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Abstract

Let X be a smooth projective curve over an algebraically closed field of characteristic $p > 0$. M. Raynaud ([Ray2]) proved that there is a **non-abelian** Galois étale covering $Y \longrightarrow X$ which is not new ordinary. In the present paper, we prove that if X is a smooth projective curve over $\overline{\mathbb{F}}_p$, then there exists an **abelian** covering $Y \longrightarrow X$ which is not new ordinary.

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Contents

1 Motivations and main theorem

Let X be a smooth projective curve over an algebraically closed field k of positive characteristic p, $g_X > 1$ the genus of X, and F_X the absolute Frobenius morphism of X. By the specialization morphism, we know that the étale fundamental group $\pi_1(X)$ is a finitely generated profinite group. Thus by the general theory of profinte groups, $\pi_1(X)$ be determined by the set of finite quotients of $\pi_1(X)$. In order to understand the structure of the étale fundamental group $\pi_1(X)$ of X, we consider the following question: for a given finite group G, when G is a quotient of $\pi_1(X)$?

By the specialization isomorphism of prime to p étale fundamental groups, we have the prime to p part of $\pi_1(X)$ is the same as the prime to p part of étale fundamental group of a smooth projective curve over $\mathbb{C}$ with genus g_X. Hence if the order of G is prime to p, G is a quotient of $\pi_1(X)$ if and only if G can be regard as a quotient of the surface group of genus g_X elements. What is the p-part of $\pi_1(X)$? The natural first step is to consider the p-rank of X defined as follows:

Definition 1.1. The p-rank $\sigma(X)$ of X is defined as $\dim_{\mathbb{F}_p} \mathrm{H}^1(X, \mathcal{O}_X)^{F_X}$, where $(-)^{F_X}$ means the F_X-invariant subspace.

Remark 1.1.1. Note that if X is a stable curve over k, we can also define the p-rank of X by $\dim_{\mathbb{F}_p} \mathrm{H}^1(X, \mathcal{O}_X)^{F_X}$.

Since the curve X is projective, by Artin-Schreier theory of étale cohomology, we have $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{H}^1(X, \mathcal{O}_X)^{F_X}$. Furthermore, $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z}/p\mathbb{Z})$. Therefore, we can also define the p-rank of X as

$$\sigma(X) := \mathrm{rank}(\pi_1^p(X)^{\mathrm{ab}}),$$

where the right hand side means the rank of abelianization of pro-p étale fundamental group of X. Furthermore, by a theorem of Shafarevich, the pro-p completion $\pi_1^p(X)$ is a free profinite group generated by $\sigma(X)$ elements. This means that a finite p-group G is a quotient of $\pi_1(X)$ if and only if G can be generated by $\sigma(X)$ elements. Furthermore, let us consider the finite group G in the more general case. Suppose that G is an extension of a prime to p group H by a p-group P. Fix a surjection $f_H : \pi_1(X) \longrightarrow H$, whether or not f_H can be lifted to a surjection $f_G : \pi_1(X) \longrightarrow G$ is called an embedding problem. Let $c_H : Y \longrightarrow X$ be the H-étale covering corresponding to the surjection f_H. The first step to solve the embedding problem is whether or not P can be generated by $\sigma(Y)$ elements. This leads us to study the p-rank of a covering of X.

Let $c_G : Y \longrightarrow X$ be a connected G-étale covering. If G is a finite p-group, then the p-rank of Y can be computed by Deuring-Shafarevich formula (cf. [Cre]). If the order of G is prime to p, the relationship between $\sigma(X)$ and $\sigma(Y)$ will be very complicated and difficult to understand. But we can consider some special coverings.

Write J_X (resp. J_Y) for the Jacobian of X (resp. Y). Thus, the étale covering c_G induces a natural morphism $g_G : J_X \longrightarrow J_Y$ of Jacobians. Write J_Y^{new} for the quotient of abelian varieties $J_Y/g_G(J_X)$, and J_Y^{new} will be called the new part of the Jacobian J_Y of Y with respect to the morphism c_G.

Definition 1.2. A connected G-étale covering $c_G : Y \longrightarrow X$ is called to be new ordinary if the new part J_Y^{new} of Jacobian of Y with respect to the morphism c_G is an ordinary abelian variety (i.e., the p-rank of J_Y^{new} is equal to the dimension of J_Y^{new}). X is called to be strongly new ordinary if any connected μ_n-torsor over X is new ordinary, where $(n, p) = 1$.

Note that if $c_G : Y \longrightarrow X$ is a new ordinary covering, then we have the following equation of p-ranks: $\sigma(Y) = \sigma(X) + g_Y - g_X$, where g_Y denotes the genus of Y. Thus, we can ask a natural question: what curves are strongly new ordinary?

Let $\overline{M}_{g_X, \overline{\mathbb{F}}_p}$ be the coarse moduli space of curves of genus g_X. Write X_{gen} for the curve corresponding to a geometric generic point of $\overline{M}_{g_X, \overline{\mathbb{F}}_p}$. The following result of S. Nakajima (cf. [Nak]) and B. Zhang (cf. [Zhang]) is well-known and we can prove Nakajima-Zhang theorem immediately by using the theory of admissible coverings. For more details on admissible coverings of stable curves, see [Moc], [Yang2] .

Proposition 1.3. *X_{gen} is strongly new ordinary.*

Proof. By considering a reduction of X_{gen} which is a chain of ordinary elliptic curves. Thus, since for any prime to p abelian admissible covering of a chain of ordinary elliptic curves is an étale covering, the proposition can be deduced by the specialization theorem of admissible fundamental groups (cf. [Yang2] Proposition 1.1) immediately.

$\square$

We use the notation $\overline{M}^{f}_{g_X,\overline{\mathbb{F}}_p}$ to denote the locally closed reduced subscheme of $\overline{M}_{g_X,\overline{\mathbb{F}}_p}$) whose geometric points correspond to curves with p-rank f. Write X^f_{gen} for a curve corresponding to a geometric generic point of $\overline{M}^{f}_{g_X,\overline{\mathbb{F}}_p}$. By considering a suitable reduction of X^f_{gen}, E. Ozman and R. Prise generalized Nakajima-Zhang theorem to X^f_{gen} (cf. [OP] Application 1.1). More precisely, they proved a result as follows:

Proposition 1.4. *X^f_{gen} is strongly new ordinary.*

Proposition 1.3 and 1.4 show that any **abelian** étale covering of X^f_{gen} is new ordinary. On the other hand, M. Raynaud constructed a **non-abelian** Galois étale covering of X_{gen} with Galois group of order prime to p which is not new ordinary (cf. [ray2, Théorème 2]). By Raynaud's theorem and specialization isomorphism of prime to p étale fundamental groups, we have for any smooth projective curve X, there is a non-abelian Galois étale covering of X which is not new ordinary. We can ask a question as follows:

Question 1.5. *whether or not exist a smooth projective curve defined over $\overline{\mathbb{F}}_p$ which is strongly new ordinary? or for any smooth projective curve over $\overline{\mathbb{F}}_p$, whether or not exist an* **abelian** *covering which is not new ordinary?*

We have the main theorem of the present paper (cf. Theorem 2.4).

Theorem 1.6. *Let X be a smooth projective curve of genus $g_X \geq 2$ over $\overline{\mathbb{F}}_p$. Then X is not strongly new ordinary.*

Remark 1.6.1. Let X be a smooth curve over an algebraically closed field k of positive characteristic. Thus we have a natural morphism of $c_X : \operatorname{Spec} k \longrightarrow M_{g_X,\overline{\mathbb{F}}_p}$. Write q_X for the set-theoretical image of c_X, k_X for an algebraic closure of the residue field $k(q_X)$ in k. The field k_X will be called a minimal field of definition of X. If the transcendence degree k_X over $\mathbb{F}_p$ is not equal to 0 (i.e., X can not be defined over $\overline{\mathbb{F}}_p$), then the theorem maybe not hold. Let q_{ell} be a point of $\overline{M}_{g_X,\overline{\mathbb{F}}_p}$ such that the curve corresponding to a geometric point of q_{ell} is a chain of elliptic curves. Moreover, by the proofs of Proposition 1.4, for a given integer $1 \leq h \leq 3g-3$, we can chose a point $q \in M^f_{g_X,\overline{\mathbb{F}}_p}$ such that the transcendence degree of $k(q)$ over $\overline{\mathbb{F}}_p$ is equal to h and the closure of q in $\overline{M}_{g_X,\overline{\mathbb{F}}_p}$ contains q_{ell}. Thus the curve corresponding to a geometric point of q is strongly new ordinary.

Remark 1.6.2. In the case of $p \geq 5$, Ozman and Pries (cf. [OP, Application 1.2]) proved that there exists a curve of given genus and given p-rank which admits a non new ordinary $\mathbb{Z}/2\mathbb{Z}$-étale covering. Moreover, the problem whether or not exists a curve of given genus and given p-rank which admits a non new ordinary $\mathbb{Z}/\ell\mathbb{Z}$-étale covering for any prime number $\ell \neq p$ is still unknown, see [OP, Question 7.4].

Let k be an algebraically closed field of characteristic $p > 0$. Thus $M_{g_X, \overline{\mathbb{F}}_p}(k)$ is the set of isomorphism classes of smooth projective curves over k of genus g_X. For any $\overline{x} \in M_{g_X, \overline{\mathbb{F}}_p}(k)$, we obtain a smooth projective curve $X_{\overline{x}}$ over k which corresponding to $\overline{x}$, and a unique point $x \in M_{g_X, \overline{\mathbb{F}}_p}$ such that $\overline{x}$ factors through x. Since the geometric fundamental groups of projective curves do not dependent on the base field, we can write $\pi_1(x)$ for $\pi_1(X_{\overline{x}})$. Thus we obtain a functor from the prime points of coarse moduli space $M_{g_X, \overline{\mathbb{F}}_p}$ to the category of profinite groups π_1 which sends $x \in M_{g_X, \overline{\mathbb{F}}_p}$ to $\pi_1(x)$.

As an application, we can re-prove a result of Pop-Saïdi (cf. [PS, Corollary]) which answered a question asked by David Harbater, see also Corollary 2.5.

Corollary 1.7. *There is no nonempty open subset $U \subseteq M_{g_X, \overline{\mathbb{F}}_p}$ such that the isomorphy type of the geometric fundamental group $\pi_1(x)$ is constant on U.*

2 Proof of the main theorem

Write $X^1 := X \times_{k, F_k} k$ for the pull-back of X by the Frobenius F_k of k. Thus, we obtain a relative Frobenius morphism $F_{X/k} : X \longrightarrow X^1$. The canonical differential $(F_{X/k})_*(d) : (F_{X/k})_*(\mathcal{O}_X) \longrightarrow (F_{X/k})_*(\Omega^1_X)$ is a morphism of $\mathcal{O}_X$-modules. Write B_X for the image of $(F_{X/k})_*(d)$ which is called the sheaf of locally exact differentials. One has the exact sequence

$$0 \longrightarrow \mathcal{O}_{X^1} \longrightarrow (F_{X/k})_*(\mathcal{O}_X) \longrightarrow B_X \longrightarrow 0,$$

and B_X is a vector bundle on X^1 of rank $p-1$. Raynaud's theorem (cf. [Ray1], Theoreme 4.1.1) shows that there is a divisor Θ_X of J^1_X, where J^1_X is the pull-back of the Jacobian J_X of X by the Frobenius F_k. Furthermore, the support of Θ_X is as follows:

$$\Theta_X(k) = \{[\mathcal{L}] \in J^1(k) \mid \mathrm{H}^1(X^1, B_X \otimes \mathcal{L}) \neq 0\}.$$

For more details on Raynaud's theory of theta divisors, see [Ray1].

Definition 2.1. Let M be a torsion abelian group. For each element $t \in M$, we define the saturation of x to be the subset of elements of in the form $i.t$, where i is an integer prime to the order of t. We use the notation $\mathrm{Sat}(t)$ to denote the saturation of t.

We have the following relationship between new ordinary and theta divisors.

Proposition 2.2. *Let $f : Y \longrightarrow X$ be a μ_n-torsor. Let t be a torsion point of $J^1_X(k)$ of order n corresponding to the μ_n-torsor $f^1 : Y^1 \longrightarrow X^1$. Then $f : Y \longrightarrow X$ is new ordinary if and only if $\mathrm{Sat}(t) \bigcap \Theta_X = \emptyset$. In particular, X is ordinary if and only if the zero point of J^1_X is not contained in Θ_X.*

Proof. See [Ray3], Proposition 2.1.4. □

Before the proof of our main theorem, we need a well-know theorem of Anderson-Indik as follows (cf. [Tam1] P704):

Proposition 2.3. *Let A be an abelian variety over $\overline{\mathbb{F}}_p$, Z an irreducible reduced closed subscheme of A. Write $Z\{p'\}$ for the set of prime to p torsion points of Z. If $Z\{p'\}$ is not dense in Z, then Z is contained in a translate of a proper abelian subvariety of A.*

Theorem 2.4. *Let X be a smooth projective curve of genus $g_X > 1$ over $\overline{\mathbb{F}}_p$. Then X is not strongly new ordinary.*

Proof. Write Θ_i for arbitrary irreducible components of Θ_X, where Θ_X denotes the Raynaud's theta divisor. If arbitrary abelian covering $Y \longrightarrow X$ is new ordinary, then we have $J^1_X\{p'\} \bigcap \Theta_X(\overline{\mathbb{F}}_p) \subseteq \{0_{J^1_X}\}$, where $J^1_X\{p'\}$ denotes the set of prime to p torsion points of $J^1_X(\overline{\mathbb{F}}_p)$. Then since $\dim(\Theta_i) > 0$, $\Theta_i\{p'\} := J^1_X\{p'\} \bigcap \Theta_i(\overline{\mathbb{F}}_p) \subseteq J^1_X\{p'\} \bigcap \Theta_X(\overline{\mathbb{F}}_p)$ is not dense in Θ_i. Furthermore, since Θ_i is defined over $\overline{\mathbb{F}}_p$, by applying Anderson-Indik's theorem, we have the Θ_i is a subvariety of a translate of a proper abelian varietry of J^1_X. But this contradict to a theorem of Raynaud (cf. [Ray3], Proposition 1.2.1) which says that there exists an irreducible component of Θ_X which is not contain in a translate of a proper sub-abelian variety of J^1_X.

□

As an application of Theorem 2.4, we can re-prove Pop-Saïdi's result as follows.

Corollary 2.5. *There is no nonempty open subset $U \subseteq M_{g_X, \overline{\mathbb{F}}_p}$ such that the isomorphy type of the geometric fundamental group $\pi_1(x)$ is constant on U.*

Proof. Let U be an arbitrary open set of $M_{g_X, \overline{\mathbb{F}}_p}$, x be a closed point of U. Note that U contains the generic point η of $M_{g_X, \overline{\mathbb{F}}_p}$. By choosing a complete discrete valuation ring R and a morphism $c_R : \mathrm{Spec}\, R \longrightarrow M_{g_X, \overline{\mathbb{F}}_p}$ such that $c_R(\eta_R) = \eta$ and $c_R(s_R) = x$, where η_R is the generic point of $\mathrm{Spec}\, R$ and s_R is the closed point of $\mathrm{Spec}\, R$. Thus we obtain a smooth projective curve $\mathcal{X}$ over $\mathrm{Spec}\, R$ and a specialization morphism $sp_R : \pi_1(\eta) \longrightarrow \pi_1(x)$. By applying Proposition 1.3 and Theorem 2.4, there exists a positive integer n and a $\mathbb{Z}/n\mathbb{Z}$-étale covering $\mathcal{Y}$ of $\mathcal{X}$ such that the geometric generic fiber $\mathcal{Y}_{\overline{\eta}_R}$ is ordinary and the geometric special fiber $\mathcal{Y}_{\overline{s}}$ is not ordinary. This means that sp_R is not an isomorphism. Thus π_1 is not a constant on U by the general theory of profinite groups. □

Remark 2.5.1. In [PS], F. Pop and M. Saïdi proved a theorem which shows the specialization morphism of fundamental groups of smooth projective curves over positive characteristic is an isomorphism under certain assumptions. Then together with a theorem of C-L. Chai and F. Oort, and a theorem of J-P. Serre, they obtained Corollary 2.5.

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