , s_\mu\ \rangle = \delta_{\lambda,\mu}$. In other words, if V and W be two $GL(N)$-modules, then $\langle V, W\rangle = \dim Hom_{GL(N)}(V,W)$.

Let us remind below one of the main result obtained in [16], namely, Theorem 6.6, which connects the principal specialization of the internal product of Schur functions with certain parabolic Kostka polynomials, and gives, via Corollary 6.7, [16], an effective method for computing the polynomials $s_\alpha * s_\beta(q,\ldots,q^{N-1})$ which, turns out to be for the first time, does not use the character table of the symmetric group S_n, $n=|\alpha|$.

Let α and β be partitions, $\ell(\alpha)=r$, $\ell(\beta)=s$ and $|\alpha|=|\beta|$. Let N be an integer such that $r+s<N$. Consider partition

$$[\alpha,\beta]_N := (\alpha_1+\beta_1, \alpha_2+\beta_1, \ldots, \alpha_r+\beta_1, \underbrace{\beta_1,\ldots,\beta_1}_{N-r-s}, \beta_1-\beta_s, \ldots, \beta_1-\beta_2].$$

Clearly, $|[\alpha,\beta]_N| = \beta_1 N$, $\ell([\alpha,\beta]_N = N-1$.

Theorem 4.1 *i) Let α,β be partitions, $|\alpha|=|\beta|$, $l(\alpha)\leq r$, and $l(\alpha)+l(\beta)\leq Nr$. Consider the sequence of rectangular shape partitions*

$$R_N = \{\underbrace{(\beta_1^r),\ldots,(\beta_1^r)}_{N}\}.$$

Then

$$s_\alpha * s_\beta(q,\ldots,q^{N-1}) \doteq K_{[\alpha,\beta]_{\mathbf{Nr}},R_N}(q). \qquad (4.7)$$

ii) (Dual form) Let α,β be partitions such that $|\alpha|=|\beta|$, $\alpha_1\leq r$ and $\beta_1\leq k$. For given integer N such that $\alpha_1+\beta_1\leq Nr$, consider partition

$$\lambda_N := (rN-\beta'_k, rN-\beta'_{k-1}, \ldots, rN-\beta'_1, \alpha')$$

and a sequence of rectangular shape partitions

$$R_N := (\underbrace{(r^k),\ldots,(r^k)}_{N}).$$

Then

$$K_{\lambda_N R_N}(q) \doteq s_\alpha * s_\beta(q,\ldots,q^{N-1}) \qquad (4.8)$$

Theorem 4.2 *(***Fermionic formula for the principal specialization of the internal product of Schur functions***).*

Let α and β be two partitions of the same size, and $r := \ell(\alpha)$ be the length of α. Then

$$s_\alpha * s_\beta(q,\ldots,q^{N-1}) = \sum_{\{\nu\}} q^{c(\{\nu\})} \prod_{k,j\geq 1} \begin{bmatrix} P_j^{(k)}(\nu) + m_j(\nu^{(k)}) + N(k-1)\delta_{j,\beta_1}\theta(r-k) \\ P_j^{(k)}(\nu) \end{bmatrix}_q, \quad (4.9)$$

where the sum runs over the set of admissible configurations $\{\nu\}$ of type $([\alpha,\beta]_N, (\beta_1)^N)$. Here for any partition λ, λ_j denotes its j-th component.

Let us explain notations have used in Theorem 3.5.

- A configuration $\{\nu\}$ of type $[\alpha,\beta]_N$ consists of a collection of partitions $\{\nu^{(1)},\ldots,\nu^{(N-1)}\}$ such that $|\nu^k| = \sum_{j>k}\Big([\alpha,\beta]_N\Big)_j$; by definition we set $\nu^{(0)} := (\beta_1)^N$;
- $P_j^{(k)}(\nu) := N\min(j,\beta_1)\ \delta_{k,1} + Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{k+1})$; here for any partition λ we set $Q_n(\lambda) := \sum_{j\leq n}\ \min(n,\lambda_j)$;
- For any partition λ, $m_j(\lambda)$ denotes the number of parts of λ are equal to j;
- A configuration $\{\nu\}$ of type $[\alpha,\beta]_N$ is called *admissible configuration of type* $([\alpha,\beta]_N, (\beta_1)^N)$, if $P_j^{(k)}(\nu) \geq 0,\ \forall j,k \geq 1$;
- Here $\delta_{n,m}$ denotes Kronecker's delta function, and we define $\theta(x) = 1,\ if\ \ x \geq 0$, and $\theta(x) = 0,\ if\ \ x < 0$;
- $c(\{\nu\}) = \sum_{n,k\geq 1} \binom{\lambda_n^{(k-1)} - \lambda_n^{(k)}}{2}$ denotes the **charge** of a configuration $\{\nu\}$; by definition, $\binom{x}{2} := x(x-1)/2,\ \forall x \in \mathbb{R}$.

Let us draw attention to the fact that the summation in (4.9) runs over the set of all admissible configurations of type $([\alpha,\beta]_N, (\beta_1)^N)$, other than that of type $([\alpha,\beta]_{Nr}, (\beta_1^r)^N)$.

Corollary 4.3 *([12],[17])*

For any partitions of the same size α and β, the polynomial $s_\alpha * s_\beta(q,\ldots,q^{N-1})$ is symmetric and **unimodal**. *In particular, the generalized Gaussian polynomial $\begin{bmatrix} N \\ \alpha \end{bmatrix}_q$ is symmetric and* **unimodal** *for any partition α.*

Indeed, in the case $\beta = (n)$, $n := |\beta|$, one has $s_\alpha * s_{(n)} = s_\alpha$.

Our proof of Corollary 3.7 is based on the following identity

$$2c(\nu) + \sum_{k,j\geq 1} P_j^{(k)}(\nu)\Big[m_j(\nu^{(k)}) + N(k-1)\delta_{j,\beta_1}\theta(r-k)\Big] = N|\alpha|,$$

which can be checked directly. This identity shows that the all polynomials associated with a given admissible configuration involved, are symmetric and have the same "symmetry center" $N|\alpha|/2$, and therefore the resulting polynomial $s_\alpha * s_\beta(q,\ldots,q^{N-1})$ is symmetric and unimodal.

Corollary 4.4 *Let α and β be partitions of the same size, and $K_{\beta,\lambda}(q,t)$ denotes the Kostka–Macdonald polynomial associated with partitions α and β, [19]. One has*

$$K_{\beta,\alpha}(q,q)=H_{\alpha}(q)\Bigg(\sum_{\{\nu\}} q^{c(\{\nu\})}\prod_{k=2}^{r}\frac{1}{\Big[m_{\beta_1}(\nu^{(k)})\Big]_q!}\prod_{\substack{k\ge 1\\ j\ge 1,\ j\ne\beta_1}}\begin{bmatrix}P_j^{(k)}(\nu)+m_j(\nu^{(k)})\\ m_j(\nu^{(k)})\end{bmatrix}_q\Bigg),$$

where the sum runs over the same set of admissible configurations as in Theorem 3.6, and $[m]_q! := \prod_{j=1}^{m}(1-q^j)$ stands for the q-factorial of an positive integer m, and by definition $[0]_q! = 1$; $H_\alpha(q) := \prod_{x\in\alpha}(1-q^{h(x)})$ denotes the hook polynomial associated with partition α, see e.g. [19].

Indeed, one can show [13], [12] that

$$\lim_{N\to\infty} s_\alpha * s_\beta(1,q,q^2,,q^N)=\frac{K_{\beta,\alpha}(q,q)}{H_\alpha(q)}=\lim_{N\to\infty}K_{[\alpha,\beta]_N,(\beta_1)^N}(q).$$

A fermionic formula for the principal specialization of the internal product of Schur functions, and therefore that for the generalized Gaussian polynomials, is a far generalization of the so-called KOH-identity [24] which is equivalent to the fermionic formula for the Kostka number $K_{(Nk,k),(k)^{N+1}}(1)$. The rigged configuration bijection gives rise to a combinatorial proof of Theorem 3.5, and therefore to a combinatorial proof of unimodality of the generalized Gaussian polynomials [12], as well as to give an interpretation of the statistics introduced in [24] in terms of rigged configurations data, see [13], Section 10.2.

Example 4.5 *Let $\alpha=(4,2)$, $\beta=\alpha^{'}=(2,2,1,1)$. We want to compute the principal specialization of the internal product of Schur functions $s_\alpha * s_\beta(q,\ldots,q^{N-1})$ by means of a fermionic formula* (3.10)*.First of all, there are 8 admissible configurations of type $((4,2),(2,2,1,1),(6)^N)$.In fact, it is a general fact that for given partitions λ and μ, the number of admissible configurations of type $(N\lambda,N\mu)$* **doesn't depend** *on N ,if $N>N_0$ i a certain number $N_0:=N_0(\lambda,\mu)$ depending on λ and μ only. This fact is a direct consequence of constraints are imposed by the set of inequalities $\{P_j^{(k)}(\nu)\ge 0,\ \ \forall j,k\ge 1\}$.*
Now let us list the **conjugate** *of the first configurations $\nu^{(1)}\in\{\nu\}$ for all admissible configurations $\{\nu\}$ of type $[(4,2),(2,2,1,1)_N]$, together with all non-zero numbers $P_j^{(k)}(\nu)$, $j,k\ge 1$.*

$$(N-3,N-3),\quad P_2^{(1)}=2, P_2^{(2)}=2,\quad c=9,$$

$$(N-2,N-4),\quad P_2^{(1)}=2, P_1^{(2)}=1, P_2^{(2)}=2,\quad 9,$$

$$(N-3,N-4,1),\quad P_1^{(1)}=2, P_1^{(2)}=4, P_1^{(3)}=2, P_2^{(2)}=1,\quad c=11,$$

$$(N-2,N-5,1),\quad P_1^{(2)}=1, P_2^{(1)}=4, P_3^{(1)}=2, P_2^{(2)}=1,\quad c=13,$$

$$(N-3,N-5,1,1),\quad P_1^{(1)}=2, P_2^{(1)}=6, (P_2^{(2)}=0), P_3^{(1)}=2,\quad c=15,$$

$$(N-3,N-5,2),\quad P_1^{(1)}=2, P_2^{(1)}=6, (P_2^{(2)}=0), P_3^{(1)}=2,\quad c=17,$$

$$(N-2,N-6,1,1),\quad P_1^{(2)}=1, P_2^{(1)}=6, (P_2^{(2)}=0), P_4^{(1)}=2,\quad c=19,$$
$$(N-2,N-6,2),\quad P_1^{(2)}=1, P_2^{(1)}=6, (P_2^{(2)}=0), P_3^{(1)}=2,\quad c=21.$$

These are data related with the first configurations $\nu^{(1)}$ in the set of all admissible configurations of type $([(42),(2211)]_N,(6)^N)$, and the set of all nonzero numbers $P_j^{(k)}(\nu)$. All other diagrams $\nu^{(k)}$, $k>1$ from the set of admissible configurations in question, are the <u>*same,*</u> *and are displayed below*

$$\nu^{(k)}=(N-2-k, max(N-4-k,0)),\quad 2\le k\le N-3,$$

so that $m_1^{(2)}=2,\quad m_2^{(2)}=N-6.$

Therefore,

$$(\clubsuit)\quad K_{[(4,2),(2,2,1,1)]_{2N},(2,2)^N}(q)\doteq q^9\begin{bmatrix}N-1\\2\end{bmatrix}\begin{bmatrix}2N-4\\2\end{bmatrix}+q^9\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}N-2\\2\end{bmatrix}\begin{bmatrix}2N-4\\2\end{bmatrix}+$$
$$q^{11}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}N-1\\4\end{bmatrix}\begin{bmatrix}2N-5\\1\end{bmatrix}+q^{13}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}N-2\\4\end{bmatrix}\begin{bmatrix}2N-5\\1\end{bmatrix}+$$
$$q^{15}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}N\\6\end{bmatrix}+q^{17}\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}N-1\\6\end{bmatrix}+q^{19}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}N-1\\6\end{bmatrix}+q^{21}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}N-2\\6\end{bmatrix}.$$

On the other hand,

$$s_{42}*s_{2211}=\frac{1}{720}\Big(81p_1^6-135p_1^4p_2+45p_1p_2^2-90p_1^2p_4+144p_1p_5-135p_2^3+90p_2p_4\Big).$$

Here $p_k:=\sum_{i\ge1}x_i^k$ *stands for the power sum symmetric functions degree of* k. *One can check that* $s_{42}*s_{2211}(q,\ldots,q^{N-1})=RHS(\clubsuit)\doteq K_{[(4,2),(2,2,1,1)]_{2N},(2,2)^N}(q)$, *as expected.*
Finally one can check that $\lim_{N\to\infty}\ s_{42}*s_{2211}(q,\ldots,q^{N-1})=q^9(2,1,2,2,1,1)=K_{2211,42}(q,q)$.

4.2 Polynomiality of stretched Kostka and Littlewood–Richardson numbers

• As it was mentioned above, for a given partitions λ and μ (resp. λ and a sequence of rectangular shape partitions $\{\mathcal{R}\}$), the number of **admissible** configurations of type $(N\lambda,N\mu)$ (resp. of type $(N\lambda,\{\mathcal{R}\})$) **doesn't** depend on $N>N_0$, where the number N_0 depends on λ and μ (resp. λ and $\{\mathcal{R}\}$) only.

• Each admissible configuration $\{\nu\}$ provides a contribution of a form

$$a_{\{\nu\}}(q)\prod_{j,k\ge1}\begin{bmatrix}b_{j,k}(\nu)N+d_{jk}(\nu)\\d_{jk}(\nu)\end{bmatrix}_q,$$

to the parabolic Kostka polynomial $K_{N\lambda,N\{\mathcal{R}\}}(q)$, where a polynomial $a_\nu(q)$ and a **finite set** of numbers $\{b_{jk}(\nu),d_{jk}(\nu)\}_{j,k\ge1}$ both doesn't depend on $N>N_0$ for some $N_0:=N_0(\lambda,\{\mathcal{R}\})$.

• It is clear that the sum $\sum_{N\ge0}\binom{aN+b}{b}t^N$ is a **rational** function of variable t. It is well-known (and easy to prove) that the *Adamar product* [13] of *rational* functions is again a

[13] See e.g. $wikipedia.org/wiki/Hadamard\_product\_(matrices)$ and the literature quoted therein.

rational function. Therefore,

Corollary 4.6 *([16],[7], [26])*

For any partition λ and a sequence of rectangular shape partitions $\{\mathcal{R}\}$, the generating function

$$\sum_{N\geq 0} K_{\lambda,\{\mathcal{R}\}}(1)\ t^N$$

is a rational function of variable t with a unique pole at $t=1$.

More generally using a q-version of Adamar's product Theorem, we can show

Theorem 4.7 *([16])*

For any partition λ and a dominant *sequence of rectangular shape partitions $\{\mathcal{R}\}$, the generating function of stretched parabolic Kostka polynomials*

$$\sum_{N\geq 0} K_{\lambda,\{\mathcal{R}\}}(q)\ t^N$$

is a rational function of variables q and t of a form $P_{\lambda,\{\mathcal{R}\}}(q,t)/Q_{\lambda,\{\mathcal{R}\}}(q,t)$, where the dominator $Q_{\lambda,\{\mathcal{R}\}}(q,t)$ has the following form

$$Q_{\lambda,\{\mathcal{R}\}}(q,t)=\prod_{s\in S}(1-q^s\ t)$$

for a certain finite set $S:=S(\lambda,\{\mathcal{R}\})$ depending on λ and $\{\mathcal{R}\}$.

Clearly that Corollary 3.9 is a special case $q=1$ of Theorem 3.10.

- (Littlewood–Richardson polynomials) Let λ be a partition and $\{\mathcal{R}\}$ be a dominant sequence of rectangular shape partitions. Write

$$K_{\lambda,\{\mathcal{R}\}}(q)=b(\lambda,\mathcal{R})\ q^{a(\lambda,\mathcal{R})}+\ \ \textit{higher degree terms.}$$

(1) (Generalized saturation theorem [17])

$$a(N\lambda,N\{\mathcal{R}\})=N\ a(\lambda,\{\mathcal{R}\}).$$

Therefore,

$$\sum_{N\geq 0} b(N\lambda,N\{\mathcal{R}\})\ t^N=\frac{P_{\lambda,\mathcal{R}}(q,q^{-a(\lambda,\{\mathcal{R}\})}\ t)}{Q_{\lambda,\mathcal{R}}(q,q^{-a(\lambda,\{\mathcal{R}\})}\ t)}\Bigg|_{q=0}$$

is a **rational** function with a unique pole at $t=1$ with multiplicity equals to $\#|s\in S(\lambda,\mathcal{R})\mid s=a(\lambda,\mathcal{R})|$.

(2) Now let λ, μ and ν be partitions such that $|\lambda|+|\mu|=|\nu|$. Consider an integer $N\geq max(\ell(\lambda),\mu_1)$, and define partition $\Lambda=\Lambda(N,\lambda,\mu):=(N^N)\oplus\lambda,\mu)$ and the dominant rearrangement of the set of rectangular shape partitions $\{(N^N),\nu_1,\ldots,\nu_{\ell(\nu)}\}$, denoted by $M:=M(N,\nu)$.

Proposition 4.8 *([17])*

One has

$$b(\Lambda, M) := c^{\nu}_{\lambda,\mu},$$

where $c^{\nu}_{\lambda,\mu}$ *denotes the Littlewood–Richardson number corresponding to partitions* λ, μ *and* ν*, that is , the multiplicity of Schur function* s_ν *in the product of Schur functions* $s_\lambda\ s_\mu$*.*

Theorem 4.9 *([17],[26])*

Given three partitions λ, μ *and* ν *such that* $|\lambda| + |\mu| = |\nu|$*. The the generating function*

$$\sum_{N \geq 0} c^{N\nu}_{N\lambda,N\mu}\ t^N$$

is a **rational** *function of variable* t *with a unique pole at* $t = 1$*. Therefore,* $c^{N\nu}_{N\lambda,N\mu}$ *is a* **polynomial** *in* N *with rational coefficients.*

Example 4.10 *([13], [16]) (MacMahon polytope and multidimensional Narayana numbers again)*

Take $\lambda = (n+k, n, n-1, \ldots, 2)$ *and* $\mu = \lambda' = (n, n, n-1, n-2, \ldots, 2, 1^k)$*. One can show [16] that if* $n \geq k \geq 1$*, then for any positive integer* N

- $a(N\lambda, N\mu) = (2k-1)N$;
- $b(N\lambda, N\mu) = \dim V^{\mathfrak{gl}(N+k-1)}_{((n-k+1)^{k-1})} = \prod_{i=1}^{k-1}\prod_{j=1}^{n-k+1} \dfrac{N+i+j-1}{i+j-1}.$

In other words, the number $b(N\lambda, N\mu)$ *is equal to the number of (weak) plane partitions of rectangular shape* $((n-k+1)^{k-1})$ *whose parts do not exceed* N*. According to Exercise 1,* **c***, [16], pp. 102–103,* $b(N\lambda, N\mu)$ *is equal also to the number* $i(\mathfrak{M}_{k-1,n-k+1}; N)$ *of rational points* $\mathbf{x}$ *in the MacMahon polytope* $\mathfrak{M}_{k-1,n-k+1}$ *such that the points* $N\mathbf{x}$ *have integer coordinates. It follows from (2.8) that the generating function for numbers* $b(n\lambda, n\mu)$ *has the following form*

$$\sum_{n \geq 0} b(n\lambda, n\mu)t^n = \left(\sum_{j=0}^{(k-2)(n-k)} N(k-1, n-k+1; j)t^j\right) / (1-t)^{(k-1)(n-k+1)+1},$$

where $N(k, n; j)$*,* $0 \leq j \leq (k-1)(n-1)$*, denote rectangular Narayana's numbers, see e.g. [21], [30].*

One can show (A.K.) that

- *if* $r := k - \binom{n+2}{2} \geq 0$*, then* $b(\lambda, \mu) = 1$*, and*

$$a(\lambda, \mu) = 2\binom{n+3}{3} + (n+1)(2r-1) + \binom{r}{2};$$

- *if* $1 \leq k < \binom{n+2}{2}$*, then there exists a unique* p*,* $1 \leq p \leq n$*, such that*

$$(p-1)(2n-p+4)/2 < k \leq p(2n-p+3)/2.$$

In this case

$$a(\lambda,\mu)=p(2k-(p-1)n-p)+2\binom{p}{3},$$

and one can take $\Gamma(\lambda,\mu)$ *to be equal to the MacMahon polytope* $\mathfrak{M}_{r(k),s(k)}$ *with* $r(k):=k-1-(p-1)(2n-p+4)/2$, *and* $s(k):=p(2n-p+3)/2-k$.

This Example gives some flavor how intricate the piecewise linear function $a(\lambda,\mu)$ *may be.*

Conjecture 4.11 *Let* λ *and* μ *be partitions of the same size. Then*

- *([19])* $a(\lambda,\mu)=a(\mu',\lambda')$,
- *([13])* $b(\lambda,\mu)=b(\mu',\lambda')$.

■

Definition 4.12 *([16]) Let* α *and* β *be partitions of the same size. Define Liskova polynomials* $L^{\mu}_{\alpha,\beta}(q)$ *through the decomposition of the internal product of Schur functions in terms of Hall-Littlewood polynomials*

$$s_\alpha * s_\beta(X)=\sum_\mu L^{\mu}_{\alpha,\beta}\ P_\mu(X;q).$$

Clearly, $L^{\mu}_{\alpha,\beta}\in\mathbb{N}[q]$, *and* $L_{\alpha,(|\alpha|)}(q)=K_{\alpha,\mu}(q)$, *so that the Liskova polynomials are natural generalization of Kostka–Foulkes polynomials.*

Problem 4.13 *Find for Liskova polynomials an analogue of a fermionic formula for Kostka–Foulkes polynomials stated, for example, in [9], [13].*

5 Appendix. Rigged Configurations: a brief review

Let λ be a partition and $R=((\mu_a^{\eta_a}))_{a=1}^{p}$ be a sequence of rectangular shape partitions such that

$$|\lambda|=\sum_a |R_a|=\sum_a \mu_a\eta_a.$$

Definition 5.1

The configuration of type (λ,R) *is a sequence of* **partitions** $\{\nu\}=(\nu^{(1)},\nu^{(2)},\ldots)$ *such that*

$$|\nu^{(k)}|=\sum_{j>k}\lambda_j-\sum_{a\geq 1}\mu_a\max(\eta_a-k,0)=-\sum_{j\leq k}\lambda_j+\sum_{a\geq 1}\mu_a\min(k,\eta_a)$$

for each $k\geq 1$.

Note that if $k\geq l(\lambda)$ and $k\geq\eta_a$ for all a, then $\nu^{(k)}$ is empty.

In the sequel we make the convention that $\nu^{(0)}$ is the empty partition [14].
For a partition μ and an integer $j \geq 1$ define the number

$$Q_j(\mu) = \mu'_1 + \cdots + \mu'_j,$$

which is equal to the number of cells in the first j columns of μ.

The *vacancy* numbers $P_j^{(k)}(\nu) := P_j^{(k)}(\nu; R)$ of the configuration $\{\nu\}$ of type (λ, R) are defined by

$$P_j^{(k)}(\nu) = Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) + \sum_{a \geq 1} \min(\mu_a, j)\delta_{\eta_a,k}$$

for $k, j \geq 1$, where $\delta_{a,b}$ is the Kronecker delta.

Definition 5.2 *The configuration $\{\nu\}$ of type (λ, R) is called* **admissible***, if*

$$P_j^{(k)}(\nu; R) \geq 0 \quad \text{for all} \quad k, j \geq 1.$$

We denote by $C(\lambda; R)$ the set of all admissible configurations of type (λ, R), and call the vacancy number $P_j^{(k)}(\nu, R)$ *essential*, if $m_j(\nu^{(k)}) > 0$.

Finally, for configuration $\{\nu\}$ of type (λ, R) let us define its **charge**

$$c(\nu) = \sum_{k,j \geq 1} \binom{\alpha_j^{(k-1)} - \alpha_j^{(k)} + \sum_a \theta(\eta_a - k)\theta(\mu_a - j)}{2},$$

and cocharge

$$\overline{c}(\nu) = \sum_{k,j \geq 1} \binom{\alpha_j^{(k-1)} - \alpha_j^{(k)}}{2},$$

where $\alpha_j^{(k)} = (\nu^{(k)})'_j$ denotes the size of the j–th column of the k–th partition $\nu^{(k)}$ of the configuration $\{\nu\}$; for any real number $x \in \mathbb{R}$ we put $\theta(x) = 1$, if $x \geq 0$, and $\theta(x) = 0$, if $x < 0$.

Theorem 5.3 (**Fermionic formula for parabolic Kostka polynomials** [13])
Let λ be a partition and R be a dominant sequence of rectangular shape partitions. Then

$$K_{\lambda R}(q) = \sum_{\nu} q^{c(\nu)} \prod_{k,j \geq 1} \begin{bmatrix} P_j^{(k)}(\nu; R) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{bmatrix}_q, \tag{5.10}$$

summed over all admissible configurations ν of type $(\lambda; R)$; $m_j(\lambda)$ denotes the number of parts of the partition λ of size j.

---

[14] However, in some cases it is more convenient to set $\nu^{(0)} = (\mu_{i_1}, \ldots, \mu_{i_s})$, where we assume that $\eta_{i_a} = 1,\ a = 1, \ldots, s$. We will give an indication of such choice if it is necessary.

Corollary 5.4 (**Fermionic formula for Kostka–Foulkes polynomials [9]**)
Let λ and μ be partitions of the same size. Then

$$K_{\lambda\mu}(q)=\sum_{\nu} q^{c(\nu)} \prod_{k,j\ge 1} \left[\begin{array}{c} P_j^{(k)}(\nu,\mu)+m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{array} \right]_q , \tag{5.11}$$

summed over all sequences of partitions $\nu=\{\nu^{(1)},\nu^{(2)},\ldots\}$ such that

- $|\nu^{(k)}|=\sum_{j>k}\lambda_j,\ k=1,2,\ldots;$
- $P_j^{(k)}(\nu,\mu):=Q_j(\nu^{(k-1)})-2Q_j(\nu^{(k)})+Q_j(\nu^{(k+1)})\ge 0$ *for all* $k,j\ge 1$, *where by definition we put* $\nu^{(0)}=\mu$;
- $$c(\nu)=\sum_{k,j\ge 1}\binom{(\nu^{(k-1)})'_j-(\nu^{(k)})'_j}{2}. \tag{5.12}$$

It is frequently convenient to represent an admissible configuration $\{\nu\}$ by a matrix $m(\nu)=(m_{ij}),\ m_{ij}\in\mathbb{Z},\forall i,j\ge 1$, which must meets certain conditions. Namely, starting from the collection of partitions $\{\nu\}=(\nu^{(1)},\nu^{(2)},\ldots,\ldots)$ corresponding to configuration $\{\nu\}$, define matrix

$$m(\nu):=(m_{ij}),\quad m_{ij}=(\nu^{(i-1)})'_j-(\nu^{(i)})'_j+\sum_{a\ge 1}\theta(\eta_a-i)\theta(\mu_a-j),\quad \nu^{(0)}:=\emptyset,$$

where we set by definition $\theta(x)=1,\ if\ x\in\mathbb{R}_{\ge 0}$ and $\theta(x)=0,\ x\in\mathbb{R}_{<0}$.
One can check that a configuration $\{\nu\}$ of type (λ,R) is **admissible** if and only if the matrix $m(\nu)$ meets the following conditions

(0) $m_{ij}\in\mathbb{Z}$,
(1) $\sum_{i\ge 1} m_{ij}=\sum_{a\ge 1}\eta_a\theta(\mu_a-j)$,
(2) $\sum_{j\ge 1} m_{ij}=\lambda_i$,
(3) $\sum_{j\le k}(m_{ij}-m_{i+1,j})\ge 0$, for all i,j,k
(4) $\sum_{a\ge 1}\min(\eta_a,k)\delta_{\mu_a,j}\ge\sum_{i\le k}(m_{ij}-m_{i,j+1})$, for all i,j,k.

One can check that if matrix (m_{ij}) satisfies the conditions $(0)-(4)$, then the set of partitions $\{\nu\}=(\nu^{(1)},\nu_{(2)},\ldots,\ldots)$

$$(\nu^{(k)})'_j:=\sum_{i>k}m_{ij}-\sum_a\max(\eta_a-k,0)\theta(\mu_a-j)$$

defines an admissible configuration of type $(\lambda,R=\{(\mu_a)^{\eta_a}\})$.

Example 5.5 *Take* $\lambda=(44332),\ \ R=\{(2^3),(2^2),(2^2),(1),(1)\}$, *so that*

$$\{\mu_a\}=(2,2,2,1,1)\ \ and\ \ \{\eta_a\}=(3,2,2,1,1),\quad a=1,\ldots,5.$$

. *Therefore* $|\nu^{(1)}|=4,\ |\nu^{(2)}|=6,\ |\nu^{(3)}|=5$, *and* $|\nu^{(4)}|=2$. *It is not hard to check that there exist* 6 *admissible configurations. They are:*

(1) $\{\nu^{(1)} = (3,1), \nu^{(2)} = (3,3), \nu^{(3)} = (3,2), \nu_{(4)} = (2)\}$,
(2) $\{\nu^{(1)} = (3,1), \nu^{(2)} = (3,2,1), \nu^{(3)} = (3,2), \nu^{(4)} = (2)\}$,
(3) $\{\nu^{(1)} = (2,2), \nu^{(2)} = (2,2,2), \nu^{(3)} = (3,2), \nu^{(4)} = (2)\}$,
(4) $\{\nu^{(1)} = (4),\ \nu^{(2)} = (3,3), \nu^{(3)} = (3,2), \nu^{(4)} = (2)\}$,
(5) $\{\nu^{(1)} = (3,1), \nu^{(2)} = (2,2,1,1), \nu^{(3)} = (2,2,1), \nu^{(4)} = (2)\}$,
(6) $\{\nu^{(1)} = (3,1), \nu^{(2)} = (2,2,1,1), \nu^{(3)} = (3,1,1), \nu^{(4)} = (2)\}$,
Let us compute the matrix (m_{ij}) *corresponding to the configuration* (2). *Clearly,* $(m_{ij}) = ((\nu^{(i-1)})'_j - (\nu^{(i)})'_j) + (\sum_{a\geq 1} \theta(\eta_a - i)\theta(\mu_a - j)) := U + W$. *One can check that*

$$U = \begin{pmatrix} -3 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 3 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Therefore,

$$m(\{\nu\}) = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ 3 & 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

One can read off directly from the matrix m_{ij} the all additional quantities need to compute the parabolic Kostka polynomial corresponding to λ and a (dominant) sequence of rectangular shape partitions R. Namely,

$$P_j^{(k)} = \sum_{i\geq j}(m_{ki} - m_{k+1,i}),\ \ m_j(\nu^{(k)}) = \sum_{a\geq 1}\min(\eta_a, k)\delta_{\mu_a,j} - \sum_{i\leq k}(m_{ij} - m_{i,j+1}), c(\nu) = \sum_{i,j\geq 1}\binom{m_{ij}}{2}.$$

For example, in our example, we have $c(\nu) = 8$, $P_1^{(1)} = 1, P_2^{(2)} = 1, P_3^{(2)} = 1, P_2^{(3)} = 1$ are all non-zero vacancy numbers, and the contribution of the configuration in question to the parabolic Kostka polynomial is equal to $q^8 \begin{bmatrix} 2 \\ 1 \end{bmatrix}^4$. Treating in a similar fashion other configurations, we come to a fermionic formula

$$K_{44332,\{(2^3),(2^2),(2^2),(1),(1)\}}(q) = q^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + q^8 \begin{bmatrix} 2 \\ 1 \end{bmatrix}^4 + q^8 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + q^{12} + q^6 \begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + q^8.$$

■

If $\eta_a = 1$, $\forall a$, then $\sum_{a\geq 1}\eta_a\theta(\mu_a - j) = \mu'_j$, and we set $\nu^{(0)} = \mu'$. In this case one can rewrite the conditions $(1) - (4)$ as follows
(1′) $\sum_{i\geq 1} m_{ij} = \mu'_j$,
(2′) $\sum_{j\geq 1} m_{ij} = \lambda_i$,
(3′) $\sum_{j\leq k}(m_{ij} - m_{i+1,j}) \geq 0$, for all i, j, k
(4′) $\sum_{i>k}(m_{ij} - m_{i,j+1}) \geq 0$, for all $i, j.k$

Let us remark that if $m_{ij} \in \mathbb{Z}_{\geq 0}$ then the matrix (m_{ij}) defines a lattice plane partition of shape λ. For example, take $\lambda = (6,4,2,2,1,1)$, $\mu = (2^8)$ and admissible configuration $\{\nu\} = \{(5,5),(4,2),(3,1),(2),(1)\}$. The corresponding matrix and lattice plane partition of shape λ are

$$(m_{ij}) = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \text{and plane partition} \quad \begin{matrix} 3 & 3 \\ 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & \\ 1 & \end{matrix}.$$

The corresponding lattice word is 111222.1222.12.12.1.1.

In the case $\eta_a = 1, \ \forall a$, there exists a unique admissible configuration of type (λ, μ) denoted by $\Delta(\lambda,\mu)$ such that $\max(c((\Delta(\lambda,\mu),J))) = n(\mu) - n(\lambda)$, where the maximum is taken over all rigged configurations associated with configuration $\Delta(\lambda,\mu)$. Recall that for any partition λ,

$$n(\lambda) = \sum_{j \geq 1} \binom{\lambda_a^{'}}{2},$$

and if $\lambda \geq \mu$ with respect to the *dominance order*, then the degree of Kostka polynomial $K_{\lambda,\mu}(q)$ is equal to $n(\mu) - n(\lambda)$, see, e.g. [19], Chapter 1, for details. Namely, the configuration $\Delta(\lambda,\mu)$ corresponds to the following matrix:

$$m_{1j} = \mu_j^{'} - \max(\lambda_j^{'} - 1, 0), j \geq 1, \quad m_{ij} = 1, \ \ if \ \ (i,j) \in \lambda, \ i \geq 2, \ \ m_{ij} = 0, \ if \ (i,j) \notin \lambda.$$

In other words, the configuration $\Delta(\lambda,\mu)$ consists of the following partitions $(\lambda[1],\lambda[2],\ldots)$, where $\lambda[k] = (\lambda_{k+1}, \lambda_{k+2}, \ldots)$. It is not difficult to see that the contribution to the Kostka polynomial $K_{\lambda,\mu}(q)$ coming from the maximal configuration, is equal to

$$K_q(\Delta(\lambda,\mu)) := q^{c(\Delta(\lambda,\mu))} \prod_{j=1}^{\lambda_2} \begin{bmatrix} Q_j(\mu) - Q_j(\lambda) + \lambda_j^{'} - \lambda_{j+1}^{'} \\ \lambda_j^{'} - \lambda_{j+1}^{'} \end{bmatrix}_q,$$

where $c(\Delta(\lambda,\mu)) = n(\lambda) + n(\mu) - \sum_{j \geq 1} \mu_J^{'}(\lambda_j^{'} - 1)$. Therefore,

$$K_{\lambda,\mu}(q) \geq K_q(\Delta(\lambda,\mu)). \tag{5.13}$$

It is clearly seen that if $\lambda \geq \mu$, then $Q_j(\mu) \geq Q_j(\lambda), \ \forall \ j \geq 1$, and thus, $K_{q=1}(\Delta(\lambda,\mu)) \geq 1$, and the inequality (5.13) can be considered as a "quantitative" generalization of the Gale–Ryser theorem, see, e.g. [19], Chapter I, Section 7, or [11] for details.

Now let us **stress** that for a fixed k, the all partitions $\nu^{(k)}$ which contribute to the set of admissible configurations of type (λ,μ) have the same size equals to $\sum_{j \geq k+1} \lambda_j$, and thus the size of each $\nu^{(k)}$ doesn't depend on μ. However the Rigged Configuration bijection

$$RC_{\lambda,\mu} : STY(\lambda,\mu) \longrightarrow RC(\lambda,\mu)$$

happens to be essentially depends on μ. One can check that the map $RC_{\lambda,\mu}$ is compatible with the familiar *Bender–Knuth* transformations on the set of semistandard Young tableaux of a fixed shape.

As it was mentioned above, for a fixed k the all (admissible) configurations have the same size. Therefore, the set of admissible configurations admits a partial ordering denoted by "$\succcurlyeq$" . Namely, if $\{\nu\}$ and $\{\xi\}$ are two admissible configurations of the same type (λ,μ), we will write $\{\nu\} \succcurlyeq \{\xi\}$, if either $\{\nu\}=\{\xi\}$ or there exists an integer ℓ such that $\nu^{(a)}=\xi^{(a)}$ if $1\le a\le \ell$, and $\nu^{(\ell+1)}>\xi^{(\ell+1)}$ with respect of the dominance order on the set of the same size partitions. It seems an interesting **Problem** to study poset structures on the set of admissible configurations of type (λ,μ), especially to investigate the posets of admissible configurations associated with the multidimensional Catalan numbers, (work in progress).

■

Theorem 5.6 *(***Duality theorem for parabolic Kostka polynomials** *[13])*

Let λ be partition and $R=\{(\mu_a^{\eta_a})\}$ be a dominant sequence of rectangular shape partitions. Denote by $\lambda^{'}$ the conjugate of λ, and by $R^{'}$ a dominant rearrangement of a sequence of rectangular shape partitions $\{(\eta_a^{\mu_a})\}$. Then

$$K_{\lambda,R}(q)=q^{n(R)}K_{\lambda^{'},R^{'}}(q^{-1}),$$

where

$$n(R)=\sum_{a<b}\min(\mu_a,\mu_b)\min(\eta_a,\eta_b).$$

A technical proof is based on checking of the statement that the map

$$\iota: m_{ij}\longrightarrow \hat{m}_{ij}=-m_{ji}+\theta(\lambda_j-i)+\sum_{a\ge 1}\theta(\mu_a-j)\theta(\eta_a-i)$$

establishes bijection between the sets of admissible configurations of types (λ,R) and $(\lambda^{'},R^{'})$, and $\iota(c(m_{ij}))=c((\hat{m}_{ij}))$.

5.1 Example

Let $n=6$, consider for example, a standard Young tableau

$$T=\begin{matrix}1&2&3&6&8&9\\4&5&7&10&11&12\end{matrix},\quad c(T)=48.$$

The corresponding rigged configuration (ν,J) is

$$\nu=(321),\quad J=(J_3=0,\ J_2=2,J_1=6),\quad (m_{ij})(\nu)=\begin{pmatrix}9&-2&-1\\3&2&1\end{pmatrix},\quad c(\nu)=44.$$

Recall that $c(T)$ and $c(\nu)$ denote the charge of tableau T and configuration ν correspondingly.

- One can see that $c(T)=c(\nu)+J_3+J_2+J_1$, as it should be in general.

• Now, the descent set and descent number of tableau T are $Des(T) = \{3, 6, 9\}$, $des(T) = 3$. One can see that $des(T) = 3 = \nu_1^{'}$, as it should be in general [15].

• One can check that our tableau T is invariant under the action of the Schützenberger involution [16] on the set of standard Young tableaux of a shape λ. It is clearly seen from the set of riggings J [17] that the rigged configuration (ν, J) corresponding to tableau T, is invariant under the Flip involution [18] on the set of rigged configurations of type $(\lambda, 1^{|\lambda|})$, as it should be in general, see [15] for a complete proof of the statement that the action of the Schützenberger transformation on a Littlewood– Richardson tableau $T \in LR(\lambda, R)$, under the Rigged Configuration Bijection transforms tableau T to a Littlewood–Richardson tableau corresponding to the rigged configuration $\nu\kappa(J))$, where (νJ) is the rigged configuration corresponding to tableau T we are started with.

References

[1] P. Alexandersson, *Gelfand–Tsetlin polytopes and the integer decomposition property*, $arXive : 1405.4718v.3(2015)$.

[2] Brylinski R.K., *Matrix concomitants with mixed tensor model*, Adv. in Math. **100** (1993), 28–52.

[3] C. Chindris, H. Derksen and J. Weyman, *Counterexamples to Okounkov's Log-Concavity Conjecture*, Compos. Math. **143** (2007) , 1545–1557.

[4] J. A. De Loera and T.B. McAllister, *Vertices of Gelfand–Tsetlin Polytopes*, Discrete Comput. Geometry **32** (2004), 459–470.

[5] Fürlinger J. and Hofbauer J., *q–Catalan numbers*, Journ. of Comb. Theory A **40** (1985), 248-264.

[6] Hibi T., *Algebraic combinatorics on convex polytope*, Carslaw Publications, 1992.

[15] In fact the shape of the first configuration $\nu^{(1)}$ of type (λ, μ) can be read off from the set of "secondary" descent sets $\{Des^{(1)}(T) = Des(T), Des^{(2)}(T), \ldots, ...)$, cf [10].

[16] *http://en.wikipedia.org/wiki/Jeu_de_taquin*

[17] In our example $J = (0, 1, 3)$.

[18] Recall that a rigging of an admissible configuration ν is a collection of integers

$$J = (\{J_{s,r}^{(k)}\} \ , \ 1 \le s \le m_r(\nu^{(k)})$$

such that for a given k, r one has

$$0 \le J_{1,r}^{(k)} \le J_{2,r}^{(k)} \le \cdots \le J_{m_r(\nu^{(k)}),r}^{(k)} \le P_r^{(k)}(\nu).$$

The Flip involution κ is defined as follows:

$$\kappa(\nu, \{J_{s,r}^{(k)}\}) = (\nu, \{J_{m_r(\nu^{(k)})-s+1,r}^{(k)}\}).$$

[7] R. C. King, C. Tollu, and F. Toumazet, *Stretched Littlewood– Richardson and Kostka coeffients*, CRM Proceedings and Lecture Notes, Vol. **34** (2004), 99–112, Amer. Math. Soc..

[8] A.N. Kirillov, *Combinatorial identities and completeness of states for the Heisenberg magnet*, (Russian), Zap. Nauch. Sem. LOMI **131** (1983), 88–105, translation in Journal of Soviet Math. **30** (1985), 2298–3310.

[9] A.N. Kirillov and N. Reshetikhin, *The Bethe ansatz and the combinatorics of Young tableaux*, (Russian), Zap. Nauch. Sem. LOMI **155** (1986), 65-115, translation in Journal of Soviet Math. **41** (1988), 925-955.

[10] A.N. Kirillov, *On some properties of the Robinson– Schensted correspondence*, Proceedings of Hayashibara conference on special functions, 1990, 122–126.

[11] A.N. Kirillov, *Generalization of the Gale-Ryser theorem*, European Journal of Combinatorics, **21** (2000), 1047–1055; $hep-th/9304099$.

[12] (a) A.N. Kirillov, *Decomposition of symmetric and exterior powers of the adjoint representation of* $\mathfrak{gl}(N)$*. 1. Unimodality of principal specialization of the internal product of the Schur functions*, Jour. Mod. Phys. A **7** (1992), 545–579.

(b) A.N.Kirillov, *Unimodality of generalized Gaussian coefficients*, C.R. Acad. Sci. Paris, Ser, I, **315** (1992), no, 5, 497–501.

[13] A. N. Kirillov, *Combinatorics of Young tableaux and rigged configurations*, (Russian), Proceedings of the St.Petersburg Math. Soc. , **7** (1999), 23-115; translation in Proceedings of the St.Petersburg Math. Soc. volume VII, Amer. Math. Soc. Transl. Ser.2, **20** (2001), 17–98, AMS, Providence, RI.

[14] A.N. Kirillov, *Bijectives correspondances for rigged configurations*, Algebra i Analiz **12** (2000), 204–240, translation in St.Petersburg Math. J. **12** (2001), no.1, 161–190

[15] A.N. Kirillov, A. Schilling and M. Schimozono, A bijection between Littlewood-Richardson tableaux and rigged configurations, Selecta Mathematica **8** (2002), 67–135.

[16] A.N. Kirillov, *Ubiquity of Kostka polynomials,* Physics and Combinatorics 1999 (Proceedings of the Nagoya 1999 International Workshop on Physics and Combinatorics, Nagoya University, August 23-27, 1999, ed. A.Kirillov, A.Tsuchiya and H.Umemura), 85-200, World Scientific, Singapore, 2001.

[17] A.N. Kirillov, *An invitation to the generalized saturation conjecture*, Publications of RIMS Kyoto University **40** (2004), 1147–1239.

[18] A.N. Kirillov, *Kostka, and higher dimensional Catalan and Narayana numbers*, preprint (2001).

[19] Macdonald I.G., *Symmetric functions and Hall polynomials,* 2nd ed., Oxford, 1995.

[20] A. I. Molev, *Littlewood-Richardson polynomials*, J. Algebra **321** (2009), 3450–3468.

[21] P. A. MacMahon, *Combinatorial Analysis*, (1915-1916) ; reprint Chelsea Pub. Co., (1960).

[22] M. Mulase, *The Laplace Transform, Mirror Symmetry, and the Topological Recursion of Eynard–Orantin*, Geometric Methods in Physics. XXXI Workshop 2012, Trends in Mathematics, 127–142, 2013. Springer, Basel.

[23] F. Müller-Hoissen, J.-M.Pallo and J. Stasheff (Eds.), *Associahedra, Tamari Lattices and Related Structures*, Tamari Memorial Festschrift. Progress in Mathematics **299** (2012), Birkhäuser, Boston, MA.

[24] K. O' Hara, *Unimodality of Gaussian coefficients: a constructive proof*, Jour. Comb. Theory A, **53** (1990), 29–52.

[25] A. Okounkov, *Why would multiplicites be log-concave?*, *in* The orbit method in geometry and physics (Marseille, 2000), Progress in Mathematics **213** (2003), 329–347. Birkhäuser, Boston, MA.

[26] E. Rassart, *A polynomiality property for Littlewood– Richardson coefficients*, Journal of Combinatorial Theory, Series A, **107** (2004), 161–179.

[27] R. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry,* Annals of the New York Academy of Sciences **576** (1989), 500-535.

[28] R. Stanley , *Enumerative Combinatorics*, vols. 1 and 2, Cambridge University Press, 1999.

[29] R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

[30] R. Sulanke, *Generalizing Narayana and Schröder numbers to higher dimensions,* Electr. Journ. Comb. **11** (2004), $\#R54$.

[31] R. Sulanke, *Three dimensional Narayana and Schröder numbers,* Theor. Comp. Sci. **346(2-3)** (2005), 455-468.