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Categorical Characterization of Strict Morphisms of Fs Log Schemes

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Abstract. In the present paper, we study a categorical characterization of strict morphisms of fs log schemes. In particular, we prove that strictness of morphisms of fs log schemes is preserved by an arbitrary equivalence of categories between suitable categories of fs log schemes. The main result of the present paper leads us to a relatively simple alternative proof of a result on a categorical representation of fs log schemes proved by S. Mochizuki.

Contents

Introduction 1
0. Notations and Conventions 2
1. Characterization of Trivial and Standard Log Points 4
2. Characterization of Fs Log Points 7
3. Characterization of Strict Morphisms 9
Appendix A. Twisted Versions of Hilbert's Theorem 90 13
References 16

Introduction

Let S be an *fs log scheme* whose underlying scheme is *locally noetherian*. Then, by considering *noetherian fs log schemes of finite type* over S, we obtain a category $\mathrm{Sch}^{\log}(S)$ [cf. §0, Log Schemes]. In the present paper, we discuss a categorical characterization of *strict morphisms* in this category $\mathrm{Sch}^{\log}(S)$. Our main result is as follows [cf. Theorem 3.7]:

Theorem. *Let S and T be* **fs log schemes** *whose underlying schemes are* **locally noetherian**,

$$\phi\colon\ \mathrm{Sch}^{\log}(S)\ \xrightarrow{\sim}\ \mathrm{Sch}^{\log}(T)$$

an **equivalence of categories**, *and f a morphism in $\mathrm{Sch}^{\log}(S)$. Then it holds that f is* **strict** *if and only if $\phi(f)$ is* **strict**.

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Note that Theorem was already essentially proved by *S. Mochizuki*. Moreover, Mochizuki also proved that a result concerning a *categorical representation* of fs log schemes follows from Theorem, together with some discussions [cf. Remark 3.7.1]. On the other hand, in the present paper, by establishing [cf. Proposition 2.5] a categorical characterization of *fs log points* [i.e., fs log schemes whose underlying schemes are isomorphic to the spectra of fields], we obtain a *simple proof* of Theorem. In particular, the proof of the main theorem of the present paper may be regarded as a *relatively simple alternative proof* of the categorical representation of fs log schemes already proved by Mochizuki.

In the proof of Theorem, we prove, by applying *Hilbert's Theorem 90*, a *sufficient condition* [cf. Proposition 1.3, Remark 1.3.1] for an fs log point to be *quasi-split* [cf. Definition 1.2, (ii)]. In Appendix of the present paper, we also discuss, by considering *twisted versions of Hilbert's Theorem 90*, further such *sufficient conditions* [cf. Theorem A.5]. Note that the proof of Theorem does *not depend* on these further sufficient conditions obtained in Appendix.

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0. Notations and Conventions

Monoids: We shall refer to a commutative semigroup with the unit element as a *monoid*. Let M be a monoid. Then we shall write $M^{\times} \subseteq M$ for the submonoid consisting of invertible elements of M, $\overline{M} \stackrel{\text{def}}{=} M/M^{\times}$, and M^{gp} for the groupification of M. Moreover,

- we shall say that M is *sharp* if $M^{\times}$ only has the unit element;
- we shall say that M is *integral* if the natural homomorphism $M \to M^{\text{gp}}$ is injective [which thus implies that M may be regarded as a submonoid of M^{gp}];
- we shall say that M is *saturated* if M is integral, and, moreover, for each $x \in M^{\text{gp}}$, it holds that $x \in M$ if the submonoid of M^{gp} generated by x intersects nontrivially $M \subseteq M^{\text{gp}}$;
- we shall say that M is *fs* if M is finitely generated and saturated.

Let $h\colon M \to N$ be a homomorphism of fs monoids. Then we shall write $h^{\text{gp}}\colon M^{\text{gp}} \to N^{\text{gp}}$ for the homomorphism between the groupifications induced by h. Moreover,

- we shall say that h is *local* if $M^{\times} = h^{-1}(N^{\times})$;

• we shall say that h is *exact* if $M = (h^{\mathrm{gp}})^{-1}(N)$ [in M^{gp}].

Log Schemes: A basic reference for the notion of *log schemes* is [Kato].

Let X be an fs log scheme. Then we shall write $\overset{\circ}{X}$ for the underlying scheme of X, $\mathcal{O}_X$ for the structure sheaf of $\overset{\circ}{X}$, $\mathcal{M}_X$ for the [étale] sheaf of monoids on $\overset{\circ}{X}$ which defines the log structure of X, and $\overline{\mathcal{M}}_X \overset{\text{def}}{=} \mathcal{M}_X/\mathcal{O}_X^{\times}$. Moreover,

• we shall say that X is *of log rank* n [where n is an integer] if the groupification [which is necessarily a *free module*] of the stalk of $\overline{\mathcal{M}}_X$ at any geometric point of $\overset{\circ}{X}$ is of rank n;

• we shall say that X is an *fs log point* if $\overset{\circ}{X}$ is isomorphic to the spectrum of a field;

• we shall say that X is a *trivial log point* if X is an fs log point and of log rank 0;

• we shall say that X is a *standard log point* if X is an fs log point and of log rank 1.

Let $f\colon X \to Y$ be a morphism of log schemes. Then we shall write $\overset{\circ}{f}\colon \overset{\circ}{X} \to \overset{\circ}{Y}$ for the underlying morphism of schemes of f. Moreover,

• we shall say that f is *strict* if the natural homomorphism $\overset{\circ}{f}{}^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$ is an isomorphism;

• we shall say that f is *exact* if the homomorphism [of fs monoids] obtained by considering the stalk of the homomorphism $\overset{\circ}{f}{}^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$ at any geometric point of $\overset{\circ}{X}$ is exact.

Let S be an *fs log scheme* whose underlying scheme $\overset{\circ}{S}$ is *locally noetherian*. Then we shall write

$$\mathrm{Sch}^{\log}(S)$$

for the category defined as follows: An object of $\mathrm{Sch}^{\log}(S)$ is a *morphism of log schemes* $X \to S$, where X is an *fs log scheme* whose underlying scheme is *noetherian*, whose underlying morphism of schemes is *of finite type*. A morphism in $\mathrm{Sch}^{\log}(S)$ [from an object $X \to S$ to an object $Y \to S$] is a *morphism of log schemes* $X \to Y$ lying over S [whose underlying morphism of schemes is necessarily *of finite type*]. To simplify the exposition, we shall often refer to the domain X of an arrow $X \to S$ which is an object of $\mathrm{Sch}^{\log}(S)$ as an "object of $\mathrm{Sch}^{\log}(S)$".

1. Characterization of Trivial and Standard Log Points

In the present §1, we give a categorical characterization of *trivial and standard log points* [cf. Proposition 1.6 below]. In the present §1, let S be an fs log scheme whose underlying scheme is *locally noetherian*.

First, let us prove some facts on *sharp fs* monoids:

Lemma 1.1. *Let M be a* **sharp fs** *monoid. Write $V \stackrel{\text{def}}{=} M^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $r \stackrel{\text{def}}{=} \dim_{\mathbb{Q}}(V)$. Then the following hold:*

(i) *For each $x \in M^{\text{gp}} \smallsetminus \{0\}$, there exists a* **local** *homomorphism $h\colon M \to \mathbb{N}$ such that $h^{\text{gp}}(x) \neq 0$.*

(ii) *Let $L \subseteq V$ be a nonzero $\mathbb{Q}$-subspace. Then there exist r* **local** *homomorphisms $h_1, \ldots, h_r\colon M \to \mathbb{N}$ which satisfy the following two conditions:*

(1) *The homomorphism $h^{\text{gp}}\colon M^{\text{gp}} \to \bigoplus_{i=1}^{r} \mathbb{Z}$ induced by the [necessarily* **local***] homomorphism $h\colon M \to \bigoplus_{i=1}^{r} \mathbb{N}$ given by mapping $x \in M$ to $(h_i(x))_{i=1}^{r} \in \bigoplus_{i=1}^{r} \mathbb{N}$ is* **injective***.*

(2) *For every $1 \leq i \leq r$, L is* **not contained** *in the kernel of the $\mathbb{Q}$-linear homomorphism $h_i^{\mathbb{Q}}\colon V \to \mathbb{Q}$ induced by h_i.*

(iii) *Suppose that a finite group G acts on M. Then there exists a homomorphism $h\colon M \to \mathbb{N}$ which is* **local** *and* ***G*-equivariant** *[with respect to the trivial action of G on $\mathbb{N}$].*

(iv) *In the situation of* (iii)*, suppose that $r \geq 2$. Then there exists a submonoid $P \subseteq M^{\text{gp}}$ such that $M \subsetneq P$, and, moreover, P is* ***G*-stable***,* **sharp***, and* **fs***.*

Proof. Assertion (i) follows from [Mzk1], Lemma 2.5, (iii). Next, we verify assertion (ii). It follows immediately from assertion (i) that there exist r *local* homomorphisms $h_1, \ldots, h_r\colon M \to \mathbb{N}$ which satisfy condition (1). Thus, there exists $1 \leq i_0 \leq r$ such that $L \not\subseteq \text{Ker}(h_{i_0}^{\mathbb{Q}})$. Then one verifies easily that, by replacing h_i [where $1 \leq i \leq r$] by $h_i + h_{i_0}$ (respectively, h_i) if $L \subseteq \text{Ker}(h_i^{\mathbb{Q}})$ (respectively, $L \not\subseteq \text{Ker}(h_i^{\mathbb{Q}})$), we obtain r homomorphisms of the desired type. This completes the proof of assertion (ii). Assertion (iii) follows [by considering the sum $\sum_{g \in G} h \circ g\colon M \to \mathbb{N}$ for some *local* homomorphism $h\colon M \to \mathbb{N}$] from assertion (i).

Finally, we verify assertion (iv). Let $h\colon M \to \mathbb{N}$ be a *G-equivariant local* homomorphism [cf. assertion (iii)]. Then since $r \geq 2$, there exists $x \in M^{\text{gp}} \smallsetminus M$ such that $h^{\text{gp}}(x) \in \mathbb{N} \smallsetminus \{0\}$. Write $P \subseteq M^{\text{gp}}$ for the *saturation* of the submonoid generated by $M \subseteq M^{\text{gp}}$ and the G-orbit of $x \in M^{\text{gp}}$. Then it is immediate that $M \subsetneq P$, and that P is *G-stable* and *fs*. Moreover, since h is *G-equivariant*, it follows immediately from our choice of x that P is *sharp*. This completes the proof of assertion (iv), hence also of Lemma 1.1. □

Definition 1.2. Let X be an fs log point. Thus, $\overset{\circ}{X}$ is isomorphic to the spectrum of a field k. Let k_{sep} be a separable closure of k. Write $G_k \overset{\text{def}}{=} \mathrm{Gal}(k_{\mathrm{sep}}/k)$, $\overline{x} \to \overset{\circ}{X}$ for the geometric point determined by the separable closure k_{sep}, and $M \overset{\text{def}}{=} \mathcal{M}_{X,\overline{x}}$. [Thus, the G_k-monoid $\overline{M}$ is naturally isomorphic to the G_k-monoid obtained by forming the stalk $\overline{\mathcal{M}}_{X,\overline{x}}$; moreover, $\overline{M}$ is sharp and fs.]

(i) We shall say that X is *split* if the action of G_k on $\overline{M}$ is trivial.

(ii) We shall say that X is *quasi-split* if the G_k-equivariant surjection $M \twoheadrightarrow \overline{M}$ has a G_k-equivariant splitting [which thus determines a G_k-equivariant isomorphism $k_{\mathrm{sep}}^{\times} \times \overline{M} \xrightarrow{\sim} M$ — i.e., an isomorphism of sheaves $\mathcal{O}_X^{\times} \times \overline{\mathcal{M}}_X \xrightarrow{\sim} \mathcal{M}_X$].

Note that one verifies easily that the issue of whether or not X is split (respectively, quasi-split) does not depend on the choice of k_{sep}.

Proposition 1.3. *Let X be an* **fs log point**. *Suppose that X is* **split**. *Then X is* **quasi-split**. *In particular, a* **standard log point** *is* **quasi-split**.

Proof. Since the monoid $\mathbb{N}$ has *no nontrivial* automorphism, the final assertion follows from the first assertion. Let us verify the first assertion. Since we are in the situation of Definition 1.2, we shall apply the notation of Definition 1.2. Then we have an exact sequence $1 \to k_{\mathrm{sep}}^{\times} \to M^{\mathrm{gp}} \to \overline{M}^{\mathrm{gp}} \to 1$ of G_k-modules. Thus, since $\overline{M}^{\mathrm{gp}}$ is a *free module*, to verify that the G_k-equivariant surjection $M^{\mathrm{gp}} \twoheadrightarrow \overline{M}^{\mathrm{gp}}$, hence also $M \twoheadrightarrow \overline{M}$, has a G_k-equivariant splitting, it suffices to verify that $H^1(G_k, \mathrm{Hom}_{\mathbb{Z}}(\overline{M}^{\mathrm{gp}}, k_{\mathrm{sep}}^{\times})) = \{0\}$. On the other hand, since the action of G_k on $\overline{M}$ is *trivial*, this follows from *Hilbert's Theorem 90*. This completes the proof of Proposition 1.3. □

Remark 1.3.1. Proposition 1.3 gives us a *sufficient condition* for an fs log point to be *quasi-split*. Now let us observe that Proposition 1.3 essentially follows from *Hilbert's Theorem 90*. In §A, we discuss, by considering *twisted versions of Hilbert's Theorem 90*, further such *sufficient conditions*.

Definition 1.4. Let X be an object of $\mathrm{Sch}^{\log}(S)$.

(i) We shall say that X is *minimal* if X is non-initial, and, moreover, every monomorphism in $\mathrm{Sch}^{\log}(S)$ from a non-initial object to X is an isomorphism [cf. [Mzk1], Proposition 2.4].

(ii) We shall say that a morphism in $\mathrm{Sch}^{\log}(S)$ is a *minimal log point* if the domain of the morphism is minimal.

(iii) We shall say that a morphism in $\mathrm{Sch}^{\log}(S)$ is an *fs* (respectively, a *trivial*; a *standard*) *log point* if the domain of the morphism is an fs (respectively, a trivial; a standard) log point.

Lemma 1.5. *Let X be an object of* $\mathrm{Sch}^{\log}(S)$*. Then it holds that X is* **minimal** *if and only if X is either a* **trivial log point** *or a* **standard log point**.

Proof. Sufficiency follows immediately from the surjectivity portion of necessity of [Mzk1], Proposition 2.3 [cf. also [Mzk2], Appendix]. Next, we verify *necessity*. Suppose that X is *minimal*. Then one verifies immediately, by considering a suitable strict closed immersion into X, that X is an *fs log point*. Now since we are in the situation of Definition 1.2, we shall apply the notation of Definition 1.2.

Assume that the free module $\overline{M}^{\mathrm{gp}}$ is *of rank* ≥ 2. Then it follows from Lemma 1.1, (iv), that there exists a G_k*-stable* submonoid $P \subseteq \overline{M}^{\mathrm{gp}}$ such that $\overline{M} \subsetneq P$, and, moreover, P is *sharp* and *fs*. Thus, we have a G_k*-stable* submonoid $N \stackrel{\mathrm{def}}{=} (M^{\mathrm{gp}} \twoheadrightarrow \overline{M}^{\mathrm{gp}})^{-1}(P)$ of M^{gp} such that $M \subsetneq N$, and, moreover, the natural homomorphism $M^{\mathrm{gp}} \to N^{\mathrm{gp}}$ is an *isomorphism*.

Next, let us observe that since P is *sharp*, by mapping each element of $N \smallsetminus M$ to $0 \in k_{\mathrm{sep}}$, we obtain a G_k*-equivariant* extension $N \to k_{\mathrm{sep}}$ of the homomorphism $M \to k_{\mathrm{sep}}$ of monoids [where we regard k_{sep} as a monoid by multiplication] which defines the log structure of X. Moreover, one verifies easily that this homomorphism $N \to k_{\mathrm{sep}}$ of monoids determines an *fs log structure* on $\overset{\circ}{X}$. Write Y for the resulting [*non-initial*] fs log scheme. Then since $M \subsetneq N$, and the natural homomorphism $M^{\mathrm{gp}} \to N^{\mathrm{gp}}$ is an *isomorphism*, the morphism $Y \to X$ [in $\mathrm{Sch}^{\log}(S)$] induced by the natural inclusion $M \hookrightarrow N$ is a *monomorphism* but *not an isomorphism*. In particular, we conclude that X is *not minimal*, in *contradiction* to our assumption that X is *minimal*. This completes the proof of Lemma 1.5. □

Proposition 1.6. *Let X be an object of* $\mathrm{Sch}^{\log}(S)$*. Then the following hold*:

(i) *The following two conditions are* **equivalent**:

(1) *X is a* **trivial log point**.

(2) *X is* **minimal**, *and, moreover, there exists a minimal log point $f\colon Y \to X$ such that Y has an endomorphism over X [relative to f] which is* **not an isomorphism**.

(ii) *The following two conditions are* **equivalent**:

(3) *X is a* **standard log point**.

(4) *X is* **minimal** *but* **not a trivial log point**.

Proof. These assertions follow immediately from Lemma 1.5. □

2. Characterization of Fs Log Points

In the present §2, we give a categorical characterization of *fs log points* [cf. Proposition 2.5 below]. In the present §2, let S be an fs log scheme whose underlying scheme is *locally noetherian* and X an object of $\mathrm{Sch}^{\log}(S)$.

Definition 2.1.

(i) We shall say that a finite set $\{f_i\colon Y_i \to X\}_{i\in I}$ consisting of standard log points whose codomains are X is an *epimorphic family* of X if the morphism $\bigsqcup_{i\in I} Y_i \to X$ from the coproduct of the Y_i's to X determined by the f_i's is an epimorphism in $\mathrm{Sch}^{\log}(S)$.

(ii) We shall say that a collection consisting of standard log points whose codomains are X is an *indispensable collection* for X if every epimorphic family of X has an element which belongs to the collection.

Definition 2.2. Suppose that X is an fs log point. Since we are in the situation of Definition 1.2, we shall apply the notation of Definition 1.2. Write $V \stackrel{\text{def}}{=} \overline{M}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $f\colon Y \to X$ be a standard log point. Then, by considering a geometric point of $\overset{\circ}{Y}$ which lifts $\overline{x} \to \overset{\circ}{X}$, we obtain a [necessarily local] homomorphism $\overline{M} \to \mathbb{N}$. We shall write $L(f) \subseteq V$ for the kernel of the $\mathbb{Q}$-linear homomorphism $V \to \mathbb{Q}$ induced by this homomorphism $\overline{M} \to \mathbb{N}$. [Note that one verifies easily that $L(f)$ does not depend on the choice of the geometric point of $\overset{\circ}{Y}$.]

Lemma 2.3. *Suppose that X is an* **fs log point**. *Let $\{f_i\colon Y_i \to X\}_{i\in I}$ be a nonempty finite set consisting of standard log points whose codomains are X. Suppose that $\bigcap_i L(f_i) = \{0\}$. Then the finite set $\{f_i\colon Y_i \to X\}_{i\in I}$ is an* **epimorphic family** *of X.*

Proof. This may be easily verified. □

Lemma 2.4. *Let $f\colon Y \to X$ be an* **epimorphism** *in $\mathrm{Sch}^{\log}(S)$. Then every* **closed point** *of $\overset{\circ}{X}$ is* **contained** *in the image of $\overset{\circ}{f}\colon \overset{\circ}{Y} \to \overset{\circ}{X}$.*

Proof. Suppose that the image of $\overset{\circ}{f}$ does *not contain* a closed point $x \in \overset{\circ}{X}$. Write Z for the object of $\mathrm{Sch}^{\log}(S)$ obtained by *glueing* two copies of X along the open subscheme $\overset{\circ}{X} \smallsetminus \{x\}$ [via the identity automorphism of $\overset{\circ}{X} \smallsetminus \{x\}$]. Then we have two *distinct* natural open immersions $X \hookrightarrow Z$ whose restrictions to $\overset{\circ}{X} \smallsetminus \{x\}$ *coincide*, which thus implies that f is *not an epimorphism* in $\mathrm{Sch}^{\log}(S)$. This completes the proof of Lemma 2.4. □

Proposition 2.5. *The following two conditions are* **equivalent**:

(1) *X is an* **fs log point**.

(2) *Every* **indispensable** *collection for X has a finite subset which forms an* **epimorphic family** *of X.*

Proof. First, we verify the implication (1) $\Rightarrow$ (2). Suppose that condition (1) is satisfied. Since we are in the situation of Definition 1.2, we shall apply the notation of Definition 1.2. Write $V \stackrel{\text{def}}{=} \overline{M}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Assume that there exists an *indispensable* collection A for X such that A does *not have* any finite subset which forms an *epimorphic family* of X. [Note that since there exists a *nonempty epimorphic family* of X by Lemma 1.1, (ii), Proposition 1.3, and Lemma 2.3, it holds that $A \neq \emptyset$.] Then it follows immediately from Lemma 2.3 that $L \stackrel{\text{def}}{=} \bigcap_f L(f) \neq \{0\}$ — where f ranges over the members of A. Thus, one verifies immediately from Lemma 1.1, (ii), that there exists a finite set $\{g_j\}_{j \in J}$ consisting of standard log points whose codomains are X such that $L \not\subseteq L(g_j)$ [cf. condition (2) of Lemma 1.1, (ii)] — which thus implies that g_j is *not contained* in A for every $j \in J$ — and, moreover, $\bigcap_j L(g_j) = \{0\}$ [cf. condition (1) of Lemma 1.1, (ii)] — which thus implies [cf. Lemma 2.3] that this finite set $\{g_j\}_{j \in J}$ is an *epimorphic family*. In particular, since A is *indispensable*, we obtain a *contradiction*. This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied. Let $x \in \overset{\circ}{X}$ be a closed point of $\overset{\circ}{X}$. Write A for the collection consisting of the standard log points whose codomains are X and images coincide with $\{x\} \subseteq \overset{\circ}{X}$. [Note that it follows, in light of Proposition 1.3, from Lemma 1.1, (i), that $A \neq \emptyset$.] Then it follows from Lemma 2.4 that A is *indispensable*. In particular, since some finite subset of A forms an *epimorphic family* of X [cf. condition (2)], again by Lemma 2.4, we conclude that x is the *unique* closed point of X, which thus implies that $\overset{\circ}{X}$ is isomorphic to the spectrum of a noetherian local ring R.

Let $\pi \in R \smallsetminus R^{\times}$. Write $\mathbb{A}^1_X$ for the fs log scheme over X obtained by equipping $\mathrm{Spec}(R[t])$ — where t is an indeterminate — with the log structure induced by the log structure of X. Then we have two strict closed immersions $f_0, f_\pi \colon X \hookrightarrow \mathbb{A}^1_X$ over X determined by the R-linear homomorphisms $R[t] \to R$ given by mapping $t \in R[t]$ to 0, $\pi \in R$, respectively. Now let us observe that one verifies immediately that, for an fs log point $f \colon Y \to X$, if the image of $\overset{\circ}{f}$ is $\{x\}$, then it holds that $f_0 \circ f = f_\pi \circ f$. Thus, since some finite subset of A forms an *epimorphic family* of X as verified above, we conclude that $f_0 = f_\pi$, hence also $\pi = 0$, which thus implies that R is a *field*. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Proposition 2.5. $\square$

3. Characterization of Strict Morphisms

In the present §3, we prove the main theorem of the present paper [cf. Theorem 3.7 below]. In the present §3, let S be an fs log scheme whose underlying scheme is *locally noetherian*, X and Y objects of $\mathrm{Sch}^{\log}(S)$, and $f\colon X \to Y$ a morphism in $\mathrm{Sch}^{\log}(S)$.

Lemma 3.1. *Suppose that X is an* **fs log point***, and that f is a* **monomorphism***. Then it holds that f is* **strict** *if and only if f is a* **terminal object** *among the fs log points $Z \to Y$ which satisfy that $X \times_Y Z$ is* **non-initial***.*

Proof. This is [Mzk1], Corollary 2.13. However, since the proof contains an *error* [cf. [Mzk2], Appendix], we give a proof as follows: Let us first observe that, to verify Lemma 3.1, we may assume without loss of generality, by replacing Y by the log scheme obtained by equipping the spectrum of the residue field of $\overset{\circ}{Y}$ at the image of $\overset{\circ}{f}$ with the log structure induced by the log structure of Y, that Y is an *fs log point*. Now, to verify *necessity*, suppose that f is *strict*. Then since f is a *strict monomorphism*, one verifies easily that f is an *isomorphism*. Thus, *necessity* is immediate. Next, we verify *sufficiency*. Since the identity automorphism of Y is an *fs log point* which satisfies that $X \times_Y Y = X$ is *non-initial*, by our condition, f has a *splitting* over Y [i.e., a morphism $s\colon Y \to X$ over Y such that the composite $X \xrightarrow{f} Y \xrightarrow{s} X$ is the identity automorphism of X]. Thus, f is an *isomorphism*. This completes the proof of *sufficiency*, hence also of Lemma 3.1. □

Lemma 3.2. *It holds that f is* **strict** *if and only if, for every commutative diagram in* $\mathrm{Sch}^{\log}(S)$

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ W & \longrightarrow & Y \end{array}$$

— where the horizontal arrows are **fs log points***,* **monomorphisms***, and* **strict** *— it holds that the left-hand vertical arrow is* **strict***.*

Proof. This is [Mzk1], Corollary 2.14. However, since the proof contains an *error* [cf. [Mzk2], Appendix], we give a proof as follows: *Necessity* may be easily verified. Next, we verify *sufficiency*. Let x be a closed point of $\overset{\circ}{X}$. Write $Z \to X$ for the *strict* morphism whose underlying morphism of schemes is given by the natural morphism from the spectrum of the residue field of $\overset{\circ}{X}$ at x and $W \to Y$ for the *strict* morphism whose underlying morphism of schemes is given by the natural morphism from the spectrum of the residue field of $\overset{\circ}{Y}$ at $\overset{\circ}{f}(x)$. Then one verifies easily that $Z \to X$, hence also $W \to Y$, is a morphism in

$\mathrm{Sch}^{\log}(S)$. Moreover, it holds that $Z \to X$ and $W \to Y$ are *monomorphisms*. Thus, by our condition, the natural morphism $Z \to W$ is *strict*, which thus implies that $\overset{\circ}{f}{}^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$ is an *isomorphism* at $x \in \overset{\circ}{X}$. This implies *sufficiency*. □

Lemma 3.3. *Suppose that both X and Y are* **fs log points**. *Then the following hold*:

(i) *It holds that f is* **exact** *if and only if, for every fs log point $Z \to Y$, the fiber product $X \times_Y Z$ is* **non-initial**.

(ii) *It holds that f is* **strict** *if and only if the following condition is satisfied*: *f is* **exact**, *and, moreover, for every minimal log point $Z \to Y$, the second projection $X \times_Y Z \to Z$ is* **strict**.

Proof. Assertion (i) is essentially [Naka], (A.1), Proposition [cf. also the proof of [Naka], (A.1), Proposition]. Next, we verify assertion (ii). *Necessity* may be easily verified. We verify *sufficiency*. Write M for the stalk of $\overline{\mathcal{M}}_X$ at a geometric point of $\overset{\circ}{X}$, N for the stalk of $\overline{\mathcal{M}}_Y$ at the geometric point of $\overset{\circ}{Y}$ determined by the geometric point of $\overset{\circ}{X}$, and $\phi\colon N \to M$ for the [necessarily *exact* — cf. our condition] local homomorphism induced by f.

Let $\psi\colon N \to \mathbb{N}$ be a local homomorphism [cf. Lemma 1.1, (i)]. Write P for the quotient by the torsion elements of the saturation of the pushout [in the category of monoids] of $M \overset{\phi}{\leftarrow} N \overset{\psi}{\to} \mathbb{N}$. Then, by our condition [cf. also Lemma 1.5], together with [Naka], Proposition (2.1.1), the natural homomorphism $\mathbb{N} \to P$ is an *isomorphism*. Thus, it follows that $\mathrm{rank}_{\mathbb{Z}}(\mathrm{Coker}(\phi^{\mathrm{gp}}\colon N^{\mathrm{gp}} \to M^{\mathrm{gp}})) = \mathrm{rank}_{\mathbb{Z}}(\mathrm{Coker}(\mathbb{Z} = \mathbb{N}^{\mathrm{gp}} \to P^{\mathrm{gp}})) = 0$.

Assume that f is *not strict*, i.e., that ϕ is *not an isomorphism*. Then since ϕ is *exact*, and $\mathrm{Coker}(\phi^{\mathrm{gp}})$ is *of rank* 0, it holds that $\mathrm{Coker}(\phi^{\mathrm{gp}})$ has a *nontrivial torsion*. In particular, there exists a homomorphism $\pi\colon N^{\mathrm{gp}} \to \mathbb{Z}$ which does *not factor* through ϕ^{gp}. Next, observe that since N is *finitely generated*, there exists a positive integer n such that the homomorphism $N^{\mathrm{gp}} \to \mathbb{Z}$ given by mapping $x \in N^{\mathrm{gp}}$ to $\pi(x) + n \cdot \psi^{\mathrm{gp}}(x) \in \mathbb{Z}$ maps $N \subseteq N^{\mathrm{gp}}$ to $\mathbb{N} \subseteq \mathbb{Z}$, and the resulting homomorphism $\psi_0\colon N \to \mathbb{N}$ is *local*. Then it follows from our choice of π [together with the fact that ψ *factors* through ϕ — cf. the above discussion concerning P] that ψ_0 does *not factor* through ϕ. Thus, by means of ψ_0, one may construct a minimal log point $Z \to Y$ such that the second projection $X \times_Y Z \to Z$ is *not strict*, in *contradiction* to our condition. This completes the proof of assertion (ii), hence also of Lemma 3.3. □

Lemma 3.4. *Suppose that Y is* **minimal**. *Then it holds that f is* **strict** *if and only if every fs log point $Z \to X$* **factors** *through a*

minimal log point $W \to X$ *such that the composite* $W \to X \xrightarrow{f} Y$ *is* **strict**.

Proof. First, we verify *necessity*. Suppose that f is *strict*. Let $Z \to X$ be an fs log point. Write W for the log scheme obtained by equipping $\overset{\circ}{Z}$ with the log structure induced by the log structure of X. Thus, we have a factorization $Z \to W \to X$, where $W \to X$ is *strict*. Since f is *strict* and Y is *minimal*, the composite $W \to X \xrightarrow{f} Y$ is *strict* and W is *minimal* [cf. Lemma 1.5]. This completes the proof of *necessity*.

Next, we verify *sufficiency*. Suppose that f is *not strict*. Then one verifies easily that there exists a closed point x of $\overset{\circ}{X}$ such that the homomorphism $\overset{\circ}{f}{}^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$ is *not an isomorphism* at $x \in \overset{\circ}{X}$. Thus, to verify *sufficiency*, we may assume without loss of generality, by replacing X by the log scheme obtained by equipping [the reduced closed subscheme determined by] $\{x\}$ with the log structure induced by the log structure of X, that X is an *fs log point*.

Assume that the identity automorphism $X \to X$ [which is in fact an *fs log point*] *factors* through a minimal log point $W \to X$ which satisfies that the composite $W \to X \xrightarrow{f} Y$ is *strict*.

If Y is a *trivial log point*, then since the composite $W \to X \xrightarrow{f} Y$ is *strict*, it follows that W is a *trivial log point*, which thus implies that X is a *trivial log point*. In particular, f is *strict*, in *contradiction* to our assumption that f is *not strict*.

Thus, it follows from Lemma 1.5 that we may assume without loss of generality that Y is a *standard log point*. Then since the composite $W \to X \xrightarrow{f} Y$ is *strict*, it follows that W is a *standard log point*. Thus, it follows immediately, by considering our factorization $X \to W \to X$ of the identity automorphism of X, that X is a *standard log point*, and, moreover, $X \to W$, hence also the composite $X \to W \to X \to Y$ [i.e., f], is *strict*, in *contradiction* to our assumption that f is *not strict*. This completes the proof of *sufficiency*, hence also of Lemma 3.4. □

Lemma 3.5. *Suppose that both* X *and* Y *are* **minimal**. *Then it holds that* f *is* **strict** *if and only if the following condition is satisfied*: *There exists a factorization* $X \to Z \to Y$ *of* f *such that* Z *is* **connected** *and either* **of log rank 0** *or* **of log rank 1**, $X \to Z$ *is a* **monomorphism**, *and* $Z \to Y$ *has a* **splitting** [*i.e., a morphism* $s\colon Y \to Z$ *such that* $Y \xrightarrow{s} Z \to Y$ *is the identity automorphism*].

Proof. This follows from [Mzk2], Proposition 2.4. □

Lemma 3.6. *It holds that* X *is* **of log rank 0** (*respectively,* **1**) *if and only if every fs log point* $Z \to X$ **factors** *through a* **trivial** (*respectively,* **standard**) *log point* $W \to X$.

Proof. *Necessity* follows by considering a suitable *strict monomorphism* $W \to X$. Next, we verify *sufficiency*. One verifies easily that X is *of log rank* 0 if every fs log point $Z \to X$ *factors* through a *trivial* log point $W \to X$. Thus, suppose that every fs log point $Z \to X$ *factors* through a *standard* log point $W \to X$. Then it follows immediately from our condition that the module $\overline{\mathcal{M}}_{X,\overline{x}}^{\mathrm{gp}}$ is *of rank* 1 for every geometric point $\overline{x} \to \overset{\circ}{X}$ whose image is *closed* in $\overset{\circ}{X}$. Write $U \subseteq \overset{\circ}{X}$ for the maximal [necessarily *open* — cf. the well-known *constructibility* of $\overline{\mathcal{M}}_X$] subset on which $\overline{\mathcal{M}}_X$ is *trivial*. If $U \neq \emptyset$, then since U has a *closed point*, we obtain a *contradiction* by our condition. This completes the proof of *sufficiency*, hence also of Lemma 3.6. □

Theorem 3.7. *Let S and T be fs log schemes whose underlying schemes are* **locally noetherian**,

$$\phi\colon\ \mathrm{Sch}^{\log}(S)\ \xrightarrow{\sim}\ \mathrm{Sch}^{\log}(T)$$

an **equivalence of categories**, *and f a morphism in* $\mathrm{Sch}^{\log}(S)$. *Then it holds that f is* **strict** *if and only if $\phi(f)$ is* **strict**.

Proof. Let X be an object of $\mathrm{Sch}^{\log}(S)$ and g a morphism in $\mathrm{Sch}^{\log}(S)$. Let us first observe that it follows from Proposition 1.6, Proposition 2.5 that

(1) it holds that X is a *trivial* (respectively, a *standard*; an *fs*) *log point* if and only if $\phi(X)$ is a *trivial* (respectively, a *standard*; an *fs*) *log point*.

Moreover, it follows from Lemma 3.1, together with assertion (1), that

(2) if g [hence also $\phi(g)$ — cf. (1)] is an *fs log point* and a *monomorphism*, then it holds that g is *strict* if and only if $\phi(g)$ is *strict*.

Thus, it follows from Lemma 3.2 that, to verify Theorem 3.7, it is enough to verify the following assertion (3):

(3) If the domain and codomain of g [hence also of $\phi(g)$ — cf. (1)] are *fs log points*, then it holds that g is *strict* if and only if $\phi(g)$ is *strict*.

Next, let us observe that, to verify assertion (3), it follows from Lemma 3.3, together with assertion (1), that it suffices to verify the following assertion (4):

(4) If the codomain of g [hence also of $\phi(g)$] is *minimal*, then it holds that g is *strict* if and only if $\phi(g)$ is *strict*.

Next, to verify assertion (4), it follows from Lemma 3.4, together with assertion (1), that it is sufficient to verify the following assertion (5):

(5) If the domain and codomain of g [hence also of $\phi(g)$] are *minimal*, then it holds that g is *strict* if and only if $\phi(g)$ is *strict*.

Thus, to verify Theorem 3.7, it follows from Lemma 3.5 that it suffices to verify the following assertion (6):

(6) It holds that X is either *of log rank* 0 or *of log rank* 1 if and only if $\phi(X)$ is either *of log rank* 0 or *of log rank* 1.

On the other hand, assertion (6) follows from Lemma 3.6, together with assertion (1). This completes the proof of Theorem 3.7. □

Remark 3.7.1.

(i) Note that Theorem 3.7 was already essentially proved by *S. Mochizuki* [cf. [Mzk1], §2; [Mzk2], §3]. Moreover, Mochizuki also proved that a result concerning a *categorical representation* of fs log schemes [i.e., [Mzk1], Theorem B; [Mzk2], Theorem A] follows from Theorem 3.7, together with some discussions [cf. the portion of [Mzk2] from the discussion preceding [Mzk2], Proposition 3.7, to the end of [Mzk2], §3 — also [Mzk2], Appendix].

(ii) On the other hand, in the present paper, by establishing a categorical characterization of *fs log points* [cf. Proposition 2.5], we obtain a *simple proof* of Theorem 3.7. In particular, the proof of the main theorem of the present paper may be regarded as a *relatively simple alternative proof* of the categorical representation of fs log schemes already proved by Mochizuki [cf. (i)].

Appendix A. Twisted Versions of Hilbert's Theorem 90

In §1, we gave a *sufficient condition* for an fs log point to be *quasi-split* [cf. Proposition 1.3, Remark 1.3.1]. In the present §A, we discuss, by considering *twisted versions of Hilbert's Theorem 90*, further such *sufficient conditions*. In the present §A, let k be a field and k_{sep} a separable closure of k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(k_{\mathrm{sep}}/k)$. Let M be a *sharp fs* monoid equipped with a continuous action of G_k [with respect to the *discrete* topology on M].

Proposition A.1. *Let X be an* **fs log point**. *Then the following three conditions are* **equivalent**:

(1) *X is* **quasi-split**.

(2) *There exist an* **fs log point** *Y and a morphism $f\colon Y \to X$ such that Y is* **quasi-split**, *and, moreover, $\overset{\circ}{f}$ is an* **isomorphism**.

(3) *There exist a* **standard log point** *Y and a morphism $f\colon Y \to X$ such that $\overset{\circ}{f}$ is an* **isomorphism**.

Proof. The implication (3) ⇒ (2) follows from Proposition 1.3. The implication (2) ⇒ (1) follows immediately from the definition [cf. also

the G_k-equivariant isomorphisms of Definition 1.2, (ii)]. The implication (1) $\Rightarrow$ (3) follows from Lemma 1.1, (iii). This completes the proof of Proposition A.1. $\square$

Definition A.2. We shall say that the pair (k, M) is *quasi-split* if the following condition is satisfied: For every fs log scheme X whose underlying scheme is the spectrum of k, if there exists a G_k-equivariant isomorphism of M with the stalk of $\overline{\mathcal{M}}_X$ at the geometric point $\overline{x} \to \overset{\circ}{X}$ determined by the separable closure k_{sep}, then X is quasi-split.

Proposition A.3. *It holds that the pair (k, M) is* **quasi-split** *if and only if $H^1(G_k, \mathrm{Hom}_{\mathbb{Z}}(M^{\mathrm{gp}}, k_{\mathrm{sep}}^{\times})) = \{0\}$ [where the action of G_k on $\mathrm{Hom}_{\mathbb{Z}}(M^{\mathrm{gp}}, k_{\mathrm{sep}}^{\times})$ is given by $g \cdot \phi \overset{\mathrm{def}}{=} g \circ \phi \circ g^{-1}$].*

Proof. Sufficiency follows immediately from the various definitions involved [cf. also the proof of Proposition 1.3]. Next, we verify *necessity*. Suppose that (k, M) is *quasi-split*. Let $1 \to k_{\mathrm{sep}}^{\times} \to E \to M^{\mathrm{gp}} \to 1$ be an exact sequence of G_k-modules corresponding to an element of $\mathrm{Ext}^1_{G_k}(M^{\mathrm{gp}}, k_{\mathrm{sep}}^{\times}) = H^1(G_k, \mathrm{Hom}_{\mathbb{Z}}(M^{\mathrm{gp}}, k_{\mathrm{sep}}^{\times}))$. Write $N \overset{\mathrm{def}}{=} (E \twoheadrightarrow M^{\mathrm{gp}})^{-1}(M) \subseteq E$. [Thus, N is isomorphic, as an abstract monoid, to $k_{\mathrm{sep}}^{\times} \times M$.] Then since M is *sharp* [which thus implies that $N^{\times} = k_{\mathrm{sep}}^{\times}$], by mapping each element of $N \smallsetminus k_{\mathrm{sep}}$ to $0 \in k_{\mathrm{sep}}$, we obtain a G_k-*equivariant* homomorphism $N \to k_{\mathrm{sep}}$ of monoids [where we regard k_{sep} as a monoid by multiplication] which is an extension of the natural inclusion $k_{\mathrm{sep}}^{\times} \hookrightarrow k_{\mathrm{sep}}$. Moreover, one verifies easily that this homomorphism $N \to k_{\mathrm{sep}}$ of monoids determines an *fs log structure* on $\mathrm{Spec}(k)$. On the other hand, since (k, M) is *quasi-split*, the resulting fs log scheme is *quasi-split*, which thus implies that the above exact sequence of G_k-modules has a G_k-*equivariant* splitting. This completes the proof of Proposition A.3. $\square$

Lemma A.4. *Let K be a finite Galois extension of k. Write $G \overset{\mathrm{def}}{=} \mathrm{Gal}(K/k)$. Let $H \subseteq N \subseteq G$ be subgroups such that N is* **normal** *in G. Let us define an action of G on the module $\mathrm{Map}(G/N, (K^H)^{\times})$ [consisting of maps of sets $G/N \to (K^H)^{\times}$] by $g \cdot \phi \overset{\mathrm{def}}{=} \phi \circ g^{-1}$; moreover, let us also define an action of G on the module $\mathrm{Map}(G/H, K^{\times})$ [consisting of maps of sets $G/H \to K^{\times}$] by $g \cdot \phi \overset{\mathrm{def}}{=} g \circ \phi \circ g^{-1}$. Then the homomorphism*

$$\begin{array}{ccc} \mathrm{Map}(G/N, (K^H)^{\times}) & \longrightarrow & \mathrm{Map}(G/H, K^{\times}) \\ \phi & \mapsto & (gH \mapsto g\phi(gN)) \end{array}$$

determines a **G-equivariant isomorphism**

$$\mathrm{Map}(G/N, (K^H)^{\times}) \xrightarrow{\sim} \mathrm{Map}(G/H, K^{\times})^N$$

of G-modules. In particular, the G/N-module $\mathrm{Map}(G/H, K^{\times})^N$ is a **coinduced module**.

Proof. This follows from a straightforward computation. □

Theorem A.5. *If one of the following three conditions is satisfied, then the pair* (k, M) *is* **quasi-split**.

(1) *The action of* G_k *on* M *is* **trivial**.

(2) *The Brauer group of every finite separable extension of* k *is* **zero**.

(3) *There exists a* [*not necessarily* G_k*-equivariant*] *isomorphism* $M \xrightarrow{\sim} \mathbb{N}^{\oplus n}$ *of monoids for some positive integer* n.

Proof. Theorem A.5 in the case where condition (1) is satisfied follows formally from Proposition 1.3. Theorem A.5 in the case where condition (2) is satisfied follows from Proposition A.3 and [Serre], Chapter X, Proposition 11, as well as [Serre], Chapter X, Corollary to Proposition 11.

Finally, we verify Theorem A.5 in the case where condition (3) is satisfied. Suppose that condition (3) is satisfied. Let us first observe that one verifies easily that each automorphism of the monoid $\mathbb{N}^{\oplus n}$ arises from some *permutation of the* n *factors*. Thus, it follows from Proposition A.3 that, to complete the verification of Theorem A.5, it suffices to verify that

> (†): for a finite set S and a finite Galois extension K of k whose Galois group $G \stackrel{\text{def}}{=} \mathrm{Gal}(K/k)$ acts on S, it holds that $H^1(G, \mathrm{Map}(S, K^\times)) = \{0\}$ [where the action of G on $\mathrm{Map}(S, K^\times)$ is given by $g \cdot \phi \stackrel{\text{def}}{=} g \circ \phi \circ g^{-1}$].

Next, let us observe that we may assume without loss of generality, by replacing G by a p-Sylow subgroup of G [where p is a prime number], that G in (†) is a [nontrivial] *p-group*. Next, let us observe that we may assume without loss of generality, by replacing S by the G-orbit of an element of S, that S in (†) is the G-set G/H for a subgroup $H \subseteq G$. Observe that if $H = G$, then (†) follows from *Hilbert's Theorem 90*; thus, we may assume without loss of generality that $H \neq G$.

Let $N \subseteq G$ be a *normal* subgroup such that $H \subseteq N$, and, moreover, $[G : N] = p$. [Note that one verifies easily that such a normal subgroup always *exists*.] Thus, by *induction* on the cardinality of G, to verify (†), it suffices to verify that

$$H^1(G/N, \mathrm{Map}(G/H, K^\times)^N) \;=\; \{0\}.$$

On the other hand, this follows from Lemma A.4. This completes the proof of Theorem A.5. □

Remark A.5.1. Examples of "k" which satisfies condition (2) in the statement of Theorem A.5 are given in the discussion entitled "Examples of Fields with Zero Brauer Group" in [Serre], p.162. For instance, a *quasi-algebraically closed field* [i.e., a field which has property C_1] —

e.g., a *finite field* — satisfies condition (2) in the statement of Theorem A.5.

Remark A.5.2. An example of an fs log point which is *not quasi-split* is given as follows: Write M for the monoid obtained by taking the quotient of $\mathbb{N}^{\oplus 3}$ by the relation $(a, a, 0) \sim (0, 0, 2a)$, where $a \in \mathbb{N}$. Let us define an action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}^\times \times M$ by $\sigma(z, [a, b, c]) \stackrel{\text{def}}{=} ((-1)^c \cdot \sigma(z), [b, a, c])$ — where we write $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ for the unique nontrivial element and "$[-]$" ($\in M$) for the image of "$(-)$" ($\in \mathbb{N}^{\oplus 3}$) in M — as well as a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant homomorphism $\mathbb{C}^\times \times M \to \mathbb{C}$ of monoids by $(z, [a, b, c]) \mapsto z$ if $(a, b, c) = (0, 0, 0)$ (respectively, $\mapsto 0$ if $(a, b, c) \neq (0, 0, 0)$). Then one verifies immediately from a straightforward computation that this $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant homomorphism $\mathbb{C}^\times \times M \to \mathbb{C}$ determines an *fs log structure* on $\mathrm{Spec}(\mathbb{R})$, and, moreover, the resulting fs log point is *not quasi-split*.

References

[Kato] K. Kato, Logarithmic structures of Fontaine-Illusie, *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, 191–224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Mzk1] S. Mochizuki, Categorical representation of locally noetherian log schemes, *Adv. Math.* **188** (2004), no. **1**, 222–246.

[Mzk2] S. Mochizuki, Monomorphisms in categories of log schemes, *Kodai Math. J.* **38** (2015), no. **2**, 365–429.

[Naka] C. Nakayama, Logarithmic étale cohomology, *Math. Ann.* **308** (1997), no. **3**, 365–404.

[Serre] J.-P. Serre, *Local fields*, Translated from the French by Marvin Jay Greenberg. Graduate Texts in Mathematics, **67**. Springer-Verlag, New York-Heidelberg-Berlin, 1979.

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