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Abstract. In this article we survey the fundamental theory of the exact WKB analysis, that is, the WKB analysis based on the Borel resummation method. Starting with the exact WKB analysis for second order linear ordinary differential equations, we explain its application to the computation of monodromy groups of Fuchsian equations and its generalization to higher order equations. Some recent developments of the theory such as the exact WKB analysis for completely integrable systems are also briefly discussed.

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1. Introduction

Since the very beginning of the quantum mechanics, the WKB (Wentzel-Kramers-Brillouin) approximation has been employed to obtain approximate eigenfunctions and solve the eigenvalue problems for Schrödinger equations. The (full-order) WKB approximations provide formal solutions (with respect to the Planck constant) of Schrödinger equations but, as they are divergent in almost all cases, they were not so often used in rigorous mathematical analysis. Around 1980, using the Borel resummed WKB solutions, Voros ([36]) successfully studied spectral functions of quartic oscillators and also Silverstone ([33]) discussed the WKB-type connection problem more rigorously. After their pioneering works, Pham, Delabaere and others (cf., e.g., [30], [9], [11], [12], [13]) have developed this new kind of WKB analysis (sometimes called "exact WKB analysis" or "complex WKB analysis") based on

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the Borel resummation technique with the aid of Ecalle's theory of resurgent functions ([14], [15], [16], see also [32]). At present it turns out that the exact WKB analysis is very efficient not only for eigenvalue problems of Schrödinger equations but also for the global study of differential equations in the complex domain.

In this article, mainly using some concrete and illuminating examples, we explain the fundamental theory of the exact WKB analysis, its application to the global study of differential equations in the complex domain, and some recent developments of the theory.

The explanation will be done basically by following our monographs [27] and [25]. To be more specific, the article is organized as follows: We first discuss the exact WKB analysis for second order linear ordinary differential equations of Schrödinger type

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi = 0, \tag{1.1}$$

where $Q(x)$ is a polynomial or a rational function and η denotes the inverse of the Planck constant (and hence a large parameter). Starting with the definition of WKB solutions, we introduce the Stokes geometry and explain the fundamental theorems of the exact WKB analysis, in particular, Voros' connection formula for Borel resummed WKB solutions, the most important result in the theory, in Section 2. Then, after illustrating an outline of the proof of the fundamental theorems in Section 3, we discuss its application to the computation of monodromy groups of Fuchsian equations (Section 4) and wall crossing formulas for WKB solutions with respect to the change of parameters contained in the equation (Section 5). In the latter part of the article, we consider generalizations of the exact WKB analysis to higher order linear ordinary differential equations of the form

$$\left(\frac{d^m}{dx^m} + \eta p_1(x)\frac{d^{m-1}}{dx^{m-1}} + \cdots + \eta^m p_m(x)\right)\psi = 0. \tag{1.2}$$

In Section 6 we discuss the problem of new Stokes curves pointed out by Berk-Nevins-Roberts ([8]), which is peculiar to higher order equations, and introduce the notion of virtual turning points with the help of the theory of microlocal analysis to treat new Stokes curves in a more intrinsic manner. Finally, in Section 7, we explain some recent developments of the theory such as the exact WKB analysis for completely integrable systems. These recent developments are also closely related to the problem of new Stokes curves and virtual turning points for higher order equations.

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2. Exact WKB analysis for second order linear ODEs of Schrödinger type

2.1. WKB solutions

Let us first discuss the exact WKB analysis for second order linear ordinary differential equations of Schrödinger type

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi = 0, \tag{2.1}$$

where $Q(x)$ is a polynomial or a rational function. Throughout this article η denotes a large parameter and is often assumed to be real and positive.

Definition 2.1. A WKB solution of (2.1) is a formal solution of the following form:

$$\psi(x,\eta) = \exp(\eta y_0(x)) \sum_{n=0}^{\infty} \psi_n(x)\eta^{-(n+\alpha)}, \tag{2.2}$$

where $y_0(x)$ and $\psi_n(x)$ are suitable analytic functions of x and $\alpha \geq 0$ is a constant.

In the case of (2.1) WKB solutions can be readily constructed in the following way: Assume that a solution of (2.1) has the form

$$\psi(x,\eta) = \exp \int^x S(x,\eta)dx, \tag{2.3}$$

then $S(x,\eta)$ should satisfy

$$S^2 + \frac{dS}{dx} = \eta^2 Q(x) \qquad (\text{“Riccati equation”}). \tag{2.4}$$

We further suppose that $S = S(x,\eta)$ can be expanded as $S = \eta S_{-1}(x) + S_0(x) + \eta^{-1}S_1(x) + \cdots$. It then follows from (2.4) that

$$(S_{-1})^2 = Q(x), \tag{2.5}$$

$$2S_{-1}S_{n+1} + \sum_{k=0}^{n} S_k S_{n-k} + \frac{dS_n}{dx} = 0 \qquad (n = -1, 0, 1, \ldots). \tag{2.6}$$

That is, once $S_{-1} = \pm\sqrt{Q(x)}$ is fixed, we obtain two solutions $S_\pm(x,\eta)$ of (2.4) in a recursive manner.

Remark 2.2. Let us denote $S_\pm$ as $S_\pm = \pm S_{\mathrm{odd}} + S_{\mathrm{even}}$, then the following relation is readily confirmed.

$$2S_{\mathrm{odd}}S_{\mathrm{even}} + \frac{dS_{\mathrm{odd}}}{dx} = 0, \quad \text{i.e.,} \quad S_{\mathrm{even}} = -\frac{1}{2}\frac{d}{dx}\log S_{\mathrm{odd}}. \tag{2.7}$$

Thus for Eq. (2.1) we obtain the following WKB solutions:

$$\begin{aligned}\psi_\pm(x,\eta) &= \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_{x_0}^{x} S_{\mathrm{odd}}\,dx\right)\\ &= \exp\Big(\pm\eta\int_{x_0}^{x}\sqrt{Q(x)}\,dx\Big)\sum_{n=0}^{\infty}\psi_{\pm,n}(x)\eta^{-(n+1/2)},\end{aligned} \tag{2.8}$$

where x_0 is an arbitrarily chosen reference point.

Unfortunately WKB solutions are, in general, divergent. In fact, the following holds.

Proposition 2.3. (i) *Each $S_n(x)$ and $\psi_{\pm,n}(x)$ are holomorphic on*

$$U := \{\, x \in \mathbb{C} \mid Q(x) \text{ is holomorphic near } x \text{ and } Q(x) \neq 0 \,\}. \tag{2.9}$$

(ii) *For any compact set K in U, there exist positive constants A_K and C_K satisfying*

$$|S_n(x)| \le A_K C_K^n n!, \quad |\psi_{\pm,n}(x)| \le A_K C_K^n n! \quad (x \in K) \tag{2.10}$$

for any n.

To give an analytic meaning to WKB solutions, we employ the Borel resummation technique (or the Borel-Laplace method) with respect to a large parameter η in the exact WKB analysis.

Definition 2.4. Let $\eta > 0$ be a large parameter. For an infinite series $f = \exp(\eta y_0)$ $\sum_{n\ge 0} f_n \eta^{-(n+\alpha)}$ ($\alpha > 0$, y_0, f_n : constants), we define

$$f_B(y) = \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(n+\alpha)}(y+y_0)^{n+\alpha-1} \quad : \text{ Borel transform of } f, \tag{2.11}$$

$$F(\eta) = \int_{-y_0}^{\infty} e^{-y\eta} f_B(y)\,dy \quad : \text{ Borel sum of } f, \tag{2.12}$$

provided that they are well-defined. Here $\Gamma(s)$ denotes Euler's Γ-function and the integration path of (2.12) is taken to be parallel to the positive real axis.

See, e.g., [7], [10] for the details of the Borel-Laplace method. Here we only refer the following very fundamental properties of the Borel transform and the Borel sum.

Proposition 2.5. (i) *If f is convergent, then $(y+y_0)^{1-\alpha} f_B(y)$ is an entire function of exponential type. In this case the Borel sum $F(\eta)$ of f is well-defined for a sufficiently large $\eta > 0$ and coincides with the original f.*
(ii) *If f is Borel summable, that is,*

(a) $\sum \frac{f_n}{\Gamma(n+\alpha)}(y+y_0)^n$ *is convergent in a neighborhood of $y = -y_0$,*

(b) *$f_B(y)$ can be analytically continued along the integration path of the Borel sum, and*

(c) $\displaystyle\int_{-y_0}^{\infty} e^{-y\eta} f_B(y)dy$ *exists for a sufficiently large* $\eta > 0$,

then the following asymptotic formula holds :

$$\exp(-\eta y_0)\eta^{\alpha} F(\eta) \sim \sum_{n=0}^{\infty} f_n \eta^{-n} \qquad (\eta > 0,\ \eta \to \infty). \tag{2.13}$$

Proposition 2.6. *For* $\psi(x,\eta) = \exp(\eta y_0(x)) \sum_{n\geq 0} \psi_n(x)\eta^{-(n+\alpha)}$ $(\alpha > 0, \alpha \notin \mathbb{Z})$ *the following formulas hold :*

(i) $\displaystyle\Big[\frac{\partial}{\partial x}\psi\Big]_B = \frac{\partial}{\partial x}\psi_B.$

(ii) $\displaystyle\Big[\eta^m \psi\Big]_B = \Big(\frac{\partial}{\partial y}\Big)^m \psi_B \qquad (m = 1, 2, \ldots).$

(iii) $\displaystyle\Big[\eta^{-m}\psi\Big]_B = \frac{1}{(m-1)!}\int_{-y_0(x)}^{y} (y-y')^{m-1}\psi_B(x,y')dy' \ \Big(=: \Big(\frac{\partial}{\partial y}\Big)^{-m}\psi_B\Big).$

Furthermore, for $\psi = \sum \psi_n(x)\eta^{-(n+\alpha)}$ *and* $\varphi = \sum \varphi_n(x)\eta^{-(n+\beta)}$ $(\alpha, \beta > 0)$ *we have*

(iv) $\displaystyle\Big[\varphi\psi\Big]_B = \varphi_B * \psi_B := \int_0^y \varphi_B(x, y-y')\psi_B(x,y')dy'.$

2.2. WKB solutions of the Airy equation and their Borel transforms

To investigate properties of the Borel transform of WKB solutions, we consider WKB solutions of the Airy equation in this subsection.

Example. **(Airy equation)** Let us consider the Airy equation

$$\left(\frac{d^2}{dx^2} - \eta^2 x\right)\psi = 0 \tag{2.14}$$

and its WKB solutions normalized at $x = 0$

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\text{odd}}}}\exp\left(\pm\int_0^x S_{\text{odd}}dx\right) = \exp\big(\pm\eta y_0(x)\big)\sum_{n=0}^{\infty}\psi_{\pm,n}(x)\eta^{-(n+1/2)}, \tag{2.15}$$

where $y_0(x) = \int_0^x \sqrt{x}dx = (2/3)x^{3/2}$. We compute the Borel transform of (2.15) explicitly.

By the recursion formulas (2.5)-(2.6) we easily find that each coefficient $S_n(x)$ of the formal power series solution of the Riccati equation associated with (2.14) has the form $S_n = c_n x^{-1-(3/2)n}$ with some constant c_n $(n = -1, 0, 1, \ldots)$. This implies that each coefficient $\psi_{\pm,n}(x)$ of (2.15) also has the form $\psi_{\pm,n} = d_{\pm,n}x^{-1/4-(3/2)n}$ with another constant $d_{\pm,n}$ $(n = 0, 1, 2, \ldots)$. Hence the Borel transform $\psi_{\pm,B}(x,y)$ can be written as

$$\psi_{\pm,B}(x,y) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{d_{\pm,n}}{\Gamma(n+1/2)}\left(\frac{y}{x^{3/2}} \pm \frac{2}{3}\right)^{n-1/2} = \frac{1}{x}\phi_{\pm}(yx^{-3/2}) \tag{2.16}$$

with $\phi_\pm(t)$ being an analytic function of one variable $t = yx^{-3/2}$. On the other hand, since $\psi_\pm$ is a solution of (2.14), it follows from Proposition 2.6, (ii) that $\psi_{\pm,B}(x,y)$ should satisfy

$$\left(\frac{\partial^2}{\partial x^2} - x\frac{\partial^2}{\partial y^2}\right)\psi_{\pm,B}(x,y) = 0. \tag{2.17}$$

Consequently we obtain the following ODE for $\phi_\pm(t)$:

$$\left(\left(1-\frac{9}{4}t^2\right)\frac{d^2}{dt^2} - \frac{27}{4}t\frac{d}{dt} - 2\right)\phi_\pm = 0, \tag{2.18}$$

or, employing a change of variable $s = 3t/4 + 1/2$,

$$\left(s(1-s)\frac{d^2}{ds^2} + \left(\frac{3}{2} - 3s\right)\frac{d}{ds} - \frac{8}{9}\right)\phi_\pm = 0. \tag{2.19}$$

Eq. (2.19) is nothing but Gauss' hypergeometric equation (with the parameter $(\alpha,\beta,\gamma) = (4/3, 2/3, 3/2)$). Thus we have the following expression for $\psi_{\pm,B}(x,y)$:

$$\psi_{+,B}(x,y) = \frac{\sqrt{3}}{2\sqrt{\pi}}\,\frac{1}{x}\,s^{-1/2}F\left(5/6,\ 1/6,\ 1/2;\ s\right), \tag{2.20}$$

$$\psi_{-,B}(x,y) = \frac{\sqrt{3}}{2\sqrt{\pi}}\,\frac{1}{x}\,(1-s)^{-1/2}F\left(5/6,\ 1/6,\ 1/2;\ 1-s\right), \tag{2.21}$$

where $F(\alpha,\beta,\gamma;z)$ denotes Gauss' hypergeometric function and $s = 3yx^{-3/2}/4 + 1/2$.

Using this expression (2.20)-(2.21), we can deduce the following important properties of WKB solutions of the Airy equation.

Property (A). In addition to the reference point (singularity) $y = -y_0(x) = -(2/3)x^{3/2}$, $\psi_{+,B}(x,y)$ has a singularity also at $y = y_0(x) = (2/3)x^{3/2}$. This singularity is sometimes called a **"movable singularity"**, since its relative location with respect to the reference point moves according as x varies.

Property (B). The Borel sum $\Psi_+(x,\eta)$ is well-defined as long as $\Im(-y_0(x)) \neq \Im y_0(x)$, that is, provided that x does not belong to the set $\{\Im x^{3/2} = 0\}$, whereas it is not defined on $\{\Im x^{3/2} = 0\}$ where the movable singularity is located on the integration path of the Borel sum.

Property (C). The set $\{\Im x^{3/2} = 0\}$ defined above consists of three half-lines emanating from the origin in x-plane. If we consider the analytic continuation of the Borel sum $\Psi_+(x,\eta)$ across one of them, say, the positive real axis, then $\Psi_+(x,\eta)$ becomes the sum of the two Laplace integrals of $\psi_{+,B}(x,y)$ along Γ_j $(j = 0,1)$ described in Figure 1.

Property (D). After the analytic continuation across the positive real axis, the Borel sum $\Psi_+(x,\eta)$ becomes the following

$$\Psi_+ \rightsquigarrow \Psi_+ + i\Psi_-, \tag{2.22}$$

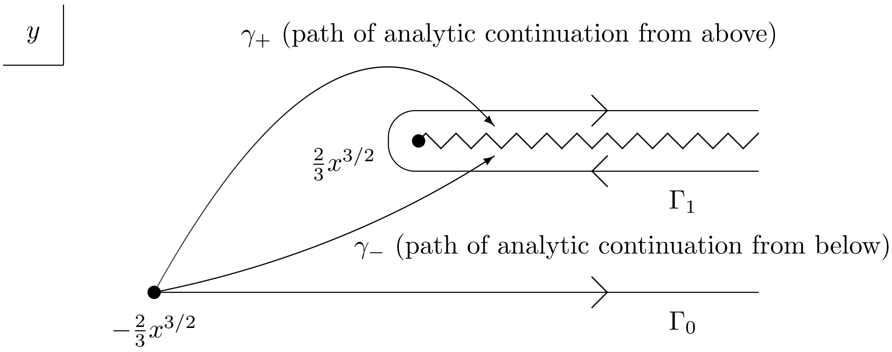

Figure 1 : Integration paths Γ_0 and Γ_1 (wiggly lines designate cuts to define a multi-valued analytic function $\psi_{+,B}(x,y)$).

that is, a Stokes phenomenon occurs for $\Psi_+(x,\eta)$ on the positive real axis. Formula (2.22) is often called the **"connection formula"** for $\Psi_+(x,\eta)$.

The connection formula (2.22) is a direct consequence of Property (C) and the following discontinuity formula for the Borel transform $\psi_{+,B}(x,y)$:

$$\triangle_{y=y_0(x)}\, \psi_{+,B}(x,y) = i\psi_{-,B}(x,y), \tag{2.23}$$

where the discontinuity (or the **"alien derivative"** in the sense of Ecalle) of $\psi_{+,B}(x,y)$ is defined as follows:

$$\triangle_{y=y_0(x)}\, \psi_{+,B}(x,y) := (\gamma_+)_*\psi_{+,B}(x,y) - (\gamma_-)_*\psi_{+,B}(x,y), \tag{2.24}$$

where $(\gamma_\pm)_*\psi_{+,B}$ denotes the analytic continuation of $\psi_{+,B}$ along $\gamma_\pm$, that is, the discontinuity is the difference between the analytic continuations of $\psi_{+,B}(x,y)$ above the cut and below the cut (cf. Figure 1). Note that in the case of the Airy equation the discontinuity formula (2.23) immediately follows from the expression (2.20)-(2.21) and Gauss' formula for hypergeometric functions:

$$\begin{aligned} s^{-1/2}F\,(5/6,\,1/6,\,1/2;\,s)\; &= \frac{1}{2}\,(1-s)^{-1/2}F\,(5/6,\,1/6,\,1/2;\,1-s) \\ &\quad + \frac{1}{\sqrt{3}}\,F\,(4/3,\,2/3,\,3/2;\,1-s) \end{aligned} \tag{2.25}$$

(cf. [17, p.105, 2.9(1)-2.9(24) and p.108, 2.10(1)]).

2.3. Stokes geometry and connection formula

Taking into account Properties (A)-(D) for the Airy equation observed in the preceding subsection, we introduce the notion of turning points and Stokes curves for (2.1) as follows:

Definition 2.7. (i) A zero of $Q(x)$ is called a turning point of (2.1). In particular, a simple zero of $Q(x)$ is called a simple turning point of (2.1).
(ii) A Stokes curve of (2.1) is, by definition, an integral curve of the vector field (or, more precisely, the direction field) $\Im \sqrt{Q(x)}dx = 0$ emanating from a turning point, that is, a curve defined by

$$\Im \int_a^x \sqrt{Q(x)}dx = 0, \tag{2.26}$$

where a is a turning point of (2.1).

In the case of the Airy equation (2.14) the origin is the unique turning point (which is simple) and the Stokes curves are given by $\{\, \Im x^{3/2} = 0 \,\}$.

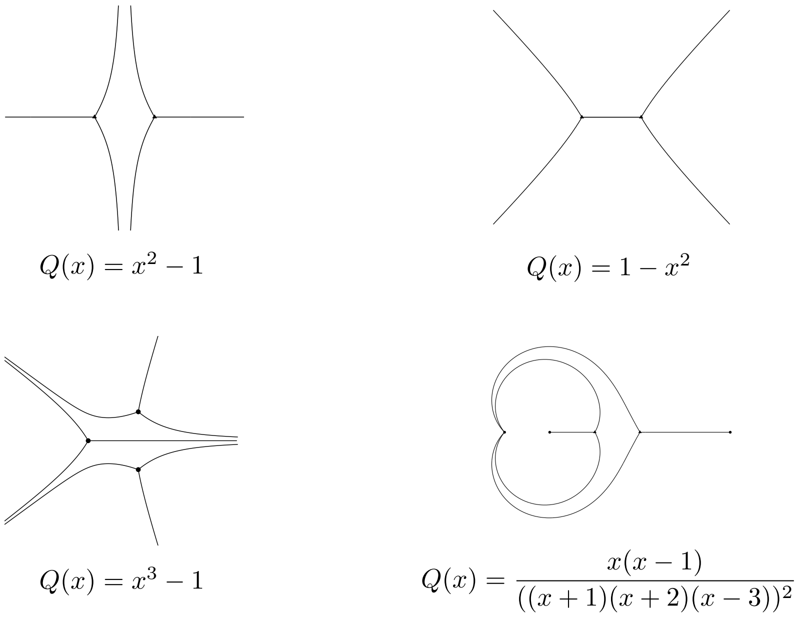

Figure 2 : Several examples of the Stokes geometry.

In what follows we usually assume the following non-degenerate condition:

Condition (ND). There is no Stokes curve of (2.1) which connects two turning points. In other words, every Stokes curve of (2.1) emanating from a turning point flows into a singular point of $Q(x)$.

Then, in parallel with the case of the Airy equation, the following fundamental theorems do hold for a second order equation (2.1) under Condition (ND).

Theorem 2.8. *Assume Condition (ND). Then WKB solutions $\psi_{\pm}(x, \eta)$ are Borel summable except on Stokes curves.*

Theorem 2.9. (Voros [36]) *Let $x = a$ be a simple turning point of* (2.1)*. Then, for WKB solutions*

$$\psi_{\pm}(x,\eta) = \frac{1}{\sqrt{S_{\rm odd}}} \exp\left(\pm \int_a^x S_{\rm odd} dx\right) \tag{2.27}$$

of (2.1) *normalized at $x = a$, the following properties hold in a neighborhood of $(x, y) = (a, 0)$:*

(i) $\psi_{\pm,B}(x, y)$ *have singularities at $y = \pm y_0(x) = \pm \int_a^x \sqrt{Q(x)}dx$ (cf. Figure 3).*

(ii)

$$\triangle_{y=y_0(x)}\, \psi_{+,B}(x, y) = i\psi_{-,B}(x, y), \quad \triangle_{y=-y_0(x)}\, \psi_{-,B}(x, y) = i\psi_{+,B}(x, y). \tag{2.28}$$

(iii) *On a Stokes curve Γ emanating from $x = a$ the following connection formula*

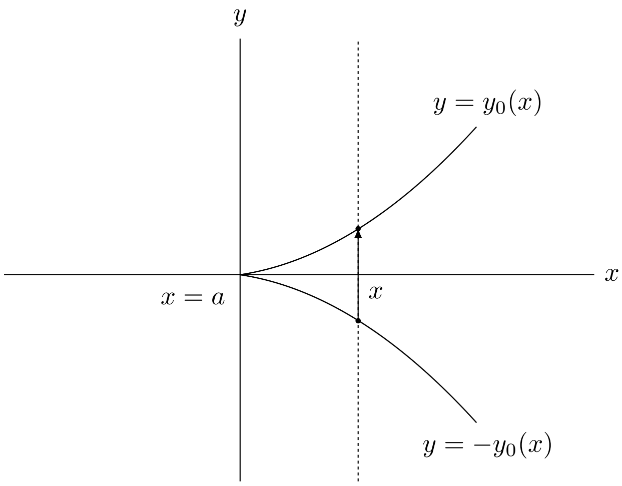

Figure 3 : Singularity locus of $\psi_{+,B}(x, y)$ near a simple turning point $x = a$.

holds for the Borel sums $\Psi_{\pm}$ of $\psi_{\pm}$.

Type (+) : *When $\Re \int_a^x \sqrt{Q(x)}dx > 0$ holds on Γ, that is, ψ_+ is dominant over ψ_- there, then*

$$\Psi_+ \rightsquigarrow \Psi_+ \pm i\Psi_-, \qquad \Psi_- \rightsquigarrow \Psi_-. \tag{2.29}$$

Type (−) : *When $\Re \int_a^x \sqrt{Q(x)}dx < 0$ holds on Γ, that is, ψ_- is dominant over ψ_+ there, then*

$$\Psi_+ \rightsquigarrow \Psi_+, \qquad \Psi_- \rightsquigarrow \Psi_- \pm i\Psi_+. \tag{2.30}$$

Here the sign $\pm$ depends on which direction one crosses the Stokes curve Γ. To be more precise, when one crosses Γ in an anticlockwise (resp., clockwise) manner viewed from $x = a$, we adopt the sign $+$ (resp., $-$).

3. Proof of the fundamental theorems

In this section we explain an outline of the proof of Theorems 2.8 and 2.9.

3.1. Proof of Theorem 2.8

First, following the argument of Koike-Schäfke [28] and using in part an idea of Costin [10], we explain the proof of Theorem 2.8.

The central step is to verify the Borel summability of formal power series solutions of the Riccati equation

$$S^2 + \frac{dS}{dx} = \eta^2 Q(x). \tag{3.1}$$

For the sake of simplicity we assume $Q(x)$ is a polynomial and consider the Borel summability of $S(x,\eta) = S_+(x,\eta)$ only. We now write $S(x,\eta)$ as

$$S(x,\eta) = \eta S_{-1}(x) + S_0(x) + T(x,\eta), \qquad T(x,\eta) = \sum_{n=1}^{\infty} S_n(x)\eta^{-n}. \tag{3.2}$$

Then $T(x,\eta)$ satisfies

$$\frac{dT}{dx} + 2S_{-1}(\eta T - S_1) + 2S_0 T + T^2 = 0. \tag{3.3}$$

In view of Proposition 2.6, the Borel transform T_B of T satisfies

$$\frac{\partial T_B}{\partial x} + 2\sqrt{Q(x)}\,\frac{\partial T_B}{\partial y} + 2S_0(x)T_B + T_B * T_B = 0, \quad T_B(x,0) = S_1(x). \tag{3.4}$$

Since holomorphic solutions of (3.4) are unique near $y = 0$, it suffices to show the existence of a global holomorphic solution of (3.4) near the positive real axis $\mathbb{R}_+ = \{\, y \geq 0 \,\}$.

Definition 3.1. Let K be a compact subset of $U = \{\, x \in \mathbb{C} \mid Q(x) \neq 0 \,\}$. For a point $x_0 \in K$ we define

$$\Gamma_{x_0} := \{\, x \in \mathbb{C} \mid \Im \int_{x_0}^{x} \sqrt{Q(x)}dx = 0 \,\},$$

$$\Gamma_{x_0}^{(\pm)} := \{\, x \in \Gamma_{x_0} \mid \Re \int_{x_0}^{x} \sqrt{Q(x)}dx \gtrless 0 \,\} \,\cup\, \{\, x_0 \,\},$$

$$\widehat{K} := \bigcup_{x \in K} \Gamma_x, \quad \widehat{K}^{(\pm)} := \bigcup_{x \in K} \Gamma_x^{(\pm)}.$$

Theorem 3.2. *Let $Q(x)$ be a polynomial of degree d and K be a compact subset of $U = \{\, x \in \mathbb{C} \mid Q(x) \neq 0 \,\}$. If $\widehat{K}^{(-)}$ does not contain a turning point in its closure, the following hold :*
(i) *$T_B(x,y)$ is holomorphic in $\Omega := \widehat{K}^{(-)} \times \{\, y \mid \mathrm{dist}(y, \mathbb{R}_+) < \delta \,\}$ for a sufficiently small number $\delta > 0$.*
(ii) *There exist positive constants C_1 and C_2 that satisfy*

$$\bigl|T_B(x,y)\bigr| \leq \frac{C_1}{1 + |x|^{d/2+2}}\, e^{C_2|y|} \quad \text{in } \Omega. \tag{3.5}$$

The proof of Theorem 3.2 consists of four steps.

Step 1. We employ the so-called Liouville transformation defined by

$$z(x) = \int_{x_0}^{x} \sqrt{Q(x)}\, dx. \tag{3.6}$$

Writing $T_B(x,y) = u(z,y)$, we find that $u(z,y)$ satisfies

$$\frac{\partial u}{\partial z} + 2\,\frac{\partial u}{\partial y} + 2A_1(z)u + A_2(z)u * u = 0, \qquad u(z,0) = A_0(z), \tag{3.7}$$

where

$$A_0(z(x)) = S_1(x), \quad A_1(z(x)) = \frac{S_0(x)}{\sqrt{Q(x)}}, \quad A_2(z(x)) = \frac{1}{\sqrt{Q(x)}}. \tag{3.8}$$

Step 2. By using a linear change of variables $s = 2z - y, t = y$ and integrating (3.7) once with respect to the variable t, we can convert (3.7) into the following integral equation:

$$\begin{aligned} u(z,y) \;&= A_0\Big(z - \frac{y}{2}\Big) - \int_0^y A_1\Big(z - \frac{y-y'}{2}\Big) u\Big(z - \frac{y-y'}{2}, y'\Big)\, dy' \\ &\quad - \frac{1}{2}\int_0^y A_2\Big(z - \frac{y-y'}{2}\Big)(u*u)\Big(z - \frac{y-y'}{2}, y'\Big)\, dy'. \end{aligned} \tag{3.9}$$

Step 3. To discuss the existence of solutions of (3.9), we introduce the following domains:

$$\Omega^-(K,\delta) \;:=\; \{\, z \in \mathbb{C} \,|\, \mathrm{dist}\big(z, z(\widehat{K}^{(-)})\big) < \delta \,\}, \tag{3.10}$$

$$\begin{aligned} \mathcal{R} \;:=\; \{\, (z,y) \in \mathbb{C}^2 \,|\, &\mathrm{dist}(y, \mathbb{R}_+) < \delta \text{ and the segment } [z, z-y/2] \\ &\text{is contained in } \Omega^-(K,\delta) \,\}. \end{aligned} \tag{3.11}$$

Note that $\mathcal{R}$ is star-shaped with respect to the variable y (i.e., $(z,y) \in \mathcal{R}$ implies $(z,\theta y) \in \mathcal{R}$ for any $\theta \in [0,1]$) and that

$$z(\widehat{K}^{(-)}) \times \{\, y \in \mathbb{C} \,|\, \mathrm{dist}(y, \mathbb{R}_+) < \delta \,\} \subset \mathcal{R}$$

holds. Furthermore, if $\delta > 0$ is sufficiently small, we may assume that $A_j(z)$ ($j = 0,1,2$) are holomorphic and bounded in the closure of $\Omega^-(K,\delta)$.

In what follows we solve the integral equation (3.9) in $\mathcal{R}$.

Step 4. Let

$$\begin{aligned} \mathcal{O}_\lambda \;:=\; \Big\{\, u(z,y) \;\Big|\; &u \text{ is holomorphic in } \mathcal{R} \text{ and} \\ &\|u\|_\lambda := \sup_{(z,y)\in\mathcal{R}} \int_0^y |u(z,y')| e^{-\lambda |y'|}\, d|y'| < \infty \Big\}, \end{aligned} \tag{3.12}$$

where $\lambda > 0$ is a parameter. Then we can prove

Proposition 3.3. *$\mathcal{O}_\lambda$ is a Banach algebra with respect to the convolution $*$, that is, $\mathcal{O}_\lambda$ is a Banach space and the following holds :*

$$\text{If } u, v \in \mathcal{O}_\lambda, \text{ then } u * v \in \mathcal{O}_\lambda \text{ and } \|u * v\|_\lambda \le \|u\|_\lambda \|v\|_\lambda. \tag{3.13}$$

Proposition 3.4. *Let $F(u)$ denote the right-hand side of (3.9), that is,*

$$\begin{aligned}(F(u))(z,y) &= A_0\Big(z-\frac{y}{2}\Big) - \int_0^y A_1\Big(z-\frac{y-y'}{2}\Big)u\Big(z-\frac{y-y'}{2}, y'\Big)\,dy' \\ &\quad - \frac{1}{2}\int_0^y A_2\Big(z-\frac{y-y'}{2}\Big)(u*u)\Big(z-\frac{y-y'}{2}, y'\Big)\,dy'.\end{aligned} \tag{3.14}$$

Then $F(u)$ defines a contractive mapping from $\{\, u \in \mathcal{O}_\lambda \mid \|u\|_\lambda \le 1 \,\}$ to itself when $\lambda > 0$ is sufficiently large.

Therefore the contractive mapping principle provides us with a (unique) holomorphic solution of (3.9) in $\mathcal{R} \supset z(\widehat{K}^{(-)}) \times \{\, y \in \mathbb{C} \mid \mathrm{dist}(y, \mathbb{R}_+) < \delta \,\}$. The exponential estimate (3.5) for its solution can be obtained by repeating the above argument after replacing T by $\tilde{T} = (x - x_1)^{d/2+2} T$ ($x_1 \in \widehat{K}^{(-)}$) and further by using the boundedness of the norm $\|u\|_\lambda$ of the solution.

Theorem 3.2 assures the Borel summability of formal power series solutions $S_\pm(x,\eta)$ of the Riccati equation. Once the Borel summability of $S_\pm(x,\eta)$ is established, then the Borel summability of WKB solutions

$$\psi_\pm(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{x_0}^x S_{\mathrm{odd}}\, dx \right)$$

of (2.1) can be confirmed by the following argument:

(I) When the integration path from x_0 to x does not cross any Stokes curve, then the Borel summability of $\psi_\pm$ immediately follows from Theorem 3.2.

(II) Even when the integration path crosses several Stokes curves, if we deform the integration path in such a way that

> at every crossing point $\hat{x}$ of the integration path with a Stokes curve Γ, we avoid the crossing with Γ and go to $x = \infty$ along one side of Γ and then return to $x = \hat{x}$ along the other side of Γ,

then the Borel summability of $\psi_\pm$ is ensured by Theorem 3.2 also in this case. Note that Condition (ND) guarantees that such a deformation of the integration path is always possible.

This is an outline of the proof of Theorem 2.8. See [28] for more details of the discussion.

3.2. Proof of Theorem 2.9

To prove Theorem 2.9, we make use of the transformation theory to the Airy equation developed in [4], [27, Chapter 2].

Let us consider

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 \tilde{Q}(\tilde{x}) \right) \tilde{\psi} = 0 \tag{$\widetilde{3.15}$}$$

and apply a change of variable $x = x(\tilde{x})$ to $(\widetilde{3.15})$. If we further employ a change of unknown function $\tilde{\psi}(\tilde{x}) = (dx/d\tilde{x})^{-1/2}\psi(x(\tilde{x}))$, then $(\widetilde{3.15})$ is transformed to

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi = 0 \tag{3.15}$$

with

$$\tilde{Q}(\tilde{x}) = \left(\frac{dx}{d\tilde{x}}\right)^2 Q(x(\tilde{x})) - \frac{1}{2}\eta^{-2}\{x;\tilde{x}\}. \tag{3.16}$$

Here $\{x;\tilde{x}\}$ stands for the Schwarzian derivative:

$$\{x;\tilde{x}\} = \frac{d^3x/d\tilde{x}^3}{dx/d\tilde{x}} - \frac{3}{2}\left(\frac{d^2x/d\tilde{x}^2}{dx/d\tilde{x}}\right)^2.$$

Taking this relation into account, we introduce the following terminology.

Definition 3.5. We say that $(\widetilde{3.15})$ is transformed (in the sense of exact WKB analysis) to (3.15) at $\tilde{x} = \tilde{x}_0$ if there exists an infinite series $x(\tilde{x},\eta) = \sum_{n\geq 0} x_n(\tilde{x})\eta^{-n}$ that satisfies the following conditions:
(i) $x_n(\tilde{x})$ is holomorphic in a fixed neighborhood $\tilde{U}$ of $\tilde{x} = \tilde{x}_0$ (i.e., $\tilde{U}$ is independent of n).
(ii) The following relation holds (as formal power series of η^{-1}):

$$\tilde{Q}(\tilde{x}) = \left(\frac{\partial x}{\partial \tilde{x}}(\tilde{x},\eta)\right)^2 Q(x(\tilde{x},\eta)) - \frac{1}{2}\eta^{-2}\{x(\tilde{x},\eta);\tilde{x}\}. \tag{3.17}$$

Under this terminology we can prove the following

Theorem 3.6. *Let $\tilde{x} = \tilde{a}$ be a simple turning point of*

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 \tilde{Q}(\tilde{x})\right)\tilde{\psi} = 0, \tag{$\widetilde{3.15}$}$$

that is, $\tilde{a}$ is a simple zero of $\tilde{Q}(\tilde{x})$. Then at $\tilde{x} = \tilde{a}$ $(\widetilde{3.15})$ can be transformed (in the sense of exact WKB analysis) to the Airy equation

$$\left(\frac{d^2}{dx^2} - \eta^2 x\right)\psi = 0. \tag{3.18}$$

Theorem 3.6 is proved by constructing $x(\tilde{x},\eta) = x_0(\tilde{x}) + x_1(\tilde{x})\eta^{-1} + \cdots$ that satisfies

$$\tilde{Q}(\tilde{x}) = \left(\frac{\partial x}{\partial \tilde{x}}(\tilde{x},\eta)\right)^2 x(\tilde{x},\eta) - \frac{1}{2}\eta^{-2}\{x(\tilde{x},\eta);\tilde{x}\} \tag{3.19}$$

in a recursive manner. For example, the top order part $x_0(\tilde{x})$ is given by

$$x_0(\tilde{x}) = \left(\frac{3}{2}\int_{\tilde{a}}^{\tilde{x}} \sqrt{\tilde{Q}(\tilde{x})}\, d\tilde{x}\right)^{2/3} \tag{3.20}$$

and the higher order part $x_n(\tilde{x})$ is determined by solving a first order ODE of the form

$$\left(2\frac{x_0}{x_0'}\frac{d}{d\tilde{x}} + 1\right)x_n = (\text{given}), \quad \text{i.e.,} \quad \left(2z\frac{d}{dz} + 1\right)x_n = (\text{given})$$

degree by degree. Here z denotes a new independent variable $z = x_0(\tilde{x})$. Note that $x_n(\tilde{x})$ identically vanishes for an odd integer n and also that $x_n(\tilde{x})$ satisfies the estimate

$$|x_n(\tilde{x})| \le AC^n n! \tag{3.21}$$

for some positive constants $A, C > 0$ in a fixed neighborhood $\tilde{U}$ of $\tilde{x} = \tilde{a}$.

Thus Eq. $(\widetilde{3.15})$ is transformed to the Airy equation (3.18) near a simple turning point $\tilde{a}$ by the formal coordinate transformation $x = x(\tilde{x}, \eta)$. Furthermore, in the current situation we can verify the following relation between WKB solutions of $(\widetilde{3.15})$ and those of (3.18) in all orders of η^{-1}:

$$\tilde{\psi}_\pm(\tilde{x}, \eta) = \Big(\frac{\partial x}{\partial \tilde{x}}\Big)^{-1/2} \psi_\pm(x(\tilde{x}, \eta), \eta), \tag{3.22}$$

where $\tilde{\psi}_\pm$ and $\psi_\pm$ are WKB solutions of $(\widetilde{3.15})$ and (3.18) normalized at the turning points in question, respectively:

$$\tilde{\psi}_\pm(\tilde{x}, \eta) = \frac{1}{\sqrt{\tilde{S}_{\rm odd}}} \exp\left(\pm \int_{\tilde{a}}^{\tilde{x}} \tilde{S}_{\rm odd} d\tilde{x}\right), \tag{3.23}$$

$$\psi_\pm(x, \eta) = \frac{1}{\sqrt{S_{\rm odd}}} \exp\left(\pm \int_0^x S_{\rm odd} dx\right). \tag{3.24}$$

Theorem 2.9 is proved by considering the Borel transform of both sides of (3.22). As a matter of fact, the multiplication operator η^{-1} turns out to be an integral operator $(\partial/\partial y)^{-1}$ via the Borel transformation in view of Proposition 2.6. Thus, using the Taylor expansion, we find that the Borel transform of (3.22) is expressed as

$$\tilde{\psi}_{\pm,B}(\tilde{x}, y) = \Big(\sum_{j\ge 0} \frac{\partial x_j}{\partial \tilde{x}} \big(\frac{\partial}{\partial y}\big)^{-j}\Big)^{-1/2} \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{j\ge 1} x_j(\tilde{x}) \big(\frac{\partial}{\partial y}\big)^{-j}\Big)^n \Big(\frac{\partial^n}{\partial x^n}\, \psi_{\pm,B}\Big)\big(x_0(\tilde{x}), y\big). \tag{3.25}$$

As [6, Appendix C] shows, if we use $(x, y) = (x_0(\tilde{x}), y)$ as new independent variables instead of $(\tilde{x}, y)$, the right-hand side of (3.25) can be expressed also as

$$\int_{-y_0}^{y} K\big(x, y - y', \frac{\partial}{\partial x}\big) \psi_{\pm,B}(x, y')\, dy' =: L\big(x, \frac{\partial}{\partial x}, \big(\frac{\partial}{\partial y}\big)^{-1}\big) \psi_{\pm,B}(x, y) \tag{3.26}$$

with some integro-differential operator $L = L(x, \partial/\partial x, (\partial/\partial y)^{-1})$. The operator L is what is called a **"microdifferential operator"** in the theory of microlocal analysis (cf. [31]). As its consequence, it turns out that L does not change the location of singularities of the operand. Hence, since the singular points of $\psi_{\pm,B}(x, y)$ are

$$y = \pm \int_0^x \sqrt{x}\, dx = \pm \frac{2}{3} x^{3/2}$$

(cf. Property (A) in Section 2.2), the singular points of $\tilde{\psi}_{\pm,B}(\tilde{x},y)$ are also confined to

$$y = \pm\frac{2}{3}x^{3/2}\Big|_{x=x_0(\tilde{x})} = \pm\int_{\tilde{a}}^{\tilde{x}}\sqrt{\tilde{Q}(\tilde{x})}\,d\tilde{x} = \pm y_0(\tilde{x}).$$

Furthermore, the discontinuity formula (2.28) of $\tilde{\psi}_{\pm,B}(\tilde{x},y)$ is also confirmed as

$$\begin{aligned}\triangle_{y=y_0(\tilde{x})}\,\tilde{\psi}_{+,B}(\tilde{x},y) &= \triangle_{y=y_0(\tilde{x})}\,(L\psi_{+,B})\Big|_{x=x_0(\tilde{x})}\\ &= L\triangle_{y=2x^{3/2}/3}\,\psi_{+,B}\Big|_{x=x_0(\tilde{x})}\\ &= L(i\psi_{-,B})\Big|_{x=x_0(\tilde{x})} = i\tilde{\psi}_{-,B}(\tilde{x},y). \qquad (3.27)\end{aligned}$$

Thus we have verified Theorem 2.9, (i),(ii). For more details we refer the reader to [27, Chapter 2]. More recently Kamimoto and Koike [26] have proved the Borel summability of the transformation series $x = x(\tilde{x},\eta)$, which guarantees that the connection formula (2.29)-(2.30) for the WKB solutions (2.27) of (2.1) is derived from that of the Airy equation. This completes the proof of Theorem 2.9.

4. Application — Computation of monodromy representations of Fuchsian equations

In this section, as an application of the exact WKB analysis, let us compute the monodromy representations of second order equations of the form

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi = 0 \qquad (4.1)$$

with

$$Q(x) = \frac{F(x)}{G(x)^2} \qquad (F(x), G(x) : \text{polynomials}). \qquad (4.2)$$

In what follows we assume that

$$\begin{aligned}\deg F &= 2g+2, \qquad & F(x) &= (x-a_0)\cdots(x-a_{2g+1}),\\ \deg G &= g+2, \qquad & G(x) &= (x-b_0)\cdots(x-b_{g+1})\end{aligned}$$

for some non-negative integer $g \geq 0$ and that all a_j and b_k are mutually distinct. Then the set of singular points of (4.1) is given by

$$\mathcal{S} = \{\, b_0, \ldots, b_{g+1}, b_{g+2} = \infty \,\}$$

and all singular points become regular singular. Thus Eq. (4.1) is the so-called Fuchsian equation.

Remark 4.1. When $g = 0$, (4.1) is equivalent to Gauss' hypergeometric equation.

For such a Fuchsian equation the monodromy representation is naturally defined: Take a base point $x_0 \in P^1(\mathbb{C}) \setminus \mathcal{S}$ and a fundamental system of solutions (ψ_0, ψ_1) around x_0. For any closed path γ in $P^1(\mathbb{C}) \setminus \mathcal{S}$ emanating from x_0 we consider analytic continuation of (ψ_0, ψ_1) along γ:

$$(\psi_0, \psi_1) \text{ near } x_0 \underset{\substack{\text{analytic continuation} \\ \text{along } \gamma}}{\rightsquigarrow} (\gamma_* \psi_0, \gamma_* \psi_1) = (\psi_0, \psi_1)\, {}^{\exists}\!A_\gamma, \tag{4.3}$$

where A_γ is an invertible 2×2 constant matrix. Then the monodromy representation of (4.1) is, by definition, the algebraic homomorphism

$$\pi_1(P^1(\mathbb{C}) \setminus \mathcal{S}, x_0) \ni [\gamma] \longmapsto A_\gamma \in GL_2(\mathbb{C}), \tag{4.4}$$

where π_1 designates the fundamental group.

From now on we compute the monodromy representation of (4.1) by making use of WKB solutions. Before doing the computation, we prepare one proposition which is concerned with the behavior of WKB solutions at a regular singular point.

Proposition 4.2. *At each regular singular point $x = b_k$, $S_{\mathrm{odd}}(x, \eta)$ has a pole of order* 1 *and its residue there is explicitly given by*

$$\operatorname*{Res}_{x=b_k} S_{\mathrm{odd}}(x, \eta) = c_k \eta \sqrt{1 + \frac{1}{4c_k^2}\eta^{-2}}, \tag{4.5}$$

where $c_k = \mathrm{Res}_{x=b_k} \sqrt{Q(x)}$. (For $k = g+2$ we define $c_{g+2} = \mathrm{Res}_{\zeta=0}(-\sqrt{Q(1/\zeta)}/\zeta)$.)

We explain the computation by using the following example discussed in [27, Chapter 3].

Example. Let us consider

$$\left(\frac{d^2}{dx^2} - \eta^2 \frac{(x^2-9)(x^2-1/9)}{(x^3 - \exp(i\pi/8))^2} \right) \psi = 0. \tag{4.6}$$

In this case $g = 1$ and we number turning points and regular singular points as follows:

$$\begin{aligned} &a_0 = -3, \quad a_1 = -1/3, \quad a_2 = 1/3, \quad a_3 = 3, \\ &b_0 = \exp(33i\pi/24), \ \ b_1 = \exp(i\pi/24), \ \ b_2 = \exp(17i\pi/24), \ \ b_3 = \infty. \end{aligned}$$

The Stokes geometry of (4.6) is described in Figure 4. As is shown in Figure 4, we take a base point x_0 between a_0 and a_1 and use (the Borel sums of) WKB solutions

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{x_0}^{x} S_{\mathrm{odd}}\, dx \right) \tag{4.7}$$

as a fundamental system of solutions around x_0. In Figure 4, for the later use, we draw (in blue) a path C_k $(0 \le k \le g+2)$ of analytic continuation which starts from x_0 and returns to x_0 after encircling a regular singular point b_k once in an anticlockwise manner, and also (in red) a path γ_j $(0 \le j \le 2g+1)$ which starts from x_0 and ends at a turning point a_j. Note that the branch of $\sqrt{Q(x)}$ is chosen

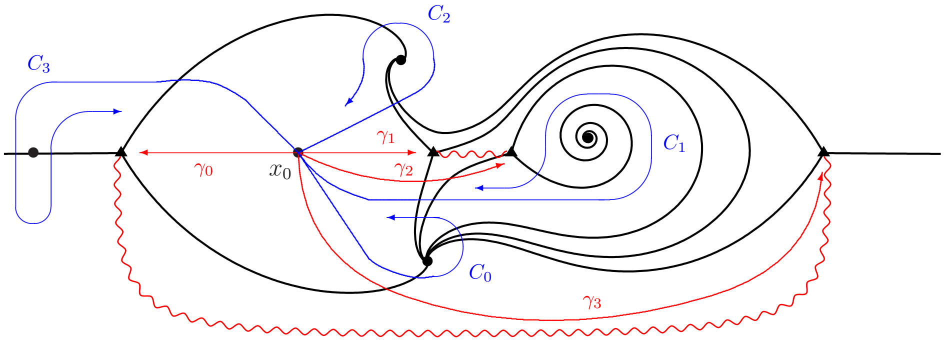

Figure 4 : Stokes geometry of Eq. (4.6)
(wiggly lines designate cuts to define $\sqrt{Q(x)}$).

here so that $\sqrt{Q(x)} \sim 1/x$ holds near $x = \infty$. This choice of the branch of $\sqrt{Q(x)}$ assures

$$\Re\, c_1, \Re\, c_2 > 0, \quad \Re\, c_0, \Re\, c_3 < 0$$

and hence it follows from Proposition 4.2 that on a Stokes curve flowing into b_1 or b_2 (resp., b_0 or b_3) the connection formula (2.30) of type $(-)$ (resp., (2.29) of type $(+)$) holds. In what follows we also use the following notations:

$$\nu_k^{\pm} := \exp\left(i\pi\left(1 \pm \sqrt{4c_k^2\eta^2 + 1}\right)\right) \quad (k = 0, \ldots, g+2), \tag{4.8}$$

$$u_j := \exp\left(2\int_{\gamma_j} S_{\mathrm{odd}}dx\right), \quad u_{jk} := u_j^{-1}u_k \quad (j, k = 0, \ldots, 2g+1). \tag{4.9}$$

Computation of monodromy matrices A_k along C_k

Since $\pi_1(P^1(\mathbb{C}) \setminus \mathcal{S}, x_0)$ is generated by C_k, it suffices to compute a monodromy matrix $A_k = A_{C_k}$ along C_k $(k = 0, 1, 2, 3)$. Let us first consider the computation of A_2 along C_2.

As is shown in Figure 4, C_2 crosses three Stokes curves and at each crossing point a Stokes phenomenon described by Theorem 2.9 occurs. For example, at the first crossing point C_2 crosses a Stokes curve emanating from a turning point a_1. Note that on this Stokes curve the connection formula (2.30) of type $(-)$ holds, as was noted before. Since (2.30) is described in terms of the WKB solutions

$$\varphi_{\pm}^{(1)} = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_{a_1}^{x} S_{\mathrm{odd}}dx\right) \tag{4.10}$$

normalized at the turning point a_1 where the Stokes curve in question emanates, we factorize the WKB solutions (4.7) as

$$\psi_\pm = \varphi_\pm^{(1)} \exp\left(\pm \int_{\gamma_1} S_{\mathrm{odd}} dx\right). \tag{4.11}$$

For the WKB solutions $\varphi_\pm^{(1)}$ normalized at a_1 we have the connection formula (2.30). Hence for $\psi_\pm$ the following holds:

$$\Psi_+ \rightsquigarrow \Psi_+, \quad \Psi_- \rightsquigarrow \Psi_- - i \exp\left(-2 \int_{\gamma_1} S_{\mathrm{odd}} dx\right) \Psi_+ = \Psi_- - i u_1^{-1} \Psi_+, \tag{4.12}$$

that is,

$$(\Psi_+, \Psi_-) \ \rightsquigarrow \ (\Psi_+, \Psi_-) \begin{pmatrix} 1 & -iu_1^{-1} \\ 0 & 1 \end{pmatrix}. \tag{4.13}$$

The Stokes phenomenon at the second crossing point can be similarly computed by using the WKB solutions

$$\varphi_\pm^{(3)} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{a_3}^{x} S_{\mathrm{odd}} dx\right) \tag{4.14}$$

normalized at a_3 and the factorization

$$\psi_\pm = \varphi_\pm^{(3)} \exp\left(\pm \int_{x_0}^{a_3} S_{\mathrm{odd}} dx\right). \tag{4.15}$$

However, in this case the integration path γ_{x_0,a_3} from x_0 to a_3 is not homotopic to γ_3; the closed path $\gamma_{x_0,a_3}(\gamma_3)^{-1}$ encircles two turning points b_0, b_1 and the cut connecting a_1 and a_2. Thus the factorization formula (4.15) reads as

$$\psi_\pm = \varphi_\pm^{(3)} \exp\left(\pm \int_{\gamma_3} S_{\mathrm{odd}} dx\right) \left(\nu_0^\pm\right)^{-1} \left(\nu_1^\pm\right)^{-1} \left(u_{21}\right)^{\pm 1} \tag{4.16}$$

and the Stokes phenomenon at the second crossing point is described by

$$(\Psi_+, \Psi_-) \ \rightsquigarrow \ (\Psi_+, \Psi_-) \begin{pmatrix} 1 & -iu_3^{-1} u_{21}^{-2} \dfrac{\nu_0^+ \nu_1^+}{\nu_0^- \nu_1^-} \\ 0 & 1 \end{pmatrix}. \tag{4.17}$$

It is now clear how to compute the Stokes phenomenon at the third crossing point and consequently we obtain

$$A_2 = \begin{pmatrix} \nu_2^+ & 0 \\ 0 & \nu_2^- \end{pmatrix} \begin{pmatrix} 1 & -iu_0^{-1} \dfrac{\nu_2^-}{\nu_2^+} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -iu_3^{-1} u_{12}^{2} \dfrac{\nu_0^+ \nu_1^+}{\nu_0^- \nu_1^-} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -iu_1^{-1} \\ 0 & 1 \end{pmatrix}. \tag{4.18}$$

Note that after the analytic continuation along C_2 S_{odd} changes its branch in view of Proposition 4.2; this is the reason why the first matrix in the right-hand side of (4.18) appears.

The computation of the other matrices A_k is the same as that of A_2. The result is as follows:

$$A_0 = \begin{pmatrix} \nu_0^+ & 0 \\ 0 & \nu_0^- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_1\dfrac{\nu_0^+}{\nu_0^-} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2\dfrac{\nu_0^+}{\nu_0^-} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2\dfrac{\nu_0^+\nu_1^+}{\nu_0^-\nu_1^-} & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 \\ -iu_1u_{12}^2\dfrac{\nu_0^+\nu_1^+}{\nu_0^-\nu_1^-} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_0 & 1 \end{pmatrix}, \tag{4.19}$$

$$A_1 = \begin{pmatrix} \nu_1^+ & 0 \\ 0 & \nu_1^- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_1\dfrac{\nu_1^+}{\nu_1^-} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2\dfrac{\nu_1^+}{\nu_1^-} & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & -iu_2^{-1}\dfrac{\nu_1^-}{\nu_1^+} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_1 & 1 \end{pmatrix}, \tag{4.20}$$

$$A_3 = \begin{pmatrix} \nu_3^+ & 0 \\ 0 & \nu_3^- \end{pmatrix} \begin{pmatrix} 1 & -iu_0^{-1}\dfrac{\nu_3^-}{\nu_3^+} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_0\dfrac{\nu_3^+}{\nu_3^-} & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 \\ -iu_3u_{21}^2\dfrac{\nu_0^-\nu_1^-\nu_2^-}{\nu_0^+\nu_1^+\nu_2^+} & 1 \end{pmatrix} \begin{pmatrix} 1 & iu_0^{-1} \\ 0 & 1 \end{pmatrix}. \tag{4.21}$$

Finally, if we change a fundamental system of solutions as

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm\int_{x_0}^x S_{\mathrm{odd}}dx\right) \longmapsto \tilde{\psi}_\pm = \exp\left(\mp\int_{\gamma_0} S_{\mathrm{odd}}dx\right)\psi_\pm, \tag{4.22}$$

then we find that every A_k can be described solely by $\{\nu_k^\pm\}$ and $\{u_{jk}\}$ (or $\{\nu_k^+\}$ and $\{u_{01}, u_{12}\}$ thanks to Remark 4.3 below).

Remark 4.3. Among $\{\nu_k^\pm\}$ and $\{u_{jk}\}$ we have the following relations:

$$\nu_k^+\nu_k^- = 1 \quad (k = 0, 1, 2, 3),$$
$$\nu_0^+\nu_1^+\nu_2^+\nu_3^+u_{12}u_{30} = 1.$$

In conclusion we have

Theorem 4.4. *Every monodromy matrix can be described in terms of the following two kinds of quantities :*
(i) *Characteristic exponents $\{\nu_k^\pm\}$ at regular singular points $\{b_k\}$.*
(ii) *Contour integrals $\{u_{jk}\}$ of S_{odd} on the Riemann surface of $\sqrt{Q(x)}$.*

5. Voros coefficients and wall-crossing formulas

As we have seen so far, the connection formula (or Theorems 2.8 and 2.9) is very powerful to study the global behavior of solutions of second order ODEs. The most important analytic ingredient of the connection formula is the movable singular points $y = \pm y_0(x) = \pm\int_a^x \sqrt{Q(x)}dx$ of the Borel transform of WKB solutions. Here we should recall that, in applying the connection formula, we have assumed Condition (ND), that is, non-existence of Stokes curves connecting two turning points. In this section we consider the situation where this non-degeneracy condition (Condition (ND)) is violated. In such a degenerate situation another kind of singularities of the Borel transform of WKB solutions may play an important role. The study of this degenerate situation is also related to the so-called **"wall-crossing formula"** discussed by Gaiotto-Moore-Neitzke ([18]).

Let us consider the problem by using a simple example.

Example. **(Weber equation)** We consider the Weber equation

$$\left(\frac{d^2}{dx^2} - \eta^2\big(c - \frac{x^2}{4}\big)\right)\psi = 0 \qquad (c > 0) \tag{5.1}$$

and its WKB solutions

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_{2\sqrt{c}}^x S_{\mathrm{odd}}dx\right). \tag{5.2}$$

Eq. (5.1) has two simple turning points $x = \pm 2\sqrt{c}$ and they are connected by a Stokes curve (**"Stokes segment"**), as is shown in Figure 2. In what follows we study the effect of this Stokes segment.

As $x = 2\sqrt{c}$ is a simple turning point, we can apply Theorem 2.9 to find that the singularity locus of $\psi_{+,B}(x,y)$ form a cusp and have two branches $y = \pm y_0(x)$ near $x = 2\sqrt{c}$, where

$$y_0(x) = \int_{2\sqrt{c}}^x \sqrt{c - \frac{x^2}{4}}\,dx$$

(cf. Figure 3). These two branches of the singularity locus can be prolonged to $x = -2\sqrt{c}$ and they form again a cusp near $x = -2\sqrt{c}$. Repeating this process, we thus obtain Figure 5 for the singularity locus of $\psi_{+,B}(x,y)$. They become a "ladder-like" set and, as

$$2\int_{-2\sqrt{c}}^{2\sqrt{c}} \sqrt{c - \frac{x^2}{4}}\,dx = 2\pi c$$

holds, the singularities of $\psi_{+,B}(x,y)$ for fixed x have the periodic structure with period $2\pi c$. Among them the singularities $-y_0(x) + 2\pi nc$ ($n \in \mathbb{Z}$) are often called **"fixed singularities"**, as their relative location with respect to the reference singularity $-y_0(x)$ does not depend on x.

The existence of fixed singularities can be rigorously confirmed by the following arguments.

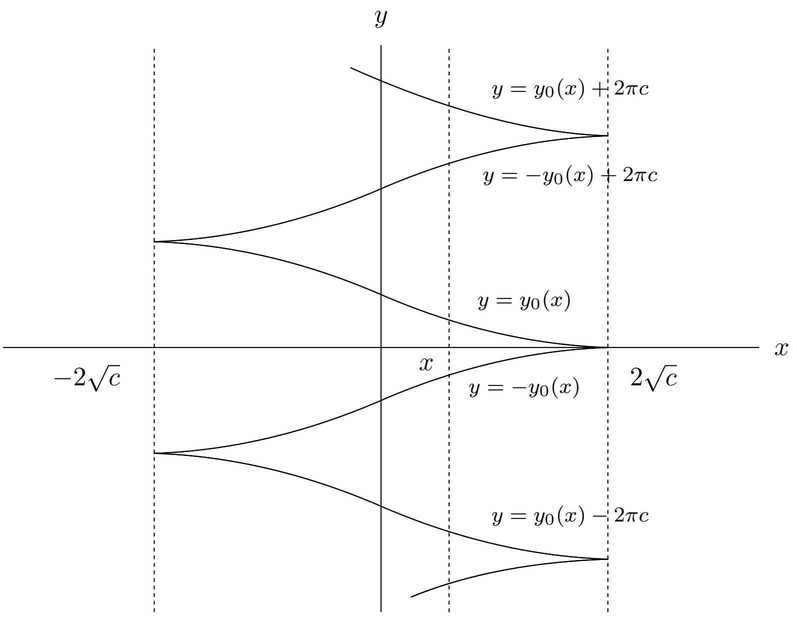

Figure 5 : Singularity locus of $\psi_{+,B}(x,y)$ for Eq. (5.1).

1st approach. We can verify the existence of fixed singularities by using the differential equation that $\psi_{\pm,B}(x,y)$ satisfy:

$$\left(\frac{\partial^2}{\partial x^2} - \left(c - \frac{x^2}{4}\right)\frac{\partial^2}{\partial y^2}\right)\psi_{\pm,B}(x,y) = 0. \tag{5.3}$$

According to the general result for the propagation of singularities of solutions for linear partial differential equations established by the theory of microlocal analysis (cf. [31]), the singularities of solutions of (5.3) propagate along a bicharacteristic flow, that is, a Hamiltonian flow of the principal symbol of (5.3):

$$\begin{cases} \dot{x} = \dfrac{\partial p_B}{\partial \xi} = 2\xi, \\ \dot{y} = \dfrac{\partial p_B}{\partial \eta} = -2\Big(c - \dfrac{x^2}{4}\Big)\eta, \\ \dot{\xi} = -\dfrac{\partial p_B}{\partial x} = -\dfrac{x}{2}\eta^2, \\ \dot{\eta} = -\dfrac{\partial p_B}{\partial y} = 0, \end{cases} \tag{5.4}$$

where $\dot{x} = dx/dt$ etc. and $p_B(x, y, \xi, \eta) = \xi^2 - (c - x^2/4)\eta^2$ denotes the principal symbol of (5.3). A solution of (5.4) with the initial condition $(x(0), y(0), \xi(0), \eta(0)) = (2\sqrt{c}, 0, 0, 1)$ is explicitly given by

$$x = 2\sqrt{c}\cos t, \quad y = -c(t - \sin t \cos t), \quad \xi = -\sqrt{c}\sin t, \quad \eta = 1 \tag{5.5}$$

and its projection to the base space $\mathbb{C}^2_{(x,y)}$, i.e.,

$$x = 2\sqrt{c}\cos t, \qquad y = -c(t - \sin t \cos t) \tag{5.6}$$

precisely describes the singularity locus of $\psi_{\pm,B}(x, y)$.

In this way the singularities of $\psi_{\pm,B}(x, y)$ can be analyzed by tracing the bicharacteristic flow of the Borel transformed equation (5.3).

2nd approach. The second approach is more WKB-theoretic and provides us with more detailed information about the fixed singularities.

We start with the following factorization of $\psi_\pm(x, \eta)$.

$$\psi_\pm(x, \eta) = \psi_\pm^{(\infty)}(x, \eta) \exp\left(\pm \int_{2\sqrt{c}}^{\infty} (S_{\mathrm{odd}} - \eta S_{-1})\, dx\right), \tag{5.7}$$

where

$$\psi_\pm^{(\infty)}(x, \eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp \pm \left(\eta \int_{2\sqrt{c}}^{x} S_{-1} dx + \int_{\infty}^{x} (S_{\mathrm{odd}} - \eta S_{-1}) dx\right) \tag{5.8}$$

is a WKB solution normalized at $x = \infty$. Thanks to Theorem 3.2, we find that $\psi_\pm^{(\infty)}(x, \eta)$ is Borel summable near $\{\, x \in \mathbb{R} \,|\, x > 2\sqrt{c}\,\}$ and hence its Borel transform $\psi_{\pm,B}^{(\infty)}(x, y)$ has no singularities near

$$\{\, y \in \mathbb{C} \mid y = \mp y_0(x) + \rho, \;\; \rho > 0 \,\}$$

for a fixed $x > 2\sqrt{c}$. On the other hand, the second factor or its exponent

$$V := \int_{2\sqrt{c}}^{\infty} (S_{\mathrm{odd}} - \eta S_{-1})\, dx, \tag{5.9}$$

which is often called the **"Voros coefficient"**, has fixed singularities on the positive real axis. As a matter of fact, V has the following expression in terms of the Bernoulli numbers.

Proposition 5.1.

$$2V = \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n} (ic\eta)^{1-2n}, \tag{5.10}$$

where B_{2n} stands for the Bernoulli numbers defined by

$$\frac{w}{e^w - 1} = 1 - \frac{w}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} w^{2n}. \tag{5.11}$$

Proposition 5.1 is related to the shift operator with respect to the parameter c and the explicit form (5.10) of V is derived from the following difference equation for the solution $S_+ = S_+(x, c, \eta)$ of the Riccati equation associated with (5.1). (To clarify the dependence on the parameter c we here use the notation $S_+(x, c, \eta)$.)

$$S_+(x, c - \eta^{-1} i, \eta) - S_+(x, c, \eta) = \frac{d}{dx} \log \left(\eta^{-1} S_+(x, c, \eta) - i \, \frac{x}{2} \right). \tag{5.12}$$

Eq. (5.12) is an immediate consequence of the following commutation relation.

$$\left(\frac{d}{dx} - i \, \frac{x}{2} \, \eta \right) \left(\frac{d^2}{dx^2} - \eta^2 (c - \frac{x^2}{4}) \right) = \left(\frac{d^2}{dx^2} - \eta^2 (c - \eta^{-1} i - \frac{x^2}{4}) \right) \left(\frac{d}{dx} - i \, \frac{x}{2} \, \eta \right). \tag{5.13}$$

Using (5.12), we can verify that $2V$ satisfies a difference equation

$$\phi(\sigma + 1) - \phi(\sigma) = 1 + \log \left(1 + \frac{1}{2\sigma} \right) - (\sigma + 1) \log \left(1 + \frac{1}{\sigma} \right), \tag{5.14}$$

where $\sigma = ic\eta$. Since the right-hand side of (5.10) is the unique formal solution of the difference equation (5.14), we obtain Proposition 5.1.

It follows from Proposition 5.1 that

$$V_B(y) = \frac{1}{4y} \left(\frac{1}{e^{y/(2ic)} - 1} + \frac{1}{e^{y/(2ic)} + 1} - \frac{2ic}{y} \right), \tag{5.15}$$

which tells us that $V_B(y)$ has simple poles at $y = 2m\pi c$ ($m \in \mathbb{Z} \setminus \{0\}$) with residues $(-1)^{m-1}/(4\pi i m)$. This verifies V_B, and hence $\psi_{\pm,B}$ as well, has fixed singularities. Furthermore, as the Borel sum of the Voros coefficient can be computed explicitly by using (5.15) (in fact, Binet's formula implies the Borel sum of $2V$ is given by

$$\log \frac{\Gamma(ic\eta + 1/2)}{\sqrt{2\pi}} - ic\eta \left(\log(ic\eta) - 1 \right) \tag{5.16}$$

for $\arg c < 0$ (cf. [17, Section 1.9]) and

$$-\log \frac{\Gamma(-ic\eta + 1/2)}{\sqrt{2\pi}} - ic\eta \left(\log(ic\eta) - 1 \right) - \pi c \eta \tag{5.17}$$

for $\arg c > 0$, respectively), we obtain

Theorem 5.2. *Let $\Psi_+(x, \eta)$ (resp., $\widetilde{\Psi}_+(x, \eta)$) denote the Borel sum of $\psi_+(x, \eta)$ for $x > 2\sqrt{c}$ when $\arg c < 0$ (resp., $\arg c > 0$). Then the following relation holds.*

$$\Psi_+ = \left(1 + \exp(-2\pi c \eta) \right)^{-1/2} \widetilde{\Psi}_+(x, \eta). \tag{5.18}$$

Thus a kind of Stokes phenomena occurs with WKB solutions of the Weber equation (5.1) even when the parameter c varies (**"parametric Stokes phenomena"**). Formula (5.18) exactly coincides with the wall-crossing formula discussed by Gaiotto-Moore-Neitzke ([18]). It has been analyzed from the viewpoint of the resurgent analysis by Pham and his collaborators (cf. [11], [13]). Note that, from the viewpoint of the resurgent analysis, (5.18) is equivalent to the following formula for the alien derivative (in the sense of Ecalle) of $\psi_{+,B}$ at the fixed singularities:

$$\triangle_{y=-y_0(x)+2m\pi c}\, \psi_{+,B}(x,y) = \frac{(-1)^m}{2m}\, \psi_{+,B}(x, y-2m\pi c). \tag{5.19}$$

For more details we refer the reader to [35].

6. Exact WKB analysis for higher order ODEs

In this section we discuss generalization of the exact WKB analysis to higher order linear ordinary differential equations

$$P\psi = \left(\frac{d^m}{dx^m} + \eta p_1(x)\frac{d^{m-1}}{dx^{m-1}} + \cdots + \eta^m p_m(x)\right)\psi = 0. \tag{6.1}$$

Here $m \geq 3$ is an integer and η denotes a large parameter.

6.1. WKB solutions, Stokes geometry

Similarly to the case of second order equations, we can construct a WKB solution of (6.1) of the form

$$\psi_j(x,\eta) = \exp\left(\eta \int_{x_0}^{x} \zeta_j(x)dx\right) \sum_{n=0}^{\infty} \psi_{j,n}(x)\eta^{-(n+1/2)}, \tag{6.2}$$

where $\zeta_j(x)$ is a root of the characteristic equation of (6.1):

$$\zeta^m + p_1(x)\zeta^{m-1} + \cdots + p_m(x) = 0. \tag{6.3}$$

For details of the construction of WKB solutions we refer the reader to [1], [2].

Definition 6.1. (i) A point $x = a$ is said to be a turning point of (6.1) if (6.3) has a multiple root at $x = a$. In other words, a turning point is a zero of the discriminant of (6.3) in ζ. In particular, a simple zero of the discriminant is called a simple turning point of (6.1). When $\zeta_j(a) = \zeta_k(a)$ $(j \neq k)$ holds at $x = a$, the turning point $x = a$ is said to be of type (j,k).
(ii) A Stokes curve of type (j,k) of (6.1) is, by definition, a curve defined by

$$\Im \int_a^x (\zeta_j(x) - \zeta_k(x))dx = 0, \tag{6.4}$$

where $x = a$ is a turning point of type (j,k). Furthermore, if $\Re \int_a^x (\zeta_j(x) - \zeta_k(x))dx > 0$ holds in addition to (6.4), the Stokes curve is said to be of type $j > k$.

In parallel with Theorem 2.9 the following theorem holds also for WKB solutions of higher order equations.

Theorem 6.2. *Let $x = a$ be a simple turning point of* (6.1) *of type* (j,k)*. Then, for suitably normalized WKB solutions ψ_j and ψ_k of* (6.1)*, the following properties hold in a neighborhood of* $(x,y) = (a,0)$:

(i) *$\psi_{j,B}(x,y)$ and $\psi_{k,B}(x,y)$ are singular only along $\Gamma_j \cup \Gamma_k$, where*

$$\Gamma_j = \{\,(x,y)\,|\, y = -\int_a^x \zeta_j(x)\,dx\,\}, \quad \Gamma_k = \{\,(x,y)\,|\, y = -\int_a^x \zeta_k(x)\,dx\,\}. \tag{6.5}$$

(ii)

$$\triangle_{y=-\int_a^x \zeta_k(x)dx}\psi_{j,B}(x,y) = i\psi_{k,B}(x,y), \quad \triangle_{y=-\int_a^x \zeta_j(x)dx}\psi_{k,B}(x,y) = i\psi_{j,B}(x,y). \tag{6.6}$$

Theorem 6.2 is proved in the following manner: We first consider the factorization of the differential operator P to reduce the problem to that for second order equations, and then use transformation theory similar to Theorem 3.6. To be more specific, we prove the following two assertions.

Proposition 6.3. *In a neighborhood of a simple turning point $x = a$, we can find differential operators Q and R of order $(m-2)$ and 2, respectively, that satisfy*

$$P = QR. \tag{6.7}$$

Here Q and R have the form

$$Q = \frac{d^{m-2}}{dx^{m-2}} + \eta q_1(x,\eta)\frac{d^{m-3}}{dx^{m-3}} + \cdots + \eta^{m-2}q_{m-2}(x,\eta), \tag{6.8}$$

$$R = \frac{d^2}{dx^2} + \eta r_1(x,\eta)\frac{d}{dx} + \eta^2 r_2(x,\eta), \tag{6.9}$$

where $q_j(x,\eta) = \sum_{n\ge 0} q_{j,n}(x)\eta^{-n}$ and $r_j(x,\eta) = \sum_{n\ge 0} r_{j,n}(x)\eta^{-n}$ are formal power series in η^{-1} with holomorphic coefficients. Furthermore, the following conditions are also satisfied.

$$\left(\zeta^{m-2} + q_{1,0}(x)\zeta^{m-3} + \cdots + q_{m-2,0}(x)\right)\Big|_{\zeta=\zeta_j(x)\ or\ \zeta_k(x)} \neq 0, \tag{6.10}$$

$$\zeta^2 + r_{1,0}(x)\zeta + r_{2,0}(x) = (\zeta - \zeta_j(x))(\zeta - \zeta_k(x)). \tag{6.11}$$

Proposition 6.4. *In a neighborhood of $x = a$, after the employment of the gauge transformation*

$$\psi \longmapsto \left(\exp\Big(-\frac{1}{2}\eta\int_a^x r_1(x,\eta)\,dx\Big)\right)\psi,$$

the second order differential equation $R\psi = 0$ in Proposition 6.3 *can be transformed* (*in the sense of exact WKB analysis*) *to the Airy equation.*

For more detailed explanation see [25], [1], [2].

6.2. BNR equation — Appearance of new Stokes curves

Theorem 6.2 asserts that, as far as the local theory near a simple turning point is concerned, the behavior of Borel resummed WKB solutions of higher order equations is the same as that of second order equations. However, the global behavior is completely different, as Berk et al [8] pointed out by using the following example.

Example. **(BNR equation)**

$$\left(\frac{d^3}{dx^3} + 3\eta^2\frac{d}{dx} + 2ix\eta^3\right)\psi = 0. \tag{6.12}$$

The characteristic equation of (6.12) is $\zeta^3 + 3\zeta + 2ix = 0$. Considering its discriminant, we find that (6.12) has two turning points $x = \pm 1$. Figure 6 indicates the configuration of Stokes curves of (6.12).

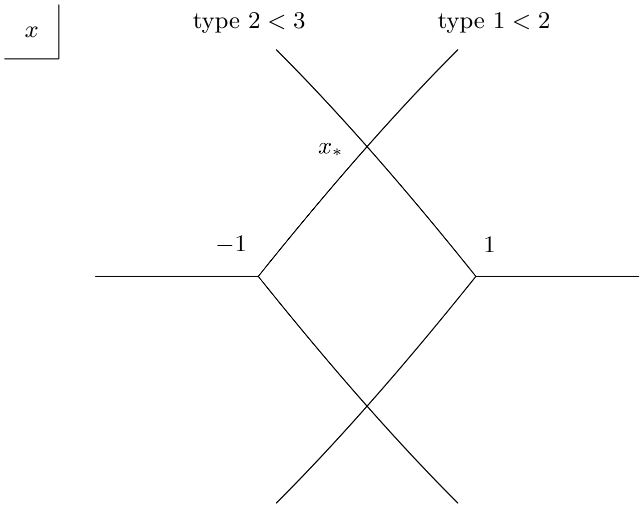

Figure 6 : Stokes curves of the BNR equation (6.12).

As Figure 6 shows, there exist crossing points of Stokes curves for (6.12). Such crossing points cause the following serious difficulty: We consider the analytic continuation of the Borel sum of a WKB solution $\Psi_3(x,\eta)$ near a crossing point x_* of Stokes curves. Assuming the Borel summability, we can expect that a Stokes phenomenon of the form

$$\Psi_3 \rightsquigarrow \Psi_3 + \alpha\Psi_2 \quad (\text{resp., } \Psi_2 \rightsquigarrow \Psi_2 + \beta\Psi_1)$$

with some suitable constant α (resp., β) occurs on a Stokes curve of type $2 < 3$ (resp., of type $1 < 2$), in view of Theorem 6.2. Hence, by the analytic continuation along γ_+ (cf. Figure 7) Ψ_3 should be changed to $\Psi_3 + \alpha(\Psi_2 + \beta\Psi_1)$, whereas by the analytic continuation along γ_- Ψ_3 should become $\Psi_3 + \alpha\Psi_2$. This is a contradiction if $\alpha\beta \neq 0$, since Eq. (6.12) does not have any singularity near x_*!

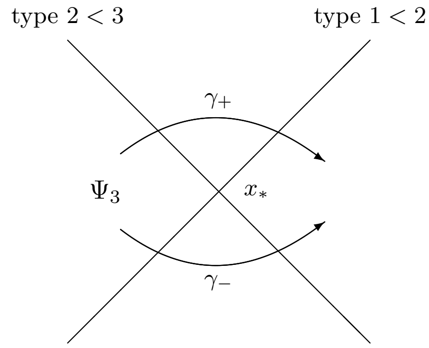

Figure 7 : Paths of analytic continuation near x_*.

To resolve this paradoxical problem Berk et al ([8]) proposed to introduce a **"new Stokes curve"** of type $1 < 3$ that emanates from x_* and tends to ∞ (cf. Figure 8). As a matter of fact, if a Stokes phenomenon of the form

$$\Psi_3 \rightsquigarrow \Psi_3 - \alpha\beta\Psi_1$$

occurs on it, the contradiction disappears. Berk et al confirmed the existence of a new Stokes curve by investigating an integral representation of solutions of (6.12) through the steepest descent method ([8], see also [34]).

If we consider the structure of singularities of $\psi_{3,B}(x,y)$ in y-plane (the so-called Borel plane) near x_*, we find that at x_* three relevant singular points $-y_j(x) := -\int^x \zeta_j(x)dx$ $(j = 1, 2, 3)$ of $\psi_{3,B}(x,y)$ have the same imaginary part. From the singularity structure the new Stokes curve is characterized as a curve where the two distant singular points $-y_3(x)$ and $-y_1(x)$ have the same imaginary part. Note that on the upper portion of the new Stokes curve $-y_1(x)$ is visible from $-y_3(x)$, whereas it is not visible on the lower portion, as is indicated in Figure 8. In this way a new Stokes curve is also related to the singularities of the Borel transform $\psi_{j,B}(x,y)$ of WKB solutions. Since the sheet structure of the Riemann surface of $\psi_{j,B}(x,y)$ is complicated, a new Stokes curve may become inert on some portion of it.

6.3. Virtual turning points

The Borel transform $\psi_{j,B}(x,y)$ of WKB solutions of the BNR equation (6.12) satisfies

$$\left(\frac{\partial^3}{\partial x^3} + 3\frac{\partial^3}{\partial x \partial y^2} + 2ix\frac{\partial^3}{\partial y^3}\right)\psi_{j,B}(x,y) = 0. \tag{6.13}$$

Since the new Stokes curve of the BNR equation is related to the singularities of $\psi_{j,B}(x,y)$, let us investigate the bicharacteristic flow of (6.13) to understand the singularity structure of $\psi_{j,B}(x,y)$ and a new Stokes curve more thoroughly.

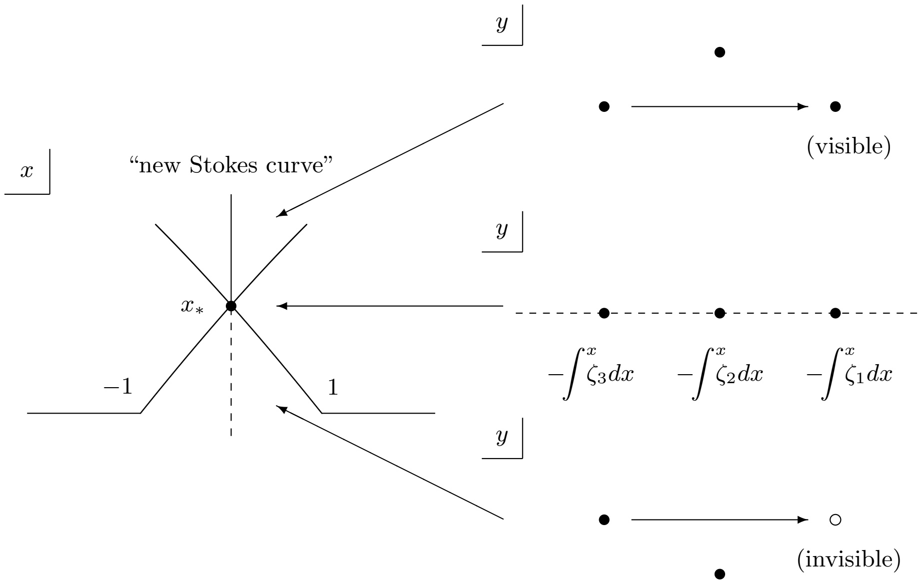

Figure 8 : New Stokes curve passing through x_* and singularities of $\psi_{3,B}(x,y)$ in the Borel plane near x_*.

The bicharacteristic flow of (6.13) is defined by

$$\begin{cases} \dot{x} = \dfrac{\partial p_B}{\partial \xi} = 3\xi^2 + 3\eta^2, \\ \dot{y} = \dfrac{\partial p_B}{\partial \eta} = 6\xi\eta + 6ix\eta^2, \\ \dot{\xi} = -\dfrac{\partial p_B}{\partial x} = -2i\eta^3, \\ \dot{\eta} = -\dfrac{\partial p_B}{\partial y} = 0, \end{cases} \tag{6.14}$$

where $p_B(x,y,\xi,\eta) = \xi^3 + 3\xi\eta^2 + 2ix\eta^3$ is the principal symbol of (6.13). A solution of (6.14) with the initial condition $(x(0), y(0), \xi(0), \eta(0)) = (1, 0, -i, 1)$ (note that $\xi(0) = -i$ is a double root of $\zeta^3 + 3\zeta + 2ix = 0$ at $x = 1$) is given by

$$\begin{aligned} &x = -4t^3 - 6t^2 + 1 = -(2t+1)(2t^2 + 2t - 1), \\ &y = -6it^4 - 12it^3 - 6it^2 = -6it^2(t+1)^2, \\ &\xi = -2it - i, \quad \eta = 1. \end{aligned} \tag{6.15}$$

Hence its projection to the base space $\mathbb{C}^2_{(x,y)}$, which describes the singularities of $\psi_{j,B}(x,y)$, becomes as is visualized in Figure 9. Near the simple turning points $x = \pm 1$ two branches of singularities coalesce and form a cuspidal singularity. We also observe that, in addition to these turning points, the singularities make a

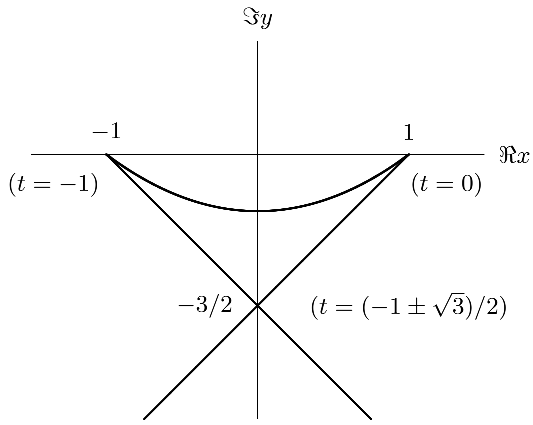

Figure 9 : Bicharacteristic curve of (6.13).

self-intersection point at $(x, y) = (0, -3i/2)$ and two branches intersect there. In fact, if we regard this self-intersection point (to be more precise, its x-component $x = 0$) as a new kind of turning points and add a Stokes curve

$$\Im \int_0^x \big(\zeta_1(x) - \zeta_3(x)\big)\, dx = 0$$

emanating from this point to Figure 6 (i.e., the original configuration of Stokes curves of (6.12)), we obtain Figure 10. Thus we can re-obtain a new Stokes curve of the BNR equation.

This consideration naturally leads to the following

Definition 6.5. Let

$$P_B\psi_B = \left(\frac{\partial^m}{\partial x^m} + p_1(x)\frac{\partial^m}{\partial x^{m-1}\partial y} + \cdots + p_m(x)\frac{\partial^m}{\partial y^m}\right)\psi_B = 0 \tag{6.16}$$

be the Borel transformed equation of (6.1) and

$$p_B(x, y, \xi, \eta) = \xi^m + p_1(x)\xi^{m-1}\eta + \cdots + p_m(x)\eta^m \tag{6.17}$$

its principal symbol. Then we call the x-component of a self-intersection point of a bicharacteristic curve of (6.16) a virtual turning point of (6.1). Here a bicharacteristic curve of (6.16) means the projection of a bicharacteristic flow of (6.16) onto the base space $\mathbb{C}^2_{(x,y)}$.

We can verify that each singularity of $\psi_{j,B}(x, y)$ (or, equivalently, a bicharacteristic curve) is locally described by $y = -\int^x \zeta_k(x)dx$ and at a virtual turning point two branches of singularities of $\psi_{j,B}(x, y)$ (for example, $y = -\int^x \zeta_k(x)dx$

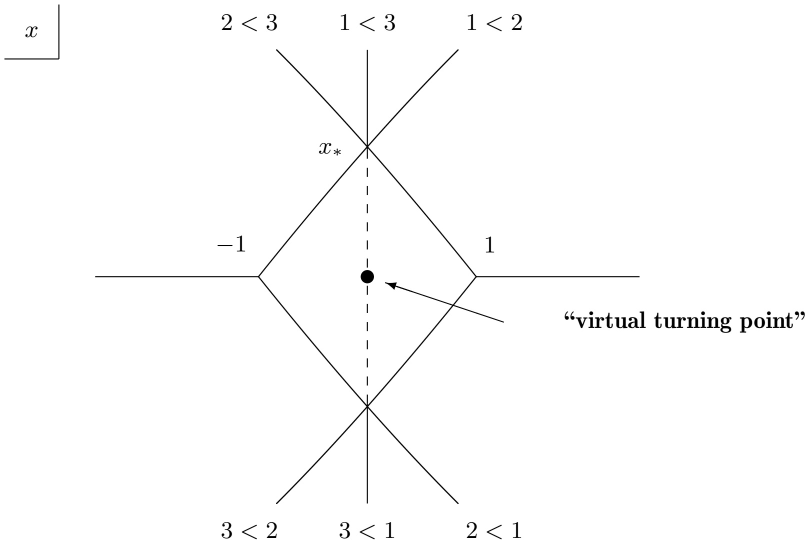

Figure 10 : Complete Stokes geometry of the BNR equation (6.12).
(A dotted line indicates the inert portion of a new Stokes curve.)

and $y = -\int^x \zeta_l(x)dx$ with $k \neq l$) cross by its definition. Thus we can naturally define a Stokes curve emanating from a virtual turning point (concretely by $\Im \int^x (\zeta_k(x) - \zeta_l(x))dx = 0$ in the above situation).

Remark 6.6. A virtual turning point was first introduced in [5] under the name of “new turning point”.

Remark 6.7. A crossing point of Stokes curves is highly dependent on the way how the Borel resummation is performed (for example, it heavily depends on $\arg \eta$), whereas the definition of a virtual turning point is independent of the way of resummation. In this sense a virtual turning point is related to the operator P in (6.1) more intrinsically than a new Stokes curve.

Once the definition of virtual turning points is provided, we obtain the following recipe for finding the proper Stokes geometry of a higher order equation (6.1).

Recipe 6.8.

(a) Draw all Stokes curves that emanate from turning points defined in Definition 6.1.

(b) Draw the new Stokes curve that emanates from a virtual turning point.

(c) As the portion of a new Stokes curve near a virtual turning point is inert, we draw the new Stokes curve in (b) by a dotted line until it hits a crossing point of (new) Stokes curves.

(d) When the new Stokes curve in (b) is of type $j > l$ and it hits a crossing point of a (new) Stokes curve of type $j > k$ and that of type $k > l$, we use a solid line to draw the portion of the new Stokes curve in (b) after passing over the crossing point.

Practically speaking, Recipe 6.8 is powerful enough to discuss the Stokes geometry (including new Stokes curves) of a higher order equation (6.1). However, it is not complete: As there exist in general infinitely many virtual turning points due to the existence of fixed singularities, it is necessary to exclude redundant virtual turning points.

Example.

$$\left(\frac{d^3}{dx^3} - 6(x+1)\eta^2\frac{d}{dx} + (4x+2i)\eta^3\right)\psi = 0. \tag{6.18}$$

As discussed in [5, Example 2.5], (6.18) has infinitely many virtual turning points. Among them at most three are non-redundant and Figure 11 describes the complete Stokes geometry of (6.18). (In Figure 11 wiggly lines designate cuts to define a root $\zeta_j(x)$ of the characteristic equation associated with (6.18) as a single-valued function.)

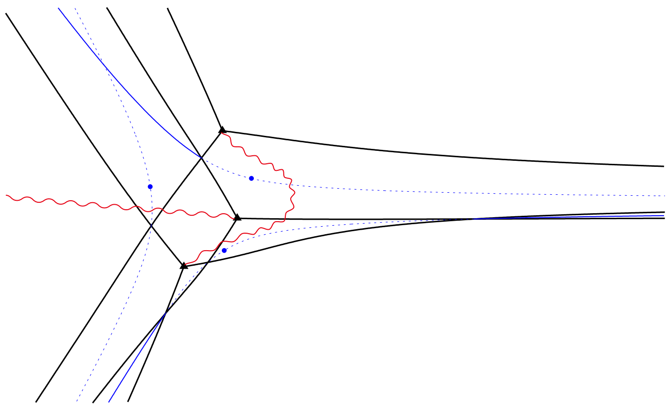

Figure 11 : Complete Stokes geometry of Eq. (6.18).

At present no complete criterion is available for the determination of redundant virtual turning points. This problem is also related to the Borel summability of WKB solutions of a higher order equation (6.1). To establish the Borel summability of WKB solutions of a higher order equation is still an open problem.

For more detailed explanation of virtual turning points and new Stokes curves including the connection formula on it, we refer the reader to [25].

Remark 6.9. Honda made a detailed study on the efficiency of Recipe 6.8 and gave a satisfactory answer to the finiteness of non-redundant virtual turning points. See [23] and [24] for his study.

Remark 6.10. Recently the Stokes geometry of higher order equations including new Stokes curves (but no virtual turning points) is investigated under the name of "spectral networks" also by Gaiotto-Moore-Neitzke ([19], [20]).

7. Some recent developments

In the final section we briefly discuss two recent developments of the exact WKB analysis, both of which are related to the problem of new Stokes curves and virtual turning points for higher order ODEs.

7.1. Borel summability of formal solutions of inhomogeneous second order equations

Let us first discuss the Borel summability of formal solutions of the following inhomogeneous second order equations:

$$P\psi = \left(\frac{d^2}{dx^2} + \eta p(x)\frac{d}{dx} + \eta^2 q(x)\right)\psi = F(x), \tag{7.1}$$

where $p(x), q(x), F(x)$ are assumed to be polynomials for the sake of simplicity. As one can easily confirm, Eq. (7.1) has a unique formal power series solution of the form

$$\widehat{\psi} = \eta^{-2}\psi_2(x) + \eta^{-3}\psi_3(x) + \cdots, \tag{7.2}$$

whose coefficients $\psi_n(x)$ $(n = 2, 3, \ldots)$ are determined by the recursive relation

$$q\psi_2 = F, \quad q\psi_3 + p\psi_2' = 0, \quad q\psi_n + p\psi_{n-1}' + \psi_{n-2}'' = 0 \ \ (n \geq 4).$$

In [29] we showed the following result for the Borel summability of $\widehat{\psi}$:

Theorem 7.1. *Let $\zeta_\pm(x)$ be the roots of $\zeta^2 + p(x)\zeta + q(x) = 0$, i.e., $\zeta_\pm(x) = (-p \pm \sqrt{p^2 - 4q})/2$.*

Case I (The case where $p(x) \equiv 0$). *Assume $p(x) \equiv 0$. Then, if a curve $\Im \int_{x_0}^{x} \sqrt{-q(x)}\, dx = 0$ passing through x_0 does not flow into a turning point (i.e., x_0 is not located on any Stokes curve of $P\psi = 0$), then $\widehat{\psi}$ is Borel summable at $x = x_0$.*

Case II (The case where $p(x) \not\equiv 0$). *Suppose that the following three conditions are satisfied:*

(i) *x_0 is not located on any Stokes curve of $P\psi = 0$, that is, a curve $\Im \int_{x_0}^{x} (\zeta_+ - \zeta_-)\, dx = \Im \int_{x_0}^{x} \sqrt{p^2 - 4q(x)}\, dx = 0$ passing through x_0 does not flow into a turning* *point.*

(ii) *A curve* $\Gamma^0_{\pm}$ *defined by*

$$\Im \int_{x_0}^{x} (-\zeta_{\mp})\, dx = \Im \int_{x_0}^{x} \frac{p \pm \sqrt{p^2 - 4q}}{2}\, dx = 0, \tag{7.3}$$

that is, a steepest descent path of $\Re \int_{x_0}^{x} (-\zeta_{\mp})\, dx$ *passing through* x_0*, can be extended to* $x = \infty$*.*

(iii) *When* $\Gamma^0_{\pm}$ *crosses a Stokes curve of* $P\psi = 0$ *of type* $\pm > \mp$ *at* $x = x_1$*, then a bifurcated path* $\Gamma^1_{\mp}$ *emanating from* $x = x_1$ *defined by*

$$\Im \int_{x_1}^{x} (-\zeta_{\pm})\, dx = 0 \tag{7.4}$$

is also extensible to $x = \infty$*.*

If these three conditions (i)-(iii) *are satisfied, then* $\widehat{\psi}$ *is Borel summable at* $x = x_0$*.*

Remark 7.2. In Case II, if $\Gamma^0_{\pm}$ and/or $\Gamma^1_{\mp}$ cross other Stokes curves of $P\psi = 0$, we further consider additional bifurcated paths emanating from crossing points similarly defined as in (iii) and impose their extensibility to $x = \infty$.

An inhomogeneous second order equation can be thought of as a special case of a third order homogeneous equation. (As a matter of fact, (7.1) can be written as $(d/dx)(F(x)^{-1}P\psi) = 0$.) Hence, Theorem 7.1 suggests the difficulty for characterizing the Borel summability of formal solutions of higher order equations.

7.2. WKB analysis for completely integrable systems

The following equation is a variant of the BNR equation.

$$\left(\frac{d^3}{dx^3} + \frac{c}{2}\eta^2 \frac{d}{dx} + \frac{x}{4}\eta^3 \right) \psi = 0, \tag{7.5}$$

where a parameter c is introduced into the coefficient of the first order term. If, in addition, we consider a differential equation in the variable c, we obtain the following holonomic system of differential equations in two variables $(x_1, x_2) = (x, c)$:

$$\begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{x_2}{2}\eta^2 \dfrac{\partial}{\partial x_1} + \dfrac{x_1}{4}\eta^3 \right) \psi = 0, \\ \left(\eta \dfrac{\partial}{\partial x_2} - \dfrac{\partial^2}{\partial x_1^2} \right) \psi = 0. \end{cases} \tag{7.6}$$

Eq. (7.6) can be written also in the form of completely integrable systems of first order equations as follows:

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x)\Psi, & P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -x_1/4 & -x_2/2 & 0 \end{pmatrix}, \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x)\Psi, & Q = P^2 + \dfrac{x_2}{3} - \dfrac{\eta^{-1}}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \end{cases} \tag{7.7}$$

where Ψ is an unknown 3-vector. Eq. (7.6) (or, equivalently, (7.7)) is called the **"Pearcey system"** as a particular solution of (7.6) is given by the Pearcey integral. In what follows we discuss the exact WKB analysis of the Pearcey system or, more generally, of holonomic systems (or completely integrable systems).

For such a holonomic system with a large parameter η, making use of the complete integrability condition, we can construct a WKB solution of the form

$$\psi^{(j)} = \exp \int^x \omega^{(j)} \tag{7.8}$$

($j = 1, 2, 3$ in the case of the Pearcey system), where $\omega^{(j)} = S^{(j)}dx_1 + T^{(j)}dx_2$ is an infinite series of the closed 1-form:

$$S^{(j)} = \eta S_{-1}^{(j)} + S_0^{(j)} + \eta^{-1}S_{-1}^{(j)} + \cdots, \quad T^{(j)} = \eta T_{-1}^{(j)} + T_0^{(j)} + \eta^{-1}T_{-1}^{(j)} + \cdots. \tag{7.9}$$

As a matter of fact, the top order part $\omega_{-1}^{(j)} = S_{-1}^{(j)}dx_1 + T_{-1}^{(j)}dx_2$ is determined by some algebraic equations (for example, in the case of the Pearcey system, $(S_{-1}^{(j)}, T_{-1}^{(j)})$ satisfies

$$(S_{-1}^{(j)})^3 + \frac{x_2}{2}S_{-1}^{(j)} + \frac{x_1}{4} = 0, \quad T_{-1}^{(j)} = (S_{-1}^{(j)})^2$$

) and, once $\omega_{-1}^{(j)}$ is fixed, the higher order parts $\omega_n^{(j)} = S_n^{(j)}dx_1 + T_n^{(j)}dx_2$ ($n \geq 0$) are uniquely determined in a recursive manner.

Turning points and Stokes surfaces are defined in parallel with Definition 6.1.

Definition 7.3. (i) A point $a = (a_1, a_2) \in \mathbb{C}^2$ is said to be a turning point of type (j, k) if

$$(\omega_{-1}^{(j)} - \omega_{-1}^{(k)})\Big|_{x=a} = 0, \quad \text{i.e.,} \quad S_{-1}^{(j)}(a) = S_{-1}^{(k)}(a) \text{ and } T_{-1}^{(j)}(a) = T_{-1}^{(k)}(a) \tag{7.10}$$

hold for some (j, k) with $j \neq k$.
(ii) A Stokes surface of type (j, k) is, by definition, a real hypersurface defined by

$$\Im \int_a^x (\omega_{-1}^{(j)} - \omega_{-1}^{(k)}) = 0, \tag{7.11}$$

where $a = (a_1, a_2)$ is a turning point of type (j, k).

For example, by straightforward computations we find that the set of turning points of the Pearcey system is explicitly given by

$$\Lambda = \{\, (x_1, x_2) \in \mathbb{C}^2 \mid 27x_1^2 + 8x_2^3 = 0 \,\}. \tag{7.12}$$

Note also that, by the definition, the section of the Stokes surfaces of the Pearcey system (7.6) with $x_2 = c$ contains the Stokes curves of the BNR equation (7.5).

The following result of Hirose impressively shows an advantage of considering the Pearcey system instead of considering the BNR equation.

Theorem 7.4. (Hirose [21]**)** *The Stokes surface of the Pearcey system* (7.6) *contains not only the Stokes curves of the BNR equation* (7.5) *but also its new Stokes curves in their section with* $x_2 = c$.

The reason why the Stokes surface of (7.6) contains the new Stokes curves of (7.5) is the following: If we change the parameter c (or x_2) in (7.5), we encounter the degenerate configuration where a Stokes curve emanating from a turning point hits another turning point with different type at some value of c. As was discussed in [3], the role of a new Stokes curve and that of an ordinary Stokes curve are interchanged through such a degenerate configuration. Otherwise stated, by the change of the parameter c a new Stokes curve of (7.5) is continuously deformed to an ordinary Stokes curve. Thus, in $\mathbb{C}^2_{(x_1,x_2)}$ new Stokes curves and ordinary Stokes curves are connected and hence the Stokes surface of (7.6) inevitably contains the new Stokes curves of (7.5).

Theorem 7.4 strongly suggests that the exact WKB analysis for holonomic systems or completely integrable systems may play an important role also in the analysis of new Stokes curves for higher order ODEs.

Another peculiar feature of the Pearcey system is that the set Λ of its turning points has a unique cuspidal singularity at the origin $(x_1, x_2) = (0, 0)$. At this cuspidal singularity two turning points with different types coalesce. Furthermore, the virtual turning point of the BNR equation (7.5) also coalesces there.

For this cuspidal singular point of the Pearcey system Hirose proves the following intriguing result.

Theorem 7.5. (Hirose [22]**)** *Under some genericity condition every completely integrable system of two independent variables can be transformed* (*in the sense of exact WKB analysis*) *to the Pearcey system at a cuspidal singularity of the set of turning points.*

For the proof of Theorem 7.5 see [22].

To clarify the implication of Theorem 7.5, let us consider, for example, the following holonomic system:

$$\begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{2}{3}x_2\eta\dfrac{\partial^2}{\partial x_1^2} + \dfrac{1}{3}x_1\eta^2\dfrac{\partial}{\partial x_1} - \dfrac{\alpha}{3}\eta^3\right)\psi = 0, \\ \left(\eta\dfrac{\partial}{\partial x_2} - \dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0. \end{cases} \tag{7.13}$$

Eq. (7.13) is the so-called "$(1,4)$-hypergeometric system", an example of (confluent) hypergeometric systems of two variables. Note that, when x_2 is fixed, (7.13) is equivalent to (6.18) discussed in Section 6.3.

By a straightforward computation we confirm that the set of turning points of (7.13) has three cuspidal singular points. Then Theorem 7.5 tells us that at each cuspidal singular point the $(1,4)$-hypergeometric system (7.13) can be transformed to the Pearcey system. In particular, there exists a virtual turning point that coalesces with two ordinary turning points with different types at each cuspidal singularity. These three virtual turning points are all non-redundant and play an important role in describing the complete Stokes geometry of the higher order ODE (6.18) discussed in Section 6.3. In this way the cuspidal singularity of the set of turning points of a completely integrable system is closely related to the problem of (non-)redundant virtual turning points of a higher order ODE.

These two results of Hirose brings a new insight to the problem of new Stokes curves and virtual turning points for higher order ODEs. It is the future problem to develop the exact WKB analysis for completely integrable systems in a systematic manner.

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