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On the indecomposability of the image of the universal pro-$\{l\}$ outer monodromy representation of the moduli stack of once-punctured elliptic curves

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ON THE INDECOMPOSABILITY OF THE IMAGE OF THE UNIVERSAL PRO-$\{l\}$ OUTER MONODROMY REPRESENTATION OF THE MODULI STACK OF ONCE-PUNCTURED ELLIPTIC CURVES

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Abstract. Minamide proved that the pro-l Grothendieck–Teichmüller group GT_l and the image of the absolute Galois group of a number field in GT_l are *indecomposable*, i.e., do *not* have a *nontrivial* direct product decomposition. This Galois image may be identified with the image of the universal pro-$\{l\}$ outer monodromy representation of the moduli stack of projective lines minus three points over the number field. In the present paper, we prove the *indecomposability* of the image of the universal pro-$\{l\}$ outer monodromy representation of the moduli stack of *once-punctured elliptic curves over either a number field or an algebraically closed field of characteristic zero.*

Contents

Introduction

Let l be a prime number, (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$, n a positive integer, k a field of characteristic zero, and $\overline{k}$ an algebraic closure of k. Write $G_k := \mathrm{Gal}(\overline{k}/k)$, $(\mathcal{M}_{g,r})_k$ for the *moduli stack of r-pointed smooth proper curves of genus g over k whose r marked points are equipped with an ordering*, $\Delta_{g,r}^{\{l\}}$ for the pro-$\{l\}$ completion of the (topological) fundamental group of a topological space obtained by removing r distinct points from a connected orientable compact topological surface of genus g, and

$$\rho_{g,r/k}^{\{l\}} \colon \pi_1((\mathcal{M}_{g,r})_k) \longrightarrow \mathrm{Out}(\Delta_{g,r}^{\{l\}})$$

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for the *universal pro-$\{l\}$ outer monodromy representation of* $(\mathcal{M}_{g,r})_k$. Note that, since $(\mathcal{M}_{0,3})_k$, $(\mathcal{M}_{0,4})_k$ are naturally isomorphic to Spec k, $\mathbb{P}^1_k \setminus \{0,1,\infty\}$, respectively, $\rho^{\{l\}}_{0,3/k}$ may be identified with the pro-$\{l\}$ outer Galois representation associated to $\mathbb{P}^1_k \setminus \{0,1,\infty\}$.

We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \xrightarrow{\sim} H \times K$, where H, K are profinite groups, it holds that either H or K is the trivial group. The notion of the indecompsability may be thought of as a sort of *rigidity* of profinite groups. It is known that the absolute Galois group of a number field is *indecomposable* (cf. [12, Corollary 1.4]). Also, Minamide proved the following result (cf. [12, Theroem F]; also Theorem 2.2):

> (M) Suppose that k is a number field. Then the pro-l Grothendieck–Teichmüller group GT_l and $\mathrm{im}(\rho^{\{l\}}_{0,3/k})$ are *indecomposable.*

Here, the pro-l Grothendieck–Teichmüller group GT_l is a closed subgroup of $\mathrm{Out}(\varDelta^{\{l\}}_{0,3})$ which may be thought of as a sort of abstract combinatorial approximation of the image of the absolute Galois group $G_{\mathbb{Q}}$ of the field $\mathbb{Q}$ of rational numbers via the pro-$\{l\}$ outer Galois representation associated to $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$. Thus, although $\rho^{\{l\}}_{0,3/\mathbb{Q}}$ is far from injective, and it is not known at the time of writing whether or not $G_{\mathbb{Q}} \to \mathrm{GT}_l$ is surjective, one may assert that $\mathrm{im}(\rho^{\{l\}}_{0,3/\mathbb{Q}})$ and GT_l satisfy an *analogous property to* $G_{\mathbb{Q}}$, i.e., the *indecomposability.*

In the present paper, we prove an analog of (M) in the case where (g,r) is equal to $(1,1)$. Write $\mathrm{Out}^{\mathrm{FC}}((\varDelta^n_{1,1})^{\{l\}})$ for the group of FC-admissible outer automorphisms of the maximal pro-$\{l\}$ quotient of the étale fundamental group of the n-th cofiguration space of a once-punctured elliptic curve over $\overline{k}$. Then $\mathrm{Out}^{\mathrm{FC}}((\varDelta^n_{1,1})^{\{l\}})$ may be regarded as a closed subgroup of $\mathrm{Out}(\varDelta^{\{l\}}_{1,1})$ which contains $\mathrm{im}(\rho^{\{l\}}_{1,1/k})$. The main result of the present paper is the following (cf. Theorem 2.8, Corollary 2.9):

Theorem A. *Suppose that $n \geq 3$, and that k is either a number field or algebraically closed. Then $\mathrm{Out}^{\mathrm{FC}}((\varDelta^n_{1,1})^{\{l\}})$ and $\mathrm{im}(\rho^{\{l\}}_{1,1/k})$ are indecomposable.*

Also, we compute the cardinality of the centralizer of $\mathrm{im}(\rho^{\{l\}}_{1,1/\overline{k}})$ (cf. Proposition 2.6), and prove an analog of Theorem A for the moduli stack of punctured elliptic curves (cf. Corollary 2.12).

The outline of the proof of Theorem A is as follows: By means of the *tripod homomorphism*, we reduce the indecomposability of $\mathrm{Out}^{\mathrm{FC}}((\varDelta^n_{1,1})^{\{l\}})$ and $\mathrm{im}(\rho^{\{l\}}_{1,1/k})$ to the indecomposability of the arithmetic parts and the geometric parts of these profinite groups. Since the indecomposability of the arithmetic parts is nothing but (M), to verify Theorem A, it suffices to verify the indecomposability of the geometric parts. In the proof of (M), the *rigidity of the image of Frobenius elements* played an essential role. In our proof of the indecomposability of the geometric parts, instead of Frobenius elements, we apply the *rigidity of profinite Dehn twists*.

the rigidity of the image of Frobenius elements
$\Downarrow$
the indecomposability of the arithemtic parts

the rigidity of profinite Dehn twists
$\Downarrow$
the indecomposability of the geometric parts

Finally, in §3, we prove the indecomposability of $\pi_1((\mathcal{M}_{1,1})_k)$ in the case where k is either a number field or algebraically closed (cf. Theorem 3.4).

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Notations and Conventions

Numbers: We shall refer to a finite extension of the field of rational numbers $\mathbb{Q}$ as a *number field*.

Profinite groups: For a profinite groups G, and a closed subgroup H of G, we shall write G^{ab} for the *abelianization* of G, $Z(G)$ for the *centralizer* of G, $Z_G(H)$ for the *centralizer* of H in G, and $N_G(H)$ for the *normalizer* of H in G, i.e., $\{g \in G \mid g \cdot H \cdot g^{-1} = H\}$. We shall say that a profinite group G is *slim* if for any open subgroup $H \subseteq G$, it holds that $Z_G(H) = \{1\}$. We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \xrightarrow{\sim} H \times K$, where H, K are profinite groups, it holds that either H or K is the trivial group. We shall say that a profinite group G is *strongly indecomposable* if any open subgroup of G is indecomposable.

For a profinite group G and a property $\mathcal{P}$ for profinite groups, we shall say that G is *almost* $\mathcal{P}$ if an open subgroup of G is $\mathcal{P}$.

For a profinite group G, $\mathrm{Aut}(G)$ for the group of (continuous) automorphisms of the topological group G, $\mathrm{Inn}(G)$ for the group of inner automorphisms of G, and $\mathrm{Out}(G)$ for the quotient of $\mathrm{Aut}(G)$ with respect to the normal subgroup $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$. If, moreover, G is *topologically finitely generated*, then one verifies that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the group $\mathrm{Aut}(G)$, hence also a *profinite topology* on the group $\mathrm{Out}(G)$. We shall refer to an element of $\mathrm{Out}(G)$ as an *outer automorphism* of G. For profinite groups G_1, G_2, we shall refer to as an *outer homomorphism* from G_1 to G_2 an equivalent class of a homomorphism of profinite groups $G_1 \to G_2$ modulo $\mathrm{Inn}(G_2)$.

Curves: Let (g, r) be a pair of nonnegative integers such that $2g-2+r>0$, and k a field of characteristic zero.

We shall write $(\mathcal{M}_{g,r})_k$ for the moduli stack of r-pointed smooth proper curves of genus g over k whose r marked points are equipped with an ordering, and $(\overline{\mathcal{M}}_{g,r})_k$ for the moduli stack of r-pointed stable curves of genus g over k whose r marked points are equipped with an ordering (cf. [11]). Then by regarding $(\mathcal{M}_{g,r})_k$ as an open substack of $(\overline{\mathcal{M}}_{g,r})_k$, we obtain a log stack $(\overline{\mathcal{M}}_{g,r}^{\log})_k$, i.e., the log stack obtained by equipping $(\overline{\mathcal{M}}_{g,r})_k$ with the log structure associated to the divisor with normal crossings $(\overline{\mathcal{M}}_{g,r})_k \setminus (\mathcal{M}_{g,r})_k$. We shall write $\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{M}_{g,r+1})_k)$ for the group of automorphisms of the algebraic stack $(\mathcal{M}_{g,r+1})_k$ over $(\mathcal{M}_{g,r})_k$ relative to the (1-)morphism $(\mathcal{M}_{g,r+1})_k \to (\mathcal{M}_{g,r})_k$ given by forgetting the last marked point.

We shall write $(\operatorname{Spec} k)^{\log}$ for the log scheme obtained by equipping $\operatorname{Spec} k$ with the log structure determined by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$. We shall refer to as a *stable log curve (of type (g,r)) over* $(\operatorname{Spec} k)^{\log}$ the pulling back of the (1-)morphism of log stacks $(\overline{\mathcal{M}}_{g,r+1}^{\log})_k \to (\overline{\mathcal{M}}_{g,r}^{\log})_k$ given by forgetting the last marked point via a (1-)morphism of log stacks $(\operatorname{Spec} k)^{\log} \to (\overline{\mathcal{M}}_{g,r}^{\log})_k$ in the category of fs log stacks, and this (1-)morphism of log stacks $(\operatorname{Spec} k)^{\log} \to (\overline{\mathcal{M}}_{g,r}^{\log})_k$ as the *classifying (1-)morphism of the stable log curve*. For a stable log curve C of type (g,r) over $(\operatorname{Spec} k)^{\log}$ and a positive integer n, we shall refer to as the *n-th log configuration space of C* the pulling back of the (1-)morphism of log stacks $(\overline{\mathcal{M}}_{g,r+n}^{\log})_k \to (\overline{\mathcal{M}}_{g,r}^{\log})_k$ given by forgetting the last n marked points via the classifying (1-)morphism $(\operatorname{Spec} k)^{\log} \to (\overline{\mathcal{M}}_{g,r}^{\log})_k$ of C in the category of fs log stacks. We shall denote by C^n the n-th log configuration space of the stable log curve C.

1. The universal pro-Σ outer monodromy representation of the moduli stack of curves

In the present §1, we recall generalities of the universal pro-Σ outer monodromy representation of the moduli stack of curves.

Let l be a prime number, Σ either the set of prime numbers or $\{l\}$, (g,r) a pair of nonnegative integers such that $2g-2+r>0$, n a positive integer, k a field of characteristic zero, and $\overline{k}$ an algebraic closure of k. Write $G_k := \mathrm{Gal}(\overline{k}/k)$.

Definition 1.1. (i) We shall write $\pi_1((\mathcal{M}_{g,r})_k)$, $\pi_1((\overline{\mathcal{M}}_{g,r}^{\log})_k)$ for the étale fundamental group of $(\mathcal{M}_{g,r})_k$, the log fundamental group of $(\overline{\mathcal{M}}_{g,r}^{\log})_k$, respectively (cf., e.g., [18], [3], respectively). (In fact, $\pi_1(-)$ is defined for the pair of " $-$ " and a base point of " $-$ ". However, since the $\pi_1(-)$ is *independent*, up to inner automorphisms, of the choice of the base point, we shall omit the base point.) Now by the log purity theorem (cf. [13, Theorem B]), we have a natural outer isomorphism

$$\pi_1((\overline{\mathcal{M}}_{g,r}^{\log})_k) \xrightarrow{\sim} \pi_1((\mathcal{M}_{g,r})_k).$$

In this paper, we shall identify $\pi_1((\overline{\mathcal{M}}^{\log}_{g,r})_k)$ with $\pi_1((\mathcal{M}_{g,r})_k)$ via the above outer isomorphism. Also, for a stable log curve C of type (g,r) over $(\operatorname{Spec} k)^{\log}$, we shall write $(\Delta^n_C)^\Sigma$ for the maximal pro-Σ quotient of the kernel of the outer homomorphism from the log fundamental group $\pi_1(C^n)$ of C^n to the log fundamental group $\pi_1((\operatorname{Spec} k)^{\log})$ of $(\operatorname{Spec} k)^{\log}$ determined by the structural morphism $C^n \to (\operatorname{Spec} k)^{\log}$.

(ii) We shall write $\Delta^n_{g,r}$ for the kernel of the natural outer surjection of profinite groups $\pi_1((\mathcal{M}_{g,r+n})_k) \twoheadrightarrow \pi_1((\mathcal{M}_{g,r})_k)$ arising from the (1-)morphism $(\mathcal{M}_{g,r+n})_k \to (\mathcal{M}_{g,r})_k$ given by forgetting the last n marked points, and $(\Delta^n_{g,r})^\Sigma$ for the maximal pro-Σ quotient of $\Delta^n_{g,r}$. We shall regard $\pi_1((\mathcal{M}_{g,r})_{\overline{k}})$ as a closed subgroup of $\pi_1((\mathcal{M}_{g,r})_k)$ by the natural injection $\pi_1((\mathcal{M}_{g,r})_{\overline{k}}) \hookrightarrow \pi_1((\mathcal{M}_{g,r})_k)$. Then we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta^n_{g,r} \longrightarrow \pi_1((\mathcal{M}_{g,r+1})_k) \longrightarrow \pi_1((\mathcal{M}_{g,r})_k) \longrightarrow 1.$$

We shall write

$$\rho^{n,\Sigma}_{g,r/k} \colon \pi_1((\mathcal{M}_{g,r})_k) \longrightarrow \operatorname{Out}((\Delta^n_{g,r})^\Sigma)$$

for the composite of the homomorphism $\pi_1((\mathcal{M}_{g,r})_k) \to \operatorname{Out}(\Delta^n_{g,r})$ determined by the above exact sequence of profinite groups and the homomorphism $\operatorname{Out}(\Delta^n_{g,r}) \to \operatorname{Out}((\Delta^n_{g,r})^\Sigma)$ arising from the natural surjection $\Delta^n_{g,r} \twoheadrightarrow (\Delta^n_{g,r})^\Sigma$. For simplicity, we shall write $\rho^\Sigma_{g,r/k}$ (respectively, $\Delta^\Sigma_{g,r}$) instead of $\rho^{1,\Sigma}_{g,r/k}$ (respectively, $(\Delta^1_{g,r})^\Sigma$). We shall refer to $\rho^\Sigma_{g,r/k}$ as the *universal pro-Σ outer monodromy representation of* $(\mathcal{M}_{g,r})_k$. We shall write $\iota^\Sigma_{g,r} \colon \operatorname{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{M}_{g,r+1})_k) \to \operatorname{Out}(\Delta^\Sigma_{g,r})$ for the composite of natural homomorphisms

$$\operatorname{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{M}_{g,r+1})_k) \to \operatorname{Aut}_{\pi_1((\mathcal{M}_{g,r})_k)}(\pi_1((\mathcal{M}_{g,r+1})_k))/\operatorname{Inn}(\Delta_{g,r})$$

$$\to \operatorname{Out}(\Delta^\Sigma_{g,r}).$$

(iii) Let C be a stable log curve of type (g,r) over $(\operatorname{Spec} k)^{\log}$. Then the classifying (1-)morphism $(\operatorname{Spec} k)^{\log} \to (\overline{\mathcal{M}}^{\log}_{g,r})_k$ of C induces an outer isomorphism

$$i^n_C \colon (\Delta^n_C)^\Sigma \xrightarrow{\sim} (\Delta^n_{g,r})^\Sigma.$$

We shall write

$$\operatorname{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^\Sigma)$$

for the image of the subgroup of FC-admissible outer automorphisms of $(\Delta^n_C)^\Sigma$ (i.e., roughly speaking, outer automorphisms that preserve the fiber subgroups of $(\Delta^n_C)^\Sigma$ and the cuspidal inertia subgroups of these fiber subgroups — cf. [15, Definition 1.1, (ii)]) via the outer isomorphism

$$\operatorname{Out}((\Delta^n_C)^\Sigma) \xrightarrow{\sim} \operatorname{Out}((\Delta^n_{g,r})^\Sigma)$$

determined by the outer isomorphism $i^n_C \colon (\Delta^n_C)^\Sigma \xrightarrow{\sim} (\Delta^n_{g,r})^\Sigma$. Note that the subgroup $\operatorname{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^\Sigma)$ of $\operatorname{Out}((\Delta^n_{g,r})^\Sigma)$ does *not depend* on the choice of a stable log curve C of type (g,r) over $(\operatorname{Spec} k)^{\log}$, and that the image of $\rho^{n,\Sigma}_{g,r/k}$ is contained in $\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^{\Sigma})$.

(iv) Suppose that

$$n \geq \begin{cases} 4 & \text{if } r = 0, \\ 3 & \text{if } r \geq 1. \end{cases}$$

Then we shall write $\Delta^{\Sigma}_{\mathrm{tpd}}$ for the *central* $\{1,2,3\}$*-tripod of* $(\Delta^n_{g,r})^{\Sigma}$ (i.e., roughly speaking, under the outer isomorphism of profinite groups $i^n_C \colon (\Delta^n_C)^{\Sigma} \xrightarrow{\sim} (\Delta^n_{g,r})^{\Sigma}$, the maximal pro-$\Sigma$ quotient of the étale fundamental group of $\mathbb{P}^1_{\overline{k}} \setminus \{0,1,\infty\}$ that arises, in the case where the given stable log curve has no nodes, by blowing up the intersection of the three diagonal divisors of the direct product of three copies of the curve over $\overline{k}$ — cf. [9, Definition 3.3, (i); Definition 3.7, (ii)]), $\mathrm{GT}_{\Sigma} \subseteq \mathrm{Out}(\Delta^{\Sigma}_{\mathrm{tpd}})$ for

$$\begin{cases} \text{the pro-}l\text{ Grothendieck–Teichmüller group} & \text{if } \Sigma = \{l\}, \\ \text{the profinite Grothendieck–Teichmüller group} & \text{otherwise} \end{cases}$$

(cf. [12, Definition 5.1], [15, Definition 1.11, (ii); Remark 1.11.1]),

$$\mathfrak{T}^{n,\Sigma}_{g,r} \colon \mathrm{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^{\Sigma}) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{\mathrm{tpd}})$$

for the *tripod homomorphism associated to* $(\Delta^n_{g,r})^{\Sigma}$ (cf. [9, Definition 3.19]), and $\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^{\Sigma})^{\mathrm{geo}}$ for the kernel of $\mathfrak{T}^{n,\Sigma}_{g,r}$. Under the natural outer isomorphism $\Delta^{\Sigma}_{\mathrm{tpd}} \xrightarrow{\sim} \Delta^{\Sigma}_{0,3}$, we shall regard $\mathrm{im}(\rho^{\Sigma}_{0,3/k})$ as a closed subgroup of GT_{Σ}.

In the study of the universal pro-Σ outer monodromy representation of the moduli stack of curves, the following theorem is fundamental.

Theorem 1.2 (Ihara, Oda, Nakamura, Takao, Hoshi–Mochizuki)**.**

(i) *The surjection* $(\Delta^{n+1}_{g,r})^{\Sigma} \twoheadrightarrow (\Delta^n_{g,r})^{\Sigma}$ *determined by the outer surjection of profinite groups* $\pi_1((\mathcal{M}_{g,r+n+1})_k) \twoheadrightarrow \pi_1((\mathcal{M}_{g,r+n})_k)$ *that arises from the (*1*-)morphism* $(\mathcal{M}_{g,r+n+1})_k \to (\mathcal{M}_{g,r+n})_k$ *given by forgetting the last marked point induces an injection of profinite groups* $\mathrm{Out}^{\mathrm{FC}}((\Delta^{n+1}_{g,r})^{\Sigma}) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^{\Sigma})$*. If, moreover,*

$$n \geq \begin{cases} 4 & \textit{if } r = 0, \\ 3 & \textit{if } r \geq 1, \end{cases}$$

then this injection is an isomorphism.

In particular, $\mathrm{im}(\rho^{n,\Sigma}_{g,r/k})$ *is naturally isomorphic to* $\mathrm{im}(\rho^{\Sigma}_{g,r/k})$*.*

(ii) *The kernel of the composite of natural outer homomorphisms*

$$\pi_1((\mathcal{M}_{0,3})_k) \xrightarrow{\sim} G_k \xrightarrow{\sim} \pi_1((\mathcal{M}_{g,r})_k)/\pi_1((\mathcal{M}_{g,r})_{\overline{k}})$$
$$\twoheadrightarrow \mathrm{im}(\rho^{\Sigma}_{g,r/k})/\rho^{\Sigma}_{g,r/k}(\pi_1((\mathcal{M}_{g,r})_{\overline{k}}))$$

is equal to $\ker(\rho^{\Sigma}_{0,3/k})$*.*

(iii) *Suppose, moreover, that*

$$n \geq \begin{cases} 4 & \textit{if } r = 0, \\ 3 & \textit{if } r \geq 1. \end{cases}$$

Then the image of the tripod homomorphism $\mathfrak{T}^{n,\Sigma}_{g,r}\colon \mathrm{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^{\Sigma}) \to \mathrm{Out}(\Delta^{\Sigma}_{\mathrm{tpd}})$ *associated to* $(\Delta^n_{g,r})^{\Sigma}$ *is equal to* $\mathrm{GT}_{\Sigma} \subseteq \mathrm{Out}(\Delta^{\Sigma}_{\mathrm{tpd}})$*, and fits into a commutative diagram of profinite groups*

$$\begin{tikzcd}
1 \arrow[r] & \rho^{n,\Sigma}_{g,r/k}(\pi_1((\mathcal{M}_{g,r})_{\overline{k}})) \arrow[r] \arrow[d, hook] & \mathrm{im}(\rho^{n,\Sigma}_{g,r/k}) \arrow[r] \arrow[d, hook] & \mathrm{im}(\rho^{\Sigma}_{0,3/k}) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^{\Sigma})^{\mathrm{geo}} \arrow[r] & \mathrm{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^{\Sigma}) \arrow[r, "\mathfrak{T}^{n,\Sigma}_{g,r}"] & \mathrm{GT}_{\Sigma} \arrow[r] & 1
\end{tikzcd}$$

where the upper horizontal sequence is an exact sequence of profinite groups determined by (ii), and the vertical arrows are natural injections.

Proof. For the first portion of assertion (i), see [7, Theorem B]. The final portion of assertion (i) follows from the first portion of assertion (i). For assertion (ii), see [19, Theorem 0.5, (2)], and [7, Corollary 6.4]. For assertion (iii), see [9, Theorem C, (iv)] (also [9, Remark 3.19.1]). □

In the rest of this paper, by means of Theorem 1.2, (i), we shall regard the profinite group $\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^{\Sigma})$ as a closed subgroup of $\mathrm{Out}^{\mathrm{FC}}(\Delta^{\Sigma}_{g,r})$, and identify $\mathrm{im}(\rho^{n,\Sigma}_{g,r/k})$ with $\mathrm{im}(\rho^{\Sigma}_{g,r/k})$.

2. The proof of the main results

In this §2, we prove Theorem A (cf. Theorem 2.8, Corollary 2.9, below).

Lemma 2.1. *Suppose that k is a number field. Then* $\mathrm{GT}_{\{l\}}$ *and* $\mathrm{im}(\rho^{\{l\}}_{0,3/k})$ *are slim.*

Proof. See [12, Lemma 5.3] and [4, Lemma 4.3, (ii)]. □

Theorem 2.2 (Minamide)**.** *Suppose that k is a number field. Then* $\mathrm{GT}_{\{l\}}$ *and* $\mathrm{im}(\rho^{\{l\}}_{0,3/k})$ *are strongly indecomposable.*

Proof. See [12, Theroem 5.4]. (Although [12, Theroem 5.4] stated only the strongly indecomposability of $\mathrm{GT}_{\{l\}}$, in fact, the proof of [12, Theroem 5.4] implies also the strongly indecomposability of $\mathrm{im}(\rho^{\{l\}}_{0,3/k})$.) □

Definition 2.3. Let X be a stable log curve of type $(1,1)$ over $(\mathrm{Spec}\,\overline{k})^{\log}$ whose underlying scheme has nodes. We shall write $\mathcal{G}_{\Sigma}$ for the *semi-graph of anabelioids of pro-Σ PSC-type* determined by the stable log curve X over $(\mathrm{Spec}\,\overline{k})^{\log}$ (i.e., roughly speaking, a system of the dual semi-graph of the stable curve X^{un} over $\overline{k}$ determined by X and Galois categories obtained from irreducible components of X^{un}, points at infinity of X^{un}, and nodes of X^{un} — cf. [14, Definition 1.1, (i); Example 2.5]), $|\mathcal{G}_{\Sigma}|$ for the underlying semi-graph of $\mathcal{G}_{\Sigma}$ (i.e., the dual semi-graph of the stable curve X^{un} over $\overline{k}$), and $\Pi_{\mathcal{G}_{\Sigma}}$ for the *PSC-fundamental group* of the semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{G}_{\Sigma}$ (i.e., roughly speaking, the maximal pro-Σ quotient of the admissible fundamental group of the stable curve X^{un} over $\overline{k}$ — cf. [14, Definition 1.1, (ii)]). Note that the isomorphic class of the semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{G}_{\Sigma}$ is *independent* on the choice of a stable log curve X of type $(1,1)$ over $(\operatorname{Spec}\overline{k})^{\log}$ whose underlying scheme has nodes. Then we have natural outer isomorphisms

$$\Pi_{\mathcal{G}_\Sigma} \xrightarrow{\sim} (\Delta_X^1)^\Sigma \xrightarrow{\sim} \Delta_{1,1}^\Sigma.$$

We shall identify $\Pi_{\mathcal{G}_\Sigma}$ with $\Delta_{1,1}^\Sigma$ via the above composite. We shall write $\mathrm{Aut}(\mathcal{G}_\Sigma) \subseteq \mathrm{Out}(\Delta_{1,1}^\Sigma)$ for the group of automorphisms of the semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{G}_\Sigma$, and $\mathrm{Aut}(|\mathcal{G}_\Sigma|)(\twoheadleftarrow \mathrm{Aut}(\mathcal{G}_\Sigma))$ for the group of automorphisms of the semi-graph $|\mathcal{G}_\Sigma|$, $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\Sigma) \subseteq \mathrm{Aut}(\mathcal{G}_\Sigma)$ for the kernel of the natural surjection $\mathrm{Aut}(\mathcal{G}_\Sigma) \twoheadrightarrow \mathrm{Aut}(|\mathcal{G}_\Sigma|)$ (cf. [9, Remark 4.1.2]), and $\mathrm{Dehn}(\mathcal{G}_\Sigma) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\Sigma)$ for the group of *profinite Dehn twists of* $\mathcal{G}_\Sigma$ (i.e., roughly speaking, the image of the local universal outer monodromy representation associated to X^{un} in $\mathrm{Out}(\Delta_{1,1}^\Sigma)$ — cf. [8, Definition 4.4]). Then by [8, Proposition 5.6, (ii)], $\mathrm{Dehn}(\mathcal{G}_\Sigma) \subseteq \mathrm{Out}(\Delta_{1,1}^\Sigma)$ is contained in $\mathrm{im}(\rho_{1,1/\overline{k}}^\Sigma)$.

Lemma 2.4. *$\iota_{1,1}^\Sigma$ is injective, and factors through $Z(\mathrm{im}(\rho_{1,1/k}^\Sigma)) \subseteq \mathrm{Out}(\Delta_{1,1}^\Sigma)$.*
Moreover, $\mathrm{im}(\iota_{1,1}^\Sigma) \subseteq \mathrm{Aut}(\mathcal{G}_\Sigma)$, *and* $\mathrm{im}(\iota_{1,1}^\Sigma) \cap \mathrm{Dehn}(\mathcal{G}_\Sigma) = \{1\}$.

Proof. First, the injectivity of $\iota_{1,1}^\Sigma$ follows from the well-known fact that any *nontrivial* automorphism of a hyperbolic curve over k induces a *nontrivial* outer automorphism of the maximal pro-Σ quotient of the geometric fundamental group of the hyperbolic curve. Next, it is well-known that there exists a natural outer isomorphism $SL_2(\mathbb{Z})^\wedge \xrightarrow{\sim} \pi_1((\mathcal{M}_{1,1})_{\overline{k}})$, where we write $SL_2(\mathbb{Z})^\wedge$ for the profinite completion of $SL_2(\mathbb{Z})$, such that the image of

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{Z}) \subseteq SL_2(\mathbb{Z})^\wedge$$

in $\mathrm{Out}(\Delta_{1,1}^\Sigma)$ via the composite of $SL_2(\mathbb{Z})^\wedge \xrightarrow{\sim} \pi_1((\mathcal{M}_{1,1})_{\overline{k}}) \to \mathrm{Out}(\Delta_{1,1}^\Sigma)$ coincides with the image of the *unique nontrivial* element of the group $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{M}_{1,2})_k) \simeq \mathbb{Z}/2$ in $\mathrm{Out}(\Delta_{1,1}^\Sigma)$. Thus, by [8, the disucussion entitled "Topological group" in §0], $\mathrm{im}(\iota_{1,1}^\Sigma) \subseteq Z(\mathrm{im}(\rho_{1,1/k}^\Sigma))$. This completes the proof of the first portion of Lemma 2.4.

Finally, since $\mathrm{Dehn}(\mathcal{G}_\Sigma) \subseteq \mathrm{im}(\rho_{1,1/k}^\Sigma)$, it follows from [8, Theorem 5.14, (ii)] that $\mathrm{im}(\iota_{1,1}^\Sigma) \subseteq \mathrm{Aut}(\mathcal{G}_\Sigma)$. Also, by the *torsion-freeness* of $\mathrm{Dehn}(\mathcal{G}_\Sigma)$ (cf. [8, Theorem 4.8, (iv)]), the intersection of the finite group $\mathrm{im}(\iota_{1,1}^\Sigma)$ and $\mathrm{Dehn}(\mathcal{G}_\Sigma)$ is *trivial*. This completes the proof of the final portion of Lemma 2.4. □

Lemma 2.5. *Suppose that $n \geq 3$, and that k is algebraically closed. Let $I \subseteq \mathrm{Dehn}(\mathcal{G}_\Sigma)$ be an open subgroup of $\mathrm{Dehn}(\mathcal{G}_\Sigma)$. Then the equalities*

$$\mathrm{im}(\iota_{1,1}^\Sigma) \times \mathrm{Dehn}(\mathcal{G}_\Sigma) = Z_{\mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^n)^\Sigma)^{\mathrm{geo}}}(I) = N_{\mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^n)^\Sigma)^{\mathrm{geo}}}(I)$$

hold.

Proof. Since $\mathrm{Dehn}(\mathcal{G}_\Sigma)$ is abelian (cf. [8, Theorem 4.8, (iv)]), and is contained in $\mathrm{im}(\rho_{1,1/k}^\Sigma)$, by Lemma 2.4, the inclusions $\mathrm{im}(\iota_{1,1}^\Sigma) \times \mathrm{Dehn}(\mathcal{G}_\Sigma) \subseteq Z_{\mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^n)^\Sigma)^{\mathrm{geo}}}(I) \subseteq N_{\mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^n)^\Sigma)^{\mathrm{geo}}}(I)$ hold. Moreover, it follows from [8, Proposition 4.10, (ii); Theorem 5.14, (ii)], and [9, Theorem 3.18, (ii)] that the inclusions $N_{\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^\Sigma)^{\mathrm{geo}}}(I) \subseteq \mathrm{Aut}(\mathcal{G}_\Sigma)$ and $\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^\Sigma)^{\mathrm{geo}} \cap \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\Sigma) \subseteq \mathrm{Dehn}(\mathcal{G}_\Sigma)$ hold. Since the cardinality of $\mathrm{Aut}(|\mathcal{G}_\Sigma|) \simeq \mathrm{Aut}(\mathcal{G}_\Sigma)/\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\Sigma)$ is *equal to* the cardinality of $\mathrm{im}(\iota^\Sigma_{1,1})$ (i.e., 2), the inclusion $\mathrm{im}(\iota^\Sigma_{1,1}) \times \mathrm{Dehn}(\mathcal{G}_\Sigma) \subseteq N_{\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^\Sigma)^{\mathrm{geo}}}(I)$ is in fact an equality. This completes the proof of Lemma 2.5. □

Proposition 2.6. *Suppose that $n \geq 3$, and that k is algebraically closed. Let H be an open subgroup of* $\mathrm{im}(\rho^\Sigma_{1,1/k})$. *Then the equality*

$$\mathrm{im}(\iota^\Sigma_{1,1}) = Z_{\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^\Sigma)^{\mathrm{geo}}}(H)$$

holds.

In particular, in this case, $\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^\Sigma)^{\mathrm{geo}}$ *and* $\mathrm{im}(\rho^\Sigma_{1,1/k})$ *are almost slim.*

Proof. First, we verify the first portion of Proposition 2.6. Now it follows from Lemma 2.4 that the inclusion $\mathrm{im}(\iota^\Sigma_{1,1}) \subseteq Z_{\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^\Sigma)^{\mathrm{geo}}}(H)$ holds. Thus, to verify the first portion of Proposition 2.6, it suffices to verify the inclusion $\mathrm{im}(\iota^\Sigma_{1,1}) \supseteq Z_{\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^\Sigma)^{\mathrm{geo}}}(H)$. Since H contains an open subgroup of $\mathrm{Dehn}(\mathcal{G}_\Sigma) \subseteq \mathrm{im}(\rho^\Sigma_{1,1/k})$, by Lemma 2.5, the inclusion $Z_{\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^\Sigma)^{\mathrm{geo}}}(H) \subseteq \mathrm{im}(\iota^\Sigma_{1,1}) \times \mathrm{Dehn}(\mathcal{G}_\Sigma)$ holds. Thus, to verify the first portion of Proposition 2.6, it suffices to verify that $Z_{\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^\Sigma)^{\mathrm{geo}}}(H) \cap \mathrm{Dehn}(\mathcal{G}_\Sigma)$ is trivial. Write $p^{\mathrm{ab}}\colon \mathrm{im}(\rho^\Sigma_{1,1/k}) \to \mathrm{Aut}((\Delta^\Sigma_{1,1})^{\mathrm{ab}})$ for the homomorphism determined by the natural surjection $\Delta^\Sigma_{1,1} \twoheadrightarrow (\Delta^\Sigma_{1,1})^{\mathrm{ab}}$. Note that $(\Delta^\Sigma_{1,1})^{\mathrm{ab}}$ is a free $\hat{\mathbb{Z}}^\Sigma$-module (cf. [16, Remark 1.2.2]). (Here, $\hat{\mathbb{Z}}^\Sigma$ is the pro-Σ completion of the ring of rational integers $\mathbb{Z}$.) Then it follows from [8, Proposition 5.6, (ii)], and the well-known criterion of the reduction of an elliptic curve that the action of $\mathrm{Dehn}(\mathcal{G}_\Sigma)$ on $(\Delta^\Sigma_{1,1})^{\mathrm{ab}}$ is *faithful* and *unipotent*. Thus, to verify the first portion of Proposition 2.6, it suffices to verify that $Z_{\mathrm{im}(p^{\mathrm{ab}})}(p^{\mathrm{ab}}(H)) \cap p^{\mathrm{ab}}(\mathrm{Dehn}(\mathcal{G}_\Sigma))$ is trivial. Now it is well-known that, by choosing a suitable basis of the free $\hat{\mathbb{Z}}^\Sigma$-module $(\Delta^\Sigma_{1,1})^{\mathrm{ab}}$, we may identify $\mathrm{im}(p^{\mathrm{ab}})$ with $SL_2(\hat{\mathbb{Z}}^\Sigma)$. In particular, since $p^{\mathrm{ab}}(H)$ is open in $SL_2(\hat{\mathbb{Z}}^\Sigma)$, we obtain the equality

$$Z_{SL_2(\hat{\mathbb{Z}}^\Sigma)}(p^{\mathrm{ab}}(H)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Therefore, since the action of $\mathrm{Dehn}(\mathcal{G}_\Sigma)$ on $(\Delta^\Sigma_{1,1})^{\mathrm{ab}}$ is *unipotent*, the profinite group $Z_{\mathrm{im}(p^{\mathrm{ab}})}(p^{\mathrm{ab}}(H)) \cap p^{\mathrm{ab}}(\mathrm{Dehn}(\mathcal{G}_\Sigma))$ is *trivial*. This completes the proof of the first portion of Proposition 2.6.

Finally, the final portion of Proposition 2.6 follows from the first portion of Proposition 2.6. This completes the proof of Proposition 2.6. □

Remark. If $(g, r) \neq (1, 1)$, then it is already known that the profinite groups $\mathrm{im}(\rho^\Sigma_{g,r/k})$ and $\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{g,r})^\Sigma)^{\mathrm{geo}}$ are *almost slim* (cf. [8, Theorem D, (i)]). Also, if $2 \in \Sigma$, and k is algebraically closed, then the almost slimness of $\mathrm{im}(\rho^\Sigma_{1,1/k})$ follows from the fact that a pro-Σ version of the congruence subgroup problem of mapping class groups of genus 1 has an affirmative answer (cf. [1, Theorem 5] and [6, Theorem A, (i)]), the fact that an almost pro-Σ quotient of $\pi_1((\mathcal{M}_{1,1})_k)$ has an open subgroup which is isomorphic to a pro-Σ surface group, and the fact that any pro-Σ surface group is slim (cf., e.g., [16, Proposition 1.4]).

Lemma 2.7. *Suppose that $n \geq 3$, and that k is algebraically closed. Write Γ for either $\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^{\Sigma})^{\mathrm{geo}}$ or $\mathrm{im}(\rho^{\Sigma}_{1,1/k})$. Then there does not exist a closed subgroup H of Γ such that the equality $\mathrm{im}(\iota^{\Sigma}_{1,1}) \times H = \Gamma$ holds.*

Proof. First, we verify that $SL_2(\mathbb{Z}) \twoheadrightarrow PSL_2(\mathbb{Z})$ does not have a section. Assume that $SL_2(\mathbb{Z}) \twoheadrightarrow PSL_2(\mathbb{Z})$ has a section $s\colon PSL_2(\mathbb{Z}) \to SL_2(\mathbb{Z})$. Then since $SL_2(\mathbb{Z})$ is equal to $\mathrm{im}(s) \times Z(SL_2(\mathbb{Z}))$, $SL_2(\mathbb{Z})^{\mathrm{ab}}$ is isomorphic to $\mathrm{im}(s)^{\mathrm{ab}} \times \mathbb{Z}/2$. Here, we denote by $SL_2(\mathbb{Z})^{\mathrm{ab}}$, $\mathrm{im}(s)^{\mathrm{ab}}$ the abelianizations of $SL_2(\mathbb{Z})$, $\mathrm{im}(s)$, respectively. This contradicts the well-known fact that $SL_2(\mathbb{Z})^{\mathrm{ab}} \simeq \mathbb{Z}/12 \simeq \mathbb{Z}/3 \times \mathbb{Z}/4$ (cf., e.g., [2, p.123]). This completes the proof of the assertion that $SL_2(\mathbb{Z}) \twoheadrightarrow PSL_2(\mathbb{Z})$ does not have a section. Next, assume that there exists a closed subgroup H such that the equality $\mathrm{im}(\iota^{\Sigma}_{1,1}) \times H = \Gamma$ holds. Note that the composite

$$SL_2(\mathbb{Z}) \to \pi_1((\mathcal{M}_{1,1})_k) \to \Gamma$$

of the outer homomorphism $SL_2(\mathbb{Z}) \to \pi_1((\mathcal{M}_{1,1})_k)$ arising from a natural outer isomorphism $SL_2(\mathbb{Z})^{\wedge} \xrightarrow{\sim} \pi_1((\mathcal{M}_{1,1})_{\overline{k}})$, where we write $SL_2(\mathbb{Z})^{\wedge}$ for the profinite completion of $SL_2(\mathbb{Z})$, and $\rho^{\Sigma}_{1,1/k}$ is *injective*. Also, note that the image of the centralizer of $SL_2(\mathbb{Z})$ via this injection is contained in $\mathrm{im}(\iota^{\Sigma}_{1,1})$. Thus, since the cardinality of $Z(SL_2(\mathbb{Z}))$, and $\mathrm{im}(\iota^{\Sigma}_{1,1})$ are equal to 2, we have a *cartesian* diagram of groups

$$\begin{tikzcd} SL_2(\mathbb{Z}) \arrow[r, hook] \arrow[d, two heads] & \Gamma \arrow[d, two heads] \\ PSL_2(\mathbb{Z}) \arrow[r, hook] & H. \end{tikzcd}$$

Therefore, since $SL_2(\mathbb{Z}) \twoheadrightarrow PSL_2(\mathbb{Z})$ does not have a section, $\Gamma \twoheadrightarrow H$ does not have a section. This *contradicts* the definition of H. This completes the proof of Lemma 2.7. ☐

Theorem 2.8. *Suppose that $n \geq 3$, and that k is algebraically closed. Then $\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^{\{l\}})^{\mathrm{geo}}$ and $\mathrm{im}(\rho^{\{l\}}_{1,1/k})$ are almost strongly indecomposable.*

Moreover, in this case, $\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^{\{l\}})^{\mathrm{geo}}$ and $\mathrm{im}(\rho^{\{l\}}_{1,1/k})$ are indecomposable.

Proof. First, we verify the first portion of Theorem 2.8. To verify the first portion of Theorem 2.8, it suffices to verify the indecomposability of any open subgroup Γ of either $\mathrm{Out}^{\mathrm{FC}}((\Delta^n_{1,1})^{\{l\}})^{\mathrm{geo}}$ or $\mathrm{im}(\rho^{\{l\}}_{1,1/k})$ which does *not* contain the finite group $\mathrm{im}(\iota^{\{l\}}_{1,1})$. Assume that there exist *nontrivial* profinite groups H, K, and an isomorphism of profinite groups $H \times K \xrightarrow{\sim} \Gamma$. In the following, we shall identify Γ with $H \times K$ via this isomorphism. Then $I := \Gamma \cap \mathrm{Dehn}(\mathcal{G}_{\{l\}})$ is an open subgroup of $\mathrm{Dehn}(\mathcal{G}_{\{l\}})$. If $I \cap H = \{1\}$ and $I \cap K = \{1\}$, then by considering the restriction of natural projections $\Gamma \twoheadrightarrow H$, $\Gamma \twoheadrightarrow K$ to $I(\simeq \mathbb{Z}_l)$, we have a free $\mathbb{Z}_l$-module of rank 2 as a subgroup of $Z_{\Gamma}(I)$. Thus, since $\mathrm{Dehn}(\mathcal{G}_{\{l\}}) \simeq \mathbb{Z}_l$, we conclude from Lemma 2.5 that either $I \cap H \neq \{1\}$ or $I \cap K \neq \{1\}$. Therefore, we may assume without loss of generality that $I \cap H \neq \{1\}$, hence also H contains an open subgroup of I. Now by Lemma 2.5, K is contained in I. Since K is nontrivial, this contradicts that $H \cap K = \{1\}$. This completes the proof of the first portion of Theorem 2.8.

Next, we verify the final portion of Theorem 2.8. Write $\mathfrak{G}$ for the profinite group either $\mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^{n})^{\Sigma})^{\mathrm{geo}}$ or $\mathrm{im}(\rho_{1,1/k}^{\{l\}})$. Assume that there exist *nontrivial* profinite groups L, M, and an isomorphism of profinite groups $L \times M \xrightarrow{\sim} \mathfrak{G}$. In the following, we shall identify $\mathfrak{G}$ with $L \times M$ via this isomorphism. Then by the first portion of Theorem 2.8, either L or M is finite. Thus, we may assume without loss of generality that L is *finite*. In particular, since K is *open* in $\mathfrak{G}$, by Proposition 2.6, the inclusion $L \subseteq \mathrm{im}(\iota_{1,1}^{\{l\}})$ holds. However, since $\mathrm{im}(\iota_{1,1}^{\{l\}}) \simeq \mathbb{Z}/2$, and L is nontrivial, this *contradicts* Lemma 2.7. This completes the proof of Theorem 2.8. $\square$

Remark. (i) Note that $\mathrm{im}(\rho_{1,1/k}^{\{l\}})$ is *not* strongly indecomposable. Indeed, let $U \subseteq \mathrm{im}(\rho_{1,1/k}^{\{l\}})$ be an open subgroup of $\mathrm{im}(\rho_{1,1/k}^{\{l\}})$ such that $U \cap \mathrm{im}(\iota_{1,1}^{\Sigma}) = \{1\}$. Then $U \times \mathrm{im}(\iota_{1,1}^{\Sigma}) \subseteq \mathrm{im}(\rho_{1,1/k}^{\{l\}})$ is open in $\mathrm{im}(\rho_{1,1/k}^{\{l\}})$, and *not* indecomposable.

(ii) Suppose that $n \geq 3$. Write

$$\mathrm{Out}_{\mathrm{Z}}^{\mathrm{FC}}((\Delta_{1,1}^{n})^{\{l\}})^{\mathrm{geo}} := Z_{\mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^{n})^{\{l\}})^{\mathrm{geo}}}(\mathrm{im}(\iota_{1,1}^{\{l\}})).$$

Now we have inclusions

$$\mathrm{im}(\rho_{1,1/\overline{k}}^{\{l\}}) \overset{\simeq ?}{\hookrightarrow} \mathrm{Out}_{\mathrm{Z}}^{\mathrm{FC}}((\Delta_{1,1}^{n})^{\{l\}})^{\mathrm{geo}} \overset{\simeq ?}{\hookrightarrow} \mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^{n})^{\{l\}})^{\mathrm{geo}}.$$

Then by the argument used in the proof of Theorem 2.8, we may check that the profinite group $\mathrm{Out}_{\mathrm{Z}}^{\mathrm{FC}}((\Delta_{1,1}^{n})^{\{l\}})^{\mathrm{geo}}$ is *indecomposable* and *almost strongly indecomposable*.

Corollary 2.9. *Suppose that $n \geq 3$, and that k is a number field. Then the profinite groups $\mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^{n})^{\{l\}})$ and $\mathrm{im}(\rho_{1,1/k}^{\{l\}})$ are indecomposable and almost strongly indecomposable.*

Proof. To verify Corollary 2.9, by Theorem 2.8, for any open subgroup Π of either $\mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^{n})^{\{l\}})$ or $\mathrm{im}(\rho_{1,1/k}^{\{l\}})$ such that $\Gamma := \ker(\mathfrak{T}_{1,1}^{n,\{l\}}) \cap \Pi$ is indecomposable, it suffices to verify that Π is indecomposable. Write $G := \mathfrak{T}_{1,1}^{n,\{l\}}(\Pi)$. Then by the definition of Π, Lemma 2.1, and Theorem 2.2, G is slim and strongly indecomposable. Thus, since Π is indecomposable, by [12, Proposition 1.8, (i)], to verify the indecomposability of Π, it suffices to verify that the image of the outer representation $G \to \mathrm{Out}(\Gamma)$ associated to the natural exact sequence of profinite groups

$$1 \longrightarrow \Gamma \longrightarrow \Pi \xrightarrow{\mathfrak{T}_{1,1}^{n,\{l\}}} G \longrightarrow 1$$

is nontrivial. Since Π is open in either $\mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^{n})^{\{l\}})$ or $\mathrm{im}(\rho_{1,1/k}^{\{l\}})$, by [8, Remark 3.8.1; Theorem 4.8, (v); Theorem 5.14, (ii)], and the nontriviality of the image of the l-adic cyclotomic character of the absolute Galois group of any number field, there exists $\sigma \in \Pi$ such that the automorphism of Γ determined by the conjugation of σ *preserves* $\mathrm{Dehn}(\mathcal{G}_{\{l\}}) \cap \Gamma$ and the restriction of this automorphism to $\mathrm{Dehn}(\mathcal{G}_{\{l\}}) \cap \Gamma$ is *nontrivial*. On the other hand, by Lemma 2.5, any inner automorphism of Γ either does not preserve $\mathrm{Dehn}(\mathcal{G}_{\{l\}}) \cap \Gamma$ or acts trivially on $\mathrm{Dehn}(\mathcal{G}_{\{l\}}) \cap \Gamma$. Thus, the image of $\mathfrak{T}_{1,1}^{n,\{l\}}(\sigma) \in G$ via the outer representation $G \to \mathrm{Out}(\Gamma)$ is *nontrivial*. This completes the proof of Corollary 2.9. $\square$

Remark. Suppose that $n \geq 3$. Write

$$\mathrm{Out}_{\mathrm{Z}}^{\mathrm{FC}}((\Delta_{1,1}^n)^{\{l\}}) := Z_{\mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^n)^{\{l\}})}(\mathrm{im}(\iota_{1,1}^{\{l\}})).$$

Now we have inclusions

$$\mathrm{im}(\rho_{1,1/\mathbb{Q}}^{\{l\}}) \overset{\cong ?}{\hookrightarrow} \mathrm{Out}_{\mathrm{Z}}^{\mathrm{FC}}((\Delta_{1,1}^n)^{\{l\}}) \overset{\cong ?}{\hookrightarrow} \mathrm{Out}^{\mathrm{FC}}((\Delta_{1,1}^n)^{\{l\}}).$$

Then by the argument used in the proof of Corollary 2.9, we may check that the profinite group $\mathrm{Out}_{\mathrm{Z}}^{\mathrm{FC}}((\Delta_{1,1}^n)^{\{l\}})$ is *indecomposable* and *almost strongly indecomposable*.

Lemma 2.10. *Let G be a slim profinite group which is almost strongly indecomposable. Then G is indecomposable.*

Proof. Assume that there exist *nontrivial* profinite groups H, K, and an isomorphism of profinite groups $H \times K \xrightarrow{\sim} G$. In the following, we shall identify G with $H \times K$ via this isomorphism. Then since G is almost strongly indecomposable, either H or K is *finite*. Thus, we may assume without loss of generality that H is *finite*. Therefore, since $H \times K = G$, K is an *open* subgroup of G. Then it follows from the slimness of G that $H \subseteq Z_G(K)$ is *trivial*. This *contradicts* that H is nontrivial. This completes the proof of Lemma 2.10. $\square$

Lemma 2.11. *Suppose that k is a number field. Let H be an open subgroup of $\mathrm{im}(\rho_{1,1/k}^{\{l\}})$. Then the equality*

$$\mathrm{im}(\iota_{1,1}^{\{l\}}) = Z_{\mathrm{im}(\rho_{1,1/k}^{\{l\}})}(H)$$

holds.

Proof. Note that, since $\mathrm{im}(\rho_{0,3/k}^{\{l\}})$ is *slim* (cf. Lemma 2.1), by Theorem 1.2, (ii), $Z_{\mathrm{im}(\rho_{1,1/k}^{\{l\}})}(H)$ is *contained in* the profinite group $\mathrm{im}(\rho_{1,1/\overline{k}}^{\{l\}})$ $(= \ker(\mathrm{im}(\rho_{1,1/k}^{\{l\}}) \twoheadrightarrow \mathrm{im}(\rho_{0,3/k}^{\{l\}})))$. Thus, Lemma 2.11 follows from the first portion of Lemma 2.4, and Proposition 2.6. $\square$

Corollary 2.12. *Let m be a positive integer. Suppose that k is either a number field or algebraically closed. Then $\mathrm{im}(\rho_{1,m/k}^{\{l\}})$ is*

$$\begin{cases} \textit{almost strongly indecomposable} & \textit{if } m \leq 2, \textit{ and } l = 2, \\ \textit{indecomposable and almost strongly indecomposable} & \textit{if } m \leq 2, \textit{ and } l \neq 2, \\ \textit{strongly indecomposable} & \textit{if } m \geq 3. \end{cases}$$

Proof. First, we verify that $\mathrm{im}(\rho^{\{l\}}_{1,m/k})$ is almost strongly indecomposable by induction on m. If $m = 1$, then the almost strongly indecomposability of $\mathrm{im}(\rho^{\{l\}}_{1,m/k})$ follows from Theorem 2.8 and Corollary 2.9. Now suppose that $m > 1$, and that the induction hypothesis is in force. Then by induction, Proposition 2.6, Lemma 2.11, and [8, Theorem D, (i)], we may find an open subgroup U_1 of $\mathrm{im}(\rho^{\{l\}}_{1,m-1/k})$ which is slim and strongly indecomposable. Also, by [5, Lemma 20], and [8, Theorem 6.12, (i)], there exist an exact sequence of profinite groups

$$1 \longrightarrow \varDelta^{\{l\}}_{1,m-1} \longrightarrow \mathrm{im}(\rho^{\{l\}}_{1,m/k}) \longrightarrow \mathrm{im}(\rho^{\{l\}}_{1,m-1/k}) \longrightarrow 1,$$

and an open subgroup U_2 of $\mathrm{im}(\rho^{\{l\}}_{1,m/k})$ such that, for any open subgroup U' of $\varDelta^{\{l\}}_{1,m-1}$, $Z_{U_2}(U' \cap U_2)$ is trivial. Write U_3 for the intersection of U_2 and the inverse image of U_1 via $\mathrm{im}(\rho^{\{l\}}_{1,m/k}) \twoheadrightarrow \mathrm{im}(\rho^{\{l\}}_{1,m-1/k})$. We verify that U_3 is strongly indecomposable. Let V be an open subgroup U_3. Write V' for the image of V via $\mathrm{im}(\rho^{\{l\}}_{1,m/k}) \twoheadrightarrow \mathrm{im}(\rho^{\{l\}}_{1,m-1/k})$, and V'' for the intersection of V and $\varDelta^{\{l\}}_{1,m-1}$. Thus, there exists an exact sequence of profinite groups

$$1 \longrightarrow V'' \longrightarrow V \longrightarrow V' \longrightarrow 1,$$

such that $Z_V(V'')$ is trivial. In particular, the outer representation $V' \to \mathrm{Out}(V'')$ associated to this exact sequence of profinite groups is *injective*. Therefore, since $\varDelta^{\{l\}}_{1,m-1}$ is strongly indecomposable (cf. [16, Proposition 3.2]), by [12, Proposition 1.8, (i)], V is *indecomposable*. This implies that U_3 is strongly indecomposable, hence also the assertion that $\mathrm{im}(\rho^{\{l\}}_{1,m/k})$ is almost strongly indecomposable.

Next, we verify Corollary 2.12 in the case where $m \leq 2$, and $l \neq 2$. If $m = 1$, then the indecomposability of $\mathrm{im}(\rho^{\{l\}}_{1,m/k})$ follows from Theorem 2.8 and Corollary 2.9. Assume that there exist *nontrivial* profinite groups H, K, and an isomorphism of profinite groups $H \times K \xrightarrow{\sim} \mathrm{im}(\rho^{\{l\}}_{1,2/k})$. In the following, we shall identify $\mathrm{im}(\rho^{\{l\}}_{1,2/k})$ with $H \times K$ via this isomorphism. Then by the almost strongly indecomposability of $\mathrm{im}(\rho^{\{l\}}_{1,2/k})$, either H or K is finite. Thus, we may assume without loss of generality that H is *finite*. In particular, since K is open in $\mathrm{im}(\rho^{\{l\}}_{1,2/k})$, and H is nontrivial, by [8, Theorem D, (i)], the cardinality of H is equal to 2. In particular, since $l \neq 2$, the image of the composite of natural homomorphisms

$$\varDelta^{\{l\}}_{1,1} \hookrightarrow \mathrm{im}(\rho^{\{l\}}_{1,2/k}) \twoheadrightarrow H$$

is *trivial*. Therefore, by means of the first display of this proof, $\mathrm{im}(\rho^{\{l\}}_{1,1/k})$ is *isomorphic to* $H \times (K/\varDelta^{\{l\}}_{1,1})$. This *contradicts* the indecomposability of $\mathrm{im}(\rho^{\{l\}}_{1,1/k})$. This completes the proof of Corollary 2.12 in the case where $m \leq 2$, and $l \neq 2$.

Finally, Corollary 2.12 in the case where $m \geq 3$ follows from the slimness of $\mathrm{im}(\rho^{\{l\}}_{1,m/k})$ (cf. [8, Theorem D, (i)]) and Lemma 2.10. This completes the proof of Corollary 2.12. $\square$

3. Appendix: The indecomposability of the étale fundamental group of the moduli stack of once-punctured elliptic curves

In this §3, we prove the indecomposability of $\pi_1((\mathcal{M}_{1,1})_k)$. In this §3, suppose that Σ is *the set of prime numbers.*

Lemma 3.1. *Let m be a positive integer. Suppose that k is either a number field or algebraically closed. Then $\rho^{\Sigma}_{1,m/k}$ is injective.*

Proof. Lemma 3.1 follows from [1, Theorem 2; Theorem 5], and [7, Corollary 6.5]. $\square$

In the rest of this paper, by means of Lemma 3.1, if k is either a number field or algebraically closed, then for a positive integer m, we shall identify the profinite group $\pi_1((\mathcal{M}_{1,m})_k)$ with $\mathrm{im}(\rho^{\Sigma}_{1,m/k}) \subseteq \mathrm{Out}(\Delta^{\Sigma}_{1,m})$.

Lemma 3.2. *Suppose that k is either a number field or algebraically closed. Let H be an open subgroup of $\pi_1((\mathcal{M}_{1,1})_k)$. Then the homomorphism of profinite groups $\iota^{\Sigma}_{1,1}\colon \mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{M}_{1,2})_k) \to \mathrm{Out}(\Delta^{\Sigma}_{1,1})$ determines an isomorphism of profinite groups*

$$\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{M}_{1,2})_k) \xrightarrow{\sim} Z_{\pi_1((\mathcal{M}_{1,1})_k)}(H) \subseteq \mathrm{Out}(\Delta^{\Sigma}_{1,1}).$$

Proof. Note that, if k is a number field, then G_k is *slim* (cf., e.g., [17, (12.1.5) Proposition]). Therefore, $Z_{\pi_1((\mathcal{M}_{1,1})_k)}(H)$ is *contained in* the profinite group $\pi_1((\mathcal{M}_{1,1})_{\overline{k}})$ $(= \ker(\pi_1((\mathcal{M}_{1,1})_k) \twoheadrightarrow G_k))$. Thus, Lemma 3.2 follows from the first portion of Lemma 2.4, and Proposition 2.6. $\square$

Lemma 3.3. *Suppose that k is either a number field or algebraically closed. Then the natural surjection $\pi_1((\mathcal{M}_{1,1})_k) \twoheadrightarrow \pi_1((\mathcal{M}_{1,1})_k)/Z(\pi_1((\mathcal{M}_{1,1})_k))$ does not have a section.*

Proof. Lemma 3.3 follows from a similar argument to the argument used in the proof of Lemma 2.7 by replacing Γ (resp. Proposition 2.6) by $\pi_1((\mathcal{M}_{1,1})_k)$ (resp. Lemma 3.2) in the proof of Lemma 2.7. $\square$

Theorem 3.4. *Suppose that k is either a number field or algebraically closed. Then $\pi_1((\mathcal{M}_{1,1})_k)$ is indecomposable and almost strongly indecomposable.*

Proof. First, we verify the almost strongly indecomposability of $\pi_1((\mathcal{M}_{1,1})_k)$. Now it is well-known that there exists a finite étale covering Y of $(\mathcal{M}_{1,1})_k$ which is representable by a *hyperbolic curve.* Since the étale fundamental group $\pi_1(Y)$ of Y is *strongly indecomposable* (cf. [12, Theorem 2.1; Corollary 3.8, (ii)]), $\pi_1((\mathcal{M}_{1,1})_k)$ (which contains $\pi_1(Y)$ as an open subgroup) is *almost strongly indecomposable.* This completes the proof of the almost strongly indecomposability of $\pi_1((\mathcal{M}_{1,1})_k)$.

Finally, the indecomposability of $\pi_1((\mathcal{M}_{1,1})_k)$ follows from a similar argument to the argument used in the proof of the final portion of Theorem 2.8 by replacing $\mathrm{im}(\rho^{\{l\}}_{1,1/k})$ (resp. Proposition 2.6; the first portion of Theorem

2.8; Lemma 2.7) by $\pi_1((\mathcal{M}_{1,1})_k)$ (resp. Lemma 3.2; the almost strongly indecomposability of $\pi_1((\mathcal{M}_{1,1})_k)$; Lemma 3.3) in the proof of the final portion of Theorem 2.8. $\square$

Remark. Write $(\mathcal{A}_g)_k$ for the moduli stack of principally polarized abelian varieties of dimension g over k. Note that, if $g > 1$, and k is algebraically closed, then the étale fundamental group $\pi_1((\mathcal{A}_g)_k)$ of $(\mathcal{A}_g)_k$ is *neither indecomposable nor almost strongly indecomposable.* Indeed, there exists a natural outer isomorphism

$$\pi_1((\mathcal{A}_g)_k) \xrightarrow{\sim} \prod_{p \in \Sigma} Sp_{2g}(\mathbb{Z}_p)$$

(cf., e.g., [10, (3.1)]). Thus, a result similar to the results stated in Theorem 3.4 *does not hold for the moduli stack of principally polarized abelian varieties of dimension* $g > 1$.

Corollary 3.5. *Let m be a positive integer. Suppose that k is either a number field or algebraically closed. Then $\pi_1((\mathcal{M}_{1,m})_k)$ is*

$$\begin{cases} \textit{almost strongly indecomposable} & \textit{if } m \leq 2, \\ \textit{strongly indecomposable} & \textit{if } m \geq 3. \end{cases}$$

Proof. Corollary 3.5 follows from a similar argument to the argument used in the first paragraph and the final paragraph of the proof of Corollary 2.12 by replacing $\mathrm{im}(\rho^{\{l\}}_{1,m/k})$ (resp. Theorem 2.8 and Corollary 2.9; Proposition 2.6 and Lemma 2.11; $\Delta^{\{l\}}_{1,m-1}$; $\mathrm{im}(\rho^{\{l\}}_{1,m-1/k})$) by $\pi_1((\mathcal{M}_{1,m})_k)$ (resp. Theorem 3.4; Lemma 3.2; $\Delta^{\Sigma}_{1,m-1}$; $\pi_1((\mathcal{M}_{1,m-1})_k)$) in the first paragraph and the final paragraph of the proof of Corollary 2.12. $\square$

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