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Rigged Configurations and Catalan, Stretched Parabolic Kostka Numbers and Polynomials : Polynomiality, Unimodality and Log-concavity

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Abstract

We will look at the Catalan numbers from the *Rigged Configurations* point of view originated [9] from an combinatorial analysis of the Bethe Ansatz Equations associated with the higher spin anisotropic Heisenberg models . Our strategy is to take a combinatorial interpretation of Catalan numbers C_n as the number of standard Young tableaux of rectangular shape (n^2), or equivalently, as the Kostka number $K_{(n^2),1^{2n}}$, as the starting point of research. We observe that the rectangular (or multidimensional) Catalan numbers $C(m,n)$ introduced and studied by P. MacMahon [21], [30], see also [31], can be identified with the Kostka number $K_{(n^m),1^{mn}}$, and therefore can be treated by Rigged Configurations technique. Based on this technique we study the stretched Kostka numbers and polynomials, and give a proof of " a strong rationality " of the stretched Kostka polynomials. This result implies a polynomiality property of the stretched Kostka and stretched Littlewood–Richardson coefficients [7], [26], [16].

Another application of the Rigged Configuration technique presented, is a new family of counterexamples to Okounkov's log-concavity conjecture [25].

Finally, we apply Rigged Configurations technique to give a combinatorial prove of the unimodality of the principal specialization of the internal product of Schur functions. In fact we prove a combinatorial formula for generalized q-Gaussian polynomials which is a far generalization of the so-called KOH-identity [24], as well as it manifests the unimodality property of the q-Gaussian polynomials.

2000 Mathematics Subject Classifications: 05E05, 05E10, 05A19.

1 Introduction

The literature devoted to the study of Catalan [1] and Narayana numbers [2] , their different combinatorial interpretations (more than 200 in fact, [29]), numerous generalizations, applications to Combinatorics, Algebraic Geometry, Probability Theory and so on and so forth, are enormous, see [29] and the literature quoted therein. There exists a wide variety of different generalizations of Catalan numbers, such as the Fuss–Catalan numbers [3] and the Schröder numbers [4] , higher genus multivariable Catalan numbers [22], higher dimensional Catalan and Narayana numbers [21], [30], and many and varied other generalizations. Each a such generalization comes from a generalization of a certain combinatorial interpretation of Catalan numbers, taken as a starting point for investigation. One a such interpretation of Catalan numbers has been taken as the starting point of the present paper, is the well-known fact that the Catalan number C_n is equal to the number of *standard* Young tableaux of the shape (n^2).

Let us look at the Catalan numbers from *Rigged Configurations* side. Since $C_n = K_{(n^2),1^{2n}}$ we can apply a *fermionic formula* for Kostka numbers [8], and come to the following combinatorial expressions for Catalan and Narayana numbers

$$C_n = \sum_{\nu \vdash n} \prod_{j \geq 1} \binom{2n - 2(\sum_{a \leq j} \nu_j) + \nu_j - \nu_{j+1}}{\nu_j - \nu_{j+1}},$$

where the sum runs over all partitions ν of size n;

$$N(n,k) = \sum_{\substack{\nu \vdash n \\ \nu_1 = k}} \prod_{j \geq 1} \binom{2n - 2(\sum_{a \leq j} \nu_j) + \nu_j - \nu_{j+1}}{\nu_j - \nu_{j+1}},$$

where the sum runs over all partitions ν of size n, $\nu_1 = k$.
A q-versions of these formulas one can find, for example, in [12].
Let us illustrate our fermionic formulas for $n = 6$. There are 11 partitions of size 6. We display below the distribution of contributions to the fermionic formulae for the Catalan and Narayana numbers presented above, which come from partitions ν of size 6 and $k = 1, 2, \ldots, 6$.
$N(1,6) = 1$,
$N2,6 = \binom{9}{1} + \binom{5}{4} + 1 = 9 + 5 + 1 = 15$,
$N(3,6) = \binom{8}{6} + \binom{3}{1}\binom{7}{1} + 1 = 28 + 21 + 1 = 50$,
$N(4,6) = \binom{7}{4} + \binom{6}{4} = 35 + 15 = 50$,
$N(5,6) = \binom{6}{4} = 15$,
$N(6,6) = 1$.

A few comments in order.

[1] *en.wikipedia.org/wiki/Catalan_number*
[2] *en.wikipedia.org/wiki/Narayana_number*
[3] *en.wikipedia.org/wiki/Fuss − Catalan_number*
[4] wolfram.com/SchröederNumber.html

• q-versions of formulas for Catalan and Narayana numbers displayed above coincide with the Carlitz–Riordan q-analog of Catalan numbers [28] and q-analog of Narayana numbers correspondingly.

• It is well-known that partitions of n with respect to the dominance ordering, form a lattice denoted by L_n. One (A.K) can define an ordering [5] on the set of admissible configurations of type (λ, μ) as well. In the case $\lambda = (n^2)$, $\mu = (1^{2n})$ the poset of admissible configurations of type (λ, μ) is essentially the same as the lattice of partitions L_n. Therefore, to each vertex ν of the lattice L_n one can attach the space of rigged configurations $RC_{\lambda,\mu}(\nu)$ associated with partition ν. Under a certain evolution a configuration (ν, J) evolves and touches the boundary of the set $RC_{\lambda,\mu}(\nu)$. When such is the case, "state" (ν, J) suffers "a phase transition" , executes the wall-crossing, and end up as a newborn state of some space $RC_{\lambda,\nu}(\nu')$. A precise description of this process is the essence of the Rigged Configuration Bijection [14], [15]. It seems an interesting task to write out in full the evolution process going on in the space of triangulations of a convex $(n+2)$-gon under the Rigged Configuration Bijection.

• It is an open **Problem** to count the number of admissible configurations associated with the multidimensional Catalan numbers $C(m, n)$, if $m \geq 3$, and a structure of the corresponding poset, and dynamics of a rigged configuration in a given set of configuration. If $m = 3$, the set of of admissible configurations consists of pairs of partitions $(\nu^{(1)}, \nu^{(2)})$ such that $\nu^{(2)} \vdash n$ and $\nu^{(1)} \geq \nu^{(2)} \vee \nu^{(2)}$ [6]. One can check that the number of admissible configurations of type $(n^3, 1^{3n})$ is equal to $1, 3, 6, 16, 33, 78$, for $n = 1, 2, 3, 4, 5, 6$.

• It is well-known that the q-Narayana numbers [7] obey the symmetry property, namely, $N(k, n) = N(n-k+1, n)$. Therefore it implies some non trivial relations among the products of q-binomial coefficients, combinatorial proofs of whose are desirable.

• It is well-known that the Narayana number $N(k, n)$ counts the number of Dyck paths of the semilength n with exactly k peaks. Therefore, the set of rigged configurations $\{\nu\}$ which associated with the Catalan number C_n and have fixed $\nu_1 = k$, is in one-to-one correspondence with the set of the semilength n Dyck paths with exactly k peaks.

Thus it looks natural to find and study combinatorial properties of the number of standard Young tableaux of an arbitrary rectangular shape (n^m),that is the Kostka number $K_{(n^m),1^{mn}}$, which are inherent in the classical Catalan and Narayana numbers. For example, one can expect that a multidimensional Catalan number is the sum of multidimensional Narayana ones (this is so !), or expect that a multidimensional Narayana polynomial is the h-vector of

[5] This ordering is inherited from an *evolution* of the maximal configuration $\Delta(\lambda, \mu)$ of type (λ, μ) under certain transformations on the set of admissible configurations $C(\lambda, \mu)$ coming from representation theory of the algebra $\mathfrak{gl}_n$. An evolution of the *maximal rigged configuration* $(\Delta(\lambda, \mu), J = \{P_j^{(k)}(\Delta(\lambda, \mu))\})$ induces a certain poset structure on the set of rigged configurations $R_{\lambda,\mu}$. We expect some connections between certain poset (lattice ?) structures arising on the set of Rigged Configurations of type $(n^2, 1^{2n})$ coming from an analysis of the Bethe Ansatz, and the known poset and lattice structures on a variety of sets counting by Catalan numbers, see e.g. [29], [23]. It is an interesting and important Problem to study poset structures on the set(s) counting by the multidimensional Catalan and Narayana numbers. Details will appear elsewhere.

[6] Recall that for any partitions λ and μ, $\lambda \vee \mu$ denotes partition corresponding to composition $(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots)$.

[7] Recall that the q- Narayana number $N(k, n \mid q) = \frac{1-q}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q$.

a certain convex lattice polytope, see e.g. [27] for the case of classical Catalan and Narayana numbers [8].

• Combinatorial analysis of the Bethe Ansatz Equations [9], gives rise to a natural interpretation of the Catalan and rectangular Catalan and Narayana numbers in terms of rigged configurations, and pose **Problem** to elaborate combinatorial structures induced by rigged configurations on any chosen combinatorial interpretation of Catalan numbers. For example, how to describe all triangulations of a convex $(n+2)$-gon which are in a "natural" bijection with the set of all rigged configurations (μ, J) corresponding to a given configuration ν of type $((n^2), 1^{2n})$? One can ask similar questions concerning Dyck paths and its multidimensional generalizations [31], and so on.

In Section 5.1 we present an example to illustrate some basic properties of the Rigged Configuration Bijection.

■

In the present paper we are interested in to investigate combinatorics related with the higher dimensional Catalan numbers, had been introduced and studied in depth by P. MacMahon [21]. It is highly possible that the starting point to introduce the higher dimensional Catalan numbers in [21] was an interpretation of classical Catalan numbers as the number of rectangular shape (n^2) standard Young tableaux mentioned above.

Our main objective in the present paper is to look on the multidimensional Catalan numbers $C(m,n) := C(m,n|1)$, defined as the value of the Kostka– Foulkes polynomials $K_{(n^m),(1^{mn})}(q)$ at $q=1$, from the point of view of Rigged Configurations Theory. In other words, we want to study the multidimensional Catalan and Narayana numbers introduced in [21], [30], by means of a fermionic formula for parabolic Kostka polynomials due to the author, e.g. [13], [16]. In particular, we apply the fermionic formula for parabolic Kostka polynomials cited above, to the study a stretched (parabolic) Kostka polynomials $K_{N\lambda,N\{\mathcal{R}\}}(q)$. At this way we obtain the following results.

Theorem 1.1 *(Polynomiality)*
Let λ be partition and $\{\mathcal{R}\}$ be a dominant sequence of rectangular shape partitions. Then

$$\sum_{N\geq 0} K_{N\lambda,N\mathcal{R}}(q)\ t^N = \frac{P_{\lambda,\mathcal{R}}(q,t)}{Q_{\lambda,\mathcal{R}}(q,t)},$$

were a polynomial $P_{\lambda,\mathcal{R}}(q,t)$ is such that $P_{\lambda,\mathcal{R}}(0,0)=1$;
a polynomial $Q_{\lambda,\mathcal{R}}(q,t)=\prod_{s\in S}(1-q^s t)$ for a some set of non-negative integers $S:=S(\lambda,\mathcal{R})$, depending on data λ and $\mathcal{R}$.

Corollary 1.2 *([7], [26], [16])*
Let λ be partition and $\{\mathcal{R}\}$ be a dominant sequence of rectangular shape partitions. Then

- *$\mathcal{K}_{\lambda,\mathcal{R}}(N) := K_{N\lambda,N\mathcal{R}}(1)$ is a polynomial of N with rational coefficients.*
- *(Littlewood–Richardson polynomials, [20],[26],[16])*

[8] The multidimensional Catalan and Narayana numbers, as well as the first *expectatio*, had been introduced and proved by P.MacMahon [21]. The second *expectation* will be treated in the present paper, Section 3.

Let λ, μ and ν be partitions such that $|\lambda| + |\mu| = |\nu|$.
The Littlewood–Richardson number $c^{\nu}_{\lambda,\mu}(N) := c^{N\nu}_{N\lambda,N\mu}$ is a polynomial of N with rational coefficients.

Problem 1.3 *Compute* [9] *the degree of polynomial $\mathcal{K}_{\lambda,,\mathcal{R}}(N)$.*

Our next objective is to define a lattice convex polytope $\mathcal{P}(n,m)$ which has the h-vector equals to the sequence of multidimensional Narayana numbers $\{N(m,n;k|1), \quad 1 \leq k \leq (m-1)(n-1)\}$.

As a preliminary step we recall the definition of a Gelfand -Tsetlin polytope.
Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be partition and $\mu = (\mu_1, \ldots, \mu_n)$ be composition, $|\lambda| = |\mu|$. The Gelfand– Tsetlin polytope of type (λ, μ), denoted by $GT(\lambda, \mu)$, is the convex hull of all points $(x_{ij})_{1 \leq i \leq j \leq n} \in \mathbb{R}_+^{\binom{n+1}{2}}$ which satisfy the following set of inequalities and equalities

$$x_{i,j+1} \geq x_{ij} \geq x_{i+1,j+1} \geq 0, \quad x_{1j} = \lambda_j, \quad 1 \leq j \leq n, \quad \sum_{a=1}^{j} x_{aj} = \sum_{a=1}^{j} \mu_a.$$

It is well-known that the number of integer points in the Gelfand–Tsetlin polytope $GT(\lambda, \mu)$, i.e. points $(x_{ij}) \in GT(\lambda, \mu)$ such that $x_{ij} \in \mathbb{Z}_{\geq 0}$, $\forall 1 \leq i \leq j \leq n$, is equal to the Kostka number $K_{\lambda,\mu}(1)$. Therefore the stretched Kostka number $K_{N\lambda,N\mu}(1)$ counts the number of integer points in the polytope $GT(N\lambda, N\mu) = N \cdot GT(\lambda, \mu)$. So As far as is we know, there is no general criterion to decide where or not the Gelfand–Tsetlin polytope $GT(\lambda, \mu)$ has only integral vertices, but see [4],[7], [1] for particular cases treated.

In the present paper we are interested in the h-vectors of Gelfand–Tsetlin polytopes $GT(n, 1^d)$ and that $GT((n^k, 1^{kd}), (1^k)^{n+d})$. We expect (cf [1]) that the polytope $GT(n, 1^d)$ is an integral one, but we don't know how to describe the set of parameters (n, k, d) such that the polytope $GT((n^k, (1^k)^{n+d})$ is an integral one.

Theorem 1.4
(1) *Let $\lambda := \lambda_{n,d} = (n, 1^d)$ and $\mu = \mu_{n,d} := (1^{n+d})$. Then*

$$\sum_{N \geq 0} K_{N\lambda,N\mu}(q) \; t^N = \frac{C_{d,n-1}(q^{\binom{n}{2}} \; t, q)}{(q^{\binom{n}{2}} \; t; q))_{d(n-1)+1}},$$

where $C_{d,m}(t,q) = \sum_{k=1}^{(d-1)(m-1)} N(d,m,k \mid q) \; t^{k-1}$ stands for a (q,t)–analogue of the rectangular (d,n)-Catalan number.
In particular, the normalized volume of the Gelfand–Tsetlin polytope $GT((n, 1^d), 1^{n+d})$ is equal to the d–dimensional Catalan number

$$C_{d,n}(1,1) := (dn)! \; \prod_{j=0}^{d-1} \frac{j!}{(n+j)!} = f^{(n^d)} = f^{(d^n)},$$

[9] It seems that the formulas for the degree of the stretched Kostka polynomials stated in [7], [26]. [16] are valid only for a special choice of λ, μ or $\mathcal{R}$.

(2) *Let* $\lambda := \lambda_{n,1,2} = (n^2, 1^2)$ *and* $\mu = ((1,1)^{n+1})$, $n \geq 2$. *Then*

$$\sum_{N \geq 0} K_{N(n,n,1,1),N(1,1)^{n+1}}(1)\ t^N = \frac{P_{2,n}(t)}{(1-t)^{4n-6}},$$

and $P_{2,n}(1) = C_{n-3}\ C_{n-2}$, *product of two Catalan numbers.*

(3) *Let* $\lambda := \lambda_{n,k,d} = (n^k, 1^{kd})$ *and* $\mu = ((1^k)^{n+d})$, $d \geq 1$. *Then*

$$\sum_{N \geq 0} K_{N(n^k,1^{kd}).N(1^k)^{n+d}}(1)\ t^N = \frac{P_{k,d,n}(t)}{Q_{k,d,n}(t)}.$$

Moreover, $P_{k,d,n}(0) = 1$,

$$Q_{k,d,n}(t) = (1-t)^{k^2(d(n-1)-1)+2+(k-1)\delta_{n,2}\ \delta_{d,1})},$$

and the polynomial $P_{k,d,n}(t)$ *is symmetric with respect to variable* t;

$$deg_t(P_{k,k,n}(t)) = (k-1)(k(n-2)+2(\delta_{n,2}-1)).$$

One can see from Theorem 1.4, (1), that the degree of the stretched Kostka polynomial $\mathcal{K}_{(n,1),1^{n+1}}(N) := K_{N(n,1),N(1^{n+1})}(1)$ is equal to $n-1$, whereas it follows from Theorem 1.4, (2) that

$$deg_N(\mathcal{K}_{2(n,1),2(1)^{n+1}}(N)) = 4n-7 > 3\ deg_N(\mathcal{K}_{(n,1),1^{n+1}}(N)), \quad if \quad N > 4.$$

Therefore one comes to an infinite family of counterexamples to Okounkov log- concavity conjecture for the Littlewood–Richardson coefficients [24].

Corollary 1.5

- *Let* $n \geq 3$. *There exists an integer* $N_0(n)$ *such that*

$$K_{2N(n,1),2N(1)^{n+1}}(1) \quad > \quad \left(K_{N(n,1),N(1)^{n+1}}(1)\right)^2 \quad for \quad all \quad N \geq N_0(n); \tag{1.1}$$

- *Let* $n \geq 5$ *be an integer, choose* $\epsilon,\ 0 \leq \epsilon < \frac{n-4}{n-1}$. *There is an integer* $N_0(n;\epsilon)$ *such that*

$$K_{2N(n,1),2N(1)^{n+1}}(1) \quad > \quad \left(K_{N(n,1),N(1)^{n+1}}(1)\right)^{3+\epsilon} \quad for \quad all \quad N \geq N_0(n;\epsilon).$$

(3) *Let* $n > 1 + \frac{k^2+2}{k^2 d}$. *There exist an integer* $N_0(n,k,d)$ *such that*

$$K_{2N(n^k,1^{kd}),2N(1^k)^{n+d}}(1) \quad > \quad \left(K_{N(n^k,1^{kd}),N(1^k)^{n+d}}(1)\right)^3 \quad for \quad all \quad N \geq N_0(n,k,d).$$

For example,

- $K_{2N(5,1),2N(1,1)^6}(1) > (K_{N(5,1),N(1^6)}(1))^3$

if and only if $N \geq 49916$.

Let us recall the well-known fact that any parabolic Kostka number $K_{\lambda,\mathcal{R}}(1)$ can be realized

as the Littlewood–Richardson coefficient $c^{\Lambda}_{\lambda,M}$ for uniquely defined partitions Λ and M, see Section 3.2 for details

It should be stressed that for $n = 3$ the example (1.1) has been discovered in [3], and independently by the author (unpublished notes [18]). In this case the minimal value of $N_0(3)$ is equal to 23; one can show (A.K.) that $N_0(4) = 8$.

Our next objective of the present paper is to prove the unimodality of the principal specialization $s_\alpha * s_\beta(q, \ldots, q^{N-1})$ of Schur functions [12],[13]. Proofs given in *loc. cit.* is based on an identification of the principal specialization of internal product of Schur functions with a certain parabolic Kostka polynomial.

Theorem 1.6 *(***Principal specialization of the internal product of Schur functions and parabolic Kostka polynomials***)*

Let α, β be partitions such that $|\alpha| = |\beta|$, $\alpha_1 \leq r$ and $\beta_1 \leq k$. For given integer N such that $\alpha_1 + \beta_1 \leq Nr$, consider partition

$$\lambda_N := (rN - \beta_k^{'}, rN - \beta_{k-1}^{'}, \ldots, rN - \beta_1^{'}, \alpha^{'})$$

and a sequence of rectangular shape partitions

$$R_N := (\underbrace{(r^k), \ldots, (r^k)}_{N}).$$

Then [10]

$$K_{\lambda_N, R_N}(q) \stackrel{\bullet}{=} s_\alpha * s_\beta(q, \ldots, q^{N-1}). \tag{1.2}$$

Now we state a fermionic formula for polynomials

$$V^N_{\alpha,\beta}(q) := s_\alpha * s_\beta(q, \ldots, q^{N-1}),$$

which is our main tool to give a combinatorial proof of the unimodality of the principal specialization of the Schur functions, and that of the generalized q-Gaussian polynomials $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q$ associated with a partition λ, as a special case.

Theorem 1.7 *Let α and β be two partitions of the same size, and $r := \ell(\alpha)$ be the length of α. Then*

$$s_\alpha * s_\beta(q, \ldots, q^{N-1}) = \sum_{\{\nu\}} q^{c(\{\nu\})} \prod_{k,j \geq 1} \begin{bmatrix} P_j^{(k)}(\nu) + m_j(\nu^{(k)}) + N(k-1)\delta_{j,\beta_1}\theta(r-k) \\ P_j^{(k)}(\nu) \end{bmatrix}_q, \tag{1.3}$$

where the sum runs over the set of admissible configurations $\{\nu\}$ of type $([\alpha, \beta]_N, (\beta_1)^N)$. Here for any partition λ, λ_j denotes its j-th component.

[10] Hereinafter we shall use the notation $A(q) \stackrel{\bullet}{=} B(q)$ to mean that the ratio $A(q)/B(q)$ is a certain power of q.

See Section 4, Theorem 4.2 for details concerning notation. An important property which is specific to admissible configurations of type $[\alpha,\beta]_N,\beta_1^n)$, is the following relations

$$2c(\nu)+\sum_{k,j\geq 1}P_j^{(k)}(\nu)\Big[m_j(\nu^{(k)})+N(k-1)\delta_{j,\beta_1}\theta(r-k)\Big]=N|\alpha|,$$

which imply the unimodality of polynomials $V^N_{\alpha,\beta}(q)$, and $\begin{bmatrix}N\\ \alpha\end{bmatrix}_q=V^N_{\alpha,(|\alpha|)}(q)$. Let us stress that the sum in the $RHS(1.3)$ runs over the set of admissible configurations of type $([\alpha,\beta]_N,(\beta_1)^N)$. Remark, that the $RHS(1.3)$ has a natural generalization to the case $|\alpha|\equiv|\beta|\ (mod\ N)$, but in this case a representation-theoretical meaning of the $LHS(1.3)$ is unclear to the author.

Acknowledgments

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2 Higher dimensional Catalan and Narayana numbers, [21], [16], [30]

2.1 Rectangular Catalan and Narayana polynomials, and MacMahon polytope, [16]

2.1.1 Rectangular Catalan and Narayana numbers and polynomials

Define **rectangular Catalan polynomial**

$$C(n,m|q)=\frac{(q;q)_{nm}}{\prod\limits_{i=1}^{n}\prod\limits_{j=1}^{m}(1-q^{i+j-1})}=[d\ n]_q!\prod_{j=0}^{d-1}\frac{[j]_q!}{[n+j]_q!},\tag{2.4}$$

where $[n]_q:=\frac{1-q^n}{1-q}$ stands for the q-analogue of an integer n, and by definition $[n]_q!:=\prod_{j=1}^n[j]_q$.

Proposition 2.1

$$q^{m\binom{n}{2}}C(n,m|q)=K_{(n^m),(1^{nm})}(q).\tag{2.5}$$

Thus, $C(n,m|q)$ is a polynomial of degree $nm(n-1)(m-1)/2$ in the variable q with non–negative integer coefficients. Moreover,

$$C(n,2|q)=C(2,n|q)=c_n(q)=\frac{1-q}{1-q^{n+1}}\begin{bmatrix}2n\\ n\end{bmatrix}_q$$

coincides with "the most obvious" q–analog of the Catalan numbers, see e.g. [5], p.255, or [28], and [21],

$$C(n,3|q) = \frac{[2]_q\ [3\ n]_q\ !}{[n]_q\ !\ [n+1]_q\ !\ [n+2]_q\ !}.$$

It follows from (2.5) that the rectangular Catalan number $C(n,m|1)$ counts the number of *lattice* words

$$w = a_1 a_2 \cdots a_{nm}$$

of weight (m^n), i.e. lattice words in which each i between 1 and m occurs exactly n times. Let us recall that a word $a_1 \cdots a_p$ in the symbols $1, \ldots, m$ is said to be a *lattice* word, if for $1 \leq r \leq p$ and $1 \leq j \leq m-1$, the number of occurrences of the symbol j in $a_1 \cdots a_r$ is not less than the number of occurrences of $j+1$:

$$\#\{i | 1 \leq i \leq r \ \text{ and } \ a_i = j\} \geq \#\{i | 1 \leq i \leq r \ \text{ and } \ a_i = j+1\}. \tag{2.6}$$

For any word $w = a_1 \cdots a_k$, in which each a_i is a positive integer, define the major index

$$\mathrm{maj}(w) = \sum_{i=1}^{k-1} i\chi(a_i > a_{i+1}),$$

and the number of descents

$$\mathrm{des}(w) = \sum_{i=1}^{k-1} \chi(a_i > a_{i+1}).$$

Finally, for any integer k between 0 and $(n-1)(m-1)$, define **rectangular q–Narayana** number

$$N(n,m;k \mid q) = \sum_w q^{\mathrm{maj}(w)},$$

where w ranges over all lattice words of weight (m^n) such that $\mathrm{des}(w) = k$.

Equivalently, $N(n,m;k)$ is equal to the number of rectangular standard Young tableaux with n rows and m columns having k descents, i.e. k occurrences of an integer j appearing in a lower row that that $j+1$.

Example 2.2 Take $n = 4$, $m = 3$, then

$$\sum_{k=0}^{6} N(3,4;k \mid 1)t^k = 1 + 22t + 113t^2 + 190t^3 + 113t^4 + 22t^5 + t^6.$$

We summarize the basic known properties of the rectangular Catalan and Narayana numbers in Proposition 2.3 below.

Proposition 2.3 *([21], [30], [13])*

(**A**) *(***Lattice words and rectangular Catalan numbers***)*

$C(n,m|q) = \sum_w q^{\mathrm{maj}(w)}$, *where w ranges over all lattice words of weight (m^n);*

(**B**) *(***Bosonic formula for multidimensional Narayana numbers***)*

$$N(n,m;k \mid q) = \sum_{a=0}^{k} (-1)^{k-a}\, q^{\binom{k-a}{2}} \begin{bmatrix} n & m+1 \\ k-a & \end{bmatrix}_q \prod_{b=0}^{n-1} \frac{[b]!\ [m+a+b]!}{[m+b]!\ [a+b]!}, \tag{2.7}$$

(**C**) *(***Summation formula***) Let r be a positive integer, then*

$$\sum_{k=0}^{r} \begin{bmatrix} n & m+r-k \\ r-k & \end{bmatrix}_q N(n,m;k \mid q) = \prod_{a=0}^{m-1} \frac{[a]!\ [n+r+a]!}{[n+a]!\ [r+a]!} =$$

$$= \prod_{a=0}^{n-1} \frac{[a]!\ [m+r+a]!}{[m+a]!\ [r+a]!} = \prod_{a=0}^{r-1} \frac{[a]!\ [n+m+a]!}{[n+a]!\ [m+a]!}.$$

(**D**) *(***Symmetry***)*

$$N(n,m;k \mid q) = q^{nm((n-1)(m-1)/2-k)} N(n,m;(n-1)(m-1)-k \mid q^{-1}) = N(m,n;k \mid q),$$

for any integer k, $0 \le k \le (n-1)(m-1)/2$;

(**E**) *(*q**-Narayana numbers***)*

$$N(2,n;k \mid q) = q^{k(k+1)} \frac{1-q}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \stackrel{\bullet}{=} \dim_q V_{(k,k)}^{\mathfrak{g}l(n-k+1)}, \quad 0 \le k \le n-1,$$

where $V_{(k,k)}^{\mathfrak{g}l(n-k+1)}$ stands for the irreducible representation of the Lie algebra $\mathfrak{g}l(n-k+1)$ corresponding to the two row partition (k,k); recall that for any finite dimensional $\mathfrak{g}l(N)$–module V the symbol $\dim_q V$ denotes its q–dimension, i.e. the principal specialization of the character of the module V:

$$\dim_q V = (\mathrm{ch}V)(1,q,\ldots,q^{N-1});$$

(**F**) $N(n,m;1 \mid 1) = \sum_{j \ge 2} \binom{n}{j} \binom{m}{j} = \binom{n+m}{n} - nm - 1$;

(**G**) *(***Fermionic formula for q–Narayana numbers***, [16])*

$$q^{m\binom{n}{2}} N(n,m;l \mid q) = \sum_{\{\nu\}} q^{c(\nu)} \prod_{k,j \ge 1} \begin{bmatrix} P_j^{(k)}(\nu) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{bmatrix}_q, \tag{2.8}$$

summed over all sequences of partitions $\{\nu\} = \{\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(m-1)}\}$ such that

- $|\nu^{(k)}| = (m-k)n$, $1 \le k \le m-1$;
- $(\nu^{(1)})'_1 = (m-1)n - l$, *i.e. the length of the first column of the diagram $\nu^{(1)}$ is equal to* $(m-1)n - l$, $l = 0, \ldots, (m-1)(n-1)$;

• $P_j^{(k)}(\nu) := Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) \geq 0$, *for all* $k, j \geq 1$,

where by definition we put $\nu^{(0)} = (1^{nm})$*; for any diagram* λ *the number* $Q_j(\lambda) = \lambda'_1 + \cdots \lambda'_j$ *is equal to the number of cells in the first* j *columns of the diagram* λ*, and* $m_j(\lambda)$ *is equal to the number of parts of* λ *of size* j*;*

• $c(\nu) = \sum_{k,j\geq 1} \left(\begin{array}{c} (\nu^{(k-1)})'_j - (\nu^{(k)})'_j \\ 2 \end{array} \right)$.

Example 2.4 *Consider the case* $m = 3$, $n = 4$. *In this case* $C(3, 4 \mid 1) = 462$, *and the sequences of Narayana numbers is* $(1, 22, 113, 190, 113, 22, 1)$. *Let us display below the distribution of Narayana numbers which is coming from the counting the number of admissible rigged configurations of type* $((4^3), (1^{12}))$ *according to the number* $(m-1)n - \ell(\nu^{(1)})$*, where* $\ell(\nu^{(1)})$ *denotes the length of the first configuration* $\nu^{(1)}$*:*

$N(3, 4; 0 \mid 1) = 1$, $N(3, 4; 1 \mid 1) = 1 + 21$, $N(3, 4; 2 \mid 1) = 15 + 35 + 63$,

$N(3, 4; 3 \mid 1) = 140 + 15 + 35$, $N(3, 4; 4 \mid 1) = 21 + 28 + 63$,

$N(3, 4; 5 \mid 1) = 6 + 16$, $N(3, 4; 6 \mid 1) = 1$.

Conjecture 2.5 *If* $1 \leq k \leq (n-1)(m-1)/2$, *then*

$$N(n, m; k-1 \mid 1) \leq N(n, m; k \mid 1),$$

i.e. the sequence of rectangular Narayana numbers $\{N(n, m; k \mid 1)\}_{k=0}^{(n-1)(m-1)}$ *is symmetric and* **unimodal**.

For definition of unimodal polynomials/sequences see e.g. [27], where one may find a big variety of examples of unimodal sequences which frequently appear in Algebra, Combinatorics and Geometry.

2.1.2 Volume of the MacMahon polytope and rectangular Catalan and Narayana numbers

Let $\mathfrak{M}_{mn}$ be the convex polytope in $\mathbb{R}^{nm}$ of all points $\mathbf{x} = (x_{ij})_{1\leq i\leq n, 1\leq j\leq m}$ satisfying the following conditions

$$0 \leq x_{ij} \leq 1, \quad x_{ij} \geq x_{i-1,j}, \quad x_{ij} \geq x_{i,j-1}, \tag{2.9}$$

for all pairs of integers (i, j) such that $1 \leq i \leq n$, $1 \leq j \leq m$, and where by definition we set $x_{i0} = 0 = x_{0j}$.

We will call the polytope $\mathfrak{M}_{nm}$ by *MacMahon polytope*. The MacMahon polytope is an integral polytope of dimension nm with $\left(\begin{array}{c} m+n \\ n \end{array} \right)$ vertices which correspond to the set of (0,1)–matrices satisfying (2.9).

If k is a positive integer, define $i(\mathfrak{M}_{nm}; k)$ to be the number of points $\mathbf{x} \in \mathfrak{M}_{nm}$ such that $k\mathbf{x} \in \mathbb{Z}^{nm}$. Thus, $i(\mathfrak{M}_{nm}; k)$ is equal to the number of plane partitions of rectangular shape (n^m) with all parts do not exceed k. By a theorem of MacMahon (see e.g. [19], Chapter I, §5, Example 13)

$$i(\mathfrak{M}_{nm}; k) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{k+i+j-1}{i+j-1}. \tag{2.10}$$

It follows from (2.10) that the Ehrhart polynomial $\mathcal{E}(\mathfrak{M}_{nm};t)$ of the MacMahon polytope $\mathfrak{M}_{nm}$ is completely resolved into linear factors:

$$\mathcal{E}(\mathfrak{M}_{nm};t)=\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{t+i+j-1}{i+j-1}.$$

Hence, the normalized volume

$$\widetilde{\mathrm{vol}}(\mathfrak{M}_{nm})=(nm)!\mathrm{vol}(\mathfrak{M}_{nm})$$

of the MacMahon polytope $\mathfrak{M}_{nm}$ is equal to the rectangular Catalan number $C(n,m|1)$, i.e. the number of standard Young tableaux of rectangular shape (n^m). We refer the reader to [28], Section 4.6, and [6], Chapter IX, for definition and basic properties of the Ehrhart polynomial $\mathcal{E}(\mathcal{P};t)$ of a convex integral polytope $\mathcal{P}$.

Proposition 2.6

$$\sum_{k\geq 0}i(\mathfrak{M}_{nm};k)z^k=\left(\sum_{j=0}^{(n-1)(m-1)}N(n,m;j)z^j\right)/(1-z)^{nm+1}, \tag{2.11}$$

where

$$N(n,m;j):=N(n,m;j|1)$$

denotes the rectangular Narayana number.

Thus, the sequence of Narayana numbers

$$(1=N(n,m;0),N(n,m;1),\ldots,N(n,m;(n-1)(m-1))=1)$$

is the *h–vector* (see e.g. [28], p. 235) of the MacMahon polytope. In the case $n=2$ (or $m=2$) all these results may be found in [28], Chapter 6, Exercise 6.31.

Question. (*Higher associahedron*) Does there exist an $(m-1)(n-1)$–dimensional integral convex (simplicial?) polytope $Q_{n,m}$ which has h–vector

$$h=(h_0(Q_{n,m}),h_1(Q_{n,m}),\ldots,h_{(n-1)(m-1)}(Q_{n,m}))$$

given by the *rectangular Narayana numbers* $N(n,m;k)$:

$$\sum_{i=0}^{(n-1)(m-1)}h_i(Q_{n,m})t^i=C(n,m|t)\ ?$$

We refer the reader to [6], Chapter I, §6 and Chapter III, for definitions and basic properties of the h–vector of a simplicial polytope; see also, R. Stanley (J. Pure and Appl. Algebra **71** (1991), 319-331).

An answer on this question is known if either n or m is equal to 2, see e.g. R. Simion (Adv. in Appl. Math. **18** (1997), 149-180, Example **4** (the Associahedron)).

Definition 2.7 *([21], [30]) Define* **rectangular** *Schröder polynomial*

$$S(n,m|t) := C(n,m|1+t),$$

and put

$$S(n,m|t) = \sum_{k\geq 0}^{(n-1)(m-1)} S(n,m||k)t^k.$$

A combinatorial interpretation of the numbers $S(n,m||k)$ and $S(n,m|1)$ has been done by R. Sulanke [30].

2.1.3 Rectangular Narayana and Catalan numbers, and d dimensional lattice paths, [30]

Let $\mathcal{C}(d,n)$ denote the set of d–dimensional lattice paths using the steps

$$X_1 = (1,0,\cdots,0),\ X_2 = (0,1,\cdots,0),\cdots,X_d = (0,0,\cdots,1),$$

running from $(0,0,\cdots,0)$ to $(n,n,\cdots,n)$, and lying in the region

$$\{(x_1,x_2,\cdots,x_d) \in \mathbb{R}^d_{\geq 0} \mid x_1 \leq x_2 \leq \cdots \leq x_d\}.$$

For each path $P := p_1p_2\cdots p_{nd} \in \mathcal{C}(d,n)$ define the statistics

$$asc(P) := \#\{j \mid p_jp_{j+1} = X_k\ X_l,\ k<l\}.$$

Definition 2.8 *The n–th d–dimensional MacMahon–Narayana number of level k, $MN(d,n,k)$ counts the paths $P \in \mathcal{C}(d,n)$ with $asc(P) = k$.*

Proposition 2.9 *(Cf [30]) For any $d \geq 2$ and for $0 \leq k \leq (d-1)(n-1)$,*

$$MN(d,n,k) = \sum_{j=0}^{k}(-1)^{k-j}\binom{dn+1}{k-j}\ \prod_{a=0}^{j-1}\frac{a!\ (d+n+a)!}{(d+a)!\ (n+a)!}.$$

Note that the product $\prod_{a=0}^{j-1}\frac{a!\ (d+n+a)!}{(d+a)!\ (n+a)!}$ is equal to the number of plane partitions of the rectangular shape (n^d), all the parts do not exceed j.

Definition 2.10 *For $d \geq 3$ and $n \geq 1$ the n-th d-Narayana polynomial defined to be*

$$N_{d,n}(t) = \sum_{k=0}^{(d-1)(n-1)} MN(d,n,k)\ t^k.$$

Corollary 2.11 *(Recurrence relations, [30]) For any integer $m \geq 0$ one has*

$$\sum_{k=0}^{m}\binom{dn+m-k}{m-k}\ MN(d,n,k) = \prod_{a=0}^{d-1}\frac{a!\ (n+m+a)!}{(n+a)!\ (m+a)!}.$$

Corollary 2.12 *The MacMahon–Narayana number $MN(d,n,k)$ is equal to the rectangular Narayana number $N(d,n;k)$.*

2.1.4 Gelfand–Tsetlin polytope $GT((n,1^d),(1)^{n+d})$ and rectangular Narayana numbers

Theorem 2.13 *Let $\lambda := \lambda_{n,d} = (n,1^d)$ and $\mu = \mu_{n,d} := (1^{n+d})$. Then*

$$\sum_{N\geq 0} K_{N\lambda,N\mu}(q)\ t^N = \frac{C_{d,n-1}(q^{\binom{n}{2}}\ t,q)}{(q^{\binom{n}{2}}\ t;q))_{d(n-1)+1}},$$

where $C_{d,m}(t,q) = \sum_{k=0}^{(d-1)(m-1)} N(d,m,k \mid q)\ t^k$ stands for a (q,t)–analog of the rectangular (d,n)-Catalan number.
In particular, the normalized volume of the Gelfand–Tsetlin polytope $GT((n,1^d),1^{n+d})$ is equal to the d–dimensional Catalan number

$$C_{d,n}(1,1) := (dn)!\ \prod_{j=0}^{d-1}\frac{j!}{(n+j)!} = f^{(n^d)} = f^{(d^n)},$$

where for any partition λ, f^λ denotes the number of standard Young tableaux of shape λ.

3 Rigged configurations, stretched Kostka numbers, log-concavity and unimodality

3.1 Stretched Kostka numbers $K_{N(n^k.1^{kd}),N(1^k)^{n+d}}(1)$

Theorem 3.1 (1)

$$\sum_{N\geq 0} K_{N(n,n,1,1)),N((1,1)^{n+1})}(1)\ t^N = \frac{P_{2,n}(t)}{(1-t)^{4n-6}},$$

and $P_{2,n}(1) = C_{n-3}\ C_{n-2}$.
(2) *Let $d \geq 1$, then*

$$\sum_{N\geq 0} K_{N(n^k,1^{kd}).N(1^k)^{n+d}}(1)\ t^N = \frac{P_{k,d,n}(t)}{Q_{k,d,n}(t)}.$$

Moreover, $P_{k,d,n}(0) = 1$,

$$Q_{k,d,n}(t) = (1-t)^{k^2(d(n-1)-1)+2+(k-1)\delta_{n,2}\ \delta_{d,1})},$$

and the polynomial $P_{k,d,n}(t)$ is symmetric with respect to variable t;

$$deg_t(P_{k,k,n}(t)) = (k-1)(k(n-2)+2(\delta_{n,2}-1)).$$

■

For example, assume that $d = 1$ and set $P_{k,n}(t) := P_{k,1,n}(t)$. Then
$P_{2,3}(t) = 1,\ P_{2,4}(t) = (1, 0, 1), P_{2,5}(t) = (1, 1, 6, 1, 1),\ \ P_{2,6}(t) = (1, 3, 21, 20, 21, 3, 1)$,
$P_{2,7}(t) = (1, 6, 56, 126, 210, 126, 56, 6, 1)$,
$P_{2,8}(t) = (1, 10, 125, 500, 1310, 1652, 1310, 500, 125, 10, 1),\ \ P_{3,3}(t) = (1, -1, 1)$,
$P_{3,4}(t) = (1, 0, 20, 20, 55, 20, 20, 0, 1), P_{3,5}(t) = (1, 6, 141, 931, 4816, 13916,$
$27531, 33391, 27531, 13916, 4816, 931, 141, 6, 1), P_{4,1,3}(t) = (1, -3, 9, -8, 9, -3, 1) = P_{4,2,2}(t)$.

It follows from the duality theorem for parabolic Kostka polynomials [16] that

$$K_{N\lambda,N\mu}(1) = K_{(Nn,N^d)',((N)^{n+d})'}(1) = K_{((d+1)^N,1^{N(n-1)}),(1^N)^{n+d})}(1),$$

and

$$K_{((2d+2)^N,2^{N(n-1)}),((2)^N)^{n+d})}(1) = K_{(Nn,Nn,N^{2d}),((N,N)^{n+d})}(1).$$

Now consider the case $d = 1$, that is $\lambda = (n, 1),\ \mu = (1^{n+1})$. Then

$$K_{N\lambda,N\mu}(1) = K_{(Nn,N),(N^{n+1})}(1) = \binom{N+n-1}{n-1}.$$

The second equality follows from a more general result [12], [16],

Proposition 3.2 *Let λ be a partition and N be a positive integer. Consider partitions $\lambda_N := (N|\lambda|, \lambda)$ and $\mu_N := (|\lambda|^{N+1}) = (\underbrace{|\lambda|, \dots, |\lambda|}_{N+1})$. Then*

$$K_{\lambda_N,\mu_N}(q) \doteq \begin{bmatrix} N \\ \lambda \end{bmatrix} = dim_q V_\lambda^{\mathfrak{gl}(N)},$$

where the symbol $P(q) \doteq R(q)$ means that the ratio $P(q)/R(q)$ is a power of q; the symbol $\begin{bmatrix} N \\ \lambda \end{bmatrix}$ stands for the generalized Gaussian coefficient corresponding to a partition λ, see [19] for example.

3.2 Counterexamples to Okounkov's log-concavity conjecture

On the other hand,

$$K_{2\lambda_N,2\mu_N}(1) = K_{N(n,n,1,1),N(1,1)^{n+1}}(1) = Coeff_{t^N}\left(\frac{P_{2,n}(t)}{(1-t)^{4n-6}}\right).$$

Therefore the number $K_{2\lambda_N,2\mu_N}(1)$ is a **polynomial** of the degree $4\ n - 7$ with respect to parameter N. Recall that the number $K_{N\lambda,N\mu}(1) = \binom{N+n-1}{n-1}$ is a polynomial of degree $n - 1$ with respect to parameter N. Therefore we come to the following infinite set of examples which violate the log-concavity Conjecture stated by A. Okounkov [25].

Corollary 3.3 *For any integer $n > 4$, there exists a constant $N_0(n)$ such that*

$$K_{2\lambda_N,2\mu_N}(1) > \Big(K_{N\lambda,N\mu}(1)\Big)^3,$$

for all $N > N_0(n)$.

Recall that $\lambda = (n,1),\ \mu = (1^{n+1})$.

Now take $n = 3$. One has [3]

$$K_{N(3,1),N(1^4)}(1) = \binom{N+2}{2}, \quad K_{N(3,3,1,1),N(1,1)^4}(1) = \binom{N+5}{5}.$$

One can check [3] that

$$K_{N(3,3,1,1),N(1,1)^4}(1) > \Big(K_{N(3,1),N(1^3)}(1)\Big)^2.$$

if (and only if) $N \geq 21$.

Indeed,

$$K_{N(3,3,1,1),N(1,1)^4}(1) - \Big(K_{N(3,1),N(1^3)}(1)\Big)^2 = \frac{N^2 - 18N - 43}{20}\ \binom{n+2}{3}..$$

Now take $n = 4$. One has

$$K_{N(4,1),N(1^5)}(1) = \binom{N+3}{3}, \quad K_{N(4,4,1,1),N(1,1)^5}(1) = \binom{N+9}{9} + \binom{N+7}{9}.$$

One can check that

$$K_{N(4,4,1,1),N(1,1)^5}(1) > \Big(K_{N(4,1),N(1^5)}(1)\Big)^2,$$

if (and only if) $N \geq 8$,

Now take $n = 5$.

Proposition 3.4 *Let $\nu_N := N(5,1)$ and $\eta_N := N(1)^6$. Then*

- $K_{2\nu_N,2\eta_N}(1) > (K_{\nu_N,\eta_N}(1))^2$

if and only if $N \geq 6$,

- $K_{2\nu_N,2\eta_N}(1) > (K_{\nu_N,\eta_N}(1))^3$

if and only if $N \geq 49916$.

Indeed,

$$K_{N(5,1),N(1^6)}(1) = \binom{N+4}{4},$$

$$K_{N(5,5,1,1),N(1,1)^6}(1) = \binom{N+13}{13} + \binom{N+12}{13} + 6\binom{N+11}{13} + \binom{N+10}{13} + \binom{N+9}{13},$$

and $51891840 \times \left[K_{N(5,5,1,1),N(1,1)^6}(1) - \left(K_{N(5,1),N(1^6)}(1)\right)^3\right] = \binom{N+4}{5} \times$
$(-78631416 - 172503780\ N - 174033932\ N^2 - 101206400\ N^3 - 35852065\ N^4 -$
$7638110\ N^5 - 899548\ N^6 - 44990\ N^7 + N^8)$.

Note, see e.g. [19], that for any set of partitions $\lambda, \mu^{(1)}, \ldots, \mu^{(p)}$ the parabolic Kostka number $K_{\lambda,\mu^{(1)},\ldots,\mu^{(p)}}(1)$ is equal to the Littlewood–Richardson number $c^{\Lambda}_{\lambda,M}$, where partitions $\Lambda \supset M$ are such that $\Lambda \setminus M = \coprod_i \mu^{(i)}$ is a disjoint union of partitions $\mu^{(i)},\ i = 1, \ldots, p$.

4 Internal product of Schur functions

The irreducible characters χ^λ of the symmetric group S_n are indexed in a natural way by partitions λ of n. If $w \in S_n$, then define $\rho(w)$ to be the partition of n whose parts are the cycle lengths of w. For any partition λ of m of length l, define the power–sum symmetric function

$$p_\lambda = p_{\lambda_1} \ldots p_{\lambda_l},$$

where $p_n(x) = \sum x_i^n$. For brevity write $p_w := p_{\rho(w)}$. The Schur functions s_λ and power–sums p_μ are related by a famous result of Frobenius

$$s_\lambda = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda(w) p_w. \tag{4.1}$$

For a pair of partitions α and β, $|\alpha| = |\beta| = n$, let us define the internal product $s_\alpha * s_\beta$ of Schur functions s_α and s_β:

$$s_\alpha * s_\beta = \frac{1}{n!} \sum_{w \in S_n} \chi^\alpha(w) \chi^\beta(w) p_w. \tag{4.2}$$

It is well–known that

$$s_\alpha * s_{(n)} = s_\alpha, \quad s_\alpha * s_{(1^n)} = s_{\alpha'},$$

where α' denotes the conjugate partition to α.

Let α, β, γ be partitions of a natural number $n \geq 1$, consider the following numbers

$$g_{\alpha\beta\gamma} = \frac{1}{n!} \sum_{w \in S_n} \chi^\alpha(w) \chi^\beta(w) \chi^\gamma(w). \tag{4.3}$$

The numbers $g_{\alpha\beta\gamma}$ coincide with the structural constants for multiplication of the characters χ^α of the symmetric group S_n:

$$\chi^\alpha \chi^\beta = \sum_\gamma g_{\alpha\beta\gamma} \chi^\gamma. \tag{4.4}$$

Hence, $g_{\alpha\beta\gamma}$ are non–negative integers. It is clear that

$$s_\alpha * s_\beta = \sum_\gamma g_{\alpha\beta\gamma} s_\gamma. \tag{4.5}$$

4.1 Internal product of Schur functions, principal specialization, fermionic formulas and unimodality

Let $N \geq 2$, consider the principal specialization $x_i = q^i$, $1 \leq i \leq N-1$, and $x_i = 0$, if $i \geq N$, of the internal product of Schur functions s_α and s_β:

$$s_\alpha * s_\beta(q, q^2, \ldots, q^{N-1}) = \frac{1}{n!} \sum_{w \in S_n} \chi^\alpha(w)\chi^\beta(w) \prod_{k \geq 1} \left(\frac{q^k - q^{kN}}{1 - q^k} \right)^{\rho_k(w)}, \tag{4.6}$$

where $\rho_k(w)$ denotes the number of the length k cycles of w.

By a result of R.-K. Brylinski [2], Corollary 5.3, the polynomials

$$s_\alpha * s_\beta(q, \ldots, q^{N-1})$$

admit the following interpretation. Let $P_{n,N}$ denote the variety of n by n complex matrices z such that $z^N = 0$. Denote by

$$R_{n,N} := \mathbb{C}[P_{n,N}]$$

the coordinate ring of polynomial functions on $P_{n,N}$ with values in the field of complex numbers $\mathbb{C}$. This is a graded ring:

$$R_{n,N} = \oplus_{k \geq 0} R^{(k)}_{n,N},$$

where $R^{(k)}_{n,N}$ is a finite dimensional $\mathfrak{g}l(n)$–module with respect to the adjoint action. Let α and β be partitions of common size. Then [2]

$$s_\alpha * s_\beta(q, \ldots, q^{N-1}) = \sum_{k \geq 0} \langle V_{[\alpha,\beta]_n}, P^{(k)}_{n,N} \rangle q^k,$$

as long as $n \geq \max(Nl(\alpha), Nl(\beta), l(\alpha) + l(\beta))$. Here the symbol $\langle \bullet \, , \bullet \, \rangle$ denotes the scalar product on the ring of symmetric functions such that $\langle s_\lambda \, , s_\mu \, \rangle = \delta_{\lambda,\mu}$. In other words, if V and W be two $GL(N)$-modules, then $\langle V, W \rangle = \dim Hom_{GL(N)}(V, W)$.

Let us remind below one of the main result obtained in [16], namely, Theorem 6.6, which connects the principal specialization of the internal product of Schur functions with certain parabolic Kostka polynomials, and gives, via Corollary 6.7, [16], an effective method for computing the polynomials $s_\alpha * s_\beta(q, \ldots, q^{N-1})$ which, turns out to be for the first time, does not use the character table of the symmetric group S_n, $n = |\alpha|$.

Let α and β be partitions, $\ell(\alpha) = r$, $\ell(\beta) = s$ and $|\alpha| = |\beta|$. Let N be an integer such that $r + s < N$. Consider partition

$$[\alpha, \beta]_N := (\alpha_1 + \beta_1, \alpha_2 + \beta_1, \ldots, \alpha_r + \beta_1, \underbrace{\beta_1, \ldots, \beta_1}_{N-r-s}, \beta_1 - \beta_s, \ldots, \beta_1 - \beta_2].$$

Clearly, $|[\alpha, \beta]_N| = \beta_1 N$, $\ell([\alpha, \beta]_N = N - 1$.

Theorem 4.1 *i) Let α, β be partitions, $|\alpha| = |\beta|$, $l(\alpha) \leq r$, and $l(\alpha) + l(\beta) \leq Nr$. Consider the sequence of rectangular shape partitions*

$$R_N = \{\underbrace{(\beta_1^r), \ldots, (\beta_1^r)}_{N}\}.$$

Then

$$s_\alpha * s_\beta(q, \ldots, q^{N-1}) \stackrel{\bullet}{=} K_{[\alpha,\beta]_{\mathbf{Nr}}, R_N}(q). \tag{4.7}$$

ii) (Dual form) Let α, β be partitions such that $|\alpha| = |\beta|$, $\alpha_1 \leq r$ and $\beta_1 \leq k$. For given integer N such that $\alpha_1 + \beta_1 \leq Nr$, consider partition

$$\lambda_N := (rN - \beta_k^{'}, rN - \beta_{k-1}^{'}, \ldots, rN - \beta_1^{'}, \alpha^{'})$$

and a sequence of rectangular shape partitions

$$R_N := (\underbrace{(r^k), \ldots, (r^k)}_{N}).$$

Then

$$K_{\lambda_N R_N}(q) \stackrel{\bullet}{=} s_\alpha * s_\beta(q, \ldots, q^{N-1}) \tag{4.8}$$

Theorem 4.2 *(***Fermionic formula for the principal specialization of the internal product of Schur functions***).*

Let α and β be two partitions of the same size, and $r := \ell(\alpha)$ be the length of α. Then

$$s_\alpha * s_\beta(q, \ldots, q^{N-1}) = \sum_{\{\nu\}} q^{c(\{\nu\})} \prod_{k,j \geq 1} \begin{bmatrix} P_j^{(k)}(\nu) + m_j(\nu^{(k)}) + N(k-1)\delta_{j,\beta_1}\theta(r-k) \\ P_j^{(k)}(\nu) \end{bmatrix}_q, \tag{4.9}$$

where the sum runs over the set of admissible configurations $\{\nu\}$ of type $([\alpha, \beta]_N, (\beta_1)^N)$. Here for any partition λ, λ_j denotes its j-th component.

Let us explain notations have used in Theorem 3.5.

- A configuration $\{\nu\}$ of type $[\alpha, \beta]_N$ consists of a collection of partitions $\{\nu^{(1)}, \ldots, \nu^{(N-1)}\}$ such that $|\nu^k| = \sum_{j>k} \Big([\alpha, \beta]_N\Big)_j$; by definition we set $\nu^{(0)} := (\beta_1)^N$;
- $P_j^{(k)}(\nu) := N \min(j, \beta_1)\ \delta_{k,1} + Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{k+1})$; here for any partition λ we set $Q_n(\lambda) := \sum_{j \leq n} \min(n, \lambda_j)$;
- For any partition λ, $m_j(\lambda)$ denotes the number of parts of λ are equal to j;
- A configuration $\{\nu\}$ of type $[\alpha, \beta]_N$ is called *admissible configuration of type* $([\alpha, \beta]_N, (\beta_1)^N)$, if $P_j^{(k)}(\nu) \geq 0,\ \forall j, k \geq 1$;
- Here $\delta_{n,m}$ denotes Kronecker's delta function, and we define $\theta(x) = 1,\ if\ \ x \geq 0$, and $\theta(x) = 0,\ if\ \ x < 0$;

• $c(\{\nu\}) = \sum_{n,k\geq 1} \binom{\lambda_n^{(k-1)} - \lambda_n^{(k)}}{2}$ denotes the **charge** of a configuration $\{\nu\}$; by definition, $\binom{x}{2} := x(x-1)/2,\ \forall x \in \mathbb{R}$.

Let us draw attention to the fact that the summation in (4.9) runs over the set of all admissible configurations of type $([\alpha,\beta]_N, (\beta_1)^N)$, other than that of type $([\alpha,\beta]_{Nr}, (\beta_1^r)^N)$.

Corollary 4.3 *([12],[17])*
For any partitions of the same size α and β, the polynomial $s_\alpha * s_\beta(q,\dots,q^{N-1})$ is symmetric and **unimodal**. *In particular, the generalized Gaussian polynomial $\begin{bmatrix} N \\ \alpha \end{bmatrix}_q$ is symmetric and* **unimodal** *for any partition α.*

Indeed, in the case $\beta = (n)$, $n := |\beta|$, one has $s_\alpha * s_{(n)} = s_\alpha$.

Our proof of Corollary 3.7 is based on the following identity

$$2c(\nu) + \sum_{k,j\geq 1} P_j^{(k)}(\nu)\Big[m_j(\nu^{(k)}) + N(k-1)\delta_{j,\beta_1}\theta(r-k)\Big] = N|\alpha|,$$

which can be checked directly. This identity shows that the all polynomials associated with a given admissible configuration involved, are symmetric and have the same "symmetry center" $N|\alpha|/2$, and therefore the resulting polynomial $s_\alpha * s_\beta(q,\dots,q^{N-1})$ is symmetric and unimodal.

Corollary 4.4 *Let α and β be partitions of the same size, and $K_{\beta,\lambda}(q,t)$ denotes the Kostka–Macdonald polynomial associated with partitions α and β, [19]. One has*

$$K_{\beta,\alpha}(q,q) = H_\alpha(q)\Big(\sum_{\{\nu\}} q^{c(\{\nu\})} \prod_{k=2}^{r} \frac{1}{\Big[m_{\beta_1}(\nu^{(k)})\Big]_q!} \prod_{\substack{k\geq 1 \\ j\geq 1,\ j\neq\beta_1}} \begin{bmatrix} P_j^{(k)}(\nu) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{bmatrix}_q\Big),$$

where the sum runs over the same set of admissible configurations as in Theorem 3.6, and $[m]_q! := \prod_{j=1}^m (1-q^j)$ stands for the q-factorial of an positive integer m, and by definition $[0]_q! = 1$; $H_\alpha(q) := \prod_{x\in\alpha}(1-q^{h(x)})$ denotes the hook polynomial associated with partition α, see e.g. [19].

Indeed, one can show [13], [12] that

$$\lim_{N\to\infty} s_\alpha * s_\beta(1,q,q^2,,q^N) = \frac{K_{\beta,\alpha}(q,q)}{H_\alpha(q)} = \lim_{N\to\infty} K_{[\alpha,\beta]_N,(\beta_1)^N}(q).$$

A fermionic formula for the principal specialization of the internal product of Schur functions, and therefore that for the generalized Gaussian polynomials, is a far generalization of the so-called KOH-identity [24] which is equivalent to the fermionic formula for the Kostka number $K_{(Nk,k),(k)^{N+1}})(1)$. The rigged configuration bijection gives rise to a combinatorial proof of Theorem 3.5, and therefore to a combinatorial proof of unimodality of the generalized Gaussian polynomials [12], as well as to give an interpretation of the statistics introduced in [24] in terms of rigged configurations data, see [13], Section 10.2.

Example 4.5 *Let* $\alpha = (4,2)$, $\beta = \alpha' = (2,2,1,1)$. *We want to compute the principal specialization of the internal product of Schur functions* $s_\alpha * s_\beta(q,\dots,q^{N-1})$ *by means of a fermionic formula* (3.10).*First of all, there are* 8 *admissible configurations of type* $((4,2),(2,2,1,1),(6)^N)$.*In fact, it is a general fact that for given partitions* λ *and* μ, *the number of admissible configurations of type* $(N\lambda, N\mu)$ **doesn't depend** *on* N ,*if* $N > N_0$ *i a certain number* $N_0 := N_0(\lambda,\mu)$ *depending on* λ *and* μ *only. This fact is a direct consequence of constraints are imposed by the set of inequalities* $\{P_j^{(k)}(\nu) \geq 0, \quad \forall j,k \geq 1\}$.
Now let us list the **conjugate** *of the first configurations* $\nu^{(1)} \in \{\nu\}$ *for all admissible configurations* $\{\nu\}$ *of type* $[(4,2),(2,2,1,1)_N]$, *together with all non-zero numbers* $P_j^{(k)}(\nu)$, $j,k \geq 1$.

$$(N-3,N-3), \quad P_2^{(1)} = 2, P_2^{(2)} = 2, \quad c = 9,$$

$$(N-2,N-4), \quad P_2^{(1)} = 2, P_1^{(2)} = 1, P_2^{(2)} = 2, \quad 9,$$

$$(N-3,N-4,1), \quad P_1^{(1)} = 2, P_1^{(2)} = 4, P_1^{(3)} = 2, P_2^{(2)} = 1, \quad c = 11,$$

$$(N-2,N-5,1), \quad P_1^{(2)} = 1, P_2^{(1)} = 4, P_3^{(1)} = 2, P_2^{(2)} = 1, \quad c = 13,$$

$$(N-3,N-5,1,1), \quad P_1^{(1)} = 2, P_2^{(1)} = 6, (P_2^{(2)} = 0), P_3^{(1)} = 2, \quad c = 15,$$

$$(N-3,N-5,2), \quad P_1^{(1)} = 2, P_2^{(1)} = 6, (P_2^{(2)} = 0), P_3^{(1)} = 2, \quad c = 17,$$

$$(N-2,N-6,1,1), \quad P_1^{(2)} = 1, P_2^{(1)} = 6, (P_2^{(2)} = 0), P_4^{(1)} = 2, \quad c = 19,$$

$$(N-2,N-6,2), \quad P_1^{(2)} = 1, P_2^{(1)} = 6, (P_2^{(2)} = 0), P_3^{(1)} = 2, \quad c = 21.$$

These are data related with the first configurations $\nu^{(1)}$ *in the set of all admissible configurations of type* $([(42),(2211)]_N,(6)^N)$, *and the set of all nonzero numbers* $P_j^{(k)}(\nu)$. *All other diagrams* $\nu^{(k)}$, $k > 1$ *from the set of admissible configurations in question, are the same, and are displayed below*

$$\nu^{(k)} = (N-2-k, max(N-4-k,0)), \quad 2 \leq k \leq N-3,$$

so that $m_1^{(2)} = 2, \quad m_2^{(2)} = N-6$.

Therefore,

$$(\clubsuit) \quad K_{[(4,2),(2,2,1,1)]_{2N},(2,2)^N}(q) \doteq q^9 \begin{bmatrix} N-1 \\ 2 \end{bmatrix} \begin{bmatrix} 2N-4 \\ 2 \end{bmatrix} + q^9 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} N-2 \\ 2 \end{bmatrix} \begin{bmatrix} 2N-4 \\ 2 \end{bmatrix} +$$

$$q^{11} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} N-1 \\ 4 \end{bmatrix} \begin{bmatrix} 2N-5 \\ 1 \end{bmatrix} + q^{13} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} N-2 \\ 4 \end{bmatrix} \begin{bmatrix} 2N-5 \\ 1 \end{bmatrix} +$$

$$q^{15} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} N \\ 6 \end{bmatrix} + q^{17} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} N-1 \\ 6 \end{bmatrix} + q^{19} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} N-1 \\ 6 \end{bmatrix} + q^{21} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} N-2 \\ 6 \end{bmatrix}.$$

On the other hand,

$$s_{42} * s_{2211} = \frac{1}{720}\Big(81p_1^6 - 135p_1^4p_2 + 45p_1p_2^2 - 90p_1^2p_4 + 144p_1p_5 - 135p_2^3 + 90p_2p_4\Big).$$

Here $p_k := \sum_{i\geq 1} x_i^k$ *stands for the power sum symmetric functions degree of* k*. One can check that* $s_{42} * s_{2211}(q,\dots,q^{N-1}) = RHS(\clubsuit) \stackrel{\bullet}{=} K_{[(4,2),(2,2,1,1)]_{2N},(2,2)^N}(q)$*, as expected.*
Finally one can check that $\lim_{N\to\infty}\ s_{42} * s_{2211}(q,\dots,q^{N-1}) = q^9(2,1,2,2,1,1) = K_{2211,42}(q,q)$.

4.2 Polynomiality of stretched Kostka and Littlewood–Richardson numbers

• As it was mentioned above, for a given partitions λ and μ (resp. λ and a sequence of rectangular shape partitions $\{\mathcal{R}\}$), the number of **admissible** configurations of type $(N\lambda, N\mu)$ (resp. of type $(N\lambda, \{\mathcal{R}\})$) **doesn't** depend on $N > N_0$, where the number N_0 depends on λ and μ (resp. λ and $\{\mathcal{R}\}$) only.

• Each admissible configuration $\{\nu\}$ provides a contribution of a form

$$a_{\{\nu\}}(q) \prod_{j,k\geq 1} \begin{bmatrix} b_{j,k}(\nu)N + d_{jk}(\nu) \\ d_{jk}(\nu) \end{bmatrix}_q,$$

to the parabolic Kostka polynomial $K_{N\lambda,N\{\mathcal{R}\}}(q)$, where a polynomial $a_\nu(q)$ and a **finite set** of numbers $\{b_{jk}(\nu), d_{jk}(\nu)\}_{j,k\geq 1}$ both doesn't depend on $N > N_0$ for some $N_0 := N_0(\lambda, \{\mathcal{R}\})$.

• It is clear that the sum $\sum_{N\geq 0} \binom{aN+b}{b} t^N$ is a **rational** function of variable t. It is well-known (and easy to prove) that the *Adamar product* [11] of *rational* functions is again a *rational* function. Therefore,

Corollary 4.6 *([16],[7], [26])*

For any partition λ *and a sequence of rectangular shape partitions* $\{\mathcal{R}\}$*, the generating function*

$$\sum_{N\geq 0} K_{\lambda,\{\mathcal{R}\}}(1)\ t^N$$

is a rational function of variable t *with a unique pole at* $t = 1$.

More generally using a q-version of Adamar's product Theorem, we can show

Theorem 4.7 *([16])*

For any partition λ *and a* <u>*dominant*</u> *sequence of rectangular shape partitions* $\{\mathcal{R}\}$*, the generating function of stretched parabolic Kostka polynomials*

$$\sum_{N\geq 0} K_{\lambda,\{\mathcal{R}\}}(q)\ t^N$$

is a rational function of variables q *and* t *of a form* $P_{\lambda,\{\mathcal{R}\}}(q,t)/Q_{\lambda,\{\mathcal{R}\}}(q,t)$*, where the dominator* $Q_{\lambda,\{\mathcal{R}\}}(q,t)$ *has the following form*

$$Q_{\lambda,\{\mathcal{R}\}}(q,t) = \prod_{s\in S}(1 - q^s\ t)$$

for a certain finite set $S := S(\lambda, \{\mathcal{R}\})$ *depending on* λ *and* $\{\mathcal{R}\}$.

[11] See e.g. *wikipedia.org/wiki/Hadamard_product_(matrices)* and the literature quoted therein.

Clearly that Corollary 3.9 is a special case $q = 1$ of Theorem 3.10.

• (Littlewood–Richardson polynomials) Let λ be a partition and $\{\mathcal{R}\}$ be a dominant sequence of rectangular shape partitions. Write

$$K_{\lambda,\{\mathcal{R}\}}(q) = b(\lambda, \mathcal{R})\ q^{a(\lambda,\mathcal{R})} + \quad higher\ \ degree\ \ terms.$$

(1) (Generalized saturation theorem [17])

$$a(N\lambda, N\{\mathcal{R}\}) = N\ a(\lambda, \{\mathcal{R}\}).$$

Therefore,

$$\sum_{N\geq 0} b(N\lambda, N\{\mathcal{R}\})\ t^N = \frac{P_{\lambda,\mathcal{R}}(q, q^{-a(\lambda,\{\mathcal{R}\})}\ t)}{Q_{\lambda,\mathcal{R}}(q, q^{-a(\lambda,\{\mathcal{R}\})}\ t)}\Big|_{q=0}$$

is a **rational** function with a unique pole at $t = 1$ with multiplicity equals to $\#|s \in S(\lambda, \mathcal{R}) \mid s = a(\lambda, \mathcal{R})|$.

(2) Now let λ, μ and ν be partitions such that $|\lambda| + |\mu| = |\nu|$. Consider an integer $N \geq max(\ell(\lambda), \mu_1)$, and define partition $\Lambda = \Lambda(N, \lambda, \mu) := (N^N) \oplus \lambda, \mu)$ and the dominant rearrangement of the set of rectangular shape partitions $\{(N^N), \nu_1, \ldots, \nu_{\ell(\nu)}\}$, denoted by $M := M(N, \nu)$.

Proposition 4.8 *([17])*

One has

$$b(\Lambda, M) := c^{\nu}_{\lambda,\mu},$$

where $c^{\nu}_{\lambda,\mu}$ denotes the Littlewood–Richardson number corresponding to partitions λ, μ and ν, that is , the multiplicity of Schur function s_ν in the product of Schur functions $s_\lambda\ s_\mu$.

Theorem 4.9 *([17],[26])*

Given three partitions λ, μ and ν such that $|\lambda| + |\mu| = |\nu|$. The the generating function

$$\sum_{N\geq 0} c^{N\nu}_{N\lambda,N\mu}\ t^N$$

is a **rational** *function of variable t with a unique pole at $t = 1$. Therefore, $c^{N\nu}_{N\lambda,N\mu}$ is a* **polynomial** *in N with rational coefficients.*

Example 4.10 *([13], [16]) (MacMahon polytope and multidimensional Narayana numbers again)*

Take $\lambda = (n+k, n, n-1, \ldots, 2)$ and $\mu = \lambda' = (n, n, n-1, n-2, \ldots, 2, 1^k)$. One can show [16] that if $n \geq k \geq 1$, then for any positive integer N

- $a(N\lambda, N\mu) = (2k-1)N$;
- $b(N\lambda, N\mu) = \dim V^{\mathfrak{gl}(N+k-1)}_{((n-k+1)^{k-1})} = \prod_{i=1}^{k-1} \prod_{j=1}^{n-k+1} \frac{N+i+j-1}{i+j-1}$.

In other words, the number $b(N\lambda, N\mu)$ is equal to the number of (weak) plane partitions of rectangular shape $((n-k+1)^{k-1})$ whose parts do not exceed N. According to Exercise 1, **c**, *[16], pp. 102–103, $b(N\lambda, N\mu)$ is equal also to the number $i(\mathfrak{M}_{k-1,n-k+1}; N)$ of rational points* $\mathbf{x}$ *in the MacMahon polytope $\mathfrak{M}_{k-1,n-k+1}$ such that the points $N\mathbf{x}$ have integer coordinates. It follows from (2.8) that the generating function for numbers $b(n\lambda, n\mu)$ has the following form*

$$\sum_{n\geq 0} b(n\lambda, n\mu)t^n = \left(\sum_{j=0}^{(k-2)(n-k)} N(k-1, n-k+1; j)t^j\right)/(1-t)^{(k-1)(n-k+1)+1},$$

where $N(k, n; j)$, $0 \leq j \leq (k-1)(n-1)$, denote rectangular Narayana's numbers, see e.g. [21], [30].

One can show (A.K.) that

- *if $r := k - \binom{n+2}{2} \geq 0$, then $b(\lambda, \mu) = 1$, and*

$$a(\lambda, \mu) = 2\binom{n+3}{3} + (n+1)(2r-1) + \binom{r}{2};$$

- *if $1 \leq k < \binom{n+2}{2}$, then there exists a unique p, $1 \leq p \leq n$, such that*

$$(p-1)(2n-p+4)/2 < k \leq p(2n-p+3)/2.$$

In this case

$$a(\lambda, \mu) = p(2k-(p-1)n-p) + 2\binom{p}{3},$$

and one can take $\Gamma(\lambda, \mu)$ to be equal to the MacMahon polytope $\mathfrak{M}_{r(k),s(k)}$ with $r(k) := k-1-(p-1)(2n-p+4)/2$, and $s(k) := p(2n-p+3)/2-k$.

This Example gives some flavor how intricate the piecewise linear function $a(\lambda, \mu)$ may be.

Conjecture 4.11 *Let λ and μ be partitions of the same size. Then*

- *([19])* $a(\lambda, \mu) = a(\mu', \lambda')$,
- *([13])* $b(\lambda, \mu) = b(\mu', \lambda')$.

■

Definition 4.12 *([16]) Let α and β be partitions of the same size. Define Liskova polynomials $L^{\mu}_{\alpha,\beta}(q)$ through the decomposition of the internal product of Schur functions in terms of Hall-Littlewood polynomials*

$$s_\alpha * s_\beta(X) = \sum_\mu L^{\mu}_{\alpha,\beta}\ P_\mu(X; q).$$

Clearly, $L^{\mu}_{\alpha,\beta} \in \mathbb{N}[q]$, and $L_{\alpha,(|\alpha|)}(q) = K_{\alpha,\mu}(q)$, so that the Liskova polynomials are natural generalization of Kostka–Foulkes polynomials.

Problem 4.13 *Find for Liskova polynomials an analogue of a fermionic formula for Kostka–Foulkes polynomials stated, for example, in [9], [13].*

5 Appendix. Rigged Configurations: a brief review

Let λ be a partition and $R = ((\mu_a^{\eta_a}))_{a=1}^p$ be a sequence of rectangular shape partitions such that

$$|\lambda| = \sum_a |R_a| = \sum_a \mu_a \eta_a.$$

Definition 5.1

The configuration of type (λ, R) *is a sequence of* **partitions** $\{\nu\} = (\nu^{(1)}, \nu^{(2)}, \ldots)$ *such that*

$$|\nu^{(k)}| = \sum_{j>k} \lambda_j - \sum_{a\geq 1} \mu_a \max(\eta_a - k, 0) = -\sum_{j\leq k} \lambda_j + \sum_{a\geq 1} \mu_a \min(k, \eta_a)$$

for each $k \geq 1$.

Note that if $k \geq l(\lambda)$ and $k \geq \eta_a$ for all a, then $\nu^{(k)}$ is empty.

In the sequel we make the convention that $\nu^{(0)}$ is the empty partition [12].

For a partition μ and an integer $j \geq 1$ define the number

$$Q_j(\mu) = \mu'_1 + \cdots + \mu'_j,$$

which is equal to the number of cells in the first j columns of μ.

The *vacancy* numbers $P_j^{(k)}(\nu) := P_j^{(k)}(\nu; R)$ of the configuration $\{\nu\}$ of type (λ, R) are defined by

$$P_j^{(k)}(\nu) = Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) + \sum_{a\geq 1} \min(\mu_a, j)\delta_{\eta_a,k}$$

for $k, j \geq 1$, where $\delta_{a,b}$ is the Kronecker delta.

Definition 5.2 *The configuration* $\{\nu\}$ *of type* (λ, R) *is called* **admissible**, *if*

$$P_j^{(k)}(\nu; R) \geq 0 \quad \text{for all} \quad k, j \geq 1.$$

We denote by $C(\lambda; R)$ the set of all admissible configurations of type (λ, R), and call the vacancy number $P_j^{(k)}(\nu, R)$ *essential*, if $m_j(\nu^{(k)}) > 0$.

Finally, for configuration $\{\nu\}$ of type (λ, R) let us define its **charge**

$$c(\nu) = \sum_{k,j\geq 1} \binom{\alpha_j^{(k-1)} - \alpha_j^{(k)} + \sum_a \theta(\eta_a - k)\theta(\mu_a - j)}{2},$$

and cocharge

$$\overline{c}(\nu) = \sum_{k,j\geq 1} \binom{\alpha_j^{(k-1)} - \alpha_j^{(k)}}{2},$$

where $\alpha_j^{(k)} = (\nu^{(k)})'_j$ denotes the size of the j–th column of the k–th partition $\nu^{(k)}$ of the configuration $\{\nu\}$; for any real number $x \in \mathbb{R}$ we put $\theta(x) = 1$, if $x \geq 0$, and $\theta(x) = 0$, if $x < 0$.

[12] However, in some cases it is more convenient to set $\nu^{(0)} = (\mu_{i_1}, \ldots, \mu_{i_s})$, where we assume that $\eta_{i_a} = 1,\ a = 1, \ldots, s$. We will give an indication of such choice if it is necessary.

Theorem 5.3 (**Fermionic formula for parabolic Kostka polynomials** [13])
Let λ be a partition and R be a dominant sequence of rectangular shape partitions. Then

$$K_{\lambda R}(q) = \sum_{\nu} q^{c(\nu)} \prod_{k,j\geq 1} \left[\begin{array}{c} P_j^{(k)}(\nu; R) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{array} \right]_q, \tag{5.10}$$

summed over all admissible configurations ν of type $(\lambda; R)$; $m_j(\lambda)$ denotes the number of parts of the partition λ of size j.

Corollary 5.4 (**Fermionic formula for Kostka–Foulkes polynomials [9]**)
Let λ and μ be partitions of the same size. Then

$$K_{\lambda\mu}(q) = \sum_{\nu} q^{c(\nu)} \prod_{k,j\geq 1} \left[\begin{array}{c} P_j^{(k)}(\nu, \mu) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{array} \right]_q, \tag{5.11}$$

summed over all sequences of partitions $\nu = \{\nu^{(1)}, \nu^{(2)}, \ldots\}$ such that

- $|\nu^{(k)}| = \sum_{j>k} \lambda_j$, $k = 1, 2, \ldots$;
- $P_j^{(k)}(\nu, \mu) := Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) \geq 0$ *for all* $k, j \geq 1$, *where by definition we put* $\nu^{(0)} = \mu$;
- $$c(\nu) = \sum_{k,j\geq 1} \left(\begin{array}{c} (\nu^{(k-1)})'_j - (\nu^{(k)})'_j \\ 2 \end{array} \right). \tag{5.12}$$

It is frequently convenient to represent an admissible configuration $\{\nu\}$ by a matrix $m(\nu) = (m_{ij})$, $m_{ij} \in \mathbb{Z}, \forall i, j \geq 1$, which must meets certain conditions. Namely, starting from the collection of partitions $\{\nu\} = (\nu^{(1)}, \nu^{(2)}, \ldots, ...)$ corresponding to configuration $\{\nu\}$, define matrix

$$m(\nu) := (m_{ij}), \quad m_{ij} = (\nu^{(i-1)})'_j - (\nu^{(i)})'_j + \sum_{a\geq 1} \theta(\eta_a - i)\theta(\mu_a - j), \quad \nu^{(0)} := \emptyset,$$

where we set by definition $\theta(x) = 1$, $if\ x \in \mathbb{R}_{\geq 0}$ and $\theta(x) = 0$, $x \in \mathbb{R}_{<0}$.
One can check that a configuration $\{\nu\}$ of type (λ, R) is **admissible** if and only if the matrix $m(\nu)$ meets the following conditions

(0) $m_{ij} \in \mathbb{Z}$,
(1) $\sum_{i\geq 1} m_{ij} = \sum_{a\geq 1} \eta_a \theta(\mu_a - j)$,
(2) $\sum_{j\geq 1} m_{ij} = \lambda_i$,
(3) $\sum_{j\leq k}(m_{ij} - m_{i+1,j}) \geq 0$, for all i, j, k
(4) $\sum_{a\geq 1} \min(\eta_a, k)\delta_{\mu_a, j} \geq \sum_{i\leq k}(m_{ij} - m_{i,j+1})$, for all i, j, k.

One can check that if matrix (m_{ij}) satisfies the conditions (0) − (4), then the set of partitions $\{\nu\} = (\nu^{(1)}, \nu_{(2)}, \ldots, ...)$

$$(\nu^{(k)})'_j := \sum_{i>k} m_{ij} - \sum_a \max(\eta_a - k, 0)\theta(\mu_a - j)$$

defines an admissible configuration of type $(\lambda, R = \{(\mu_a)^{\eta_a}\})$.

Example 5.5 *Take* $\lambda = (44332)$, $R = \{(2^3), (2^2), (2^2), (1), (1)\}$, *so that*

$$\{\mu_a\} = (2,2,2,1,1) \;\; \textit{and} \;\; \{\eta_a\} = (3,2,2,1,1), \quad a = 1, \ldots, 5.$$

. Therefore $|\nu^{(1)}| = 4$, $|\nu^{(2)}| = 6$, $|\nu^{(3)}| = 5$, *and* $|\nu^{(4)}| = 2$. *It is not hard to check that there exist* 6 *admissible configurations. They are:*
(1) $\{\nu^{(1)} = (3,1), \nu^{(2)} = (3,3), \nu^{(3)} = (3,2), \nu_{(4)} = (2)\}$,
(2) $\{\nu^{(1)} = (3,1), \nu^{(2)} = (3,2,1), \nu^{(3)} = (3,2), \nu^{(4)} = (2)\}$,
(3) $\{\nu^{(1)} = (2,2), \nu^{(2)} = (2,2,2), \nu^{(3)} = (3,2), \nu^{(4)} = (2)\}$,
(4) $\{\nu^{(1)} = (4), \; \nu^{(2)} = (3,3), \nu^{(3)} = (3,2), \nu^{(4)} = (2)\}$,
(5) $\{\nu^{(1)} = (3,1), \nu^{(2)} = (2,2,1,1), \nu^{(3)} = (2,2,1), \nu^{(4)} = (2)\}$,
(6) $\{\nu^{(1)} = (3,1), \nu^{(2)} = (2,2,1,1), \nu^{(3)} = (3,1,1), \nu^{(4)} = (2)\}$,
Let us compute the matrix (m_{ij}) *corresponding to the configuration* (2). *Clearly,* $(m_{ij}) = ((\nu^{(i-1)})'_j - (\nu^{(i)})'_j) + (\sum_{a\geq 1} \theta(\eta_a - i)\theta(\mu_a - j)) := U + W$. *One can check that*

$$U = \begin{pmatrix} -3 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 3 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Therefore,

$$m(\{\nu\}) = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ 3 & 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

One can read off directly from the matrix m_{ij} the all additional quantities need to compute the parabolic Kostka polynomial corresponding to λ and a (dominant) sequence of rectangular shape partitions R. Namely,

$$P_j^{(k)} = \sum_{i\geq j} (m_{ki} - m_{k+1,i}), \;\; m_j(\nu^{(k)}) = \sum_{a\geq 1} \min(\eta_a, k)\delta_{\mu_a, j} - \sum_{i\leq k} (m_{ij} - m_{i,j+1}), c(\nu) = \sum_{i,j\geq 1} \binom{m_{ij}}{2}.$$

For example, in our example, we have $c(\nu) = 8$, $P_1^{(1)} = 1, P_2^{(2)} = 1, P_3^{(2)} = 1, P_2^{(3)} = 1$ are all non-zero vacancy numbers, and the contribution of the configuration in question to the parabolic Kostka polynomial is equal to $q^8 \begin{bmatrix} 2 \\ 1 \end{bmatrix}^4$. Treating in a similar fashion other configurations, we come to a fermionic formula

$$K_{44332,\{(2^3),(2^2),(2^2),(1),(1)\}}(q) = q^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + q^8 \begin{bmatrix} 2 \\ 1 \end{bmatrix}^4 + q^8 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + q^{12} + q^6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + q^8.$$

■

If $\eta_a = 1$, $\forall a$, then $\sum_{a\geq 1} \eta_a \theta(\mu_a - j) = \mu'_j$, and we <u>set</u> $\nu^{(0)} = \mu'$. In this case one can rewrite the conditions $(1) - (4)$ as follows

$(1^{'})$ $\sum_{i\geq 1} m_{ij} = \mu_j^{'}$,
$(2^{'})$ $\sum_{j\geq 1} m_{ij} = \lambda_i$,
$(3^{'})$ $\sum_{j\leq k}(m_{ij} - m_{i+1,j}) \geq 0$, for all i, j, k
$(4^{'})$ $\sum_{i>k}(m_{ij} - m_{i,j+1}) \geq 0$, for all $i, j.k$

Let us remark that if $m_{ij} \in \mathbb{Z}_{\geq 0}$ then the matrix (m_{ij}) defines a lattice plane partition of shape λ. For example, take $\lambda = (6, 4, 2, 2, 1, 1)$, $\mu = (2^8)$ and admissible configuration $\{\nu\} = \{(5, 5), (4, 2), (3, 1), (2), (1)\}$. The corresponding matrix and lattice plane partition of shape λ are

$$(m_{ij}) = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \textit{and plane partition} \quad \begin{array}{cc} 3 & 3 \\ 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & \\ 1 & \end{array}.$$

The corresponding lattice word is 111222.1222.12.12.1.1.

In the case $\eta_a = 1$, $\forall a$, there exists a unique admissible configuration of type (λ, μ) denoted by $\Delta(\lambda, \mu)$ such that $\max(c((\Delta(\lambda, \mu), J))) = n(\mu) - n(\lambda)$, where the maximum is taken over all rigged configurations associated with configuration $\Delta(\lambda, \mu)$. Recall that for any partition λ,

$$n(\lambda) = \sum_{j\geq 1} \binom{\lambda_a^{'}}{2},$$

and if $\lambda \geq \mu$ with respect to the *dominance order*, then the degree of Kostka polynomial $K_{\lambda,\mu}(q)$ is equal to $n(\mu) - n(\lambda)$, see, e.g. [19], Chapter 1, for details. Namely, the configuration $\Delta(\lambda, \mu)$ corresponds to the following matrix:

$$m_{1j} = \mu_j^{'} - \max(\lambda_j^{'} - 1, 0), j \geq 1, \quad m_{ij} = 1, \ \ if \ \ (i, j) \in \lambda, \ i \geq 2, \ \ m_{ij} = 0, \ if \ (i, j) \notin \lambda.$$

In other words, the configuration $\Delta(\lambda, \mu)$ consists of the following partitions $(\lambda[1], \lambda[2], \ldots)$, where $\lambda[k] = (\lambda_{k+1}, \lambda_{k+2}, \ldots)$. It is not difficult to see that the contribution to the Kostka polynomial $K_{\lambda,\mu}(q)$ coming from the maximal configuration, is equal to

$$K_q(\Delta(\lambda, \mu)) := q^{c(\Delta(\lambda,\mu))} \prod_{j=1}^{\lambda_2} \begin{bmatrix} Q_j(\mu) - Q_j(\lambda) + \lambda_j^{'} - \lambda_{j+1}^{'} \\ \lambda_j^{'} - \lambda_{j+1}^{'} \end{bmatrix}_q,$$

where $c(\Delta(\lambda, \mu)) = n(\lambda) + n(\mu) - \sum_{j\geq 1} \ \mu_J^{'}(\lambda_j^{'} - 1)$. Therefore,

$$K_{\lambda,\mu}(q) \geq K_q(\Delta(\lambda, \mu)). \tag{5.13}$$

It is clearly seen that if $\lambda \geq \mu$, then $Q_j(\mu) \geq Q_j(\lambda)$, $\forall\ j \geq 1$, and thus, $K_{q=1}(\Delta(\lambda, \mu)) \geq 1$, and the inequality (5.13) can be considered as a "quantitative" generalization of the Gale–

Ryser theorem, see, e.g. [19], Chapter I, Section 7, or [11] for details.

Now let us **stress** that for a fixed k, the all partitions $\nu^{(k)}$ which contribute to the set of admissible configurations of type (λ, μ) have the same size equals to $\sum_{j \geq k+1} \lambda_j$, and thus the size of each $\nu^{(k)}$ doesn't depend on μ. However the Rigged Configuration bijection

$$RC_{\lambda,\mu} : STY(\lambda, \mu) \longrightarrow RC(\lambda, \mu)$$

happens to be essentially depends on μ. One can check that the map $RC_{\lambda,\mu}$ is compatible with the familiar *Bender–Knuth* transformations on the set of semistandard Young tableaux of a fixed shape.

As it was mentioned above, for a fixed k the all (admissible) configurations have the same size. Therefore, the set of admissible configurations admits a partial ordering denoted by "$\succcurlyeq$" . Namely, if $\{\nu\}$ and $\{\xi\}$ are two admissible configurations of the same type (λ, μ), we will write $\{\nu\} \succcurlyeq \{\xi\}$, if either $\{\nu\} = \{\xi\}$ or there exists an integer ℓ such that $\nu^{(a)} = \xi^{(a)}$ if $1 \leq a \leq \ell$, and $\nu^{(\ell+1)} > \xi^{(\ell+1)}$ with respect of the dominance order on the set of the same size partitions. It seems an interesting **Problem** to study poset structureds on the set of admissible configurations of type (λ, μ), especiallyto investigate the posets of admissible configurations associated with the multidimensional Catalan numbers, (work in progress).

■

Theorem 5.6 *(***Duality theorem for parabolic Kostka polynomials** *[13])*

Let λ be partition and $R = \{(\mu_a^{\eta_a})\}$ be a dominant sequence of rectangular shape partitions. Denote by $\lambda^{'}$ the conjugate of λ, and by $R^{'}$ a dominant rearrangement of a sequence of rectangular shape partitions $\{(\eta_a^{\mu_a})\}$. Then

$$K_{\lambda,R}(q) = q^{n(R)} K_{\lambda',R'}(q^{-1}),$$

where

$$n(R) = \sum_{a<b} \min(\mu_a, \mu_b) \min(\eta_a, \eta_b).$$

A technical proof is based on checking of the statement that the map

$$\iota : m_{ij} \longrightarrow \hat{m}_{ij} = -m_{ji} + \theta(\lambda_j - i) + \sum_{a \geq 1} \theta(\mu_a - j)\theta(\eta_a - i)$$

establishes bijection between the sets of admissible configurations of types (λ, R) and $(\lambda^{'}, R^{'})$, and $\iota(c(m_{ij})) = c((\hat{m}_{ij}))$.

5.1 Example

Let $n = 6$, consider for example, a standard Young tableau

$$T = \begin{array}{cccccc} 1 & 2 & 3 & 6 & 8 & 9 \\ 4 & 5 & 7 & 10 & 11 & 12 \end{array}, \quad c(T) = 48.$$

The corresponding rigged configuration (ν, J) is

$$\nu = (321), \quad J = (J_3 = 0,\ J_2 = 2, J_1 = 6), \quad (m_{ij})(\nu) = \begin{pmatrix} 9 & -2 & -1 \\ 3 & 2 & 1 \end{pmatrix}, \quad c(\nu) = 44.$$

Recall that $c(T)$ and $c(\nu)$ denote the charge of tableau T and configuration ν correspondingly.

- One can see that $c(T) = c(\nu) + J_3 + J_2 + J_1$, as it should be in general.
- Now, the descent set and descent number of tableau T are $Des(T) = \{3, 6, 9\}$, $des(T) = 3$. One can see that $des(T) = 3 = \nu_1^{'}$, as it should be in general [13].
- One can check that our tableau T is invariant under the action of the Schützenberger involution [14] on the set of standard Young tableaux of a shape λ. It is clearly seen from the set of riggings J [15] that the rigged configuration (ν, J) corresponding to tableau T, is invariant under the Flip involution [16] on the set of rigged configurations of type $(\lambda, 1^{|\lambda|})$, as it should be in general, see [15] for a complete proof of the statement that the action of the Schützenberger transformation on a Littlewood– Richardson tableau $T \in LR(\lambda, R)$, under the Rigged Configuration Bijection transforms tableau T to a Littlewood–Richardson tableau corresponding to the rigged configuration $\nu\kappa(J))$, where (νJ) is the rigged configuration corresponding to tableau T we are started with.

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[13] In fact the shape of the first configuration $\nu^{(1)}$ of type (λ, μ) can be read off from the set of "secondary" descent sets $\{Des^{(1)}(T) = Des(T), Des^{(2)}(T), \ldots, ...)$, cf [10].

[14] *http* : *//en.wikipedia.org/wiki/Jeu_de_taquin*

[15] In our example $J = (0, 1, 3)$.

[16] Recall that a rigging of an admissible configuration ν is a collection of integers

$$J = (\{J_{s,r}^{(k)}\} \ , \ 1 \leq s \leq m_r(\nu^{(k)})$$

such that for a given k, r one has

$$0 \leq J_{1,r}^{(k)} \leq J_{2,r}^{(k)} \leq \ldots \leq J_{m_r(\nu^{(k)}),r}^{(k)} \leq P_r^{(k)}(\nu).$$

The Flip involution κ is defined as follows:

$$\kappa(\nu, \{J_{s,r}^{(k)}\}) = (\nu, \{J_{m_r(\nu^{(k)})-s+1,r}^{(k)}\}).$$

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