RIMS-1828

Difference between l-adic Galois representations and pro-l outer Galois representations associated to hyperbolic curves

By

Yu IIJIMA

June 2015

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

DIFFERENCE BETWEEN l-ADIC GALOIS REPRESENTATIONS AND PRO-l OUTER GALOIS REPRESENTATIONS ASSOCIATED TO HYPERBOLIC CURVES

YU IIJIMA

Abstract. Let l be a prime number, and k a field of characteristic zero. In the present paper, we consider the issue of whether or not the image of the pro-l outer Galois representation associated to a hyperbolic curve over k is an l-adic Lie group. In particular, we prove that, if k satisfies a mild assumption concerning l, then the image of the pro-l outer Galois representation associated to a hyperbolic curve over k is *not* an l-adic Lie group. Also, we consider the issue of whether or not the image of the universal pro-l outer monodromy representation of the moduli stack of hyperbolic curves is an l-adic Lie group.

Contents

Introduction

Let l be a prime number, Σ a set of prime numbers containing l, k a field of characteristic zero, $\overline{k}$ an algebraic closure of k, and C a *hyperbolic curve* over k. Write $G_k := \mathrm{Gal}(\overline{k}/k)$, $\Delta_C^{\{l\}}$ for the *pro-$\{l\}$ geometric étale fundamental group* of C, i.e., the maximal pro-$\{l\}$ quotient of the étale fundamental group $\pi_1(C \otimes_k \overline{k})$ of $C \otimes_k \overline{k}$,

$$\rho_C^{\{l\}} : G_k \longrightarrow \mathrm{Out}(\Delta_C^{\{l\}})$$

for the *pro-$\{l\}$ outer Galois representation associated to* C, and

$$\rho_C^{\{l\}\text{-ab}} : G_k \longrightarrow \mathrm{Out}((\Delta_C^{\{l\}})^{\mathrm{ab}})$$

2010 *Mathematics Subject Classification.* Primary 14H30; Secondary 14H25.

Key words and phrases. pro-$\{l\}$ outer Galois representation, hyperbolic curve, l-adic Lie group, l-cyclotomically inertially full, universal pro-$\{l\}$ outer monodromy representation, quasi-$\{l\}$-monodromically full.

for the homomorphism obtained from $\rho_C^{\{l\}}$ and the maximal abelian quotient $\Delta_C^{\{l\}} \twoheadrightarrow (\Delta_C^{\{l\}})^{\mathrm{ab}}$ of $\Delta_C^{\{l\}}$. (Although "pro-$\{l\}$" is often written "pro-l", since we also consider "pro-Σ", we use this notation.) Note that, if C is proper, then $\rho_C^{\{l\}\text{-ab}}$ may be regarded as the l-adic Galois representation obtained from the l-adic Tate module of the Jacobian variety of C.

In the present paper, we consider the natural surjection

$$\mathrm{im}(\rho_C^{\{l\}}) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{im}(\rho_C^{\{l\}\text{-ab}}) .$$

In the early 1990's, research of the kernel of this surjection $\mathrm{im}(\rho_C^{\{l\}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}\text{-ab}})$ was used to study the anabelian geometry (cf., e.g., [22], [20]). If the surjection $\mathrm{im}(\rho_C^{\{l\}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}\text{-ab}})$ is injective, then $\mathrm{im}(\rho_C^{\{l\}})$ is an l-adic Lie group. From this observation, we may pose the following question:

> Do there exist a positive integer n and an *l-adic Galois representation*
> $$\rho_n^l \colon G_k \longrightarrow GL_n(\mathbb{Z}_l)$$
> such that $\ker(\rho_n^l)$ is *equal to* $\ker(\rho_C^{\{l\}})$? In other words, is $\mathrm{im}(\rho_C^{\{l\}})$ an *l-adic Lie group*?

If k is a *number field* and ρ_n^l is obtained from the l-adic Tate module of an *abelian variety* over k, then it was known that $\ker(\rho_n^l)$ is *not equal to* $\ker(\rho_C^{\{l\}})$ (cf. [10, Corollary 1.3]).

The first main result of the present paper is as follows (cf. Theorem 1.10):

Theorem A. *Suppose that k is l-cyclotomically inertially full, i.e., there exists a pair of an injection $\overline{\mathbb{Q}} \hookrightarrow \overline{k}$ and a prime $\mathfrak{l}$ of $\overline{\mathbb{Q}}$ over l such that the intersection of* $\mathrm{im}(G_{k(\mu_{l^\infty})} \to G_{\mathbb{Q}(\mu_{l^\infty})})$ *and the inertia subgroup $I_{\mathfrak{l}} \subseteq G_{\mathbb{Q}}$ of $\mathfrak{l}$ is an open subgroup of $I_{\mathfrak{l}} \cap G_{\mathbb{Q}(\mu_{l^\infty})}$, where $\mu_{l^\infty} \subseteq \overline{\mathbb{Q}}$ is the group of roots of l-power order of unity (cf. Definition 1.8). Then* $\mathrm{im}(\rho_C^{\{l\}})$ *is not an l-adic Lie group.*

In particular, in this case, the natural surjection

$$\mathrm{im}(\rho_C^{\{l\}}) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{im}(\rho_C^{\{l\}\text{-ab}})$$

is not injective.

Theorem A follows from the analysis of the pro-$\{l\}$ outer Galois representation associated to a *split tripod*, i.e., $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Also, we prove a partial generalization of Theorem A for pro-Σ outer Galois representations (cf. Corollary 1.12).

Next, we consider a geometric version of the above question. Let (g, r) be a pair of nonnegative integers such that $2g-2+r > 0$. Write $(\mathcal{M}_{g,r})_k$ for the *moduli stack of r-pointed smooth proper curves of genus g over k whose r marked points are equipped with an ordering*, $\Delta_{g,r}^{\{l\}}$ for the pro-$\{l\}$ completion of the (topological) fundamental group of a topological space obtained by removing r distinct points from a connected orientable compact topological surface of genus g,

$$\rho_{g,r/k}^{\{l\}} \colon \pi_1((\mathcal{M}_{g,r})_k) \longrightarrow \mathrm{Out}(\Delta_{g,r}^{\{l\}})$$

for the *universal pro-$\{l\}$ outer monodromy representation of* $(\mathcal{M}_{g,r})_k$, and

$$\rho^{\{l\}\text{-ab}}_{g,r/k} \colon \pi_1((\mathcal{M}_{g,r})_k) \longrightarrow \mathrm{Aut}((\Delta^{\{l\}}_{g,r})^{\mathrm{ab}})$$

for the homomorphism determined by $\rho^{\{l\}}_{g,r/k}$ and the natural surjection $\Delta^{\{l\}}_{g,r} \twoheadrightarrow (\Delta^{\{l\}}_{g,r})^{\mathrm{ab}}$.

The second main result of the present paper is as follows (cf. Corollary 2.2, Proposition 2.9):

Theorem B. *Suppose that* $3g-3+r>0$. *Then the natural surjection*

$$\mathrm{im}(\rho^{\{l\}}_{g,r/k}) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{im}(\rho^{\{l\}\text{-ab}}_{g,r/k})$$

is not injective.

Suppose, moreover, that either $(g,r) \neq (1,1)$ *or* $l=2$. *Then* $\mathrm{im}(\rho^{\{l\}}_{g,r/k})$ *is not an* l*-adic Lie group.*

The final portion of Theorem B follows from Theorem A and [14, Theorem 3.4]. Also, by means of the results of the *classical anabelian geometry*, we prove the first portion of Theorem B in the case where $(g,r)=(1,1)$ and $l>2$. Finally, we prove a partial generalization of Theorem B for universal pro-Σ outer monodromy representations (cf. Corollary 2.11), and a corollary to Theorem B, which is a partial strengthening of Theorem A (cf. Corollary 2.15).

Acknowledgments. The author would like to thank *Yuichiro Hoshi* for inspiring the author by means of his result given in [10], helpful suggestions, and advice; *Mamoru Asada* and *Hiroaki Nakamura* for explaining to him the unpublished result of Spencer Bloch concerning Proposition 2.9; *Akio Tamagawa* for carefully reading preliminary versions of this paper, giving many suggestions including Lemma 1.7, Corollary 1.12, Remark 2.8, and heartfelt encouragement. This research was partially supported by Grant-in-Aid for JSPS Fellows (KAKENHI No. 14J01306).

Notations and Conventions

Sets: If S is a set, then we shall denote by $\sharp(S)$ the *cardinality* of S.

Numbers: The notation $\mathbb{Z}$ (respectively, $\mathbb{Q}$) will be used to denote the ring of rational integers (respectively, the field of rational numbers). The notation $\mathbb{Z}_{>0}$ will be used to denote the (additive) monoid of positive integers. Let l be a prime number. The notation $\mathbb{Z}_l$ (respectively, $\mathbb{Q}_l$) will be used to denote the l-adic completion of $\mathbb{Z}$ (respectively, $\mathbb{Q}$). We shall refer to a finite extension of $\mathbb{Q}$ as a *number field.*

Profinite groups: Let G be a profinite group. For $x, y \in G$, we shall write $[x,y] := x^{-1}y^{-1}xy \in G$ for the *commutator* of x and y. We shall write G^{ab} for the *abelianization* of G (i.e., the quotient of G by the closure of the commutator subgroup $[G,G]$ of G), and G^{nilp} for the *maximal pro-nilpotent quotient* of G.

If G is a profinite group, then we shall denote by $\mathrm{Aut}(G)$ the group of (continuous) automorphisms of the topological group G, by $\mathrm{Inn}(G)$ the group of inner automorphisms of G, and by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ with respect to the normal subgroup $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$. If, moreover, G is *topologically finitely generated*, then one verifies that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the group $\mathrm{Aut}(G)$, hence also a *profinite topology* on the group $\mathrm{Out}(G)$.

Curves: Let k be a field, X a scheme over k, and (g, r) a pair of nonnegative integers. Then we shall say that X is a *curve* (*of type* (g, r)) over k if there exist a scheme X^{cpt} over k which is smooth, proper, geometrically connected and whose geometric fibers are of dimension 1 and of genus g, and a closed subscheme $D \subseteq X^{\mathrm{cpt}}$ of X^{cpt} which is finite and étale over k of degree r, such that X is isomorphic to $X^{\mathrm{cpt}} \setminus D$ over k. In this case, these X^{cpt} and D are *uniquely determined* by X up to unique canonical isomorphism over k, and we shall refer to X^{cpt} as the *smooth compactification* of X and D as the *divisor of infinity* of X. we shall say that a curve X over k is *split* if the divisor of infinity of X is isomorphic to a disjoint union of copies of $\mathrm{Spec}\, k$ over k.

Let k be a field. We shall say that a scheme X over k is a *hyperbolic curve* over k if there exists a pair (g, r) of nonnegative integers such that $2g - 2 + r > 0$, and that X is a curve of type (g, r) over k.

1. The pro-$\{l\}$ outer Galois representations associated to hyperbolic curves

In the present §1, we recall generalities on the outer Galois representations associated to hyperbolic curves, and prove Theorem A (cf. Theorem 1.10, below) by means of the analysis of the pro-$\{l\}$ outer Galois representation associated to a *split tripod*, i.e., $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Also, we prove a partial generalization of Theorem A for pro-Σ outer Galois representations (cf. Corollary 1.12, below).

Throughout the present paper, let Σ be a nonempty set of prime numbers, l a prime number, k a field of characteristic zero, and $\overline{k}$ an algebraic closure of k. For any extension $k' \subseteq \overline{k}$ of k, write $G_{k'} := \mathrm{Gal}(\overline{k}/k')$. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. For any subfield $K \subseteq \overline{\mathbb{Q}}$ of $\overline{\mathbb{Q}}$, write $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. For a positive integer m, let $\mu_m \subseteq \overline{\mathbb{Q}}$ be the group of m-th roots of unity, and write

$$\mu_{m^\infty} := \bigcup_{n \in \mathbb{Z}_{>0}} \mu_{m^n}.$$

We shall denote by $\chi_l \colon G_{\mathbb{Q}} \to \mathbb{Z}_l^\times$ the *l-adic cyclotomic character* of $G_{\mathbb{Q}}$, i.e., the composite of

$$G_{\mathbb{Q}} \to \mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty})/\mathbb{Q}) \tilde{\to} \mathbb{Z}_l^\times$$

where the left-hand arrow is the natural surjection, and the right-hand arrow is the isomorphism obtained by sending $n \in \mathbb{Z}_l^\times$ to the automorphism of $\mathbb{Q}(\mu_{l^\infty})$ determined by $\mu_{l^\infty} \ni \zeta \mapsto \zeta^n \in \mu_{l^\infty}$. For an m and a $\mathbb{Z}_l$-module A, we shall denote by $A(m)$ the *Tate twist* of A, i.e., $A(m)$ is the $G_{\mathbb{Q}}$-module for which the base module is equal to A and the action of $G_{\mathbb{Q}}$ is determined by, for any $\sigma \in G_{\mathbb{Q}}$ and any $a \in A(m)$, $\sigma \cdot a = \chi_l(\sigma)^m a$. By means of an injection $\overline{\mathbb{Q}} \hookrightarrow \overline{k}$, let us regard $\overline{\mathbb{Q}}$ as a subfield of $\overline{k}$. We shall write $T := \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$, and $T_k := T \otimes_{\mathbb{Q}} k$.

Definition 1.1. Let X be a scheme of finite type, separated, and geometrically connected over k.

(i) We shall write
$$\Delta_X^\Sigma$$
for the *pro-Σ geometric étale fundamental group* of X, i.e., the maximal pro-Σ quotient of the étale fundamental group $\pi_1(X \otimes_k \overline{k})$ of $X \otimes_k \overline{k}$, and
$$\Pi_X^\Sigma$$
for the *geometrically pro-Σ étale fundamental group* of X, i.e., the quotient of the étale fundamental group $\pi_1(X)$ of X by the kernel of the natural surjection $\pi_1(X \otimes_k \overline{k}) \twoheadrightarrow \Delta_X^\Sigma$. (The étale fundamental group of X is defined for the pair of X and a base point of X. However, since the étale fundamental group of X is *independent* of the choice of the base point — up to inner automorphism —, we shall omit the base point.) Thus, we have a natural exact sequence of profinite groups
$$1 \longrightarrow \Delta_X^\Sigma \longrightarrow \Pi_X^\Sigma \longrightarrow G_k \longrightarrow 1 \ .$$

(ii) We shall write
$$\rho_X^\Sigma \colon G_k \longrightarrow \mathrm{Out}(\Delta_X^\Sigma)$$
for the outer Galois representation determined by the exact sequence of (i). We shall refer to ρ_X^Σ as the *pro-Σ outer Galois representation associated to X*. We shall write
$$\rho_X^{\Sigma\text{-}\mathrm{ab}} \colon G_k \longrightarrow \mathrm{Aut}((\Delta_X^\Sigma)^{\mathrm{ab}})$$
$$(\text{resp. } \rho_X^{\Sigma\text{-}\mathrm{nilp}} \colon G_k \longrightarrow \mathrm{Out}((\Delta_X^\Sigma)^{\mathrm{nilp}}))$$
for the Galois representation determined by the exact sequence of (i) and the natural surjection $\Delta_X^\Sigma \twoheadrightarrow (\Delta_X^\Sigma)^{\mathrm{ab}}$ (respectively, the natural surjection $\Delta_X^\Sigma \twoheadrightarrow (\Delta_X^\Sigma)^{\mathrm{nilp}}$).

(iii) We shall write
$$\Omega_X^\Sigma := \overline{k}^{\ker(\rho_X^\Sigma)}, \ \Omega_X^{\Sigma\text{-}\mathrm{ab}} := \overline{k}^{\ker(\rho_X^{\Sigma\text{-}\mathrm{ab}})}, \ \Omega_X^{\Sigma\text{-}\mathrm{nilp}} := \overline{k}^{\ker(\rho_X^{\Sigma\text{-}\mathrm{nilp}})}.$$
Note that, by the definitions of Ω_X^Σ, $\Omega_X^{\Sigma\text{-}\mathrm{ab}}$, and $\Omega_X^{\Sigma\text{-}\mathrm{nilp}}$, the inclusions
$$\Omega_X^{\Sigma\text{-}\mathrm{ab}} \subseteq \Omega_X^{\Sigma\text{-}\mathrm{nilp}} \subseteq \Omega_X^\Sigma$$
hold.

Remark 1.2. Let (g, r) be a pair of nonnegative integers such that $2g-2+r > 0$ and $r \leq 1$, and C a hyperbolic curve over k of type (g, r).

(i) Write J_C for the Jacobian variety of C^{cpt}. Then it follows immediately from the discussion given in [19, §18] and [16, Proposition 9.1] that there exists a natural isomorphism of $(\Delta_C^\Sigma)^{\mathrm{ab}}$ with
$$\Delta_{J_C}^\Sigma = \prod_{p \in \Sigma} T_p(J_C)$$

where $T_p(J_C)$ is the *p-adic Tate module* of J_C. Moreover, one verifies that the Galois representation

$$\rho_C^{\Sigma\text{-}\mathrm{ab}}\colon G_k \longrightarrow \mathrm{Aut}((\Delta_C^{\Sigma})^{\mathrm{ab}})$$

coincides, relative to this isomorphism $(\Delta_C^{\Sigma})^{\mathrm{ab}} \xrightarrow{\sim} \prod_{p\in\Sigma} T_p(J_C)$, with the usual Galois representation $G_k \to \mathrm{Aut}(\prod_{p\in\Sigma} T_p(J_C))$ associated to the abelian variety J_C. Therefore, the equality

$$\Omega_C^{\Sigma\text{-}\mathrm{ab}} = \Omega_{J_C}^{\Sigma}$$

holds, and the Galois extension $\Omega_C^{\Sigma\text{-}\mathrm{ab}}$ of k is generated by the coordinates of all torsion points of J_C of which the prime factors of the order are contained in Σ. In particular, we have an *explicit description* of generators of the Galois extension $\Omega_C^{\Sigma\text{-}\mathrm{ab}}$ over k.

(ii) Suppose that k is a number field. Then it was known that $\Omega_C^{\{l\}}$ does *not coincide* with $\Omega_C^{\{l\}\text{-}\mathrm{ab}}$ (cf. [26, Corollary 4.1, and Remark 4.4]). Also, more strongly, Hoshi proved that, for any abelian variety A over k, $\Omega_C^{\{l\}}$ does *not coincide* with $\Omega_A^{\{l\}}$ (cf. [10, Corollary 1.3]).

Theorem 1.3 (Takao, Hoshi-Mochizuki)**.** *Let C be a hyperbolic curve over k. Then the inclusion*

$$\ker(\rho_C^{\{l\}}) \subseteq \ker(\rho_{T_k}^{\{l\}}),$$

hence also

$$\Omega_{T_k}^{\{l\}} \subseteq \Omega_C^{\{l\}},$$

holds.

Proof. This is a consequence of [26, Theorem 0.5, (2), Remark 0.3, (1), (2)] or [11, Theorem C, (i)]. ☐

Remark 1.4.

(i) In [1, Theorem B], Anderson and Ihara proved that the Galois extension $\Omega_{T_k}^{\{l\}}$ of k is generated by all *higher circular l-units* (cf. [1, p.284, Definition]). In particular, we have an *explicit description* of generators of the Galois extension $\Omega_{T_k}^{\{l\}}$ over k.

(ii) Let (g,r) be a pair of nonnegative integers such that $2g-2+r>0$. Suppose that k is a number field. Then it follows from [8, Theorem C] that there are *only finitely many* isomorphism classes over k of hyperbolic curves C *of type* (g,r) for which $\Omega_C^{\{l\}}$ coincides with $\Omega_{T_k}^{\{l\}}$.

Definition 1.5.

(i) We define the filtration

$$\{\Delta_T^{\{l\}}(m)\}\ (m \in \mathbb{Z}_{>0})$$

of $\Delta_T^{\{l\}}$ by

$$\Delta_T^{\{l\}}(1) := \Delta_T^{\{l\}};$$

$$\Delta_T^{\{l\}}(m) := \text{the closure of } [\Delta_T^{\{l\}}, \Delta_T^{\{l\}}(m-1)] \text{ for } m>1.$$

(ii) Let $\mathfrak{l}$ be a prime of $\overline{\mathbb{Q}}$ over l. Write $I_{\mathfrak{l}} \subseteq G_{\mathbb{Q}}$ for the inertia subgroup of $\mathfrak{l}$. For a positive integer m, we shall write

$$\rho_T^{\{l\},m} \colon G_{\mathbb{Q}} \longrightarrow \mathrm{Out}(\Delta_T^{\{l\}}/\Delta_T^{\{l\}}(m+1))$$

for the outer Galois representation determined by $\rho_T^{\{l\}}$ and the natural surjection $\Delta_T^{\{l\}} \twoheadrightarrow \Delta_T^{\{l\}}/\Delta_T^{\{l\}}(m+1)$,

$$\mathbb{Q}(m) := \overline{\mathbb{Q}}^{\ker(\rho_T^{\{l\},m})}, \ \mathbb{Q}(m)_{\mathfrak{l}} := \overline{\mathbb{Q}}^{\ker(\rho_T^{\{l\},m}) \cap I_{\mathfrak{l}}},$$

$$\mathrm{gr}^m \mathfrak{g} := \mathrm{Gal}(\mathbb{Q}(m+1)/\mathbb{Q}(m)),$$

and

$$\mathrm{gr}^m \mathfrak{h}_{\mathfrak{l}} := \mathrm{Gal}(\mathbb{Q}(m+1)_{\mathfrak{l}}/\mathbb{Q}(m)_{\mathfrak{l}}).$$

Let m be a positive integer. By definition, we may regard $\mathrm{gr}^m \mathfrak{h}_{\mathfrak{l}}$ as a subgroup of $\mathrm{gr}^m \mathfrak{g}$. Also, since $G_{\mathbb{Q}(m)}$ (respectively, $G_{\mathbb{Q}(m)} \cap I_{\mathfrak{l}}$) is a normal subgroup of $G_{\mathbb{Q}}$ (respectively, $I_{\mathfrak{l}}$), we regard $\mathrm{gr}^m \mathfrak{g}$ (respectively, $\mathrm{gr}^m \mathfrak{h}_{\mathfrak{l}}$) as a group with $G_{\mathbb{Q}}$-action (respectively, $I_{\mathfrak{l}}$-action) by the conjugation action of $G_{\mathbb{Q}}$ (respectively, $I_{\mathfrak{l}}$).

Lemma 1.6 (Ihara)**.**

(i) *The equalities*

$$\mathbb{Q}(1) = \mathbb{Q}(\mu_{l^\infty}), \ \bigcup_{m \in \mathbb{Z}_{>0}} \mathbb{Q}(m) = \Omega_T^{\{l\}}$$

hold.

(ii) $\Omega_T^{\{l\}}$ *is a pro-$\{l\}$ extension of $\mathbb{Q}(\mu_{l^\infty})$ which is unramified at every nonarchimedean prime whose residue characteristic is $\neq l$.*

(iii) *For a positive integer m, $\mathrm{gr}^m \mathfrak{g}$ is isomorphic to a finite direct sum of $\mathbb{Z}_l(m)$ as a group with $G_{\mathbb{Q}}$-action.*

Proof. Assertion (i) follows immediately from [12, I, §4], [21, (2.5) Corollary], and [26, Lemma 2.9]. Assertion (ii) follows from [12, I, §3, Theorem 1, (i), and I, §5, Proposition 7, (ii)]. Assertion (iii) follows immediately from [12, I, §5, Proposition 7, (ii), and I, §5, Proposition 8]. □

Lemma 1.7. *Let $\mathfrak{l}$ be a prime of $\overline{\mathbb{Q}}$ over l. Write $I_{\mathfrak{l}} \subseteq G_{\mathbb{Q}}$ for the inertia subgroup of $\mathfrak{l}$. Then there exists a positive integer m_0 such that, for any odd integer $m \geq m_0$, $\mathrm{gr}^m \mathfrak{h}_{\mathfrak{l}} \neq \{0\}$.*

Proof. Let m be an odd integer > 1. We regard $\mathrm{Gal}(\mathbb{Q}(m)/\mathbb{Q}(\mu_{l^\infty}))^{\mathrm{ab}}$ as a $G_{\mathbb{Q}}$-module by the conjugation action of $G_{\mathbb{Q}}$. We write $\Lambda(l) \subseteq \overline{\mathbb{Q}}$ for the maximal pro-$\{l\}$ extension of $\mathbb{Q}(\mu_{l^\infty})$ which is unramified at every nonarchimedean prime whose residue characteristic is $\neq l$, and

$$\mathrm{Res}_m \colon \mathrm{Hom}_{G_{\mathbb{Q}}}(\mathrm{Gal}(\Lambda(l)/\mathbb{Q}(\mu_{l^\infty}))^{\mathrm{ab}}, \mathbb{Z}_l(m)) \to \mathrm{Hom}_{I_{\mathfrak{l}}}(J_{\mathfrak{l}}, \mathbb{Z}_l(m))$$

for the homomorphism obtained by the restriction from $\mathrm{Gal}(\Lambda(l)/\mathbb{Q}(\mu_{l^\infty}))^{\mathrm{ab}}$ to $J_{\mathfrak{l}} := \mathrm{im}(I_{\mathfrak{l}} \cap G_{\mathbb{Q}(\mu_{l^\infty})} \to \mathrm{Gal}(\Lambda(l)/\mathbb{Q}(\mu_{l^\infty}))^{\mathrm{ab}})$. Now we claim that

> There exists a positive integer m_0 such that, for any integer $m \geq m_0$, Res_m is injective.

Indeed, write $X(l)$ for the maximal pro-$\{l\}$ quotient of $\mathrm{Gal}(\Lambda(l)/\mathbb{Q}(\mu_{l^\infty}))^{\mathrm{ab}}/J_{\mathfrak{l}}$. It follows from [23, Propositin 11.1.4] that $X(l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is *of finite dimension over* $\mathbb{Q}_l$. Therefore, by consideration of the *weights* of $X(l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, there exists a positive integer m_0 such that, for any integer $m \geq m_0$,

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(X(l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \mathbb{Q}_l(m)) = \{0\}.$$

This implies the above claim.

Suppose that $m \geq m_0$. Now it follows from [13, Proposition 1] that there exists a *nonzero* element

$$\kappa_m \in \mathrm{Hom}_{G_{\mathbb{Q}}}(G^{\mathrm{ab}}_{\mathbb{Q}(\mu_{l^\infty})}, \mathbb{Z}_l(m))$$

such that $\ker(\kappa_m)$ contains $\mathrm{im}(G_{\mathbb{Q}(m+1)} \to G^{\mathrm{ab}}_{\mathbb{Q}(\mu_{l^\infty})})$. By means of Lemma 1.6, (i), (ii), it follows from the above claim that there exists a *nonzero* element

$$\kappa'_m \in \mathrm{Hom}_{I_{\mathfrak{l}}}(J_{\mathfrak{l}}, \mathbb{Z}_l(m))$$

such that $\ker(\kappa'_m)$ contains $\mathrm{im}(G_{\mathbb{Q}(m+1)} \cap I_{\mathfrak{l}} \to J_{\mathfrak{l}})$. Also, since, for any positive integer $n < m$, $\mathrm{gr}^n \mathfrak{h}_{\mathfrak{l}}$ is isomorphic to a finite direct sum of $\mathbb{Z}_l(n)$ as a $I_{\mathfrak{l}}$-module (cf. Lemma 1.6, (iii)), by consideration of the *weights* of the modules involved, the equality

$$\mathrm{Hom}_{I_{\mathfrak{l}}}((G_{\mathbb{Q}(1)} \cap I_{\mathfrak{l}}/G_{\mathbb{Q}(m)} \cap I_{\mathfrak{l}})^{\mathrm{ab}}, \mathbb{Z}(m)) = \{0\}$$

holds. Thus, the restriction of κ'_m to $\mathrm{im}(G_{\mathbb{Q}(m)} \cap I_{\mathfrak{l}} \to J_{\mathfrak{l}})$, hence also $(G_{\mathbb{Q}(m)} \cap I_{\mathfrak{l}})/(G_{\mathbb{Q}(m+1)} \cap I_{\mathfrak{l}}) = \mathrm{gr}^m \mathfrak{h}_{\mathfrak{l}}$, is *nontrivial.* This complete the proof of Lemma 1.7. ☐

Definition 1.8. We shall say that k is *l-cyclotomically inertially full* if there exists a prime $\mathfrak{l}$ of $\overline{\mathbb{Q}}$ over l such that the intersection of $\mathrm{im}(G_{k(\mu_{l^\infty})} \to G_{\mathbb{Q}(\mu_{l^\infty})})$ and the inertia subgroup $I_{\mathfrak{l}} \subseteq G_{\mathbb{Q}}$ of $\mathfrak{l}$ is an open subgroup of $I_{\mathfrak{l}} \cap G_{\mathbb{Q}(\mu_{l^\infty})}$.

Remark 1.9.

(i) One may verify easily that whether or not k is l-cyclotomically inertially full is *independent* of the choice of an injection $\overline{\mathbb{Q}} \hookrightarrow \overline{k}$.
(ii) If k is *generalized sub-l-adic* (i.e., may be embedded as a subfield of a finitely generated extension of the field of fractions of the ring of Witt vectors with coefficients in an algebraic closure of the finite field of l elements), then k is *l-cyclotomically inertially full.* On the other hand, by an elementary theory of cyclotomic fields, the maximal abelian extension $\mathbb{Q}_{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ of $\mathbb{Q}$ is *l-cyclotomically inertially full, but not generalized sub-l-adic.*

The following result is the main result of the present §1.

Theorem 1.10. *Let C be a hyperbolic curve over k. Suppose that k is l-cyclotomically inertially full (cf. Definition 1.8). Then $\mathrm{im}(\rho_C^{\{l\}})$ is not an l-adic Lie group.*

In particular, in this case, $\Omega_C^{\{l\}}$ does not coincide with $\Omega_C^{\{l\}\text{-ab}}$, and, for any abelian variety A over k, $\Omega_C^{\{l\}}$ does not coincide with $\Omega_A^{\{l\}}$.

Proof. First, we verify the first portion of Theorem 1.10. It follows from Theorem 1.3 that we have a surjection $\mathrm{im}(\rho_C^{\{l\}}) \twoheadrightarrow \mathrm{im}(\rho_{T_k}^{\{l\}})$. Thus, by [5, 9.6 Theorem, (ii)], to verify the first portion of Theorem 1.10, it suffices to verify that $\mathrm{im}(\rho_{T_k}^{\{l\}})$ is not an l-adic Lie group. Next, the natural projection $T_k \to T$ induces the following commutative diagram of profinite groups

$$\begin{tikzcd} G_k \arrow[r] \arrow[d, "\rho_{T_k}^{\{l\}}"'] & G_{\mathbb{Q}} \arrow[d, "\rho_T^{\{l\}}"] \\ \mathrm{Out}(\Delta_{T_k}^{\{l\}}) \arrow[r, "\sim"] & \mathrm{Out}(\Delta_T^{\{l\}}) \end{tikzcd}$$

where the upper arrow is the natural homomorphism, and the lower arrow is the isomorphism determined by the isomorphism $\Delta_{T_k}^{\{l\}} \tilde{\to} \Delta_T^{\{l\}}$ obtained by the natural projection $T_k \to T$. Also, since k is l-cyclotomically inertially full, there exists a prime $\mathfrak{l}$ of $\overline{\mathbb{Q}}$ over l such that the intersection of $\mathrm{im}(G_{k(\mu_{l^\infty})} \to G_{\mathbb{Q}(\mu_{l^\infty})})$ and the inertia subgroup $I_{\mathfrak{l}} \subseteq G_{\mathbb{Q}}$ of $\mathfrak{l}$ is an open subgroup of $I_{\mathfrak{l}} \cap G_{\mathbb{Q}(\mu_{l^\infty})}$. Therefore, by [5, 9.7 Theorem], to verify the first portion of Theorem 1.10, it suffices to verify that $\rho_T^{\{l\}}(I_{\mathfrak{l}} \cap G_{\mathbb{Q}(\mu_{l^\infty})})$ is not an l-adic Lie group. Assume that $\rho_T^{\{l\}}(I_{\mathfrak{l}} \cap G_{\mathbb{Q}(\mu_{l^\infty})})$ is an l-adic Lie group. Then the dimension of $\rho_T^{\{l\}}(I_{\mathfrak{l}} \cap G_{\mathbb{Q}(\mu_{l^\infty})})$ as an l-adic analytic manifold is *finite*. On the other hand, by Lemma 1.7, there exists a positive integer m_0 such that, for any positive odd integer $m \geq m_0$, $\mathrm{gr}^m\, \mathfrak{h}_{\mathfrak{l}} \neq \{0\}$. In particular, by Lemma 1.6, (iii), for any odd integer $m \geq m_0$, the dimension of $\mathrm{gr}^m\, \mathfrak{h}_{\mathfrak{l}}$ as an l-adic analytic manifold is *positive*. Therefore, since $\rho_T^{\{l\}}(I_{\mathfrak{l}} \cap G_{\mathbb{Q}(\mu_{l^\infty})})$ is a *pro-$\{l\}$ group* (cf. [25, Lemma 4.5.5]), it follows immediately from [5, 4.8 Theorem, and 8.36 Theorem] that the dimension of $\rho_T^{\{l\}}(I_{\mathfrak{l}} \cap G_{\mathbb{Q}(\mu_{l^\infty})})$ is *infinite*. This *contradicts* that $\rho_T^{\{l\}}(I_{\mathfrak{l}} \cap G_{\mathbb{Q}(\mu_{l^\infty})})$ is an l-adic Lie group. This completes the proof of the first portion of Theorem 1.10.

Finally, since there exists a positive integer n (respectively, n') which $\mathrm{Aut}((\Delta_C^{\{l\}})^{\mathrm{ab}})$ (respectively, $\mathrm{Aut}(\Delta_A^{\{l\}})$) is isomorphic to $GL_n(\mathbb{Z}_l)$ (respectively, $GL_{n'}(\mathbb{Z}_l)$) (cf., e.g., [18, Remark 1.2.2] (respectively, [19, §18])), $\mathrm{im}(\rho_C^{\{l\}\text{-}\mathrm{ab}})$ (respectively, $\rho_A^{\{l\}}$) is an l-adic Lie groups. Thus, the final portion of Theorem 1.10 follows from the first portion of Theorem 1.10. This completes the proof of Theorem 1.10. □

Remark 1.11. In the notation of Theorem 1.10, our proof of the fact that

$$\Omega_C^{\{l\}\text{-}\mathrm{ab}} \subsetneq \Omega_C^{\{l\}}$$

depends on the analysis of the profinite group $\mathrm{Gal}(\Omega_{T_k}^{\{l\}}/k)$. Thus, a question that may occur to some readers is the following:

(Q): Is $\Omega_C^{\{l\}}$ equal to the composite of $\Omega_C^{\{l\}\text{-}\mathrm{ab}}$ and $\Omega_{T_k}^{\{l\}}$?

If either C is proper or $\sharp(C^{\mathrm{cpt}}(k) \setminus C(k)) = 1$, then it follows from Remark 1.2, (i) and Remark 1.4, (i) that question (Q) is equivalent to the following question:

> Is $\Omega_C^{\{l\}}$ generated by the coordinates of torsion points of l-power order of the Jacobian variety of C^{cpt} and all higher circular l-units over k?

However, in general, question (Q) has a *negative answer* (cf. Corollary 2.15, below).

Corollary 1.12. *Let C be a hyperbolic curve over k. Suppose that l is contained in Σ, and that one of the following conditions is satisfied:*

(a) $\sharp(\Sigma) < \infty$ *and k is l-cyclotomically inertially full.*
(b) *k is a finitely generated extension of $\mathbb{Q}$.*

Then $\Omega_C^{\Sigma\text{-}\mathrm{nilp}}$ does not coincide with $\Omega_C^{\Sigma\text{-}\mathrm{ab}}$.

In particular, in this case, Ω_C^{Σ} does not coincide with $\Omega_C^{\Sigma\text{-}\mathrm{ab}}$.

Proof. For any finite extension K of k, we have the following commutative diagram of profinite groups

$$\begin{tikzcd}
\mathrm{im}(\rho_{C\otimes_k K}^{\Sigma\text{-}\mathrm{nilp}}) \arrow[r, two heads] \arrow[d, hook] & \mathrm{im}(\rho_{C\otimes_k K}^{\Sigma\text{-}\mathrm{ab}}) \arrow[d, hook] \\
\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}}) \arrow[r, two heads] & \mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}})
\end{tikzcd}$$

where the vertical arrows are *injective*. Thus, to verify the first portion of Corollary 1.12, we may replace k by a finite extension of k.

Note that, since $(\Delta_C^{\Sigma})^{\mathrm{nilp}} = \prod_{p\in\Sigma}\Delta_C^{\{p\}}$ (respectively, $(\Delta_C^{\Sigma})^{\mathrm{ab}} = \prod_{p\in\Sigma}(\Delta_C^{\{p\}})^{\mathrm{ab}}$), the natural homomorphism

$$\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}}) \longrightarrow \prod_{p\in\Sigma}\mathrm{im}(\rho_C^{\{p\}})$$

$$(\text{resp. } \mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}}) \longrightarrow \prod_{p\in\Sigma}\mathrm{im}(\rho_C^{\{p\}\text{-}\mathrm{ab}}))$$

induced by the natural surjection $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\{p\}})$ (respectively, $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\{p\}\text{-}\mathrm{ab}})$) for $p\in\Sigma$ is *injective*. Let $S_l^{\Sigma\text{-}\mathrm{nilp}}$ be an l-Sylow subgroup of $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}})$. Write $S_l^{\Sigma\text{-}\mathrm{ab}}$ for the l-Sylow subgroup of $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}})$ which is the image of $S_l^{\Sigma\text{-}\mathrm{nilp}}$ by the the natural surjection $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}})$.

First, we verify the first portion of Corollary 1.12 in the case where condition (a) is satisfied. Assume that $\ker(\rho_C^{\Sigma\text{-}\mathrm{nilp}})$ is equal to $\ker(\rho_C^{\Sigma\text{-}\mathrm{ab}})$. Then, since, for any $p\in\Sigma$, $\mathrm{im}(\rho_C^{\{p\}})$ is an *almost pro-$\{p\}$ group*, i.e., $\mathrm{im}(\rho_C^{\{p\}})$ has an open subgroup which is a *pro-$\{p\}$ group* (cf. [25, Lemma 4.5.5]), and $\sharp(\Sigma) < \infty$, by replacing k by a finite extension of k, we may assume that, for any $p\in\Sigma$, $\mathrm{im}(\rho_C^{\{p\}})$, hence also $\mathrm{im}(\rho_C^{\{p\}\text{-}\mathrm{ab}})$, is a *pro-$\{p\}$ group*. Therefore, it follows from the injection $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}}) \hookrightarrow \prod_{p\in\Sigma}\mathrm{im}(\rho_C^{\{p\}})$ (respectively, $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}}) \hookrightarrow \prod_{p\in\Sigma}\mathrm{im}(\rho_C^{\{p\}\text{-}\mathrm{ab}})$) that the natural surjection $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}})$ (respectively, $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}\text{-}\mathrm{ab}})$) induces an *isomorphism* $S_l^{\Sigma\text{-}\mathrm{nilp}} \xrightarrow{\sim} \mathrm{im}(\rho_C^{\{l\}})$ (respectively, $S_l^{\Sigma\text{-}\mathrm{ab}} \xrightarrow{\sim} \mathrm{im}(\rho_C^{\{l\}\text{-}\mathrm{ab}})$). In particular, since the natural surjection $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}})$ is an *isomorphism*, the natural surjection $\mathrm{im}(\rho_C^{\{l\}}) \xrightarrow{\sim} \mathrm{im}(\rho_C^{\{l\}\text{-}\mathrm{ab}})$ is an *isomorphism* . This contradicts Theorem 1.10. This completes the proof of the first portion of Corollary 1.12 in the case where condition (a) is satisfied.

Next, we verify the first portion of Corollary 1.12 in the case where condition (b) is satisfied. Assume that $\ker(\rho_C^{\Sigma\text{-}\mathrm{nilp}})$ is equal to $\ker(\rho_C^{\Sigma\text{-}\mathrm{ab}})$. Then, by replacing k by a finite extension of k, it follows from [25, Lemma 4.5.5], and [17, Theorem 4.12] that we may assume that $\mathrm{im}(\rho_C^{\{l\}})$ is a *pro-*$\{l\}$ *group* which is *slim* (i.e., any open subgroup of $\mathrm{im}(\rho_C^{\{l\}})$ is *center-free*). Also, since, for $p \in \Sigma$, $(\Delta_C^{\{p\}})^{\mathrm{ab}}$ is *torsion-free* (cf., e.g., [18, Remark 1.2.2]), it follows from [4, Corollary 4.6] or [6, Theorem 1.1] that, by replacing k by a finite extension of k, we may assume that the natural injection $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}}) \hookrightarrow \prod_{p\in\Sigma} \mathrm{im}(\rho_C^{\{p\}\text{-}\mathrm{ab}})$ is an *isomorphism.* In particular, by replacing $S_l^{\Sigma\text{-}\mathrm{nilp}}$ by a suitable l-Sylow subgroup of $\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}})$, we may assume that there exists an l-Sylow subgroup $S_l^{\{p\}}$of $\mathrm{im}(\rho_C^{\{p\}\text{-}\mathrm{ab}})$ for each $p \in \Sigma$ such that

$$S_l^{\Sigma\text{-}\mathrm{ab}} = \prod_{p\in\Sigma} S_l^{\{p\}}.$$

Note that, since $\mathrm{im}(\rho_C^{\{l\}})$ is a *pro-*$\{l\}$ *group*, the restriction of the composite of

$$\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}}) \xrightarrow{\sim} \mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}})$$

to $S_l^{\Sigma\text{-}\mathrm{ab}}$ is *surjective.* Also, for $p \in \Sigma \setminus \{l\}$, since $S_l^{\{p\}} \times \prod_{q\in\Sigma\setminus\{p\}}\{1\}$ is a *finite normal subgroup* (cf. [25, Lemma 4.5.5]) of $S_l^{\Sigma\text{-}\mathrm{ab}}$, the image of the composite of

$$\rho_p\colon S_l^{\{p\}} \times \prod_{q\in\Sigma\setminus\{p\}} \{1\} \hookrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}}) \xrightarrow{\sim} \mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}})$$

is a *finite normal subgroup* of $\mathrm{im}(\rho_C^{\{l\}})$. Thus, since a finite normal subgroup of a slim profinite group is *trivial* (cf., e.g., [18, §0]), for $p \in \Sigma \setminus \{l\}$, the image of ρ_p is *trivial.* Therefore, the restriction of the composite of

$$\mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{ab}}) \xrightarrow{\sim} \mathrm{im}(\rho_C^{\Sigma\text{-}\mathrm{nilp}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}})$$

to $S_l^{\{l\}} \times \prod_{p\in\Sigma\setminus\{l\}}\{1\}$ is *surjective.* On the other hand, since there exists a positive integer n which $\mathrm{Aut}((\Delta_C^{\{l\}})^{\mathrm{ab}})$ is isomorphic to $GL_n(\mathbb{Z}_l)$ (cf., e.g., [18, Remark 1.2.2]), $S_l^{\{l\}} \simeq S_l^{\{l\}} \times \prod_{p\in\Sigma\setminus\{l\}}\{1\}$ ($\hookrightarrow \mathrm{Aut}((\Delta_C^{\{l\}})^{\mathrm{ab}})$) is an *$l$-adic Lie group.* This and [5, Theorem 9.6, (ii)] *contradict* Theorem 1.10 . This completes the proof of the first portion of Corollary 1.12 in the case where condition (b) is satisfied.

Finally, the final portion of Corollary 1.12 follows from the first portion of Corollary 1.12 (cf. Definition 1.1, (iii)). This completes the proof of Corollary 1.12. □

By modifying the argument used in the proof of Corollary 1.12 in the case where condition (b) is satisfied, we may obtain the following proposition which is a partial result in the case where k is a *finitely generated extension of an algebraically closed field*. This result will be used in §2.

Proposition 1.13. *Let C be a hyperbolic curve over k. Suppose that l is contained in Σ, that k is a finitely generated extension of an algebraically closed field, and that $\Omega_C^{\{l\}}$ does not coincide with $\Omega_C^{\{l\}\text{-ab}}$. Then $\Omega_C^{\Sigma\text{-nilp}}$ does not coincide with $\Omega_C^{\Sigma\text{-ab}}$.*

In particular, in this case, Ω_C^{Σ} does not coincide with $\Omega_C^{\Sigma\text{-ab}}$.

Proof. First, by a standard argument in algebraic geometry, we may find a quadruplet (F, F_0, k_0, C_0) where $F \subseteq k$ is an algebraically closed field which k is a finitely generated extension of F, $F_0 \subseteq k$ is a finitely generated extension of $\mathbb{Q}$, $k_0 \subseteq k$ is a finitely generated extension of F_0 which is linearly disjoint with F over F_0 and the composite of k_0 and F is equal to k, and C_0 is a hyperbolic curve over k_0 which $C_0 \otimes_{k_0} k$ is *isomorphic to C* over k. Let $\overline{F_0} \subseteq F$ be an algebraic closure of F_0. Write k_1 for the composite of k_0 and $\overline{F_0}$, and $C_1 := C_0 \otimes_{k_0} k_1$. Then the natural morphism $C_0 \otimes_{k_0} k \to C_0 \otimes_{k_0} k_1$ induces the following commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \Delta_C^{\Sigma} \arrow[r] \arrow[d, "\wr"] & \Pi_C^{\Sigma} \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \Delta_{C_1}^{\Sigma} \arrow[r] & \Pi_{C_1}^{\Sigma} \arrow[r] & G_{k_1} \arrow[r] & 1
\end{tikzcd}$$

where the horizontal sequences are exact, and the left-hand vertical arrow is an isomorphism. This commutative diagram of profinite groups induces natural injections $\mathrm{im}(\rho_C^{\Sigma\text{-nilp}}) \hookrightarrow \mathrm{im}(\rho_{C_1}^{\Sigma\text{-nilp}})$, $\mathrm{im}(\rho_C^{\Sigma\text{-ab}}) \hookrightarrow \mathrm{im}(\rho_{C_1}^{\Sigma\text{-ab}})$. Note that, since k_1 is linearly disjoint with F over $\overline{F_0}$, $G_k \to G_{k_1}$ is *surjective*. Thus, $\mathrm{im}(\rho_C^{\Sigma\text{-nilp}}) \hookrightarrow \mathrm{im}(\rho_{C_1}^{\Sigma\text{-nilp}})$ and $\mathrm{im}(\rho_C^{\Sigma\text{-ab}}) \hookrightarrow \mathrm{im}(\rho_{C_1}^{\Sigma\text{-ab}})$ are *isomorphisms*. Also, by replacing Σ by $\{l\}$ in above argument, the natural injections $\mathrm{im}(\rho_C^{\{l\}}) \hookrightarrow \mathrm{im}(\rho_{C_1}^{\{l\}})$ and $\mathrm{im}(\rho_C^{\{l\}\text{-ab}}) \hookrightarrow \mathrm{im}(\rho_{C_1}^{\{l\}\text{-ab}})$ are *isomorphisms*. In particular, since $\mathrm{im}(\rho_C^{\{l\}})$ is not an *l-adic Lie group*, $\mathrm{im}(\rho_{C_1}^{\{l\}})$ is also not an *l-adic Lie group*. Therefore, to verify that the natural surjection $\mathrm{im}(\rho_C^{\Sigma\text{-nilp}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-ab}})$ is not injective, we may replace C (respectively, k) by C_1 (respectively, k_1). Suppose that $C = C_1$ and $k = k_1$. Also, it follows from [25, Lemma 4.5.5] and [17, Theorem 4.12] that, by replacing k_0 by a finite extension of k_0 (cf. the first paragraph of the proof of Corollary 1.12), we may assume that $\mathrm{im}(\rho_{C_0}^{\{l\}})$ is a *pro-$\{l\}$ group* and *slim*. Moreover, since, for $p \in \Sigma$, $(\Delta_C^{\{p\}})^{\mathrm{ab}}$ is *torsion-free* (cf., e.g., [18, Remark 1.2.2]), it follows from [4, Corollary 4.6] or [3, Theorem 1.2] that, by replacing k_0 by a finite extension of k_0, the natural injection $\mathrm{im}(\rho_C^{\Sigma\text{-ab}}) \hookrightarrow \prod_{p\in\Sigma} \mathrm{im}(\rho_C^{\{p\}\text{-ab}})$ is an *isomorphism*. Since we may regard G_k as a normal closed subgroup of G_{k_0}, we may regard $\mathrm{im}(\rho_C^{\{l\}})$ (respectively, $\mathrm{im}(\rho_C^{\Sigma\text{-ab}})$; $\mathrm{im}(\rho_C^{\{p\}\text{-ab}})$ for each $p \in \Sigma$) as a normal closed subgroup $\mathrm{im}(\rho_{C_0}^{\{l\}})$ (respectively, $\mathrm{im}(\rho_{C_0}^{\Sigma\text{-ab}})$; $\mathrm{im}(\rho_{C_0}^{\{p\}\text{-ab}})$ for each $p \in \Sigma$). In particular, $\mathrm{im}(\rho_C^{\{l\}})$, hence also $\mathrm{im}(\rho_C^{\{l\}\text{-ab}})$, is also a *pro-$\{l\}$ group*.

Next, assume that the natural surjection $\mathrm{im}(\rho_C^{\Sigma\text{-nilp}}) \twoheadrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-ab}})$ is an isomorphism. Let p be an element of $\Sigma\setminus\{l\}$. Since $\mathrm{im}(\rho_C^{\{p\}\text{-ab}})\times\prod_{q\in\Sigma\setminus\{p\}}\{1\}$ is a normal closed subgroup of $\prod_{q\in\Sigma}\mathrm{im}(\rho_{C_0}^{\{q\}\text{-ab}})$, $\mathrm{im}(\rho_C^{\{p\}\text{-ab}})\times\prod_{q\in\Sigma\setminus\{p\}}\{1\}$ is a normal closed subgroup of $\mathrm{im}(\rho_{C_0}^{\Sigma\text{-ab}})(\subseteq \prod_{q\in\Sigma}\mathrm{im}(\rho_{C_0}^{\{q\}\text{-ab}}))$. Thus, the image of the composite of

$$\mathrm{im}(\rho_C^{\{p\}\text{-ab}})\times\prod_{q\in\Sigma\setminus\{p\}}\{1\}\hookrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-ab}})\xleftarrow{\sim}\mathrm{im}(\rho_C^{\Sigma\text{-nilp}})\twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}})$$

where the right-hand arrow is the natural surjection, is a normal closed subgroup of $\mathrm{im}(\rho_{C_0}^{\{l\}})$. On the other hand, since $\mathrm{im}(\rho_C^{\{l\}})$ is a pro-$\{l\}$ group, the image of the composite of

$$\mathrm{im}(\rho_C^{\{p\}\text{-ab}})\times\prod_{q\in\Sigma\setminus\{p\}}\{1\}\hookrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-ab}})\xleftarrow{\sim}\mathrm{im}(\rho_C^{\Sigma\text{-nilp}})\twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}})$$

is equal to the image of the composite of

$$S_l^{\{p\}}\times\prod_{q\in\Sigma\setminus\{p\}}\{1\}\hookrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-ab}})\xleftarrow{\sim}\mathrm{im}(\rho_C^{\Sigma\text{-nilp}})\twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}})$$

(where $S_l^{\{p\}}$ is an l-Sylow subgroup of $\mathrm{im}(\rho_C^{\{p\}\text{-ab}})$). Also, since $S_l^{\{p\}}$ is a *finite group* (cf. [25, Lemma 4.5.5]), the image of composite of

$$\mathrm{im}(\rho_C^{\{p\}\text{-ab}})\times\prod_{q\in\Sigma\setminus\{p\}}\{1\}\hookrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-ab}})\xleftarrow{\sim}\mathrm{im}(\rho_C^{\Sigma\text{-nilp}})\twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}})$$

is a *finite normal subgroup* of $\mathrm{im}(\rho_{C_0}^{\{l\}})$. Thus, since a finite normal subgroup of a slim profinite group is *trivial* (cf., e.g., [18, §0]) and $\mathrm{im}(\rho_{C_0}^{\{l\}})$ is slim, the image of the composite of

$$\mathrm{im}(\rho_C^{\{p\}\text{-ab}})\times\prod_{q\in\Sigma\setminus\{p\}}\{1\}\hookrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-ab}})\xleftarrow{\sim}\mathrm{im}(\rho_C^{\Sigma\text{-nilp}})\twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}})$$

is *trivial*. Therefore, since

$$\mathrm{im}(\rho_C^{\{l\}\text{-ab}})\times\prod_{q\in\Sigma\setminus\{l\}}S_l^{\{q\}}$$

(where, for $q\in\Sigma\setminus\{l\}$, $S_l^{\{q\}}$ is an l-Sylow subgroup of $\mathrm{im}(\rho_C^{\{q\}\text{-ab}})$) is an l-Sylow subgroup of $\mathrm{im}(\rho_C^{\Sigma\text{-ab}})$, the composite of

$$\mathrm{im}(\rho_C^{\{l\}\text{-ab}})\times\prod_{q\in\Sigma\setminus\{l\}}\{1\}\hookrightarrow \mathrm{im}(\rho_C^{\Sigma\text{-ab}})\xleftarrow{\sim}\mathrm{im}(\rho_C^{\Sigma\text{-nilp}})\twoheadrightarrow \mathrm{im}(\rho_C^{\{l\}})$$

is *surjective*. In particular, we have the surjection $\mathrm{im}(\rho_C^{\{l\}\text{-ab}})\twoheadrightarrow\mathrm{im}(\rho_C^{\{l\}})$. However, we may verify immediately that this surjection $\mathrm{im}(\rho_C^{\{l\}\text{-ab}})\twoheadrightarrow\mathrm{im}(\rho_C^{\{l\}})$ is the *inverse map* of the natural surjection $\mathrm{im}(\rho_C^{\{l\}})\twoheadrightarrow\mathrm{im}(\rho_C^{\{l\}\text{-ab}})$. This *contradict* the condition that $\Omega_C^{\{l\}}$ does not coincide with $\Omega_C^{\{l\}\text{-ab}}$. This completes the proof of the first portion of Proposition 1.13.

Finally, the final portion of Proposition 1.13 follows from the first portion of Proposition 1.13 (cf. Definition 1.1, (iii)). This completes the proof of Proposition 1.13. ∎

2. The universal pro-$\{l\}$ outer monodromy representation of the moduli stack of hyperbolic curves

In the present §2, we recall generalities on the universal pro-$\{l\}$ outer monodromy representation of the moduli stack of hyperbolic curves, and prove Theorem B (cf. Corollary 2.2, Proposition 2.9, below). Moreover, we prove a partial generalization of Theorem B for universal pro-Σ outer monodromy representations (cf. Corollary 2.11, below), and a corollary to Theorem B, which is a partial strengthening of Theorem A (cf. Corollary 2.15, below).

We maintain the notation of the preceding §1. Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$.

Definition 2.1.

(i) We shall denote by $(\mathcal{M}_{g,r})_k$ the moduli stack of r-pointed smooth proper curves of genus g over k whose r marked points are equipped with an ordering. We shall regard $(\mathcal{M}_{g,r+1})_k$ as an algebraic stack over $(\mathcal{M}_{g,r})_k$ by the (1-)morphism $(\mathcal{M}_{g,r+1})_k \to (\mathcal{M}_{g,r})_k$ obtained from forgetting the last marked point.

(ii) We shall write $\Delta_{g,r}$ for the kernel of the surjection of profinite groups $\pi_1((\mathcal{M}_{g,r+1})_k) \to \pi_1((\mathcal{M}_{g,r})_k)$ determined by $(\mathcal{M}_{g,r+1})_k \to (\mathcal{M}_{g,r})_k$. It is well-known that $\Delta_{g,r}$ is naturally isomorphic to the profinite completion of the (topological) fundamental group $\Delta^{\mathrm{top}}_{g,r}$ of a topological space obtained by removing r distinct points from a connected orientable compact topological surface of genus g. Thus, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_{g,r} \longrightarrow \pi_1((\mathcal{M}_{g,r+1})_k) \longrightarrow \pi_1((\mathcal{M}_{g,r})_k) \longrightarrow 1 \; .$$

We shall write

$$\Delta^{\Sigma}_{g,r}$$

for the maximal pro-Σ quotient of $\Delta_{g,r}$,

$$\rho^{\Sigma}_{g,r/k} \colon \pi_1((\mathcal{M}_{g,r})_k) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{g,r})$$

for the natural homomorphism determined by the above exact sequence of profinite groups and the natural surjection $\Delta_{g,r} \twoheadrightarrow \Delta^{\Sigma}_{g,r}$,

$$\rho^{\Sigma\text{-}\mathrm{ab}}_{g,r/k} : \pi_1((\mathcal{M}_{g,r})_k) \longrightarrow \mathrm{Aut}((\Delta^{\Sigma}_{g,r})^{\mathrm{ab}})$$

for the natural homomorphism determined by $\rho^{\Sigma}_{g,r/k}$ and the natural surjection $\Delta^{\Sigma}_{g,r} \to (\Delta^{\Sigma}_{g,r})^{\mathrm{ab}}$, and

$$\rho^{\Sigma\text{-}\mathrm{nilp}}_{g,r/k} : \pi_1((\mathcal{M}_{g,r})_k) \longrightarrow \mathrm{Aut}((\Delta^{\Sigma}_{g,r})^{\mathrm{nilp}})$$

for the natural homomorphism determined by $\rho^{\Sigma}_{g,r/k}$ and the natural surjection $\Delta^{\Sigma}_{g,r} \to (\Delta^{\Sigma}_{g,r})^{\mathrm{nilp}}$. We shall refer to $\rho^{\Sigma}_{g,r/k}$ as the *universal pro-Σ outer monodromy representation of* $(\mathcal{M}_{g,r})_k$.

(iii) We shall write

$$\Gamma_{g,r}$$

for the kernel of the surjection of profinite groups $\pi_1((\mathcal{M}_{g,r})_k) \to G_k$ determined by the natural morphism $(\mathcal{M}_{g,r})_k \to \operatorname{Spec} k$. Thus, a natural exact sequence of profinite groups

$$1 \longrightarrow \varGamma_{g,r} \longrightarrow \pi_1((\mathcal{M}_{g,r})_k) \longrightarrow G_k \longrightarrow 1 .$$

Note that it follows from the argument used in [24] that $\varGamma$ is naturally isomorphic to the profinite completion of the (*pure*) *mapping class group* $\mathrm{MCG}_{g,r}$ *of type* (g,r), i.e., the group of isotopy classes of orientation-preserving automorphisms of a topological space obtained by removing r distinct points from a connected orientable compact topological surface of genus g that fix each removed point, and the restriction of $\rho^{\varSigma}_{g,r/k}$ (respectively, $\rho^{\varSigma\text{-}\mathrm{ab}}_{g,r/k}$; $\rho^{\varSigma\text{-}\mathrm{nilp}}_{g,r/k}$) to $\varGamma_{g,r}$ may be regarded as the homomorphism obtained from the natural homomorphism

$$\mathrm{MCG}_{g,r} \longrightarrow \mathrm{Out}(\varDelta^{\mathrm{top}}_{g,r})$$

and the pro-$\varSigma$ completion of $\varDelta^{\mathrm{top}}_{g,r}$ (respectively, the abelianization of the pro-$\varSigma$ completion of $\varDelta^{\mathrm{top}}_{g,r}$; the maximal pro-nilpotent quotient of the pro-$\varSigma$ completion of $\varDelta^{\mathrm{top}}_{g,r}$). In particular, the properties of the profinite group $\rho^{\varSigma}_{g,r/k}(\varGamma_{g,r})$, the natural surjection $\rho^{\varSigma}_{g,r/k}(\varGamma_{g,r}) \twoheadrightarrow \rho^{\varSigma\text{-}\mathrm{ab}}_{g,r/k}(\varGamma_{g,r})$, and $\rho^{\varSigma\text{-}\mathrm{nilp}}_{g,r/k}(\varGamma_{g,r}) \twoheadrightarrow \rho^{\varSigma\text{-}\mathrm{ab}}_{g,r/k}(\varGamma_{g,r})$ as topological groups are *independent* of the choice of the field k of characteristic zero.

Now we prove a part of the main result of the present §2 as a corollary of Theorem 1.10 and [14, Theorem 3.4].

Corollary 2.2. *Suppose that* $3g-3+r>0$*, and that either* $(g,r) \neq (1,1)$ *or* $l=2$*. Then an open subgroup of the profinite group* $\mathrm{im}(\rho^{\{l\}}_{g,r/k})$ *is not an* l*-adic Lie group.*

In particular, in this case, the restriction of the natural surjection

$$\mathrm{im}(\rho^{\{l\}}_{g,r/k}) \twoheadrightarrow \mathrm{im}(\rho^{\{l\}\text{-}\mathrm{ab}}_{g,r/k})$$

to an open subgroup of $\mathrm{im}(\rho^{\{l\}}_{g,r/k})$ *is not injective.*

Proof. First, we verify the first portion of Corollary 2.2. By means of [5, 9.7 Theorem], to verify the first portion of Corollary 2.2, it suffices to show that $\mathrm{im}(\rho^{\{l\}}_{g,r/k})$ is not an l-adic Lie group. Moreover, since a closed subgroup of an l-adic Lie group is also an l-adic Lie group (cf. [5, 9.6 Theorem, (i)]), to verify the first portion of Corollary 2.2, it suffices to show that $\rho^{\{l\}}_{g,r/k}(\varGamma_{g,r})$ is not an l-adic Lie group. Assume that $\rho^{\{l\}}_{g,r/k}(\varGamma_{g,r})$ is an l-adic Lie group. Then, since $\rho^{\{l\}}_{g,r/k}(\varGamma_{g,r})$ is compact, $\rho^{\{l\}}_{g,r/k}(\varGamma_{g,r})$ is a *compact* l*-adic Lie group*. Thus, it follows from [5, 8.35 Corollary, and 9.6 Theorem, (i), (ii)] that $\mathrm{Out}(\rho^{\{l\}}_{g,r/k}(\varGamma_{g,r}))$ is an l-adic Lie group. On the other hand, by [14, Theorem 3.4], we have an *injection*

$$\mathrm{Gal}(\varOmega^{\{l\}}_T/\mathbb{Q}) \hookrightarrow \mathrm{Out}(\rho^{\{l\}}_{g,r/k}(\varGamma_{g,r})) .$$

However, by Theorem 1.10, $\mathrm{Gal}(\Omega_T^{\{l\}}/\mathbb{Q})$ is *not* an l-adic Lie group. This and [5, 9.6 Theorem, (i)] *contradict* that $\mathrm{Out}(\rho_{g,r/k}^{\{l\}}(\Gamma_{g,r}))$ is an l-adic Lie group. This completes the first portion of Corollary 2.2.

Finally, since there exists a positive integer n which $\mathrm{Aut}((\Delta_{g,r}^{\{l\}})^{\mathrm{ab}})$ is isomorphic to $GL_n(\mathbb{Z}_l)$ (cf., e.g., [18, Remark 1.2.2]), the final portion of Corollary 2.2 follows from the first portion of Corollary 2.2 in the case where k is algebraically closed. This completes the proof of Corollary 2.2. $\square$

Remark 2.3.

(i) In the notation of Corollary 2.2, suppose that g is not equal to 1. Then it seems that Corollary 2.2 follows from [2, Theorem B] and the computation of the rank of the graded Lie algebra associated to a central filtration of mapping class groups, *without using* Theorem 1.10.

(ii) In the notation of Corollary 2.2, since $\Gamma_{0,3} = \{1\}$, it is immediate that a result similar to the results stated in Corollary 2.2 *does not hold in the case where* $(g,r) = (0,3)$ *and* k *is algebraically closed.* On the other hand, it is not clear to the author at the time of writing whether or not a result similar to the results stated in the first portion of Corollary 2.2 *holds in the case where* $(g,r) = (1,1)$ *and* $l > 2$. Nevertheless, we are able to obtain a result similar to the results stated in the final portion of Corollary 2.2*, even in the case where* $(g,r) = (1,1)$ *and* $l > 2$ (cf. Proposition 2.9, below).

Definition 2.4.

(i) We shall denote by

$$\mathcal{S}(G_k, (\Delta_{g,r}^{\Sigma})^{\mathrm{nilp}})$$

$$(\text{resp. } \mathcal{S}(G_k, (\Delta_{g,r}^{\Sigma})^{\mathrm{ab}}))$$

the set of homomorphisms $G_k \to \mathrm{im}(\rho_{g,r/k}^{\Sigma\text{-}\mathrm{nilp}})$ (respectively, $G_k \to \mathrm{im}(\rho_{g,r/k}^{\Sigma\text{-}\mathrm{ab}})$) obtained by the composite of a section of $\pi_1((\mathcal{M}_{g,r})_k) \twoheadrightarrow G_k$ and the homomorphism $\rho_{g,r/k}^{\Sigma\text{-}\mathrm{nilp}}$ (respectively, $\rho_{g,r/k}^{\Sigma\text{-}\mathrm{ab}}$) considered up to $\rho_{g,r/k}^{\Sigma\text{-}\mathrm{nilp}}(\Gamma_{g,r})$-inner automorphism (respectively, $\rho_{g,r/k}^{\Sigma\text{-}\mathrm{ab}}(\Gamma_{g,r})$-inner automorphism).

(ii) Let C be a split hyperbolic curve of type (g,r) over k, and $\mathfrak{O}_C$ an ordering of $C^{\mathrm{cpt}}(k)\setminus C(k)$, i.e., a bijection $C^{\mathrm{cpt}}(k)\setminus C(k) \xrightarrow{\sim} \{1,2,\dots,r\}$. The classifying (1-)morphism $\mathrm{Spec}\, k \to (\mathcal{M}_{g,r})_k$ of the pair $(C, \mathfrak{O}_C)$ determines — up to $\Gamma_{g,r}$-inner automorphism — a section of the natural surjection $\pi_1((\mathcal{M}_{g,r})_k) \twoheadrightarrow G_k$. We shall write

$$s_{(C,\mathfrak{O}_C)} \colon G_k \longrightarrow \pi_1((\mathcal{M}_{g,r})_k)$$

for this section. If $r \leq 1$, then the choice of the ordering of $C^{\mathrm{cpt}}(k) \setminus C(k)$ is *unique*. Therefore, if $r \leq 1$, we shall write s_C for $s_{(C,\mathfrak{O}_C)}$.

(iii) We shall write

$$\varphi_{g,r}^{\Sigma\text{-}\mathrm{nilp}} \colon (\mathcal{M}_{g,r})_k(k) \longrightarrow \mathcal{S}(G_k, (\Delta_{g,r}^{\Sigma})^{\mathrm{nilp}})$$

$$(\text{resp. } \varphi_{g,r}^{\Sigma\text{-}\mathrm{ab}} \colon (\mathcal{M}_{g,r})_k(k) \longrightarrow \mathcal{S}(G_k, (\Delta_{g,r}^{\Sigma})^{\mathrm{ab}}))$$

for the map determined by sending the pair of a split hyperbolic curve C of type (g, r) over k and an ordering $\mathfrak{O}_C$ of $C^{\mathrm{cpt}}(k) \setminus C(k)$ to the composite of $s_{(C,\mathfrak{O}_C)}$ (cf. (ii)) and the homomorphism $\rho^{\Sigma\text{-}\mathrm{nilp}}_{g,r/k}$ (respectively, $\rho^{\Sigma\text{-}\mathrm{ab}}_{g,r/k}$). (Here, by consideration of isomorphic classes, we regard $(\mathcal{M}_{g,r})_k(k)$ as a set.)

Remark 2.5. Let C be a split hyperbolic curve of type (g, r) over k, and $\mathfrak{O}_C$ an ordering of $C^{\mathrm{cpt}}(k) \setminus C(k)$.

(i) Then it follows immediately from [9, Lemma 20] that the cartesian (1-)diagram of algebraic stacks

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\ (\mathcal{M}_{g,r+1})_k \arrow[r] & (\mathcal{M}_{g,r})_k \end{tikzcd}$$

determined by the classifying (1-)morphism $\operatorname{Spec} k \to (\mathcal{M}_{g,r})_k$ of the pair of $(C, \mathfrak{O}_C)$ induces the following commutative diagram of profinite groups

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(C \otimes_k \overline{k}) \arrow[r] \arrow[d, "\wr"] & \pi_1(C) \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d, "s_{(C,\mathfrak{O}_C)}"] & 1 \\ 1 \arrow[r] & \Delta_{g,r} \arrow[r] \arrow[d, two heads] & \pi_1((\mathcal{M}_{g,r+1})_k) \arrow[r] \arrow[d, two heads, "\rho^{\{l\}}_{g,r+1/k}"] & \pi_1((\mathcal{M}_{g,r})_k) \arrow[r] \arrow[d, two heads, "\rho^{\{l\}}_{g,r/k}"] & 1 \\ 1 \arrow[r] & \Delta^{\{l\}}_{g,r} \arrow[r] & \mathrm{im}(\rho^{\{l\}}_{g,r+1/k}) \arrow[r] & \mathrm{im}(\rho^{\{l\}}_{g,r/k}) \arrow[r] & 1 \end{tikzcd}$$

where the horizontal sequences are exact, the lower vertical arrows are surjective, and the upper left-hand vertical arrow is an *isomorphism*, and the following commutative diagram of profinite groups

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(C \otimes_k \overline{k}) \arrow[r] \arrow[d, two heads] & \pi_1(C) \arrow[r] \arrow[d, two heads] & G_k \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \Delta^{\{l\}}_C \arrow[r] \arrow[d, "\wr"] & \Pi^{\{l\}}_C \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d, "\rho^{\{l\}}_{g,r/k} \circ s_{(C,\mathfrak{O}_C)}"] & 1 \\ 1 \arrow[r] & \Delta^{\{l\}}_{g,r} \arrow[r] & \mathrm{im}(\rho^{\{l\}}_{g,r+1/k}) \arrow[r] & \mathrm{im}(\rho^{\{l\}}_{g,r/k}) \arrow[r] & 1 \end{tikzcd}$$

where the horizontal sequences are exact, the upper vertical arrows are surjective, and the lower left-hand vertical arrow is an *isomorphism.* In particular, we have an *isomorphism*

$$\Pi^{\{l\}}_C \xrightarrow{\sim} \mathrm{im}(\rho^{\{l\}}_{g,r+1/k}) \times_{\mathrm{im}(\rho^{\{l\}}_{g,r/k})} G_k$$

over G_k.

(ii) Moreover, it follows from [26, Theorem 0.5, (2)] that there exists a surjection $\mathrm{im}(\rho^{\{l\}}_{g,r/k}) \twoheadrightarrow \mathrm{im}(\rho^{\{l\}}_{T_k})$ which *fits into* the following commutative diagram of profinite groups

$$\begin{tikzcd} G_k \arrow[r, two heads] \arrow[d, "s_{\mathfrak{O}_C}"'] & \mathrm{im}(\rho^{\{l\}}_C) \arrow[r, two heads] \arrow[d, hook] & \mathrm{im}(\rho^{\{l\}}_{T_k}) \\ \pi_1((\mathcal{M}_{g,})_k) \arrow[r, two heads] & \mathrm{im}(\rho^{\{l\}}_{g,r/k}) \arrow[ur] & \end{tikzcd}$$

where the middle vertical arrow is the homomorphism determined by the above second commutative diagram of profinite groups, and the right-hand horizontal arrow is the homomorphism determined by the inclusion $\ker(\rho^{\{l\}}_C) \subseteq \ker(\rho^{\{l\}}_{T_k})$ (cf. Theorem 1.3).

Lemma 2.6. *Suppose that k is generalized sub-l-adic (i.e., may be embedded as a subfield of a finitely generated extension of the field of fractions of the ring of Witt vectors with coefficients in an algebraic closure of the finite field of l elements), and that l is contained in Σ. Then the map*

$$\varphi^{\Sigma\text{-}\mathrm{nilp}}_{g,r} \colon (\mathcal{M}_{g,r})_k(k) \longrightarrow \mathcal{S}(G_k, (\Delta^{\Sigma}_{g,r})^{\mathrm{nilp}})$$

(cf. Definition 2.4, (iii)) is injective.

Proof. First, one may verify that the natural map

$$\mathcal{S}(G_k, (\Delta^{\Sigma}_{g,r})^{\mathrm{nilp}}) \to \mathcal{S}(G_k, \Delta^{\{l\}}_{g,r})$$

determined by the natural surjection $\mathrm{im}(\rho^{\Sigma\text{-}\mathrm{nilp}}_{g,r/k}) \twoheadrightarrow \mathrm{im}(\rho^{\{l\}}_{g,r/k})$ *fits into* the following commutative diagram of sets

$$\begin{tikzcd} (\mathcal{M}_{g,r})_k(k) \arrow[r, "\varphi^{\Sigma\text{-}\mathrm{nilp}}_{g,r}"] \arrow[dr, "\varphi^{\{l\}\text{-}\mathrm{nilp}}_{g,r}"'] & \mathcal{S}(G_k, (\Delta^{\Sigma}_{g,r})^{\mathrm{nilp}}) \arrow[d] \\ & \mathcal{S}(G_k, \Delta^{\{l\}}_{g,r}). \end{tikzcd}$$

Thus, to verify Lemma 2.6, one may assume that Σ is equal to $\{l\}$. Next, let s and s' be elements of $(\mathcal{M}_{g,r})_k(k)$ such that $\varphi^{\{l\}\text{-}\mathrm{nilp}}_{g,r}(s)$ is equal to $\varphi^{\{l\}\text{-}\mathrm{nilp}}_{g,r}(s')$. Write $(C_s, \mathfrak{O}_s)$ (respectively, $(C_{s'}, \mathfrak{O}_{s'})$) for the pair of the hyperbolic curve C_s (respectively, $C_{s'}$) of type (g, r) over k and the ordering of $C^{\mathrm{cpt}}_s(k) \setminus C_s(k)$ (respectively, $C^{\mathrm{cpt}}_{s'}(k) \setminus C_{s'}(k)$) determined by the pulling back the (1-)morphism $(\mathcal{M}_{g,r+1})_k \to (\mathcal{M}_{g,r})_k$ via $s\colon \operatorname{Spec} k \to (\mathcal{M}_{g,r})_k$ (respectively, $s'\colon \operatorname{Spec} k \to (\mathcal{M}_{g,r})_k$). Then, since $\varphi^{\{l\}\text{-}\mathrm{nilp}}_{g,r}(s)$ is equal to $\varphi^{\{l\}\text{-}\mathrm{nilp}}_{g,r}(s')$, it follows from Remark 2.5, (i) that we obtain an *isomorphism*

$$\Pi^{\{l\}}_{C_s} \xrightarrow{\sim} \Pi^{\{l\}}_{C_{s'}}$$

over G_k. Thus, by consideration of the correspondence of the *cuspidal inertia subgroups* determined by the above isomorphism, it follows from [17, Theorem 4.12] that s is *equal to* s'. This completes the proof of Lemma 2.6. $\square$

Lemma 2.7. *Suppose that there exists a prime number p which is not contained in Σ. Then there exists a finite extension K of k such that the map*

$$\varphi_{1,1}^{\Sigma\text{-ab}} \colon (\mathcal{M}_{1,1})_K(K) \longrightarrow \mathcal{S}(G_K, (\Delta_{1,1}^{\Sigma})^{\mathrm{ab}})$$

(cf. Definition 2.4, (iii)) is not injective.

Proof. First, by replacing k by a finite extension of k, one may assume that there exists an elliptic curve (E, O) over k such that $\mathrm{End}_k(E)$ is *isomorphic to* $\mathbb{Z}$, and that any torsion point of E of order p is *k-rational*. Write (E', O') for the elliptic curve over k determined by the quotient of E by the subgroup scheme D generated by $t \in E(k)$ which is of order p, and $f \colon (E, O) \to (E', O')$ for the isogeny over k determined by the quotient map.

Next, write $\mathcal{M}_0(p)$ for the moduli stack of pairs $(\mathfrak{E}, \mathfrak{D})$ where $\mathfrak{E}$ is an elliptic curve and $\mathfrak{D}$ is a cyclic subgroup scheme of $\mathfrak{E}$ of order p over k; $\mathcal{C}_0(p) \to \mathcal{M}_0(p)$ for the the family of hyperbolic curves of type $(1,1)$ determined by pulling back the (1-)morphism $(\mathcal{M}_{1,2})_k \to (\mathcal{M}_{1,1})_k$ via the natural (1-)morphism $\mathcal{M}_0(p) \to (\mathcal{M}_{1,1})_k$; the $\mathcal{E} \to \mathcal{M}_0(p)$ for the family of elliptic curves determined by the family of hyperbolic curves $\mathcal{C}_0(p) \to \mathcal{M}_0(p)$ of type $(1,1)$; $\mathcal{D} \subseteq \mathcal{E}$ for the universal cyclic subgroup of order p over $\mathcal{M}_0(p)$; $\mathcal{E}' \to \mathcal{M}_0(p)$ for the quotient of $\mathcal{E}$ by $\mathcal{D} \subseteq \mathcal{E}$ over $\mathcal{M}_0(p)$; $\mathcal{U} := \mathcal{E} \setminus \mathcal{D}$; $\mathcal{C}_0(p)' \to \mathcal{M}_0(p)$ for the family of hyperbolic curves of type $(1,1)$ determined by the image of the restriction of the quotient (1-)morphism $\mathcal{E} \to \mathcal{E}'$ to $\mathcal{U}$; $i \colon \operatorname{Spec} k \to \mathcal{M}_0(p)$ for the classifying (1-)morphism of the pair (E, D); $s \colon G_k \to \pi_1(\mathcal{M}_0(p))$ for the homomorphism determined by i. Thus, by consideration of the definitions of $\mathcal{C}_0(p)'$, i, and s, there exist the following *cartesian* (1-)diagrams of algebraic stacks

$$\begin{tikzcd} E \setminus \{O\} \arrow[r] \arrow[d] & \mathcal{C}_0(p) \arrow[d] \\ \operatorname{Spec} k \arrow[r, "i"] & \mathcal{M}_0(p), \end{tikzcd} \qquad \begin{tikzcd} E' \setminus \{O'\} \arrow[r] \arrow[d] & \mathcal{C}_0(p)' \arrow[d] \\ \operatorname{Spec} k \arrow[r, "i"] & \mathcal{M}_0(p), \end{tikzcd}$$

hence also the following *cartesian* diagrams of profinite groups

$$\begin{tikzcd} \pi_1(E \setminus \{O\}) \arrow[r] \arrow[d] & \pi_1(\mathcal{C}_0(p)) \arrow[d] \\ G_k \arrow[r, "s"] & \pi_1(\mathcal{M}_0(p)), \end{tikzcd} \qquad \begin{tikzcd} \pi_1(E' \setminus \{O'\}) \arrow[r] \arrow[d] & \pi_1(\mathcal{C}_0(p)') \arrow[d] \\ G_k \arrow[r, "s"] & \pi_1(\mathcal{M}_0(p)). \end{tikzcd}$$

Also, the following commutative (1-)diagram of algebraic stacks

$$\begin{tikzcd} \mathcal{C}_0(p) \arrow[dr] & \\ \mathcal{U} \arrow[u, hook] \arrow[r] \arrow[d] & \mathcal{M}_0(p) \\ \mathcal{C}_0(p)' \arrow[ur] & \end{tikzcd}$$

where the upper vertical arrow is the natural embedding, induces the following commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \Delta_{1,1} \arrow[r] & \pi_1(\mathcal{C}_0(p)) \arrow[r] & \pi_1(\mathcal{M}_0(p)) \arrow[r] & 1 \\
1 \arrow[r] & \Delta_{1,p} \arrow[r] \arrow[u, two heads] \arrow[d, hook] & \pi_1(\mathcal{U}) \arrow[r] \arrow[u, two heads] \arrow[d, hook] & \pi_1(\mathcal{M}_0(p)) \arrow[r] \arrow[u, equal] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta_{1,1} \arrow[r] & \pi_1(\mathcal{C}_0(p)') \arrow[r] & \pi_1(\mathcal{M}_0(p)) \arrow[r] & 1
\end{tikzcd}$$

where the horizontal sequences are exact, the upper vertical arrows are surjective, and the lower vertical arrows are injective. Moreover, since p is *not contained in* Σ, one may verify that the left-hand vertical arrows of the above commutative diagram of profinite groups induce the following commutative diagram of profinite groups

$$\begin{tikzcd}
\Delta_{1,1} \arrow[d, two heads] & \Delta_{1,p} \arrow[l, two heads] \arrow[r, hook] \arrow[d, two heads] & \Delta_{1,1} \arrow[d, two heads] \\
\Delta_{1,1}^{\Sigma} \arrow[d, two heads] & \Delta_{1,p}^{\Sigma} \arrow[l, two heads] \arrow[r, two heads] & \Delta_{1,1}^{\Sigma} \arrow[d, two heads] \\
(\Delta_{1,1}^{\Sigma})^{\mathrm{ab}} \arrow[rr, equal] & & (\Delta_{1,1}^{\Sigma})^{\mathrm{ab}}.
\end{tikzcd}$$

Thus, it follows from the above commutative diagrams of profinite groups that $\varphi_{1,1}^{\Sigma\text{-}\mathrm{ab}}(s_{E\setminus\{O\}})$ is *equal to* $\varphi_{1,1}^{\Sigma\text{-}\mathrm{ab}}(s_{E'\setminus\{O'\}})$. Assume that $E \setminus \{O\}$ is isomorphic to $E' \setminus \{O'\}$. Then we have an isomorphism of elliptic curves $i\colon (E', O') \xrightarrow{\sim} (E, O)$ over k. Thus, we have the isogeny $i \circ f\colon (E, O) \to (E, O)$ over k of degree p. However, this *contradicts* that $\mathrm{End}_k(E)$ is isomorphic to $\mathbb{Z}$. Therefore, $E \setminus \{O\}$ is *not isomorphic to* $E' \setminus \{O'\}$, and $\varphi_{1,1}^{\Sigma\text{-}\mathrm{ab}}(s_{E\setminus\{O\}})$ is *equal to* $\varphi_{1,1}^{\Sigma\text{-}\mathrm{ab}}(s_{E'\setminus\{O'\}})$. This completes the proof of Lemma 2.7. $\square$

Remark 2.8. In Lemma 2.7, Tamagawa pointed out that the condition that $p \notin \Sigma$ is *not necessary.* Indeed, write $\hat{\mathbb{Z}}^{\Sigma}$ for the pro-Σ completion of $\mathbb{Z}$. First, by means of the natural isomorphism

$$SL_2(\hat{\mathbb{Z}}^{\Sigma}) \xrightarrow{\sim} \prod_{q\in\Sigma} SL_2(\mathbb{Z}_q),$$

we may verify that the natural map

$$\phi^{\Sigma\text{-}\mathrm{ab}}\colon \mathcal{S}(G_k, (\Delta_{1,1}^{\Sigma})^{\mathrm{ab}}) \longrightarrow \prod_{q\in\Sigma} \mathcal{S}(G_k, (\Delta_{1,1}^{\{q\}})^{\mathrm{ab}})$$

determined by the natural surjection $\mathrm{im}(\rho_{g,r/k}^{\Sigma\text{-}\mathrm{ab}}) \twoheadrightarrow \mathrm{im}(\rho_{g,r/k}^{\{q\}\text{-}\mathrm{ab}})$ for each $q \in \Sigma$ is *injective.* On the other hand, by the *theory of complex multiplications of elliptic curves,* and replacing k by a finite extension of k, there exists a quadruplet

$$((E, O), (E', O'), f\colon (E, O) \to (E', O'), g\colon (E, O) \to (E', O'))$$

where (E,O) and (E',O') are elliptic curves over k which E is *not isomorphic to* E' over k, $f\colon (E,O) \to (E',O')$ and $g\colon (E,O) \to (E',O')$ are isogenies whose kernels are cyclic, and $\deg(f)$ is *prime to* $\deg(g)$. Write Σ' (respectively, Σ'') for the set of prime numbers which are not prime factors of $\deg(f)$ (respectively, $\deg(g)$). In particular, $\Sigma := \Sigma' \cup \Sigma''$ is *equal to* the set of prime numbers. Then, by replacing $\mathcal{M}_0(p)$ by $\mathcal{M}_0(\deg(f))$ (respectively, $\mathcal{M}_0(\deg(g))$) (which is the moduli stack of pairs $(\mathfrak{E},\mathfrak{D})$ where $\mathfrak{E}$ is an elliptic curve and $\mathfrak{D}$ is a cyclic subgroup scheme of $\mathfrak{E}$ of order $\deg(f)$ (respectively, $\deg(g)$) over k) in the argument used in the proof of Lemma 2.7, we may verify that $\varphi_{1,1}^{\Sigma'\text{-}\mathrm{ab}}(s_{E\setminus\{O\}})$ (respectively, $\varphi_{1,1}^{\Sigma''\text{-}\mathrm{ab}}(s_{E\setminus\{O\}})$) is equal to $\varphi_{1,1}^{\Sigma'\text{-}\mathrm{ab}}(s_{E'\setminus\{O'\}})$ (respectively, $\varphi_{1,1}^{\Sigma''\text{-}\mathrm{ab}}(s_{E'\setminus\{O'\}})$). Therefore, it follows immediately from the *injectivity* of $\phi^{\Sigma\text{-}\mathrm{ab}}$ that $\varphi_{1,1}^{\Sigma\text{-}\mathrm{ab}}(s_{E\setminus\{O\}})$ is equal to $\varphi_{1,1}^{\Sigma\text{-}\mathrm{ab}}(s_{E'\setminus\{O'\}})$. Since E is *not isomorphic to* E' over k, this completes the proof of Lemma 2.7 in the case where Σ is the set of prime numbers.

Proposition 2.9. *The restriction of the natural surjection*

$$\mathrm{im}(\rho_{1,1/k}^{\{l\}}) \twoheadrightarrow \mathrm{im}(\rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}})$$

to an open subgroup of $\mathrm{im}(\rho_{1,1/k}^{\{l\}})$ *is not injective.*

Proof. Since $\ker(\mathrm{im}(\rho_{1,1/k}^{\{l\}}) \twoheadrightarrow \mathrm{im}(\rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}}))$ is *torsion-free* (cf. [2, Theorem 4, (ii), (iii)]), to verify Proposition 2.9, it suffices to verify that the natural surjection $\mathrm{im}(\rho_{1,1/k}^{\{l\}}) \twoheadrightarrow \mathrm{im}(\rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}})$ is not injective. Therefore, to verify Proposition 2.9, it suffices to verify that the natural surjection $\rho_{1,1/k}^{\{l\}}(\Gamma_{1,1}) \twoheadrightarrow \rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}}(\Gamma_{1,1})$ is not injective.

Note that, to verify Proposition 2.9, we may assume without loss of generality that k is a *number field* (cf. Definition 2.1). Assume that the natural surjection $\rho_{1,1/k}^{\{l\}}(\Gamma_{1,1}) \twoheadrightarrow \rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}}(\Gamma_{1,1})$ is injective. Write

$$\mathrm{ar}(\rho_{1,1/k}^{\{l\}}) := \mathrm{im}(\rho_{1,1/k}^{\{l\}})/\rho_{1,1/k}^{\{l\}}(\Gamma_{1,1}),$$

and

$$\mathrm{ar}(\rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}}) := \mathrm{im}(\rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}})/\rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}}(\Gamma_{1,1}).$$

Then the natural surjection $\mathrm{im}(\rho_{1,1/k}^{\{l\}}) \twoheadrightarrow \mathrm{im}(\rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}})$ induces the the following commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \rho_{1,1/k}^{\{l\}}(\Gamma_{1,1}) \arrow[r] \arrow[d] & \mathrm{im}(\rho_{1,1/k}^{\{l\}}) \arrow[r] \arrow[d] & \mathrm{ar}(\rho_{1,1/k}^{\{l\}}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}}(\Gamma_{1,1}) \arrow[r] & \mathrm{im}(\rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}}) \arrow[r] & \mathrm{ar}(\rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}}) \arrow[r] & 1
\end{tikzcd}$$

where the horizontal sequences are exact, and the vertical arrows are surjective. Now since the natural surjection $\rho_{1,1/k}^{\{l\}}(\Gamma_{1,1}) \twoheadrightarrow \rho_{1,1/k}^{\{l\}\text{-}\mathrm{ab}}(\Gamma_{1,1})$ is injective, the right square of the above commutative diagram is *cartesian*. Note that, by the definition of $\varGamma_{1,1}$, for a section s of the natural homomorphism $\pi_1((\mathcal{M}_{1,1})_k) \to G_k$, the composite of

$$G_k \overset{s}{\to} \pi_1((\mathcal{M}_{1,1})_k) \to \mathrm{ar}(\rho^{\{l\}}_{1,1/k}),$$

hence also

$$G_k \overset{s}{\to} \pi_1((\mathcal{M}_{1,1})_k) \to \mathrm{ar}(\rho^{\{l\}\text{-}\mathrm{ab}}_{1,1/k}),$$

is *independent* of the choice of s. In particular, by means of the above *cartesian* square

$$\begin{tikzcd}
\mathrm{im}(\rho^{\{l\}}_{1,1/k}) \arrow[r] \arrow[d] & \mathrm{ar}(\rho^{\{l\}}_{1,1/k}) \arrow[d] \\
\mathrm{im}(\rho^{\{l\}\text{-}\mathrm{ab}}_{1,1/k}) \arrow[r] & \mathrm{ar}(\rho^{\{l\}\text{-}\mathrm{ab}}_{1,1/k}),
\end{tikzcd}$$

we may verify that the natural map $\mathcal{S}(G_k, \varDelta^{\{l\}}_{1,1}) \to \mathcal{S}(G_k, (\varDelta^{\{l\}}_{1,1})^{\mathrm{ab}})$ determined by the natural surjection $\mathrm{im}(\rho^{\{l\}}_{1,1/k}) \twoheadrightarrow \mathrm{im}(\rho^{\{l\}\text{-}\mathrm{ab}}_{1,1/k})$ is *injective.* Since this injection *fits into* the following commutative diagram of sets

$$\begin{tikzcd}
(\mathcal{M}_{1,1})_k(k) \arrow[r, "\varphi^{\{l\}\text{-}\mathrm{nilp}}_{1,1}"] \arrow[dr, "\varphi^{\{l\}\text{-}\mathrm{ab}}_{1,1}"'] & \mathcal{S}(G_k, \varDelta^{\{l\}}_{1,1}) \arrow[d] \\
& \mathcal{S}(G_k, (\varDelta^{\{l\}}_{1,1})^{\mathrm{ab}}),
\end{tikzcd}$$

it follows from Lemma 2.6 that $\varphi^{\{l\}\text{-}\mathrm{ab}}_{1,1}$ is *injective.* However, by replacing k by a finite extension of k, this *contradicts* Lemma 2.7. This completes the proof of Proposition 2.9. $\square$

Remark 2.10. As written in [2, p.34], Proposition 2.9 is well-known at least for experts. On the other hand, our proof of Proposition 2.9 differs somewhat from the proof expected in [2, p.34] which was explained to the author by Asada and Nakamura. Also, at the time of writing, the proof of Proposition 2.9 was not available in published form. Thus, the author gave the proof of Proposition 2.9 in this paper.

Corollary 2.11. *Suppose that* $3g - 3 + r > 0$*, i.e.,* (g, r) *is not equal to* $(0, 3)$*. Then the natural surjection*

$$\mathrm{im}(\rho^{\varSigma\text{-}\mathrm{nilp}}_{g,r/k}) \twoheadrightarrow \mathrm{im}(\rho^{\varSigma\text{-}\mathrm{ab}}_{g,r/k})$$

is not injective.

In particular, in this case, the natural surjection

$$\mathrm{im}(\rho^{\varSigma}_{g,r/k}) \twoheadrightarrow \mathrm{im}(\rho^{\varSigma\text{-}\mathrm{ab}}_{g,r/k})$$

is not injective.

Proof. To verify the first portion of Corollary 2.11, it suffices to verify that the natural surjection $\rho^{\varSigma\text{-}\mathrm{nilp}}_{g,r/k}(\varGamma_{g,r}) \twoheadrightarrow \rho^{\varSigma\text{-}\mathrm{ab}}_{g,r/k}(\varGamma_{g,r})$ is not injective. First, we verify Corollary 2.11 in the case where either $(g, r) \neq (1, 1)$ or $2 \in \varSigma$. We may assume without loss of generality that k is *algebraically closed.*

Let $\mathcal{M} \to (\mathcal{M}_{g,r})_k$ be a connected finite étale covering of $(\mathcal{M}_{g,r})_k$ which is representable by a *scheme* (cf., e.g., [7, Proposition 7.2]). Write $k(\mathcal{M})$ for the function field of $\mathcal{M}$, $\overline{k(\mathcal{M})}$ for an algebraic closure of $k(\mathcal{M})$, and $\mathcal{C}$ for the hyperbolic curve over $k(\mathcal{M})$ of type (g, r) determined by the pulling-back of the composite of

$$\operatorname{Spec} k(\mathcal{M}) \to \mathcal{M} \to (\mathcal{M}_{g,r})_k$$

via the (1-)morphism $(\mathcal{M}_{g,r+1})_k \to (\mathcal{M}_{g,r})_k$. Then we have the following commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathcal{C} \otimes_{k(\mathcal{M})} \overline{k(\mathcal{M})}) \arrow[r] \arrow[d, "\wr"] & \pi_1(\mathcal{C}) \arrow[r] \arrow[d] & G_{k(\mathcal{M})} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \varDelta_{g,r} \arrow[r] \arrow[d, two heads] & \varGamma_{g,r+1} \arrow[r] \arrow[d, two heads] & \varGamma_{g,r} \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & (\varDelta_{g,r}^{\varSigma})^{\mathrm{nilp}} \arrow[r] & \rho_{g,r+1/k}^{\varSigma\text{-}\mathrm{nilp}}(\varGamma_{g,r}) \arrow[r] & \rho_{g,r/k}^{\varSigma\text{-}\mathrm{nilp}}(\varGamma_{g,r}) \arrow[r] & 1
\end{tikzcd}$$

where the horizontal sequences are exact, the upper vertical arrows have open images, the left-hand upper vertical arrow is an isomorphism, the lower vertical arrows are surjective. Thus, there exist *injections* $\mathrm{im}(\rho_{\mathcal{C}}^{\varSigma\text{-}\mathrm{nilp}}) \hookrightarrow \rho_{g,r/k}^{\varSigma\text{-}\mathrm{nilp}}(\varGamma_{g,r})$ and $\mathrm{im}(\rho_{\mathcal{C}}^{\varSigma\text{-}\mathrm{ab}}) \hookrightarrow \rho_{g,r/k}^{\varSigma\text{-}\mathrm{ab}}(\varGamma_{g,r})$ which have *open images*, and that *fit into* the following commutative diagram of profinite groups

$$\begin{tikzcd}
\mathrm{im}(\rho_{\mathcal{C}}^{\varSigma\text{-}\mathrm{nilp}}) \arrow[r, two heads] \arrow[d, hook] & \mathrm{im}(\rho_{\mathcal{C}}^{\varSigma\text{-}\mathrm{ab}}) \arrow[d, hook] \\
\rho_{g,r/k}^{\varSigma\text{-}\mathrm{nilp}}(\varGamma_{g,r}) \arrow[r, two heads] & \rho_{g,r/k}^{\varSigma\text{-}\mathrm{ab}}(\varGamma_{g,r}).
\end{tikzcd}$$

Let q be a prime number which is contained in $\varSigma$. If $(g, r) = (1, 1)$, then we replace q by 2. By replacing $\varSigma$ by $\{q\}$ in above argument, there exists an *injection* $\mathrm{im}(\rho_{\mathcal{C}}^{\{q\}}) \hookrightarrow \rho_{g,r/k}^{\{q\}}(\varGamma_{g,r})$ which has an *open image*. In particular, by Corollary 2.2, $\mathrm{im}(\rho_{\mathcal{C}}^{\{q\}})$ is *not* a *q-adic Lie group*. Therefore, since $k(\mathcal{M})$ is a *finitely generated extension over an algebraically closed field*, it follows from Proposition 1.13 that the natural surjection $\mathrm{im}(\rho_{\mathcal{C}}^{\varSigma\text{-}\mathrm{nilp}}) \twoheadrightarrow \mathrm{im}(\rho_{\mathcal{C}}^{\varSigma\text{-}\mathrm{ab}})$, hence also $\rho_{g,r/k}^{\varSigma\text{-}\mathrm{nilp}}(\varGamma_{g,r}) \twoheadrightarrow \rho_{g,r/k}^{\varSigma\text{-}\mathrm{ab}}(\varGamma_{g,r})$, is *not injective*. This completes the proof of the first portion of Corollary 2.11 in the case where either $(g, r) \neq (1, 1)$ or $2 \in \varSigma$.

Next, we verify the first portion of Corollary 2.11 in the case where $(g, r) = (1, 1)$ and $2 \notin \varSigma$. We may assume without loss of generality that k is a *number field*. Then, by replacing $\{l\}$ by $\varSigma$ in the second paragraph the proof of Proposition 2.9, Corollary 2.11 in the case where $(g, r) = (1, 1)$ and $2 \notin \varSigma$ follows from Lemmas 2.6, 2.7. This completes the proof of the first portion of Corollary 2.11.

Finally, since the natural surjection $\mathrm{im}(\rho^{\Sigma}_{g,r/k}) \twoheadrightarrow \mathrm{im}(\rho^{\Sigma\text{-}\mathrm{ab}}_{g,r/k})$ factors through the natural surjection $\mathrm{im}(\rho^{\Sigma\text{-}\mathrm{nilp}}_{g,r/k}) \twoheadrightarrow \mathrm{im}(\rho^{\Sigma\text{-}\mathrm{ab}}_{g,r/k})$, the final portion of Corollary 2.11 follows from the first portion of Corollary 2.11. This completes the proof of Corollary 2.11 $\square$

Remark 2.12. In the notation of Corollary 2.11, if $g \neq 1$ and $r \leq 1$, Corollary 2.11 follows immediately from the *injectivity* of the natural homomorphism $\mathrm{Out}(\Delta^{\mathrm{top}}_{g,r}) \to \mathrm{Out}(\Delta^{\Sigma}_{g,r})$ (cf., e.g., [14, Lemma 3.2]) and the *nontriviality* of the *Torelli subgroup* of $\mathrm{MCG}_{g,r}$ (cf. Definition 2.1).

Definition 2.13. Let C be a split hyperbolic curve of type (g, r) over k, and $\mathfrak{O}_C$ an ordering of $C^{\mathrm{cpt}}(k) \setminus C(k)$. We shall say that C is *quasi-*$\{l\}$*-monodromically full* if $\rho^{\{l\}}_{g,r/k} \circ s_{(C,\mathfrak{O}_C)}(G_k)$ is an *open subgroup of* $\mathrm{im}(\rho^{\{l\}}_{g,r/k})$. Note that whether or not C is quasi-$\{l\}$-monodromically full is *independent* of the choice of the ordering $\mathfrak{O}_C$.

Remark 2.14.

(i) The notion of the quasi-$\{l\}$-monodromic fullness is introduced by Hoshi (cf. [8, Definition 2.2]). In fact, the notion of the quasi-$\{l\}$-monodromic fullness is defined for the general hyperbolic curves which are *not necessarily split*.

(ii) Suppose that k is a *finitely generated extension of* $\mathbb{Q}$. Then, by [8, Corollary 2.6] (cf. also [15, Theorem 1.2]), there exist *infinitely many* (geometrically non-isomorphic) pairs of a finite extension K of k and a split hyperbolic curve C of type (g, r) over K such that C is quasi-$\{l\}$-monodromically full. In particular, by replacing k by a finite extension of k, there exists a split hyperbolic curve *of type* (g, r) over k which is quasi-$\{l\}$-monodromically full.

Corollary 2.15. *Let C be a split hyperbolic curve of type (g, r) over k which is quasi-$\{l\}$-monodromically full (cf. Definition 2.13; also Remark 2.14, (ii)). Suppose that $3g - 3 + r > 0$, i.e., (g, r) is not equal to $(0, 3)$. Then $\Omega^{\{l\}}_C$ is not equal to the composite of $\Omega^{\{l\}\text{-}\mathrm{ab}}_C$ and $\Omega^{\{l\}}_{T_k}$.*

Proof. Assume that $\Omega^{\{l\}}_C$ is equal to the composite of $\Omega^{\{l\}\text{-}\mathrm{ab}}_C$ and $\Omega^{\{l\}}_{T_k}$. This induces that the natural homomorphism determined by the inclusion $\ker(\rho^{\{l\}}_C) \subseteq \ker(\rho^{\{l\}}_{T_k})$ (cf. Theorem 1.3)

$$\mathrm{im}(\rho^{\{l\}}_C) \longrightarrow \mathrm{im}(\rho^{\{l\}\text{-}\mathrm{ab}}_C) \times \mathrm{im}(\rho^{\{l\}}_{T_k})$$

is *injective*. Thus, it follows from the *quasi-*$\{l\}$*-monodromic fullness of* C that there exists an *open subgroup* U *of* $\mathrm{im}(\rho^{\{l\}}_{g,r/k})$ such that the composite of

$$U \hookrightarrow \mathrm{im}(\rho^{\{l\}}_{g,r/k}) \to \mathrm{im}(\rho^{\{l\}\text{-}\mathrm{ab}}_{g,r/k}) \times \mathrm{im}(\rho^{\{l\}}_{T_k})$$

— where the right arrow is the homomorphism determined by the natural surjection $\mathrm{im}(\rho^{\{l\}}_{g,r/k}) \twoheadrightarrow \mathrm{im}(\rho^{\{l\}\text{-}\mathrm{ab}}_{g,r/k})$ and the homomorphism $\mathrm{im}(\rho^{\{l\}}_{g,r/k}) \twoheadrightarrow \mathrm{im}(\rho^{\{l\}}_{T_k})$ (cf. Remark 2.5, (ii)) — is *injective*. Then, by [26, Theorem 0.5, (2)], the kernel of the composite of

$$U \hookrightarrow \mathrm{im}(\rho^{\{l\}}_{g,r/k}) \to \mathrm{im}(\rho^{\{l\}\text{-}\mathrm{ab}}_{g,r/k}) \times \mathrm{im}(\rho^{\{l\}}_{T_k}) \to \mathrm{im}(\rho^{\{l\}}_{T_k})$$

is *equal to* $(U \cap \rho^{\{l\}}_{g,r/k}(\Gamma_{g,r}))$. Therefore, the natural homomorphism $(U \cap \rho^{\{l\}}_{g,r/k}(\Gamma_{g,r})) \to \rho^{\{l\}\text{-ab}}_{g,r/k}(\Gamma_{g,r})$ is *injective*. However, since $(U \cap \rho^{\{l\}}_{g,r/k}(\Gamma_{g,r}))$ is an open subgroup of $\rho^{\{l\}}_{g,r/k}(\Gamma_{g,r})$, this *contradicts* Corollary 2.2 and Proposition 2.9. This completes the proof of Corollary 2.15. □

References

[1] Greg W Anderson and Yasutaka Ihara. Pro-l branched coverings of $\mathbf{P}^1$ and higher circular l-units. *Ann. of Math. (2)*, 128(2):271–293, 1988.

[2] Mamoru Asada. On the filtration of topological and pro-l mapping class groups of punctured Riemann surfaces. *J. Math. Soc. Japan*, 48(1):13–36, 1996.

[3] Gebhard Böckle, Wojciech Gajda, and Sebastian Petersen. Independence of ℓ-adic representations of geometric Galois groups. to apper in *J. Reine Angew. Math.*

[4] Anna Cadoret and Akio Tamagawa. On subgroups of $GL_r(\mathbb{F}_l)$ and representations of étale fundamental groups. *Preprint.* `http://www.math.polytechnique.fr/~cadoret/Travaux.html`.

[5] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic pro-p groups*, volume 61 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1999.

[6] Wojciech Gajda and Sebastian Petersen. Independence of ℓ-adic Galois representations over function fields. *Compos. Math.*, 149(7):1091–1107, 2013.

[7] Richard Hain and Makoto Matsumoto. Relative pro-l completions of mapping class groups. *J. Algebra*, 321(11):3335–3374, 2009.

[8] Yuichiro Hoshi. Galois-theoretic characterization of isomorphism classes of monodromically full hyperbolic curves of genus zero. *Nagoya Math. J.*, 203:47–100, 2011.

[9] Yuichiro Hoshi. On monodromically full points of configuration spaces of hyperbolic curves. In *The arithmetic of fundamental groups—PIA 2010*, volume 2 of *Contrib. Math. Comput. Sci.*, pages 167–207. Springer, Heidelberg, 2012.

[10] Yuichiro Hoshi. On the kernels of the pro-l outer Galois representations associated to hyperbolic curves over number fields. to appear in *Osaka J. Math.*

[11] Yuichiro Hoshi and Shinichi Mochizuki. On the combinatorial anabelian geometry of nodally nondegenerate outer representations. *Hiroshima Math. J.*, 41(3):275–342, 2011.

[12] Yasutaka Ihara. Profinite braid groups, Galois representations and complex multiplications. *Ann. of Math. (2)*, 123(1):43–106, 1986.

[13] Yasutaka Ihara. The Galois representation arising from $\mathbf{P}^1 - \{0, 1, \infty\}$ and Tate twists of even degree. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 299–313. Springer, New York, 1989.

[14] Yu Iijima. Galois action on mapping class groups. to appear in *Hiroshima Math. J.*

[15] Makoto Matsumoto and Akio Tamagawa. Mapping-class-group action versus Galois action on profinite fundamental groups. *Amer. J. Math.*, 122(5):1017–1026, 2000.

[16] J. S. Milne. Jacobian varieties. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 167–212. Springer, New York, 1986.

[17] Shinichi Mochizuki. Topics surrounding the anabelian geometry of hyperbolic curves. In *Galois groups and fundamental groups*, volume 41 of *Math. Sci. Res. Inst. Publ.*, pages 119–165. Cambridge Univ. Press, Cambridge, 2003.

[18] Shinichi Mochizuki and Akio Tamagawa. The algebraic and anabelian geometry of configuration spaces. *Hokkaido Math. J.*, 37(1):75–131, 2008.

[19] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Published for the Tata Institute of Fundamental Research, Bombay, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

[20] Hiroaki Nakamura. On exterior Galois representations associated with open elliptic curves. *J. Math. Sci. Univ. Tokyo*, 2(1):197–231, 1995.

[21] Hiroaki Nakamura, Naotake Takao, and Ryoichi Ueno. Some stability properties of Teichmüller modular function fields with pro-l weight structures. *Math. Ann.*, 302(2):197–213, 1995.

[22] Hiroaki Nakamura and Hiroshi Tsunogai. Some finiteness theorems on Galois centralizers in pro-l mapping class groups. *J. Reine Angew. Math.*, 441:115–144, 1993.

[23] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2008.

[24] Takayuki Oda. Etale homotopy type of the moduli spaces of algebraic curves. In *Geometric Galois actions, 1*, volume 242 of *London Math. Soc. Lecture Note Ser.*, pages 85–95. Cambridge Univ. Press, Cambridge, 1997.

[25] Luis Ribes and Pavel Zalesskii. *Profinite groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2010.

[26] Naotake Takao. Braid monodromies on proper curves and pro-ℓ Galois representations. *J. Inst. Math. Jussieu*, 11(1):161–181, 2012.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: iijima@kurims.kyoto-u.ac.jp