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Abstract. The alternate discrete PI equation (or alt-dPI) arises from the continuous second Painlevé equation (PII) through Bäcklund transformations. In this announcement paper we consider Stokes phenomena for (alt-dPI) from the viewpoint of exact WKB analysis. After constructing transseries solutions and defining the Stokes geometry of (alt-dPI), we derive explicit connection formulas that describe Stokes phenomena for transseries solutions of (alt-dPI) on its Stokes curves. The derivation is based on the computation of Stokes multipliers of the Lax pair associated with (PII) and (alt-dPI). The detailed proof and computations will be discussed in our forthcoming paper.

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1. Introduction

As is well-known, solutions of the second Painlevé equation

$$\frac{d^2u}{dz^2} = 2u^3 + zu + c \tag{PII}$$

admit a Bäcklund transformation, that is, if u is a solution of (PII), then $\overline{u}$ (resp., $\underline{u}$) defined by

$$\overline{u} = -u - \frac{c+1/2}{u^2 + du/dz + z/2} \quad \left(\text{resp., } \underline{u} = -u - \frac{c-1/2}{u^2 - du/dz + z/2}\right) \tag{1.1}$$

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satisfies the same equation (PII) with the parameter c being shifted by 1 (resp., -1). Hence, regarding $\overline{u}$ (resp., $\underline{u}$) as a shift of u by 1 (resp., -1) with respect to the parameter c

$$\overline{u} = u\Big|_{c\mapsto c+1} \quad \left(\text{resp., } \underline{u} = u\Big|_{c\mapsto c-1}\right) \tag{1.2}$$

and eliminating du/dz from the defining equations (1.1) of $\overline{u}$ and $\underline{u}$, we obtain a discrete Painlevé equation known as (alt-dPI):

$$\frac{c+1/2}{\overline{u}+u} + \frac{c-1/2}{\underline{u}+u} + 2u^2 + z = 0. \tag{alt-dPI}$$

Note that the roles of the variables z and c are interchanged here from the original equation (PII), that is, the independent variable of (alt-dPI) is c and z is just a parameter there. Now the purpose of this paper is to make an asymptotic study of solutions of (alt-dPI) for an (arbitrarily) fixed z, in particular, to discuss Stokes phenomena for (alt-dPI) for fixed z from the viewpoint of exact WKB analysis; To be more specific, we will present explicit connection formulas that describe Stokes phenomena for formal transseries solutions of (alt-dPI).

There are not so many known results about Stokes phenomena for discrete Painlevé equations. One exception is a joint paper [10] of the first author with Lustri; In [10] Stokes phenomena for the discrete PI equation were investigated from the viewpoint of exponential asymptotics or the hyperasymptotic analysis (cf. [4]).

In this paper we employ the exact WKB analysis to analyze Stokes phenomena for (alt-dPI). To this aim we first introduce a large parameter η (i.e., inverse of the semi-classical parameter) into the equations by scaling of variables

$$u = \eta^{1/3}\lambda, \quad z = \eta^{2/3}t, \quad c = \eta\zeta. \tag{1.3}$$

This scaling of variables transforms (PII), (1.1), (1.2) and (alt-dPI) into

$$\eta^{-2}\frac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \zeta, \tag{PII}$$

$$\overline{\lambda} = \lambda\Big|_{\zeta\mapsto\zeta+\eta^{-1}} = -\lambda - \frac{\zeta+\eta^{-1}/2}{\lambda^2+\eta^{-1}(d\lambda/dt)+t/2}, \tag{1.4}$$

$$\left(\text{resp., } \underline{\lambda} = \lambda\Big|_{\zeta\mapsto\zeta-\eta^{-1}} = -\lambda - \frac{\zeta-\eta^{-1}/2}{\lambda^2-\eta^{-1}(d\lambda/dt)+t/2},\right) \tag{1.5}$$

$$\frac{\zeta+\eta^{-1}/2}{\overline{\lambda}+\lambda} + \frac{\zeta-\eta^{-1}/2}{\underline{\lambda}+\lambda} + 2\lambda^2 + t = 0. \tag{alt-dPI}$$

(To the scaled equations, for the sake of simplicity, we attach the same symbols (PII) and (alt-dPI) as to the original ones. We hope there will be no fear of confusions.) The exact WKB analysis, i.e., WKB analysis based on the Borel resummation technique, was initiated by Silverstone ([15]) and Voros ([19]) for one-dimensional stationary Schrödinger equations and later developed by the French school and the Japanese school ([5], [6], [13] and references cited therein). The exact WKB analysis was generalized also to the continuous Painlevé equations with a large parameter in a series of papers ([11], [2], [12], [16]) of the second author with Aoki and Kawai and, as its consequence, the connection formula for Ablowitz-Segur's connection problem for the second Painlevé equation (PII) was reobtained by the exact WKB approach ([17]). We should also mention that, by using the exact WKB analysis, Iwaki studied in [8], which is closely related to this paper, some Stokes phenomena (called "parametric Stokes phenomena") for solutios of (PII) that are observed when the parameter ζ (or c) changes. In this paper, extending the exact WKB analysis to discrete equations with a large parameter, we explicitly discuss Stokes phenomena for (alt-dPI). Here we only make an announcement of our results. The details will be discussed in our forthcoming paper.

The plan of the paper is as follows: In Section 2 we construct formal transseries solutions of (alt-dPI) that will be used for the description of Stokes phenomena. Then we define the Stokes geometry, that is, turning points and Stokes curves of (alt-dPI) in Section 3. Finally, in Section 4, we present explicit connection formulas describing Stokes phenomena for the transseries solutions of (alt-dPI) constructed in Section 2.

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2. Formal solutions of (alt-dPI)

The most important ingredients of the exact WKB analysis for differential equations are formal solutions (e.g., WKB solutions in the case of one-dimensional stationary Schrödinger equations), Stokes geometry (i.e., turning points and Stokes curves) and connection formulas that describe Stokes phenomena on Stokes curves. The situation is the same also for the discrete Painlevé equation (alt-dPI). Let us start with the construction of formal solutions of (alt-dPI).

2.1. Formal power series solution

We first note that, replacing the shift operators $\overline{\lambda}$ and $\underline{\lambda}$ by $\sum_{n\geq 0} \frac{\eta^{-n}}{n!}\frac{\partial^n \lambda}{\partial \zeta^n}$ and $\sum_{n\geq 0}(-1)^n \frac{\eta^{-n}}{n!}\frac{\partial^n \lambda}{\partial \zeta^n}$, respectively, we can regard (alt-dPI) as an ∞-order differential equation of WKB type. Then, in parallel to the case of continuous Painlevé equations (cf. [11]), we can readily construct the following formal power series solution of (alt-dPI):

$$\lambda^{(0)} = \lambda_0(\zeta) + \eta^{-1}\lambda_1(\zeta) + \eta^{-2}\lambda_2(\zeta) + \cdots , \tag{2.1}$$

where the top order term $\lambda_0(\zeta)$ satisfies

$$2\lambda_0^3 + t\lambda_0 + \zeta = 0 \tag{2.2}$$

and the lower order terms $\lambda_j(\zeta)$ $(j \geq 1)$ are uniquely determined in a recursive manner. Here, although each $\lambda_j(\zeta)$ depends also on t, we do not specify the dependence of $\lambda_j(\zeta)$ on t as we are considering (alt-dPI) with keeping t fixed.

For the formal power series solution the following holds.

Proposition 2.1. *The formal power series solution* $\lambda^{(0)}$ *of* (alt-dPI) *coincides with the formal power series solution of* (PII).

Proposition 2.1 suggests that we should consider not only (alt-dPI) but also (PII) simultaneously or, equivalently, we should consider a system of differential equations

$$\begin{cases} \eta^{-2}\dfrac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \zeta, \\ \overline{\lambda} \left(\stackrel{\mathrm{def}}{=} \displaystyle\sum_{n=0}^{\infty} \frac{\eta^{-n}}{n!}\frac{\partial^n \lambda}{\partial \zeta^n} \right) = -\lambda - \dfrac{\zeta + \eta^{-1}/2}{\lambda^2 + \eta^{-1}(d\lambda/dt) + t/2}. \end{cases} \tag{2.3}$$

In what follows we deal with this system (2.3) to construct formal solutions of (alt-dPI).

2.2. Transseries solution

It is not possible to describe Stokes phenomena solely in terms of formal power series solutions as they are almost unique. To describe Stokes phenomena we need more general solutions, called transseries solutions, which contain free parameters.

Transseries solutions of (2.3) are solutions of the following form:

$$\lambda = \lambda^{(0)} + \lambda^{(1)} + \cdots, \tag{2.4}$$

where $\lambda^{(0)}$ is a formal power series solution and $\lambda^{(1)} + \cdots$ are assumed to be exponentially small compared to $\lambda^{(0)}$. Then the subleading term $\lambda^{(1)}$ should satisfy the Fréchet derivative (i.e., variational equation) of (2.3) along $\lambda^{(0)}$, that is, $\lambda^{(1)}$ satisfies

$$\begin{cases} \eta^{-2}\dfrac{d^2}{dt^2}\lambda^{(1)} = \left(6(\lambda^{(0)})^2 + t\right)\lambda^{(1)}, \\ \overline{\lambda^{(1)}} = -\lambda^{(1)} + \left(\zeta + \dfrac{\eta^{-1}}{2}\right) \dfrac{2\lambda^{(0)}\lambda^{(1)} + \eta^{-1}(d\lambda^{(1)}/dt)}{((\lambda^{(0)})^2 + \eta^{-1}(d\lambda^{(0)}/dt) + t/2)^2}. \end{cases} \tag{2.5}$$

In particular, $\lambda^{(1)}$ is a WKB solution of this linear system (2.5). Furthermore, the remainder part (i.e., lower order terms $\lambda^{(k)}$ $(k \geq 2)$ of $\lambda^{(0)} + \lambda^{(1)} + \lambda^{(2)} + \cdots$) are recursively determined. Thus we obtain

Proposition 2.2. *For a given infinite series* α *specified by* (2.9) *below the system* (2.3) *has the following transseries solution:*

$$\lambda(t, \zeta, \eta; \alpha) = \lambda^{(0)} + \eta^{-1/2}\alpha\,\lambda^{(1)} + (\eta^{-1/2}\alpha)^2\,\lambda^{(2)} + \cdots, \tag{2.6}$$

where $\lambda^{(0)}$ *is a formal power series solution and* $\lambda^{(k)}$ $(k \geq 1)$ *is of the form*

$$\exp\Big(k\eta \int_{(t_0,\zeta_0)}^{(t,\zeta)} \omega_{-1}\Big) \sum_{n=0}^{\infty} \eta^{-n} \lambda_n^{(k)}(t,\zeta). \tag{2.7}$$

Here $\omega_{-1} = S_{-1}(t,\zeta)\, dt + Z_{-1}(t,\zeta)\, d\zeta$ *is a closed* 1*-form explicitly given by*

$$S_{-1} = \sqrt{6\lambda_0^2 + t}, \quad Z_{-1} = \cosh^{-1}\Big(\frac{8\lambda_0^3 - \zeta}{\zeta}\Big), \tag{2.8}$$

and α *is an infinite series of the form*

$$\alpha = \sum_{l=0}^{\infty} \alpha_l e^{2\pi i l \eta \zeta} \qquad (\alpha_l \in \mathbb{C}). \tag{2.9}$$

Remark 2.3. (i) $\overrightarrow{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ gives a set of free parameters of a transseries solution of (2.3).
(ii) If we fix t and regard $\lambda(t,\zeta,\eta;\alpha)$ as a formal series depending on the variable ζ, then we obtain a formal solution of (alt-dPI).

3. Stokes geometry of (alt-dPI)

We next consider the Stokes geometry of (alt-dPI). In what follows, as we are interested in the analysis of (alt-dPI), we let t be fixed.

The transseries solution of (alt-dPI) is provided by (2.6) (with t being fixed). There the phase factor of $\lambda^{(1)}$ (in the ζ-direction) is given by

$$\begin{aligned} Z_{-1,(\pm,l)}(\zeta) &= \operatorname{Cosh}^{-1}\Big(\frac{8\lambda_0^3 - \zeta}{\zeta}\Big) + 2\pi i l \\ &= \operatorname{Log}\left(\frac{8\lambda_0^3 - \zeta}{\zeta} \pm \frac{4\lambda_0^2}{\zeta}\sqrt{6\lambda_0^2 + t}\right) + 2\pi i l. \end{aligned} \tag{3.1}$$

Here we use the suffix $(\pm, l)$ $(l \in \mathbb{Z}_{\geq 0})$ to specify the branch of $Z_{-1}(\zeta)$. The Stokes geometry, that is, turning points and Stokes curves of (alt-dPI) are defined by this phase factor $Z_{-1,(\pm,l)}(\zeta)$ as follows:

Definition 3.1. (i) A point $\zeta = \widehat{\zeta}$ is said to be a turning point of (alt-dPI) if there exist two suffices $(*, l) \neq (*', l')$ for which

$$Z_{-1,(*,l)}(\widehat{\zeta}) = Z_{-1,(*',l')}(\widehat{\zeta}) \tag{3.2}$$

holds.
(ii) A Stokes curve of (alt-dPI) is defined by

$$\Im \int_{\widehat{\zeta}}^{\zeta} (Z_{-1,(*,l)} - Z_{-1,(*',l')}) d\zeta = 0, \tag{3.3}$$

where $\widehat{\zeta}$ is a turning point at which (3.2) is satisfied.

Example. Let t be fixed at $t = e^{\pi i/6}$. Then the Stokes geometry of (alt-dPI) (for $t = e^{\pi i/6}$) is given by Figure 1. Note that, as turning points and Stokes curves are

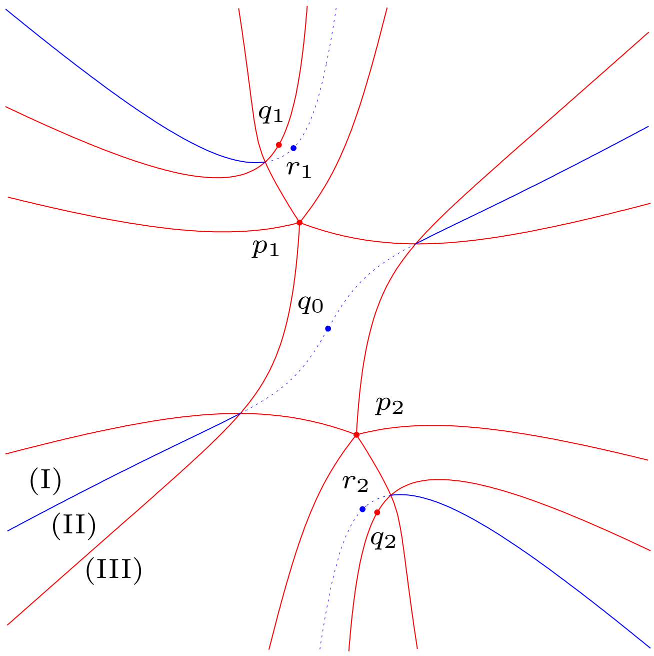

FIGURE 1. Stokes geometry of (alt-dPI) for $t = e^{\pi i/6}$.

defined through the top order term λ_0 of the formal power series solution and λ_0 is an algebraic function satisfying (2.2), Figure 1 is written on the Riemann surface of λ_0. To be more precise, Figure 1 is written on λ_0-plane since λ_0 itself gives a global coordinate of the Riemann surface.

When $t = e^{\pi i/6}$, there exist seven turning points. Among them p_1 and p_2 are turning points where the relation $Z_{-1,(+,l)}(p_k) = Z_{-1,(-,l)}(p_k)$ holds. Such a turning point is said to be of type $((+,l),(-,l))$. Similarly, q_k $(k = 0,1,2)$ is a turning point of type $((+,l),(+,l'))$ and r_k $(k = 1,2)$ is a turning point of type $((+,l),(-,l+2))$.

Remark 3.2. As is easily seen in Figure 1, there are several crossing points of Stokes curves for (alt-dPI). This is because (alt-dPI) is considered to be an ∞-order differential equation. As a matter of fact, such crossing points of Stokes curves often appear for higher order ordinary differential equations with a large parameter. Thus some Stokes curves of Figure 1 can be regarded as nonlinear analogue of "new Stokes curves" emanating from "virtual turning points" introduced by [3] and [1], respectively. We refer the reader to [7] for more details of new Stokes curves and virtual turning points.

4. Stokes phenomena for (alt-dPI)

In the preceding section we defined and presented an example (Figure 1) of the Stokes geometry of (alt-dPI). On each Stokes curve a Stokes phenomenon is expected to occur with transseries solutions of (alt-dPI). In this section, using linear differential-difference equations ("Lax pair") associated with (PII) and (alt-dPI), we analyze such Stokes phenomena and seek for connection formulas describing them in an explicit manner.

4.1. Lax pair associated with (PII) and (alt-dPI) and its Stokes geometry

The following system of linear differential-difference equations are associated with (PII) and (alt-dPI), that is, (PII) and (alt-dPI) describe its compatibility condition ([9], [14]). The system (4.1)-(4.3) is often called the "Lax pair".

$$\left(\eta^{-2}\frac{\partial^2}{\partial x^2} - Q_{\rm II}\right)\psi = 0, \tag{4.1}$$

$$\eta^{-1}\frac{\partial\psi}{\partial t} = A_{\rm II}\,\eta^{-1}\frac{\partial\psi}{\partial x} - \frac{\eta^{-1}}{2}\frac{\partial A_{\rm II}}{\partial x}\psi, \tag{4.2}$$

$$\overline{\psi} = g_{\rm II}\,\eta^{-1}\frac{\partial\psi}{\partial x} + f_{\rm II}\,\psi. \tag{4.3}$$

Here the explicit form of the coefficient functions $Q_{\rm II}$, $A_{\rm II}$, $f_{\rm II}$ and $g_{\rm II}$ is given as follows:

$$Q_{\rm II} = x^4 + tx^2 + 2\zeta x + \nu^2 - (\lambda^4 + t\lambda^2 + 2\zeta\lambda) - \eta^{-1}\frac{\nu}{x-\lambda} + \eta^{-2}\frac{3}{4(x-\lambda)^2}, \tag{4.4}$$

$$A_{\rm II} = \frac{1}{2(x-\lambda)}, \tag{4.5}$$

$$g_{\rm II} = \left((2\nu + 2\lambda^2 + t)(x-\lambda)(x-\overline{\lambda})\right)^{-1/2}, \tag{4.6}$$

$$f_{\rm II} = \left(x^2 - \lambda^2 - \nu + \eta^{-1}\frac{1}{2(x-\lambda)}\right) g_{\rm II}. \tag{4.7}$$

We use this system (4.1)-(4.3), especially the first equation (4.1) in the x-variable, to analyze the Stokes phenomena for (alt-dPI).

The following properties of the Lax pair play an important role in our discussion:

Fundamental properties of the Lax pair.

(i) *Suppose that analytic solutions of* (2.5)*, i.e., solutions of the simultaneous equations* (PII) *and* (alt-dPI) *are substituted into the coefficients of the Lax pair* (4.1)-(4.3)*. Then the Stokes multipliers of* (4.1) *become analytic functions of* (t, ζ)*. In particular, they are independent of* t *thanks to the isomonodromic property.*

(ii) *Stokes multipliers of* (4.1) *can be explicitly computed by applying the exact WKB analysis to* (4.1)*. The computation is based* (*hence heavily depends*) *on the Stokes geometry of* (4.1)*.*

(iii) *On each Stokes curve of* (alt-dPI) *some degenerate configuration is observed for the Stokes geometry of* (4.1)*. That is, if* ζ*, or more precisely a point* $\lambda_0(\zeta)$ *on the Riemann surface of* λ_0 *corresponding to* ζ*, is located on a Stokes curve of* (alt-dPI)*, there exist two turning points of* (4.1) *that are connected by a Stokes curve of* (4.1)*.*

Among these three properties the third one is crucially important. To see the relationship between the Stokes geometry of (alt-dPI) and that of (4.1) more concretely, let us take three regions (I), (II) and (III) on the Riemann surface of λ_0 specified in Figure 1 and observe the configuration of Stokes curves of (4.1) when $\lambda_0(\zeta)$ belongs to each Region (J) $(J = \mathrm{I}, \mathrm{II}, \mathrm{III})$. In what follows we take a turning point as an endpoint (t_0, ζ_0) of the integral in the definition of a transseries solution $\lambda(t, \zeta, \eta; \alpha)$ and substitute $\lambda(t, \zeta, \eta; \alpha)$ thus normalized into the coefficients of the Lax pair (4.1)-(4.3).

Example. Figure 2 shows the Stokes geometry of (4.1) when $\lambda_0(\zeta)$ belongs to each Region (J) $(J = \mathrm{I}, \mathrm{II}, \mathrm{III})$. Equation (4.1) has one double turning point at $x = \lambda_0$ and two simple turning points at a_1 and a_2. As is easily surmised from comparison between (I) and (II) of Figure 2, two simple turning points a_1 and a_2 are connected by a Stokes curve in the transition from Region (I) to Region (II). This degenerate configuration is observed exactly on a Stokes curve of (alt-dPI) separating two Regions (I) and (II) in Figure 1. Similarly, in the transition from Region (II) to Region (III) a double turning point λ_0 and a simple turning point a_1 are connected on a Stokes curve of (alt-dPI) separating Regions (II) and (III).

4.2. Stokes multipliers of (4.1)

Applying the exact WKB analysis to the linear equation (4.1) or, more specifically, using the connection formulas for second order linear ordinary differential equations in view of the configuration of Stokes curves given by Figure 2, we can compute the Stokes multipliers of (4.1). Such computation of Stokes multipliers through the exact WKB analysis was done in [16] for the linear equation associated with the first Painlevé equation and in [8] for that associated with the second Painlevé equation. Adjusting the computations in [16] and [8] to the current situation, we obtain the following explicit formulas for Stokes multipliers of (4.1). Note that in the case of (4.1) there exist six Stokes multipliers s_k $(k = 1, \ldots, 6)$ corresponding to six Stokes directions along which Stokes curves asymptotically

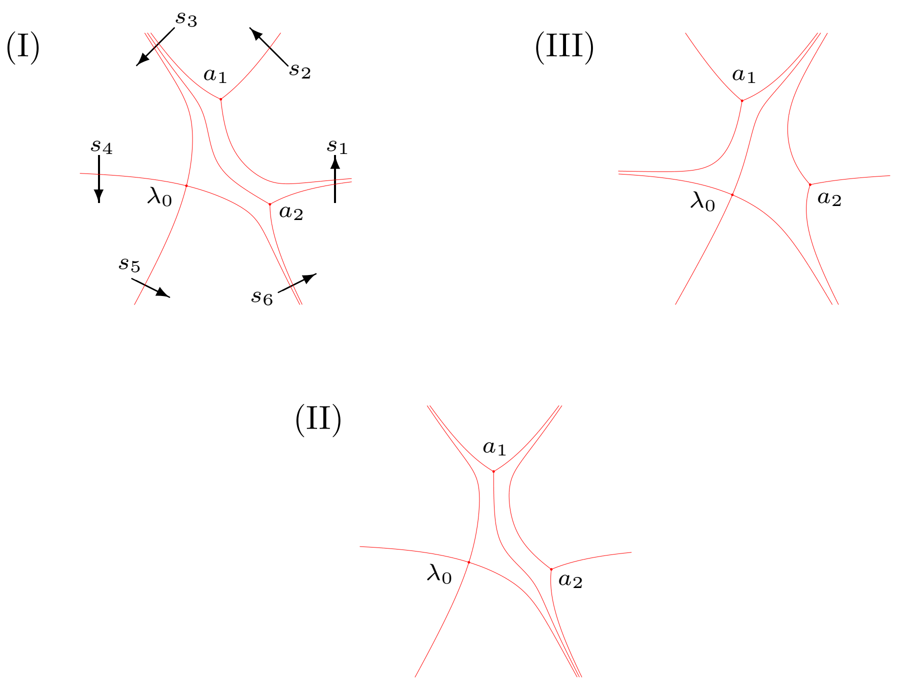

FIGURE 2. Stokes geometry of (4.1) in Region (J) $(J = \mathrm{I}, \mathrm{II}, \mathrm{III})$.

tend to an irregular singular point $x = \infty$ (cf. Figure 2,(I)). Since the computations depend on the Stokes geometry, the expressions for s_j differ according as $\lambda_0(\zeta)$ belongs to Region (I), (II) or (III). Thus in the following formulas we use the notation $s_k^{(J)}$ $(k = 1, \ldots, 6,\ J = \mathrm{I}, \mathrm{II}, \mathrm{III})$ to denote the Stokes multipliers when $\lambda_0(\zeta)$ belongs to Region (J).

Stokes multipliers of (4.1) **when** $\lambda_0(\zeta)$ **belongs to Region (I).**

$$\begin{cases} s_1^{(\mathrm{I})} = ie^{V}(1 + e^{2\pi i\eta\zeta}), \\ s_2^{(\mathrm{I})} = ie^{-V}e^{-2\pi i\eta\zeta}, \\ s_3^{(\mathrm{I})} = ie^{V}(1 + e^{-2\pi i\eta\zeta}), \\ s_4^{(\mathrm{I})} = -2\sqrt{\pi}\alpha_{\mathrm{I}}e^{-V}, \\ s_5^{(\mathrm{I})} = 0, \\ s_6^{(\mathrm{I})} = (2\sqrt{\pi}\alpha_{\mathrm{I}} + i)e^{-V}. \end{cases} \tag{4.8}$$

Stokes multipliers of (4.1) **when** $\lambda_0(\zeta)$ **belongs to Region (II).**

$$\begin{cases} s_1^{(\mathrm{II})} = ie^{V}, \\ s_2^{(\mathrm{II})} = ie^{-V}(1+e^{-2\pi i\eta\zeta}), \\ s_3^{(\mathrm{II})} = ie^{V}e^{-2\pi i\eta\zeta}, \\ s_4^{(\mathrm{II})} = -2\sqrt{\pi}\alpha_{\mathrm{II}}e^{-V}, \\ s_5^{(\mathrm{II})} = 0, \\ s_6^{(\mathrm{II})} = (2\sqrt{\pi}\alpha_{\mathrm{II}} + i(1+e^{2\pi i\eta\zeta}))e^{-V}. \end{cases} \tag{4.9}$$

Stokes multipliers of (4.1) **when** $\lambda_0(\zeta)$ **belongs to Region (III).**

$$\begin{cases} s_1^{(\mathrm{III})} = ie^{V}, \\ s_2^{(\mathrm{III})} = ie^{-V}(1+e^{-2\pi i\eta\zeta}), \\ s_3^{(\mathrm{III})} = ie^{V}e^{-2\pi i\eta\zeta}, \\ s_4^{(\mathrm{III})} = (-2\sqrt{\pi}\alpha_{\mathrm{III}} + ie^{2\pi i\eta\zeta})e^{-V}, \\ s_5^{(\mathrm{III})} = 0, \\ s_6^{(\mathrm{III})} = (2\sqrt{\pi}\alpha_{\mathrm{III}} + i)e^{-V}. \end{cases} \tag{4.10}$$

In these formulas α_J ($J = \mathrm{I}, \mathrm{II}, \mathrm{III}$) denotes the free parameter of a transseries solution given by (2.9) when $\lambda_0(\zeta)$ belongs to Region (J), and V designates the following formal series:

$$V = \sum_{n=1}^{\infty} \frac{2^{1-2n}-1}{2n(2n-1)} B_{2n}(\eta\zeta)^{1-2n}, \tag{4.11}$$

where B_{2n} stands for the Bernoulli number defined by

$$\frac{w}{e^w - 1} = 1 - \frac{w}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} w^{2n}. \tag{4.12}$$

The formal series V is called the "Voros coefficient", which appears from the comparison between WKB solutions of (4.1) normalized at a simple turning point and those normalized at $x = \infty$. For more details see [8]. (See also [18].)

4.3. Connection formula for (alt-dPI)

Let us assume that the two transseries solutions $\lambda(t, \zeta, \eta; \alpha_J)$ in Region (J) and $\lambda(t, \zeta, \eta; \alpha_{J+1})$ in Region ($J+1$) should define the same analytic solution of (alt-dPI). Then, thanks to the fundamental property (i), $s_k^{(J)} = s_k^{(J)}(\alpha_J)$ and $s_k^{(J+1)} = s_k^{(J+1)}(\alpha_{J+1})$ should be the same analytic function of ζ. This gives a constraint on α_J and α_{J+1}:

$$s_k^{(J)}(\alpha_J) = s_k^{(J+1)}(\alpha_{J+1}) \quad (k = 1, 2, \ldots, 6). \tag{4.13}$$

The relation (4.13) describes the Stokes phenomena for transseries solutions of (alt-dPI). Making use of the explicit formulas (4.8)-(4.10) for $s_k^{(J)}$, we thus obtain the following connection formula for (alt-dPI).

Connection formula for (alt-dPI).

Suppose that the transseries solutions $\lambda(t, \zeta, \eta; \alpha_J)$ *in Region* (J) $(J = \mathrm{I}, \mathrm{II}, \mathrm{III})$ *should define the same analytic solution of* (alt-dPI). *Then the following relations hold among the free parameters* α_J*.*

$$\alpha_{\mathrm{II}} = \alpha_{\mathrm{I}}(1 + e^{2\pi i \eta\zeta}), \tag{4.14}$$

$$\alpha_{\mathrm{III}} = \alpha_{\mathrm{II}} + \frac{i}{2\sqrt{\pi}} e^{2\pi i \eta \zeta}. \tag{4.15}$$

Remark 4.1. In terms of $\overrightarrow{\alpha_J} = (\alpha_0^{(J)}, \alpha_1^{(J)}, \alpha_2^{(J)}, \ldots)$ (cf. Remark 2.3), the formulas (4.14) and (4.15) can be expressed also as

$$\alpha_l^{(\mathrm{II})} = \alpha_l^{(\mathrm{I})} + \alpha_{l-1}^{(\mathrm{I})}, \tag{4.16}$$

$$\alpha_l^{(\mathrm{III})} = \alpha_l^{(\mathrm{II})} + \frac{i}{2\sqrt{\pi}} \delta_{l1} \tag{4.17}$$

$(l = 0, 1, 2, \ldots)$, where δ_{jk} denotes Kronecker's delta and $\alpha_{-1}^{(\mathrm{I})} = 0$.

Remark 4.2. The formula (4.15) immediately follows from comparison between (4.9) and (4.10). On the other hand, derivation of (4.14) is not so straightforward, since a Stokes phenomenon for the Voros coefficient V also occurs on a Stokes curve in question. Making comparison between (4.8) and (4.9) and taking the effect of a Stokes phenomenon of V into account, we obtain (4.14).

We finally remark that the relation (4.15) between α_{II} and α_{III} is the same as the connection formula for Stokes phenomena of the continuous first Painlevé equation (PI) discussed in [16]. Similarly, the relation (4.14) between α_{I} and α_{II} is the same as the connection formula for parametric Stokes phenomena of the continuous second Painlevé equation (PII) studied by Iwaki [8]. Thus both Stokes phenomena of (PI) type and those of (PII) type occur with the discrete Painlevé equation (alt-dPI).

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