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Abstract

We consider on-diagonal heat kernel estimates and the laws of the iterated logarithms for a switch-walk-switch random walk on a lamplighter graph under the condition that the random walk on the underlying graph enjoys sub-Gaussian heat kernel estimates.

Key words: Random walks; wreath products; fractals; heat kernels; sub-Gaussian estimates; LILs.
MSC2010: 60J10, 60J35, 60J55

1 Introduction

Let G be a connected infinite graph and consider the situation that on each vertex of G there is a lamp. Consider a lamplighter on the graph that makes the following random movements; first, the lamplighter turns on or off the lamp on the site with equal probability, then he/she moves to the nearest neighbor of G with equal probability, and turns on or off the lamp on the new site with equal probability. The lamplighter repeats this random movement. Such a movement can be considered as a random walk on the wreath product of graphs $\mathbb{Z}_2 \wr G$ which is roughly a graph putting $\mathbb{Z}_2 = \{0,1\}$ on each vertex of G (see Definition 2.1 for precise definition), and it is called a "switch-walk-switch walk" or "lamplighter walk" on $\mathbb{Z}_2 \wr G$. We are interested in the long time behavior of the walk. Some results are known when G is a specific graph. Pittet and Saloff-Coste [13] established on-diagonal heat kernel asymptotics of the random walk on $\mathbb{Z}_2 \wr \mathbb{Z}^d$ and they obtained the following estimates; there exist positive constants $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 \exp\left[-c_2 n^{\frac{d}{d+2}}\right] \le h_{2n}(g,g) \le c_3 \exp\left[-c_4 n^{\frac{d}{d+2}}\right]. \tag{1.1}$$

holds for all $g \in \mathbb{Z}_2 \wr \mathbb{Z}^d$, where $h_n(\cdot,\cdot)$ is the heat kernel (see [17, 18] for earlier results which are for the case of $G = \mathbb{Z}$ and [5] for the case that G is a finitely generated group with polynomial volume growth). Revelle [16] considered the lamplighter walk on the wreath product $H \wr \mathbb{Z}$ when H is either a finite set or it is in a class of groups. He obtained some relations between the rate of escape of random walks on H and the law of the iterated logarithm (LIL in short) on $H \wr \mathbb{Z}$. In particular, when $H = \mathbb{Z}_2$ he proved that there exist (non-random) constants $c_1, c_2, c_3, c_4 > 0$ such that the following hold for all $\boldsymbol{x} \in \mathbb{Z}_2 \wr \mathbb{Z}$:

$$c_1 \le \limsup_{n\to\infty} \frac{d(Y_0,Y_n)}{n^{1/2}(\log\log n)^{1/2}} \le c_2, \quad c_3 \le \liminf_{n\to\infty} \frac{d(Y_0,Y_n)}{n^{1/2}(\log\log n)^{-1/2}} \le c_4, \qquad P_{\boldsymbol{x}}\text{-a.s.}, \tag{1.2}$$

where $\{Y_n\}$ is the lamplighter random walk and $d(\cdot,\cdot)$ is a graph distance on $\mathbb{Z}_2 \wr \mathbb{Z}$.

We are interested in the following question:

(Question) How does the exponents in (1.1), (1.2) change when the graph G is more general?

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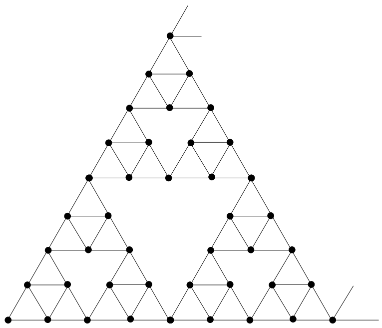

Figure 1: The Sierpinski gasket graph.

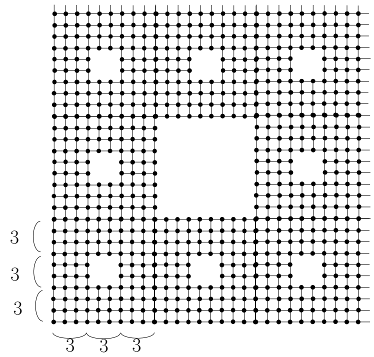

Figure 2: The Sierpinski carpet graph.

In this paper, we will consider the question when G is typically a fractal graph. Figure 1 and Figure 2 illustrate concrete examples of fractal graphs. It is known that the random walk on such a fractal graph behaves anomalously in that it diffuses slower than that of a simple random walk on $\mathbb{Z}^d$. It was proved that the heat kernel $h_n(x,y)$ of the random walk $\{X_n\}_{n\geq 0}$ enjoys the following sub-Gaussian estimates; there exist positive constants $c_1, c_2, c_3, c_4 > 0$ such that

$$\frac{c_1}{n^{d_f/d_w}} \exp\left(-c_2\left(\frac{d(x,y)^{d_w}}{n}\right)^{1/(d_w-1)}\right) \leq h_{2n}(x,y) \leq \frac{c_3}{n^{d_f/d_w}} \exp\left(-c_4\left(\frac{d(x,y)^{d_w}}{n}\right)^{1/(d_w-1)}\right) \tag{1.3}$$

holds for all $d(x,y) \leq 2n$ (note that $h_{2n}(x,y) = 0$ when $d(x,y) > 2n$), where $d(\cdot,\cdot)$ is the graph distance, d_f is the volume growth exponent of the fractal graph and d_w is called a walk dimension which expresses how the random walk on the fractal spreads out. Indeed, by integrating (1.3), one can obtain the following estimates; there exist positive constants $c_1, c_2 > 0$ such that

$$c_1 n^{1/d_w} \leq Ed(X_{2n}, X_0) \leq c_2 n^{1/d_w}$$

holds for all $n > 0$. For more details on diffusions on fractals and random walks on fractal graphs, see [1], [9] and [11]. As we see, properties of random walks on graphs are related to the geometric properties of the graphs. The volume growth is one of such properties. For the graphs with polynomial volume growth, there are well-established general methods to analyze the properties of random walks on them. But for the graphs with exponential volume growth, these methods are not applicable. In this sense, the graphs with exponential volume growth give us interesting research subject. The wreath product $\mathbb{Z}_2 \wr G$ is one of the models which belongs to this category, and this is another reason why we are interested in the lamplighter random walks on fractal graphs.

We consider the random walk on $\mathbb{Z}_2 \wr G$, where the random walk on G enjoys the sub-Gaussian heat kernel estimates (1.3). The main results of this paper are the following;

(1) Sharp on-diagonal heat kernel estimates for the random walk on $\mathbb{Z}_2 \wr G$ (Theorem 2.3),

(2) LILs for the random walk on $\mathbb{Z}_2 \wr G$ (Theorem 2.4).

The on-diagonal heat kernel estimates are heavily related to the asymptotic properties of the spectrum of the corresponding discrete operator. We can obtain the exponent $d_f/(d_f + d_w)$ in our framework as the generalization of $d/(d+2)$.

For the LILs, we establish the LIL for $d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)$, where $\{Y_n\}_{n\geq 0}$ is the random walk on $\mathbb{Z}_2 \wr G$, and the so-called another law of the iterated logarithm that gives the almost sure asymptotic behavior of the liminf of $d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)$. Note that in (1.2), various properties that are specific for $\mathbb{Z}$ were used, so the generalization is highly non-trivial. We have overcome the difficulty by finding some relationship between the range of the random walk on G and $d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)$. To our knowledge, these are the first results on the LILs for the wreath product except for $\mathbb{Z}$.

The outline of this paper is as follows. In section 2, we explain the framework and the main results of this paper. In section 3, we give some consequences of sub-Gaussian heat kernel estimates. These are preliminary results for section 4 and section 5, where we mainly treat the lamplighter random walks on fractal graphs. In section 4, we prove the on-diagonal heat kernel estimates. Section 5 has three subsections. In subsection 5.1, we give a relation between the range of random walk on G and $d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)$. Here, one of the keys is to prove the existence of a path that covers a subgraph of G with the length of the path being (uniformly) comparable to the volume of the subgraph (Lemma 5.3). In subsection 5.2, we state the LILs for the range of the random walk on fractal graphs and prove the LILs for the random walk on $\mathbb{Z}_2 \wr G$ when G is a strongly recurrent graph. In subsection 5.3, we prove the LILs for the random walk on $\mathbb{Z}_2 \wr G$ when G is a transient graph. In the Appendix A, we give an outline of the proof of the LILs for the range of the random walk.

Throughout this paper, we use the following notation.

Notation. (1) *For two non-negative sequences $\{a(n)\}_{n \geq 0}$ and $\{b(n)\}_{n \geq 0}$, we write*

- *$a(n) \asymp b(n)$ if there exist positive constants $c_1, c_2 > 0$ such that $c_1 a(n) \leq b(n) \leq c_2 a(n)$ holds for for all n.*
- *$a(n) \approx b(n)$ if there exist positive constants $c_1, c_2, c_3, c_4 > 0$ such that $c_1 a(c_2 n) \leq b(n) \leq c_3 a(c_4 n)$ holds for all n.*

(2) *We use $c, C, c_1, c_2, \cdots$ to denote deterministic positive finite constants whose values are insignificant. These constants do not depend on time parameters $n, k, \cdots$, distance parameters $r, \cdots$, and vertices of graphs.*

2 Framework and main results

In this section, we introduce the framework and the main results of this paper.

2.1 Framework

Let $G = (V(G), E(G))$ be an infinite, locally finite, connected graph. We assume $V(G)$ is a countable set. We say that G is a graph of bounded degree if

$$M = \sup_{v \in V(G)} \deg v < \infty \tag{2.4}$$

holds. We denote $d(x, y)$ the graph distance of x, y in G, i.e. the shortest length of paths between x and y. If we want to emphasize the graph G, we write $d_G(x, y)$ instead of $d(x, y)$.

Next, we introduce a wreath product of the graph.

Definition 2.1 (Wreath product)**.** *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. We define the wreath product of G and H (denoted by $H \wr G$) in the following way. We define the vertex set of the wreath product as*

$$V(H \wr G) = \left\{ (f, v) \in \left(\prod_{z \in V(G)} H \right) \times V(G) \mid \sharp \operatorname{Supp}(f) < \infty \right\},$$

where $\operatorname{Supp} f = \{x \in V(G) \mid f(x) \neq 0\}$. *For $(f, u), (g, v) \in V(H \wr G)$, $((f, u), (g, v)) \in E(H \wr G)$ if either (a) or (b) are satisfied:*

(a) $f = g$ *and* $(u,v) \in E(G)$,

(b) $u = v$, $f(x) = g(x)$ $(\forall x \in V(G) \setminus \{u\})$ *and* $(f(u), g(u)) \in E(H)$.

We call G *the underlying graph of* $H \wr G$ *and* H *the fiber graph of* $H \wr G$.

Throughout the paper, we will only consider the case $H = \mathbb{Z}_2$ that consists of two vertices $\{0,1\}$, say, and one edge that connects the two vertices. (As in Remark 2.5(2), the results in this paper hold when H is a finite graph.) We denote the elements of $V(\mathbb{Z}_2 \wr G)$ by bold alphabets $\boldsymbol{x}, \boldsymbol{y} \cdots$ and the elements of $V(G)$ by standard alphabets $x, y, \cdots$.

Next, we introduce the notion of weighted graphs. Let $\mu : V(G) \times V(G) \to [0,\infty)$ be a symmetric function such that $\mu_{xy} = \mu(x,y) > 0$ if and only if $(x,y) \in E(G)$. We call the pair (G,μ) a weighted graph. For a weighted graph (G,μ), we define a measure $m = m_G$ on $V(G)$ by $m(A) = \sum_{z \in A} m(x)$ for $A \subset V(G)$ where $m(x) = \sum_{y:y\sim x} \mu_{xy}$. We will write $V(x,r) = V_G(x,r) = m(B(x,r))$, where $B(x,r) = \{y \in V(G) \mid d(x,y) \le r\}$.

Let $\{X_n\}_{n\ge 0}$ be the (time-homogeneous) random walk on G whose transition probability is $P = (p(x,y))_{x,y \in V(G)}$, where $p(x,y) = \mu_{xy}/m(x)$. We call $\{X_n\}_{n\ge0}$ the random walk associated with the weighted graph (G,μ). $\{X_n\}_{n\ge0}$ is reversible w.r.t. m, i.e. $m(x)p(x,y) = m(y)p(y,x)$ for all $x,y \in V(G)$. Define

$$p_n(x,y) := P_x(X_n = y), \qquad \forall x,y \in V(G).$$

$p_n(x,y)/m(y)$ is called the heat kernel of the random walk.

Throughout this paper, we assume the following conditions for the graph and the random walks.

Assumption 2.2. *Let* (G,μ) *be a weighted graph. We assume the following for* (G,μ).

(1) *(p_0-condition) :* (G,μ) *satisfies* p_0*-condition, i.e. there exists* $p_0 > 0$ *such that* $\mu_{xy}/m(x) \ge p_0$ *holds for all* $x,y \in V(G)$.

(2) *(d_f-set condition) : There exist positive constants* $c_1, c_2 > 0$ *such that*

$$c_1 r^{d_f} \le V(x,r) \le c_2 r^{d_f} \tag{2.5}$$

holds for all $x \in V(G), r \ge 0$. *Here, we regard* 0^{d_f} *as* 1.

(3) *(Sub-Gaussian heat kernel estimates) : The heat kernel* $\dfrac{p_n(x,y)}{m(y)}$ *of* $\{X_n\}_{n\ge0}$ *satisfies the following estimates:*

$$\frac{p_n(x,y)}{m(y)} \le \frac{c_1}{V(x, n^{\frac{1}{d_w}})} \exp\left(-c_2 \left(\frac{d(x,y)^{d_w}}{n}\right)^{\frac{1}{d_w - 1}}\right) \tag{2.6}$$

holds for all $x,y \in V(G), n \ge 1$, *and*

$$\frac{p_n(x,y)}{m(y)} + \frac{p_{n+1}(x,y)}{m(y)} \ge \frac{c_3}{V(x, n^{\frac{1}{d_w}})} \exp\left(-c_4 \left(\frac{d(x,y)^{d_w}}{n}\right)^{\frac{1}{d_w - 1}}\right) \tag{2.7}$$

holds for $x,y \in V(G), n \ge 1$ *with* $d(x,y) \le n$. *We define*

$$d_s/2 = d_f/d_w \tag{2.8}$$

and call it the spectral dimension.

The fractal graphs such as the Sierpinski gasket graph and the Sierpinski carpet graph given in section 1 satisfy Assumption 2.2. Note that from Assumption 2.2 (2), we have $c_1 \le m_G(x) \le c_2$ for all $x \in V(G)$. Hence

$$c_1 \sharp A \le m(A) \le c_2 \sharp A, \qquad \forall A \subset V(G) \tag{2.9}$$

holds, where $\sharp A$ is the cardinal number of A. Also, note that under (2.4) and Assumption 2.2, we have

$$0 < \inf_{x,y \in V(G), x \sim y} \mu_{xy} \le \sup_{x,y \in V(G), x \sim y} \mu_{xy} < \infty. \tag{2.10}$$

Next, we define the lamplighter walk on $\mathbb{Z}_2 \wr G$. We denote the transition probability on $\mathbb{Z}_2$ by $P^{(\mathbb{Z}_2)} = (p^{(\mathbb{Z}_2)}(a,b))_{a,b \in \mathbb{Z}_2}$, where $P^{(\mathbb{Z}_2)}$ is given by

$$p^{(\mathbb{Z}_2)}(a,b) = \frac{1}{2}, \qquad \text{for all } a, b \in \mathbb{Z}_2 \ .$$

We can lift $P = (p(x,y))_{x,y \in G}$ and $P^{(\mathbb{Z}_2)} = (p^{(\mathbb{Z}_2)}(a,b))_{a,b \in \mathbb{Z}_2}$ on $\mathbb{Z}_2 \wr G$, by

$$\tilde{p}^{(G)}((f,x),(g,y)) = \begin{cases} p(x,y) & \text{if } f = g \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{p}^{(\mathbb{Z}_2)}((f,x),(g,y)) = \begin{cases} \frac{1}{2} & \text{if } x = y \text{ and } f(v) = g(v) \text{ for all } v \ne x \\ 0 & \text{otherwise} \end{cases}.$$

Let $Y_n = \{(\eta_n, X_n)\}_{n \ge 0}$ be a random walk on $\mathbb{Z}_2 \wr G$ whose transition probability $\tilde{p}$ is given by

$$\begin{aligned} &\tilde{p}((f,x),(g,y)) = \tilde{p}^{(\mathbb{Z}_2)} * \tilde{p}^{(G)} * \tilde{p}^{(\mathbb{Z}_2)}((f,x),(g,y)) \\ = &\sum_{(h_1,w_1),(h_2,w_2)} \tilde{p}^{(\mathbb{Z}_2)}((f,x),(h_1,w_1)) \tilde{p}^{(G)}((h_1,w_1),(h_2,w_2)) \tilde{p}^{(\mathbb{Z}_2)}((h_2,w_2),(g,y)). \end{aligned}$$

Note that if $(f,x),(g,y) \in V(\mathbb{Z}_2 \wr G)$ satisfy $x \sim y$ and $f(z) = g(z)$ for all $z \ne x, y$ then

$$P(Y_{n+1} = (g,y) \mid Y_n = (f,x)) = \frac{1}{4} p(x,y),$$

and otherwise it is zero.

This random walk moves in the following way. Let X_n be the site of lamplighter at time n and η_n be the on-lamp state at time n. The lamplighter changes the lamp at X_n with probability $1/2$, moves on G according to the transition probability $P = (p(x,y))_{x,y \in G}$, and then changes the lamp at X_{n+1} with probability $1/2$. The lamplighter repeats this procedure. (In the first paragraph of section 1, we discussed the case when $\{X_n\}$ is a simple random walk on G.)

Note that $\{Y_n\}_{n \ge 0}$ is reversible w.r.t. $m_{\mathbb{Z}_2 \wr G}$, where

$$m_{\mathbb{Z}_2 \wr G}((\eta, x)) = m(x).$$

We denote the transition probability of $\{Y_n\}_{n \ge 0}$ as $p(\boldsymbol{x}, \boldsymbol{y})$ (cf. $p(x,y)$ is the transition probability of $\{X_n\}_{n \ge 0}$). We sometimes write $m(\boldsymbol{x})$ instead of $m_{\mathbb{Z}_2 \wr G}(\boldsymbol{x})$.

2.2 Main results

In this subsection, we state the main results of this paper.

First, we state the result of on-diagonal heat kernel estimates.

Theorem 2.3. *Suppose that Assumption* 2.2 *holds. Then the following holds;*

$$\frac{p_{2n}(\boldsymbol{x},\boldsymbol{x})}{m_{\mathbb{Z}_2\wr G}(\boldsymbol{x})} \approx \exp[-n^{\frac{d_f}{d_f+d_w}}], \qquad \forall \boldsymbol{x} \in \mathbb{Z}_2 \wr G. \tag{2.11}$$

Next we state the results of the LIL when $d_s/2 < 1$ and $d_s/2 > 1$ respectively.

Theorem 2.4. *Let G be a graph of bounded degree.*
(I) Assume that Assumption 2.2 and $d_s/2 < 1$ hold. Then there exist (non-random) constants $c_1, c_2, c_3, c_4 > 0$ such that the following hold for all $\boldsymbol{x} \in V(\mathbb{Z}_2 \wr \mathbb{Z})$:

$$c_1 \le \limsup_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr G}(Y_0, Y_n)}{n^{d_s/2}(\log\log n)^{1-d_s/2}} \le c_2, \qquad P_{\boldsymbol{x}}\text{-a.s.} \tag{2.12}$$

$$c_3 \le \liminf_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr G}(Y_0, Y_n)}{n^{d_s/2}(\log\log n)^{-d_s/2}} \le c_4, \qquad P_{\boldsymbol{x}}\text{-a.s.} \tag{2.13}$$

(II) Assume that Assumption 2.2(1),(2), (2.6), *and $d_s/2 > 1$ hold. Then there exist (non-random) positive constants $c_1, c_2 > 0$ such that the following hold for all $\boldsymbol{x} \in V(\mathbb{Z}_2 \wr G)$:*

$$c_1 \le \liminf_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr G}(Y_0, Y_n)}{n} \le \limsup_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr G}(Y_0, Y_n)}{n} \le c_2, \qquad P_{\boldsymbol{x}}\text{-a.s.} \tag{2.14}$$

Remark 2.5. (1) *Note that* (2.7) *is not needed for Theorem 2.4(II). Since the transient case is discussed under a general framework in [12] (see subsection 5.3), we do not pursue the minimum assumption for* (2.14) *to hold.*

(2) *We can obtain the same results (by the same proof) if we replace $\mathbb{Z}_2$ to a finite graph H with $\sharp H \ge 2$ and $p^{(\mathbb{Z}_2)}$ to $p^{(H)}$, where $p^{(H)}$ is the transition probability on H given by*

$$p^{(H)}(a,b) = \frac{1}{\sharp H}, \qquad \text{for all } a, b \in V(H). \tag{2.15}$$

(3) *For each $0 < \alpha < 1$, Rau [15, Proposition 1.2] constructed the graph G_α such that the random walk on G_α satisfies the following heat kernel estimates :*

$$p_{2n}(x,x) \approx \exp(-n^\alpha).$$

For the case $1/3 \le \alpha < 1$, the graphs constructed by Rau are the wreath product on $\mathbb{Z}$, but the fiber graphs are different site by site. (The definition of wreath product given by Rau is more general than ours.)

On the other hand, for each d_f, d_w such that $2 \le d_w \le 1 + d_f$ and $d_f \ge 1$, Barlow [2, Theorem 2] constructed weighted graphs that satisfy Assumption 2.2. Combining this and Theorem 2.3, we can give an alternative example where the heat kernel enjoys (2.15) *for any given $1/3 \le \alpha < 1$.*

(4) *For the case of $d_s/2 = 1$, we could not obtain the LIL in general. However, one can obtain the LIL for the case of $\mathbb{Z}^2$ as follows. (Note that $d_s/2 = 1$ in this case since $d_f = d_w = 2$.)*

Define $R_n = \sharp\{X_0, \cdots, X_n\}$. Dvoretzky and Erdős [7, Theorem 1,4] proved the following law of large number of R_n:

$$\lim_{n\to\infty} \frac{R_n}{\pi n/\log n} = 1, \qquad P\text{-a.s.}$$

In Proposition 5.1 and 5.2, we will show that $\frac{1}{4}R_n \le d_{\mathbb{Z}_2\wr\mathbb{Z}^2}(Y_0, Y_n)$ holds for all but finitely many n and there exist a positive constant $C > 0$ such that $d_{\mathbb{Z}_2\wr\mathbb{Z}^2}(Y_0, Y_n) \le CR_n$ holds for all n. Using these facts, we see that there exist positive constants $c_1, c_2 > 0$ such that for all $\boldsymbol{x} \in V(\mathbb{Z}_2 \wr G)$

$$c_1 \le \liminf_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr\mathbb{Z}^2}(Y_0, Y_n)}{n/\log n} \le \limsup_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr\mathbb{Z}^2}(Y_0, Y_n)}{n/\log n} \le c_2, \qquad P_{\boldsymbol{x}}\text{-a.s.}$$

hold. As we see, the exponents differ from those of $d_s/2 < 1$ and $d_s/2 > 1$.

3 Consequences of heat kernel estimates

In this section, we give preliminary results obtained from the sub-Gaussian heat kernel estimates (2.6), (2.7).

First, the following can be obtained by a simple modification of [1, Lemma 3.9]. (Note that only (2.6) is needed here.)

Lemma 3.1. *There exist positive constants* $c_1, c_2 > 0$ *such that*

$$P_y\left(\max_{0\le j\le n} d(x,X_j)\ge 3r\right)\le c_1\exp\left(-c_2\left(\frac{r^{d_w}}{n}\right)^{\frac{1}{d_w-1}}\right) \tag{3.16}$$

holds for all $n\ge 1, r\ge 1, x,y\in V(G)$ *with* $d(x,y)\le r$.

The following lemma will be used in subsection 5.1. Again, only (2.6) is needed for the lemma.

Lemma 3.2. *There exist positive constants* $c_1, c_2 > 0$ *such that*

$$P_x\left(\max_{0\le j\le n} d(x,X_j)\le r\right)\le c_1\exp\left(-c_2\frac{n}{r^{d_w}}\right) \tag{3.17}$$

holds for all $x\in V(G), n, r\ge 1$.

Proof. We first show that there exists a positive constant $c_1>0$ such that

$$P_x\left(\max_{0\le j\le r^{d_w}} d(x,X_j)\le 2c_1 r\right)\le\frac{1}{2} \tag{3.18}$$

holds for all $x\in V(G)$ and all $r\ge 1$. Using heat kernel estimates, we have

$$\begin{aligned}
P_x\left(\sup_{0\le j\le r^{d_w}} d(x,X_j)\le 2c_1 r\right) &\le P_x(d(x,X_{r^{d_w}})\le 2c_1 r)\\
&\le c_2\frac{1}{(r^{d_w})^{d_f/d_w}}\sum_{y\in B(x,2c_1r)} m(y)\exp\left(-c_3\left(\frac{d(x,y)^{d_w}}{r}\right)^{1/(d_w-1)}\right)\\
&\le c_2\frac{1}{(r^{d_w})^{d_f/d_w}} m(B(x,2c_1r))\le c_4(2c_1)^{d_f}.
\end{aligned}$$

Taking c_1 small, we obtain (3.18).

We now prove (3.17). It is enough to consider the case $n\ge r^{d_w}$ since otherwise (3.17) by adjusting constants. Let $k\ge 1$ be such that $kr^{d_w}\le n<(k+1)r^{d_w}$ and let $t_i = ir^{d_w}$. Then by the Markov property and (3.18) we have

$$\begin{aligned}
P_x\left(\max_{0\le j\le n} d(x,X_j)\le c_1 r\right) &\le P_x\left(\bigcap_{0\le i\le k-1}\left\{\max_{t_i\le j\le t_{i+1}} d(X_{t_i},X_j)\le 2c_1 r\right\}\right)\\
&\le\left\{\sup_y P_y\left(\max_{0\le j\le r^{d_w}} d(y,X_j)\le 2c_1 r\right)\right\}^k\\
&\le\left(\frac{1}{2}\right)^k\le c_5\exp(-c_6 k)\le c_7\exp(-c_8 n r^{-d_w}).
\end{aligned}$$

Hence we obtain (3.17) by adjusting constants. □

In the next proposition, we show that Lemma 3.2 is sharp up to constants if we assume both (2.6) and (2.7). The idea of the proof is based on [14, Lemma 7.4.3], where a similar estimate was given for a class of random walks on $\mathbb{Z}^d$.

Proposition 3.3. *There exist positive constants $c_1, c_2 > 0$ such that*

$$P_x\left(\max_{0\le j\le n} d(x,X_j)\le r\right)\ge c_1\exp\left(-c_2\frac{n}{r^{d_w}}\right)$$

holds for all $n, r \ge 1$ with $r \le n$.

The proof consists of the following two lemmas.

Lemma 3.4. *There exists $\epsilon \in (0,1)$ such that*

$$P_y(d(x,X_n)\ge r)\le 1-\epsilon$$

holds for all $r, n \ge 1$ with $n \le r^{d_w}$ and $x, y \in V(G)$ with $d(x,y)\le r$.

Proof. We follow the argument in [14, Lemma 7.4.7]. Let $\gamma = 1/(d_w-1)$. Let $\ell_{x,y}$ be a geodesic path from x to y in G. Let $x_n \in \ell_{x,y}$ be the $[n^{1/d_w}]$-th vertex from y. Then $B(x_n,[n^{1/d_w}])\subset B(x,r)$ holds since for all $z\in B(x_n,[n^{1/d_w}])$ we have $d(x,z)\le d(x,x_n)+d(x_n,z)\le (d(x,y)-[n^{1/d_w}])+[n^{1/d_w}]\le r$. Also for all $z\in B(x_n,[n^{1/d_w}])$, we have $d(y,z)\le d(y,x_n)+d(x_n,z)\le 2[n^{1/d_w}]$. Hence, by (2.7) and (2.5) we have

$$\begin{aligned}
P_y\left(d(x,X_n)\le r\right)\ge& c_1\sum_{z\in B(x,r)}\frac{m(z)}{V(y,n^{1/d_w})}\exp\left[-c_2\left(\frac{d(y,z)^{d_w}}{n}\right)^{\gamma}\right]\\
\ge& c_1\sum_{z\in B(x_n,[n^{1/d_w}])}\frac{m(z)}{V(y,n^{1/d_w})}\exp\left[-c_2\left(\frac{d(y,z)^{d_w}}{n}\right)^{\gamma}\right]\\
\ge& c_3\left(\sum_{z\in B(x_n,[n^{1/d_w}])}\frac{m(z)}{n^{d_f/d_w}}\right)\exp\left[-c_4 2^{d_w\gamma}\right]\ge c_5\exp\left[-c_4 2^{d_w\gamma}\right].
\end{aligned}$$

The proof completes by taking $\epsilon = c_5\exp\left[-c_4 2^{d_w\gamma}\right]$ (note that we may take $c_5<1$). □

Lemma 3.5. *Let ϵ as in Lemma 3.4. Then there exists $\eta\ge 1$ such that for all $x, y\in V(G)$ with $d(x,y)\le r$ and for all ℓ with $k[r^{d_w}]\le \ell\le (k+1)[r^{d_w}]$, we have*

$$P_y\left(\max_{0\le j\le \ell} d(x,X_j)\le 3\eta r, d(x,X_\ell)\le r\right)\ge\left(\frac{\epsilon}{2}\right)^{k+1}.$$

Proof. We follow the argument in the proof of [14, Lemma 7.4.3]. We prove the assertion by induction for k.

Step I: We first prove the case $k=0$. Let $\gamma = 1/(d_w-1)$. In general, $1\le P(A)+P(B)+P((A\cup B)^c)$ holds for any events A, B. So take A, B as $A=\{\sup_{0\le j\le\ell} d(x,X_j)>3\eta r\}$, $B=\{d(x,X_\ell)>r\}$. Let $\ell\le r^{d_w}$. By Lemma 3.1 and Lemma 3.4 we have

$$\begin{aligned}
1\le& P_y\left(\max_{0\le j\le\ell} d(x,X_j)>3\eta r\right)+P_y\left(d(x,X_\ell)>r\right)+P_y\left(\max_{0\le j\le\ell} d(x,X_j)\le 3\eta r, d(x,X_\ell)\le r\right)\\
\le& c_1\exp\left[-c_2\left(\frac{(\eta r)^{d_w}}{\ell}\right)^{\gamma}\right]+(1-\epsilon)+P_y\left(\max_{0\le j\le\ell} d(x,X_j)\le 3\eta r, d(x,X_\ell)\le r\right).
\end{aligned}$$

From above and using $\ell \le r^{d_w}$ we have

$$P_y\left(\max_{0\le j\le \ell} d(x,X_j)\le 3\eta r, d(x,X_\ell)\le r\right)\ge \epsilon - c_1\exp\left[-c_2\left(\frac{(\eta r)^{d_w}}{\ell}\right)^{\gamma}\right] \quad \ge \epsilon - c_1\exp\left[-c_2\eta^{d_w\gamma}\right].$$

Taking $\eta > \left\{\frac{1}{c_2}\log\left(2c_1/\epsilon\right)\right\}^{\gamma d_w} \vee 1$, we obtain

$$P_y\left(\max_{0\le j\le \ell} d(x,X_j)\le 3\eta r, d(x,X_\ell)\le r\right)\ge \frac{\epsilon}{2}.$$

for $\ell \le r^{d_w}$.

Step II: Assume that the result holds up to k. Let ℓ satisfy $k[r^{d_w}]\le \ell \le (k+1)[r^{d_w}]$. Define $\ell' = k[r^{d_w}]$. Then using the Markov property and induction hypothesis we have

$$\begin{aligned}
&P_y\left(\max_{0\le j\le \ell} d(x,X_j)\le 3\eta r, d(x,X_\ell)\le r\right)\\
\ge &P_y\left(\max_{0\le j\le \ell} d(x,X_j)\le 3\eta r, d(x,X_\ell)\le r, d(x,X_{\ell'})\le r\right)\\
= &E_y\left[1_{\{\max_{0\le j\le \ell'} d(x,X_j)\le 3\eta r, d(x,X_{\ell'})\le r\}}P_{X_{\ell'}}\left(d(x,X_{\ell-\ell'})\le r, \max_{0\le j\le \ell-\ell'} d(x,X_j)\le 3\eta r\right)\right]\\
\ge &\frac{\epsilon}{2}P_y\left(\max_{0\le j\le \ell'} d(x,X_j)\le 3\eta r, d(x,X_{\ell'})\le r\right)\ge \left(\frac{\epsilon}{2}\right)^{k+1}.
\end{aligned}$$

We thus complete the proof. □

Given Lemma 3.5, it is straightforward to obtain Proposition 3.3.

4 On diagonal heat kernel

In this section, we give the proof of Theorem 2.3.

The lower bound follows by the same approach as in [14, Section 7] (cf. [13, Section 7] and [19, Section 15.D]). We use Proposition 3.3 for the proof.

Proof of the lower bound of Theorem 2.3. For notational simplicity, we assume $\eta = \mathbf{0}$. As we said before we write $m_{\mathbb{Z}_2\wr G}$ as m. For any finite subset $A\subset V(\mathbb{Z}_2\wr G)$, using the Cauchy-Schwarz inequality we have

$$\frac{p_{2n}(\boldsymbol{x},\boldsymbol{x})}{m(\boldsymbol{x})} = \sum_{\boldsymbol{y}\in V(\mathbb{Z}_2\wr G)}\frac{p_n(\boldsymbol{x},\boldsymbol{y})p_n(\boldsymbol{y},\boldsymbol{x})}{m(\boldsymbol{x})} = \sum_{\boldsymbol{y}\in V(\mathbb{Z}_2\wr G)}\frac{p_n(\boldsymbol{x},\boldsymbol{y})^2}{m(\boldsymbol{y})}\ge \sum_{\boldsymbol{y}\in A}\frac{p_n(\boldsymbol{x},\boldsymbol{y})^2}{m(\boldsymbol{y})}\ge \frac{1}{m(A)}P_{\boldsymbol{x}}\left(Y_n\in A\right)^2. \tag{4.19}$$

Set $A := \{\boldsymbol{y} = (f,y)\in \mathbb{Z}_2\wr G \mid y\in B_G(x,r), f(z)=0 \text{ for all } z\in V(G) \text{ such that } d(x,z)>r\}$. Using (2.5) and (2.9), we have

$$m_{\mathbb{Z}_2\wr G}(A) = \sum_{y\in B_G(x,r)} m_G(y)2^{\sharp B_G(x,r)}\le c_1 r^{d_f}2^{c_2 r^{d_f}}.$$

and using Proposition 3.3 we have

$$P_{\boldsymbol{x}}\left(Y_n\in A\right)\ge P_x\left(\max_{0\le j\le n} d(x,X_j)\le r\right)\ge c_3\exp\left[-c_4\frac{n}{r^{d_w}}\right].$$

Hence, by (4.19) we have

$$\frac{p_{2n}(\boldsymbol{x},\boldsymbol{x})}{m(\boldsymbol{x})} \geq c_5 \exp\left[-c_6\left(d_f \log r + r^{d_f} + \frac{n}{r^{d_w}}\right)\right].$$

Optimize right hand side (take r as $n^{1/(d_f+d_w)}$), then we obtain

$$\frac{p_{2n}(\boldsymbol{x},\boldsymbol{x})}{m(\boldsymbol{x})} \geq c \exp\left[-Cn^{\frac{d_f}{d_f+d_w}}\right].$$

We thus complete the proof. □

We next prove the upper bound of Theorem 2.3 (cf. [13, Section 8] and [19, Section 15.D]).

Proof of the upper bound of Theorem 2.3. For the switch-walk-switch random walk $\{Y_n = (\eta_n, X_n)\}_{n\geq 0}$ on $\mathbb{Z}_2 \wr G$, η_n is equi-distributed on $\{f \in \prod_{z\in V(G)} \mathbb{Z}_2 \mid \operatorname{Supp} f \subset \bar{R}_n\}$, where $\bar{R}_n = \{X_0, X_1, \cdots, X_n\}$. Hence, setting $R_n = \sharp \bar{R}_n$, we have

$$P_{\boldsymbol{x}}\left(Y_n = \boldsymbol{x}\right) = \sum_{k=0}^{n} P_{\boldsymbol{x}}\left(Y_n = \boldsymbol{x}, R_n = k\right) \leq \sum_{k=0}^{n} E_{\boldsymbol{x}}\left[1_{\{R_n=k\}} 2^{-k}\right] \leq E_{\boldsymbol{x}}\left[2^{-R_n}\right].$$

In [8, Theorem 1.2], Gibson showed the following Donsker-Varadhan type range estimate: for any $\nu > 0$ and any $x \in V(G)$,

$$-\log E_x\left[\exp\left\{-\nu m(\bar{R}_{n^{d_w}V(x,n)})\right\}\right] \asymp V(x,n).$$

Note that $V(x,n) \asymp n^{d_f}$. Replacing n to $n^{1/(d_f+d_w)}$ we have

$$E_x\left[\exp\left\{-\nu m(\bar{R}_n)\right\}\right] \approx \exp\left[-n^{d_f/(d_f+d_w)}\right].$$

Since $cm(\bar{R}_n) \geq R_n$ (due to (2.9)), by the above estimates, we obtain the upper estimate, i.e.

$$\frac{p_{2n}(\boldsymbol{x},\boldsymbol{x})}{m(\boldsymbol{x})} \leq c \exp\left[-Cn^{\frac{d_f}{d_f+d_w}}\right].$$

We thus complete the proof. □

5 Law of the iterated logarithm

Throughout this section, we assume that G is of bounded degree. In this section, we will prove Theorem 2.4.

We first explain the idea of the proof. For notational simplicity, let $o \in V(G)$ be a distinguished point and $\mathbf{0}$ be the element of $\prod_{v\in V(G)} \mathbb{Z}_2$ such that $\mathbf{0}(v) = 0$ for all $v \in V(G)$. In order to realize a given lamp state $(\eta, x) \in \mathbb{Z}_2 \wr G$ beginning from the lamp state $(\mathbf{0}, o) \in \mathbb{Z}_2 \wr G$, we need to visit all the "on-lamp vertices". So

$$\sum_{i\in V(G)} \eta(i) \leq d_{\mathbb{Z}_2\wr G}((\mathbf{0}, o), (\eta, x))$$
$$\leq \text{(the minimum number of steps to visit all the "on-lamp vertices" from } o \text{ to } x) + \sum_{i\in V(G)} \eta(i). \quad (5.20)$$

Note that the lamp at a certain vertex of G (say z) cannot be changed without making the lamplighter visit at z. From this and (5.20), we see that $d_{\mathbb{Z}_2\wr G}(Y_0, Y_n)$ is heavily related to the range of random walk

$\{X_n\}_{n\geq 0}$ on G. Set $R_n = \sharp\{X_0, X_1, \cdots, X_n\}$. Intuitively, $\sum_{i\in V(G)} \eta_n(i)$ is close to $\frac{1}{2}R_n$. Indeed, we will show the following in Proposition 5.1 and Proposition 5.2:

$$\frac{1}{4}R_n \leq d_{\mathbb{Z}_2\wr G}(Y_0, Y_n) \quad a.s. \quad \text{for all but finitely many } n, \tag{5.21}$$

$$d_{\mathbb{Z}_2\wr G}(Y_0, Y_n) \leq (2M+1)R_n, \qquad \text{for all } n, \tag{5.22}$$

where M is defined by (2.4). We will prove (5.21) and (5.22) in subsection 5.1. The behavior of R_n differs for $d_s/2 < 1$ and $d_s/2 > 1$. In subsection 5.2 (resp. 5.3), we prove the LILs of R_n and $d_{\mathbb{Z}_2\wr G}(Y_0, Y_n)$ for $d_s/2 < 1$ (reps. for $d_s/2 > 1$).

5.1 Relations between the distance and the range

The main goal of this subsection is to prove (5.21) and (5.22).

Proposition 5.1. *For all but finitely many n, we have*

$$\frac{1}{4}R_n \leq \sum_{i\in\{X_0, X_1, \cdots, X_n\}} \eta_n(i) \qquad P_x\text{-}a.s. \text{ for all } x \in V(G). \tag{5.23}$$

Proof. We fix $x \in V(G)$ and write P instead of P_x. Define $S_n = \sum_{i\in\{X_0, X_1, \cdots, X_n\}} \eta_n(i)$. It is easy to see that

$$P(S_n = l \mid R_n = k) = \left(\frac{1}{2}\right)^k \binom{k}{l}.$$

for $0 \leq \ell \leq k$. Then we have

$$\begin{aligned}
P\left(S_n \leq \frac{1}{4}R_n\right) &= \sum_{l=0}^{n} P\left(S_n \leq \frac{1}{4}l, R_n = l\right) = \sum_{l=0}^{n}\sum_{m=0}^{\frac{1}{4}l} \left(\frac{1}{2}\right)^{\ell} \binom{l}{m} P(R_n = l) \\
&\leq \sum_{l=0}^{n} \exp\left(-\frac{1}{16}l\right) P(R_n = l) \qquad \text{by the Chernoff bound} \\
&\leq P\left(R_n \leq n^{\frac{1}{2d_w}}\right) + \exp\left(-\frac{1}{16}n^{\frac{1}{2d_w}}\right) P(R_n \geq n^{\frac{1}{2d_w}}) \\
&\leq P\left(\sup_{0\leq \ell\leq n} d(X_0, X_\ell) \leq n^{\frac{1}{2d_w}}\right) + \exp\left(-\frac{1}{16}n^{\frac{1}{2d_w}}\right) \\
&\leq c_1 \exp(-c_2 n^{\frac{1}{2}}) + \exp\left(-\frac{1}{16}n^{\frac{1}{2d_w}}\right), \qquad \text{by Lemma 3.2.}
\end{aligned}$$

Using the Borel-Cantelli lemma, we complete the proof. □

Proposition 5.2. *There exists a constant $C > 0$ such that*

$$d_{\mathbb{Z}_2\wr G}(Y_0, Y_n) \leq CR_n$$

holds for all $n \geq 0$ and $\omega \in \Omega$.

To prove the above proposition, we need the following lemma. To exclude ambiguity, we first introduce some terminologies. Let H be a connected subgraph of G.

- A path γ on H is a sequence of vertices $v_0v_1\cdots v_k$ such that $v_i \in V(H)$, $v_iv_{i+1} \in E(H)$ hold for all i. For a path γ, we set $V(\gamma) = \{v_0, v_1, \cdots, v_k\}$, and define the length of γ as $|\gamma| = k$. $e_j = v_jv_{j+1} (j = 0, 1, \cdots, k-1)$ are said to be the edges of γ.

- For the path γ given above and the given edge $e \in E(G)$, we define $F(\gamma, e)$ as $F(\gamma, e) = \{e_l \mid e_l = e, l \in \{0, 1, \cdots, k-1\}\}$.
- We denote $\overrightarrow{e} = \overrightarrow{uv}$ if the edge e is directed from u to v and say $\overrightarrow{e}$ a directed edge. For two directed edges $\overrightarrow{e_1} = \overrightarrow{u_1 v_1}$ and $\overrightarrow{e_2} = \overrightarrow{u_2 v_2}$, $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are equal if and only if $u_1 = u_2, v_1 = v_2$.

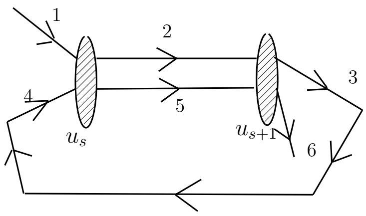

Figure 3: an example of the path η.

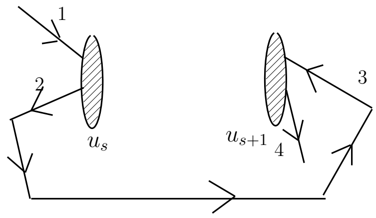

Figure 4: an example of the surgery of η.

Recall that G is of bounded degree and $M = \sup_{v \in V(G)} \deg v (< \infty)$.

Lemma 5.3. *Let H be a finite connected subgraph of G. Define $V(H) = \{v_1, \cdots, v_l\}$, $E(H) = \{e_1, \cdots, e_m\}$, and suppose that all the element of $V(H)$ and $E(H)$ are distinct.*

(1) *There exists a path $\gamma = w_0 w_1 \cdots w_k$ in H such that*

  (a) $\{w_0, w_1, w_2, \cdots, w_k\} = \{v_1, v_2, \cdots, v_l\}$,

  (b) *Define $\tilde{e}_j = w_j w_{j+1}$ for $j = 0, 1, \cdots, k-1$. Then $\sharp\{\tilde{e}_j \mid \tilde{e}_j = \tilde{e}_s, j = 0, 1, \cdots, k-1\} \leq 2$ holds for all $s = 0, 1, \cdots, k-1$.*

(2) *Let γ be as in* (1). *Then $|\gamma| \leq 2lM (= 2M\sharp V(H))$.*

Proof. (1) Take a path $\eta = u_0 u_1 \cdots u_n$ on H such that $\{u_0, u_1, \cdots, u_n\} = V(H)$. Define $f_j = u_j u_{j+1}$. If each edge f_j satisfies $\sharp\{l \in \{0, 1, \cdots, n\} \mid f_l = f_j\} \leq 2$ for $j = 0, 1, \cdots, n-1$, then η satisfies the conditions $(a), (b)$. So we may assume that there exists f_j such that $\sharp\{l \in \{0, 1, \cdots, n-1\} \mid f_l = f_j\} \geq 3$ holds. For such a edge f_j, there exist at least two edges $f_j^{(s)} = u_s u_{s+1}, f_j^{(t)} = u_t u_{t+1} \in F(\eta, f_j)$ such that $\overrightarrow{f_j^{(s)}} = \overrightarrow{f_j^{(t)}}$. Let $s < t$. Define $\eta_{st} = u_s u_{s+1} u_{s+2} \cdots u_{t-1} u_t$ (see Figure 3). Replace $\eta = u_0 \cdots u_{s-1} \eta_{st} u_{t+1} \cdots u_n$ to $\tilde{\eta} = \tilde{u_0} \tilde{u_1} \cdots \tilde{u_n} = u_0 \cdots u_{s-1} \tilde{\eta}_{st} u_{t+1} \cdots u_n$ where $\tilde{\eta}_{st} = u_s u_{t-1} u_{t-2} \cdots u_{s+2}$. $\tilde{\eta}$ is again a path, $V(\tilde{\eta}) = V(H)$ and $\sharp F(\tilde{\eta}, f_j) = \sharp F(\eta, f_j) - 2$ (see Figure 4). Repeat this operation to $f_0, f_1, \cdots, f_{n-1}$ inductively until obtaining the path satisfying $(a), (b)$.

(2) Note that w_j appears in $V(\gamma)$ at most $2\deg(w_j)$ times for each vertex $w_j \in V(H)$. The conclusion can be verified easily. □

Proof of Proposition 5.2 . $\{X_0, X_1, \cdots, X_n\}$ is itself a connected subgraph of G. So applying Lemma 5.3 for $\{X_0, X_1, \cdots, X_n\}$, we have

$$\min\{|\gamma| \mid \gamma \text{ is a path starting at } X_0, V(\gamma) = \{X_0, X_1, \cdots, X_n\}\} \leq 2MR_n.$$

By this and (5.20), we obtain $d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n) \leq (2M + 1)R_n$. □

5.2 Proof of Theorem 2.4(I)

In this subsection, we prove the LILs for $\{Y_n\}_{n \geq 0}$ when $d_s/2 < 1$.

Theorem 5.4. *Assume that Assumption* 2.2 *and* $d_s/2 < 1$ *hold. Then there exist (non-random) constants* $c_1, c_2 > 0$ *such that the following hold:*

$$\limsup_{n\to\infty} \frac{R_n}{n^{d_s/2}(\log\log n)^{1-d_s/2}} = c_1, \qquad P_x\text{-}a.s.\ \ \forall x \in V(G), \tag{5.24}$$

$$\liminf_{n\to\infty} \frac{R_n}{n^{d_s/2}(\log\log n)^{-d_s/2}} = c_2, \qquad P_x\text{-}a.s.\ \ \forall x \in V(G). \tag{5.25}$$

This is a discrete analog of [4, Proposition 4.9, 4.10]. Note that the proof of the propositions relies on the self-similarity of the process. Since our random walk does not satisfy this property, we need non-trivial modifications for the proof. Quite recently, Kim, Kumagai and Wang [10, Theorem 4.14] proved the LIL of the range for jump process without using self-similarity of process. By easy modifications, we can apply their argument for our random walk. The proof of Theorem 5.4 will be given in Appendix A.

Proof of Theorem 2.4(I). By (2.9), (5.20), Proposition 5.1, Proposition 5.2 and Theorem 5.4, we obtain (2.12) and (2.13). □

5.3 Proof of Theorem 2.4(II)

In this subsection, we prove the LILs for $\{Y_n\}_{n\geq 0}$ when $d_s/2 > 1$.

First, we explain the notion of "uniform condition" defined in [12]. We define the Dirichlet form $\mathcal{E}$ on the weighted graph (G, μ) by

$$\mathcal{E}(f,g) = \sum_{x,y\in V(G)} (f(x)-f(y))(g(x)-g(y))\mu_{xy},$$

for $f, g : V(G) \to \mathbb{R}$, and the effective resistance $R_{\text{eff}}(\cdot,\cdot)$ as

$$R_{\text{eff}}(A,B)^{-1} = \inf\{\mathcal{E}(f,f); f|A = 1, f|B = 0\}$$

for $A, B \subset V(G)$ with $A \cap B = \emptyset$. Denote $\rho(x,n) = R_{\text{eff}}(\{x\}, B(x,n)^c)$ for any $x \in V(G), n \in \mathbb{N}$ and $\rho(x) = \lim_{n\to\infty} \rho(x,n)$.

Definition 5.5 (Okamura [12])**.** *We say that a weighted graph* (G,μ) *satisfies the uniform condition* (U) *if* $\rho(x,n)$ *converges uniformly to* $\rho(x)$ *as* $n \to \infty$.

For $A \subset G$, define

$$T_A^+ = \inf\{n \geq 1 \mid X_n \in A\}.$$

We write T_x^+ in stead of $T_{\{x\}}^+$.

The following is an improvement of [12, Corollary 2.3].

Proposition 5.6. *Let* G *be a graph of bounded degree and* (G,μ) *be a weighted graph satisfying* (U) *and* (2.10)*. If* $\sup_x P_x(M < T_x^+ < \infty) = O(M^{-\delta})$ *for some* $\delta > 0$*, then*

$$1 - F_2 \leq \liminf_{n\to\infty} \frac{R_n}{n} \leq \limsup_{n\to\infty} \frac{R_n}{n} \leq 1 - F_1, \qquad P_x\text{-}a.s. \tag{5.26}$$

holds for all $x \in V(G)$*, where* $F_1 = \inf_{x\in V(G)} T_x^+$ *and* $F_2 = \sup_{x\in V(G)} T_x^+$.

Remark 5.7. *In [12, Corollary 2.3], a stronger condition* $\sup_x P_x(M < T_x^+ < \infty) = O(M^{-1-\delta})$ *for some* $\delta > 0$ *is imposed to prove* $1 - F_2 \leq \liminf_{n\to\infty} \frac{R_n}{n}$*. As we prove below, it is enough to assume* $\sup_x P_x(M < T_x^+ < \infty) = O(M^{-\delta})$.

Proof of Proposition 5.6 . For the upper bound $\limsup_{n\to\infty} \frac{R_n}{n} \le 1 - F_1$, the proof in [12, Corollary 2.3] goes through without any modifications.

Hence we prove $1 - F_2 \le \liminf_{n\to\infty} \frac{R_n}{n}$ under our assumption. Fix $\epsilon > 0$. By [12, (2.5), (2.6), (2.7)] there exists $a \in (0, 1)$ such that for any n and $M < n$ we have

$$P_x(\frac{R_n}{n} \le 1 - F_2 - \epsilon) \le \frac{2}{\epsilon} \sup_{x\in V(G)} P_x(M < T_x^+ < \infty) + a^{n/(M+1)}. \tag{5.27}$$

Choose $k > 2/\delta$. Replacing n to n^k in (5.27), we have

$$\begin{aligned} P_x(\frac{R_{n^k}}{n^k} \le 1 - F_2 - \epsilon) &\le \frac{2}{\epsilon} \sup_{x\in V(G)} P_x(M < T_x^+ < \infty) + a^{n^k/(M+1)} \\ &= \frac{2}{\epsilon} O(M^{-\delta}) + a^{n^k/(M+1)}. \end{aligned} \tag{5.28}$$

Take $M = M(n) = n^{k/2} - 1$ and we have

$$P_x(\frac{R_{n^k}}{n^k} \le 1 - F_2 - \epsilon) \le \frac{2}{\epsilon} O(\frac{1}{n^{k\delta/2}}) + a^{n^{k/2}}. \tag{5.29}$$

Since $k\delta/2 > 1$, we can apply the Borel-Cantelli lemma and we obtain

$$1 - F_2 \le \liminf_{n\to\infty} \frac{R_{n^k}}{n^k}.$$

For any m, choose n as $n^k \le m < (n + 1)^k$, we then have

$$\frac{R_m}{m} \ge \frac{n^k}{m}\frac{R_{n^k}}{n^k} = \left(\frac{n}{n+1}\right)^k \frac{(n+1)^k}{m}\frac{R_{n^k}}{n^k} \ge \left(\frac{n}{n+1}\right)^k \frac{R_{n^k}}{n^k}.$$

Take $\liminf_{m\to\infty}$ and we obtain $1 - F_2 \le \liminf_{n\to\infty} \frac{R_n}{n}$. □

Proof of Theorem 2.4(II). Note that the uniform condition (U) is satisfied in our framework by [12, Proposition 4.6].

Since $d_s/2 > 1$, we have

$$P_x(M < T_x^+ < \infty) \le \sum_{n=M+1}^{\infty} p_n(x, x) \le \sum_{n=M+1}^{\infty} n^{-d_s/2} = O(M^{1-d_s/2}).$$

By this and Proposition 5.6, we have

$$1 - F_2 \le \liminf_{n\to\infty} \frac{R_n}{n} \le \limsup_{n\to\infty} \frac{R_n}{n} \le 1 - F_1, \qquad P_x\text{-a.s.} \tag{5.30}$$

Define $G(x, x) = \sum_{n=0}^{\infty} p_n(x, x)$, and $F(x, x) = \sum_{n=1}^{\infty} P_x(T_x^+ = n) = P_x(T_x^+ < \infty)$. It is well known that

$$G(x, x) - 1 = F(x, x)G(x, x) \tag{5.31}$$

holds. Since $d_s/2 > 1$, we have

$$\sup_{x\in V(G)} \sum_{n=0}^{\infty} p_n(x, x) \le \sum_{n=0}^{\infty} \frac{1}{n^{d_s/2}} < \infty.$$

By this and (5.31) we have

$$F_2 = \sup_x F(x,x) < 1. \tag{5.32}$$

Thus, by (5.20), Proposition 5.1, Proposition 5.2, (5.30) and (5.32), we conclude that

$$\begin{aligned} 0 < \frac{1}{4}(1-F_2) &\le \liminf_{n\to\infty} \frac{d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)}{n} \\ &\le \limsup_{n\to\infty} \frac{d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)}{n} \le (2M+1)(1-F_1) < \infty, \qquad a.s. \end{aligned}$$

Hence we complete the proof. □

Appendix A Proof of Theorem 5.4

In this section, we will explain briefly the essential part of the proof of Theorem 5.4 which is a discrete analog of [4, Proposition 4.9, 4.10]. Note that the results in [4] are for the range of Brownian motion on fractals, and the proof heavily relies on the self-similarity of Brownian motion. Quite recently, Kim, Kumagai and Wang [10, Theorem 4.14] obtained the LIL of the range for jump processes on metric measure spaces without scaling law of the process. We employ the results and techniques in [4], [6] and [10], and prove the LIL for the range of the random walk without scaling law of the process and heat kernel.

The key to prove the LILs for the range of the process is to establish those for the maximum of local times. We assume $d_f < d_w$ and define the local times at $x \in V(G)$ up to the time n as

$$L_n(x) = \begin{cases} \frac{1}{m(x)} \sum_{k=0}^{n-1} 1_{\{X_k = x\}} & \text{if } n \ge 1, \\ 0 & \text{if } n = 0. \end{cases} \tag{A.1}$$

and the maximum of the local times up to the time n as

$$L_n^* = \sup_{x \in V(G)} L_n(x). \tag{A.2}$$

Let $\theta = (d_w - d_f)/2$. We start with the following lemma.

Lemma A.1. *There exist constants $c_0, c_1, c_2 > 0$ such that*

$$\begin{aligned} &\sup_{i\ge 1} \max_{\substack{x,y,z \in V(G) \\ d(x,y) \le i}} P_z \left(\max_{0 \le k \le T i^{d_w}} |L_k(x) - L_k(y)| \ge \lambda (i d(x,y))^\theta \right) \\ &\quad \le c_0 \exp(c_1 T) \exp(-c_2 \lambda) \end{aligned} \tag{A.3}$$

holds for all $T \ge 1$ and $\lambda > 0$.

Proof. This can be proved by easy modifications of the proof of [6, Proposition 6.3(a)]. The necessary changes are the following; (i) to chase the dependence on T explicitly, (ii) to use the following relations between the resistance metric and graph distance

$$R(x,y) \asymp d(x,y)^{d_w - d_f}, \qquad \forall x, y \in V(G), \tag{A.4}$$

which is a consequence of Assumption 2.2 (see [3]). □

The next theorem is the analogue of [10, Proposition 4.5]. Since our proof is different from that of [10, Proposition 4.5] which uses a scaling argument, we give the proof below.

Theorem A.2 (Moduli of continuity of local times)**.** *There exist constants c, $C > 0$ such that*

$$P_o\left(\max_{\substack{x,y\in B_d(o,\kappa u^{1/d_w})\\ d(x,y)\le L}}\max_{0\le t\le u}|L_t(x)-L_t(y)|\ge A\right)\le c\frac{(u^{1/d_w}\kappa)^{2d_f}}{L^{2d_f}}\exp\left(-\frac{CA}{(\kappa u^{1/d_w}L)^{\theta}}\right). \tag{A.5}$$

holds for all $o\in V(G)$, $u\ge 1$, $\kappa\ge 1$ and $A, L>0$.

Proof. Let $G^{(i)}$ be a graph with $V(G^{(i)}) = B_d(o,6i)$ and $E(G^{(i)}) = \{(x,y)\in E(G) \mid x,y\in V(G^{(i)})\}$. We denote $m_i(\cdot) = m_i(\cdot\cap V(G^{(i)}))$ as the weight of $G^{(i)}$. Then the following holds by the proof of [6, Theorem 6.1]; There exists a positive constant c_1 (not depending on i) such that

$$m_i(B_d(x,r))\ge c_1 r^{d_f} \tag{A.6}$$

for all $x\in V(G^{(i)})$ and $r\in[1,12i]$. Set $6i=\kappa u^{1/d_w}$. By (A.6), we have

$$\min_{x\in V(G^{(i)})} m_i(B_{d_i}(x,r)) = \min_{x\in V(G^{(i)})} m_i(B_d(x,ir))\ge c_1 i^{d_f} r^{d_f}.$$

We now apply a discrete version of Garsia's Lemma (see [6, Proposition 3.1, Remark 3.2]) for the graph $G^{(i)}$ with distance $d_i=\frac{1}{i}d$, $p(x)=x^{\theta}$, $\psi(x)=\exp(c_5|x|)-1$, and the function on $V(G^{(i)})$ as $f(x)=\dfrac{1}{i^{2\theta}}L_t(x)$ where $0\le t\le u$. For $x,y\in V(G^{(i)})=B_d(o,6i)$ with $d(x,y)\le L$ and $t\in[0,u]$, we have

$$\begin{aligned}\frac{1}{i^{2\theta}}|L_t(x)-L_t(y)| &\le 4\int_0^{2d_i(x,y)} s^{\theta-1}\log\left(\frac{\Gamma\left(\frac{1}{i^{2\theta}}L_t\right)}{c_1 i^{2d_f}s^{2d_f}/2^{2d_f}}+1\right)ds\\ &\le 4\int_0^{2L/i} s^{\theta-1}\log\left(\frac{\tilde\Gamma\left(\frac{1}{i^{2\theta}}L_u\right)}{c_2 i^{2d_f}s^{2d_f}}+1\right)ds,\end{aligned} \tag{A.7}$$

where

$$\begin{aligned}\Gamma\left(\frac{1}{i^{2\theta}}L_t\right) &:= \sum_{x,y\in V(G^{(i)})}\exp\left(c_*\frac{|L_t(x)-L_t(y)|}{(id(x,y))^{\theta}}\right)m(x)m(y),\\ \tilde\Gamma\left(\frac{1}{i^{2\theta}}L_u\right) &:= \sum_{x,y\in V(G^{(i)})}\exp\left(c_*\frac{\sup_{0\le t\le u}|L_t(x)-L_t(y)|}{(id(x,y))^{\theta}}\right)m(x)m(y),\end{aligned}$$

for some $c_*>0$ that will be chosen later. Define $v=\dfrac{\tilde\Gamma\left(\frac{1}{i^{2\theta}}L_u\right)}{c_2 i^{2d_f}s^{2d_f}}$. Then by (A.7), we have

$$\begin{aligned}\frac{1}{i^{2\theta}}|L_t(x)-L_t(y)| &\le 4\frac{c_3^{\theta}}{2d_f}\frac{1}{i^{\theta}}\tilde\Gamma\left(\frac{1}{i^{2\theta}}L_u\right)\int_b^{\infty}\left(\frac{1}{v}\right)^{\theta/(2d_f)+1}\log(v+1)dv\\ &= c_5\frac{1}{i^{\theta}}\tilde\Gamma\left(\frac{1}{i^{2\theta}}L_u\right)\int_b^{\infty}\left(\frac{1}{v}\right)^{\theta/(2d_f)+1}\log(v+1)dv,\end{aligned}$$

where $b=\dfrac{\tilde\Gamma\left(\frac{1}{i^{2\theta}}L_u\right)}{c_4 L^{2d_f}}$, $c_3=(1/c_2)^{1/(2d_f)}$, $c_4=c_2 2^{2d_f}$. By easy calculus we have

$$\int_b^{\infty}\left(\frac{1}{v}\right)^{\theta/(2d_f)+1}\log(v+1)dv\le\frac{\log(b+1)+2d_f/\theta}{\frac{\theta}{2d_f}\cdot b^{\theta/2d_f}}.$$

Thus we have

$$\frac{1}{i^{2\theta}}|L_t(x)-L_t(y)| \le c_6 \left(\frac{L}{i}\right)^{\theta} \left\{\log(b+1)+\frac{2d_f}{\theta}\right\}.$$

where $c_6 = 8d_f c_5/\theta$, so

$$\begin{aligned} P_o\left(\max_{\substack{x,y\in B_d(o,\kappa u^{1/d_w})\\ d(x,y)\le L}} \max_{0\le t\le u} |L_t(x)-L_t(y)| \ge A\right) &\le P_o\left(\log(b+1) \ge \frac{A}{c_6(iL)^{\theta}} - \frac{2d_f}{\theta}\right) \\ &\le c_7 \frac{E_0\left[\tilde{\Gamma}\left(\frac{1}{i^{2\theta}}L_u\right)\right]}{L^{2d_f}} \exp(-c_8\frac{A}{(iL)^{\theta}}). \end{aligned}$$

By Lemma A.1, taking $c_* < c_2$, noting $\kappa \ge 1$ and $6i = \kappa u^{1/d_w}$, we have

$$\begin{aligned} E_o\left[\tilde{\Gamma}\left(\frac{1}{i^{2\theta}}L_u\right)\right] &= \sum_{x,y\in V(G^{(i)})} E_o\left[\exp\left(c_*\frac{\sup_{0\le t\le u}|L_t(x)-L_t(y)|}{(id(x,y))^{\theta}}\right)\right] m(x)m(y) \\ &\le \sum_{x,y\in V(G^{(i)})} \sum_n \exp(c_*(n+1)) P_o\left(\frac{\sup_{0\le t\le u}|L_t(x)-L_t(y)|}{(id(x,y))^{\theta}} \ge n\right) m(x)m(y) \\ &\le \sum_{x,y\in V(G^{(i)})} \sum_n \exp(c_*(n+1)) \exp\left(c_9\left(\frac{1}{\kappa}\right)^{d_w}\right) \exp(-c_2 n) m(x)m(y) \\ &\le c_{10} i^{2d_f} \le c_{11} u^{2d_f/d_w}\kappa^{2d_f}. \end{aligned}$$

Therefore we have

$$P_o\left(\max_{\substack{x,y\in B_d(o,\kappa u^{1/d_w})\\ d(x,y)\le L}} \max_{0\le t\le u} |L_t(x)-L_t(y)| \ge A\right) \le c_{11}\frac{u^{2d_f/d_w}\kappa^{2d_f}}{L^{2d_f}} \exp\left(-c_{12}\frac{A}{(\kappa u^{1/d_w}L)^{\theta}}\right).$$

Thus we complete the proof. □

Given Theorem A.2, the following theorem can be proved similarly to the proof in [10, Theorem 4.10, 4.13]. (See also [4, Proposition 4.7, 4.8].)

Theorem A.3 (LILs for the local times)**.** *There exist positive constants c_1, c_2 such that the following hold.*

$$\limsup_{n\to\infty} \frac{L_n^*}{n^{1-d_s/2}(\log\log n)^{d_s/2}} = c_1 \qquad P_x\text{-a.s. for } \forall x \in V(G), \tag{A.8}$$

$$\liminf_{n\to\infty} \frac{L_n^*}{n^{1-d_s/2}(\log\log n)^{d_s/2-1}} = c_2 \qquad P_x\text{-a.s. for } \forall x \in V(G). \tag{A.9}$$

Given Theorem A.3, the proof of Theorem 5.4 can be done similarly as in [10, Theorem 4.14] by using the relation $n = \sum_{x\in R_n} L_n(x) \le R_n L_n^*$. (See also [4, Proposition 4.9, 4.10]).

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References

[1] Barlow, M.T.: Diffusions on fractals. Lect. Notes in Math. **1690**, Ecole d'été de probabilités de Saint-Flour XXV–1995, Springer, New York 1998.

[2] Barlow, M.T.: *Which values of the volume growth and escape time exponent are possible for a graph?*. Rev. Mat. Iberoamericana **20** (2004), no. 1, 1-31.

[3] Barlow, M.T., Coulhon, T., Kumagai, T.: *Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs*. Comm. Pure Appl. Math. **58** (2005), no. 12, 1642-1677.

[4] Bass, R.F., Kumagai, T.: *Laws of the iterated logarithm for some symmetric diffusion processes.* Osaka J. Math. **37** (2000), no. 3, 625-650.

[5] Coulhon, T., Grigor'yan, A., Pittet, C.: *A geometric approach to on-diagonal heat kernel lower bounds on groups*. Ann. Inst. Fourier (Grenoble) **51** (2001), no. 6, 1763-1827.

[6] Croydon, D.A.: *Moduli of continuity of local times of random walks on graphs in terms of the resistance metric*. Trans. London Math. Soc., to appear.

[7] Dvoretzky, A., Erdős, P.: *Some problems on random walk in space.* Proc. Second Berkeley Symp. Math. Statist. Probab. University of California Press, (1951), 353-367.

[8] Gibson, L.R.: *The mass of sites visited by a random walk on an infinite graph.* Electro. J. Probab. **13** (2008), 1257-1282.

[9] Kigami, J.: Analysis on fractals, Cambridge Tracts in Math. **143**, Cambridge University Press, Cambridge, 2001.

[10] Kim, P., Kumagai, T., Wang, J.: *Laws of the iterated logarithm for symmetric jump processes.* Preprint 2015, available at `arXiv:1504.06210`.

[11] Kumagai, T.: Random walks on disordered media and their scaling limits. Lect. Notes in Math. **2101**, Ecole d'été de probabilités de Saint-Flour XL–2010, Springer, New York 2014.

[12] Okamura, K.: *On the range of random walk on graphs satisfying a uniform condition.* ALEA Lat. Am. J. Probab. Math. Stat. **11** (2014), no. 1, 341-357.

[13] Pittet, C., Saloff-Coste, L.: *Amenable groups, isoperimetric profiles and random walks.* Geometric group theory down under (Canberra, 1996), 293-316, de Gruyter, Berlin, 1999.

[14] Pittet, C., Saloff-Coste, L.: *A survey on the relationships between volume growth, isoperimetry, and the behavior of simple random walk on Cayley graphs, with examples.* Preprint 2001, available at `http://www.math.cornell.edu/~lsc/articles.html`.

[15] Rau, C.: *Existence of graphs with sub exponential transitions probability decay and applications.* Bull. Soc. Math. France **138** (2010), no. 4, 491-542.

[16] Revelle, D.: *Rate of escape of random walks on wreath products and related groups.* Ann. Probab. **31** (2003), no. 4, 1917-1934.

[17] Varopoulos, N.: *Random walks on soluble groups.* Bull. Sci. Math. **107** (1983), 337-344.

[18] Varopoulos, N.: *A potential theoretic property of soluble groups.* Bull. Sci. Math. **108** (1984), 263-273.

[19] Woess, W.: Random walks on infinite graphs and groups. Cambridge Tracts in Math. **138**, Cambridge University Press, Cambridge, 2000.