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Finding a Maximum 2-Matching Excluding Prescribed Cycles in Bipartite Graphs

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Abstract

We introduce a new framework of restricted 2-matchings close to Hamilton cycles. Given a family $\mathcal{U}$ of vertex subsets, a 2-matching F is called $\mathcal{U}$-free if, for each $U \in \mathcal{U}$, F contains at most $|U|-1$ edges in the subgraph induced by U. Our framework includes C_k-free 2-matchings, i.e., 2-matchings without cycles of at most k edges, and 2-factors covering prescribed edge cuts, both of which are intensively studied as relaxations of Hamilton cycles.

The problem of finding a maximum $\mathcal{U}$-free 2-matching is NP-hard. In this paper, we prove that the problem is tractable when the graph is bipartite and each $U \in \mathcal{U}$ induces a Hamilton-laceable graph. This case generalizes the C_4-free 2-matching problem in bipartite graphs. We establish a min-max theorem, a combinatorial polynomial-time algorithm, and decomposition theorems by extending the theory of C_4-free 2-matchings. Our result provides the first polynomially solvable case for the maximum C_k-free 2-matching problem for $k \geq 5$.

1 Introduction

The Hamilton cycle problem is one of the most fundamental NP-hard problems in various research fields such as combinatorial optimization, graph theory, and computational complexity. One successful approach to the Hamilton cycle problem is to utilize matching theory. In a graph $G = (V, E)$, an edge set $F \subseteq E$ is a *2-matching* (resp., *2-factor*) if it has at most (resp., exactly) two edges incident to each vertex in V. Since a Hamilton cycle is a special kind of a 2-matching (or 2-factor) and a 2-matching of maximum size is found in polynomial time, it is reasonable to put restrictions on 2-matchings to provide a tight relaxation of Hamilton cycles to which matching theory can be applied. Examples include the following two kinds of restricted 2-matchings.

C_k-free 2-matchings. For a positive integer k, a 2-matching is called *C_k-free* if it contains no cycles of length at most k. The larger k becomes, the closer a C_k-free 2-factor becomes to a Hamilton cycle. If $k \geq |V|/2$, a C_k-free 2-factor is a Hamilton cycle, whereas a C_2-free 2-matching is nothing other than a 2-matching.

2-factors covering prescribed edge cuts. An *edge cut* is a set of edges having exactly one endpoint in some vertex subset $X \subset V$. Given a family $\mathcal{K}$ of edge cuts, an edge subset is called $\mathcal{K}$*-covering* if it intersects every edge cut in $\mathcal{K}$. A Hamilton cycle is exactly a $\mathcal{K}$-covering 2-factor, where $\mathcal{K}$ is the family of all edge cuts.

Recently, both C_k-free and $\mathcal{K}$-covering 2-factors are intensively studied and applied to approximation algorithm design for NP-hard problems related to the Hamilton cycle problem, such as the graph-TSP and the minimum 2-edge connected spanning subgraph problem [4, 5, 8, 12, 20, 30].

1.1 Previous Work

In general graphs, the C_k-free 2-matching problem is much more difficult than the 2-matching problem. For the cases $k \geq 3$, no algorithm is known other than Hartvigsen's C_3-free 2-matching algorithm [14]. NP-hardness for the case $k \geq 5$ is proved by Papadimitriou (see [7]). More generally, Hell et al. [17] proved that the problem is NP-hard, unless the excluded length of a cycle is a subset of $\{3, 4\}$. The case $k = 4$ is still open, and conjectured to be solved in polynomial time [9]. Discrete convexity shown in [22] supports this conjecture.

While only a few positive results are known for the C_k-free 2-matching problem in general graphs, in bipartite graphs the C_4-free 2-matching problem is efficiently solvable, and fundamental theorems in matching theory are extended. Motivated by a stimulating paper of Hartvigsen [15], Király [21] gave a min-max theorem for the C_4-free 2-matching problem in bipartite graphs, followed by a different min-max theorem by Frank [11]. Comparison of these two theorems is discussed in [31], together with decomposition theorems corresponding to the Dulmage-Mendelsohn and Edmonds-Gallai decompositions. Polynomial combinatorial algorithms are designed by Hartvigsen [16] and Pap [25], which are again slightly different and followed by an improvement in time complexity by Babenko [1]. For the weighted version, while the NP-hardness of the weighted C_4-free 2-matching problem in bipartite graphs is proved by Király (see [11]), positive results such as a linear programming formulation with dual integrality [23], a combinatorial algorithm [29], and discrete convexity [22] are established when the edge weight satisfies a certain property. Since the C_6-free 2-matching problem is NP-hard even in bipartite graphs [13], the C_4-free 2-matching problem in bipartite graphs is one of the few cases where the C_k-free 2-matching problem is tractable.

For a set of integers $A \subseteq \mathbf{Z}$, denote the set of edge cuts whose size belongs to A by $\mathcal{K}_A$. Kaiser and Škrekovski [18] proved that every bridgeless planar cubic graph has a $\mathcal{K}_{\{3,4\}}$-covering 2-factor, which is extended to a stronger result that every bridgeless cubic graph has a $\mathcal{K}_{\{3,4\}}$-covering 2-factor [19]. While the proof in [19] was not algorithmic, Boyd, Iwata, and Takazawa [4] designed a combinatorial algorithm for finding a $\mathcal{K}_{\{3,4\}}$-covering 2-factor in bridgeless cubic graphs, together with a combinatorial algorithm for finding a minimum-weight $\mathcal{K}_{\{3\}}$-covering 2-factor in bridgeless cubic graphs. Čada et al. [6] exhibited a family of graphs which has no $\mathcal{K}_{\{4,5\}}$-covering edge subset with even degree at every vertex, disproving a conjecture in [19].

1.2 Our Contribution

In the present paper, we introduce a new framework of restricted 2-matchings which commonly generalizes C_k-free 2-matchings and $\mathcal{K}$-covering 2-factors. For $U \subseteq V$, let $G[U] = (U, E[U])$ denote the subgraph induced by U, i.e., $E[U] = \{uv \in E \mid u, v \in U\}$. For $F \subseteq E$, let $F[U] = F \cap E[U] = \{uv \in F \mid u, v \in U\}$.

Definition 1 ($\mathcal{U}$-free 2-matchings)**.** Let $\mathcal{U} \subseteq 2^V$ be a family of vertex subsets. A 2-matching $F \subseteq E$ is called *$\mathcal{U}$-free* if $|F[U]| \leq |U| - 1$ for each $U \in \mathcal{U}$.

Equivalently, a 2-matching F is $\mathcal{U}$-free if and only if $|F[U]| \neq |U|$ for each $U \in \mathcal{U}$, i.e., F excludes cycles through U. Moreover, if F is a 2-factor, then F is $\mathcal{U}$-free if and only if $F \cap \delta(U) \neq \emptyset$ for every $U \in \mathcal{U}$, where $\delta(U)$ denotes the set of edges having exactly one endpoint in U. From these viewpoints, it is not difficult to see that $\mathcal{U}$-free 2-matchings include Hamilton cycles, C_k-free 2-matchings, and $\mathcal{K}$-covering 2-factors as special cases: put $\mathcal{U} = \{U \subseteq V \mid |U| \leq |V|/2\}$, $\mathcal{U} = \{U \subseteq V \mid |U| \leq k\}$, and $\mathcal{U} = \{U \subseteq V \mid \delta(U) \in \mathcal{K}\}$, respectively.

The *$\mathcal{U}$-free 2-matching problem* is defined as a problem of finding a $\mathcal{U}$-free 2-matching of maximum size for given G and $\mathcal{U}$. Since the $\mathcal{U}$-free 2-matching problem includes the Hamilton cycle problem, the $\mathcal{U}$-free 2-matching problem is NP-hard in general. Thus, we need some assumption in order to obtain a tractable class of the $\mathcal{U}$-free 2-matching problem, such as the cases where G is bipartite and $\mathcal{U} = \{U \subseteq V \mid |U| \leq 4\}$, and G is bridgeless cubic and $\mathcal{U} = \{U \subseteq V \mid \delta(U) \in \mathcal{K}_{\{3,4\}}\}$.

A main objective of this paper is to provide a broader well-solved class of the $\mathcal{U}$-free 2-matching problem by extending the theory of C_4-free 2-matchings in bipartite graphs. For this purpose, we exploit a graph-theoretic concept of *Hamilton-laceable graphs.* For a bipartite graph (V, E), we denote the two color classes by V^+ and V^-. For $X \subseteq V$, let $X^+ = X \cap V^+$ and $X^- = X \cap V^-$.

Definition 2 (Hamilton-laceable graphs [26])**.** A bipartite graph $G = (V, E)$ is *Hamilton-laceable* if

(i) $|V^+| = |V^-|$ and G has a Hamilton path between an arbitrary pair of $u \in V^+$ and $v \in V^-$, or

(ii) $|V^+| = |V^-| - 1$ and G has a Hamilton path between an arbitrary pair of distinct vertices $u, v \in V^-$.

In what follows, we work on the $\mathcal{U}$-free 2-matching problem under the assumption that G is bipartite and $G[U]$ is Hamilton-laceable for each $U \in \mathcal{U}$. We note that, for a 2-factor F, $|F[U]| = |U|$ implies that $|U^+| = |U^-|$. Thus, we assume $|U^+| = |U^-|$ for each $U \in \mathcal{U}$, and hence only the case (i) in Definition 2 occurs in our argument.

The smallest nontrivial example of a Hamilton-laceable graph would be a cycle of length four, and hence our assumption includes the C_4-free 2-matching problem in bipartite graphs as a special case. Further examples and previous work of Hamilton-laceable graphs are exhibited in § 2.

In the present paper, we exhibit that the theory of C_4-free 2-matching problem in bipartite graphs satisfactorily extends when $G[U]$ is Hamilton-laceable for each $U \in \mathcal{U}$. We first present a min-max theorem extending Király's min-max theorem [21]. We then design a combinatorial algorithm for finding a maximum $\mathcal{U}$-free 2-matching, which provides a constructive proof for our min-max theorem. In the design of our algorithm, we make use of both of Hartvigsen's and Pap's algorithms [16, 25]: the shrinking technique comes from Pap's algorithm; and the construction of a minimizer of the min-max theorem derives from Hartvigsen's method. Finally, we describe decomposition theorems extending those in [31] and corresponding to the Dulmage-Mendelsohn and Edmonds-Gallai decompositions.

It is noteworthy that, unlike the literature of C_k-free 2-matchings and $\mathcal{K}$-covering 2-factors, our assumption that each $G[U]$ is Hamilton-laceable does not depend on the size of the forbidden structures. One benefit of this is that our result provides the first polynomially solvable case of the C_k-free 2-matching problem for $k \geq 5$, and thus has a potential to provide better approximation ratios for the graph-TSP and the minimum 2-edge connected subgraph problem.

We further remark that our framework contains both cases where multiplicities on edges are forbidden or allowed. That is, in the former case we only deal with simple 2-matchings and one edge can only contribute one to the degree of its endpoints. In the latter case, we can put multiplicity two on one edge to have degree two on the endpoints of the edges. Actually, in the literature of C_k-free 2-matching problem, these two cases have formed different streams. The aforementioned results are of the former case, and results for the latter case include [2, 7, 24]. To the best of our knowledge, not much connection between these two cases is found. In our framework, forbidding multiplicity on an edge $uv \in E$ corresponds to have $\{u, v\}$ in $\mathcal{U}$, and it is clear that $G[\{u, v\}]$ is Hamilton-laceable if $uv \in E$. While in this paper we mainly keep the former case in mind, we note that our framework can represent the both cases.

1.3 Organization of the paper

The rest of the paper is organized as follows. In § 2, we show some previous work, observation, and examples of Hamilton-laceable graphs. After that, we work on $\mathcal{U}$-free 2-matchings in bipartite graphs where $G[U]$ is Hamilton-laceable for each $U \in \mathcal{U}$. We present a min-max theorem in § 3. Section 4 is devoted to describing a combinatorial algorithm for finding a maximum $\mathcal{U}$-free 2-matching, which provides a constructive proof for the min-max theorem. Finally, in § 5, we exhibit decomposition theorems corresponding to the Dulmage-Mendelsohn and Edmonds-Gallai decompositions.

2 Hamilton-Laceable Graphs

This section is devoted to a discussion on Hamilton-laceable graphs. We first note that the concept of Hamilton-laceable graphs is a bipartite analogue of that of *Hamilton-connected graphs*, which is well-known in the field of graph theory [3]. A graph is *Hamilton-connected* if it has a Hamilton path between an arbitrary pair of distinct vertices. Thus, a Hamilton-connected graph is nonbipartite if it has at least three vertices.

In what follows, we always assume that $G = (V, E)$ is bipartite. Trivial examples of a Hamilton-laceable graph are the case where $V^+ = \emptyset$ or $V^- = \emptyset$, and a graph of two vertices connected by an edge. It is also clear that a complete bipartite graph on $2t$ vertices, denoted by $K_{t,t}$, is Hamilton-laceable. Recall that a special case $K_{2,2}$, a cycle of length four, is an example of a Hamilton-laceable graph.

If $G = (V, E)$ is Hamilton-laceable, a graph $(V, \tilde{E})$ satisfying $\tilde{E} \supseteq E$ is also Hamilton-laceable. Thus, it would be of interest to find Hamilton-laceable graphs with as few edges as possible. Indeed, the concept of Hamilton-laceable graphs was introduced as a generalized property of Hamiltonicity of d-dimensional rectangular lattices by Simmons [26], who proved that all d-rectangular lattices are Hamilton-laceable except for the two-dimensional lattices of order $2 \times r$ ($r \neq 2$) and $3 \times 2r$. This result provides a class of Hamilton-laceable graphs (V, E) with $|E| \approx d|V|$. For instance, every hypercube is Hamilton-laceable.

Furthermore, Simmons [27] discussed the minimum number l_t of the edges of Hamilton-laceable graphs with $|V^+| = t$. It holds that $3t - \lceil t/3 \rceil \leq l_t \leq 3t - 1$ for the case (i) in Definition 2, and $l_t = 3t + 1$ for the case (ii) in Definition 2. Simmons [28] also showed that deleting fewer than $t - 1$ edges from $K_{t,t}$ or $K_{t,t+1}$ maintains Hamilton-laceability.

The motivation of introducing Hamilton-laceable graph in this paper comes from an analysis in [31], which reveals that cycles of length four in the C_4-free 2-matching problem in bipartite graphs serve as factor-critical components for the nonbipartite matching problem: if $U \subseteq V$ induces a cycle of length four in a bipartite graph, for an arbitrary pair $u \in U^+$ and $v \in U^-$, $G[U]$ contains a 2-matching of size three in which only u and v has degree one. Observe that the definition of Hamilton-laceable graphs generalizes this property. In the following sections we reveal that the property in Definition 2(i) plays a key role to provide a tractable class of restricted 2-matchings in bipartite graphs.

3 A Min-Max Theorem

In this section, we describe a min-max theorem for the $\mathcal{U}$-free 2-matching problem in bipartite graphs where each $U \in \mathcal{U}$ induces a Hamilton-laceable graph. Our theorem is an extension of Király's min-max theorem [21] for the C_4-free 2-matching problem in bipartite graphs. For $X \subseteq V$, let $\bar{X} = V \setminus X$ and $c'(X)$ denote the number of components in $G[X]$ consisting of a single vertex, a single edge, or a single cycle of length four.

Theorem 3 ([21]). *Let $G = (V, E)$ be a bipartite graph. Then, it holds that*

$$\max\{|F| \mid F \text{ is a } C_4\text{-free } 2\text{-matching}\} = \min\{|V| + |X| - c'(\bar{X}) \mid X \subseteq V\}.$$

Observe that every component contributing to $c'(\bar{X})$ is Hamilton-laceable. We now exhibit our theorem extending Theorem 3. For $X \subseteq V$, let $c(X)$ denote the number of components in $G[X]$ whose vertex set belongs to $\mathcal{U}$.

Theorem 4. *Let $G = (V, E)$ be a bipartite graph and $\mathcal{U} \subseteq 2^V$ be a family of vertex subsets in G such that $G[U]$ is Hamilton-laceable for each $U \in \mathcal{U}$. Then, it holds that*

$$\max\{|F| \mid F \text{ is a } \mathcal{U}\text{-free } 2\text{-matching}\} = \min\{|V| + |X| - c(\bar{X}) \mid X \subseteq V\}. \tag{1}$$

Before proving Theorem 4, we first show that the inequality max $\leq$ min in (1) holds for an arbitrary G and $\mathcal{U}$, i.e., G may not be bipartite and $G[U]$ may not be Hamilton-laceable for $U \in \mathcal{U}$. For disjoint vertex sets $X, Y \subseteq V$, let $E[X, Y]$ denote the set of edges connecting X and Y, $G[X, Y] = (X \cup Y, E[X, Y])$, and $F[X, Y] = F \cap E[X, Y]$.

Lemma 5. *Let $G = (V, E)$ be a graph and $\mathcal{U} \subseteq 2^V$ be a family of vertex subsets in G. For an arbitrary $\mathcal{U}$-free 2-matching F and $X \subseteq V$, it holds that $|F| \leq |V| + |X| - c(\bar{X})$.*

Proof. Since F is a 2-matching, $2|F[X]| + |F[X, \bar{X}]| \leq 2|X|$ follows. Moreover, since F is $\mathcal{U}$-free, it holds that $|F[\bar{X}]| \leq |\bar{X}| - c(\bar{X})$. □

The following lemma directly follows from the proof for Lemma 5. For $F \subseteq E$ and $u \in V$, denote the number of edges in F incident to u by $\deg_F(u)$.

Lemma 6. *If a $\mathcal{U}$-free 2-matching F and $X \subseteq V$ attain the equality in (1), it holds that*

- $F[X] = \emptyset$,
- $\deg_{F[\{u\}, \bar{X}]}(u) = 2$ *for each $u \in X$, and*
- *for each component Q in $G[\bar{X}]$,*

$$|F[V(Q)]| = \begin{cases} |V(Q)| - 1 & \text{if } V(Q) \in \mathcal{U}, \\ |V(Q)| & \text{otherwise.} \end{cases}$$

We complete a proof of Theorem 4 by establishing an algorithm for finding a $\mathcal{U}$-free 2-matching F and $X \subseteq V$ attaining equality in (1) in § 4.

4 A Combinatorial Algorithm

In this section, we describe a combinatorial polynomial-time algorithm for finding a maximum $\mathcal{U}$-free 2-matching in bipartite graphs where each $U \in \mathcal{U}$ induces a Hamilton-laceable graph. Our algorithm employs ideas both of the C_4-free 2-matching algorithms of Hartvigsen [16] and Pap [25].

4.1 Algorithm Description

Roughly speaking, our algorithm resembles Edmonds' algorithm for nonbipartite matchings [10]. One main feature in our algorithm comes from Pap's algorithm [25]: we shrink $U \in \mathcal{U}$ after we find an alternating path, whereas in Edmonds' and Hartvigsen's algorithms shrinking occurs in the middle of construction of alternating forests. Another feature derives from Hartvigsen's algorithm [16]. A minimizer $X \subseteq V$ of the right-hand side of (1) is basically determined as the set of vertices reachable from the deficient vertices. In our algorithm, if a vertex resulting from shrinking $U \in \mathcal{U}$ satisfies certain properties, it is regarded as reachable even if it is not reachable.

Before describing the entire algorithm, we present how to shrink and expand $U \in \mathcal{U}$. In order to provide concise notation, in the rest of this section we denote the input of the algorithm by $\hat{G} = (\hat{V}, \hat{E})$ and $\hat{\mathcal{U}} \subseteq 2^{\hat{V}}$, and the graph obtained by repeated shrinkings by $G = (V, E)$. In the algorithm, we maintain a $\mathcal{U}$-free b-matching F in G, where $\mathcal{U} \subseteq 2^V$ and $b \in \{1, 2\}^V$, which can be extended to a $\hat{\mathcal{U}}$-free 2-matching in $\hat{G}$. For $b \in \{1, 2\}^V$, a b-matching F is *$\mathcal{U}$-free* if $F[U]$ is not a b-factor in $G[U]$ for every $U \in \mathcal{U}$. Initially, $G = \hat{G}$, $\mathcal{U} = \hat{\mathcal{U}}$, $b_v = 2$ for each $v \in V$, and F is an arbitrary $\mathcal{U}$-free b-matching, e.g., $F = \emptyset$.

For two edge sets $F_1, F_2 \subseteq E$, denote the symmetric difference of F_1 and F_2 by $F_1 \triangle F_2$, i.e., $F_1 \triangle F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. Define the set of source vertices by $S^+ = \{u \in V^+ \mid \deg_F(u) < b_u\}$ and sink vertices $S^- = \{v \in V^- \mid \deg_F(v) < b_v\}$. Suppose that we have found an alternating path P from S^+ to S^- such that $F \triangle E(P)$ is not a $\mathcal{U}$-free b-matching. We then apply the following shrinking procedure.

Procedure $\mathsf{Shrink}(F, P)$**.** Denote the number of edges in P by $2l + 1$. Let P_i be a path consisting of the first $2i$ edges in P for $i \in [1, l]$, P_0 be an empty graph, and $P_{l+1} = P$. Let i^* be the smallest index i such that $F \triangle E(P_i)$ contains a b-factor in $G[U]$ for some $U \in \mathcal{U}$, and let $F' = F \triangle E(P_{i^*-1})$. If more than one such $U \in \mathcal{U}$ exists, choose an arbitrary U. We then update G, b, $\mathcal{U}$, and F as follows. Let u_U^+ and v_U^- be new vertices, which are obtained by identifying the vertices in U^+ and U^-, respectively. Then, reset V, b, E, F, and $\mathcal{U}$ as

$$
\begin{aligned}
&V := \bar{U} \cup \{u_U, v_U\}, \\
&b_v := \begin{cases} 1 & \text{if } v = u_U^+, v_U^-, \\ b_v & \text{otherwise}, \end{cases} \\
&E := E[\bar{U}] \cup \{u_U^+ v \mid uv \in E, u \in U^+, v \in \bar{U}^-\} \cup \{u v_U^- \mid uv \in E, u \in \bar{U}^+, v \in U^-\}, \\
&F := F'[\bar{U}] \cup \{u_U^+ v \mid uv \in F', u \in U^+, v \in \bar{U}^-\} \cup \{u v_U^- \mid uv \in F', u \in \bar{U}^+, v \in U^-\}, \\
&\mathcal{U} := \{U' \mid U' \in \mathcal{U}, U' \cap U = \emptyset\} \cup \{(U' \setminus U) \cup \{u_U, v_U\} \mid U' \in \mathcal{U}, U \subsetneq U'\}.
\end{aligned}
$$

See Figure 1 for an illustration. Observe that F is still a b-matching in G after the update. We then again search an alternating path P from S^+ to S^-, and repeat the above procedure.

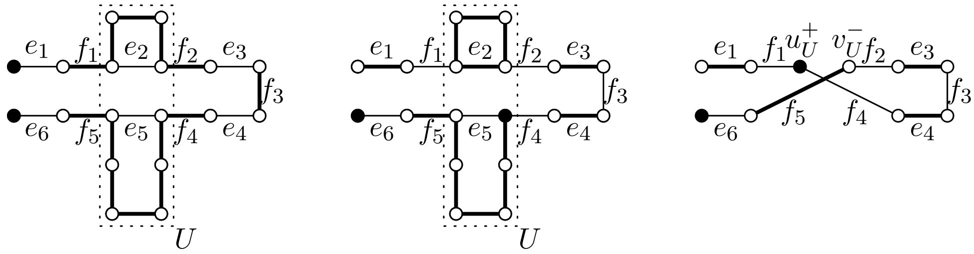

Figure 1: $b_v = 2$ for each v. The thick edges are in F, thin edges in $E \setminus F$, and the vertices in black are in S^+ or S^-. In the figure on the left, we have found P consisting of $e_1, f_1, e_2, \ldots, e_5, f_5, e_6$, and $F \triangle E(P)$ contains a 2-factor in $G[U]$ for $U \in \mathcal{U}$. In this case $i^* = 5$, and the figure in the middle shows $F \triangle E(P_4)$. The figure on the right shows the graph after $\mathsf{Shrink}(F, P)$.

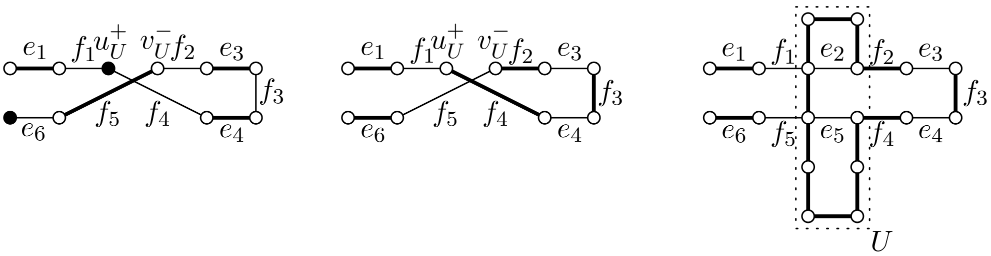

Figure 2: The graph in the middle results from an augmentation in the graph on the left, where the augmenting path P consists of $f_4, e_4, f_3, e_3, f_2, f_5, e_6$. We then expand U, where $\hat{f}_U^+ = f_4$ and $\hat{f}_U^- = f_2$. to obtain the graph on the right.

If an alternating path P from S^+ to S^- such that $F \triangle E(P)$ is $\mathcal{U}$-free b-matching is found, then we reset $F := F \triangle E(P)$ to augment the current solution, and expand the shrunk vertex sets to return to the original graph $\hat{G}$ as follows. First note that the shrunk vertex sets in $\hat{\mathcal{U}}$ form a laminar family, and it suffices to expand the maximal shrunk vertex sets. For a maximal shrunk vertex set $U \subseteq \hat{V}$, denote the unique edge in F incident to u_U^+ by f_U^+, and to v_U^- by f_U^-, if exist. Let $\hat{f}_U^+, \hat{f}_U^- \in \hat{E}$ be the edges corresponding to $f_U^+, f_U^- \in E$, respectively. Denote the vertex in U^+ incident to $\hat{f}_U^+$ by $\hat{u}_U^+$, and that in U^- incident to $\hat{f}_U^-$ by $\hat{v}_U^-$. If f_U^+ (resp., f_U^-) does not exist, let $\hat{u}_U^+$ (resp., $\hat{v}_U^-$) be an arbitrary vertex in U^+ (resp., U^-). Now, since $\hat{G}[U]$ is Hamilton-laceable, $\hat{G}[U]$ has a Hamilton path P_U between $\hat{u}_U^+$ and $\hat{v}_U^-$. In expanding U, we add $E(P_U)$ to F. That is, $\hat{F} := F \cup \bigcup_{U \in \mathcal{U}^*} E(P_U)$. See Figure 2 for an illustration of augmentation and expansion. It is not difficult to see that $\hat{F}$ is a $\mathcal{U}$-free 2-matching.

The entire algorithm is described as follows.

Input: A bipartite graph $\hat{G} = (\hat{V}, \hat{E})$ and $\hat{\mathcal{U}} \subseteq 2^{\hat{V}}$ such that $\hat{G}[U]$ is Hamilton-laceable for each $U \in \hat{\mathcal{U}}$.

Output: A maximum $\hat{\mathcal{U}}$-free 2-matching $\hat{F}$ in $\hat{G}$.

Step 0: Put $G = \hat{G}$ and $\mathcal{U} = \hat{\mathcal{U}}$. Let F be an arbitrary $\mathcal{U}$-free 2-matching in G and then go to Step 1.

Step 1: Let $S^+ = \{u \in V^+ \mid \deg_F(u) < b_u\}$ and $S^- = \{v \in V^- \mid \deg_F(v) < b_v\}$. Orient each edge in $E \setminus F$ from V^+ to V^- and each edge in F from V^- to V^+ to obtain a directed graph D. If D has a directed path P from S^+ to S^-, then go to Step 2. Otherwise, go to Step 5.

Step 2: Let $E_P \subseteq E$ be the set of edges corresponding to the directed edges in P. If $F' = F \triangle E_P$ is a $\mathcal{U}$-free b-matching, then go to Step 3. Otherwise, go to Step 4.

Step 3 (Augmentation): Reset $F := F'$, expand all maximal shrunk vertex sets, and then go back to Step 1.

Step 4 (Shrinking): Apply $\mathsf{Shrink}(F, P)$, and then go back to Step 1.

Step 5 (Termination): Expand all maximal shrunk vertex sets and return $\hat{F}$.

4.2 Proof for Optimality

In the termination of the algorithm, we have a digraph D in which no directed path from S^+ to S^- exists. Let $R \subseteq V$ denote the set of vertices reachable from S^+ in D, and define $R' \subseteq V$ by

$$R' = R \cup \{v \in \bar{R}^- \mid v \text{ is not a shrunk vertex}, \deg_{F[R^+,\{v\}]}(v) = 2\}$$
$$\cup \{v \in \bar{R}^- \mid v = v_U^- \text{ for some } U \in \mathcal{U},\, uv \in F \text{ for some } u \in R^+\}.$$

Finally, define $X \subseteq \hat{V}$ by the set of vertices corresponding to $(\bar{R}')^+ \cup (R')^-$.

Lemma 7. *The output $\hat{F}$ of the algorithm and X defined above attain the equality in* (1).

Proof. It is not difficult to see that $\hat{F}[X] = \emptyset$. Moreover, since every $v \in X$ satisfies $\deg_{\hat{F}[\{v\},\bar{X}]} = 2$, we have that $|\hat{F}[X, \bar{X}]| = 2|X|$. Finally, in G, all edges in $E[\bar{X}]$ belong to F. Thus, each edge in $\hat{E}[X]$ is in $\hat{F}$ or belongs to $\hat{E}[U]$ for some $U \in \mathcal{U}$ shrunk in G. By the definition of Q, it holds that v_U^- has no adjacent edge in $E[X]$, which implies that $\hat{G}[U]$ forms a component in $\hat{G}[\bar{X}]$. Thus, it follows that $|\hat{F}[\bar{X}]| = |\bar{X}| - c(\bar{X})$. Therefore, we conclude

$$|\hat{F}| = |\hat{F}[X]| + |\hat{F}[X, \bar{X}]| + |\hat{F}[\bar{X}]| = 2|X| + |\bar{X}| - c(\bar{X}) = |V| + |X| - c(\bar{X}).$$

□

Now Theorem 4 immediately follows from Lemmas 5 and 7. Thus, our algorithm provides a constructive proof for Theorem 4.

4.3 Complexity

In this subsection, we show that the time complexity of our algorithm is polynomial in the size of the input of the algorithm. Denote $n = |\hat{V}|$ and $m = |\hat{E}|$. We should notice that the input size of the algorithm depends on how $\hat{\mathcal{U}}$ is given, and $\hat{\mathcal{U}}$ might have a exponential size in n, e.g., $\hat{\mathcal{U}} = \{U \subseteq \hat{V} \mid |U| \leq n/2\}$. Nevertheless, in many cases determining if a given edge set is $\hat{\mathcal{U}}$-free is done efficiently, such as the C_k-free 2-matching case and the $\mathcal{K}$-covering 2-factor case. Therefore, we denote by γ the time for determining if an edge set is $\hat{\mathcal{U}}$-free, and use γ in complexity analysis of the algorithm instead of $|\hat{\mathcal{U}}|$.

It is not difficult to see that shrinkings occur $\mathrm{O}(n)$ times between augmentations. Since augmentations occur $\mathrm{O}(n)$ times, shrinkings occur $\mathrm{O}(n^2)$ times in total.

After each shrinking, we search an alternating path, which takes $\mathrm{O}(m)$ time. Moreover, we determine if $F \triangle E(P_i)$ is $\mathcal{U}$-free $\mathrm{O}(n)$ times for each shrinking. Thus, the time complexity between shrinkings is $\mathrm{O}(n\gamma + m)$.

Therefore, the total complexity of our algorithm is $\mathrm{O}(n^3\gamma + n^2 m)$.

Theorem 8. *Our algorithm finds an optimal solution in* $\mathrm{O}(n^3\gamma + n^2 m)$ *time.*

5 Decomposition Theorems

This section is devoted to decomposition theorems for the $\mathcal{U}$-free 2-matching problem in bipartite graphs where each $U \in \mathcal{U}$ induces a Hamilton-laceable graph. These theorems correspond to the Dulmage-Mendelsohn and Edmonds-Gallai decompositions, and extend decomposition theorems for the C_4-free 2-matchings in bipartite graphs [31].

Let $X_1 \subseteq V$ be a minimizer of (1) obtained by the algorithm in § 4. By exchanging the roles of V^+ and V^-, i.e., searching alternating paths from S^- to S^+, we obtain another minimizer $X_2 \subseteq V$ of (1). Now partition V into three sets $D, A, C \subseteq V$, where

$$D = \bar{X}_1^+ \cup \bar{X}_2^-, \quad A = X_2^+ \cup X_1^-, \quad C = V \setminus (D \cup A).$$

We first provide a characterization of D. Note that such characterization appears in both of the Dulmage-Mendelsohn and Edmonds-Gallai decompositions.

Theorem 9. *The vertex set* D *is characterized as*

$$D = \{v \mid \exists a \text{ maximum } \mathcal{U}\text{-free 2-matching } F \text{ with } \deg_F(v) \leq 1\}.$$

Proof. It suffices to discuss V^+. Let $u \in D^+ = \bar{X}_1^+$. By the definition of X_1, at the last stage of the algorithm where no path between S^+ and S^- is found, u is reachable from S^+ or is shrunk into a vertex reachable from S^+. Denote a path from S^+ to u by P. Then, it is not difficult to see that $F \triangle E_P$ provides a maximum $\mathcal{U}$-free 2-matching F with $\deg_F(u) = 1$. For $u \in \bar{D}^+ = X_1^+$, it holds that $\deg_F(u) = 2$ by Lemma 6. □

The following theorem, corresponding to the Dulmage-Mendelsohn decomposition, suggests that X_1 and X_2 are canonical minimizers of (1).

Theorem 10. *For an arbitrary minimizer* $Y \subseteq V$ *of* (1)*, it holds that* $X_2^+ \subseteq Y^+ \subseteq X_1^+$ *and* $X_1^- \subseteq Y^- \subseteq X_2^+$*.*

Proof. It suffices to prove $Y^+ \subseteq X_1^+$ and $X_1^- \subseteq Y^-$. For each $u \in Y^+$, by Lemma 6, $\deg_F(u) = 2$ holds for every maximum $\mathcal{U}$-free 2-matching F. Thus, $u \in \bar{D}^+ = X_1^+$ follows from Theorem 9.

We next prove $X_1^- \subseteq Y^-$. Suppose to the contrary that there exists $v \in X_1^- \setminus Y^-$. Let F be an arbitrary maximum $\mathcal{U}$-free 2-matching. Since $v \in X_1^-$, by Lemma 6 there exist two vertices $u_1, u_2 \in \bar{X}_1^+ \subseteq \bar{Y}^+$ such that $u_1 v, u_2 v \in F$. Denote the component in $G[\bar{Y}]$ containing u_1, u_2, v by Q. Since u_1, u_2 are reachable from S^+, we have that $|F[V(Q)]| < |V(Q)|$, and thus $|F[V(Q)]| = |V(Q)| - 1$ and $Q \in \mathcal{U}$ by Lemma 6.

Denote $Q_X = V(Q) \cap X_1$ and $Q_{\bar{X}} = V(Q) \cap \bar{X}_1$. We now show that there exists an edge $e \in E[Q_{\bar{X}}^+, Q_{\bar{X}}^-] \setminus F$. By Lemma 6, every vertex in Q_X^- is connected to two vertices in $Q_{\bar{X}}^+$ by two edges in F, implying that $|Q_{\bar{X}}^+| > |Q_X^-|$ and

$$1 \leq |F[Q_{\bar{X}}^+, Q_{\bar{X}}^-]| \leq 2|Q_{\bar{X}}^+| - 2|Q_X^-| - 1.$$

Let $xy \in F[Q_{\bar{X}}^+, Q_{\bar{X}}^-]$, where $x \in Q_{\bar{X}}^+$ and $y \in Q_{\bar{X}}^-$. Since $G[V(Q)]$ is Hamilton-laceable, $G[V(Q)]$ has a Hamilton path P between x and y. Then it follows that $|E(P)[Q_{\bar{X}}^+, Q_{\bar{X}}^-]| \geq 2|Q_{\bar{X}}^+| - 2|Q_X^-| - 1$, and hence

$$|E[Q_{\bar{X}}^+, Q_{\bar{X}}^-]| \geq |E(P)[Q_{\bar{X}}^+, Q_{\bar{X}}^-] \cup \{xy\}| \geq 2|Q_{\bar{X}}^+| - 2|Q_X^-|.$$

Thus, $E[Q_{\bar{X}}^+, Q_{\bar{X}}^-] \setminus F \neq \emptyset$. Then, the endpoint v_e of e in V^- should be reachable from S^+, contradicting that $v_e \in \bar{X}$. □

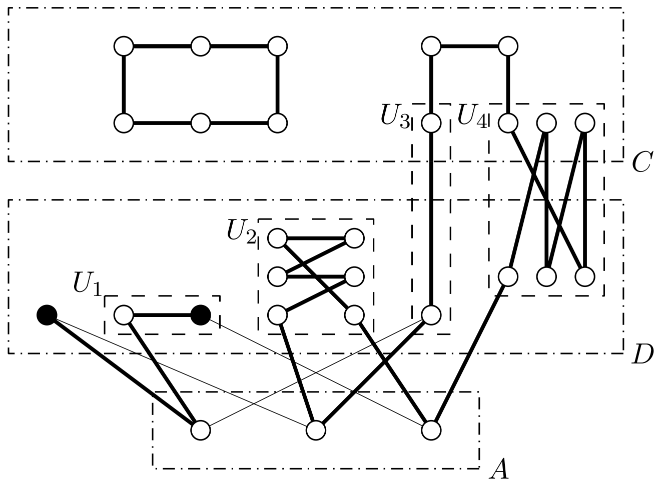

Figure 3: The thick lines are edges in a maximum $\mathcal{U}$-free 2-matching F, and the thin lines are edges in $E \setminus F$. The two vertices in black are those at which the degree of F is not two. The vertex sets U_1, U_2, U_3, and U_4 are in $\mathcal{U}$. Some edges in $E \setminus F$ are omitted.

Finally, we establish the theorem below, which corresponds to the Edmonds-Gallai decomposition. Figure 3 would help understanding the statements.

Theorem 11. *The following statements hold.*

(i) *For each $e \in E[D, A]$, there exists a maximum $\mathcal{U}$-free 2-matching containing e.*

(ii) *The vertex set of each component in $G[D]$ and $G[D, C]$ is a singleton or belongs to $\mathcal{U}$.*

(iii) *Shrink the components in $G[D]$ and $G[D, C]$ in the manner of* $\mathsf{Shrink}(F, P)$ *to obtain a new graph $G' = (V', E')$, denote the vertex subsets of V' corresponding to D, C by D', C', and define $b' \in \{1, 2\}^{D' \cup C'}$ by*

$$b'_v = \begin{cases} 1 & \text{if } v = u_U^+ \text{ or } v = v_U^- \text{ for some } U \in \mathcal{U}, \\ 2 & \text{otherwise.} \end{cases}$$

Then,

(a) *$G'[U']$ has a b'-factor, and*

(b) *for arbitrary $A' \subseteq A$, it holds that $b'(\Gamma(A') \cap D') > 2|A'|$, where $\Gamma(A')$ is the set of vertices in $V \setminus A'$ adjacent to some vertex in A'.*

(iv) *An arbitrary maximum $\mathcal{U}$-free 2-matching F is composed of the following edges.*

(a) *In $G[D]$ and $G[D, C]$, F contains $|V(Q)| - 1$ edges in $E[V(Q)]$ for each component Q.*

(b) *For $u \in A$, F contains two edges connecting u and distinct components in $G[D]$.*

(c) *In $G[U]$, $F[U]$ corresponds to a b'-factor in $G'[U']$.*

(v) *Both $A \cup C^+$ and $A \cup C^-$ minimize (1).*

Proof. **Assertion** (v)**.** This is clear from $A \cup C^+ = X_1$ and $A \cup C^- = X_2$.

Assertion (i)**.** By symmetry, it suffices to discuss $e = uv \in E[D^+, A^-]$. Let F be a maximum $\mathcal{U}$-free 2-matching found by the algorithm and suppose $e \notin F$. Then, in the last Step 1 of the algorithm, u is reachable from S^+ or u is shrunk in some u_U^+ reachable from S^+. Let P be a path starting from S^+ and reaching u. Then, from the current solution, we can obtain a new maximum $\mathcal{U}$-free 2-matching containing e by taking the symmetric difference with $E(P)$, adding e, deleting an appropriate edge in $\delta(\{v\})$, and expanding the shrunk vertex sets.

Assertions (ii) **and** (iv)(a)**.** Let Q be a component in $G[\bar{X}_1^+, \bar{X}_1^-]$ which is not a singleton. Since $\bar{X}_1^+$ is the set of vertices reachable from S^+, it follows that $F[V(Q)] < |V(Q)|$. By Lemma 6, it suffices to show that Q does not intersect both D^- and C^-. Suppose otherwise. Then, there exists one vertex in $V(Q) \cap D^-$ such that $\deg_{F[Q]}(v) = 1$, and $\deg_{F[Q]}(v') = 2$ holds for every vertex $v' \in V(Q) \cap D^-$. This implies $v' \in X_1^-$, a contradiction.

Assertions (iii)(a)**,** (iv)(b)**, and** (iv)(c)**.** These assertions are now clear from Lemma 6 and Assertion (iv)(a).

Assertion (iii)(b)**.** It suffices to discuss $A' \subseteq A^- = X_1^-$. Then the assertion follows that A' is reachable from S^+ and Assertion (iv)(b).

□

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