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On the Existence of Vertical Fibers of Coverings of Curves over a Complete Discrete Valuation Ring

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Abstract

Let X be a stable curve over a complete discrete valuation ring R of mixed characteristic or positive characteristic. In the present paper, we study geometry of coverings of X. Under certain assumptions, we prove that by replacing R by a finite extension of R, there exists a morphism from a stable curve to X such that the morphism of generic fibers is finite étale and the morphism of special fibers is non-finite.

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Contents

introduction

Let R be a complete discrete valuation ring with algebraically closed residue field, X a stable curve over $S := \operatorname{Spec} R = \{\eta, s\}$, where η (resp. s) stands for the generic point (resp. closed point). Write X_η (resp. X_s) for the generic fiber (resp. special fiber). Suppose that X_η is smooth. After choosing base points, we obtain the (surjective) specialization morphism of fundamental groups

$$sp : \pi_1(X_{\overline{\eta}}) \longrightarrow \pi_1^{\mathrm{adm}}(X_s),$$

where the left (resp. right) hand side denotes the étale (resp. admissible (cf. Notations and Conventions)) fundamental group of $X_{\overline{\eta}} := X_\eta \times_\eta \overline{\eta}$ (resp. X_s).

In the present paper, we study geometry of coverings of curves from the point of view of the specialization morphism of fundamental groups. A closed point x of X is called a vertical point if, by replacing S by a finite extension of S, there is a stable covering $f : Y \longrightarrow X$ (cf. Definition 2.1) such that the inverse image $f^{-1}(x)$ is not a finite set. We use the notation X_s^{ver} to denote the set of vertical points of X_s. We may post a question as follows:

Question: What is X_s^{ver} ?

If R has equal characteristic $(0, 0)$, then *sp* is an isomorphism. Thus, the admissible coverings of X_s can be determined by the étale coverings of $X_{\overline{\eta}}$ (cf. [17, Proposition 1.1]). This means that for any finite étale covering of generic fiber $X_{\overline{\eta}}$, by stable reduction theorem, by replacing R by a finite extension of R, the morphism of special fibers induced by the étale covering of generic fibers is an admissible covering. Then $X_s^{\mathrm{ver}} = \emptyset$.

If R has mixed characteristic $(0, p)$ or equal characteristic (p, p). We can consider that whether or not the set X_s^{ver} is empty. Moreover, we can ask that whether or not X_s^{ver} contains a smooth point.

The problem that whether or not X_s^{ver} contains a smooth point is called resolution of nonsingularities. The motivation of resolution of nonsingularities is partly came from anabelian geometry. The technique of resolution of nonsingularities in the case of p-adic number fields was first introduced by S. Mochizuki (cf. [7, the proof of Theorem 9.2]). In the situation of Mochizuki, X is a smooth curve over the valuation ring of a p-adic number field. By applying the technique of resolution nonsingularities, Mochizuki reduced the Grothendieck conjecture for proper, hyperbolic curves over number fields to the Grothendieck conjecture for proper, singular, stable curves over finite fields, which is then reduced to the Grothendieck conjecture for affine curves over finite fields which had been proven by A. Tamagawa. Afterward, in [17], Tamagawa introduced the problem of resolution of nonsingularities and proved a theorem in the case of mixed characteristic. More precisely, Tamagawa's theorem (cf. [17, Theorem 0.2 (v)]) is essentially as follows: if R is strictly of mixed characteristic with residue field $\overline{\mathbb{F}}_p$ and X is non-isotrivial, then $X_s^{\mathrm{ver}} = X_s^{\mathrm{cl}}$, where X_s^{cl} denotes the set of the closed points of X_s.

In the present paper, we consider **Question** in the cases of mixed characteristic and equal characteristic. If R is an arbitrary complete DVR of mixed characteristic, we have a theorem as follows (see also Theorem 2.5).

Theorem 0.1. *X_s^{ver} is an infinite countable set which contains all the nodes of X_s, and the closure of X_s^{ver} in X_s is equal to X_s.*

On the other hand, if R has equal characteristic (p, p), in the case of good reduction, we have a theorem as follows (see also Theorem 2.7).

Theorem 0.2. *Suppose that X is a non-isotrivial smooth curve over S, and X_s can be defined over $\overline{\mathbb{F}}_p$. Then, X_s^{ver} is not empty.*

In particular, Theorem 0.2 can be regarded as a certain analogue of Mochizuki's result for the case of positive characteristic. In the case of bad reduction, we have the following theorem (see also Theorem 2.8).

Theorem 0.3. *Suppose that X_s is an irreducible, singular curve. Then, X_s^{ver} is not empty.*

Moreover, by applying Theorem 0.3, we have the following corollary.

Corollary 0.4. *Suppose that X_s is an irreducible, singular curve. Let $Y \longrightarrow X$ be a stable covering over S. Then Y_s^{ver} is not empty.*

Notations and Conventions

Curves and their moduli stacks:

By a curve over a field, we mean a finite type, separated, connected, one dimensional reduced scheme over a field.

An *r-pointed stable curve* (X, D_X) of type (g, r) over a scheme S consists of a flat, proper morphism $X \longrightarrow S$, together with a closed subscheme $D_X \subseteq X$ such that for each geometric point $\overline{s}$ of S:

(i) The geometric fiber $X_{\overline{s}}$ is a reduced and connected curve of genus g with at most ordinary double points (i.e., nodes).

(ii) $X_{\overline{s}}$ is smooth at the points of D_X.

(iii) The composite morphism $D_X \subseteq X \longrightarrow S$ is finite étale of degree r.

(iv) Let E be an irreducible component of $X_{\overline{s}}$ of genus g_E. Then the sum of the degree of the restriction of D_X to E and the number of points where E meets the closure of the complement of E in $X_{\overline{s}}$ is $\geq 3 - 2g_E$.

(v) $\dim(H^1(X_{\overline{s}}, \mathcal{O}_{X_{\overline{s}}})) = g$.

In this situation, one verifies easily that $2g - 2 + r$ is ≥ 1.

We shall say that an S-scheme X is a *stable curve* of genus g over S if $(X, \emptyset)$ is a 0-pointed stable curve of genus g over S.

We shall say that a pointed stable curve (X, D_X) over a scheme S is *smooth* if the morphism of schemes $X \longrightarrow S$ is smooth.

We denote (X, D_X) a pointed stable curve over S with divisor of marked points D_X and underlying scheme X. For simplicity we also use the notation X to denote the pointed stable curve (X, D_X) when there is no confusion.

Let $\overline{\mathcal{M}}_{g,r}$ be the moduli stack of stable curves of type(g, r) over $\operatorname{Spec}\mathbb{Z}$, $\mathcal{M}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ parametrizing pointed smooth curves. Then $\overline{\mathcal{M}}_{g,r}^{\log}$ is the log moduli stack obtained by equipping $\overline{\mathcal{M}}_g$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subset \overline{\mathcal{M}}_{g,r}$ relative to $\operatorname{Spec}\mathbb{Z}$. Let $\overline{\mathcal{X}}_{g,r} \longrightarrow \overline{\mathcal{M}}_{g,r}$ be the universal stable curve over $\overline{\mathcal{M}}_g$, and $\mathcal{D}_g \subset \overline{\mathcal{X}}_{g,r}$ the divisor given by the inverse image in $\overline{\mathcal{X}}_{g,r}$ of the divisor $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subset \overline{\mathcal{M}}_{g,r}$. $\mathcal{D}_{g,r}$ determines a log structure on $\overline{\mathcal{X}}_{g,r}$; denote the resulting log stack by $\overline{\mathcal{X}}_{g,r}^{\log}$. Thus, we obtain a morphism of log stacks $\overline{\mathcal{X}}_{g,r}^{\log} \longrightarrow \overline{\mathcal{M}}_{g,r}^{\log}$. In particular, if $r = 0$ (i.e., stable curve), we use notation $\overline{\mathcal{M}}_g$ (resp. $\overline{\mathcal{M}}_g^{\log}$, $\overline{\mathcal{X}}_g$, $\overline{\mathcal{X}}_g^{\log}$) to denote the stack $\overline{\mathcal{M}}_{g,0}$ (resp. $\overline{\mathcal{M}}_{g,0}^{\log}$, $\overline{\mathcal{X}}_{g,0}$, $\overline{\mathcal{X}}_{g,0}^{\log}$).

For more details on stable curves, pointed stable curves and their moduli stacks, see [3], [4].

Galois categories and their fundamental groups:

We denote the categories of finite étale, finite Kummer log étale, finite tame, and finite admissible coverings of "$(-)$" by $\mathrm{Cov}(-)$, $\mathrm{Cov}((-)^{\log})$, $\mathrm{Cov}_{\mathrm{tame}}(-)$, $\mathrm{Cov}_{\mathrm{adm}}(-)$, respectively.

The notations $\pi_1(-), \pi_1((-)^{\log}), \pi_1^{\mathrm{tame}}(-), \pi_1^{\mathrm{adm}}(-)$ will be used to denote the étale, Kummer log étale, tame, and admissible fundamental groups of "$(-)$", respectively.

The notation $(-)^{\mathrm{ab}}$ denotes the abelianization of the group $(-)$.

For more details on admissible coverings and admissible fundamental groups for (pointed) stable curves, see [6], [7], [18].

1 P-ranks and specialization homomorphism

First, let us fix the notation. In this section, let k be an algebraically closed field of characteristic $p > 0$.

Definition 1.1. Let X a stable curve of genus g_X over k, and F_X the absolute Frobenius morphism of X. The p-rank $\sigma(X)$ of X is defined as $\dim_{\mathbb{F}_p} \mathrm{H}^1(X, \mathcal{O}_X)^{F_X}$, where $(-)^{F_X}$ means the F_X-invariant subspace.

By Artin-Schreier theory of étale cohomology, we have $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{H}^1(X, \mathcal{O}_X)^{F_X}$. Furthermore, $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z}/p\mathbb{Z})$. Therefore, we can also define the p-rank of X as

$$\sigma(X) := \mathrm{rank}(\pi_1^p(X)^{\mathrm{ab}}),$$

where the right hand side means the rank of abelianization of pro-p étale fundamental group of X.

From now on, in this section, we assume that X is smooth over k. Write $X^1 := X \times_{k, F_k} k$ for the pull-back of X by the Frobenius F_k of k. Thus, we obtain a relative Frobenius morphism $F_{X/k} : X \longrightarrow X^1$. The canonical differential $(F_{X/k})_*(d) : (F_{X/k})_*(\mathcal{O}_X) \longrightarrow (F_{X/k})_*(\Omega^1_X)$ is a morphism of $\mathcal{O}_X$-modules. Write B_X for the image of $(F_{X/k})_*(d)$ which is called the sheaf of locally exact differentials. One has the exact sequence

$$0 \longrightarrow \mathcal{O}_{X^1} \longrightarrow (F_{X/k})_*(\mathcal{O}_X) \longrightarrow B_X \longrightarrow 0,$$

and B_X is a vector bundle on X^1 of rank $p-1$. Raynaud's theorem (cf. [11, Theoreme 4.1.1]) shows that there is a divisor Θ_X of J^1_X, where J^1_X is the pull-back of the Jacobian J_X of X by the Frobenius F_k. Furthermore, the support of Θ_X is as follows:

$$\Theta_X(k) = \{[\mathcal{L}] \in J^1(k) \mid \mathrm{H}^1(X^1, B_X \otimes \mathcal{L}) \neq 0\}.$$

Let J_X (resp. J_Y) be the Jacobian of X (resp. Y). Thus, the étale covering f induces a natural morphism $g : J_X \longrightarrow J_Y$ of Jacobians. Write J_Y^{new} for the quotient of abelian varieties $J_Y/g(J_X)$, and we call J_Y^{new} the new part of the Jacobian J_Y of Y with respect to the morphism f.

Definition 1.2. The μ_n-torsor $f : Y \longrightarrow X$ is called to be new ordinary if the new part J_Y^{new} of Jacobian of Y with respect to the morphism f is an ordinary abelian variety (i.e., the p-rank of J_Y^{new} is equal to the dimension of J_Y^{new}).

Definition 1.3. Let M be a torsion abelian group. For each element $x \in M$, we define the saturation of x to be the subset of elements of in the form $i.x$, where i is an integer prime to the order of x. We use the notation $\mathrm{Sat}(x)$ to denote the saturation of x.

We have a relationship between new ordinary and theta divisors as the following.

Proposition 1.4. *Let $f : Y \longrightarrow X$ be a μ_n-torsor. Let y be a torsion point of $J_X^1(k)$ of order n corresponding to the μ_n-torsor $f^1 : Y^1 \longrightarrow X^1$. Then $f : Y \longrightarrow X$ is new ordinary if and only if $\mathrm{Sat}(y) \bigcap \Theta_X = \emptyset$.*

Proof. See [13, Proposition 2.1.4]. □

Let $R := \overline{\mathbb{F}}_p[[t]]$ be a complete discrete valuation ring, $\mathcal{X}$ a smooth projective hyperbolic curve over $S := \mathrm{Spec}\, R = \{\eta, s\}$, where η (resp. s) stands for the generic point (resp. closed point) of S. Suppose that $\mathcal{X}$ is non-isotrivial over S (i.e., there dose not exist a proper and smooth k-curve X_0, such that X is isomorphic to $X_0 \times_k S$ over S). Let $\mathcal{X}^1$ be the Frobenius twist of $\mathcal{X}$ over S and $\mathcal{J}_{\mathcal{X}}^1$ the Jacobian of $\mathcal{X}^1$ over S. This is an abelian scheme over S and can be regarded as the Néron model of $\mathcal{J}_{\mathcal{X},\eta}^1 := \mathcal{J}_{\mathcal{X}}^1 \times_S \eta$. Write $\mathcal{J}_{\mathcal{X},\overline{\eta}}^1\{p'\}$ and $\mathcal{J}_{\mathcal{X},s}^1\{p'\}$ for the set of prime to p torsion points of $\mathcal{J}_{\mathcal{X},\overline{\eta}}^1$ and $\mathcal{J}_{\mathcal{X},s}^1$, respectively, where $\mathcal{J}_{\mathcal{X},s}^1 := \mathcal{J}_{\mathcal{X}}^1 \times_S s$. By the specialization isomorphism of prime to p étale fundamental groups, we have

$$\mathcal{J}_{\mathcal{X},\overline{\eta}}^1\{p'\} \longrightarrow \mathcal{J}_{\mathcal{X},s}^1\{p'\}$$

is an isomorphism of abelian groups. Identifying the two abelian groups with each other by the specialization isomorphism, we write $\mathcal{J}_{\mathcal{X}}^1\{p'\} := \mathcal{J}_{\mathcal{X},\overline{\eta}}^1\{p'\} = \mathcal{J}_{\mathcal{X},s}^1\{p'\}$. Consider the sets of prime to p torsion points of Raynaud theta divisors of geometric generic fiber $\mathcal{X}_{\overline{\eta}} := \mathcal{X}_\eta \times_\eta \overline{\eta}$ and special fiber $\mathcal{X}_s$, respectively. We have

$$\Theta_{\mathcal{X}_{\overline{\eta}}}\{p'\} \subseteq \Theta_{\mathcal{X}_s}\{p'\}.$$

Furthermore, A. Tamagawa proved a theorem as follows (cf. [16]):

Proposition 1.5. *Let $\mathcal{X}$ be a smooth, non-isotrivial (i.e., the morphism $S \longrightarrow M_{g_X,\overline{\mathbb{F}}_p}$ determined by $\mathcal{X} \longrightarrow S$ is not a constant morphism, where $M_{g_X,\overline{\mathbb{F}}_p}$ denotes the coarse moduli space of $\mathcal{M}_{g_X,\overline{\mathbb{F}}_p}$) projective hyperbolic curve over S. Then there exists a finite étale covering $\mathcal{Y} \longrightarrow \mathcal{X}$ whose Galois closure is of degree prime to p, such that*

$$\mathrm{Sat}(\Theta_{\mathcal{Y}_{\overline{\eta}}}\{p'\}) \subsetneq \mathrm{Sat}(\Theta_{\mathcal{Y}_s}\{p'\})$$

holds in $\mathcal{J}_{\mathcal{Y}}^1\{p'\}$, where $\mathcal{J}_{\mathcal{Y}}^1$ denotes the Jacobian of $\mathcal{Y}^1$ over S. In particular, we have the specialization morphism $\pi_1(\mathcal{X}_{\overline{\eta}}) \longrightarrow \pi_1(\mathcal{X}_s)$ is not an isomorphism.

Proof. See [16, Section 7]. □

Remark 1.5.1. This theorem was proved by Pop-Saïdi (cf. [10]) and Raynaud (cf. [13]) under certain assumptions of Jacobian, and then, by Tamagawa in general case (cf. [16]). In fact, Tamagawa proved much more. Tamagawa showed that if $(\mathcal{X}, D_{\mathcal{X}})$ is a projective smooth hyperbolic curve over S with divisor $D_{\mathcal{X}}$, then the specializtion morphism of tame fundamental groups along the divisors is not isomorphism. As a corollary, we have the following beautiful statement: over $\overline{\mathbb{F}}_p$, only finitely many isomorphism classes of smooth hyperbolic curves have the same tame fundamental groups.

By Tamagawa's theorem, we have a corollary as follows:

Corollary 1.6. *Let $\mathcal{X}$ be a smooth, non-isotrivial projective curve over S. Then there exists a finite étale covering $\mathcal{Z}$ such that*

$$\sigma(\mathcal{Z}_{\overline{\eta}}) - \sigma(\mathcal{Z}_s) > 0,$$

where $\mathcal{Z}_{\overline{\eta}}$ and $\mathcal{Z}_s$ denote the geometric fiber and special fiber of $\mathcal{Z}$, respectively.

Proof. By Proposition 1.5, we chose a finite étale covering $\mathcal{Y} \longrightarrow \mathcal{X}$ whose Galois closure is of degree prime to p, such that

$$\mathrm{Sat}(\Theta_{\mathcal{Y}_{\overline{\eta}}}\{p'\}) \subsetneq \mathrm{Sat}(\Theta_{\mathcal{Y}_s}\{p'\}).$$

So, we can choose an element z of $J^1_{\mathcal{Y}}\{p'\}$ such that $z \in \mathrm{Sat}(\Theta_{\mathcal{Y}_s}\{p'\})$ and $z \notin \mathrm{Sat}(\Theta_{\mathcal{Y}_{\overline{\eta}}}\{p'\})$. Then we obtain the étale covering $\mathcal{Z} \longrightarrow \mathcal{Y}$ corresponding to z. Moreover, by Proposition 1.4, $\mathcal{Z}_{\overline{\eta}} \longrightarrow \mathcal{X}_{\overline{\eta}}$ is new ordinary and $\mathcal{Z}_s \longrightarrow \mathcal{X}_s$ is not new ordinary. Thus, we have $\sigma(\mathcal{Z}_{\overline{\eta}}) - \sigma(\mathcal{Z}_s) > 0$.

□

2 Geometry of coverings of curves

In this section, we discuss geometry of coverings of stable curves. First, let us fix the notation. Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$, X a stable curve over $S := \mathrm{Spec}\, R = \{\eta, s\}$, where η (resp. s) stands for the generic point (resp. closed point) of S. Write X_η (resp. $X_{\overline{\eta}}$, X_s) for the generic fiber (resp. geometric generic fiber, special fiber) of X. Write Γ_{X_s} for the dual graph of X_s, $v(\Gamma_{X_s})$ (resp. $e(\Gamma_{X_s})$) for the set of vertices (resp. edges) of Γ_{X_s}, and X_v for the irreducible component corresponding to $v \in v(\Gamma_{X_s})$. Moreover, we assume that X_η is smooth.

Definition 2.1. Let Y be a stable curve over S. A morphism $f : Y \longrightarrow X$ is called stable covering of X if the morphism of generic fibers $f_\eta : Y_\eta \longrightarrow X_\eta$ is a finite étale morphism. Let G be a finite group. f is called G-stable covering if f is a stable covering and f_η is a G-étale covering (i.e., Galois étale covering whose Galois group is G).

Definition 2.2. Let x be a closed point of special fiber X_s. x is called a vertical point if by replacing S by a finite extension of S, there exists a stable covering $f : Y \longrightarrow X$ over S such that $\dim(f_s^{-1}(x)) = 1$. We use the notation X_s^{ver} to denote the set of vertical points of X.

There is a criterion for the existence of vertical points of a given stable covering.

Proposition 2.3. *Let $x \in X_s$ be a closed point. Suppose that $f : Y \longrightarrow X$ is a G-stable covering over S such that for each irreducible component $Z := \overline{\{z\}}$ of $\mathrm{Spec}\, \widehat{\mathcal{O}}_{X_s,x}$, and each point w of the fiber $Y \times_X z$, the natural morphism from the integral closure W^s of Z in $k(w)^s$ to Z is wildly ramified, where $k(w)^s$ denotes the maximal separable subextension of $k(w)$ in $k(z)$. Then $\dim f_s^{-1}(x) = 1$.*

Proof. See [17] Proposition 4.3 (ii). □

2.1 Existence of vertical components: mixed characteristic case

In this subsection, we assume that R has mixed characteristic $(0, p)$. Let $M_{g,r}$ be the coarse moduli space of $\mathcal{M}_{g,r} \times_{\mathrm{Spec}\, \mathbb{Z}} \mathrm{Spec}\, k$. Given a point $x \in M_{g,r}$, choose a geometric point $\overline{x}$ above x and let $(C_{\overline{x}}, D_{\overline{x}})$ be a pointed curve corresponding to the point x (well-defined up to isomorphism). Then the isomorphism type of the (geometric) tame fundamental group $\pi_1^{\mathrm{tame}}((C_{\overline{x}}, D_{\overline{x}}))$ is independent of the choice of $\overline{x}$ and $(C_{\overline{x}}, D_{\overline{x}})$ (and the implicit base point on $(C_{\overline{x}}, D_{\overline{x}})$ used to define $\pi_1^{\mathrm{tame}}((C_{\overline{x}}, D_{\overline{x}}))$. We have a result proved by Saïdi and Tamagawa.

Proposition 2.4. *Let $U \subseteq M_{g,r}$ a subvariety of positive dimension. Then the geometric tame fundamental group π_1^{tame} is not constant on U (i.e., there exist two points b and a of U, such that $a \in \overline{\{b\}}$ holds, the specialisation homomorphism $sp_{b,a} : \pi_1^{\mathrm{tame}}((C_{\overline{b}}, D_{\overline{b}})) \longrightarrow \pi_1^{\mathrm{tame}}((C_{\overline{a}}, D_{\overline{a}}))$ is not an isomorphism).*

Proof. See [14, Theorem 3.12]. □

Theorem 2.5. *X_s^{ver} is an infinite countable set which contains all the nodes of X_s, and the closure of X_s^{ver} in X_s is equal to X_s.*

Proof. Replacing X by a finite admissible covering, we can assume that X_s is sturdy and untangled (each irreducible component is smooth and genus is greater than 2, see [8, Section 0 Curves]).

Let $S^{\log}$ be a log regular scheme whose underlying scheme is S and the log structure is determined by the closed point. There is a natural morphism from $S^{\log}$ to the log moduli stack $\overline{\mathcal{M}}_{g_X}^{\log}$, where g_X is the genus of X_η. Thus, we obtain a stable log curve $X^{\log}$ whose underlying scheme is X and the log structure of $X^{\log}$ is the pulling-back log structure of $\overline{\mathcal{M}}_{g_X,1}^{\log}$.

Let x be a closed point of X_s. Write X_v for an irreducible component which contains x. We can regard X_v as a pointed smooth curve of type (g_{X_v}, r_v) with marked point $X_v \bigcap X_s^{\mathrm{Sing}}$, where X_s^{Sing} denotes the set of singular points of X_s.

Write η_X (resp. $\widehat{\eta}_X$, η_v, $\widehat{\eta}_v$) for the generic point of X (resp. generic point of $\operatorname{Spec}\widehat{\mathcal{O}}_{X,\eta_v}$, generic point of X_v, generic point of $\operatorname{Spec}\widehat{\mathcal{O}}_{X_s,x}$).

Consider the 2-th log configuration space $\mathcal{X}^{\log} := \overline{\mathcal{M}}^{\log}_{g_X,2} \times_{\overline{\mathcal{M}}^{\log}_{g_X}} S^{\log}$ of X. Write $\eta_v^{\log}$ for the log scheme whose underlying scheme is η_v and the log structure is the pulling back log structure of $X^{\log}$. Write $(\mathcal{X}'_v)^{\log}$ for the 2-th log configuration space of pointed smooth curve X_v, $\mathcal{X}_v^{\log}$ for $(\mathcal{X}'_v)^{\log} \times_{\eta_v} \eta_v^{\log}$. By the specialization theorem of log étale fundamental groups (cf. [18, Proposition 1]), we obtain a commutative diagram of fundamental groups and all seven rows are exact.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\mathcal{X}_{\overline{\eta}_X}) & \longrightarrow & \pi_1(\mathcal{X}_{X_\eta}) & \longrightarrow & \pi_1(X_\eta) & \longrightarrow & 1 \\
 & & \| & & S.\uparrow & & S.\uparrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{X}_{\overline{\eta}_X}) & \longrightarrow & \pi_1(\mathcal{X}_{\eta_X}) & \longrightarrow & \pi_1(\eta_X) & \longrightarrow & 1 \\
 & & \| & & I.\uparrow & & I.\uparrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{X}_{\overline{\widehat{\eta}_X}}) & \longrightarrow & \pi_1(\mathcal{X}_{\widehat{\eta}_X}) & \longrightarrow & \pi_1(\widehat{\eta}_X) & \longrightarrow & 1 \\
 & & S.\downarrow & & S.\downarrow & & S.\downarrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{X}^{\log}_{\tilde{\eta}_v^{\log}}) & \longrightarrow & \pi_1(\mathcal{X}^{\log}_{\eta_v^{\log}}) & \longrightarrow & \pi_1(\eta_v^{\log}) & \longrightarrow & 1 \\
 & & I.\uparrow & & I.\uparrow & & \| & & \\
1 & \longrightarrow & \pi_1((\mathcal{X}_v)^{\log}_{\tilde{\eta}_v^{\log}}) & \longrightarrow & \pi_1((\mathcal{X}_v)^{\log}_{\eta_v^{\log}}) & \longrightarrow & \pi_1(\eta_v^{\log}) & \longrightarrow & 1 \\
 & & \| & & S.\downarrow & & S.\downarrow & & \\
1 & \longrightarrow & \pi_1((\mathcal{X}'_v)^{\log}_{\overline{\eta}_v}) & \longrightarrow & \pi_1((\mathcal{X}'_v)^{\log}_{\eta_v}) & \longrightarrow & \pi_1(\eta_v) & \longrightarrow & 1 \\
 & & \| & & I.\uparrow & & I.\uparrow & & \\
1 & \longrightarrow & \pi_1((\mathcal{X}'_v)^{\log}_{\overline{\widehat{\eta}_v}}) & \longrightarrow & \pi_1((\mathcal{X}'_v)^{\log}_{\widehat{\eta}_v}) & \longrightarrow & \pi_1(\widehat{\eta}_v) & \longrightarrow & 1,
\end{array}$$

where $S.$ (resp. $I.$) means surjection (resp. injection) and $\tilde{\eta}_v^{\log}$ denotes the log geometric point of $\eta_v^{\log}$.

For each $i = 1, ..., 7$, write $1 \longrightarrow \Delta_i \longrightarrow \Pi_i \longrightarrow G_i \longrightarrow 1$ for the i-th row of the above commutative diagram, $\rho_i : G_i \longrightarrow \operatorname{Out}(\Delta_i)$ for the outer representation, and Im_i for the image of ρ. Then by [17, Remark 2.3, Lemma 5.2], we obtain

$$\operatorname{Im}_1 = \operatorname{Im}_2 \hookleftarrow \operatorname{Im}_3 \twoheadrightarrow \operatorname{Im}_4 \twoheadrightarrow \operatorname{Im}_5 = \operatorname{Im}_6 \hookleftarrow \operatorname{Im}_7.$$

Write $D_{\widehat{\eta}_X}$ (resp. I_{η_v}) for the image (resp. the kernel) of $\pi_1(\widehat{\eta}_X) \longrightarrow \pi_1(X_\eta)$ (resp. $\pi_1(\widehat{\eta}_X) \longrightarrow \pi_1(\eta_v)$). Write I_x for $\pi_1(\widehat{\eta}_v)$. We have a commutative diagram as follows:

$$\begin{array}{ccccc}
\pi_1(\widehat{\eta}_X) & \longrightarrow & D_{\widehat{\eta}_X}/I_{\eta_X} & \longrightarrow & \mathrm{Im}_6 \\
\downarrow & & \uparrow & & \Vert \\
\pi_1(\eta_v) & = & \pi_1(\eta_v) & \longrightarrow & \mathrm{Im}_6 \\
 & & \uparrow & & \uparrow \\
 & & I_x & \longrightarrow & \mathrm{Im}_7.
\end{array}$$

Suppose that the specialization morphism

$$sp_x : \pi_1((\mathcal{X}'_v)^{\log}_{\overline{\widehat{\eta}}_v}) \longrightarrow \pi_1((\mathcal{X}'_v)^{\log}_x)$$

is not an isomorphism. Thus, by applying [17, Proposition 4.1 (ii)], we have the image of wild inertia subgroup I_x^{w} in $D_{\widehat{\eta}_X}/I_{\eta_X}$ is infinite. Then, by Proposition 2.3, we have $x \in X_s^{\mathrm{ver}}$.

If x is a node of X_s, then $(\mathcal{X}'_v)_x$ is a singular curve. Thus, by [17, Corollary 3.11], sp_x is not an isomorphism. This means that X_s^{ver} contains all the nodes of X_s. If x is a smooth closed point of X_s, then $(\mathcal{X}'_v)_x$ is a smooth curve over x. By applying Proposition 2.4, the closure of X_s^{ver} in X_s is equal to X_s.

On the other hand, $\pi_1(X_{\overline{\eta}})$ is topologically finitely generated, then the set of open subgoups is a countable set. In particular, X_s^{ver} is a countable set. This complete the proof of theorem.

□

2.2 Existence of vertical components: equal characteristic case

In this subsection, we assume that R has characteristic $p > 0$.

Definition 2.6. Let $f : Y \longrightarrow X$ a G-stable covering over S, v an element of $v(\Gamma_{X_s})$. Suppose G is a p-group. f is called a v-wildly ramified covering if there exists a point $\eta_{Y_v} \in f_s^{-1}(\eta_{X_v})$, where η_{X_v} denotes the generic point of X_v, such that the extension of residue fields $k(\eta_{Y_v})/k(\eta_{X_v})$ is not separable. f is called a wildly ramified covering if f is a v-wildly ramified covering for some $v \in v(\Gamma_{X_s})$.

Theorem 2.7. *Suppose that X is a non-isotrivial smooth curve over S, and X_s can be defined over $\overline{\mathbb{F}}_p$. Then, X_s^{ver} is not empty.*

Proof. By using Corollary 1.6, by replacing X by a finite étale covering of X, we may assume that $\sigma(X_{\overline{\eta}}) - \sigma(X_s) > 0$.

Let G be a p-group. If $X_s^{\mathrm{ver}} \neq \emptyset$, for any G-Galois étale covering $Z_{\overline{\eta}} \longrightarrow X_{\overline{\eta}}$, by replacing S by a finite base change of S, the morphism of stable models $Z \longrightarrow X$ induced by $Z_{\overline{\eta}} \longrightarrow X_{\overline{\eta}}$ is a finite morphism. Since G is a p-group, by [12, Proposition 1 (i)], Z_s is a smooth curve. Write η_{Z_s} for the generic point of Z_s, $I \subseteq G$ for the inertia subgroup of η_{Z_s}. By [12, Proposition 5], we have the quotient scheme Z/I is a smooth curve over S. Since I is the inertia subgroup of η_{Z_s}, $Z_s \longrightarrow Z_s/I$ is a homeomorphism. Moreover, since the natural morphism $(Z/I)_s \longrightarrow Z_s/I$ is a homeomorphism (cf. [4, Corollary A7.2.2]), the natural morphism of special fibers $Z_s \longrightarrow (Z/I)_s$ induced by the quotient $Z \longrightarrow Z/I$ is a homeomorphism. By the computation of genera of special fiber and generic fiber of stable curve Z/I, we have I is trivial. Thus, Z_s is an étale covering over X_s. Thus, the specialization morphism of pro-p étale fundamental groups $\pi_1^p(Z_{\overline{\eta}}) \longrightarrow \pi_1^p(Z_s)$ is an isomorphism. This is a contradiction, then the theorem follows.

□

Theorem 2.8. *Suppose that X_s is an irreducible, singular curve. Then, X_s^{ver} is not empty.*

Proof. For proving the theorem, we only need to prove that there exists an admissible covering of X whose set of vertical points is not empty. On the other hand, there exists an admissible covering X_s' of X_s such that X_s' is untangled and sturdy (cf. [8, Section 0 Curves]). Thus, for proving the theorem, by the assumption X_s is irreducible and applying [17, Corollary 3.11], we may assume that X is a stable curve over S such that the following conditions holds: (1) X_s is untangled and sturdy; (2) $\pi_1(X_{\overline{\eta}}) \otimes \mathbb{F}_p \longrightarrow \pi_1^{\mathrm{adm}}(X_s) \otimes \mathbb{F}_p$ is not an isomorphism; (3) X_η endowed with the action of a finite group H, and the quotient X/H is a stable curve over S such that the morphism of generic fibers $X_\eta \longrightarrow (X/H)_\eta$ (resp. the morphism of special fibers $X_s \longrightarrow (X/H)_s$) induced by the natural morphism $X \longrightarrow X/H$ is an étale covering (resp. an admissible covering) with Galois group H and the special fiber of X/H is an irreducible, singular curve.

From now on, we suppose that Theorem 2.8 does not hold and by replacing S by a finite extension of S, we may assume that all the étale $\mathbb{Z}/p\mathbb{Z}$-coverings of X_η have stable reductions over S. Note that for any finite $\mathbb{Z}/p\mathbb{Z}$-stable covering, the image of nodes and smooth points are nodes and smooth points, respectively (cf. [19, Proposition 2.1]).

Claim 1: There exists a $\mathbb{Z}/p\mathbb{Z}$-stable covering of $f : Y \longrightarrow X$ such that f is a wildly ramified covering.

If Claim 1 does not hold, we have that for any $\mathbb{Z}/p\mathbb{Z}$-stable covering $f : Y \longrightarrow X$, the morphism of special fibers $f_s : Y_s \longrightarrow X_s$ is generically étale. Then by [19, Proposition 2.4], f_s is an admissible covering. This contradicts our assumption (2). We completes the proof of Claim 1.

For a $\mathbb{Z}/p\mathbb{Z}$-stable covering $f : Y \longrightarrow X$, by the computation of genera of generic fiber and special fiber of Y, $f_s : Y_s \longrightarrow X_s$ is not purely inseparable. Thus, if we suppose that f_s is not an admissible covering, by Claim 1, there exist two vertices v_1 and v_2 of $v(\Gamma_{X_s})$ which linked by an edge $e \in e(\Gamma_{X_s})$ such that f is a v_1-wildly ramified covering and f is not a v_2-wildly ramified ramified covering. Moreover, we have the following claim.

Claim 2: There exists a $\mathbb{Z}/p\mathbb{Z}$-stable covering $g : Z \longrightarrow X$ such that g is a v_2-wildly ramified covering and g is not a v_1-wildly ramified covering.

Let us prove Claim 2. Let $v, v' \in v(\Gamma_{X_s})$ be two vertices which linked by an edge. we define the relation $v \rightsquigarrow v'$ if all the v-wildly ramified coverings are v'-wildly ramified coverings. Write Γ_v for the maximal subgraph of Γ_{X_s} under the relation " $\rightsquigarrow$ ". The set of vertices of $v(\Gamma_v)$ consists of the vertices satisfy the following condition hold: $v'' \in v(\Gamma_v)$ if there is a chain $v_0e_{01}v_1e_{12}...v_{n-1}e_{(n-1)n}v_n$ such that (a) $v_0 = v$ and $v_n = v''$; (b) $e_{i(i+1)}$ links v_i and v_{i+1}; (c) $v_0 \rightsquigarrow v_1, ..., v_{n-1} \rightsquigarrow v_n$. The set of edges of $v(\Gamma_v)$ consists of the edges satisfy the following condition hold: $e \in e(\Gamma_v)$ linked v_1^e and v_2^e is contained in $e(\Gamma_v)$ if $v_1^e, v_2^e \in v(\Gamma_v)$. If Claim 2 does not hold, by the definition of Γ_{v_1} and Γ_{v_2}, we have $\Gamma_{v_1} \subseteq \Gamma_{v_2}$. But note that the $v(\Gamma_{X_s})$ is transitive under the action of H, we obtain $\Gamma_{v_1} = \Gamma_{v_2}$. In particular, we have $v_1 \rightsquigarrow v_2$. This is a contradiction, then Claim 2 follows.

By replacing S by a finite extension of S, we may assume that $Y_\eta \times_{X_\eta} Z_\eta$ admits a stable model W over S. The natural morphisms $W_\eta = Y_\eta \times_{X_\eta} Z_\eta \longrightarrow Y_\eta$ and $W_\eta = Y_\eta \times_{X_\eta} Z_\eta \longrightarrow Z_\eta$ induce two morphisms of stable curves $W \longrightarrow Y$ and $W \longrightarrow Z$ over S, respectively. Write T for the fiber product $Y \times_X Z$. We obtain a natural morphism $n : W \longrightarrow T$ by the universal property of fiber products. Write h for the stable covering $W \longrightarrow X$ induced by the natural morphism $W_\eta \longrightarrow X_\eta$, h' for the natural morphism $T \longrightarrow X$. Note that we have $h = h' \circ n$ and h is finite. Thus, n is a finite morphism. Then W is the normalization of T. Write X_{v_1} (resp. X_{v_2}) for the irreducible component of X_s corresponding to v_1 (resp. v_2), Y_1 (resp. Y_2) for the closed subscheme $Y \times_X X_{v_1} \subset Y$ (resp. $Y \times_X X_{v_2} \subset Y$), Z_1 (resp. Z_2) for the closed subscheme $Z \times_X X_{v_1} \subset Z$ (resp. $Z \times_X X_{v_2} \subset Z$), T_1 (resp. T_2) for the closed subscheme $T \times_X X_{v_1} \subset T$ (resp. $T \times_X X_{v_2} \subset T$), W_1 (resp. W_2) for the closed subscheme $W \times_X X_{v_1} \subset W$ (resp. $W \times_X X_{v_2} \subset W$). By the construction of Y and Z, we have $T \longrightarrow Y$ is étale at the generic point of Y_1 and $T \longrightarrow Z$ is étale at the generic point of Z_2. Thus, $\mathcal{O}_{T,\eta_{T_1}}$ and $\mathcal{O}_{T,\eta_{T_2}}$ are normal, where η_{T_1} and η_{T_2} denote the respective generic points of T_1 and T_2. Then $n|_{W_1} : W_1 \longrightarrow T_1$ and $n|_{W_2} : W_2 \longrightarrow T_2$ are birational. Moreover, since T_1 and T_2 are smooth, W_1 and W_2 are smooth too. Then $n|_{W_1}$ and $n|_{W_2}$ are isomorphisms.

Write q_e for the node corresponding to e which links X_{v_1} and X_{v_2}. The inverse image $h^{-1}(q_e)$ only consists of one point which is denoted by w_e. Write $\widehat{X}_{v_1} := \overline{\{\widehat{\eta}_{X_{v_1}}\}}$ and $\widehat{X}_{v_2} := \overline{\{\widehat{\eta}_{X_{v_2}}\}}$ for the irreducible components of $\mathrm{Spec}\,\widehat{\mathcal{O}}_{X_s,q_e}$, respectively, where $\widehat{\eta}_{X_{v_1}}$ and $\widehat{\eta}_{X_{v_2}}$ denote the generic points of $\mathrm{Spec}\,\widehat{\mathcal{O}}_{X_s,q_e}$. Note that $\mathcal{O}_{\widehat{X}_{v_1}}$ and $\mathcal{O}_{\widehat{X}_{v_2}}$ are DVRs. Write $\widehat{h_s}$ for the morphism $\mathrm{Spec}\,\widehat{\mathcal{O}}_{W_s,w_e} \longrightarrow \mathrm{Spec}\,\widehat{\mathcal{O}}_{X_s,q_e}$ induced by h, $\widehat{\eta}_{W_1} := (\widehat{h_s})^{-1}(\widehat{\eta}_{X_{v_1}})$ and $\widehat{\eta}_{W_2} := (\widehat{h_s})^{-1}(\widehat{\eta}_{X_{v_2}})$ for the generic points of the irreducible components of $\mathrm{Spec}\,\widehat{\mathcal{O}}_{W_s,w_e}$, $k(\widehat{\eta}_{W_1})$ and $k(\widehat{\eta}_{W_2})$ for the residue fields, respectively. Write $\widehat{W}_1^s$ and $\widehat{W}_2^s$ for the respective integral closure of $\widehat{X}_{v_1}$ and $\widehat{X}_{v_2}$ in $k(\widehat{\eta}^1_{w_e})^s$ and $k(\widehat{\eta}^1_{w_e})^s$, where $k(\widehat{\eta}_{W_1})^s$ and $k(\widehat{\eta}_{W_2})^s$ denote the respective maximal separable subextension of $k(\widehat{\eta}_{X_{v_1}})$ and

$k(\widehat{\eta}_{X_{v_2}})$ in $k(\widehat{\eta}_{W_1})$ and $k(\widehat{\eta}_{W_2})$. Note that $\mathcal{O}_{\widehat{W}_{v_1}}$ and $\mathcal{O}_{\widehat{W}_{v_2}}$ are DVRs.

Claim 3: The morphism $\mathcal{O}_{\widehat{X}_{v_1}} \longrightarrow \mathcal{O}_{\widehat{W}_{v_1}}$ (resp. $\mathcal{O}_{\widehat{X}_{v_2}} \longrightarrow \mathcal{O}_{\widehat{W}_{v_2}}$) induced by the natural morphism $\widehat{W}_1^s \longrightarrow \widehat{X}_{v_1}$ (resp. $\widehat{W}_2^s \longrightarrow \widehat{X}_{v_2}$) is a wildly ramified extension.

Write $\eta_{X_{v_1}}$ (resp. $\eta_{X_{v_2}}$) for the generic point of irreducible component X_{v_1} (resp. X_{v_2}). Write $t_e \in T_1 \bigcap T_2$ (resp. η_{T_1}, η_{T_2}) for the inverse image of $(h')^{-1}(q_e)$ (resp. $(h')^{-1}(\eta_{X_{v_1}})$, $(h')^{-1}(\eta_{X_{v_2}})$). We have

$$T_1 \longrightarrow T_1^s \longrightarrow X_{v_1}$$

and

$$T_2 \longrightarrow T_2^s \longrightarrow X_{v_2},$$

where T_1^s and T_2^s are smooth projective curves whose function fields are the maximal separable subextensions of $k(\eta_{T_1})/k(\eta_{X_{v_1}})$ and $k(\eta_{T_2})/k(\eta_{X_{v_2}})$, respectively. Then by the construction of T, we have T_1^s and T_2^s are isomorphic to Z_1 and Y_2, respectively. Thus, $T_1^s \longrightarrow X_{v_1}$ and $T_2^s \longrightarrow X_{v_2}$ are wildly ramified at the image of t of the morphism $T_1 \longrightarrow T_1^s$ and the image of t of the morphism $T_2 \longrightarrow T_2^s$, respectively. Since the $n|_{W_1}$ and $n|_{W_2}$ are isomorphisms, we have $\widehat{W}_1^s \cong \mathrm{Spec}\, \widehat{\mathcal{O}}_{T_1^s, t_e}$ and $\widehat{W}_2^s \cong \mathrm{Spec}\, \widehat{\mathcal{O}}_{T_2^s, t_e}$. Then Claim 3 follows.

Then Proposition 2.3 and Claim 3 imply X_s^{ver} is not empty. This is a contradiction. Then the theorem follows.

□

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