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Abstract

In the present paper, we study the ordinariness of coverings of stable curves. Let $f : Y \longrightarrow X$ be a morphism of stable curves over a complete discrete valuation ring with algebraically closed residue field of characteristic $p > 0$. Suppose that the generic fiber X_η of X is smooth and the morphism of generic fibers f_η is a Galois étale covering whose Galois group is a solvable group G. We prove that if the special fiber X_s is sturdy (i.e., the genus of the normalization of each irreducible component of $X_s \geq 2$) and f_s is not an admissible covering, then f_s is not new-ordinary. This result extends a result of M. Raynaud concerning the ordinariness of coverings to the case of stable curves.

Introduction

Let R be a complete valuation ring with algebraically closed residue field k of characteristic $p > 0$, K the quotient field and $\overline{K}$ an algebraic closure of K. We use the notation S to denote the spectrum of R. Write $\eta, \overline{\eta}$ and s for the generic point, the geometric generic point and the closed point corresponding to the natural morphisms $\mathrm{Spec}\, K \longrightarrow S$, $\mathrm{Spec}\, \overline{K} \longrightarrow S$ and $\mathrm{Spec}\, k \longrightarrow S$, respectively. Let X be a stable curve of genus g_X over S. Write X_η, $X_{\overline{\eta}}$ and X_s for the generic fiber, the geometric generic fiber and the special fiber, respectively. Moreover, we suppose that X_η is smooth.

Let Y_η be a geometrically connected curve, $f_\eta : Y_\eta \longrightarrow X_\eta$ a Galois étale covering with Galois group G. By replacing S by a finite extension of S, Y_η admits a stable model

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over S and f_η can be extended to a unique G-stable covering $f : Y \longrightarrow X$ over S (cf. Definition 1.10). It is natural to pose the following question:

QUESTION 0.1. *What is the special fiber Y_s of Y (e.g. the p-rank, the dual graph, properties of morphism of special fibers f_s and so on) ?*

In order to approach this question, first, let us consider the case that $(\sharp G, p) = 1$, where $\sharp G$ denotes the order of G. By the specialization theorem of prime to p log étale fundamental groups (cf. [18, Proposition 1.1]), we have f_s is an admissible covering (cf. [7] for the definition of (log) admissible coverings). Thus, the covering f_s is simple. But the p-rank and the ordinariness of prime to p covering Y_s is to become complicated. For example, there are well-known results as follows. If G is abelian, and Y_s is a curve corresponding to a geometric generic point of the moduli space, then f_s is new-ordinary (cf. Definition 1.5) (cf. [10], [22]). If $G \cong \mathbb{Z}/\ell\mathbb{Z}$, where ℓ is a prime number, this result be generalized to the case of the geometric curves corresponding to a geometric generic point of an irreducible component of the p-strata of the moduli space (cf. [11]). On the other hand, if Y_s can be defined over $\overline{\mathbb{F}}_p$, then there exists an abelian G-stable covering (i.e., G is abelian) f such that f_s is not new-ordinary (cf. [19]). Moreover, there exists a non-abelian solvable G-stable covering (i.e., G is non-abelian solvable) such that f_s is not new-ordinary (cf. [14]).

In the case that $p|\sharp G$. If G is a p-group and f_s is an admissible covering, then the p-rank of Y_s can be calculated by using Deuring-Shafarevich formula (cf. Proposition 1.3). However, f_s is not an admissible covering in general. The problem that whether or not f_s is a finite morphism, moreover, an admissible covering was studied by A. Tamagawa and the author (cf. [17], [20], [21]). On the other hand, for the p-rank of Y_s, M. Raynaud proved that if X is smooth and f_s is not an admissible covering (note that if X_s is smooth, then the definitions of étale coverings and admissible coverings are equivalent), then f_s is not new-ordinary (cf. [12], [13]).

In the present paper, if G is a solvable group, we generalize Raynaud's theorem to the case of stable curves as follows, see also Theorem 3.4.

THEOREM 0.2. *Let G be a finite solvable group, $f : Y \longrightarrow X$ a G-stable covering. Suppose that X_s is sturdy (i.e., the genus of the normalization of each irreducible component of $X_s \geq 2$) and the morphism of special fibers f_s is not an admissible covering. Then f_s is not new-ordinary.*

Notations and Conventions

Curves:

By a curve over a field, we mean a finite type, separated, connected, one dimensional reduced scheme over a field.

An semi-stable curve X over a scheme S consists of a flat, proper morphism $X \longrightarrow S$ such that for each geometric point $\overline{s}$ of S, the geometric fiber $X_{\overline{s}}$ is a reduced and connected curve of genus g with at most ordinary double points (i.e., nodes).

An n-pointed stable curve (X, D_X) of type (g, n) over a scheme S consists of a flat, proper morphism $X \longrightarrow S$, together with a closed subscheme $D_X \subseteq X$ such that for each geometric point $\overline{s}$ of S:

(i) The geometric fiber $X_{\overline{s}}$ is a reduced and connected curve of genus g with at most ordinary double points (i.e., nodes).

(ii) $X_{\overline{s}}$ is smooth at the points of D_X.

(iii) The composite morphism $D_X \subseteq X \longrightarrow S$ is finite étale of degree n.

(iv) Let E be an irreducible component of $X_{\overline{s}}$ of genus g_E. Then the sum of the degree of the restriction of D_X to E and the number of points where E meets the closure of the complement of E in $X_{\overline{s}}$ is $\geq 3 - 2g_E$.

(v) $\dim(H^1(X_{\overline{s}}, \mathcal{O}_{X_{\overline{s}}})) = g$.

In this situation, one verifies easily that $2g - 2 + n$ is ≥ 1.

We shall say that an S-scheme X is a stable curve of genus g over S if $(X, \emptyset)$ is a 0-pointed stable curve of genus g over S.

We shall say that a pointed stable curve (X, D_X) over a scheme S is smooth if the morphism of schemes $X \longrightarrow S$ is smooth.

We denote $X^\bullet := (X, D_X)$ a pointed stable curve over S with divisor of marked points D_X and underlying scheme X.

For more details on stable curves, pointed stable curves and their moduli stacks, see [3], [4].

Galois categories and their fundamental groups:

We denote the categories of finite étale and finite admissible coverings of "$(-)$" by $\mathrm{Cov}(-)$ and $\mathrm{Cov}_{\mathrm{adm}}(-)$, respectively.

The notations $\pi_1(-)$ and $\pi_1^{\mathrm{adm}}(-)$ will be used to denote the étale and admissible fundamental groups of "$(-)$", respectively; the notation $(-)^{\mathrm{ab}}$ denotes the abelianization of the group $(-)$

For more admissible coverings and their fundamental groups for (pointed) stable curves, see [7], [8], [18].

1 Preliminary

In this section, we give some definitions and propositions which will be used in the present paper.

DEFINITION 1.1. Let X be a semi-stable curve over an algebraically closed field of characteristic $p > 0$, $\pi_1^p(X)$ the maximal pro-p quotient of the admissible fundamental group $\pi_1^{\mathrm{adm}}(X)$ (by choosing a base point). It is well-known that $\pi_1^p(X)$ is a finitely generated free profinite group. We define the p-rank $\sigma(X)$ of X as follows:

$$\sigma(X) := \mathrm{rank}(\pi_1^p(X)).$$

REMARK 1.2. If X is smooth, then the p-rank $\sigma(X)$ is equal to the dimension of the p-torsion points of the Jacobian J_X as a $\mathbb{F}_p$-vector space.

If X is a singular curve. Write Γ_X for the dual graph of X, $v(\Gamma_X)$ for the set of vertices, X_v for the irreducible component of X associated to $v \in v(\Gamma_X)$, $\widetilde{X_v}$ for the normalization of X_v, respectively. Then the p-rank $\sigma(X)$ of X is equal to

$$\sum_{v \in v(\Gamma_X)} \sigma(\widetilde{X_v}) + \mathrm{rank}(\pi_1(\Gamma_X)),$$

where $\mathrm{rank}(\pi_1(\Gamma_X))$ denotes the rank of the topological fundamental group of the dual graph Γ_X.

The p-rank of a p-Galois covering (i.e., the Galois group is a p-group) of a smooth projective curve can be calculated by Deuring-Shafarevich formula as follows (cf. [2]).

PROPOSITION 1.3. *Let $f : Y \longrightarrow X$ be a Galois covering (possibly ramified) of smooth projective curves over an algebraically closed field of characteristic $p > 0$, whose Galois group is a finite p-group G. Then*

$$\sigma(Y) - 1 = (\sharp G)(\sigma(X) - 1) + \sum_{y \in Y}(e_y - 1),$$

where e_y denotes the ramification index at y.

DEFINITION 1.4. Let X be a semi-stable curve of genus g_X over an algebraically closed field of characteristic $p > 0$. X is called ordinary if $\sigma(X) = g_X$.

DEFINITION 1.5. Let $f : Y \longrightarrow X$ be a non-constant morphism (which is not necessarily finite) of semi-stable curves over an algebraically closed field of characteristic $p > 0$. Write g_X and g_Y for the genera of X and Y, respectively. f is called new-ordinary if $g_Y - g_X = \sigma(Y) - \sigma(X)$.

REMARK 1.6. The original definition of new-ordinary is as follows. Let X and Y be two smooth projective curves over an algebraically closed field of characteristic $p > 0$, g_X and g_Y the genera of X and Y, respectively, $f : Y \longrightarrow X$ a finite étlae covering with Galois group $\mathbb{Z}/n\mathbb{Z}$. f is called new-ordinary if the new part of the Jacobian J_Y respect to the morphism f is ordinary. Moreover, this definition is equivalent to $g_Y - g_X = \sigma(Y) - \sigma(X)$.

REMARK 1.7. Let $f : Y \longrightarrow X$ be a Galois covering (possibly ramified) over an algebraically closed field of characteristic $p > 0$, whose Galois group is a finite p-group G, n the cardinality of the set of branch points of f, g_X and g_Y the genera of X and Y, respectively. By Hurwitz formula, we have

$$2g_Y - 2 = (\sharp G)(2g_X - 2) + \deg(\mathcal{R}),$$

where $\mathcal{R}$ denotes the ramification divisor of f. By applying Deuring-Shafarevich fomula, we have f is new-ordinary if and only if $\deg(\mathcal{R}) = 2n(p-1)$.

DEFINITION 1.8. Let X be a projective curve over an algebraically closed field of characteristic $p > 0$, x a closed point of X. x is called geometrically unibranch if $\mathrm{Spec}\,\mathcal{O}_{X,x}^{\mathrm{hs}}$ is irreducible, where $\mathcal{O}_{X,x}^{\mathrm{hs}}$ denotes a strict henselization of $\mathcal{O}_{X,x}$.

REMARK 1.9. Suppose the singular points of X are either nodes or geometrically unibranch. By taking the normalization of X at the geometrically unibranch singular points, there is a unique semi-stable curve X^{ss} such that the normalization morphism $\delta : X^{\mathrm{ss}} \longrightarrow X$ is a homeomorphism. We call X^{ss} the semi-stable curve associated to X.

From now on, we fix some notation as follows. Let R be a complete valuation ring with algebraically closed residue field k of characteristic $p > 0$, K the quotient field and $\overline{K}$ an algebraic closure of K. We use the notation S to denote the spectrum of R, $\eta, \overline{\eta}$ and s stand for the generic point, the geometric generic point, the closed point corresponding to the natural morphisms $\mathrm{Spec}\,K \longrightarrow S$, $\mathrm{Spec}\,\overline{K} \longrightarrow S$ and $\mathrm{Spec}\,k \longrightarrow S$, respectively. Let X be a semi-stable curve over S, X_η, $X_{\overline{\eta}}$ and X_s the generic fiber, the geometric generic fiber and the special fiber, respectively. Moreover, we suppose that X_η is smooth.

DEFINITION 1.10. Let $f : Y \longrightarrow X$ be a morphism of semi-stable curves over S, G a finite group. f is called a semi-stable covering (resp. G-semi-stable covering) if the morphism of generic fibers f_η is an étale covering (resp. an étale covering with Galois group G). f is called a stable covering (resp. G-stable covering) if X and Y are stable curves.

REMARK 1.11. For any G-étale covering $f_\eta : Y_\eta \longrightarrow X_\eta$ of smooth, geometrically connected projective curves over $\mathrm{Spec}\,K$. By applying the semi-stable reduction theorem of curves and Proposition 4.4, by replacing S by a finite extension of S, f_η can be extended to a G-semi-stable covering, moreover a G-stable covering $f : Y \longrightarrow X$ over S.

DEFINITION 1.12. Let $f : Y \longrightarrow X$ be a semi-stable covering. Suppose that the morphism of special fibers $f_s : Y_s \longrightarrow X_s$ is not finite. A closed point $x \in X$ is called a vertical point associated to f, or for simplicity, a vertical point when there is no fear of confusion, if $\dim(f^{-1}(x)) = 1$. The inverse image $f^{-1}(x)$ is called the vertical fiber associated to x.

The following proposition was proved in [21, Proposition 2.4].

PROPOSITION 1.13. *Let $f : Y \longrightarrow X$ be a G-stable covering, x a vertical point of X. If x is a smooth point or a node which is contained in only one irreducible component (resp. a node which is contained in two different irreducible components), we use the notation X_v (resp. X_{v_1}, X_{v_2}) to denote the irreducible component (resp. irreducible components) which contains x. Write y and Y_v (resp. Y_{v_1}, Y_{v_2}) for a point of the inverse image of x and an irreducible component (resp. the irreducible components) of Y such that $f_s(Y_v) = X_v$ and $y \in Y_v$ (resp. $f_s(Y_{v_1}) = X_{v_1}, f_s(Y_{v_2}) = X_{v_2}$ such that $y \in Y_{v_1}$ or $y \in Y_{v_2}$). Write $I_v \subseteq G$ (resp. $I_{v_1} \subseteq G$ and $I_{v_2} \subseteq G$) for the inertia subgroup of Y_v (resp. the inertia subgroups of Y_{v_1} and Y_{v_2} respectively). Then, $I_v \neq \{1\}$ (resp. $I_{v_1} \neq \{1\}$ or $I_{v_2} \neq \{1\}$). In particular, if f_s is generically étale, then f_s is an admissible covering.*

PROOF. By using [1, 6.7 Proposition 4], we can contract $f_s^{-1}(x)$ and obtain a contraction morphism $c : Y \longrightarrow Y'$. Since Y' is a blowing-up of the integral closure of X in the function field of Y, Y' is a fiber surface over S (i.e., normal and flat over S) and there is natural commutative diagram as follows:

$$\begin{array}{ccc} Y_\eta & \longrightarrow & Y \\ {\scriptstyle c_\eta}\downarrow & & {\scriptstyle c}\downarrow \\ Y'_\eta & \longrightarrow & Y' \\ {\scriptstyle f'_\eta}\downarrow & & {\scriptstyle f'}\downarrow \\ X_\eta & \longrightarrow & X, \end{array}$$

where c_η is an identity morphism.

If x is a smooth point and I_v is trivial. Then, f' is étale at the generic point of $c(Y_v)$. By applying Zariski-Nagata purity, we have the image $c(y)$ is a smooth point.

If x is a node, and I_v (resp. I_{v_1} and I_{v_2}) is (resp. are) trivial. Then, f' is étale at the generic point (resp. generic points) of $c(Y_v)$ (resp. $c(Y_{v_1})$ and $c(Y_{v_2})$). The completion of the local ring at x is $\hat{\mathcal{O}}_{X,x} \cong R[[u,v]]/(uv - \pi^{p^e n'})$, where π denotes an uniformizer of R and $(n', p) = 1$. Since the étale fundamental group of $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \setminus \{\hat{x}\}$ is isomorphic to $\mathbb{Z}/n'\mathbb{Z}$, where $\hat{x}$ denotes the closed point of $\operatorname{Spec} \hat{\mathcal{O}}_{X,x}$, we have the image $c(y)$ is a node.

Then Y' is the stable model of Y'_η over S in either case, so that $Y = Y'$. This contradict to the assumption $\dim(f^{-1}(x)) = 1$. Thus $I_v \neq \{1\}$ (resp. $I_{v_1} \neq \{1\}$ or $I_{v_2} \neq \{1\}$). This completes the proof of the proposition.

$\square$

The following result was proved by Raynaud (cf. [12, Théorème 1 and Proposition 1]).

PROPOSITION 1.14. *Let G be a finite p-group, $f : Y \longrightarrow X$ a G-semi-stable covering and x a vertical point associated to f. If x is a smooth point of X_s, then the p-rank of each connected component of the vertical fiber $f^{-1}(x)$ associated to x is equal to 0. Furthermore, by contracting the vertical fibers $f^{-1}(x)$, we obtain a curve Y^{c} over S. Write $c : Y \longrightarrow Y^{\mathrm{c}}$ for the contracting morphism. Then the points of $c(f^{-1}(x))$ are geometrically unibranch.*

2 P-ranks of vertical fibers of a stable covering

From now on, we only consider stable coverings. In this section, we study the p-ranks of vertical fibers of a stable covering.

Let $f : Y \longrightarrow X$ be a $\mathbb{Z}/p\mathbb{Z}$-stable covering , x a vertical point. Moreover, we suppose that x is a singular point of X_s. Then there are two irreducible components X_{v_1} and X_{v_2} (which may be equal) such that $x \in X_{v_1} \bigcap X_{v_2}$. Write $V_x \subseteq f^{-1}(x)$ for a connected component of the vertical fiber associated to x (in fact, $f^{-1}(x)$ is connected), Γ_x for the dual graph of V_x. Write Y_{v_1} (resp. Y_{v_2}) for an irreducible component of Y_s such that $f_s(Y_{v_1}) = X_{v_1}$ (resp. $f_s(Y_{v_2}) = X_{v_2}$) and $Y_{v_1} \bigcap V_x$ (resp. $Y_{v_2} \bigcap V_x$) is not empty. The action of $\mathbb{Z}/p\mathbb{Z}$ on the generic fiber Y_η induces an action of $\mathbb{Z}/p\mathbb{Z}$ on Y_{v_1} (resp. Y_{v_2}). We use the notation $I_{Y_{v_1}}$ (resp. $I_{Y_{v_2}}$) to denote the inertia subgroup of $\mathbb{Z}/p\mathbb{Z}$. Since x is a vertical point, by Proposition 1.13, we obtain either $I_{Y_{v_1}} = \mathbb{Z}/p\mathbb{Z}$ or $I_{Y_{v_2}} = \mathbb{Z}/p\mathbb{Z}$. Thus, we may assume that $I_{Y_{v_1}} = \mathbb{Z}/p\mathbb{Z}$. Moreover, we have a lemma as follows:

LEMMA 2.1. *(a) If $I_{Y_{v_1}} = \mathbb{Z}/p\mathbb{Z}$ and $I_{Y_{v_2}}$ is trivial, then $\sigma(V_x) = 0$. In particular, V_x is not ordinary. (b) If $I_{Y_{v_1}} = I_{Y_{v_2}} = \mathbb{Z}/p\mathbb{Z}$, then one of the following conditions will be satisfied: (i) $\sigma(V_x) = 0$; (ii) $\sigma(V_x) = p-1$ and $\pi_1(\Gamma_x) = p-1$; (iii) $\sigma(V_x) = p-1$ and Γ_x is a tree.*

PROOF. Write $\widetilde{X}$ for the quotient $Y/\mathbb{Z}/p\mathbb{Z}$ of Y. Note that $\widetilde{X}$ is a semi-stable curve over S whose generic fiber $\widetilde{X}_\eta$ is isomorphic to X_η (cf. [12, Proposition 5]). We obtain two morphisms of semi-stable curves $\widetilde{q} : Y \longrightarrow \widetilde{X}$ and $\widetilde{b} : \widetilde{X} \longrightarrow X$ such that $\widetilde{b} \circ \widetilde{q} = f$. Write $\widetilde{X}_{v_1}$ (resp. $\widetilde{X}_{v_2}$) for the irreducible component of the inverse image $\widetilde{b}^{-1}(X_{v_1})$ (resp. $\widetilde{b}^{-1}(X_{v_2})$) such that $\widetilde{X}_{v_1}$ (resp. $\widetilde{X}_{v_2}$) is finite over X_{v_1} (resp. X_{v_2}). There exists a chain of projective lines $\widetilde{C} := \widetilde{P}_1\widetilde{P}_2, ..., \widetilde{P}_n$ of the special fiber $\widetilde{X}_s$ of $\widetilde{X}$ that meet

$\widetilde{b}^{-1}(X_s - \{x\})$ at two points. More precisely, $\{P_i\}_i$ are closed subschemes of $\widetilde{X}_s$ satisfy the following conditions: (1) $P_i \neq P_j$ if $i \neq j$; (2) $\sharp(\widetilde{P}_1 \bigcap \widetilde{X}_{v_1}) = 1$; (3) $\sharp(\widetilde{P}_n \bigcap \widetilde{X}_{v_2}) = 1$; (4) $\sharp(\widetilde{P}_i \bigcap \widetilde{P}_{i+1}) = 1$. Write $\widetilde{B}_x$ for $\widetilde{b}^{-1}(x)$, $\overline{\{\widetilde{B}_x - \widetilde{C}\}}$ for the closure of $\widetilde{B}_x - \widetilde{C}$ in $\widetilde{B}_x$, $\widetilde{B}_1, ..., \widetilde{B}_m$ for the connected components of $\overline{\{\widetilde{B}_x - \widetilde{C}\}}$. By the general theory of semi-stable models, for any i, $\widetilde{B}_i$ is a tree which consists of projective lines and $\widetilde{B}_i \bigcap \widetilde{C}$ is a smooth point of $\widetilde{C}$. By contracting $\{\widetilde{B}_i\}_i$, we obtain a semi-stable curve Z whose generic fiber Z_η is isomorphic to X_η (cf. [1, 6.7 Theorem 1] and [6, Lemma 10.3.31]), a contracting morphism of semi-stable curves $c_X : \widetilde{X} \longrightarrow Z$ and a natural morphism $\widetilde{q}' : Z \longrightarrow X$ such that $\widetilde{q} = \widetilde{q}' \circ c_X$. On the other hand, by contracting $\widetilde{q}^{-1}(\bigcup_i \widetilde{B}_i)$, we obtain a curve W over S whose generic fiber W_η is isomorphic to Y_η, and a contracting morphism $c_Y : Y \longrightarrow W$. Note that, by Proposition 1.14, the singular points of W_s are either nodes or geometrically unibranch. Write W_{v_1} (resp. W_{v_2}) for the image $c_Y(Y_{v_1})$ (resp. $c_Y(Y_{v_2})$). Moreover, by the construction of the contracting morphism (cf. [1, 6.7 Theorem 1]), the action of $\mathbb{Z}/p\mathbb{Z}$ on Y induces an action of $\mathbb{Z}/p\mathbb{Z}$ on W. Write $I_1 \subseteq \mathbb{Z}/p\mathbb{Z}$ (resp. $I_2 \subseteq \mathbb{Z}/p\mathbb{Z}$) for the inertia subgroup of W_{v_1} (resp. W_{v_2}). Then we have $I_1 = I_{Y_{v_1}}$ and $I_2 = I_{Y_{v_2}}$. Moreover, by the constructions of W and Z, we obtain a natural morphism $g : W \longrightarrow Z$ induced by $\widetilde{q}$ satisfies the following commutative diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{c_Y} & W & & \\ {\scriptstyle \widetilde{q}}\downarrow & & {\scriptstyle g}\downarrow & & \\ \widetilde{X} & \xrightarrow{c_X} & Z & \xrightarrow{\widetilde{q}'} & X, \end{array}$$

where $\widetilde{q}' \circ c_X \circ \widetilde{q} = f$, $\widetilde{q}$ and g are finite morphisms.

Write Z_{v_1} (resp. Z_{v_2}, C, P_i) for the image $c_X(\widetilde{X}_{v_1})$ (resp. $c_X(\widetilde{X}_{v_2})$, $c_X(\widetilde{C})$, $c_X(\widetilde{P}_i)$). Note that C is a chain of projective lines that meet the other irreducible components of the chain at most two points, and for any i, $c_X(\widetilde{B}_i)$ is a smooth point of the special fiber Z_s. By Proposition 1.8, the p-ranks of all the connected components of $(c_X \circ \widetilde{b} \circ \widetilde{q})^{-1}(c_X(\bigcup_i \widetilde{B}_i)) = \widetilde{q}^{-1}(\bigcup_i \widetilde{B}_i)$ are equal to 0. Thus, the lemma is equivalent to the following form: (a') If $I_1 = \mathbb{Z}/p\mathbb{Z}$ and I_2 is trivial, then the $\sigma(g_s^{-1}(C)^{\mathrm{ss}}) = 0$, where $g_s^{-1}(C)^{\mathrm{ss}}$ denotes the unique semi-stable curve associated to $g_s^{-1}(C)$ defined in Remark 1.9. Moreover, either $\widetilde{q}^{-1}(\bigcup_i \widetilde{B}_i)$ is not empty or $g_s^{-1}(C)^{\mathrm{ss}}$ is not ordinary. (b') If $I_1 = I_2 = \mathbb{Z}/p\mathbb{Z}$, then one of the following conditions will be satisfied: (i) $\sigma(g_s^{-1}(C)^{\mathrm{ss}}) = 0$; (ii) $\sigma(g_s^{-1}(C)^{\mathrm{ss}}) = p - 1$ and $\pi_1(\Gamma_{g_s^{-1}(C)^{\mathrm{ss}}}) = p - 1$; (iii) $\sigma(g_s^{-1}(C)^{\mathrm{ss}}) = p - 1$ and $\Gamma_{g_s^{-1}(C)^{\mathrm{ss}}}$ is a tree, where $\Gamma_{g_s^{-1}(C)^{\mathrm{ss}}}$ denotes the dual graph of semi-stable curve $g_s^{-1}(C)^{\mathrm{ss}}$.

Let us prove (a'). If there exists an irreducible component P_i of C such that $1 \leq i \leq n - 1$ and the morphism of special fibers $g_s : W_s \longrightarrow Z_s$ is generically étale over P_i. Contacting $P_{i+1}, P_{i+2}, ..., P_n$ and $(g \circ c_Y)^{-1}(\bigcup_{i+1 \leq j \leq n} P_j)$, we obtain two curves Z' and Y' over S, respectively. Since C is a chain of projective lines that meet the other irreducible components of the chain at most two points, by [6, Lemma 10.3.31], Z' is a semi-stable curve. By Proposition 1.13, we have Y' is a stable curve over S. This is a contradiction. Thus, g_s is purely inseparable over P_i (i.e., the extension of residue fields at generic points induced by g_s is purely inseparable extension) for each $1 \leq i \leq n-1$.

If g_s is purely inseparable over P_n. Then $g_s^{-1}(C) \subset W_s$ is a chain of rational curves and the p-rank of $\sigma(g_s^{-1}(C)^{\mathrm{ss}})$ is 0. Moreover, since Y is a stable curve, we have $\widetilde{q}^{-1}(\bigcup_i \widetilde{B}_i)$ is not empty.

If g_s is generically étale over P_n and purely inseparable over P_j for each $1 \leq j \leq n-1$. Then the p-rank $\sigma(g_s^{-1}(C)^{\mathrm{ss}})$ is equal to the p-rank $\sigma(g_s^{-1}(P_n))$. Since $I_2 = 0$, we have $g_s|_{g_s^{-1}(P_n)} : g_s^{-1}(P_n) \longrightarrow P_n$ is a generically étale morphism. By Proposition 1.13, we have $(\bigcup_i \widetilde{B}_i) \bigcap \widetilde{P}_n = \emptyset$, then $g_s^{-1}(P_n)$ is smooth. Thus, $g_s|_{g_s^{-1}(P_n)}$ has only one branch point. Thus, by Proposition 1.2, $\sigma(g_s^{-1}(P_n)) = 0$. Moreover, since $g_s^{-1}(P_n)$ is not ordinary, $g_s^{-1}(C)^{\mathrm{ss}}$ is not ordinary. This completes the proof of (a').

Let us prove (b'). If g_s is purely inseparable over C, then $\sigma(g_s^{-1}(C)^{\mathrm{ss}}) = 0$. This is Case (i).

If g_s is not purely inseparable over C, then there exists an i such that g_s is generically étale over P_i and purely inseparable over P_j for all the $i+1 \leq j \leq n$. The following two cases will appear: (1) g_s is purely inseparable over P_j if $j \neq i$; (2) there exists $i' < i$ such that g_s is generically étale over $P_{i'}$.

In (1), the dual graph of $g_s^{-1}(C)^{\mathrm{ss}}$ is a tree and the p-rank $\sigma(g_s^{-1}(C)^{\mathrm{ss}})$ is equal to the p-rank $\sigma(g^{-1}(P_i))$. Note that by Proposition 1.13, we have $(\bigcup_j \widetilde{B}_j) \bigcap \widetilde{P}_i = \emptyset$, then $g_s^{-1}(P_i)$ is smooth. Moreover, since the morphism $g_s|_{g_s^{-1}(P_i)} : g_s^{-1}(P_i) \longrightarrow P_i$ is generically étale with two branch points, by Proposition 1.2, we have $\sigma(g_s^{-1}(P_i)) = p-1$. This is Case (iii).

In (2), if $i' \neq i-1$. Contacting $P_{i'+1}, ..., P_{i-1}$ and $(g \circ c_Y)^{-1}(\bigcup_{i'+1 \leq j \leq i-1} P_j)$, we obtain two curves Z'' and Y'' over S. Since C is a chain of projective lines that meet the other irreducible components of the chain at most two points, by [6, Lemma 10.3.31], we have Z'' is a semi-stable curve. By applying Proposition 1.13, we obtain Y'' is a stable curve over S whose generic fiber is isomorphic to Y_η. This is a contradiction. Thus, we have $i' = i-1$ and g_s is purely inseparable over P_j if $j \neq i-1, i$. By Proposition 1.13, g_s is étale over $P_{i-1} \bigcap P_i$. Thus, we obtain that the rank of dual graph of $g_s^{-1}(C)$ is equal to $p-1$. On the other hand, by Proposition 1.13, we have $(\bigcup_j \widetilde{B}_j) \bigcap (\widetilde{P}_i \bigcup \widetilde{P}_{i-1}) = \emptyset$, then $g_s^{-1}(P_i)$ and $g_s^{-1}(P_{i-1})$ are smooth. Since $g_s|_{g_s^{-1}(P_{i-1})} : g_s^{-1}(P_{i-1}) \longrightarrow P_{i-1}$ (resp. $g_s|_{g_s^{-1}(P_i)} : g_s^{-1}(P_i) \longrightarrow P_i$) is generically étale morphism with one branch point, by Porposition 1.2, we have the p-rank of all the irreducible components of $g_s^{-1}(C)^{\mathrm{ss}}$ is equal to 0. This is Case (ii). This completes the proof of the lemma.

$\square$

REMARK 2.2. Note that similar arguments to the arguments given in the proof of the lemma imply that the lemma also holds for the case of $\mathbb{Z}/p\mathbb{Z}$-semi-stable coverings.

REMARK 2.3. In Section 4 of the present paper, we construct some examples of $\mathbb{Z}/p\mathbb{Z}$-stable coverings which satisfy the conditions of Lemma 2.1.

3 New-ordinariness of stable coverings of a sturdy stable curve

In this section, we prove the main theorem of the present paper.

LEMMA 3.1. *Let $f : Y \longrightarrow X$ be a $\mathbb{Z}/p\mathbb{Z}$-stable covering. Then f_s is new-ordinary if and only if for each irreducible component $Y_v \subseteq Y_s$, one of the following conditions hold: (i) if $f_s|_{Y_v}$ is a constant morphism (i.e., $f(Y_v)$ is a point), then Y_v is ordinary; (ii) if $f_s|_{Y_v}$ is finite, then $f_s|_{Y_v}$ is new-ordinary.*

PROOF. The lemma follows from the definition of new-ordinary.

□

DEFINITION 3.2. Let Z be a stable curve over an algebraically closed field. Z is called sturdy if the genus of the normalization of each irreducible component of Z is ≥ 2.

REMARK 3.3. For any stable curve Z over an algebraically closed field, we have an admissible covering $W \longrightarrow Z$ such that W is sturdy (cf. [9, Section 0 Curves]).

Now, let us prove the main theorem.

THEOREM 3.4. *Let G be a finite solvable group, $f : Y \longrightarrow X$ a G-stable covering. Suppose that X_s is sturdy and the morphism of special fibers f_s is not an admissible covering. Then f_s is not new-ordinary.*

PROOF. Since G is a finite solvable group, we have the derived series of G as follows:

$$\{1\} \subset G^{(m)} \subset G^{(m-1)} \subset ...... \subset G^{(0)} := G.$$

Then we obtain the following sequence of stable coverings:

$$Y =: Y_0 \longrightarrow Y_1 \longrightarrow ...... \longrightarrow Y_m \longrightarrow X$$

such that $f_i : Y_i \longrightarrow Y_{i+1}$ is a $G^{(i)}/G^{(i+1)}$-stable covering and $f_m \circ ...... \circ f_0 = f$. Since f_s is not an admissible covering, there is a $0 \leq w \leq m$ such that $(f_i)_s$ is an admissible covering for each $i \geq w+1$ and $(f_w)_s$ is not an admissible covering. Note that since an admissible covering of a sturdy stable curve is sturdy, we have Y_{w+1} is sturdy. Since f_s is new-ordinary if and only if f_i is new-ordinary for each i, we only need to prove that $(f_w)_s$ is not new-ordinary. Thus, for proving our main theorem, we can assume that G is an abelian group. Write $G_{p'}$ (resp. G_p) for the prime to p part (resp. p-part) of G. Since the morphisms of special fibers of $G_{p'}$-stable coverings are admissible coverings (i.e., the specialization isomorphism of log étale fundamental groups (cf. [18, Proposition 1.1])), we can assume that $G = G_p$ is a p-group. Furthermore, by Deuring-Shafarevich formula, we can assume that $G = \mathbb{Z}/p\mathbb{Z}$.

Write V for the set of vertical points associated to f. If V contains a smooth point of X_s, then by applying Proposition 1.14 and Lemma 3.1, f_s is not new-ordinary. Thus, we can assume that either $V = \emptyset$ or V consists of singular points.

If f_s is new-ordinary. Write $\{X_i^{\text{ét}}\}_i$ (resp. $\{X_j^{\text{in}}\}_j$) for the set of semi-stable sub-curves of X_s such that the following conditions: (a) for each i (resp. j), f_s is generically étale over $X_i^{\text{ét}}$ (resp. purely inseparable over X_j^{in}); (b) if an irreducible component $X_v \subseteq X_s$ such that $X_v \bigcap X_i^{\text{ét}} \neq \emptyset$ and $X_v \not\subseteq X_i^{\text{ét}}$ (resp. $X_v \bigcap X_j^{\text{in}} \neq \emptyset$ and $X_v \not\subseteq X_j^{\text{in}}$), then f_s is purely inseparable (resp. generically étale) over X_v. Note that since f_s is not an admissible covering, f_s is not generically étale. Then $\{X_j^{\text{in}}\}_j$ is not empty. Write $g_{X_i^{\text{ét}}}$ (resp. $g_{X_j^{\text{in}}}$) for the genus of $X_i^{\text{ét}}$ (resp. X_j^{in}) for each i (resp. j). Note that for a stable curve over an algebraically closed field of characteristic $p > 0$, in order to calculate the p-rank and the genus of a generically étale covering with Galois group $\mathbb{Z}/p\mathbb{Z}$ of the given stable curve, we can assume that the given stable curve is smooth. Thus, for proving our main theorem, we may assume that $X_i^{\text{ét}}$ are smooth for all the i. Write $r_X := \text{rank}(\pi_1(\Gamma_{X_s}))$ for the rank of the dual graph of X_s. Write n_i for the cardinality of the set

$$X_i^{\text{ét}} \bigcap (\bigcup_j X_j^{\text{in}}).$$

For each i (resp. j), we write $Y_i^{\text{ét}}$ (resp. Y_j^{in}) for the semi-stable sub-curve of Y_s which is generically étale over $X_i^{\text{ét}}$ (resp. which is purely inseparable over X_j^{in}), and $g_{Y_i^{\text{ét}}}$ (resp. $g_{Y_j^{\text{in}}}$) for the genus. By Hurwitz formula, we have

$$g_{Y_i^{\text{ét}}} = p(g_{X_i^{\text{ét}}} - 1) + \frac{1}{2} \cdot \deg(\mathcal{R}_i) + 1,$$

where $\mathcal{R}_i$ denotes the ramification divisor of $f_s|_{Y_i^{\text{ét}}}$. By Lemma 3.1 and Remark 1.7, we obtain $\deg(\mathcal{R}_i) = 2n_i(p-1)$.

If $V = \emptyset$. Note that the natural morphism of dual graphs $\Gamma_{Y_s} \longrightarrow \Gamma_{X_s}$ induced by f_s is an isomorphism. Write g_{Y_s} for the genus of Y_s, we have

$$g_{Y_s} = \sum_i g_{Y_i^{\text{ét}}} + \sum_j g_{X_j^{\text{in}}} + r_X$$

$$= \sum_i (p(g_{X_i^{\text{ét}}} - 1) + n_i(p-1) + 1) + \sum_j g_{X_j^{\text{in}}} + r_X.$$

On the other hand, by the computation of the genus g_{Y_η} of generic fiber Y_η, we have

$$g_{Y_\eta} = p((\sum_i g_{X_i^{\text{ét}}} + \sum_j g_{X_j^{\text{in}}} + r_X) - 1) + 1.$$

Since $g_{Y_\eta} = g_{Y_s}$, we obtain

$$(1-p)(\sum_j g_{X_j^{\text{in}}} - 1 + r_X - \sum_i (n_i - 1)) = 0.$$

By the assumption that X_s is sturdy and $r_X - \sum_i (n_i - 1) \geq 0$, we have $\sum_j g_{X_j^{\text{in}}} - 1 + r_X - \sum_i (n_i - 1) \neq 0$. This is a contradiction. Then if $V = \emptyset$, the theorem holds.

If $V \neq \emptyset$. Write m_i for the cardinality of $V \bigcap X_j^{\text{in}}$, and V_x for the vertical fiber associated to a vertical point $x \in V$. By Lemma 2.1 (a) and Lemma 3.1, we have $V \subset \bigcup_j X_j^{\text{in}}$, $V \bigcap X_j^{\text{ét}} = \emptyset$ and moreover, the genus g_{V_x} of V_x is equal to $p-1$ for each $x \in V$. Then we have

$$g_{Y_s} = \sum_i g_{Y_i^{\text{ét}}} + \sum_j g_{X_j^{\text{in}}} + \sum_{x \in V} g_{V_x} + r_X$$

$$= \sum_i (p(g_{X_i^{\text{ét}}} - 1) + n_i(p-1) + 1) + \sum_j g_{X_j^{\text{in}}} + \sum_j m_j(p-1) + r_X.$$

On the other hand, by the computation of the genus g_{Y_η} of generic fiber Y_η, we have

$$g_{Y_\eta} = p((\sum_i g_{X_i^{\text{ét}}} + \sum_j g_{X_j^{\text{in}}} + r_X) - 1) + 1.$$

Since $g_{Y_\eta} = g_{Y_s}$, we obtain

$$(p-1)(\sum_j (g_{X_j^{\text{in}}} - m_j) - 1 + r_X - \sum_i (n_i - 1)) = 0.$$

Write $\Gamma_{X_j^{\text{in}}}$ for the dual graph of X_j^{in}, $v(\Gamma_{X_j^{\text{in}}})$ and $e(\Gamma_{X_j^{\text{in}}})$ for the set of vertices and the set of edges, respectively. By the assumptions that X_s is sturdy, we have

$$g_{X_j^{\text{in}}} = \sum_{v \in v(\Gamma_{X_j^{\text{in}}})} g_{X_v} + \text{rank}(\pi_1(\Gamma_{X_j^{\text{in}}}))$$

$$\geq 2 \cdot \sharp v(\Gamma_{X_j^{\text{in}}}) + \text{rank}(\pi_1(\Gamma_{X_j^{\text{in}}})) = \sharp v(\Gamma_{X_j^{\text{in}}}) + \sharp e(\Gamma_{X_j^{\text{in}}}) + 1.$$

Since $\sharp e(\Gamma_{X_j^{\text{in}}}) \geq m_j$ and $\{X_j^{\text{in}}\}_j$ is not empty, we have $\sum_j (g_{X_j^{\text{in}}} - m_j) - 1 > 0$. Then we obtain a contradiction. This completes the proof of our main theorem.

$\square$

REMARK 3.5. In the case that X is smooth, Raynaud proved the following result (cf. [13]). Let G be a finite group and $f : Y \longrightarrow X$ a G-stable covering. Suppose that X is smooth and the morphism of special fibers f_s is not an admissible covering. Then f_s is not new-ordinary.

REMARK 3.6. On the other hand, M. Saïdi extended Raynaud's theorem to the case of Galois coverings (cf. [15, Theorem]). More precisely, Saïdi proved the following result. Let X be smooth and $f : Y \longrightarrow X$ is a morphism of stable curves over S. Suppose that f is a Galois covering with Galois group $\mathbb{Z}/p\mathbb{Z}$ (i.e., the extension of function fields $K(Y)/K(X)$ induced by f is a Galois extension with Galois group $\mathbb{Z}/p\mathbb{Z}$) and f_s is not generically étale. Then f_s is not new-ordinary.

4 Constructions

The author does not know that whether or not exist a $\mathbb{Z}/p\mathbb{Z}$-stable covering over a spectrum of DVR and a vertical point associated to the stable covering such that the vertical fiber of the vertical point satisfies (i) or (iii) of (b) of Lemma 2.1. In this section, we construct some examples of $\mathbb{Z}/p\mathbb{Z}$-stable coverings which satisfy (a) and (ii) of (b) of Lemma 2.1.

Let $\mathcal{F}$ be a Deligne-Mumford stack. We use the notation $|\mathcal{F}|$ to denote the underlying topological space of $\mathcal{F}$ (cf. [5, Chapter 5]). Let $m : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ be a morphism of Deligne-Mumford stacks. We use the notation $|m|$ to denotes the morphism of underlying topological spaces $|\mathcal{F}_1| \longrightarrow |\mathcal{F}_2|$ induced by m.

DEFINITION 4.1. Let $\overline{\mathcal{M}}_g$ be the moduli stack of stable curves of genus g over $\operatorname{Spec} \overline{\mathbb{F}}_p$. Let X_a (resp. X_b) be a stable curve of genus g over an algebraically closed field k_a (resp. k_b) of characteristic $p > 0$, and $c_a : \operatorname{Spec} k_a \longrightarrow \overline{\mathcal{M}}_g$ (resp. $c_b : \operatorname{Spec} k_b \longrightarrow \overline{\mathcal{M}}_g$) the classifying morphism determined by X_a (resp. X_b). We say that X_a and X_b are equivalent if the images of $|c_a|$ and $|c_b|$ are equal.

DEFINITION 4.2. Let $d_1 : W_1 \longrightarrow Z_1$ (resp. $d_2 : W_2 \longrightarrow Z_2$) be a morphism of schemes over an algebraically closed field l_1 (resp. l_2). We say that d_1 is equivalent to d_2 if there exists an algebraically closed field l_3 which contains l_1 and l_2 such that the following commutative diagram:

$$\begin{array}{ccc} W_1 \times_{l_1} l_3 & \xrightarrow{h_1} & W_2 \times_{l_2} l_3 \\ {\scriptstyle d_1 \times_{l_1} l_3}\downarrow & & \downarrow{\scriptstyle d_2 \times_{l_2} l_3} \\ Z_1 \times_{l_1} l_3 & \xrightarrow{h_2} & Z_2 \times_{l_2} l_3, \end{array}$$

where h_1 and h_2 are isomorphisms (which are not necessarily l_3-isomorphisms).

Let $c_{\mathcal{B}} : \mathcal{B} \longrightarrow \overline{\mathcal{M}}_g$ be a morphism of Deligne-Mumford stacks. Write $\overline{\mathcal{C}}_g$ for the universal stable curve over $\overline{\mathcal{M}}_g$, $\mathcal{A}_{\mathcal{B}}(\overline{\mathcal{C}}_g)$ for the fiber product

$$\overline{\mathcal{M}}_g \times_{\Delta, \overline{\mathcal{M}}_g \times \overline{\mathcal{M}}_g, c_{\mathcal{B}}} \mathcal{B},$$

where Δ denotes the diagonal morphism of $\overline{\mathcal{M}}_g$. Note that $\mathcal{A}_{\mathcal{B}}(\overline{\mathcal{C}}_g)$ is finite, unramified over $\mathcal{B}$.

Let q_1, q_2 be two points of $|\mathcal{A}_{\overline{\mathcal{M}}_g}(\overline{\mathcal{C}}_g)|$ such that $q_1 \in V(q_2)$, where $V(q_2)$ denotes the closure of $\{q_2\}$ in $|\mathcal{A}_{\overline{\mathcal{M}}_g}(\overline{\mathcal{C}}_g)|$. Let k_{q_1} and k_{q_2} be two algebraically closed field of characteristic $p > 0$, $a_1 : \operatorname{Spec} k_{q_1} \longrightarrow \mathcal{A}_{\overline{\mathcal{M}}_g}(\overline{\mathcal{C}}_g)$ and $a_2 : \operatorname{Spec} k_{q_2} \longrightarrow \mathcal{A}_{\overline{\mathcal{M}}_g}(\overline{\mathcal{C}}_g)$ two morphisms. Suppose that the images of $|a_1|$ and $|a_2|$ are q_1 and q_2, respectively. By the elementary theory of algebraic geometry, we have the following lemma.

LEMMA 4.3. *There exists a complete DVR A with algebraically closed residue field and a morphism $a_A : \operatorname{Spec} A \longrightarrow \mathcal{A}_{\overline{\mathcal{M}}_g}(\overline{\mathcal{C}}_g)$ such that the image of $|a_A|$ is $\{q_1, q_2\}$. Write k_A, (resp. K_A, $\overline{K}_A$) for the residue field (resp. the quotient field, an algebraic closure of K_A), s_A (resp. $\eta_A, \overline{\eta}_A$) for the closed point (resp. the generic point, the geometric generic point) of* $\operatorname{Spec} A$. *Moreover, we have the natural morphism $a_{\overline{\eta}_A} : \overline{\eta}_A \longrightarrow \mathcal{A}_{\overline{\mathcal{M}}_g}(\overline{\mathcal{C}}_g)$ induced by a_A is equivalent to a_2, and the natural morphism $a_{s_A} : s_A \longrightarrow \mathcal{A}_{\overline{\mathcal{M}}_g}(\overline{\mathcal{C}}_g)$ induced by a_A is equivalent to a_1.*

4.1 Stable reduction of admissible coverings

Let $W_\eta^\bullet := (W_\eta, D_{W_\eta})$ (resp. $Z_\eta^\bullet := (Z_\eta, D_{Z_\eta})$) be a pointed stable curves over the generic point η of S. Suppose that $W_\eta^\bullet$ (resp. $Z_\eta^\bullet$) admits a pointed stable model $W^\bullet$ (resp. $Z^\bullet$) over S and all the nodes and the support of D_{W_η} (resp. D_{Z_η}) are η-rational points. We call a finite morphism $f_\eta^\bullet : W_\eta^\bullet \longrightarrow Z_\eta^\bullet$ is a G-admissible covering if $f_{\overline{\eta}}^\bullet$ is an admissible covering with Galois group G. We have a proposition as follows.

PROPOSITION 4.4. *$f_\eta^\bullet$ can be extended to a unique morphism $f^\bullet : W^\bullet \longrightarrow Z^\bullet$ over S.*

PROOF. By the uniqueness of pointed stable models, the action of G on the generic fiber $W_\eta^\bullet$ extends to an action of G on $W^\bullet$. Then we have the quotient morphism $q^\bullet : W^\bullet \longrightarrow W^\bullet/G$, where $W^\bullet/G$ is a pointed semi-stable model of $Z_\eta^\bullet$ (cf. [12, Proposition 5]). Thus, by the repeated contraction of the chains of projective lines of $W^\bullet/G$, we obtain a contracting morphism $c^\bullet : W^\bullet/G \longrightarrow Z^\bullet$ (cf. Remark 4.5 and [6, Lemma 10.3.31]). Then we obtain $f^\bullet = c^\bullet \circ q^\bullet$.

Let $g^\bullet : W^\bullet \longrightarrow Z^\bullet$ be an extension of f_η. Since $g^\bullet$ is a G-equivalent morphism, $g^\bullet$ factors through $q^\bullet$. Then we obtain a contracting morphism $d^\bullet : W^\bullet/G \longrightarrow Z^\bullet$ such that $g^\bullet = d^\bullet \circ f^\bullet$. Thus, by applying [6, Proposition 8.3.28] for all the irreducible components of $W^\bullet/G$, the contracting morphism $c^\bullet$ coincides with the contracting morphism $d^\bullet$. We complete the proof of the lemma.

□

REMARK 4.5. In order to obtain the contracting morphism $c^\bullet$, we want to apply [1, 6.7 Theorem 1]. But in the assumptions of [1, 6.7 Theorem 1], we need to assume that the curves over S is normal. In our case, we can also apply [1, 6.7 Theorem 1] to W/G (i.e., the underlying scheme of $W^\bullet/G$) as follows.

For any irreducible component $E_v \cong \mathbb{P}^1$ of the special fiber of W/G, there exists an irreducible semi-stable subcurve $H \subseteq W/G$ such that E_v is an irreducible component of the special fiber H_s. Write H'_η for the normalization of E_η. Thus, there exist a semi-stable model of H'_s and a natural finite morphism morphism $H' \longrightarrow H$. Write E'_v for $H' \times_H E_v$. Note that H' is a normal curve over S. Let $\mathcal{O}_{H'}(E'_v)$ (resp. $\mathcal{O}_{W/G}(E_v)$) be a line bundle over H' (resp. W/G) induced by the divisor E'_v (resp. E_v). Thus, we obtain the following commutative digram:

$$\begin{array}{ccc} H' & \xrightarrow{c_{H'}} & \mathrm{Proj}(\bigoplus_m \Gamma(H', \mathcal{O}_{H'}(mE'_v))) \\ \downarrow & & \downarrow \\ W/G & \xrightarrow{c_{W/G}} & \mathrm{Proj}(\bigoplus_m \Gamma(W/G, \mathcal{O}_{W/G}(mE_v))). \end{array}$$

Then $c_{W/G}$ is a contacting morphism such that $c_{W/G}(E_v)$ is a point.

4.2 Examples of Lemma 2.1 (a)

Let A_1 be a complete DVR of characteristic $p > 0$ with algebraically closed residue field k_1, K_1 the quotient field, $\overline{K}_1$ an algebraic closure of K_1. Let C (resp. E) be a smooth projective hyperbolic curve over $S_1 := \mathrm{Spec}\, A_1 = \{\eta_1, s_1\}$, where η_1 and s_1 stand for the generic point and closed point of S_1, respectively. Write C_{η_1}, $C_{\overline{\eta}_1}$ and C_{s_1} (resp. E_{η_1}, $E_{\overline{\eta}_1}$ and E_{s_1}) for the generic fiber, geometric generic fiber, special fiber of C (resp. E), respectively.

Suppose that $\sigma(C_{\overline{\eta}_1}) - \sigma(C_{s_1}) > 0$. Then by replacing S_1 by a finite extension of S_1, there is a $\mathbb{Z}/p\mathbb{Z}$-stable covering $\phi : D \longrightarrow C$ which is not finite. By replacing S_1 by a finite extension of S_1, we can choose a marked point D_C of C such that $D_C \bigcap C_{s_1}$ is a vertical point associated to ϕ. By replacing S_1 by a finite extension of S_1, we can assume that $D_D := \phi^{-1}(D_C)$ are S_1-rational points of D. Then we obtain two pointed stable curves $C^\bullet := (C, D_C)$ and $D^\bullet := (D, D_D)$. Moreover, we obtain a natural morphism of pointed stable curves $\phi^\bullet : C^\bullet \longrightarrow D^\bullet$ induced by ϕ.

On the other hand, for E, we suppose that $\sigma(E_{s_1}) > 0$. Then there exists an étale covering $\psi : F \longrightarrow E$ of Galois group $\mathbb{Z}/p\mathbb{Z}$. By replacing S_1 by a finite extension of

S_1, we can choose a marked point D_E of E such that $\psi^{-1}(D_E)$ are S_1-rational points. Write D_F for $\psi^{-1}(D_E)$. Thus, we obtain two pointed stable curves $E^\bullet := (E, D_E)$ and $F^\bullet := (F, D_F)$. Moreover, we obtain a natural morphism of the pointed stable curves $\psi^\bullet : F^\bullet \longrightarrow E^\bullet$ induced by ψ.

By gluing $C^\bullet$ and $E^\bullet$ (resp. $D^\bullet$ and $F^\bullet$) along the marked points, we obtain a new stable curve X_1 (resp. Y_1) over S_1. By Proposition 4.4, we can gluing $\phi^\bullet$ and $\psi^\bullet$. Then we obtain a morphism $f_1 : Y_1 \longrightarrow X_1$ over S_1 such that $f_1|_{D^\bullet} = \phi^\bullet$ and $f_1|_{F^\bullet} = \psi^\bullet$. Write $(f_1)_{\overline{\eta}_1}$ for the morphism of geometric generic fibers $(f_1) \times_{\eta_1} \overline{\eta}_1 : (Y_1)_{\overline{\eta}_1} \longrightarrow (X_1)_{\overline{\eta}_1}$. Note that $(f_1)_{\overline{\eta}_1}$ is an étale covering whose Galois group is $\mathbb{Z}/p\mathbb{Z}$.

Let A_2 be a complete DVR of characteristic $p > 0$ with algebraically closed residue field $k_2 = \overline{K}_1$, K_2 the quotient field, $\overline{K}_2$ an algebraic closure of K_2. Write S_2 for the spectrum $\mathrm{Spec}\, A_2 = \{\eta_2, s_2\}$, where η_2 and s_2 stand for the generic point and closed point of S_2, respectively. Since $(f_1)_{\overline{\eta}_1}$ is an étale covering, by deformation theory, $(f_1)_{\overline{\eta}_1}$ can be lifted to a $\mathbb{Z}/p\mathbb{Z}$-étale covering of stable curves $f_2 : Y_2 \longrightarrow X_2$ over S_2 such that the generic fiber $(X_2)_{\eta_2}$ is smooth and the morphism of special fibers $(f_2)_{s_2}$ is isomorphic to $(f_1)_{\overline{\eta}_1}$. Write $(f_2)_{\overline{\eta}_2}$ for the morphism of geometric generic fibers $(f_2) \times_{\eta_2} \overline{\eta}_2 : (Y_2)_{\overline{\eta}_2} \longrightarrow (X_2)_{\overline{\eta}_2}$.

Write $g_{\mathcal{Y}}$ (resp. $g_{\mathcal{X}}$) for the genus of $(Y_2)_{\overline{\eta}_2}$ (resp. $(X_2)_{\overline{\eta}_2}$). Let $\overline{\mathcal{M}}_{g_{\mathcal{Y}}}$ (resp. $\overline{\mathcal{M}}_{g_{\mathcal{X}}}$) be the moduli stack of stable curves of genus $g_{\mathcal{Y}}$ (resp. $g_{\mathcal{X}}$) over $\mathrm{Spec}\, \overline{\mathbb{F}}_p$. The curve $(Y_2)_{\overline{\eta}_2} \longrightarrow \overline{\eta}_2$ (resp. $(Y_1)_{s_1} \longrightarrow s_1$) determines a classifying morphism $\alpha_2 : \overline{\eta}_2 \longrightarrow \overline{\mathcal{M}}_{g_{\mathcal{Y}}}$ (resp. $\alpha_1 : s_1 \longrightarrow \overline{\mathcal{M}}_{g_{\mathcal{Y}}}$). The curve $(X_2)_{\overline{\eta}_2} \longrightarrow \overline{\eta}_2$ (resp. $(X_1)_{s_1} \longrightarrow s_1$) determines a classifying morphism $\beta_2 : \overline{\eta}_2 \longrightarrow \overline{\mathcal{M}}_{g_{\mathcal{X}}}$ (resp. $\beta_1 : s_1 \longrightarrow \overline{\mathcal{M}}_{g_{\mathcal{X}}}$). On the other hand, the action of $\mathbb{Z}/p\mathbb{Z}$ on $(Y_2)_{\overline{\eta}_2}$ (resp. $(Y_1)_{s_1}$) induces an injective group homomorphism $\gamma_{\overline{\eta}_2} : \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathcal{A}_{\overline{\eta}_2}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})(\overline{\eta}_2)$ (resp. $\gamma_{s_1} : \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathcal{A}_{s_1}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})(s_1)$). Let τ be a generator of $\mathbb{Z}/p\mathbb{Z}$. Then by the uniqueness of stable models Y_2 and Y_1, the action of $a_2 := \gamma_{\overline{\eta}_2}(\tau)$ on $(Y_2)_{\overline{\eta}_2}$ induces an element $a_1 \in \gamma_{s_1}(\mathbb{Z}/p\mathbb{Z})$ which acts on $(Y_1)_{s_1}$.

Write $q_2' \in |\mathcal{A}_{\overline{\eta}_2}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})|$ (resp. $q_1' \in |\mathcal{A}_{s_1}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})|$) for the point determined by a_2 (resp. a_1). Then we obtain a point $q_2 \in |\mathcal{A}_{\overline{\mathcal{M}}_{g_{\mathcal{Y}}}}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})|$ (resp. $q_1 \in |\mathcal{A}_{\overline{\mathcal{M}}_{g_{\mathcal{Y}}}}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})|$) via the natural morphism $|\mathcal{A}_{\overline{\eta}_2}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})| \longrightarrow |\mathcal{A}_{\overline{\mathcal{M}}_{g_{\mathcal{Y}}}}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})|$ (resp. $|\mathcal{A}_{s_1}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})| \longrightarrow |\mathcal{A}_{\overline{\mathcal{M}}_{g_{\mathcal{Y}}}}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})|$). Note that the image of q_2 (resp. q_1) of the natural morphism $|\mathcal{A}_{\overline{\mathcal{M}}_{g_{\mathcal{Y}}})}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})| \longrightarrow |\overline{\mathcal{M}}_{g_{\mathcal{Y}}}|$ is equal to the image of $|\alpha_2|$ (resp. $|\alpha_1|$). Thus, by Lemma 4.3, we obtain a complete DVR A and a morphism $a_A : \mathrm{Spec}\, A \longrightarrow \mathcal{A}_{\overline{\mathcal{M}}_{g_{\mathcal{Y}}}}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}})$ which satisfy the conditions of Lemma 4.3. Write c_Y for the composite of a_A and $\mathcal{A}_{\overline{\mathcal{M}}_{g_{\mathcal{Y}}}}(\overline{\mathcal{C}}_{g_{\mathcal{Y}}}) \longrightarrow \overline{\mathcal{M}}_{g_{\mathcal{Y}}}$. We obtain a stable curve $Y \longrightarrow \mathrm{Spec}\, A$ determined by c_Y, and an action of $\mathbb{Z}/p\mathbb{Z}$ on Y determined by a_A. Note that the quotient $Y/\mathbb{Z}/p\mathbb{Z}$ is a semi-stable model over $\mathrm{Spec}\, A$. Contracting the chains of projective lines of the semi-stable model $Y/\mathbb{Z}/p\mathbb{Z}$, we obtain a stable covering $f : Y \longrightarrow X$ over $\mathrm{Spec}\, A$. Note that the stable curve $X \longrightarrow \mathrm{Spec}\, A$ induces a classifying morphism $c_X : \mathrm{Spec}\, A \longrightarrow \overline{\mathcal{M}}_{g_{\mathcal{X}}}$ such that the image of $|c_X|$ is equal to $\mathrm{Im}(|\beta_1|) \bigcup \mathrm{Im}(|\beta_2|)$.

By the construction, we see that $f_{\overline{\eta}_A}$ is equivalent to $(f_2)_{\overline{\eta}_2}$ and f_{s_A} is equivalent to $(f_1)_{s_1}$ in the sense of Definition 4.2. Then f is a $\mathbb{Z}/p\mathbb{Z}$-stable covering which satisfies the conditions of Lemma 2.1 (a).

4.3 Examples of Lemma 2.1 (b)-(ii)

By replacing E of Section 4.2 by a copy of C and gluing two copies of $C^{\bullet}$ along the marked points, similar arguments to the arguments given in Section 4.1, we obtain a $\mathbb{Z}/p\mathbb{Z}$-stable covering $f : Y \longrightarrow X$ over a complete DVR A which satisfies the conditions of (ii) of Lemma 2.1 (b).

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