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STABILITY AND INSTABILITY OF GAUSSIAN HEAT KERNEL ESTIMATES FOR RANDOM WALKS AMONG TIME-DEPENDENT CONDUCTANCES

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ABSTRACT. We consider time-dependent random walks among time-dependent conductances. For discrete time random walks, we show that, unlike the time-independent case, two-sided Gaussian heat kernel estimates are not stable under perturbations. This is proved by giving an example of a ballistic and transient time-dependent random walk on $\mathbb{Z}$ among uniformly elliptic time-dependent conductances. For continuous time random walks, we show the instability when the holding times are i.i.d. exp(1), and in contrast, we prove the stability when the holding times change by sites in such a way that the base measure is a uniform measure.

1. INTRODUCTION

The study of heat kernels of diffusions on manifolds and Markov chains on graphs has a very long and fruitful history. One of the motivations was to obtain a priori estimates such as the estimates of the Hölder continuity for the solutions of heat equations. In the framework of the divergence operator $\mathcal{L} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ on $\mathbb{R}^d$ where $a_{ij}(\cdot)$ is measurable and symmetric, there are significant work by De Giorgi, Nash and Moser around late 50s to early 60s. For the divergence form satisfying a uniform elliptic condition, Aronson [Ar] proved the following two-sided Gaussian heat kernel estimates for all $t > 0, x, y \in \mathbb{R}^d$:

$$c_1 t^{-d/2} \exp\Big(-\frac{c_2 d(x,y)^2}{t}\Big) \le p_t(x,y) \le c_3 t^{-d/2} \exp\Big(-\frac{c_4 d(x,y)^2}{t}\Big). \tag{1.1}$$

Later in the last century, the two-sided Gaussian estimates were obtained for many operators in many spaces and the heat kernel estimates were investigated from various aspects. One of the important directions is to establish the stability of the estimates, namely to show that the estimates are preserved when the operator (or the corresponding Dirichlet form) is perturbed in a suitable way. Consider the Laplace-Beltrami operator on a complete Riemannian manifold with $d(\cdot,\cdot)$ and μ being the Riemannian metric and the Riemannian measure. Early in 90s, Grigor'yan [Gr] and Saloff-Coste [SC] independently proved that for the Laplace-Beltrami operator, a variant of (1.1) (i.e. changing $t^{-d/2}$ into $\mu(\mathbb{B}(x,t^{1/2}))^{-1}$) is equivalent to a volume doubling condition (VD) plus Poincaré inequalities (PI(2)) via the

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equivalence to parabolic Harnack inequalities –see Theorem 1.2 for definitions of the terminologies. Since (VD) and (PI(2)) are stable under the perturbations, one can obtain the stability of the heat kernel estimates. The results were later extended to the framework of Dirichlet forms on metric measure spaces by Sturm [St1, St2] and graphs by Delmotte [De]. We note that such a stability theory has been extended to the sub-Gaussian heat kernel estimates, and also the theory has been very useful in the recent developments of the random walk among random conductances (see for example [MB, Kum]).

In this note, we are mainly interested in cases when the edge conductances of the graph are themselves changing in time, independently of the walk. We will consider the stability of the two-sided Gaussian heat kernel estimates in this setting. A naive guess is that the stability holds at least when the time-dependent conductance is bounded from above and below uniformly by positive constants. However, this naive guess is completely wrong for discrete time (time-dependent) random walks. Indeed, in Proposition 1.3(i), we give an example of a ballistic and transient time-dependent random walk on $\mathbb{Z}$ among uniformly elliptic time-dependent conductances. We also give a counter example in the setting of continuous time (time-dependent) random walks, called constant speed random walks, when the holding times are i.i.d. exp(1) (Proposition 1.3(ii)). Contrary to the above, when the holding times change by sites in such a way that the base measure is a uniform measure (called a variable speed random walk), we can prove the stability by proving the equivalence of the heat kernel estimates to (VD) and (PI(2)) –see Theorem 1.2. We note that the stability of parabolic Harnack inequalities and estimates of the heat kernel were already established in the framework of time-dependent Dirichlet forms on metric measure spaces by Sturm [St1, St2], and in [DD, GOS] it was proved that for random walks on $\mathbb{Z}^d$ among uniformly elliptic time-dependent conductances, the two-sided Gaussian heat kernel estimates hold. Also in [GP], some criteria was given on the recurrence and transience of a set using the heat kernel estimates. These are results for variable speed random walks. The purpose of this note is to demonstrate a fundamental difference between discrete time random walks (or constant speed random walks) and variable speed random walks even in the framework that the random walks are uniformly elliptic and uniformly lazy. In particular, we find that both the upper and lower Gaussian bounds can be violated in these situations (see Proposition 1.3 and 1.4). This contrasts to the above mentioned situation on graphs with time-independent conductances, where the discrete time and the two types of continuous time random walks share the same long-time properties at least in the uniformly elliptic setting.

Let us mention some related works. [ABGK, DHS] study recurrence versus transience of discrete time simple random walks on graphs with monotonically changing conductances. To be fair, we note that our example in Proposition 1.3(i) borrowed an idea from [ABGK, Example 3.5]. In [GPZ], they consider controlled random walks, namely random walks that are martingales with uniformly bounded increments and nontrivial jump probabilities (that may depend on the behavior of the random walks), and show that anomalous behavior of the heat kernels can occur in the framework. The readers may find further related works in the references of the above papers.

1.1. **Framework and main results.** Let $\mathbb{G} = (V, E)$ be a locally finite connected graph. Assume that for each $t \geq 0$, the graph $\mathbb{G}$ is endowed with a conductance (weight) $\mu^{(t)}(x, y)$

which is a symmetric nonnegative deterministic function on $V \times V$ such that $\mu^{(t)}(x,y) > 0$ if and only if $\{x,y\} \in E$. Suppose further that the map $t \mapsto \mu^{(t)}(x,y)$ is right continuous and has left limit (RCLL for short) for each $\{x,y\} \in E$. We call $(\mathbb{G}, \{\mu^{(t)}(x,y)\})$ a time-dependent weighted graph. Let $\mu^{(t)}(x) := \sum_y \mu^{(t)}(x,y)$ for each x and define a measure $\mu^{(t)}$ on V by setting $\mu^{(t)}(A) = \sum_{x\in A} \mu^{(t)}(x)$ for each $A \subset V$. Let ν be a uniform measure on V, that is $\nu(A) = |A|$ for $A \subset V$ where $|A|$ is a cardinality of A. Throughout the paper, we assume the following: there exist $\mu(x,y) = \mu(y,x) > 0$ ($\{x,y\} \in E$) and $c_0, p_0 \in (0,1]$ such that

$$c_0\mu(x,y) \le \mu^{(t)}(x,y) \le c_0^{-1}\mu(x,y), \qquad \forall \{x,y\} \in E, \tag{1.2}$$

$$\frac{\mu(x,y)}{\sum_{z:\{x,z\}\in E} \mu(x,z)} \ge p_0, \qquad \forall \{x,y\} \in E. \tag{1.3}$$

We say time-dependent conductances are uniformly elliptic if there exists $c_1 \in (0,1]$ such that

$$c_1 \le \mu^{(t)}(x,y) \le c_1^{-1}, \qquad \forall \{x,y\} \in E.$$

We now define a quadratic form on $(\mathbb{G}, \{\mu^{(t)}(x,y)\})$ as follows:

$$\mathcal{E}_t(f,g) = \frac{1}{2} \sum_{x,y\in V} (f(x)-f(y))(g(x)-g(y))\mu^{(t)}(x,y)$$

for each $f, g \in H_t^2$, where

$$H_t^2 = \{f : V \to \mathbb{R} : \sum_{x,y\in V} (f(x)-f(y))^2 \mu^{(t)}(x,y) < \infty\}.$$

Define discrete Laplace operators as follows:

$$\begin{aligned} \mathcal{L}_t^C f(x) &= \sum_y (f(y)-f(x)) \frac{\mu^{(t)}(x,y)}{\mu^{(t)}(x)}, \\ \mathcal{L}_t^V f(x) &= \sum_y (f(y)-f(x))\mu^{(t)}(x,y), \ \mathcal{L}^V f(x) = \sum_y (f(y)-f(x))\mu(x,y). \end{aligned}$$

For each f, g that has finite support, we have

$$\mathcal{E}_t(f,g) = -(\mathcal{L}_t^V f, g)_\nu = -(\mathcal{L}_t^C f, g)_{\mu^{(t)}},$$

where $(f,g)_\theta := \sum_{x,y} f(x)g(x)\theta(x)$ for a measure θ.

We next provide definitions for discrete time and continuous time constant/variable speed random walks on $(\mathbb{G}, \{\mu^{(t)}(x,y)\})$. One way to construct such processes is through the theory of time-dependent Dirichlet forms (see [O]), but this will require some knowledge of probabilistic potential theory and some more notation. Here we give a more direct definition. For $x, y \in V$, we define

$$P^{(t)}(x,y) := \mu^{(t)}(x,y)/\mu^{(t)}(x).$$

Definition 1.1. *(i) The V-valued stochastic process $\{X_t\}_{t\in\mathbb{N}}$ is called a discrete time random walk on $\mathbb{G}$, if its transition probabilities at time $t \in \mathbb{N}$ are given by $P(t,x;t+1,y) = P^{(t)}(x,y)$, for any $\{x,y\} \in E$.*

(ii) The V-valued stochastic process $\{Y_t\}_{t\in\mathbb{R}_+}$ of RCLL *sample path $t \mapsto Y_t$ is called a constant speed random walk (in short* CSRW*), if it waits i.i.d.* $\exp(1)$ *times between successive jumps, and if $Y_{T^-} = x$ just prior to the current random jump time T, then the process jumps across each $\{x, y\} \in E$ with probability $P^{(T)}(x, y)$.*
(iii) The V-valued stochastic process $\{Y_t\}_{t\in\mathbb{R}_+}$ of RCLL *sample path $t \mapsto Y_t$ is called a variable speed random walk (in short* VSRW*), if the holding time of the particle at $x \in V$ at time $t \in \mathbb{R}_+$ is independent with the law* $\exp(\mu^{(t)}(x))$*, and if $Y_{T^-} = x$ just prior to the current random jump time T, then the process jumps across each $\{x, y\} \in E$ with probability $P^{(T)}(x, y)$.*

We first show that, for the VSRW we have the stability of Gaussian heat kernel estimates as expected. While we could not find out the precise statement as given below, the proof is a careful line by line modifications of the known proof (such as the proof in [De]). Once again we note that it is proved in [DD, Sect. 4] and [GOS, Appendix B] that any VSRW on $\mathbb{Z}^d$ among uniformly elliptic time-dependent conductances must satisfy the two-sided Gaussian heat kernel bounds. In the setting of time-dependent local regular Dirichlet forms on metric measure spaces, similar results are given in [St1] and the equivalence of the parabolic Harnack inequalities and the volume doubling property plus the Poincaré inequalities are given in [St2].

Note that for the VSRW $\{X_t\}_{t\geq 0}$, the heat kernel $p(s, x; t, y)$ is equal to $P(X_t = y|X_s = x)$ since the base measure is a uniform measure.

Theorem 1.2. *Let $\mathbb{G} = (V, E)$ be a locally finite connected graph with conductances $\{\mu(x, y) : x, y \in V\}$ satisfying* (1.3)*. Then the following are equivalent:*
(a) The graph $\mathbb{G}$ satisfies the volume doubling with constant $C_1 < \infty$, namely

$$\nu(\mathbb{B}(x, 2r)) \leq C_1 \nu(\mathbb{B}(x, r)) \tag{1.4}$$

for all $x \in \mathbb{G}$, $r > 0$; and the Poincaré inequality holds for some (thus for all) time-dependent conductances that satisfy (1.2) *with constant $C_2 < \infty$, namely*

$$\sum_{x\in\mathbb{B}(x_0,r)} |f(x) - f_{\mathbb{B}}|^2 \leq C_2 r^2 \sum_{x,y\in\mathbb{B}(x_0,2r)} (f(x) - f(y))^2 \mu^{(t)}(x, y), \tag{1.5}$$

for all $f : V \to \mathbb{R}$, $x_0 \in \mathbb{G}$, $r > 0$, where $f_{\mathbb{B}} = \sum_{x\in\mathbb{B}(x_0,r)} f(x)/\nu(\mathbb{B}(x_0, r))$.
(b) The parabolic Harnack inequality holds for all time-dependent conductances that satisfy (1.2)*, and all non-negative solutions of equation*

$$\frac{\partial u(t, x)}{\partial t} = \mathcal{L}^V_t u(t, x).$$

That is, set $\eta \in (0, 1)$ and $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4$, we have for all x_0, s, r, all $\mathcal{L}^V_t$ satisfying (1.2)*, and every non-negative solution on cylinder $Q = [s, s + \theta_4 r^2] \times \mathbb{B}(x_0, r)$,*

$$\sup_{Q_-} u \leq C_3 \inf_{Q_+} u,$$

where $Q_- = [s + \theta_1 r^2, s + \theta_2 r^2] \times \mathbb{B}(x_0, \eta r)$ and $Q_+ = [s + \theta_3 r^2, s + \theta_4 r^2] \times \mathbb{B}(x_0, \eta r)$, with some $C_3 < \infty$.

(b^*) The parabolic Harnack inequality holds for all non-negative solutions of equation

$$\frac{\partial u(t,x)}{\partial t} = \mathcal{L}^V u(t,x).$$

(c) The following two-sided heat kernel estimates hold for all VSRW*s with time-dependent conductances that satisfy* (1.2)*: there exist positive constants $C_4, C_5, c_6, c_7 < \infty$ such that*

$$p(0,x;t,y) \le \frac{C_4}{\nu(\mathbb{B}(x,t^{1/2}))} \exp\Big(- C_5\Big(d(x,y)^2/t \wedge d(x,y)(1 \vee \log(d(x,y)/t))\Big)\Big), \quad (1.6)$$

$$\frac{c_6}{\nu(\mathbb{B}(x,t^{1/2}))} \exp\Big(- c_7 d(x,y)^2/t\Big) \le p(0,x;t,y), \quad \forall t \ge d(x,y). \quad (1.7)$$

for all $x, y \in V$, $t > 0$ (with a restriction $t \ge d(x,y)$ in (1.7)*).*

Since the proof is similar to that of time-independent case, we will simply give a sketch of the proof in the next section. As a consequence of this theorem, we can see that the VSRW on $\mathbb{Z}^d$ among uniformly elliptic time-dependent conductances enjoys the Gaussian heat kernel estimates (1.6) and (1.7).

In contrast to the above theorem, for the discrete time random walk and the CSRW, one can construct a transient random walk on $\mathbb{Z}$ among uniformly elliptic time-dependent conductances as in the next proposition (cf. [ABGK, Example 3.5]).

Let $\gamma < 1$. We say a time-dependent discrete time random walk is γ-lazy if $P^{(t)}(x,x) \ge \gamma$ for all $x \in V$ and all $t \ge 0$.

Proposition 1.3. *(i) For any $\gamma < 1$ and $\varepsilon > 0$ there exist time-dependent conductances $\{\mu^{(t)}(x, x \pm 1), \mu^{(t)}(x,x) : x \in \mathbb{Z}\}$ on $\mathbb{Z}$ with*

$$1 - \varepsilon \le \mu^{(t)}(x, x \pm 1) \le 1 + \varepsilon, \;\; \forall x \in \mathbb{Z}, \, t \in \mathbb{N}$$

such that the corresponding discrete time random walk $\{X_t\}_{t \in \mathbb{N}}$ is γ-lazy, and it is ballistic and transient almost surely (i.e. it returns to starting point finitely often).

(ii) For any $\varepsilon, c > 0$, there exist time-dependent conductances $\{\mu^{(t)}(x, x \pm 1) : x \in \mathbb{Z}\}$ on $\mathbb{Z}$ with

$$1 - \varepsilon \le \mu^{(t)}(x, x \pm 1) \le 1 + \varepsilon, \;\; \forall x \in \mathbb{Z}, \, t \in \mathbb{R}_+$$

and the times $\{t_n\}$ at which the conductances change satisfying $t_n/n \to 1/c$, such that the corresponding CSRW *$\{Y_t\}_{t \in \mathbb{R}_+}$ is ballistic and transient almost surely.*

In particular, both walks violate the Gaussian heat-kernel on-diagonal lower bound as well as off-diagonal upper bound on $\mathbb{Z}$.

In this proposition, we give examples where the edge conductances are periodically fluctuating in time. It remains open whether adding monotone condition on the edge conductances would recover the expected Gaussian lower bound.

In a similar manner, we also give in the next proposition examples of discrete time random walks and CSRW on $\mathbb{Z}^2 \times \mathbb{Z}_{\ge 0}$ with uniformly elliptic time dependent conductances that violate the on-diagonal Gaussian upper bound. Let $\{e_1, e_2, e_3\}$ be the Cartesian standard basis of $\mathbb{Z}^3$.

Proposition 1.4. *(i) For any $\gamma < 1$ and $\epsilon > 0$ there exist time-dependent conductances $\{\mu^{(t)}(x, x \pm e_i), \mu^{(t)}(x, x) : x \in \mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}, i = 1, 2, 3\}$ on $\mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}$ with*

$$1 - \epsilon \leq \mu^{(t)}(x, x \pm e_i) \leq 1 + \epsilon, \quad i = 1, 2, 3, \forall x \in \mathbb{Z}^2 \times \mathbb{Z}_{>0}, t \in \mathbb{N}$$

such that the corresponding discrete time random walk $\{X_t\}_{t\in\mathbb{N}}$ is γ-lazy and recurrent almost surely (i.e. it returns to starting point infinitely often).

(ii) For any $\epsilon, c > 0$, there exist time-dependent conductances $\{\mu^{(t)}(x, x \pm e_i) : x \in \mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}, i = 1, 2, 3\}$ on $\mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}$ with

$$1 - \epsilon \leq \mu^{(t)}(x, x \pm e_i) \leq 1 + \epsilon, \quad i = 1, 2, 3, \forall x \in \mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}, t \in \mathbb{R}_+$$

and the times $\{t_n\}$ at which the conductances change satisfying $t_n/n \to 1/c$, such that the corresponding CSRW *$\{Y_t\}_{t\in\mathbb{R}_+}$ is recurrent almost surely.*

In particular, both walks violate the Gaussian heat kernel on-diagonal upper bound on $\mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}$.

2. Proof.

Sketch of the proof of Theorem 1.2.

(a) $\Rightarrow$ (b) $\Rightarrow$ (c): As explained in [DD, pg 374-375], it is possible to adapt Delmotte's argument ([De]) here by setting, in their notation, $\mu_{xy} = a(t, x, y) =: \mu^{(t)}(x, y)$ and $m(x) =: 1$. (Note that Delmotte's proof is for the discrete time random walk and CSRW.) In [DD], there is an extra assumption that the conductances are uniform elliptic. However, this extra assumption is not needed by the following reasons. First, concerning the point mentioned after the statement of [DD, Proposition 4.2], there is no need to put the weight $a(t, x, y)$ in the computations. In other word, [De, Theorem 2.3] holds simply by putting $m(x) = 1$. Second, concerning the heat kernel lower bound discussed in the proof of [DD, Proposition 4.3], we do not need to obtain the lower bound when $\|x\|_\Gamma \geq t$ since (1.7) is for $d(x, y) \leq t$. Note that although the framework of [DD] is $\mathbb{Z}^d$, the same modification can be employed for general $\mathbb{G}$. In both [De, DD] the term $\mathcal{E}(t, D) = \exp(-D \operatorname{arg\,sinh} \frac{D}{t} + t(\sqrt{1 + D^2/t^2} - 1))$ appears in the off-diagonal bounds, but simple computations show that it is comparable with the exponential parts of (1.6), (1.7). Let us now overview the proof. Assuming (a), (1.2) translates to a Poincaré inequality that holds uniformly for all t, as well as a weighted Poincaré inequality and a Sobolev-Poincaré inequality needed along the way ([De, Proposition 2.2, 2.4]). Since [De, Section 2] is itself in continuous time, one thus can re-produce the entire section, resulting in a parabolic Harnack inequality (b). Now (b) implies the on-diagonal upper bound and the near diagonal (i.e. for $d(x, y)^2 \leq t$) lower bound for the heat kernel of $\{X_t\}$ and its dual process. (Note that unlike the time-independent case, $p(0, x, t, y)$ is no longer equal to $p(0, y, t, x)$. However, it holds that $p(0, x, t, y) = p^*(0, y, t, x)$ where $p^*(\cdot, \cdot, \cdot, \cdot)$ is the heat kernel for the time reversal conductances, i.e. $\{\mu^{(t-\cdot)}(x, y)\}$; cf. [St1, Lemma 1.5].) The off-diagonal upper bound can be deduced from the on-diagonal one and the integrated maximum principle using the Davies' argument. The off-diagonal lower bound (in the range $d(x, y) \leq t$) follows from the near diagonal one by the usual chain argument. See [De, Section 3.1] for details.

(a) $\Leftrightarrow$ (b*): Note that (1.5) is equivalent to the inequality where the right hand side is changed to $C_2''r^2\sum_{x,y\in\mathbb{B}(x_0,2r)}(f(x)-f(y))^2\mu(x,y)$. So, the equivalence for time-independent case (that can be proved similarly to [De]) implies the desired equivalence.

(b) $\Rightarrow$ (b*): This is trivial.

(c) $\Rightarrow$ (b): This can be proved using the Balayage argument as in [De, Theorem 3.10] (see the proof of [BKM, Theorem 1.5] for more details on the Balayage argument in the setting of continuous time Markov chains). In order to apply the Balayage argument, the existence of the space-time dual process is required –in this case, we know the existence by using the time reversal conductances mentioned above. In the proof, we need the following estimate

$$\sup_{0<s\le R^2} p(0,x,s,y)\le \frac{c_1}{\nu(\mathbb{B}(x_0,R))}\qquad \text{for all}\ \ x,y\in\mathbb{B}(x_0,2R)\ \text{with}\ \ d(x,y)\ge R,$$

which can be deduced by (1.6) and the fact that there exists $\beta>0$ such that $\nu(\mathbb{B}(x,R))\le cR^\beta$ for all $x\in V, R\ge 1$. The last inequality is a consequence of (1.4) and (1.3) (from which one knows the degrees of the vertices are uniformly bounded by $1/p_0$). We note that we do not need the heat kernel lower bound for $t\le d(x,y)$ to establish (b). □

Proof of Proposition. 1.3. (i) Here $\mathbb{G}=\mathbb{Z}$ and we set the edge conductances to be

$$\mu^{(t)}(i,i-1)=1-\varepsilon,\ \mu^{(t)}(i,i)=b,\ \mu^{(t)}(i,i+1)=1+\varepsilon,\ \text{when}\ t+i\ \text{is even;}$$
$$\mu^{(t)}(i,i-1)=1+\varepsilon,\ \mu^{(t)}(i,i)=b',\ \mu^{(t)}(i,i+1)=1-\varepsilon,\ \text{when}\ t+i\ \text{is odd}\,,$$

with $b/(b+2)=\gamma$ and $b'/(b'+2)=\gamma'>\gamma$. We start at $X_0=0$ and notice that this random walk has two possible states: either X_t is at state A_+ with his right edge having conductance $1+\varepsilon$, or it is at state A_- with his right edge having conductance $1-\varepsilon$. Whenever the random walk X_t moves either to its left or right vertex, it keeps the current state, while if it stays put (i.e. $X_{t+1}=X_t$), then due to the change of conductance values, it moves to the opposite state. Let $\{Z_t\}_{t\in\mathbb{N}}$ be the $\{A_\pm\}$-valued Markov chain describing the state of $\{X_t\}$, then the transition probabilities of Z_t are thus

$$q(A_+,A_+)=1-\gamma,\ \ q(A_+,A_-)=\gamma,\ \ q(A_-,A_-)=1-\gamma',\ \ q(A_-,A_+)=\gamma'\,. \tag{2.1}$$

and its invariant measure is

$$\pi(A_+)=\frac{\gamma'}{\gamma'+\gamma},\qquad \pi(A_-)=\frac{\gamma}{\gamma'+\gamma}\,, \tag{2.2}$$

whereas by the strong law for occupation time $N_t(\cdot):=\sum_{i=0}^{t-1}\mathbb{I}_{\{Z_i=\cdot\}}$ (Cf. [Du, (5.5) pg 320]),

$$N_t(A_\pm)/t\overset{a.s.}{\to}\pi(A_\pm). \tag{2.3}$$

Further, whenever at state A_+ the random walk has drift $\Delta(A_+)=\varepsilon(1-\gamma)$ to its right while at state A_- it has drift $\Delta(A_-)=-\varepsilon(1-\gamma')$. We enumerate sequentially the random times $m_1<m_2<...$ when the random walk is at state A_+, and similarly enumerate the random times $n_1<n_2<...$ when the random walk is at state A_-, then $\mathcal{S}_+:=\{D_i:=X_{i+1}-X_i,\ i\in\{m_1,m_2,...\}\}$ are i.i.d. with drift $\Delta(A_+)$, and $\mathcal{S}_-:=\{D_i:\ i\in\{n_1,n_2,...\}\}$ are i.i.d. with drift $\Delta(A_-)$, while $\mathcal{S}_\pm$ are also mutually independent. Hence by the strong law of large numbers (SLLN) and (2.3), we have that

$$\begin{aligned}\frac{X_t}{t} &= \frac{\sum_{i=0}^{t-1} D_i \mathbb{I}_{\{D_i\in\mathcal{S}_+\}}}{N_t(A_+)}\frac{N_t(A_+)}{t} + \frac{\sum_{i=0}^{t-1} D_i \mathbb{I}_{\{D_i\in\mathcal{S}_-\}}}{N_t(A_-)}\frac{N_t(A_-)}{t} \\ &\overset{a.s.}{\to} \Delta(A_+)\pi(A_+) + \Delta(A_-)\pi(A_-) = \varepsilon\frac{\gamma'(1-\gamma)-\gamma(1-\gamma')}{\gamma'+\gamma} = \varepsilon\frac{\gamma'-\gamma}{\gamma'+\gamma} =: \beta > 0. \quad (2.4)\end{aligned}$$

It is thus ballistic and transient almost surely. If the Gaussian heat kernel off-diagonal upper bound (1.6) holds, then integrating over the region $y \in [(\beta-\epsilon)t, (\beta+\epsilon)t]$ for any $\epsilon \in (0,\beta)$, we see that $\mathbb{P}(X_t \in [(\beta-\epsilon)t, (\beta+\epsilon)t])$ decays exponentially for t, which contradicts (2.4). So this walk violates the Gaussian heat kernel off-diagonal upper bound (1.6).

To have a non-lazy example, set $b = b' = 0$ and observe that $\{X_t\}$ then keeps the state A_+ at all times.

(ii) Here again $\mathbb{G} = \mathbb{Z}$, and let $\{\tau_k\}_{k\in\mathbb{N}}$ be the successive jump times of a Poisson process of intensity $c-1 \in (0,\infty)$, with $\tau_0 = 0$, independent of the CSRW $\{Y_t\}$, and then we set the edge conductances to be

$$\mu^{(t)}(i,i+1) = 1-\epsilon, \quad \mu^{(t)}(i+1,i+2) = 1, \quad \mu^{(t)}(i+2,i+3) = 1+\epsilon,$$
$$\text{when } t \in [\tau_k, \tau_{k+1}), \text{ and } i \equiv k \bmod 3.$$

We start at $Y_0 = 0$ and notice that this CSRW has three possible states: either it is at state A_1 with left/right (L/R for short) edge conductances $1+\epsilon$, $1-\epsilon$, or it is at state A_2 with L/R edge conductances $1-\epsilon$, 1, or at state A_3 with L/R edge conductances 1, $1+\epsilon$. On the other hand, there are two independent Poisson clocks, one ("$\mathscr{C}_E$") governing the environment shift which has intensity $c-1$, and one ("$\mathscr{C}_J$") governing jumps of CSRW which has intensity 1. Denote $\{T_k\}_{k\in\mathbb{N}}$ the sequence of times when the state of $\{Y_t\}$ changes, then it is the successive jump times of a Poisson process of intensity $c \in (1,\infty)$. Let $\{Z_k\}_{k\in\mathbb{N}}$ be the $\{A_1, A_2, A_3\}$-valued process describing the state of Y_{T_k}, then the transition from Z_k to Z_{k+1} is determined by which clock rings first, and in case $\mathscr{C}_J$ does, what are the adjacent edge conductances, but not on $\{Z_0, ..., Z_{k-1}\}$. In other words, the process $\{Z_k\}$ is a time-homogeneous Markov chain with state space $\{A_1, A_2, A_3\}$.

Using properties of exponential distribution (i.e. if ξ_1 and ξ_2 are independent $\exp(\gamma_1)$ and $\exp(\gamma_2)$ random variables, then $\mathbb{P}(\xi_1 < \xi_2) = \gamma_1/(\gamma_1+\gamma_2)$), one can calculate the transition probabilities of $\{Z_k\}$:

$$q(A_1,A_2) = \frac{1-\epsilon}{2c},\ q(A_1,A_3) = 1 - \frac{1-\epsilon}{2c},\ q(A_2,A_3) = \frac{1}{(2-\epsilon)c},\ q(A_2,A_1) = 1 - \frac{1}{(2-\epsilon)c},$$
$$q(A_3,A_1) = \frac{1+\epsilon}{(2+\epsilon)c},\ q(A_3,A_2) = 1 - \frac{1+\epsilon}{(2+\epsilon)c}.$$

and its invariant measure is proportional to

$$\begin{aligned}\pi = \big[\, 2[(-4c^2+2c-1) &+ (c-1)\epsilon + c^2\epsilon^2],\ (2-\epsilon)[(4c^2-2c+1) + (2c^2-2c)\epsilon - \epsilon^2], \\ &(2+\epsilon)[(4c^2-2c+1) + (-2c^2+3c-1)\epsilon - c\epsilon^2]\,\big].\end{aligned}$$

Further, the drift Y_t is subject to when at states A_i, $i = 1, 2, 3$ for its immediate next change of state is $\Delta = \big[-\frac{\epsilon}{c}, \frac{\epsilon}{(2-\epsilon)c}, \frac{\epsilon}{(2+\epsilon)c} \big]$. By SLLN the speed of $\{Y_t\}$ is proportional to $\pi \cdot \Delta$, and one can check that the speed is positive when $\epsilon \in (-1, -\frac{3}{2c+1}) \cup (0, 1)$, and negative when $\epsilon \in (-\frac{3}{2c+1}, 0)$ (see also Remark 2.1). This implies that for arbitrary $\epsilon \in (0, 1)$, $c > 1$, $\{Y_t\}$ has non-zero speed, w.p.1 under the annealed measure on the environment.

Furthermore, the annealed result implies that for a.e. realization of the isolated Poisson jump times $\{\tau_n\}$, $\{Y_t\}$ is w.p.1. transient, ballistic and violates the Gaussian heat kernel off-diagonal upper bound (1.6) in the quenched sense. That is, there exist some (in fact uncountably many) choices of non-random $\{t_n\}$ with $t_n/n \to 1/(c-1)$, so that with the conductances changing at times $\{t_n\}$ the corresponding CSRW on $\mathbb{Z}$ is transient, ballistic and violates the off-diagonal upper bound (1.6). □

Remark 2.1. An intuitive explanation of the phenomenon concerning the region of positive/negative speed in the above example is as follows. Its asymmetry comes from the fact that, although the conductances are symmetric in both directions, the environment is shifting only to the right, and this breaks the symmetry of ϵ. Also, when ϵ is sufficiently close to -1, the speed becomes positive again. Take the special case when $c \to \infty$, then the shift of conductances is so quick that at every time the CSRW jumps (which happens independently at rate 1), its neighborhood is one of the three choices with almost equal probability. But the drift at these three neighborhoods are $-\epsilon$, $\epsilon/(2-\epsilon)$, $\epsilon/(2+\epsilon)$ respectively, one can check that their average is positive regardless of $\epsilon \in (-1, 0) \cup (0, 1)$.

Remark 2.2. The effect of oscillating edge conductances can be mapped to monotone but unboundedly increasing or decreasing conductances. Take one dimension and discrete time for example, with $a > 0$ the oscillating conductances $\{.., 1, a, 1, a...\}$ on $\mathbb{Z}$ shifting at speed 1, is equivalent to setting at time $2n$ the conductances $\{.., a^{2n}, a^{2n+1}, a^{2n}, a^{2n+1}...\}$ and at time $2n+1$ $\{.., a^{2n+2}, a^{2n+1}, a^{2n+2}, a^{2n+1}..\}$ etc. However, we expect that among monotone and uniformly elliptic conductances, the walks follow recurrence/transience of the starting and ending graphs. This has been proved for discrete time non-lazy walks on trees in [ABGK, Theorems 5.1, 5.2].

Proof of Proposition 1.4. (i) Here the vertex set is $\mathbb{V} = \mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}$ and we set the edge conductances to be

$$\begin{aligned}
&\mu^{(t)}(\xi, \xi + e_3) = 1 + \epsilon,\ \mu^{(t)}(\xi, \xi - e_3) = 1 - \epsilon,\ \mu^{(t)}(\xi, \xi) = b, \\
&\qquad \text{when } \xi = (i, j, k) \in \mathbb{Z}^2 \times \mathbb{Z}_{>0},\ t + i + j + k \text{ is odd;} \\
&\mu^{(t)}(\xi, \xi + e_3) = 1 - \epsilon,\ \mu^{(t)}(\xi, \xi - e_3) = 1 + \epsilon,\ \mu^{(t)}(\xi, \xi) = b', \\
&\qquad \text{when } \xi = (i, j, k) \in \mathbb{Z}^2 \times \mathbb{Z}_{>0},\ t + i + j + k \text{ is even;} \\
&\mu^{(t)}(\xi, \xi \pm e_l) = 1,\ l = 1, 2,\ \text{ for all } \xi \in \mathbb{Z}^2 \times \mathbb{Z}_{>0},\ \text{ and all } t\,; \\
&\mu^{(t)}(\xi, \xi \pm e_l) = 0,\ l = 1, 2,\ \text{ for all } \xi = (i, j, 0),\ \text{ and all } t, \\
&\mu^{(t)}(\xi, \xi) = f, \text{ when } \xi = (i, j, 0),\ t + i + j \text{ is odd;} \\
&\mu^{(t)}(\xi, \xi) = f', \text{ when } \xi = (i, j, 0),\ t + i + j \text{ is even.}
\end{aligned}$$

with $b/(b+6) = f/(f+1-\epsilon) = \gamma$ and $b'/(b'+6) = f'/(f'+1+\epsilon) =: \gamma' < \gamma$.

Starting at $X_0 = \underline{0}$ the random walk has two possible states. Either it is at state A_+ with upper edge conductance $1+\epsilon$, or it is at state A_- with upper edge conductance $1-\epsilon$. Whenever the random walk moves, it keeps the same state; and whenever it stays put, it changes to the opposite state. Let $\{Z_t\}_{t\in\mathbb{N}}$ denote the state of $\{X_t\}$. Define the sequence of stopping times $\{\sigma_n\}$ starting from $\sigma_0 = 0$, and for $i \geq 1$, $\sigma_i := \inf\{t > \sigma_{i-1} : R_t = 0\}$, and let $M_n := \big((X_{\sigma_n})_1, (X_{\sigma_n})_2\big)$ be the two-dimensional random walk on $\mathbb{Z}^2 \times \{0\}$. When $R_t := (X_t)_3 > 0$, the state transition probabilities $\{q(\cdot,\cdot)\}$ are given by (2.1) and they have an invariant measure $\pi(\cdot)$ given by (2.2); whereas at state A_+, the random walk has drift $\Delta(A_+) = (2\epsilon)/(6+b)$ and at state A_- it has drift $\Delta(A_-) = -(2\epsilon)/(6+b')$. Let

$$\beta := \Delta(A_+)\pi(A_+) + \Delta(A_-)\pi(A_-) = \frac{2\epsilon}{6+b}\frac{\gamma'}{\gamma+\gamma'} - \frac{2\epsilon}{6+b'}\frac{\gamma}{\gamma+\gamma'} < 0.$$

Because $\beta < 0$, by the large deviation arguments, there exists some positive constant $c_1 = c_1(\beta)$ such that for all k large enough and every n,

$$\mathbb{P}(||D_n|| > \sqrt{2}k|\mathcal{F}_n) \leq \mathbb{P}(\sigma_{n+1} - \sigma_n > k|\mathcal{F}_n) = \mathbb{P}(\min_{1\leq i\leq k} R_{\sigma_n+i} > 0|\mathcal{F}_n) \leq c_1^{-1}e^{-c_1 k}, \tag{2.5}$$

where $\mathcal{F}_n := \mathcal{F}^X_{\sigma_n}$ is the canonical filtration of $\{X_t\}$ stopped at σ_n, and $D_n := M_{n+1} - M_n$. We enumerate sequentially the random times $m_1 < m_2 < ...$ when the state of M_n is A_+, and similarly the random times $n_1 < n_2 < ...$ when the state of M_n is A_-. Then the collection $\mathcal{S}_+ := \{D_i : i \in \{m_1, m_2...\}\}$ are i.i.d. with some law ν_+, and the collection $\mathcal{S}_- := \{D_i : i \in \{n_1, n_2...\}\}$ are i.i.d. with some law ν_-, while $\mathcal{S}_\pm$ are mutually independent. Also, the sequence of states $\{Z_{\sigma_n}\}$ approach an invariant measure which we denote by $\widetilde{\pi}(\cdot)$ (different from $\pi(\cdot)$). Let $p(\cdot,\cdot)$ be the heat kernel of $\{M_n\}$, and $p^\pm(\cdot,\cdot)$ the heat kernels of aperiodic random walks of i.i.d. increments with law $\nu_\pm$, which have all moments finite by (2.5), we have by the strong law for state occupation time of $\widetilde{\pi}(\cdot)$ that

$$\lim_{n\to\infty} \frac{p(n,\underline{0})}{\big(p^+(\widetilde{\pi}(A_+)n,\cdot) * p^-(\widetilde{\pi}(A_-)n,\cdot)\big)(\underline{0})} = 1, \tag{2.6}$$

where $*$ denotes convolution on $\mathbb{Z}^2$. By the local central limit theorem for $p^\pm(\cdot,\cdot)$ ([LL, Theorem 2.3.5]), there exists some positive constant c_2 such that for all z satisfying $||z|| \leq \sqrt{n}$, we have that $p^\pm(\widetilde{\pi}(A_\pm)n, z) \geq c_2/n$, therefore

$$\big(p^+(\widetilde{\pi}(A_+)n,\cdot) * p^-(\widetilde{\pi}(A_-)n,\cdot)\big)(\underline{0}) \geq \int_{\{||z||\leq\sqrt{n}\}} p^+(\widetilde{\pi}(A_+)n,z)p^-(\widetilde{\pi}(A_-)n,-z)dz \geq c_2^2/n, \tag{2.7}$$

which by (2.6) then implies that $\{M_n\}$ is recurrent almost surely, and the same for $\{X_t\}$.

To have a non-lazy example, set $b = b' = f = f' = 0$ and observe that $\{X_t\}$ then keeps the state A_- at all times.

(ii) Here again $\mathbb{V} = \mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}$, and let $\{\tau_n\}_{n\in\mathbb{N}}$ be the successive jump times of a Poisson process of intensity $c - 1 \in (0,\infty)$, with $\tau_0 = 0$, independent of the CSRW $\{Y_t\}$, and then we set the edge conductances to be

$$
\begin{aligned}
&\mu^{(t)}(\xi,\xi+e_3)=1+\epsilon, \mu^{(t)}(\xi,\xi-e_3)=1-\epsilon,\ \mu^{(t)}(\xi,\xi\pm e_l)=1+\epsilon/2,\ l=1,2\\
&\qquad\qquad \text{when } \xi=(i,j,k)\in\mathbb{Z}^2\times\mathbb{Z}_{>0},\ t\in[\tau_n,\tau_{n+1}),\ n+k \text{ is odd;}\\
&\mu^{(t)}(\xi,\xi+e_3)=1-\epsilon, \mu^{(t)}(\xi,\xi-e_3)=1+\epsilon,\ \mu^{(t)}(\xi,\xi\pm e_l)=1-\epsilon/2,\ l=1,2\\
&\qquad\qquad \text{when } \xi=(i,j,k)\in\mathbb{Z}^2\times\mathbb{Z}_{>0},\ t\in[\tau_n,\tau_{n+1}),\ n+k \text{ is even;}\\
&\mu^{(t)}(\xi,\xi\pm e_l)=1+\epsilon/2, \text{when } \xi=(i,j,0),\ l=1,2,\ t\in[\tau_n,\tau_{n+1}),\ n \text{ is odd;}\\
&\mu^{(t)}(\xi,\xi\pm e_l)=1-\epsilon/2, \text{when } \xi=(i,j,0),\ l=1,2,\ t\in[\tau_n,\tau_{n+1}),\ n \text{ is even.}
\end{aligned}
$$

Starting at $Y_0=\underline{0}$ the CSRW has two possible states: either it is at state A_+ with its upper edge conductance $1+\epsilon$, or at state A_- with its upper edge conductance $1-\epsilon$. Let $\{T_n\}_{n\in\mathbb{N}}$ be the sequence of times when the state of $\{Y_t\}$ changes, then it is the successive jump times of a Poisson process of intensity $c\in(1,\infty)$. Let $\{Z_n\}_{n\in\mathbb{N}}$ be the $\{A_\pm\}$-valued time-homogeneous Markov chain describing the state of $\{Y_{T_n}\}$. When $R_{T_n}:=(Y_{T_n})_3>0$, the transition probabilities of $\{Z_n\}$ are given by

$$
\begin{aligned}
q(A_+,A_-)&=\frac{c-1}{c}+\frac{1}{(3+\epsilon)c},\ q(A_+,A_+)=\frac{2+\epsilon}{(3+\epsilon)c},\\
q(A_-,A_+)&=\frac{c-1}{c}+\frac{1}{(3-\epsilon)c},\ q(A_-,A_-)=\frac{(2-\epsilon)}{(3-\epsilon)c},
\end{aligned}
$$

and they have an invariant measure (where $\mathcal{Z}$ is the normalizing constant)

$$
\pi(A_+)=\Big(\frac{c-1}{c}+\frac{1}{(3-\epsilon)c}\Big)/\mathcal{Z},\quad \pi(A_-)=\Big(\frac{c-1}{c}+\frac{1}{(3+\epsilon)c}\Big)/\mathcal{Z};
$$

whereas at state A_+, the CSRW has drift (for its immediate next change of state) $\Delta(A_+)=(2\epsilon)/((6+2\epsilon)c)$, while at state A_-, it has drift $\Delta(A_-)=-(2\epsilon)/((6-2\epsilon)c)$.

Hence working analogously to part (i) and defining $\{\sigma_n\}$ for the embedded Markov chain $\{Y_{T_n}\}$, with

$$
\begin{aligned}
\widehat{\beta}&:=\Delta(A_+)\pi(A_+)+\Delta(A_-)\pi(A_-)\\
&=\frac{\epsilon}{(3+\epsilon)c}\frac{c-1+(3-\epsilon)^{-1}}{c\mathcal{Z}}-\frac{\epsilon}{(3-\epsilon)c}\frac{c-1+(3+\epsilon)^{-1}}{c\mathcal{Z}}<0
\end{aligned}
$$

we thus have the tail bound (2.5) holding and can carry out the rest of the proof resulting in the heat kernel estimates (2.6)-(2.7) as well as almost sure recurrence of $\{Y_t\}$ under the annealed measure on the environment $\{\tau_n\}$. By the same argument as in the proof of Proposition 1.3 (ii), this implies that there exists some non-random $\{t_n\}$ with $t_n/n\to 1/(c-1)$, such that almost surely under the quenched measure with the conductances changing at $\{t_n\}$, Y_t is recurrent. □

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References

[ABGK] G. Amir, I. Benjamini, O. Gurel-Gurevich and G. Kozma. Random walk in changing environment. *arXiv:1504.04870*.

[Ar] D.G. Aronson. Bounds for the fundamental solution of a parabolic equation. *Bulletin of the American Mathematical society* **73** (1967), 890-896.

[MB] M. Biskup. Recent progress on the Random Conductance Model. *Prob. Surveys* **8** (2011) 294–373.

[BKM] O. Boukhadra, T. Kumagai and P. Mathieu. Harnack inequalities and local central limit theorem for the polynomial lower tail random conductance model. *arXiv:1308.1067*.

[DD] T. Delmotte and J.-D. Deuschel. On estimating the derivatives of symmetric diffusions in stationary random environment, with applications to $\nabla\phi$ interface model. *Probab. Theory Relat. Fields* **133** (2005), 358-390.

[De] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoam.* **15** (1999), 181-232.

[DHS] A. Dembo, R. Huang and V. Sidoravicius. Walking within growing domains: recurrence versus transience. *Electron. J. Probab.* Vol. 19, Paper 106 (2014).

[Du] R. Durrett. *Probability: theory and examples.* Third edition. Duxbury advanced series, Belmont, CA, 2004.

[GOS] G. Giacomin, S. Olla and H. Spohn. Equilibrium fluctuation for $\nabla\phi$ interface model. *Ann. Probab.* **29** (2001), 1138-1172.

[GP] G. Giacomin and G. Posta. On recurrent and transient sets of inhomogeneous symmetric random walks. *Elect. Comm. in Probab.* **6** (2001) 39–53.

[Gr] A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. (in Russian) *Matem. Sbornik.* **182** (1991), 55–87. (English transl.) *Math. USSR Sbornik* **72** (1992), 47–77.

[GPZ] O. Gurel-Gurevich, Y. Peres and O. Zeitouni. Localization for controlled random walks and martingales. *Elect. Comm. in Probab.* **19** (2014), no. 24, 1–8.

[Kum] T. Kumagai. Random walks on disordered media and their scaling limits, *Lect. Notes in Math.* **2101**, Ecole d'été de probabilités de Saint-Flour XL–2010, Springer, New York 2014.

[LL] G. Lawler and V. Limic. *Random walk: a modern introduction.* Cambridge studies in advanced mathematics **123**. Cambridge University Press, Cambridge 2010.

[O] Y. Oshima. *Semi-Dirichlet forms and Markov processes.* De Gruyter Studies in Math. **48**. Walter de Gruyter & Co., Berlin, 2013.

[SC] L. Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. *Inter. Math. Res. Notices* **2** (1992), 27–38.

[St1] K.T. Sturm. Analysis on local Dirichlet spaces II. Upper Gaussian estimates for the fundamental solutions of parabolic equations. *Osaka J. Math.* **32** (1995), 275-312.

[St2] K.T. Sturm. Analysis on local Dirichlet spaces III. The parabolic Harnack inequality. *J. Math. Pures Appl.* **75** (1996), 273-297.

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