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THE EXISTENCE OF GT-SECTIONS FOR FUNDAMENTAL GROUPS OF CONFIGURATION SPACES OF HYPERBOLIC CURVES

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Abstract. Hoshi and Mochizuki constructed a surjection from a subgroup of the group of outomorphisms of étale fundamental groups of configuration spaces of hyperbolic curves to the Grothendieck-Teichmüller group by means of the combinatorial anabelian geometry developed by them. In the present paper, we prove that this surjection is *split surjective*. Also, in order to give this splitting, we prove that an exact sequence associated to *rationally degenerate* semi-graphs of anabelioids of PSC-type is *split surjective*.

Contents

Introduction

Let Σ be a nonempty set of prime numbers which either contains all prime numbers or satisfies $\sharp(\Sigma) = 1$, and n a positive integer. Write

$$\Pi_n$$

for the maximal pro-Σ quotient of the étale fundamental group of the n-th configuration space of a hyperbolic curve C over an algebraically closed field of characteristic zero, and

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

for the (closed) subgroup of $\mathrm{Out}(\Pi_n)$ consisting of FC-admissible outomorphisms of Π_n (i.e., arising from automorphisms of Π_n that preserve the fiber subgroups of Π_n and the cuspidal inertia subgroups of the fiber subgroups

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(cf. [15, Definition 1.1, (ii)])). Suppose that

$$n \geq \begin{cases} 4 & \text{if } C \text{ is proper,} \\ 3 & \text{if } C \text{ is affine.} \end{cases}$$

Write GT for the *pro-Σ Grothendieck-Teichmüller group*. In [7], Hoshi and Mochizuki constructed the *tripod homomorphism*

$$\mathfrak{T}_T \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \longrightarrow \mathrm{GT},$$

and proved that the tripod homomorphism is *surjective*. This result may be regarded as a *combinatorial group-theoretic version* of the fact that the natural outer homomorphism

$$\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \longrightarrow\!\!\!\!\!\rightarrow G_{\mathbb{Q}}$$

is surjective where $(\mathcal{M}_{g,r})_{\mathbb{Q}}$ is the moduli stack of r-pointed smooth proper curves of genus g over $\mathbb{Q}$, and $G_{\mathbb{Q}}$ is the absolute Galois group of $\mathbb{Q}$. In the present paper, we prove the following result (cf. Corollary 3.4, Remark 3.3):

Theorem A. *The tripod homomorphism*

$$\mathfrak{T}_T \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \longrightarrow \mathrm{GT}$$

is split surjective.

In particular, by Theorem A, we have an outer action of GT on Π_n which is faithful. Also, Theorem A may be regarded as a *combinatorial group-theoretic version* of the fact that the natural surjective outer homomorphism

$$\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \longrightarrow\!\!\!\!\!\rightarrow G_{\mathbb{Q}}$$

is *split surjective*. (For example, by means of a totally degenerate stable curve over $\mathbb{Q}$, we may verify that the surjection $\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \twoheadrightarrow G_{\mathbb{Q}}$ is split surjective (cf. [9])).

In order to prove Theorem A, we also consider a variant of Theorem A, as follows: Here, we do not put the assumption that either Σ contains all prime numbers or satisfies $\sharp(\Sigma) = 1$. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type, i.e., roughly speaking, a system of the dual (semi-)graph of a pointed stable curve X over an algebraically closed field of characteristic zero and Galois categories obtained from irreducible components of X, marked points of X, and nodes of X (cf. [14, Definition 1.1, (i)]; also §1). For a vertex $v \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$, we shall denote by $\mathcal{G}|_v$ a certain semi-graph of anabelioids of pro-Σ PSC-type with $\mathrm{Vert}(\mathcal{G})$ obtained as [6, Definition 2.1, (iii)] (cf. also Definition 1.1, (i)). Write

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$$

for the group of automorphisms of $\mathcal{G}$ which induce the identity automorphism on the underlying semi-graph of $\mathcal{G}$, and

$$\mathrm{Glu}(\mathcal{G}) \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$$

for the closed subgroup of *glueable* collections of outomorphisms of the direct product $\prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$ consisting of elements $(\alpha_v)_{v \in \mathrm{Vert}(\mathcal{G})}$ such that the image of α_v in $(\hat{\mathbb{Z}}^{\Sigma})^{\times}$ by the cyclotomic character does not depend on $v \in \mathrm{Vert}(\mathcal{G})$ (cf. [6, Definition 4.9]; also Definition 1.2, (iv), Definition 1.3). In [6], Hoshi and Mochizuki proved that the image of the natural homomorphism

$$\rho_{\mathcal{G}}^{\mathrm{Vert}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$$

is *equal to* $\mathrm{Glu}(\mathcal{G})$. In §2, we prove the following result (cf. Theorem 2.5):

Theorem B. *Let $\mathcal{G}$ be a rationally degenerate semi-graph of anabelioids of pro-Σ PSC-type, i.e., roughly speaking, a semi-graph of anabelioids of pro-Σ PSC-type obtained from a pointed stable curve whose irreducible components are rational (cf. Definition 1.1, (viii)). Then the surjection*

$$\rho_{\mathcal{G}}^{\mathrm{Vert}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Glu}(\mathcal{G})$$

is split surjective.

In §3, by means of Theorem B, we prove Theorem A.

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Notations and Conventions

Sets: For a set A, we shall write $\sharp(A)$ for the *cardinality* of A, and 2^A for the *power set* of A.

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of all prime numbers. The notation $\mathbb{Z}$ will be used to denote the ring of rational integers. For a nonempty subset Σ of $\mathfrak{Primes}$, the notation $\hat{\mathbb{Z}}^{\Sigma}$ will be used to denote the pro-Σ completion of $\mathbb{Z}$, and the notation $(\hat{\mathbb{Z}}^{\Sigma})^{\times}$ will be used to denote the multiplicative group of $\hat{\mathbb{Z}}^{\Sigma}$.

Profinite groups: For a profinite group G and a closed subgroup $H \subseteq G$ of G, we shall write G^{ab} for the *abelianization* of G (i.e., the quotient of G by the closure of the commutator subgroup $[G, G]$ of G), and $Z_G(H)$ (respectively, $N_G(H)$) for the *centralizer* (respectively, *normalizer*) of H in G, i.e.,

$$Z_G(H) := \{g \in G \mid g \cdot h \cdot g^{-1} = h \text{ for any } h \in H\} \subseteq G$$

$$(\text{respectively, } N_G(H) := \{g \in G \mid g \cdot H \cdot g^{-1} = H\} \subseteq G).$$

It is immediate from the definitions that

$$Z_G(H) \subseteq N_G(H) \ ; \ H \subseteq N_G(H).$$

For a profinite group G and a closed subgroup $H \subseteq G$ of G, we shall denote by $\mathrm{Aut}(G)$ the group of (continuous) automorphisms of the topological group G, by $\mathrm{Inn}(H \subseteq G)$ the image of the homomorphism obtained by sending $h \in H$ to the inner automorphism $\mathrm{Inn}(h \in G)$ of G determined by $h \in G$, and by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ with respect to the normal subgroup $\mathrm{Inn}(G \subseteq G) \subseteq \mathrm{Aut}(G)$. We shall refer to an element of $\mathrm{Out}(G)$ as an *outomorphism* of G. If, moreover, G is *topologically finitely generated*, then one verifies that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the group $\mathrm{Aut}(G)$, hence also a *profinite topology* on the group $\mathrm{Out}(G)$.

For profinite groups G_1, G_2 and a homomorphism $f\colon G_1 \to G_2$ of profinite groups, we shall say that f is *split surjective* if there exists a homomorphism of profinite groups $g\colon G_2 \to G_1$ such that $f \circ g\colon G_2 \to G_2$ is the identity automorphism of G_2.

1. The exact sequence relating glueable outomorphisms

In the present §1, we review some notions of the combinatorial anabelian geometry developed by Hoshi and Mochizuki, including the exact sequence relating glueable outomorphisms associated to a semi-graph of anabelioids of pro-Σ PSC-type (cf. Theorem 1.4, below). Throughout the present paper, let Σ be a nonempty subset of $\mathfrak{Primes}$.

First, we recall the notion of semi-graphs (cf. [13, §1]). We shall say that the collection $\mathbb{G}$ of the following date is a *semi-graph*:

(i) a set $\mathcal{V}$ — whose elements we refer to as *vertices*;
(ii) a set $\mathcal{E}$ — whose elements we refer to as *edges* — each of whose elements e is a *set of cardinality* 2 satisfying the property $e \neq e' \in \mathcal{E} \Longrightarrow e \cap e' = \emptyset$;
(iii) a collection ζ of maps ζ_e for $e \in \mathcal{E}$ — which we refer to as the *coincidence maps* — such that $\zeta_e\colon e \to \mathcal{V} \cup \{\mathcal{V}\}$ is a map from the set e to the set $\mathcal{V} \cup \{\mathcal{V}\}$.

For a semi-graph $\mathbb{G}$, we shall refer to an element $b \in e$ of an edge e of $\mathbb{G}$ as a *branch of the edge* e, and shall say that an edge e of $\mathbb{G}$ is *open* (respectively, *closed*) if $\zeta_e^{-1}(\{\mathcal{V}\}) \neq \emptyset$ (respectively, $= \emptyset$). If $v = \zeta_e(b)$, for a branch b of an edge e of $\mathbb{G}$, then we shall say that the edge e *abuts to* the vertex v, and that the branch b of the edge e *abuts to* the vertex v. For a semi-graph $\mathbb{G}$ which has at least one vertex and one edge (respectively, does not have an edge; does not have a vertex), we shall say that $\mathbb{G}$ is *connected* if any edge e of $\mathbb{G}$ abuts to a vertex of $\mathbb{G}$, and for any vertices v and v' of $\mathbb{G}$, there exist a finite sequence

$$v = v_1, v_2, \ldots, v_n = v'$$

of vertices of $\mathbb{G}$ and a finite sequence

$$e_1, e_2, \ldots, e_{n-1}$$

of edges of $\mathbb{G}$ such that e_i abuts to v_i and v_{i+1} (respectively, the cardinality of the set of vertices of $\mathbb{G}$ is equal to 1; the cardinality of the set of edges of $\mathbb{G}$ is equal to 1). A *sub-semi-graph* $\mathbb{H}$ of a semi-graph $\mathbb{G}$ is a semi-graph satisfying the following properties:

(i) the set of *vertices* (respectively, *edges*) of $\mathbb{H}$ is a subset of the set of *vertices* (respectively, *edges*) of $\mathbb{G}$;
(ii) every branch of an edge of $\mathbb{H}$ that abuts, relative to $\mathbb{G}$, to a vertex v of $\mathbb{G}$ lying in $\mathbb{H}$ also abuts to v, relative to $\mathbb{H}$;
(iii) if a branch of an edge of $\mathbb{H}$ *either* abuts, relative to $\mathbb{G}$, to a vertex v of $\mathbb{G}$ that does *not* lie in $\mathbb{H}$, *or* does not abut to a vertex, relative to $\mathbb{G}$, then this branch does not abut to a vertex, relative to $\mathbb{H}$.

For a pointed stable curve X over an algebraically closed field, we shall refer to as the *dual semi-graph of* X the semi-graph whose the vertices (respectively, closed edges; open edges; branches of a closed edge) are precisely the irreducible components (respectively, nodes; marked points; branches of a node) of X, and coincidence maps are determined by the geometry of the pointed stable curve X.

Next, we recall the notion of semi-graphs of anabelioids of pro-Σ PSC-type (cf. [11, Appendix], [12, §1], [13, §2], [14, Definition 1.1]). We shall refer to a Galois category as a *connected anabelioid.* For connected anabelioids $\mathcal{A}$ and $\mathcal{B}$, we shall define a *morphism* $\mathcal{A} \to \mathcal{B}$ *of connected anabelioids* to be an exact functor $\mathcal{B} \to \mathcal{A}$ as Galois categories (cf. [1, Exposé V, Proposition 6.1]). For a connected anabelioid $\mathcal{A}$, we shall refer to as the *pro-*Σ *completion of* $\mathcal{A}$ the connected anabelioid constituted by the full subcategory of $\mathcal{A}$ determined by the objects dominated by a Galois covering of the final object of $\mathcal{A}$ whose the prime factors of degree are contained in Σ, and the *fundamental group* $\Delta_{\mathcal{A}}$ *of* $\mathcal{A}$ the fundamental group as a Galois category relative to some base point. Note that, by definition of a morphisms of connected anabelioids, a morphisms $\mathcal{A} \to \mathcal{B}$ of connected anabelioids induces an outer homomorphism from the fundamental group of $\mathcal{A}$ to the fundamental group of $\mathcal{B}$. We shall say that the collection $\mathcal{G}$ of the following date is a semi-graph of anabelioids:

(i) a *semi-graph* $|\mathcal{G}|$ — which is referred as the *underlying semi-graph of* $\mathcal{G}$;
(ii) for each vertex v of $|\mathcal{G}|$, a *connected anabelioid* $\mathcal{G}_v$;
(iii) for each edge e of $|\mathcal{G}|$, a *connected anabelioid* $\mathcal{G}_e$, together with, for each branch $b \in e$ abutting to a vertex v, a *morphism of connected anabelioids* $b_* \colon \mathcal{G}_e \to \mathcal{G}_v$.

In the above notation, we shall refer to $\mathcal{G}_v$, $\mathcal{G}_e$ as the *constituent anabelioids of* $\mathcal{G}$. We shall say that a semi-graph of anabelioids is *connected* if the underlying semi-graph is connected. For a connected semi-graph of anabelioids $\mathcal{G}$ which $|\mathcal{G}|$ has at least one vertex, we shall denote by

$$\mathcal{B}(\mathcal{G})$$

the category of objects given by date

$$\{S_v, \phi_e\}$$

where v (respectively, e) ranges over the vertices (respectively, edges) of $|\mathcal{G}|$; for each vertex v, S_v is a object of $\mathcal{G}_v$; for each edge e, with branches b_1, b_2 abutting to vertices v_1, v_2, respectively, $\phi_e \colon \{(b_1)_*\}^* S_{v_1} \xrightarrow{\sim} \{(b_2)_*\}^* S_{v_2}$ is an isomorphism in $\mathcal{G}_e$, and morphisms given by morphisms between such date. For a semi-graph of anabelioids $\mathcal{G}$ which $|\mathcal{G}|$ has the unique edge e and does not have a vertex, we shall write

$$\mathcal{B}(\mathcal{G}) := \mathcal{G}_e.$$

One verifies immediately that this category $\mathcal{B}(\mathcal{G})$ is a *connected anabelioid.* We shall refer to the fundamental group of $\mathcal{B}(\mathcal{G})$ as the *fundamental group of* $\mathcal{G}$. For a semi-graphs $\mathcal{G}$ of anabelioids, we shall refer to as the *pro-*Σ *completion of* $\mathcal{G}$ the semi-graph of anabelioids by replacing the constituent anabelioids of $\mathcal{G}$ by its pro-Σ completion. We shall refer to as an *automorphism of a semi-graph of anabelioids* a collection of an automorphism of the underlying semi-graph, together with a compatible system of isomorphisms between the various anabelioids at each of the vertices and edges of the underlying semi-graph, which are compatible with the various morphisms of anabelioids associated to the branches of the underlying semi-graph. For a pointed stable curve X over an algebraically closed field of characteristic zero, we shall refer to as the semi-graph of anabelioids arising from X the following semi-graphs $\mathcal{G}$ of anabelioids:

(i) $|\mathcal{G}|$ is the dual semi-graph of X;

(ii) for each vertex v, $\mathcal{G}_v$ is the connected anabelioid determined by the category of étale coverings of the irreducible component X_v of $X \setminus (\{\text{the marked points of } X\} \cup \{\text{the nodes of } X\})$ corresponding to v;

(iii) for each open edge e of $|\mathcal{G}|$ which corresponds to the marked point x of X, we denote the vertex v of $|\mathcal{G}|$ which abuts to e, and X_x the scheme-theoretic intersection of $X \setminus (\{\text{the marked points of } X\} \cup \{\text{the nodes of } X\})$ and the completion of X at x. Then $\mathcal{G}_e$ is the connected anabelioid determined by the category of étale coverings of X_x, together with, for each branch $b \in e$, a morphism of connected anabelioids $b_* \colon \mathcal{G}_e \to \mathcal{G}_v$ determined by the natural morphism $X_x \to X_v$;

(iv) for each closed edge e of $|\mathcal{G}|$ which corresponds to a node ν_e of X, we denote the vertices v_1, v_2 of $|\mathcal{G}|$ which abut to e, ν_e^1 (respectively, ν_e^2) is the branch of ν_e corresponding to v_1 (respectively, v_2), and $X_{e\cap v_1}$ (respectively, $X_{e\cap v_2}$) the scheme-theoretic intersection of $X \setminus (\{\text{the marked points of } X\} \cup \{\text{the nodes of } X\})$ and the completion of the branch ν_e^1 (respectively, ν_e^2) at the node ν_e. We shall fix a (non-canonical) isomorphism $X_{e\cap v_1} \simeq X_{e\cap v_2}$ over the base field, and denote the resulting object by X_e. Then $\mathcal{G}_e$ is the connected anabelioid determined by the category of étale coverings of X_e, together with, for each branch $b \in e$ that $\zeta_e(b) = v_i$, a morphism of connected anabelioids $b_* \colon \mathcal{G}_e \to \mathcal{G}_{v_i}$ determined by the natural morphism $X_e \to X_{v_i}$.

We shall say that $\mathcal{G}$ is a *semi-graph of anabelioids of pro-Σ PSC-type* if $\mathcal{G}$ is the pro-Σ completion of a semi-graph of anabelioids arising from a pointed stable curve over an algebraically closed field of characteristic zero. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type. We shall refer to the maximal pro-Σ quotient of the fundamental group of $\mathcal{G}$ as the *PSC-fundamental group of* $\mathcal{G}$, and denote by $\Pi_{\mathcal{G}}$ the PSC-fundamental group of $\mathcal{G}$. We shall refer to an open edge (respectively, a closed edge) of $\mathcal{G}$ as a *cusp* (respectively, a *node*) of $\mathcal{G}$. We shall denote by $\mathrm{Vert}(\mathcal{G})$ (respectively, $\mathrm{Cusp}(\mathcal{G})$; $\mathrm{Node}(\mathcal{G})$) the set of vertices (respectively, cusps; nodes) of $\mathcal{G}$. We shall write $\mathrm{Edge}(\mathcal{G}) := \mathrm{Cusp}(\mathcal{G}) \sqcup \mathrm{Node}(\mathcal{G})$, $r(\mathcal{G})$ for $\sharp(\mathrm{Cusp}(\mathcal{G}))$, and $\mathcal{G}_z$ for the connected anabelioid corresponding to $z \in \mathrm{Vert}(\mathcal{G}) \cup \mathrm{Cusp}(\mathcal{G}) \cup \mathrm{Node}(\mathcal{G})$. We shall write

$$\mathcal{V} \colon \mathrm{Edge}(\mathcal{G}) \longrightarrow 2^{\mathrm{Vert}(\mathcal{G})}$$

$$(\text{respectively, } \mathcal{E} \colon \mathrm{Vert}(\mathcal{G}) \longrightarrow 2^{\mathrm{Edge}(\mathcal{G})};$$

$$\mathcal{N}\colon \operatorname{Vert}(\mathcal{G}) \longrightarrow 2^{\operatorname{Edge}(\mathcal{G})})$$

for the map obtained by sending $e \in \operatorname{Edge}(\mathcal{G})$ (respectively, $v \in \operatorname{Vert}(\mathcal{G})$; $v \in \operatorname{Vert}(\mathcal{G})$) to the set of vertices (respectively, edges; nodes) of $\mathcal{G}$ to which e abuts (respectively, which abut to v; which abut to v). For a vertex v (respectively, an edge e; a node ν; a cusp c), we shall refer to as a *verticial subgroup of* v (respectively, an *edge-like subgroup of* e; a *nodal subgroup of* ν; a *cuspidal subgroup of* c) the image of a homomorphism $\Delta_{\mathcal{G}_v} \to \Pi_{\mathcal{G}}$ (respectively, $\Delta_{\mathcal{G}_e} \to \Pi_{\mathcal{G}}$; $\Delta_{\mathcal{G}_\nu} \to \Pi_{\mathcal{G}}$; $\Delta_{\mathcal{G}_c} \to \Pi_{\mathcal{G}}$) determined by the natural morphism $\mathcal{G}_v \to \mathcal{B}(\mathcal{G})$ (respectively, $\mathcal{G}_e \to \mathcal{B}(\mathcal{G})$; $\mathcal{G}_\nu \to \mathcal{B}(\mathcal{G})$; $\mathcal{G}_c \to \mathcal{B}(\mathcal{G})$). We shall denote by $\Pi_{\mathcal{G}}^{\mathrm{cpt}}$ the quotient of $\Pi_{\mathcal{G}}$ by the normal closed subgroup generated by the cuspidal subgroups of $\Pi_{\mathcal{G}}$. We shall write

$$g(\mathcal{G}) := \frac{1}{2} \cdot \operatorname{rank}_{\hat{\mathbb{Z}}^{\Sigma}}(\Pi_{\mathcal{G}}^{\mathrm{cpt}})^{\mathrm{ab}}.$$

For a pair (g, r) of nonnegative integers such that $2g - 2 + r > 0$, we shall say that $\mathcal{G}$ is *of type* (g, r) if $g = g(\mathcal{G})$ and $r = r(\mathcal{G})$. We shall denote by $\operatorname{Aut}(\mathcal{G})$ the group of automorphisms of the semi-graph $\mathcal{G}$ of anabelioids, and by $\operatorname{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ the subgroup of $\operatorname{Aut}(\mathcal{G})$ of automorphisms of $\mathcal{G}$ which induce the identity automorphism on the underlying semi-graph of $\mathcal{G}$ (cf. [7, Remark 4.1.2]). Then the natural homomorphism

$$\operatorname{Aut}(\mathcal{G}) \longrightarrow \operatorname{Out}(\Pi_{\mathcal{G}})$$

is an *injection with closed image* (cf. [14, §2]). Thus, we shall regard $\operatorname{Aut}(\mathcal{G})$ as a closed subgroup of $\operatorname{Out}(\Pi_{\mathcal{G}})$ by the above injection. For an outomorphism $\alpha \in \operatorname{Out}(\Pi_{\mathcal{G}})$ of $\Pi_{\mathcal{G}}$, we shall say that α is *graphic* if α is contained in $\operatorname{Aut}(\mathcal{G})$.

Now we recall various operations of semi-graphs of anabelioids of pro-Σ PSC-type.

Definition 1.1 (cf. [6, §2])**.** Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type.

(i) For $v \in \operatorname{Vert}(\mathcal{G})$, we shall write $\mathcal{G}|_v$ for the semi-graph of anabelioids of pro-Σ PSC-type defined as follows: We take $\operatorname{Vert}(\mathcal{G}|_v)$ to consist of the single element v, $\operatorname{Cusp}(\mathcal{G}|_v)$ to be the set of branches of $\mathcal{G}$ which abut to v, and $\operatorname{Node}(\mathcal{G}|_v)$ to be the empty set. We take the connected anabelioid of $\mathcal{G}|_v$ corresponding to the unique vertex v to be $\mathcal{G}_v$. For each edge $e \in \mathcal{E}(v)$ of $\mathcal{G}$ and each branch b of e that abuts to the vertex v, we take the connected anabelioid of $\mathcal{G}|_v$ corresponding to the branch b to be a copy of the connected anabelioid $\mathcal{G}_e$. For each edge $e \in \mathcal{E}(v)$ of $\mathcal{G}$ and each branch b of e that abuts, relative to $\mathcal{G}$, to the vertex v, we take the morphism of connected anabelioids $(\mathcal{G}|_v)_{e_b} \to (\mathcal{G}|_v)_v$ — where we write e_b for the cusp of $\mathcal{G}|_v$ corresponding to b — to be the morphism of connected anabelioids $\mathcal{G}_e \to \mathcal{G}_v$ associated, relative to $\mathcal{G}$, to the branch b.

(ii) Let $\mathbb{H}$ be a sub-semi-graph of $|\mathcal{G}|$. Then we shall say that $\mathbb{H}$ is *of PSC-type* if the following two conditions are satisfied:

(1) $\mathbb{H}$ has *at least one vertex*.

(2) If v is a vertex of $\mathbb{H}$, and e is an edge of $|\mathcal{G}|$ that abuts to v, then e is an edge of $\mathbb{H}$.

(iii) Let $\mathbb{H}$ be a sub-semi-graph of PSC-type of $|\mathcal{G}|$ (cf. (ii)). We shall write $\mathcal{G}|_{\mathbb{H}}$ for the semi-graph of anabelioids defined as follows: the underlying semi-graph is $\mathbb{H}$; for each vertex v (respectively, edge e) of $\mathbb{H}$, the connected anabelioid corresponding to v (respectively, e) is $\mathcal{G}_v$ (respectively, $\mathcal{G}_e$); for each branch b of an edge e of $\mathbb{H}$ that abuts to a vertex v of $\mathbb{H}$, the morphism associated to b is the morphism $\mathcal{G}_e \to \mathcal{G}_v$ associated to the branch of $|\mathcal{G}|$ corresponding to b. Then we may verify that $\mathcal{G}|_{\mathbb{H}}$ is a semi-graph of anabelioids of pro-Σ PSC-type. We shall refer to $\mathcal{G}|_{\mathbb{H}}$ as the *semi-graph of anabelioids of pro-Σ PSC-type obtained by restricting $\mathcal{G}$ to $\mathbb{H}$*.

(iv) We shall say that a subset $S \subseteq \mathrm{Cusp}(\mathcal{G})$ is *omittable* if the following condition is satisfied: For each vertex $v \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$, if $\mathcal{G}|_v$ is of type (g, r), then it holds that $2g - 2 + r - \sharp(\mathcal{E}(v) \cap S) > 0$.

(v) Let $S \subseteq \mathrm{Cusp}(\mathcal{G})$ be a subset of $\mathrm{Cusp}(\mathcal{G})$ which is omittable (cf. (iv)). Then, by eliminating the cusps contained in S, and for each vertex v of $\mathcal{G}$, replacing the connected anabelioid $\mathcal{G}_v$ corresponding to v by the connected anabelioid of objects of $\mathcal{G}_v$ that restrict to a trivial covering over the cusps contained in S that abut to v, we obtain a semi-graph of anabelioids

$$\mathcal{G}_{\bullet S}$$

of *pro-Σ PSC-type*. We shall refer to $\mathcal{G}_{\bullet S}$ as the *partial compactification of $\mathcal{G}$ with respect to S*.

(vi) We shall say that a subset $S \subseteq \mathrm{Node}(\mathcal{G})$ is of *separating type* if the semi-graph obtained by removing the closed edge corresponding to the elements of S from $|\mathcal{G}|$ is not connected. Moreover, for each node $e \in \mathrm{Node}(\mathcal{G})$, we shall say that e is of *separating type* if $\{e\}$ is of separating type.

(vii) Suppose that $S \subseteq \mathrm{Node}(\mathcal{G})$ is *not of separating type* (cf. (vi)). Then one may define a semi-graph of anabelioids of pro-Σ PSC-type as follows: We take the underlying semi-graph $\mathbb{H}_{\succ S}$ to be the semi-graph obtained by replacing each node e of $|\mathcal{G}|$ contained in S such that $\mathcal{V}(e) = \{v_1, v_2\} \subseteq \mathrm{Vert}(\mathcal{G})$ — where v_1, v_2 are *not necessary distinct* — by two cusps that abut to v_1, $v_2 \in \mathrm{Vert}(\mathcal{G})$, respectively. We take the connected anabelioid corresponding to a vertex v (respectively, node e) of $\mathbb{H}_{\succ S}$ to be $\mathcal{G}_v$ (respectively, $\mathcal{G}_e$). We take the connected anabelioid corresponding to a cusp of $\mathbb{H}_{\succ S}$ arising from a cusp e of $\mathcal{G}$ to be $\mathcal{G}_e$. We take the connected anabelioid corresponding to a cusp of $\mathbb{H}_{\succ S}$ arising from a node e of $\mathcal{G}$ to be $\mathcal{G}_e$. For each branch b of $\mathbb{H}_{\succ S}$ that abuts to a vertex v of a node e (respectively, of a cups e that does not arise from a node of $|\mathcal{G}|$), we take the morphism associated to b to be the morphism $\mathcal{G}_e \to \mathcal{G}_v$ associated to the branch of $|\mathcal{G}|$ corresponding to b. For each branch b of $\mathbb{H}_{\succ S}$ that abuts to a vertex v of a cusp of $\mathbb{H}_{\succ S}$ that arises from a node e of $|\mathcal{G}|$, we take the morphism associated to b to be the morphism $\mathcal{G}_e \to \mathcal{G}_v$ associated to the branch of $|\mathcal{G}|$ corresponding to b. We shall denote by the resulting semi-graph of anabelioids of pro-Σ PSC-type

$$\mathcal{G}_{\succ S}$$

and refer to $\mathcal{G}_{\succ S}$ as the *semi-graph of anabelioids of pro-Σ PSC-type obtained from $\mathcal{G}$ resolving S*. Let $v \in \mathrm{Vert}(\mathcal{G})$ be a vertex of $\mathcal{G}$. Write $\mathbb{H}_v$ for the *unique* sub-semi-graph of *PSC-type* of $|\mathcal{G}|$ (cf. (ii)) whose set of vertices is $\{v\}$. Then one may verify easily that $\mathrm{Node}(\mathcal{G}|_{\mathbb{H}_v})$ is *not of separating type* (cf. (vi)), and

$$(\mathcal{G}|_{\mathbb{H}_v})_{\succ \mathrm{Node}(\mathcal{G}|_{\mathbb{H}_v})}$$

(cf. (iii)) is *naturally isomorphic to* $\mathcal{G}|_v$ (cf. (i)).

(viii) We shall say that $\mathcal{G}$ is *totally degenerate* if $\mathcal{G}|_v$ (cf. (ii)) is of type $(0,3)$ for any $v \in \mathrm{Vert}(\mathcal{G})$, and that $\mathcal{G}$ is *rationally degenerate* if $g(\mathcal{G}|_v)$ (cf. (ii)) is equal to 0 for any $v \in \mathrm{Vert}(\mathcal{G})$.

(ix) Let $e \in \mathrm{Node}(\mathcal{G})$ be a node of $\mathcal{G}$. Write $\mathbb{H}_e$ for the *unique* sub-semi-graph of *PSC-type* of $|\mathcal{G}|$ (cf. (i)) whose set of vertices is $\mathcal{V}(e)$. Then one may verify easily that $S := \mathrm{Node}(\mathcal{G}|_{\mathbb{H}_e}) \setminus \{e\}$ is *not of separating type* (cf. (vi)). We shall write

$$\mathcal{G}|_e := (\mathcal{G}|_{\mathbb{H}_e})_{\succ S}$$

(cf. (iii), (vii)).

(x) Let $v \in \mathrm{Vert}(\mathcal{G})$ be a vertex of $\mathcal{G}$. We shall say that v is *terminal* if $\sharp(\mathcal{N}(v)) = 1$ and, for the node $e \in \mathcal{N}(v)$ which abuts to v, $\sharp(\mathcal{V}(e)) = 2$. Suppose that v is terminal. Write $\mathbb{H}_{\setminus\{v\}}$ for the *unique* sub-semi-graph of *PSC-type* of $|\mathcal{G}|$ (cf. (ii)) whose set of vertices is $\mathrm{Vert}(\mathcal{G}) \setminus \{v\}$. We shall write

$$\mathcal{G}_{\setminus\{v\}} := \mathcal{G}|_{\mathbb{H}_{\setminus\{v\}}}$$

(cf. (iii)).

(xi) Let $\mathbb{H}$ be a sub-semi-graph of PSC-type (cf. (ii)), $S \subseteq \mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ a subset of $\mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ that is not of separating type (cf. (vi)), and $T \subseteq \mathrm{Cusp}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ an omittable subset of $\mathrm{Cusp}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ (cf. (iv)). Then, by the definition of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$, we obtain a natural homomorphism

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T})$$

(cf. (iii), (v), (vii)). In particular, for a vertex $v \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$, we shall denote by $\alpha|_v \in \mathrm{Aut}(\mathcal{G}|_v)$ the image of $\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ by the natural homomorphism

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v).$$

Moreover, we recall the cyclotomic characters of semi-graphs of anabelioids of pro-Σ PSC-type.

Definition 1.2 (cf. [6, §3]). Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type. For any $e \in \mathrm{Cusp}(\mathcal{G})$, we fix a cuspidal subgroup Π_e of e.

(i) Given a central extension of profinite groups

$$1 \longrightarrow \hat{\mathbb{Z}}^{\Sigma} \longrightarrow E \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1 \ ,$$

and a cusp $e \in \mathrm{Cusp}(\mathcal{G})$, we shall refer to a section of this extension over $\Pi_e \subseteq \Pi_{\mathcal{G}}$ as a *trivialization of this extension at the cusp e*. We shall write

$$H^2_c(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$$

for the set of equivalence classes

$$[E, (\iota_e \colon \Pi_e \to E)_{e\in\mathrm{Cusp}(\mathcal{G})}]$$

of collections of date $(E, (\iota_e \colon \Pi_e \to E)_{e\in\mathrm{Cusp}(\mathcal{G})})$ as follows:

(1) E is a central extension of profinite groups

$$1 \longrightarrow \hat{\mathbb{Z}}^{\Sigma} \longrightarrow E \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1\ ;$$

(2) for each $e \in \mathrm{Cusp}(\mathcal{G})$, ι_e is a trivialization of this extension at the cusp e. The equivalence relation "$\sim$" is then defined as follows: for two collections of date $(E, (\iota_e))$ and $(E', (\iota'_e))$, we shall write $(E, (\iota_e)) \sim (E', (\iota'_e))$ if there exists an isomorphism of profinite groups $\alpha \colon E \xrightarrow{\sim} E'$ over $\Pi_{\mathcal{G}}$ which induces the identity automorphism of $\hat{\mathbb{Z}}^{\Sigma}$, and, moreover, for each $e \in \mathrm{Cusp}(\mathcal{G})$, maps ι_e to ι'_e.

We shall refer to $H^2_c(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$ as the *second cohomology group with compact supports of* $\mathcal{G}$.

(ii) For a vertex $v \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$, we shall write

$$H^2_c(v, \hat{\mathbb{Z}}^{\Sigma}) := H^2_c(\mathcal{G}|_v, \hat{\mathbb{Z}}^{\Sigma})$$

and refer to $H^2_c(v, \hat{\mathbb{Z}}^{\Sigma})$ as the *second cohomology group with compact supports of* v.

(iii) The set $H^2_c(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$ (cf. (i)) is equipped with a *natural structure of* $\hat{\mathbb{Z}}^{\Sigma}$*-module* defined as follows:

- Let $[E, (\iota_e)]$, $[E', (\iota'_e)] \in H^2_c(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$ be elements of $H^2_c(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$. Then the fiber product $E \times_{\Pi_{\mathcal{G}}} E'$ of structures $E \twoheadrightarrow \Pi_{\mathcal{G}}$, $E' \twoheadrightarrow \Pi_{\mathcal{G}}$ is an extension of $\Pi_{\mathcal{G}}$ by $\hat{\mathbb{Z}}^{\Sigma} \times \hat{\mathbb{Z}}^{\Sigma}$. Thus, the quotient S of $E \times_{\Pi_{\mathcal{G}}} E'$ by the image of the composite of

$$\hat{\mathbb{Z}}^{\Sigma} \hookrightarrow \hat{\mathbb{Z}}^{\Sigma} \times \hat{\mathbb{Z}}^{\Sigma} \hookrightarrow E \times_{\Pi_{\mathcal{G}}} E'$$

$$m \longmapsto (m, -m)$$

is an extension of $\Pi_{\mathcal{G}}$ by $\hat{\mathbb{Z}}^{\Sigma}$. On the other hand, it follows from the definition of S that for each $e \in \mathrm{Cusp}(\mathcal{G})$, the sections ι_e and ι'_e naturally determine a section $\iota^S_e \colon \Pi_e \to S$ over Π_e. Thus, we define

$$[E, (\iota_e)] + [E', (\iota'_e)] := [S, (\iota^S_e)].$$

Here, one may verify easily that the equivalence class $[S, (\iota^S_e)]$ depends only on the equivalence classes $[E, (\iota_e)]$, $[E', (\iota'_e)]$, and that this definition of "+" determined a module structure on $H^2_c(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$.
- Let $[E, (\iota_e)] \in H^2_c(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$ be an element of $H^2_c(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$ and $a \in \hat{\mathbb{Z}}^{\Sigma}$. Now the composite of

$$E \times \hat{\mathbb{Z}}^{\Sigma} \overset{\mathrm{pr}_1}{\twoheadrightarrow} E \twoheadrightarrow \Pi_{\mathcal{G}}$$

determines an extension of $\Pi_{\mathcal{G}}$ by $\hat{\mathbb{Z}}^{\Sigma} \times \hat{\mathbb{Z}}^{\Sigma}$. Thus, the quotient P of $E \times \hat{\mathbb{Z}}^{\Sigma}$ by the image of the composite of

$$\begin{array}{ccccc} \hat{\mathbb{Z}}^{\Sigma} & \hookrightarrow & \hat{\mathbb{Z}}^{\Sigma} \times \hat{\mathbb{Z}}^{\Sigma} & \hookrightarrow & E \times \hat{\mathbb{Z}}^{\Sigma} \\ m & \longmapsto & (m, -am) & & \end{array}$$

is an extension of $\Pi_{\mathcal{G}}$ by $\hat{\mathbb{Z}}^{\Sigma}$. On the other hand, it follows from the definition of P that for each $e \in \mathrm{Cusp}(\mathcal{G})$, the sections ι_e and the zero homomorphism $\Pi_e \to \hat{\mathbb{Z}}^{\Sigma}$ naturally determine a section $\iota_e^P : \Pi_e \to P$ over Π_e. Thus, we define

$$a \cdot [E, (\iota_e)] := [P, (\iota_e^P)].$$

Here, one may verify easily that the equivalence class $[P, (\iota_e^P)]$ depends only on the equivalence class $[E, (\iota_e)]$ and $a \in \hat{\mathbb{Z}}^{\Sigma}$, and that this definition of "$\cdot$" determines a $\hat{\mathbb{Z}}^{\Sigma}$-module structure on $H_c^2(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$.

It follows from [6, Lemma 3.2] that the $\hat{\mathbb{Z}}^{\Sigma}$-module "$H_c^2(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$" does not depend on the choice of $\{\Pi_e\}_{e \in \mathrm{Cusp}(\mathcal{G})}$. (More precisely, the $\hat{\mathbb{Z}}^{\Sigma}$-module "$H_c^2(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$" is uniquely determined by $\mathcal{G}$ up to the natural isomorphism obtained by [6, Lemma 3.2].) Also, for a vertex $v \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$, $H_c^2(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$ and $H_c^2(v, \hat{\mathbb{Z}}^{\Sigma})$ are *free* $\hat{\mathbb{Z}}^{\Sigma}$*-modues of rank* 1 (cf. [6, Theorem 3.7, (ii)]).

(iv) We shall write

$$\Lambda_{\mathcal{G}} := \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Sigma}}(H_c^2(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma}), \hat{\mathbb{Z}}^{\Sigma})$$

(cf. (i), (iii)) and refer to $\Lambda_{\mathcal{G}}$ as the *cyclotome associated to* $\mathcal{G}$. For a vertex $v \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$, we shall write

$$\Lambda_v := \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Sigma}}(H_c^2(v, \hat{\mathbb{Z}}^{\Sigma}), \hat{\mathbb{Z}}^{\Sigma})$$

(cf. (ii), (iii)) and refer to Λ_v as the *cyclotome associated to* v.

(v) We shall write

$$\chi_{\mathcal{G}} : \mathrm{Aut}(\mathcal{G}) \longrightarrow \mathrm{Aut}(\Lambda_{\mathcal{G}}) \simeq (\hat{\mathbb{Z}}^{\Sigma})^{\times}$$

for the natural homomorphism induced by the natural action of $\mathrm{Aut}(\mathcal{G})$ on $H_c^2(\mathcal{G}, \hat{\mathbb{Z}}^{\Sigma})$ and refer to $\chi_{\mathcal{G}}$ as the *pro-*Σ *cyclotomic character of* $\mathcal{G}$. For a vertex $v \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$, we shall write

$$\chi_v : \mathrm{Aut}(\mathcal{G}|_v) \longrightarrow \mathrm{Aut}(\Lambda_v) \simeq (\hat{\mathbb{Z}}^{\Sigma})^{\times}$$

for the natural homomorphism induced by the natural action of $\mathrm{Aut}(\mathcal{G}|_v)$ on $H_c^2(v, \hat{\mathbb{Z}}^{\Sigma})$ and refer to χ_v as the *pro-*Σ *cyclotomic character of* v. Then it follows from [6, Corollary 3.9, (ii), (iv)] that there exists a natural isomorphism of $\hat{\mathbb{Z}}^{\Sigma}$-modules

$$\Lambda_v \xrightarrow{\sim} \Lambda_{\mathcal{G}},$$

and, under this isomorphism, for any $\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$,

$$\chi_{\mathcal{G}}(\alpha) = \chi_v(\alpha|_v).$$

Finally, we recall the subgroup of "glueable" collections of outomorphisms.

Definition 1.3. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type. We shall write

$$\rho_{\mathcal{G}}^{\mathrm{Vert}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$$

for the homomorphism determined by

$$\alpha \longmapsto (\alpha|_v)_{v \in \mathrm{Vert}(\mathcal{G})}$$

(cf. Definition 1.1, (xi)). We shall denote by $\mathrm{Dehn}(\mathcal{G})$ the kernel of $\rho_{\mathcal{G}}^{\mathrm{Vert}}$, and by

$$\mathrm{Glu}(\mathcal{G}) \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$$

the closed subgroup of *glueable* collections of outomorphisms of the direct product $\prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$ consisting of elements $(\alpha_v)_{v \in \mathrm{Vert}(\mathcal{G})}$ such that $\chi_v(\alpha_v) = \chi_w(\alpha_w)$ for any $v, w \in \mathrm{Vert}(\mathcal{G})$ (cf. Definition 1.2, (v)).

Theorem 1.4 (Hoshi-Mochizuki)**.** *Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type. Then the image of the homomorphism*

$$\rho_{\mathcal{G}}^{\mathrm{Vert}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$$

(cf. Definition 1.3) is equal to $\mathrm{Glu}(\mathcal{G})$*.*

In particular, we obtain the following exact sequence of profinite groups

$$1 \longrightarrow \mathrm{Dehn}(\mathcal{G}) \longrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \xrightarrow{\rho_{\mathcal{G}}^{\mathrm{Vert}}} \mathrm{Glu}(\mathcal{G}) \longrightarrow 1.$$

Proof. Theorem 1.4 follows from [6, Theorem B, (iii)]. □

Remark 1.5. In the notation of Theorem 1.4, it is not clear to the author at the time of writing whether or not $\rho_{\mathcal{G}}^{\mathrm{Vert}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \twoheadrightarrow \mathrm{Glu}(\mathcal{G})$ is split surjective. Nevertheless, if $\mathcal{G}$ is *rationally degenerate* (cf. Definition 1.1, (viii)), then we are able to obtain the result that $\rho_{\mathcal{G}}^{\mathrm{Vert}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \twoheadrightarrow \mathrm{Glu}(\mathcal{G})$ is *split surjective* (cf. Theorem 2.5, below).

2. A splitting of the exact sequence relating glueable outomorphisms in the rationally degenerate case

In the present §2, we prove that the exact sequence of profinite groups appearing in Theorem 1.4 is *split* in the *rationally degenerate* case (cf. Theorem 2.5, below).

In the present §2, we maintain the notation of the preceding §1.

Lemma 2.1. *Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type which is of type* $(0, 3)$*, and e a cusp of $\mathcal{G}$. Write $\Pi_e \subseteq \Pi_{\mathcal{G}}$ for a cuspidal subgroup associated to e, and*

$$\mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e)$$

for the intersection of

$$\{\sigma \in \mathrm{Aut}(\Pi_{\mathcal{G}}) \mid \sigma(\Pi_e) = \Pi_e\}$$

and the subgroup of $\mathrm{Aut}(\Pi_{\mathcal{G}})$ *given by the inverse image of*

$$\mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}}).$$

Then the following hold:

(i) *The kernel of the natural homomorphism*

$$\mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e) \longrightarrow \mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G})$$

is equal to $\mathrm{Inn}(\Pi_e \subseteq \Pi_{\mathcal{G}}) \subseteq \mathrm{Aut}(\Pi_{\mathcal{G}})$.

(ii) *The natural homomorphism* $\mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e) \to \mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G})$ *is split surjective.*

In particular, we obtain the following split exact sequence of profinite groups

$$1 \longrightarrow \mathrm{Inn}(\Pi_e \subseteq \Pi_{\mathcal{G}}) \longrightarrow \mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e) \longrightarrow \mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G}) \longrightarrow 1.$$

Proof. Assertion (i) follows immediately from [14, Proposition 1.2, (ii)]. Next, by means of [14, Proposition 1.2, (ii)], assertion (ii) follows immediately from the argument used in the proof of [8, §I, Proposition 3]. More precisely, for cusps e_1, $e_2 \in \mathrm{Cusp}(\mathcal{G}) \setminus \{e\}$ of $\mathcal{G}$, write Π_{e_1}, Π_{e_2} for a cuspidal subgroup of e_1, e_2, respectively. Since a cuspidal subgroup is isomorphic to $\hat{\mathbb{Z}}^{\Sigma}$, for $i \in \mathrm{Cusp}(\mathcal{G})$, we have a topologically generator $g_i \in \Pi_i$ of Π_i. Write

$$\Phi^* := \{\sigma \in \mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e) \mid \sigma(g_{e_2}) = c \cdot (g_{e_2})^{\alpha} \cdot c^{-1},$$
$$\text{with some } \alpha \in (\hat{\mathbb{Z}}^{\Sigma})^{\times}, c \in [\Pi_{\mathcal{G}}, \Pi_{\mathcal{G}}]\} \subseteq \mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e),$$

and $p^{\mathrm{ab}} \colon \Pi_{\mathcal{G}} \twoheadrightarrow (\Pi_{\mathcal{G}})^{\mathrm{ab}}$ is the natural surjection. Then, to verify assertion (ii), it suffices to show that the restriction of $\mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e) \to \mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G})$ to Φ^* induces an *isomorphism*

$$\Phi^* \xrightarrow{\sim} \mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G}).$$

First, we verify that the restriction of $\mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e) \to \mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G})$ to Φ^* is injective. Now by means of assertion (i), to verify the injectivity of the restriction of $\mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e) \to \mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G})$ to Φ^*, it suffices to show that $\Phi^* \cap \mathrm{Inn}(\Pi_e \subseteq \Pi_{\mathcal{G}}) = \{1\}$. Let $\sigma \in \Phi^* \cap \mathrm{Inn}(\Pi_e \subseteq \Pi_{\mathcal{G}})$ be a element of $\Phi^* \cap \mathrm{Inn}(\Pi_e \subseteq \Pi_{\mathcal{G}})$. Then, since σ is contained in $\mathrm{Inn}(\Pi_e \subseteq \Pi_{\mathcal{G}})$, there exists an element $d \in \hat{\mathbb{Z}}^{\Sigma}$ of $\hat{\mathbb{Z}}^{\Sigma}$ such that $\mathrm{Inn}((g_e)^d \in \Pi_{\mathcal{G}}) = \sigma$. In particular, for any $g \in \Pi_{\mathcal{G}}$, $p^{\mathrm{ab}}(\sigma(g)) = p^{\mathrm{ab}}(g)$. Also, by the definition of Φ^*, there exists a pair (d', c) of an element $d' \in \hat{\mathbb{Z}}^{\Sigma}$ of $\hat{\mathbb{Z}}^{\Sigma}$ and an element $c \in [\Pi_{\mathcal{G}}, \Pi_{\mathcal{G}}]$ of $[\Pi_{\mathcal{G}}, \Pi_{\mathcal{G}}]$ such that $c \cdot (g_{e_2})^{d'} \cdot c^{-1} = \sigma(g_{e_2})$. Note that, since $p^{\mathrm{ab}}(\sigma(g_{e_2})) = p^{\mathrm{ab}}(g_{e_2})$, and $(\Pi_{\mathcal{G}})^{\mathrm{ab}}$ is a free $\hat{\mathbb{Z}}^{\Sigma}$-module of rank 2 with a free generating set $\{p^{\mathrm{ab}}(g_e), p^{\mathrm{ab}}(g_{e_2})\}$, $d' = 0$. Thus, it follows from [14, Proposition 1.2, (ii)] that there exists an element $d'' \in \hat{\mathbb{Z}}^{\Sigma}$ of $\hat{\mathbb{Z}}^{\Sigma}$ such that $(g_e)^d \cdot (g_{e_2})^{-d''} = c \in [\Pi_{\mathcal{G}}, \Pi_{\mathcal{G}}]$. In particular, since $(\Pi_{\mathcal{G}})^{\mathrm{ab}}$ is a free $\hat{\mathbb{Z}}^{\Sigma}$-module of rank 2 with a free generating set $\{p^{\mathrm{ab}}(g_e), p^{\mathrm{ab}}(g_{e_2})\}$, $d = d'' = 0$. This completes the proof of that $\Phi^* \cap \mathrm{Inn}(\Pi_e \subseteq \Pi_{\mathcal{G}}) = \{1\}$. Next, we verify that the restriction of $\mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e) \to \mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G})$ to Φ^* is surjective. Let $\delta \in \mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G})$ be an element of $\mathrm{Aut}^{|\,\mathrm{grph}\,|}(\mathcal{G})$, and $\delta' \in \mathrm{Aut}(\Pi_{\mathcal{G}})$ a pre-image of δ by the natural surjection $\mathrm{Aut}(\Pi_{\mathcal{G}}) \twoheadrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$. Then, by replacing δ' by a composite of δ' and an element of $\mathrm{Inn}(\Pi_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}})$, and means of the definition of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$, we may assume that $\delta' \in \mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e)$. Also, by the definition of $\mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e)$, there exists a pair of $\alpha' \in (\hat{\mathbb{Z}}^{\Sigma})^{\times}$ and $t \in \Pi_{\mathcal{G}}$ such that $\delta'(g_{e_2}) = t \cdot (g_{e_2})^{\alpha'} \cdot t^{-1}$. Since $(\Pi_{\mathcal{G}})^{\mathrm{ab}}$ is a free $\hat{\mathbb{Z}}^{\Sigma}$-module of rank 2 with a free generating set $\{p^{\mathrm{ab}}(g_e), p^{\mathrm{ab}}(g_{e_2})\}$, by replacing δ' by a composite of δ' and an element of $\mathrm{Inn}(\Pi_e \subseteq \Pi_{\mathcal{G}})$, we may assume that there exists an element $d''' \in \hat{\mathbb{Z}}^{\Sigma}$ of $\hat{\mathbb{Z}}^{\Sigma}$ such that $p^{\mathrm{ab}}(t) = p^{\mathrm{ab}}(g_{e_2})^{d'''}$. Therefore, since $\delta'(g_{e_2}) = (t \cdot (g_{e_2})^{-d'''}) \cdot (g_{e_2})^{\alpha'} \cdot (t \cdot (g_{e_2})^{-d'''})^{-1}$ and $(t \cdot (g_{e_2})^{-d'''}) \in [\Pi_{\mathcal{G}}, \Pi_{\mathcal{G}}]$, δ' is contained in Φ^*. Thus, the restriction of $\mathrm{Aut}^{|\mathrm{C}|}(\Pi_{\mathcal{G}}, \Pi_e) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ to Φ^* is *surjective*, hence also induces an *isomorphism*

$$\Phi^* \xrightarrow{\sim} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}).$$

This completes the proof of assertion (ii). Finally, the final portion of Lemma 2.1 follows from assertions (i), (ii). This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let $\mathcal{G}$ be a rationally degenerate semi-graph of anabelioids of pro-Σ PSC-type. Suppose that either $\mathcal{G}$ has no nodes or is not cyclically primitive, i.e.,* $\sharp(\mathrm{Node}(\mathcal{G})) = 1$*, and the unique node of $\mathcal{G}$ is of separating type (cf. Definition 1.1, (vi)). Then the homomorphism*

$$\rho^{\mathrm{Vert}}_{\mathcal{G}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Glu}(\mathcal{G})$$

is split surjective.

Proof. First, since $\mathcal{G}$ is a rationally degenerate semi-graph of anabelioids of pro-Σ PSC-type which $\sharp(\mathrm{Node}(\mathcal{G})) \leq 1$, we may check easily that there exists an omittable subset $S \subseteq \mathrm{Cusp}(\mathcal{G})$ (cf. Definition 1.1, (v)) such that the partial compactification $\mathcal{G}_{\bullet S}$ of $\mathcal{G}$ with respect to S (cf. Definition 1.1, (vi)) is *totally degenerate*. Then it follows from [6, Theorem 4.8, (iii), (iv)] that we obtain the following commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d] & \mathrm{Glu}(\mathcal{G}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}_{\bullet S}) \arrow[r] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\bullet S}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}_{\bullet S}}"] & \mathrm{Glu}(\mathcal{G}_{\bullet S}) \arrow[r] & 1
\end{tikzcd}$$

where the horizontal sequences are exact (cf. Theorem 1.4), and the left-hand vertical arrow is an *isomorphism*. In particular, the commutative diagram of profinite groups

$$\begin{tikzcd}
\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d] & \mathrm{Glu}(\mathcal{G}) \arrow[d] \\
\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\bullet S}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}_{\bullet S}}"] & \mathrm{Glu}(\mathcal{G}_{\bullet S})
\end{tikzcd}$$

is *cartesian*. Therefore, to verify Lemma 2.2, by replacing $\mathcal{G}$ by $\mathcal{G}_{\bullet S}$, we may assume that $\mathcal{G}$ is *totally degenerate*.

Next, note that, if $\sharp(\mathrm{Node}(\mathcal{G})) = 0$, then Lemma 2.2 follows from Theorem 1.4. Thus, we may assume that $\sharp(\mathrm{Node}(\mathcal{G})) \neq 0$. Suppose that $\sharp(\mathrm{Node}(\mathcal{G})) = 1$, and that the unique node of $\mathcal{G}$ is of separating type. Let e be the unique node of $\mathcal{G}$, v, w the vertices of $\mathcal{G}$, and c a cusp of $\mathcal{G}|_w$. Write $\Pi_e \subseteq \Pi_{\mathcal{G}}$ for a nodal subgroup associated to e. Then it follows from [5, Lemma 1.10] there exists a *unique* vertical subgroup $\Pi_v \subseteq \Pi_{\mathcal{G}}$ (respectively, $\Pi_w \subseteq \Pi_{\mathcal{G}}$) such that Π_e is contained in $\Pi_v \subseteq \Pi_{\mathcal{G}}$ (respectively, $\Pi_w \subseteq \Pi_{\mathcal{G}}$). Note that Π_v (respectively, Π_w) may be identified with $\Pi_{\mathcal{G}|_v}$ (respectively, $\Pi_{\mathcal{G}|_w}$). Write $\Pi_c \subseteq \Pi_w$ for a cuspidal subgroup associated to c, and $\mathrm{Aut}^{|\mathrm{C}|}(\Pi_v, \Pi_e)$ for the intersection of

$$\{\sigma \in \mathrm{Aut}(\Pi_v) \mid \sigma(\Pi_e) = \Pi_e\},$$

and the subgroup of $\mathrm{Aut}(\Pi_v)$ given by the inverse image of

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v) \subseteq \mathrm{Out}(\Pi_v).$$

Now it follows from [14, Proposition 1.2, (ii)] and [6, Lemma 2.12, (i)] that, for any $\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$, there exists a lifting $\widetilde{\alpha} \in \mathrm{Aut}(\Pi_{\mathcal{G}})$ such that $\widetilde{\alpha}(\Pi_c) = \Pi_c$, and $\widetilde{\alpha} \in \mathrm{Aut}(\Pi_{\mathcal{G}})$ is *uniquely determined* up to composition with an element of $\mathrm{Inn}(\Pi_c \subseteq \Pi_{\mathcal{G}})$. Note that, since α is *graphic* and $\Pi_c \subseteq \Pi_w$, $\widetilde{\alpha}(\Pi_w) = \Pi_w$ (cf. [5, Lemma 1.7]). If $N_c \subseteq \Pi_{\mathcal{G}}$ is the closed normal subgroup of $\Pi_{\mathcal{G}}$ normally generated by Π_c, then it follows immediately from [6, Lemma 4.2] and the *well-known structure of the fundamental group of a hyperbolic curve* that the natural surjection $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}/N_c$ induces an *isomorphism*

$$\Pi_{\mathcal{G}}/N_c \xrightarrow{\sim} \Pi_v,$$

and the composite of $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}/N_c \xrightarrow{\sim} \Pi_v$ induces a surjection $\Pi_w \twoheadrightarrow \Pi_e$. Thus, by identifying $\Pi_{\mathcal{G}}/N_c$ with Π_v by the above isomorphism, the correspondence $\alpha \mapsto \widetilde{\alpha}$ induces the natural homomorphism

$$\delta\colon\ \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Aut}(\Pi_v)$$

which *factors through* $\mathrm{Aut}^{|\mathrm{C}|}(\Pi_v, \Pi_e)$. Moreover, by the definition of $\rho^{\mathrm{Vert}}_{\mathcal{G}}$, we have the following commutative diagram of profinite groups

$$\begin{tikzcd}
\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d, "\delta"'] & \mathrm{Glu}(\mathcal{G}) \arrow[d] \\
\mathrm{Aut}^{|\mathrm{C}|}(\Pi_v, \Pi_e) \arrow[r] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)
\end{tikzcd}$$

where the right-hand vertical arrow is the natural projection, and the lower horizontal arrow is the natural homomorphism induced by $\mathrm{Aut}(\Pi_v) \twoheadrightarrow \mathrm{Out}(\Pi_v)$.

Now, by means of [6, Lemma 4.6], $\delta|_{\mathrm{Dehn}(\mathcal{G})}$ is injective, and $\delta(\mathrm{Dehn}(\mathcal{G}))$ is equal to $\mathrm{Inn}(\Pi_e \subseteq \Pi_v)$. In particular, we obtain the following commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d, "\delta"] & \mathrm{Glu}(\mathcal{G}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Inn}(\Pi_e \subseteq \Pi_v) \arrow[r] & \mathrm{Aut}^{|\mathrm{C}|}(\Pi_v, \Pi_e) \arrow[r] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v) \arrow[r] & 1
\end{tikzcd}$$

where the horizontal sequences are exact (cf. Theorem 1.4, Lemma 2.1), and the left-hand vertical arrow is an *isomorphism*. In particular, the commutative diagram of profinite groups

$$\begin{tikzcd} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d, "\delta"'] & \mathrm{Glu}(\mathcal{G}) \arrow[d] \\ \mathrm{Aut}^{|\mathrm{C}|}(\Pi_v, \Pi_e) \arrow[r] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v) \end{tikzcd}$$

is *cartesian*. Therefore, Lemma 2.2 follows from Lemma 2.1, (ii). This completes the proof of Lemma 2.2. $\square$

Lemma 2.3. *Let $\mathcal{G}$ be a rationally degenerate semi-graph of anabelioids of pro-Σ PSC-type of which any node is of separating type (cf. Definition 1.1, (vi)). Then the homomorphism*

$$\rho^{\mathrm{Vert}}_{\mathcal{G}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Glu}(\mathcal{G})$$

is split surjective.

Proof. We verify Lemma 2.3 by *induction on* $\sharp(\mathrm{Node}(\mathcal{G}))$. If $\sharp(\mathrm{Node}(\mathcal{G})) \leq 1$, then Lemma 2.3 follows from Lemma 2.2. Now suppose that $\sharp(\mathrm{Node}(\mathcal{G})) > 1$, and the *induction hypothesis* is in force. Then, since any node of $\mathcal{G}$ is of separating type, one may verify that there exists a vertex v of $\mathcal{G}$ which is *terminal* (cf. Definition 1.1, (x)). Let $e \in \mathcal{N}(v)$ be the *unique* node which abut to v. One may verify that any node of $\mathcal{G}_{\setminus\{v\}}$ (cf. Definition 1.1, (x)) is *of separating type*. Then, by applying *induction on* $\sharp(\mathrm{Node}(\mathcal{G}))$, the homomorphism $\rho^{\mathrm{Vert}}_{\mathcal{G}_{\setminus\{v\}}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\setminus\{v\}}) \to \mathrm{Glu}(\mathcal{G}_{\setminus\{v\}})$ is *split surjective*, i.e., there exists a homomorphism of profinite groups $s_v \colon \mathrm{Glu}(\mathcal{G}_{\setminus\{v\}}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\setminus\{v\}})$ such that $\rho^{\mathrm{Vert}}_{\mathcal{G}_{\setminus\{v\}}} \circ s_v \colon \mathrm{Glu}(\mathcal{G}_{\setminus\{v\}}) \to \mathrm{Glu}(\mathcal{G}_{\setminus\{v\}})$ is the identity automorphism of $\mathrm{Glu}(\mathcal{G}_{\setminus\{v\}})$. Write $\mathcal{S}_v \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ for the subgroup of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ given by the inverse image of $\mathrm{im}(s_v) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\setminus\{v\}})$ under the natural homomorphism $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\setminus\{v\}})$ (cf. [6, Definition 2.14, (ii)]), and $\mathcal{K}_v \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ for the subgroup of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ given by the intersection of $\mathrm{Dehn}(\mathcal{G})$ and $\mathcal{S}_v$. Then it follows immediately from [6, Theorem 4.8, (iii), and Proposition 4.10, (iv)] that the restriction of the homomorphism $\rho^{\mathrm{Vert}}_{\mathcal{G}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Glu}(\mathcal{G})$ to $\mathcal{S}_v \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ is *surjective*. Moreover, by [6, Theorem 4.8, (iii)], the homomorphism

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\setminus\{v\}}) \times \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_e)$$

given by the natural homomorphisms $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\setminus\{v\}})$ and $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_e)$ (cf. [6, Definition 2.14, (ii)], Definition 1.1, (ix)) induces an *isomorphism*

$$\mathrm{Dehn}(\mathcal{G}) \xrightarrow{\sim} \mathrm{Dehn}(\mathcal{G}_{\setminus\{v\}}) \times \mathrm{Dehn}(\mathcal{G}|_e).$$

Therefore, it follows immediately from Theorem 1.4 that the natural homomorphism $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_e)$ induces the following commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{K}_v \arrow[r] \arrow[d, "\wr"] & \mathcal{S}_v \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d] & \mathrm{Glu}(\mathcal{G}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}|_e) \arrow[r] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_e) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}|_e}"] & \mathrm{Glu}(\mathcal{G}|_e) \arrow[r] & 1
\end{tikzcd}$$

where the horizontal sequences are exact (cf. Theorem 1.4), and the left-hand vertical arrow is an *isomorphism.* In particular, the commutative diagram of profinite groups

$$\begin{tikzcd}
\mathcal{S}_v \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d] & \mathrm{Glu}(\mathcal{G}) \arrow[d] \\
\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_e) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}|_e}"] & \mathrm{Glu}(\mathcal{G}|_e)
\end{tikzcd}$$

is *cartesian.* Thus, it follows from Lemma 2.2 that $\mathcal{S}_v \to \mathrm{Glu}(\mathcal{G})$, hence also $\rho^{\mathrm{Vert}}_{\mathcal{G}}\colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Glu}(\mathcal{G})$, is *split surjective.* This completes the proof of Lemma 2.3. $\square$

Lemma 2.4. *Let $\mathcal{G}$ be a rationally degenerate semi-graph of anabelioids of pro-Σ PSC-type which is cyclically primitive, i.e.,* $\sharp(\mathrm{Node}(\mathcal{G})) = 1$*, and the unique node of $\mathcal{G}$ is not of separating type (*[6, Definition 4.1]*). Then the homomorphism*

$$\rho^{\mathrm{Vert}}_{\mathcal{G}}\colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Glu}(\mathcal{G})$$

is split surjective.

Proof. First, since $\mathcal{G}$ is rationally degenerate which $\sharp(\mathrm{Node}(\mathcal{G})) = 1$, we may check easily that there exists an omittable subset $S \subseteq \mathrm{Cusp}(\mathcal{G})$ (cf. Definition 1.1, (v)) such that the partial compactification $\mathcal{G}_{\bullet S}$ of $\mathcal{G}$ with respect to S (cf. Definition 1.1, (vi)) is *totally degenerate.* Then it follows from [6, Theorem 4.8, (iii), (iv)] that we obtain the following commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d] & \mathrm{Glu}(\mathcal{G}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}_{\bullet S}) \arrow[r] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\bullet S}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}_{\bullet S}}"] & \mathrm{Glu}(\mathcal{G}_{\bullet S}) \arrow[r] & 1
\end{tikzcd}$$

where the horizontal sequences are exact (cf. Theorem 1.4), and the left-hand vertical arrow is an *isomorphism.* In particular, the commutative diagram of profinite groups

$$\begin{tikzcd}
\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d] & \mathrm{Glu}(\mathcal{G}) \arrow[d] \\
\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\bullet S}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}_{\bullet S}}"] & \mathrm{Glu}(\mathcal{G}_{\bullet S})
\end{tikzcd}$$

is *cartesian.* Therefore, to verify Lemma 2.4, by replacing $\mathcal{G}$ by $\mathcal{G}_{\bullet S}$, we may assume that $\mathcal{G}$ is *totally degenerate.*

Next, let $v \in \mathrm{Vert}(\mathcal{G})$ be the unique vertex of $\mathcal{G}$, $e \in \mathrm{Node}(\mathcal{G})$ the unique node of $\mathcal{G}$, and $c \in \mathrm{Cusp}(\mathcal{G})$ the unique cusp of $\mathcal{G}$. Write $\Pi_e \subseteq \Pi_{\mathcal{G}}$ for a nodal subgroup associated to e, $M_{\mathcal{G}} := (\Pi_{\mathcal{G}})^{\mathrm{ab}}$, $M_{\mathcal{G}}^{\mathrm{vert}} \subseteq M_{\mathcal{G}}$ for the $\hat{\mathbb{Z}}^{\Sigma}$-submodule of $M_{\mathcal{G}}$ topologically generated by the images of the vertical subgroups of $\Pi_{\mathcal{G}}$, $M_{\mathcal{G}}^{\mathrm{comb}} := M_{\mathcal{G}}/M_{\mathcal{G}}^{\mathrm{vert}}$, $M_{\mathcal{G}}^{\mathrm{edge}} \subseteq M_{\mathcal{G}}$ for the $\hat{\mathbb{Z}}^{\Sigma}$-submodule of $M_{\mathcal{G}}$ topologically generated by the images of the edge-like subgroups of $\Pi_{\mathcal{G}}$, and $\mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}})$ for

$$\left\{\sigma \in \mathrm{Aut}(M_{\mathcal{G}}) \;\middle|\; \sigma(M_{\mathcal{G}}^{\mathrm{vert}}) = M_{\mathcal{G}}^{\mathrm{vert}},\ \text{and } \sigma \text{ induces the identity of } M_{\mathcal{G}}^{\mathrm{comb}}\right\}.$$

Then it follows from [5, Remark 1.1.1] and the definition of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ that the natural homomorphism $\mathrm{Out}(\Pi_{\mathcal{G}}) \to \mathrm{Aut}(M_{\mathcal{G}})$ induces a homomorphism $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}})$. Also, since $\Pi_e \simeq \hat{\mathbb{Z}}^{\Sigma} \simeq M_{\mathcal{G}}^{\mathrm{comb}}$ and $M_{\mathcal{G}}$ is a *free* $\hat{\mathbb{Z}}^{\Sigma}$-module of rank 2 (cf. [14, Remark 1.1.4]), the composite of $\Pi_e \hookrightarrow \Pi_{\mathcal{G}} \twoheadrightarrow M_{\mathcal{G}}$ induces an isomorphism $\Pi_e \xrightarrow{\sim} M_{\mathcal{G}}^{\mathrm{edge}} = M_{\mathcal{G}}^{\mathrm{vert}}$. In particular, by [6, Corollary 3.9, (iv), (v)], the composite of the natural homomorphism $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}})$ and the natural restricting homomorphism $\mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}}) \to \mathrm{Aut}(M_{\mathcal{G}}^{\mathrm{edge}}) \simeq (\hat{\mathbb{Z}}^{\Sigma})^{\times}$ is *equal to* the composite of $\rho_{\mathcal{G}}^{\mathrm{Vert}}$ and

$$\chi_v \colon \mathrm{Glu}(\mathcal{G}) = \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v) \to (\hat{\mathbb{Z}}^{\Sigma})^{\times}.$$

Thus, we have the following commutative diagram of profinite groups

$$\begin{tikzcd}
\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \arrow[rr, "\rho_{\mathcal{G}}^{\mathrm{Vert}}"] \arrow[d] & & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v) \arrow[d, "\chi_v"] \\
\mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}}) \arrow[r] & \mathrm{Aut}(M_{\mathcal{G}}^{\mathrm{edge}}) \arrow[r, "\sim"] & (\hat{\mathbb{Z}}^{\Sigma})^{\times},
\end{tikzcd}$$

and the natural homomorphism

$$f \colon \mathrm{Dehn}(\mathcal{G}) \longrightarrow \ker(\mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}}) \to \mathrm{Aut}(M_{\mathcal{G}}^{\mathrm{edge}})).$$

Now, by considering the difference of the element of $\ker(\mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}}) \to \mathrm{Aut}(M_{\mathcal{G}}^{\mathrm{edge}}))$ and the identity automorphism of $M_{\mathcal{G}}$, we obtain an *injection*

$$g \colon \ker(\mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}}) \to \mathrm{Aut}(M_{\mathcal{G}}^{\mathrm{edge}})) \to \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Sigma}}(M_{\mathcal{G}}^{\mathrm{comb}}, M_{\mathcal{G}}).$$

Moreover, since $M_{\mathcal{G}}^{\mathrm{edge}}$ is equal to $M_{\mathcal{G}}^{\mathrm{vert}}$, it follows immediately from the definition of $\mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}})$ that g factors through

$$\mathrm{Hom}_{\hat{\mathbb{Z}}^{\Sigma}}(M_{\mathcal{G}}^{\mathrm{comb}}, M_{\mathcal{G}}^{\mathrm{edge}}) \subseteq \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Sigma}}(M_{\mathcal{G}}^{\mathrm{comb}}, M_{\mathcal{G}}).$$

Then, since $\mathcal{G}$ is a semi-graph of anabelioids of pro-Σ PSC-type arising from a 1-pointed singular stable curve of genus 1 which has the unique connected component, by [6, Proposition 5.6, (ii)] and [2, Exposé IX, Théorèm 11.5], the composite $g \circ f$ induces an *isomorphism*

$$\mathrm{Dehn}(\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Sigma}}(M_{\mathcal{G}}^{\mathrm{comb}}, M_{\mathcal{G}}^{\mathrm{edge}}).$$

Therefore, we obtain the following commutative diagram of profinite groups

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}) \arrow[r] \arrow[d, equal] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}) \arrow[r] & \mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}}) \arrow[r] & \mathrm{Aut}(M^{\mathrm{edge}}_{\mathcal{G}}) \arrow[r] & 1 \end{tikzcd}$$

where the horizontal sequences are exact (cf. Theorem 1.4). In particular, the commutative diagram of profinite groups

$$\begin{tikzcd} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v) \arrow[d] \\ \mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}}) \arrow[r] & \mathrm{Aut}(M^{\mathrm{edge}}_{\mathcal{G}}) \end{tikzcd}$$

is *cartesian*. On the other hand, since $M_{\mathcal{G}}$ and $M^{\mathrm{comb}}_{\mathcal{G}} \simeq M_{\mathcal{G}}/M^{\mathrm{edge}}_{\mathcal{G}}$ are *free* $\hat{\mathbb{Z}}^{\Sigma}$-modules (cf. [14, Remark 1.1.4]), by consideration of the definition of $\mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}})$, one may verify easily that

$$\mathrm{Aut}^{|\mathrm{grph}|}(M_{\mathcal{G}}) \longrightarrow \mathrm{Aut}(M^{\mathrm{edge}}_{\mathcal{G}})$$

is *split surjective*. This completes the proof of Lemma 2.4. □

Theorem 2.5. *Let $\mathcal{G}$ be a rationally degenerate semi-graph of anabelioids of pro-Σ PSC-type (cf. Definition 1.1, (iv)). Then the homomorphism*

$$\rho^{\mathrm{Vert}}_{\mathcal{G}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Glu}(\mathcal{G})$$

is split surjective.

Proof. First, we claim that

> Let $e \in \mathrm{Node}(\mathcal{G})$ be a node of $\mathcal{G}$ which is *not of separating type*. Suppose that $\rho^{\mathrm{Vert}}_{\mathcal{G}_{\succ\{e\}}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\succ\{e\}}) \to \mathrm{Glu}(\mathcal{G}_{\succ\{e\}})$ (cf. Definition 1.1, (vii)) is *split surjective*. Then
>
> $$\rho^{\mathrm{Vert}}_{\mathcal{G}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Glu}(\mathcal{G})$$
>
> is *split surjective*.

Indeed, since $\rho^{\mathrm{Vert}}_{\mathcal{G}_{\succ\{e\}}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\succ\{e\}}) \to \mathrm{Glu}(\mathcal{G}_{\succ\{e\}})$ is *split surjective*, there exists a homomorphism of profinite groups $s_e \colon \mathrm{Glu}(\mathcal{G}_{\succ\{e\}}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\succ\{e\}})$ such that $\rho^{\mathrm{Vert}}_{\mathcal{G}_{\succ\{e\}}} \circ s_e \colon \mathrm{Glu}(\mathcal{G}_{\succ\{e\}}) \to \mathrm{Glu}(\mathcal{G}_{\succ\{e\}})$ is the identity automorphism of $\mathrm{Glu}(\mathcal{G}_{\succ\{e\}})$. Write $\mathcal{S}_e \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ for the subgroup of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ given by the inverse image of $\mathrm{im}(s_e) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\succ\{e\}})$ under the natural homomorphism $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\succ\{e\}})$ (cf. Definition 1.1, (vii)), and $\mathcal{K}_e \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ for the subgroup of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ given by the intersection $\mathrm{Dehn}(\mathcal{G})$ and $\mathcal{S}_e$. Then it follows immediately from [6, Theorem 4.8, (iii), and Proposition 4.10, (iv)] that the restriction of the homomorphism $\rho^{\mathrm{Vert}}_{\mathcal{G}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Glu}(\mathcal{G})$ to $\mathcal{S}_e \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ is *surjective*. Moreover, by [6, Theorem 4.8, (iii)], the homomorphism

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\succ\{e\}}) \times \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_e)$$

given by the natural homomorphisms $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\succ\{e\}})$ and $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_e)$ (cf. Definition 1.1, (ix), (xi)) induces an *isomorphism*

$$\mathrm{Dehn}(\mathcal{G}) \xrightarrow{\sim} \mathrm{Dehn}(\mathcal{G}_{\succ\{e\}}) \times \mathrm{Dehn}(\mathcal{G}|_e).$$

Therefore, it follows immediately from Theorem 1.4 that the natural homomorphism $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_e)$ induces the following commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{K}_e \arrow[r] \arrow[d, "\wr"] & \mathcal{S}_e \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d] & \mathrm{Glu}(\mathcal{G}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}|_e) \arrow[r] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_e) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}|_e}"] & \mathrm{Glu}(\mathcal{G}|_e) \arrow[r] & 1
\end{tikzcd}$$

where the horizontal sequences are exact, and the left-hand vertical arrow is an *isomorphism.* In particular, the commutative diagram of profinite groups

$$\begin{tikzcd}
\mathcal{S}_e \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}}"] \arrow[d] & \mathrm{Glu}(\mathcal{G}) \arrow[d] \\
\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_e) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}|_e}"] & \mathrm{Glu}(\mathcal{G}|_e)
\end{tikzcd}$$

is *cartesian.* Thus, it follows from Lemma 2.2 and Lemma 2.4 that $\mathcal{S}_e \to \mathrm{Glu}(\mathcal{G})$, hence also $\rho^{\mathrm{Vert}}_{\mathcal{G}}\colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Glu}(\mathcal{G})$, is *split surjective.* This completes the proof of the claim.

Finally, by the above claim and induction, we may assume that any node of $\mathcal{G}$ is *of separating type.* On the other hand, if any node of $\mathcal{G}$ is *of separating type*, then Theorem 2.5 follows from Lemma 2.3. This completes the proof of Theorem 2.5. $\square$

3. A splitting of the tripod homomorphism

In the present §3, by means of Theorem 2.5, we prove that the tripod homomorphism (cf. Theorem 3.3, below) is *split surjective* (cf. Corollary 3.4, below).

In the present §3, let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$; n a positive integer; Σ a nonempty subset of $\mathfrak{Primes}$ which is equal to $\mathfrak{Primes}$ or satisfies $\sharp(\Sigma) = 1$; k an algebraically closed field of characteristic zero; $(\mathrm{Spec}\, k)^{\log}$ the log scheme obtained by equipping $\mathrm{Spec}\, k$ with the log structure determined by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$; $X^{\log} = X_1^{\log}$ a stable log curve of type (g, r) over $(\mathrm{Spec}\, k)^{\log}$ (cf. [14, §0]). For each $1 \leq m \leq n$, write

$$X_m^{\log}$$

for the m-th *log configuration space* of the stable log curve $X^{\log}$ (cf. [6, §0]), where we think of the factors as being labeled by the element of $\{1, \ldots, m\}$;

$$\Pi_m$$

for the maximal pro-Σ quotient of the kernel of the natural surjection $\pi_1(X_m^{\log}) \twoheadrightarrow \pi_1((\mathrm{Spec}\, k)^{\log})$. (The *theory of fundamental groups of log schemes* is discussed in [10, §4]; [3].) Thus, for each $1 \leq m \leq m' \leq n$, we have a projection

$$p^{\log}_{m'/m} \colon X^{\log}_{m'} \longrightarrow X^{\log}_{m}$$

obtained by forgetting the factors that belong to $\{m+1, \ldots, m'\}$. For each $1 \leq m \leq m' \leq n$, we shall write

$$p^{\Pi}_{m'/m} \colon \Pi_{m'} \longrightarrow \Pi_m$$

for the surjection induced by $p^{\log}_{m'/m}$;

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_m) \subseteq \mathrm{Out}(\Pi_m)$$

for the (closed) subgroup of $\mathrm{Out}(\Pi_m)$ consisting of FC-admissible outomorphisms of Π_m (i.e., arising from automorphisms of Π_m which, for any fiber subgroup $F \subseteq \Pi_m$ of Π_m, preserve F and the set of cuspidal inertia subgroups of F (cf. [15, Definition 1.1, (ii)]));

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_m)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_m)$$

for the (closed) subgroup of $\mathrm{Out}^{\mathrm{FC}}(\Pi_m)$ consisting FC-admissible outomorphisms of Π_m that determine (via the surjection $p^{\Pi}_{n/1} \colon \Pi_n \twoheadrightarrow \Pi_1$) an outomorphism of Π_1 that induces the identity permutation of the set of conjugacy classes of cuspidal inertia subgroups of Π_1 (cf. [15, Definition 1.1, (v)]).

Definition 3.1. We shall write

$$\mathcal{G}_{X^{\log}}$$

for the semi-graph of anabelioids of pro-Σ PSC-type arising from the pointed stable curve X over k determined by the stable log curve $X^{\log}$ over $(\mathrm{Spec}\, k)^{\log}$ (cf. [15, Example 2.5]). Thus, we have a natural outer isomorphism

$$\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}_{X^{\log}}}.$$

Theorem 3.2 (Hoshi-Mochizuki)**.** *Suppose that*

$$n \geq \begin{cases} 4 & \textit{if } r = 0, \\ 3 & \textit{if } r \geq 1. \end{cases}$$

Let $T \subseteq \Pi_3$ be the central $\{1,2,3\}$-tripod, i.e., roughly speaking, the tripod that arises, in the case where the given log stable curve has no nodes, by blowing up the intersection of the three diagonal divisors of the direct product of three copies of the curve (cf. [7, Definition 3.7, (ii)]*). Write*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

for the (closed) subgroup of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ consisting of FC-admissible outomorphism α of Π_n such that the outomorphism of Π_3 determined (via the surjection $p^{\Pi}_{n/3} \colon \Pi_n \twoheadrightarrow \Pi_3$) by α preserves the Π_3-conjugacy class of $T \subseteq \Pi_3$. Then it holds that

$$N_{\Pi_3}(T) = T \times Z_{\Pi_3}(T),$$

and that

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T] = \mathrm{Out}^{\mathrm{FC}}(\Pi_n).$$

Thus, by applying [7, Lemma 3.10, (i)] *to outomorphisms of* Π_3 *determined (via the surjection* $p^{\Pi}_{n/3}\colon \Pi_n \twoheadrightarrow \Pi_3$*) by elements of* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$*, one obtains a natural homomorphism*

$$\mathfrak{T}_T\colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \longrightarrow \mathrm{Out}(T).$$

Write

$$\mathrm{Out}(T)^{\Delta+}$$

for the image of the homomorphism $\mathfrak{T}_T\colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}(T)$*. We shall refer to the resulting surjection* $\mathfrak{T}_T\colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}(T)^{\Delta+}$ *as the tripod homomorphism associated to* Π_n*.*

Proof. Theorem 3.2 follows from [7, Theorem A, (ii), and Theorem C, (i), (ii)]. □

Remark 3.3. In the notation of Theorem 3.2, $\mathrm{Out}(T)^{\Delta+}$ may be naturally identified with the pro-Σ *Grothendieck-Teichmüller group* GT (cf. [7, Theorem C, (iv)], [15, Remark 3.19.1]). In particular, (the isomorphic class of) $\mathrm{Out}(T)^{\Delta+}$ is *independent* of the choice of the triple (g, r, n) and $X^{\log}$.

Corollary 3.4. *Suppose that*

$$n \geq \begin{cases} 4 & \textit{if } r = 0, \\ 3 & \textit{if } r \geq 1. \end{cases}$$

Let $T \subseteq \Pi_n$ *be the central* $\{1, 2, 3\}$*-tripod (cf.* [7, Definition 3.7, (ii)]*). Then the tripod homomorphism*

$$\mathfrak{T}_T\colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \longrightarrow \mathrm{Out}(T)^{\Delta+}$$

is split surjective.

Proof. First, by considering a suitable stable log curve of type (g, r) over $(\mathrm{Spec}\, k)^{\log}$, applying a suitable specialization isomorphism, and means of [7, Propositon 3.24, (i)], to verify Corollary 3.4, we may assume without loss of generality that $\mathcal{G}_{X^{\log}}$ is *totally degenerate.* Moreover, It follows immediately from [4, Lemma 1.4, (i)] and the commutative diagram of [7, Remark 3.19.1] that there exist elements of $\ker(\mathfrak{T}_T)$ that induce, relative to $p^{\Pi}_{n/1}\colon \Pi_n \twoheadrightarrow \Pi_1$, *arbitrary permutations* of the set of conjugacy classes of cuspidal inertia group of Π_1. Therefore, the restriction of the $\mathfrak{T}_T\colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}(T)^{\Delta+}$ to $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is *surjective.* Thus, to verify that the tripod homomorphism $\mathfrak{T}_T\colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}(T)^{\Delta+}$ is *split surjective*, it suffices to show that the restriction $\mathfrak{T}_T\colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}(T)^{\Delta+}$ to $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is *split surjective.*

Next, we consider the homomorphism

$$\rho^{\mathrm{brch}}_n\colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}} \longrightarrow \mathrm{Glu}(\Pi_n)$$

(i.e., roughly speaking, the homomorphism obtained by naturally extending $\rho^{\mathrm{Vert}}_{\mathcal{G}}$ to the closed subgroup of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ consisting of FC-admissible outomorphisms α of Π_n such that the outomorphism of Π_1 determined by α induces the identity automorphism of $\mathrm{Vert}(\mathcal{G})$, $\mathrm{Node}(\mathcal{G})$, and, moreover, fixes each of the branches of every node of $\mathcal{G}$ (cf. [7, Definition 4.11])). Write

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}}_{\mathrm{cusp}} := \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}} \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}},$$

and

$$\mathrm{Glu}(\Pi_n)^{\mathrm{cusp}} := \mathrm{im}(\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}}_{\mathrm{cusp}} \overset{\rho_n^{\mathrm{brch}}}{\to} \mathrm{Glu}(\Pi_n)).$$

(Here, for $v \in \mathrm{Vert}(\mathcal{G}_{X^{\log}})$, we denote by $(\Pi_v)_n \subseteq \Pi_n$ a *configuration space subgroup associated to* v (cf. [7, Definition 4.3]).) Since $\rho_n^{\mathrm{brch}} \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}} \to \mathrm{Glu}(\Pi_n)$ is *surjective* (cf. [7, Theorem 4.14, (iii)]), one may verify from the various definitions involved that $\mathrm{Glu}(\Pi_n)^{\mathrm{cusp}}$ is *equal to*

$$\mathrm{Glu}(\Pi_n) \cap \prod_{v \in \mathrm{Vert}(\mathcal{G}_{X^{\log}})} \mathrm{Out}^{\mathrm{FC}}((\Pi_v)_n)^{\mathrm{cusp}}.$$

Then, since $\mathcal{G}_{X^{\log}}$ is *totally degenerate*, it follows immediately from [7, Theorem 3.18] that there exists an *isomorphism* $\mathrm{Glu}(\Pi_n)^{\mathrm{cusp}} \xrightarrow{\sim} \mathrm{Out}(T)^{\Delta+}$ which *fits into* the following commutative diagram of profinite groups

$$\begin{tikzcd}
\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}}_{\mathrm{cusp}} \arrow[r, two heads, "\rho_n^{\mathrm{brch}}"] \arrow[d, hook] & \mathrm{Glu}(\Pi_n)^{\mathrm{cusp}} \arrow[d, "\wr"] \\
\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \arrow[r, two heads, "\mathfrak{T}_T"] & \mathrm{Out}(T)^{\Delta+}
\end{tikzcd}$$

where the horizontal arrows are surjective, the left-hand vertical arrow is injective, and the right-hand vertical arrow is an isomorphism. Thus, to verify that the restriction of the tripod homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \to \mathrm{Out}(T)^{\Delta+}$ is *split surjective*, it suffices to show that the surjection

$$\rho_n^{\mathrm{brch}} \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}}_{\mathrm{cusp}} \to \mathrm{Glu}(\Pi_n)^{\mathrm{cusp}}$$

is *split surjective*.

Finally, one may verify from [5, Theorem B] and the various definitions involved that the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ determined by $p^{\Pi}_{n/1} \colon \Pi_n \twoheadrightarrow \Pi_1$ induces an *inclusion* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}}_{\mathrm{cusp}} \hookrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{X^{\log}})$, and that this inclusion *fits into* the following commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \arrow[r] \arrow[d, equal] & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}}_{\mathrm{cusp}} \arrow[r, "\rho_n^{\mathrm{brch}}"] \arrow[d, hook] & \mathrm{Glu}(\Pi_n)^{\mathrm{cusp}} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \arrow[r] & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{X^{\log}}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}_{X^{\log}}}"] & \mathrm{Glu}(\mathcal{G}_{X^{\log}}) \arrow[r] & 1
\end{tikzcd}$$

where the horizontal sequences are exact (cf. [7, Theorem 4.14, (iii)], Theorem 1.4), and the vertical arrows are injective. In particular, the commutative diagram of profinite groups

$$\begin{tikzcd}
\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}}_{\mathrm{cusp}} \arrow[r, "\rho_n^{\mathrm{brch}}"] \arrow[d, hook] & \mathrm{Glu}(\Pi_n)^{\mathrm{cusp}} \arrow[d, hook] \\
\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{X^{\log}}) \arrow[r, "\rho^{\mathrm{Vert}}_{\mathcal{G}_{X^{\log}}}"] & \mathrm{Glu}(\mathcal{G}_{X^{\log}})
\end{tikzcd}$$

is *cartesian*. Thus, since $\mathcal{G}_{X^{\log}}$ is *totally degenerate*, it follows from Theorem 2.5 that $\rho_n^{\mathrm{brch}} \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{brch}}_{\mathrm{cusp}} \to \mathrm{Glu}(\Pi_n)^{\mathrm{cusp}}$, hence also $\mathfrak{T}_T \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}(T)^{\Delta+}$, is *split surjective*. This completes the proof of Corollary 3.4. $\square$

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