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Stokes geometry of higher order linear ordinary differential equations and middle convolution

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Abstract

The middle convolution, introduced by Katz and developed by Dettweiler-Reiter, Oshima and others, defines an operation of reduction of linear ordinary differential equations with polynomial coefficients. In this paper, employing an idea of the exact steepest descent method proposed by Aoki-Kawai-Takei, we study how to determine the complete Stokes geometry and how to obtain the Borel summability of WKB solutions of a higher order linear ordinary differential equation with a large parameter when it is reduced to a second order equation via middle convolution. To show the practical usefulness of the method, we also investigate some concrete examples numerically.

1 Introduction

Let

$$P\varphi(x,\eta) = \sum_{j=0}^{n} a_j(x)(\eta^{-1}\partial_x)^j \varphi(x,\eta) = 0 \tag{1.1}$$

be an ordinary differential equation with polynomial coefficients containing a large parameter $\eta > 0$. Here n is a positive integer, $\partial_x = \partial/\partial x$, $a_n(x) \equiv a_{n0} \neq 0$ (non-zero complex constant) and $a_j(x)$ $(j = 1, \ldots, n-1)$ is a polynomial

$$a_j(x) = \sum_{k=0}^{\deg a_j} a_{jk} x^k \tag{1.2}$$

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with a_{jk} being a complex constant. For Eq. (1.1) there exists a formal (divergent) solution, called a WKB solution, of the following form:

$$\begin{aligned}\varphi(x,\eta) &= \eta^{-1/2}\exp\left(\int^x S\,dx\right)\\ &= \eta^{-1/2}\exp\left(\int^x (\eta S_{-1}(x)+S_0(x)+\eta^{-1}S_1(x)+\cdots)\,dx\right)\\ &= \exp\left(\eta\int^x S_{-1}(x)\,dx\right)\sum_{m=0}^{\infty}\varphi_m(x)\eta^{-m-1/2}, \qquad (1.3)\end{aligned}$$

where the top order term $S_{-1}(x)$ is a root of the characterstic equation

$$\sigma_0(P)(x,\zeta)=\sum_{j=0}^{n}a_j(x)\zeta^j=0 \qquad (1.4)$$

of (1.1) and lower order terms $S_m(x)$ ($m \geq 0$) are uniquely determined once $S_{-1}(x)$ is fixed. In the case of second order equations (i.e., $n = 2$), as is discussed in, e.g., [8], [14], etc., the exact WKB analysis, that is, the analysis based on the use of Borel resummed WKB solutions, provides a very powerful tool for analyzing the global behavior of solutions of (1.1) in the complex domain. The most important ingredients there are the Stokes geometry and the connection formulas: The Stokes geometry describes regions (sometimes called Stokes regions) where WKB solutions are Borel summable and the connection formulas describe relations between the Borel sums of WKB solutions in different Stokes regions. In the case of higher order equations (i.e., $n \geq 3$), however, the complete structure of the Stokes geometry is not fully understood yet. As was first pointed out by Berk-Nevins-Roberts ([6]), Stokes curves of a higher order equation may cross in general and what they call a new Stokes curve appears from some crossing points of Stokes curves. To determine the complete structure of the Stokes geometry for higher order equations is one of the most important problems in the exact WKB analysis.

On the other hand, recently Oshima ([17]) has developed a systematic study of ordinary differential equations with polynomial coefficients and produced many remarkable results by using middle convolutions and additions. The middle convolution is an operation of reduction and plays a central role in his study. It was first introduced by Katz ([13]) and reformulated as an operation on Fuchsian systems by Dettweiler-Reiter ([9]). The following definition, a modified one of that given in [17], of the middle convolution for Eq. (1.1) containing a large parameter η is given by Iwaki-Koike ([11]), where they apply the middle convolution to the computation of the so-called Voros coefficients.

Definition 1.1. Let $\mu \in \mathbb{C}\setminus\{0\}$ be a non-zero complex constant. For a differential operator P of the form (1.1) we define its middle convolution (with a large parameter η) by

$$mc_{\mu\eta}P = (\eta^{-1}\partial_x)^l \circ \mathrm{Ad}(\partial_x^{-\mu\eta})P,$$

where $l = \max\{k - j; a_{jk} \neq 0\}$ and

$$\mathrm{Ad}(\partial_x^{-\mu\eta})P = \partial_x^{-\mu\eta} \circ P \circ \partial_x^{\mu\eta}.$$

Since

$$\mathrm{Ad}(\partial_x^{-\mu\eta})x = x - \mu\eta\partial_x^{-1}, \quad \mathrm{Ad}(\partial_x^{-\mu\eta})x^k = (x - \mu\eta\partial_x^{-1})^k, \quad \mathrm{Ad}(\partial_x^{-\mu\eta})\partial_x^j = \partial_x^j$$

hold, the middle convolution of an operator P of the form (1.1) is explicitly given by

$$mc_{\mu\eta}P = (\eta^{-1}\partial_x)^l \sum_{j=0}^{n} \sum_{k=0}^{\deg a_j} a_{jk}(x - \mu\eta\partial_x^{-1})^k(\eta^{-1}\partial_x)^j. \tag{1.5}$$

In particular, $mc_{\mu\eta}P$ is a differential operator of order $n + l$. In what follows we denote $mc_{\mu\eta}P$ simply by $\tilde{P}$. Note that, if $\varphi(x, \eta)$ is a solution of Eq. (1.1), then a solution $\tilde{\varphi}(x, \eta)$ of $\tilde{P}\tilde{\varphi} = 0$ is provided by the Euler transform

$$\tilde{\varphi}(x, \eta) = \frac{1}{\Gamma(\mu\eta)} \int_c^x (x - z)^{\mu\eta-1}\varphi(z, \eta)\, dz, \tag{1.6}$$

where c is a suitably chosen point. The right-hand side of (1.6) is known to be the Riemann-Liouville integral or the fractional derivation of $\varphi(x, \eta)$.

Then, a natural question arises:

> *If $\tilde{P}$ is the middle convolution of a differential operator P, how is the Stokes geometry of $\tilde{P}$ related to that of P ?*

The purpose of this paper is to consider this problem by employing an idea of the exact steepest descent method for the Laplace transform developed by Aoki-Kawai-Takei ([3, 4]). As its consequence we will show that, when a higher order ordinary differential equation is reduced to a second order equation via middle convolution, we can obtain important information (almost the characterization) for the Stokes geometry of the higher order equation from that of the second order equation.

The paper is organized as follows: In Section 2, after preparing some notions and notations, we state our main theorem. Then we explain our proof of the main theorem in Sections 3 and 4. In Section 3 we deal with the local aspect of the problem and in Section 4 we consider the problem in the global setting. Finally, in Section 5 we study some concrete examples with the aid of a computer to examine the main result practically.

2 Main result

In this section we formulate and state our main result. Before stating the main theorem, we review some basic properties of WKB solutions to prepare some notions and notations.

We denote the roots of the characteristic equation (1.4) by $\zeta_1(x), \ldots, \zeta_n(x)$ and let $\varphi_j(x,\eta)$ be a WKB solution of Eq. (1.1) determined by $S_{-1}(x) = \zeta_j(x)$, that is,

$$\varphi_j(x,\eta) = \exp\left(\eta \int^x \zeta_j(x)\,dx\right) \sum_{m=0}^{\infty} \varphi_{j,m}(x)\eta^{-m-1/2}. \tag{2.1}$$

As $\varphi_j(x,\eta)$ is not convergent in general, in the exact WKB analysis we consider the Borel sum of $\varphi_j(x,\eta)$, which is defined as follows: Let $\varphi_{j,B}(x,y)$ (or $\mathcal{B}[\varphi_j](x,y)$) be the Borel transform of $\varphi_j(x,\eta)$, i.e.,

$$\varphi_{j,B}(x,y) = \mathcal{B}[\varphi_j](x,y) = \sum_{m=0}^{\infty} \frac{\varphi_{j,m}(x)}{\Gamma(n+1/2)}(y+s_j(x))^{m-1/2}, \tag{2.2}$$

where $s_j(x) = \int^x \zeta_j(x)dx$ and $\Gamma(z)$ denotes Euler's gamma function. Then the Borel sum $\Phi_j(x,\eta)$ of $\varphi_j(x,\eta)$ is, by definition,

$$\Phi_j(x,\eta) = \int_{-s_j(x)}^{\infty} e^{-\eta y}\varphi_{j,B}(x,y)\,dy. \tag{2.3}$$

Here the path of integration is taken to be parallel to the positive real axis. For the basic properties of the Borel sum and the Borel transform we refer the reader to [5], [7]. When the Borel sum (2.3) is well-defined, $\varphi_j(x,\eta)$ is said to be Borel summable. To be more precise, $\varphi_j(x,\eta)$ is Borel summable when there exists a positive constant δ so that $\varphi_{j,B}(x,y)$ can be analytically continued to a tubular domain

$$E^{\delta}_{-s_j(x)} := \bigcup_{t\geq 0}\{y \in \mathbb{C}; |y - t + s_j(x)| < \delta\}$$

in y-plane and further $\varphi_{j,B}(x,y)$ satisfies an exponential estimate

$$|\varphi_{j,B}(x,y)| \leq C_1 \exp(C_2|y|)$$

in $E^{\delta}_{-s_j(x)}$ with some positive constants C_1, C_2.

To describe the region where a WKB solution is Borel summable, we introduce the following

Definition 2.1. (i) When the characteristic equation (1.4) has a multiple root (for ζ) at $x = a$, a is said to be a turning point. In other words, a turning point is a zero of the discriminant of (1.4). In particular, a simple zero of the discriminant is called a simple turning point. When two roots $\zeta_j(x)$ and $\zeta_{j'}(x)$ ($j \neq j'$) of (1.4) merge at a turning point $x = a$, a is said to be of type (j, j').
(ii) Let $x = a$ be a turning point of type (j, j'). Then, a curve emanating from a defined by

$$\Im \int_a^x (\zeta_j(x) - \zeta_{j'}(x))\,dx = 0$$

is called a Stokes curve of type (j, j'). In particular, a Stokes curve is said to be of type $j > j'$ (resp., $j < j'$) when $\Re \int_a^x (\zeta_j(x) - \zeta_{j'}(x))\,dx > 0$ (resp., < 0) holds on it.

A region surrounded by Stokes curves is called a Stokes region. Note that at a simple turning point $x = a$ there exist exactly two roots $\zeta_j(x)$ and $\zeta_{j'}(x)$ of (1.4) that merge at $x = a$ and

$$\left.\frac{\partial \sigma_0(P)}{\partial x}\right|_{(x,\zeta)=(a,\zeta_j(a))} \neq 0$$

holds.

In the case of second order equations (i.e., $n = 2$), the region where a WKB solution is Borel summable is completely described by Stokes curves. That is, the following Theorem 2.1 holds. Here, and throughout this paper, we assume the following condition to avoid some difficulties caused by degenerate situations:

There exists no Stokes curve that connects two turning points. (ND)

Note that the condition (ND) means that every Stokes curve emanating from a turning point flows into the point at infinity.

Theorem 2.1 ([15])**.** *Let $n = 2$ and suppose the condition* (ND)*. Then a WKB solution is Borel summable except on Stokes curves. That is, a WKB solution is Borel summable in each Stokes region.*

Furthermore, in the case of second order equations WKB solutions enjoy the following properties (Theorems 2.2 and 2.3) on a Stokes curve emanating from a simple turning point.

Theorem 2.2 ([18],[1],[12])**.** *Let $n = 2$ and suppose* (ND)*. We denote the roots of the characteristic equation* (1.4)*, which is quadratic, by $\zeta = \zeta_\pm(x)$. Let $x = 0$ be a simple turning point and $\varphi_\pm$ be WKB solutions of* (1.1) *normalized at $x = 0$. Then, when x lies on a Stokes curve of type $+ > -$ emanating from $x = 0$, the Borel transform $\varphi_{+,B}(x, y)$ of a WKB solution φ_+ has a singularity at $y = -\int_0^x \zeta_- dx$. This is the unique singularity of $\varphi_{+,B}(x, y)$ in a neighborhood of*

$$\left\{ y \in \mathbb{C}; y = -\int_0^x \zeta_+ dx + v,\ v > 0 \right\}.$$

Furthermore, the Borel transform $\varphi_{-,B}(x, y)$ of φ_- has no singularity in a neighborhood of

$$\left\{ y \in \mathbb{C}; y = -\int_0^x \zeta_- dx + v,\ v > 0 \right\},$$

and hence φ_- is Borel summable on a Stokes curve of type $+ > -$.

Here WKB solutions $\varphi_\pm$ of (1.1) normalized at a simple turning point $x = a$ mean WKB solutions of the form

$$\varphi_\pm(x, \eta) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(-\eta \int_a^x \frac{a_1(x)}{2a_2(x)}\, dx \pm \int_a^x S_{\text{odd}}\, dx \right), \tag{2.4}$$

where S_{odd} (resp., S_{even}) denotes the odd (resp., even) part of $(\partial/\partial x)\log\varphi_{\pm}$, i.e., $(\partial/\partial x)\log\varphi_{\pm} = \pm S_{\mathrm{odd}} + S_{\mathrm{even}}$, and the integral of S_{odd} in the right-hand side of (2.4) is considered to be a contour integral starting from x and returning to x after encircling a. Note that, since

$$S_{\mathrm{even}} = -\frac{1}{2}\frac{\partial}{\partial x}\log S_{\mathrm{odd}} - \eta\frac{a_1(x)}{2a_2(x)}$$

holds in the case of second order equations, (2.4) actually gives a WKB solution of (1.1).

Theorem 2.3 ([18],[1]). *Let $n = 2$ and suppose* (ND). *Let $x = 0$ be a simple turning point and C a Stokes curve of type $+ > -$ emanating from $x = 0$. Then for the Borel transforms $\varphi_{\pm,B}(x, y)$ of WKB solutions $\varphi_{\pm}$ normalized at $x = 0$ the following relation holds:*

$$\Delta_{y=-\int_0^x \zeta_- dx}\, \varphi_{+,B}(x, y) = \sqrt{-1}\varphi_{-,B}(x, y), \tag{2.5}$$

where $\Delta_{y=\sigma(x)} f(x, y)$ stands for the discontinuity of $f(x, y)$ along the cut $\{y; y = \sigma(x) + v, v > 0\}$, that is, the difference of the boundary value of $f(x, y)$ from upper-side of the cut and that from lower-side of the cut. Furthermore, if we denote two Stokes regions neighboring along C by U_k $(k = 1, 2)$ and the Borel sums of $\varphi_{\pm}$ in U_k by $\Phi_{\pm}^{(k)}$, we have the following relation ("*connection formula*"):

$$\begin{cases} \Phi_+^{(1)} &= \Phi_+^{(2)} \pm \sqrt{-1}\Phi_-^{(2)}, \\ \Phi_-^{(1)} &= \Phi_-^{(2)}. \end{cases} \tag{2.6}$$

Here we take the $+$-sign (*resp., $-$-sign*) *in the right-hand side of* (2.6) *when the path of analytic continuation from U_1 to U_2 crosses C in an anti-clockwise* (*resp., clockwise*) *manner viewed from $x = 0$.*

In the case of higher order equations (i.e., $n \geq 3$), however, the situation becomes much more complicated. Stokes curves are not sufficient to describe the region where a WKB solution is Borel summable. As a matter of fact, as already explained in Introduction, Berk et al. ([6]) first pointed out that Stokes curves of a higher order equation may cross in general and that the Borel summability of a WKB solution may break down on what they call a new Stokes curve emanating from a crossing point of Stokes curves, in particular, an ordered crossing point of Stokes curves which is defined as follows:

Definition 2.2. Let j, k, l be a triple of mutually distinct indices. Then a crossing point of a Stokes curve of type $j > k$ and a Stokes curve of type $k > l$ is called an ordered crossing point. A crossing point which is not an ordered crossing point is called a non-ordered crossing point.

Later, by making use of microlocal analysis, Aoki-Kawai-Takei ([2]) introduced the notion of virtual turning points and defined a new Stokes curve to be a Stokes curve emanating from a virtual turning point. Note that the Borel summability of a WKB solution does not break down on all portions of new Stokes curves. For example, the Borel summability does hold near a virtual turning point; the Borel summability breaks down only on the portion of a new Stokes curve after passing over an ordered crossing point. In this way the Borel summability of WKB solutions of higher order equations is a very subtle problem. For the precise definition of virtual turning points and more detailed properties of virtual turning points and new Stokes curves we refer the reader to [10]. In what follows, if there is no fear of confusions, new Stokes curves are also called Stokes curves for short.

Now, in this paper, to discuss the Borel summability of WKB solutions of a higher order equation, we consider the relationship between the Stokes geometry of $\tilde{P}$ and that of P when $\tilde{P}$ is the middle convolution of P. In particular, we investigate the case where a higher order equation $\tilde{P}\psi = 0$ is obtained from a second order equation $P\varphi = 0$ via middle convolution. For that purpose we employ an idea of the exact steepest descent method developed by Aoki-Kawai-Takei ([3, 4]).

In [3, 4], for a differential equation of the form (1.1), its solution given by the integral

$$\varphi(x, \eta) = \int e^{\eta x \xi} \hat{\varphi}_k \, d\xi \tag{2.7}$$

is studied, where $\hat{\varphi}_k$ is a WKB solution of the Laplace transform (with a large parameter with respect to the independent variable x)

$$\hat{P}\hat{\varphi}(\xi, \eta) = \sum_{j=0}^{n} \sum_{k=0}^{\deg a_j} a_{jk} \left(-\eta^{-1} \frac{\partial}{\partial \xi} \right)^k \xi^j \hat{\varphi}(\xi, \eta) = 0$$

of $P\varphi = 0$. The integrand of (2.7) has the form $\exp(\eta F_k + O(\eta^0))$ with some function F_k and hence, if we take a steepest descent path of $\Re F_k$ passing through a saddle point $\xi_j(x)$ of F_k (i.e., a zero of $\partial F_k / \partial \xi$ in ξ-plane) as the integration path of (2.7), we can expect that the integral (2.7) gives a WKB solution of $P\varphi = 0$. Here one important point is that, when the steepest descent path in question crosses a Stokes curve of $\hat{P}$ of type $k > k'$ ($k \neq k'$), we have to bifurcate another steepest descent path of $\Re F_{k'}$ passing through the crossing point. We repeat this bifurcation process of steepest descent paths until no further new crossing point of a steepest descent path and a Stokes curve appears and call the totality of such steepest descent paths "*an exact steepest descent path*" passing through a saddle point $\xi_j(x)$ when the process terminates. The exact steepest descent method deals with the integral of the form (2.7) along such an exact steepest descent path. In [3, 4] it is examined that the integral (2.7) along an exact steepest descent path gives the Borel sum of a WKB solution of $P\varphi = 0$ and further that a Stokes phenomenon occurs with the Borel sum of a WKB solution if and only if an exact steepest descent path passes through another saddle point. Note that such an exact steepest descent path also appears in the study of the Borel summability of some formal solutions of second order inhomogeneous linear ordinary differential equations (cf. [16]).

We apply this idea of the exact steepest descent method to our problem. In our current situation a solution of $\tilde{P}\psi = 0$, obtained from $P\varphi = 0$ via middle convolution, is given by

$$\psi(x, \eta) = \int (x - z)^{\mu\eta - 1} \varphi_k(z, \eta)\, dz, \tag{2.8}$$

where φ_k is a WKB solution of the original equation $P\varphi = 0$. Since the integrand of (2.8) again has the form $\exp(\eta f_k + O(\eta^0))$ with

$$f_k = \mu \log(x - z) + \int^z \zeta_k(z)\, dz,$$

where $\zeta_k(z)$ denotes a root of the characteristic equation (1.4), we take a steepest descent path of $\Re f_k$ passing through a saddle point $z_j(x)$ of f_k as the integration path of (2.8). Then we can expect that the integral (2.8) gives a suitably normalized WKB solution $\eta^{-1/2}\psi_j$ of $\tilde{P}\psi = 0$. (For the precise definition of ψ_j see (3.13) below.) Furthermore, when the steepest descent path in question crosses a Stokes curve of $P\varphi = 0$ of type $k > k'$ $(k \neq k')$, we bifurcate another steepest descent path of $\Re f_{k'}$ passing through the crossing point and repeat this bifurcation process of steepest descent paths until the process terminates, that is, until no further new crossing point of a steepest descent path and a Stokes curve appears. Following the terminology of [3, 4], we call the totality of such steepest descent paths "*an exact steepest descent path*" of (2.8) passing through a saddle point $z_j(x)$.

Then our main theorem can be formulated as follows:

Main Theorem. *Let $\tilde{P}\psi = 0$ be obtained from $P\varphi = 0$ via middle convolution. Suppose that P is of second order and that all the turning points are simple and the condition* (ND) *holds for $P\varphi = 0$. Let $z_j(x)$ be a saddle point of* (2.8) *which does not coincide with a turning point. We further assume that the above bifurcation process terminates in finite steps so that an exact steepest descent path passing through $z_j(x)$ is well-defined. Then, for a given x, if the exact steepest descent path passing through $z_j(x)$ does not pass through any other saddle point, a WKB solution $\eta^{-1/2}\psi_j(x, \eta)$ of $\tilde{P}\psi = 0$ corresponding to the integral* (2.8) *along an* (*exact*) *steepest descent path passing through $z_j(x)$ is Borel summable at x.*

Corollary 2.1. *Under the situation of Main Theorem, a point x is not located on any Stokes curve of $\tilde{P}\psi = 0$, if any exact steepest descent path passing through a saddle point of the integral* (2.8) *does not pass through any other saddle point.*

Remark 2.1. As we will see in Section 5, it sometimes happens that, although the exact steepest descent path passing through $z_j(x)$ hits another saddle point, a WKB solution $\eta^{-1/2}\psi_j(x, \eta)$ becomes Borel summable due to some subtle cancellations of connection coefficients. However, except for such cases where subtle cancellations occur, it is expected that the converse of Main Theorem is also true. In other words, the geometry of exact steepest descent paths of the integral (2.8) almost characterizes the Stokes geometry of $\tilde{P}\psi = 0$.

Remark 2.2. As mentioned in Introduction, the Borel summability of WKB solutions and the complete structure of Stokes geometry are well understood only for second order equations at the present stage. This is the reason why in Main Theorem we impose the constraint that P is of second order. However it is expected that Main Theorem also holds without this constraint. For example, as is clear from the discussion in the subsequent sections, Main Theorem does hold also for a higher order operator P if the structure of its complete Stokes geometry is known.

In the subsequent two sections we will give a proof of Main Theorem. In Section 3 we consider the relationship between the integral (2.8) and a WKB solution $\eta^{-1/2}\psi_j(x,\eta)$ locally near a saddle point and then discuss their global correspondence in Section 4 by taking the effect of an exact steepest descent path into account.

3 Proof of Main Theorem, I — Local analysis near saddle points

In this section, as the first step toward the proof of Main Theorem, we consider the relationship between the integral (2.8) and a WKB solution $\eta^{-1/2}\psi_j(x,\eta)$ of $\tilde{P}\psi = 0$ locally near a saddle point.

Let $P\varphi = 0$ be a differential equation of the form (1.1). Let $x = a$ be a simple turning point of this differential equation and $\varphi_k(x,\eta)$ be a WKB solution normalized at $x = a$ with $S_{-1}(x) = \zeta_k(x)$. Then we can rewrite (2.8) as follows:

$$\begin{aligned}\int (x-z)^{\mu\eta-1}\varphi_k(z,\eta)dz &= \int (x-z)^{\mu\eta-1}\exp\left(\int_a^z (\eta\zeta_k(z)+\cdots)dz\right)dz\\ &= \int \exp\left(\eta\left(\mu\log(x-z)+\int_a^z \zeta_k(z)dz\right)+\cdots\right)dz,\end{aligned}$$

where $\cdots$ denotes lower order terms with respect to η. Here, and throughout this paper, we define f_k as

$$f_k = \mu\log(x-z)+\int_a^z \zeta_k(z)dz. \tag{3.1}$$

Let $z_j(x)$ be a saddle point of f_k and $C^{(k)}_{z_j(x)}$ be the steepest descent path of $\Re f_k$ passing through $z_j(x)$.

There is the following relation between $z_j(x)$ and characteristic roots of $\tilde{P}$.

Proposition 3.1. *The following statements (i) and (ii) are equivalent.*
(i) There exists k such that z is a saddle point of f_k.
(ii) $\frac{\mu}{x-z}$ is a characteristic root of $\tilde{P}$.

To prove Proposition 3.1 we make use of the following lemma verified in [11].

Lemma 3.1 ([11], Lemma 2.5)**.** *Let P be a differential operator of the form* (1.1) *and $\tilde{P} = mc_{\mu\eta}P$ be its middle convolution. Then the total symbol of $\tilde{P}$ is given by*

$$\sigma(\tilde{P})(x,\zeta) = \sum_{k\geq 0} \frac{(-1)^k}{k!}[\mu - \eta^{-1}l]_k \frac{\partial^k \sigma_0(P)}{\partial x^k}\zeta^{l-k}, \tag{3.2}$$

where $\sigma_0(P)(x,\zeta) = 0$ denotes the characteristic equation of P and

$$[\lambda]_k = \begin{cases} 1 & (k=0), \\ \lambda(\lambda+\eta^{-1})\cdots(\lambda+(k-1)\eta^{-1}) & (k\geq 1). \end{cases}$$

In particular, by taking the leading part of (3.2) *with respect to η, we find that the characteristic equation $\sigma_0(\tilde{P})(x,\zeta) = 0$ of $\tilde{P}$ is given as follows:*

$$\sigma_0(\tilde{P})(x,\zeta) = \zeta^l \sigma_0(P)\left(x - \frac{\mu}{\zeta}, \zeta\right).$$

Proof of Proposition 3.1. By the definition of $z_j(x)$, (i) is equivalent to $\frac{-\mu}{x-z} + \zeta_k(z) = 0$. In terms of $w = \frac{\mu}{x-z}$ we can rewrite this as $\zeta_k(x-\mu/w) = w$, that is, $\sigma_0(P)(x-\mu/w, w) = 0$. It then follows from Lemma 3.1 that this is equivalent to $\sigma_0(\tilde{P})(x,w) = 0$, that is, $\sigma_0(\tilde{P})(x, \frac{\mu}{x-z}) = 0$. Hence we obtain (ii). □

Corollary 3.1. *There exist $n+l$ saddle points of f_k.*

From now on we assume that the order of all zeros of $\partial f_k/\partial z$ is 1.

Taking the steepest descent path $C^{(k)}_{z_j(x)}$ as the integration path of (2.8), we now consider the integral

$$\int_{C^{(k)}_{z_j(x)}} (x-z)^{\mu\eta-1}\Phi_k(z,\eta)dz, \tag{3.3}$$

where $\Phi_k(z,\eta)$ is the Borel sum of a WKB solution $\varphi_k(z,\eta)$ of $P\varphi = 0$. In what follows we will see that (3.3) is related to a WKB solution of $\tilde{P}\psi = 0$ of the form

$$\psi_j = \eta^{-1/2}\exp\left(\eta\int^x \frac{\mu dx}{x - z_j(x)} + \cdots\right). \tag{3.4}$$

We let $f_{k,0} = f_k(z_j(x))$ and parametrize $C^{(k)}_{z_j(x)}$ by $f_k = f_{k,0} - u^2$ $(u \in \mathbb{R})$. Then we have

$$\begin{aligned} &\int_{C^{(k)}_{z_j(x)}} (x-z)^{\mu\eta-1}\Phi_k(z)dz \\ &= \int_{C^{(k)}_{z_j(x)}} (x-z)^{\mu\eta-1} \int_{\substack{\tilde{y} = -\int_a^z \zeta_k(z)dz + v \\ v\geq 0}} e^{-\eta\tilde{y}}\varphi_{k,B}(z,\tilde{y})d\tilde{y}dz \\ &= \int_{C^{(k)}_{z_j(x)}} \int_{\substack{\tilde{y} = -\int_a^z \zeta_k(z)dz + v \\ v\geq 0}} \exp(\eta(\mu\log(x-z) - \tilde{y})\varphi_{k,B}(z,\tilde{y})(x-z)^{-1}d\tilde{y}dz \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&= \int_{\substack{f_k=f_{k,0}-u^2\\ u\in\mathbb{R}}} \int_{\substack{y=-f_k+v\\ v\geq 0}} e^{-\eta y}\varphi_{k,B}(z, y+\mu\log(x-z))(x-z)^{-1}dydz\\
&= \int_{\substack{f_k=f_{k,0}-u^2\\ u\in\mathbb{R}}} \int_{\substack{y=-f_{k,0}+w\\ w\geq u^2}} e^{-\eta y}\varphi_{k,B}(z, y+\mu\log(x-z))(x-z)^{-1}dydz\\
&= \int_{\substack{y=-f_{k,0}+w\\ w\geq 0}} e^{-\eta y} \int_{\substack{f_k=f_{k,0}-u^2\\ -\sqrt{w}\leq u\leq\sqrt{w}}} \varphi_{k,B}(z, y+\mu\log(x-z))(x-z)^{-1}dzdy.
\end{aligned}
$$

Here we have used a change of variable $y=\tilde{y}-\mu\log(x-z)$ in the third equality and put $w=u^2+v$ in the fourth equality.

On the other hand, since

$$
\begin{aligned}
\frac{df_{k,0}}{dx} &= \frac{d}{dx}\left(\mu\log(x-z_j(x)) + \int_a^{z_j(x)} \zeta_k(z)dz\right)\\
&= \frac{\mu(1-z_j'(x))}{x-z_j(x)} + z_j'(x)\zeta_k(z_j(x))\\
&= \frac{\mu}{x-z_j(x)} + z_j'(x)\frac{\partial}{\partial z}\left(\mu\log(x-z) + \int_a^z \zeta_k(z)dz\right)\bigg|_{z=z_j(x)}\\
&= \frac{\mu}{x-z_j(x)}
\end{aligned}
$$

holds, we can write

$$
f_{k,0} = \int_{\tilde{a}}^x \frac{\mu dx}{x-z_j(x)}
$$

with an appropriately chosen point $\tilde{a}$ satisfying $f_k(z_j(\tilde{a}))=0$. Hence we obtain

$$
\int_{C^{(k)}_{z_j(x)}} (x-z)^{\mu\eta-1}\Phi_k(z)dz = \int_{\substack{y=-\int_{\tilde{a}}^x \frac{\mu dx}{x-z_j(x)}+w\\ w\geq 0}} e^{-\eta y}\chi(x,y)dy, \tag{3.6}
$$

where

$$
\chi(x,y) = \int_{\substack{f_k=f_{k,0}-u^2\\ -\sqrt{w}\leq u\leq\sqrt{w}}} \varphi_{k,B}(z, y+\mu\log(x-z))(x-z)^{-1}dz. \tag{3.7}
$$

Taking Proposition 3.1 into account, we find that the integration path in the right-hand side of (3.6) coincides with the integration path of the Borel sum of (3.4). Thus it is expected that $\chi(x,y)$ is equal to the Borel transform of a suitably normalized WKB solution ψ_j of $\tilde{P}\psi=0$. As a matter of fact, we can prove the following theorem.

Theorem 3.1. *The function $\chi(x,y)$ given by (3.7) is equal to the Borel transform of $\eta^{-1/2}\psi_j$ in a neighborhood of $y=-\int_{\tilde{a}}^x \frac{\mu dx}{x-z_j(x)}$.*

Before we give a proof of Theorem 3.1, we prepare some lemmas. Let $w = y+\int_{\tilde{a}}^x \frac{\mu dx}{x-z_j(x)}$ and assume that $|w|$ is sufficiently small. Let $z^{(\pm)}$ be points satisfying $f_k(z^{(\pm)}) = f_{k,0} - w$. Since the integrand of the right-hand side of (3.7) has square-root type singularities at $z = z^{(\pm)}$ by the definition of $\varphi_{k,B}$, $\chi(x,y)$ can be considered as the following contour integral:

$$\chi(x,y) = \frac{1}{4}\int_C \varphi_{k,B}(z, y + \mu \log(x-z))(x-z)^{-1} dz.$$

Here C is a path of Jordan-Pochhammer type shown in Figure 1. If there is no fear of confusions, we omit C and $1/4$ in what follows.

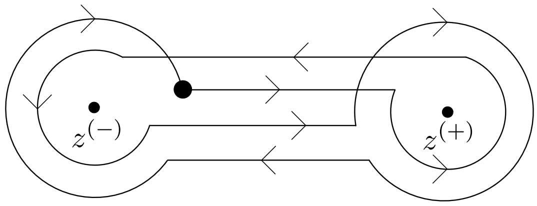


Figure 1: A path C.

Lemma 3.2. *For all $j \in \mathbb{N}$, $\chi(x,y)$ satisfies the following equalities:*

$$\partial_y^j \chi(x,y) = \int \left(\frac{\partial}{\partial t}\right)^j \varphi_{k,B}(z,t)\bigg|_{t=y+\mu\log(x-z)} (x-z)^{-1} dz, \tag{3.8}$$

$$\partial_x^j \chi(x,y) = \int \left(\frac{\partial}{\partial z}\right)^j \varphi_{k,B}(z,t)\bigg|_{t=y+\mu\log(x-z)} (x-z)^{-1} dz, \tag{3.9}$$

$$(x - \mu\partial_y\partial_x^{-1})^j \chi(x,y) = \int \left[z^j \varphi_{k,B}(z,t)\right]\big|_{t=y+\mu\log(x-z)} (x-z)^{-1} dz + F_j. \tag{3.10}$$

Here we use the notations $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ and F_j denotes a function satisfying $\partial_x^j F_j = 0$.

Proof. We first note that (3.8) is trivial since $\chi(x,y)$ is a contour integral.

To prove (3.9), it suffices to prove the case $j = 1$. By integration by parts, we have

$$\begin{aligned}
&\int \varphi_{k,B}(z, y+\mu\log(x-z))(x-z)^{-2} dz \\
&= \int \varphi_{k,B}(z, y+\mu\log(x-z)) \frac{\partial}{\partial z}(x-z)^{-1} dz \\
&= -\int \frac{\partial}{\partial z}(\varphi_{k,B}(z, y+\mu\log(x-z)))(x-z)^{-1} dz \\
&= -\int \left[\frac{\partial}{\partial z}\varphi_{k,B}(z,t)\bigg|_{t=y+\mu\log(x-z)} (x-z)^{-1} \right.
\end{aligned}$$

$$- \left.\frac{\partial}{\partial t}\varphi_{k,B}(z,t)\right|_{t=y+\mu\log(x-z)} \mu(x-z)^{-2}\Bigg] dz.$$

Hence we have

$$\begin{aligned}\partial_x\chi(x,y) =& \int \left[\left.\frac{\partial}{\partial t}\varphi_{k,B}(z,t)\right|_{t=y+\mu\log(x-z)} \mu(x-z)^{-2} \right.\\ & \left. -\varphi_{k,B}(z,y+\mu\log(x-z))(x-z)^{-2}\right] dz\\ =& \int \left.\frac{\partial}{\partial z}\varphi_{k,B}(z,t)\right|_{t=y+\mu\log(x-z)} (x-z)^{-1}dz.\end{aligned}$$

This verifies (3.9).

We prove (3.10) by induction on j. First, since

$$\begin{aligned}\mu\partial_y\partial_x^{-1}\chi(x,y) =& \mu\partial_x^{-1}\int \left.\frac{\partial}{\partial t}\varphi_{k,B}(z,t)\right|_{t=y+\mu\log(x-z)} (x-z)^{-1}dz\\ =& \partial_x^{-1}\int \frac{\partial}{\partial x}\varphi_{k,B}(z,y+\mu\log(x-z))dz\\ =& \int \varphi_{k,B}(z,y+\mu\log(x-z))dz + F_1,\end{aligned}$$

we have

$$\begin{aligned}(x-\mu\partial_y\partial_x^{-1})\chi(x,y) =& \int \varphi_{k,B}(z,y+\mu\log(x-z))\left(\frac{x}{x-z}-1\right)dz + F_1\\ =& \int \left.[z\varphi_{k,B}(z,t)]\right|_{t=y+\mu\log(x-z)} (x-z)^{-1}dz + F_1.\end{aligned}$$

This shows that (3.10) is true in the case $j=1$. We next assume that (3.10) is true for $j \geq 1$. Then we have

$$\begin{aligned}&(x-\mu\partial_y\partial_x^{-1})^{j+1}\chi(x,y)\\ =&(x-\mu\partial_y\partial_x^{-1})\left(\int \left.[z^j\varphi_{k,B}(z,t)]\right|_{t=y+\mu\log(x-z)} (x-z)^{-1}dz + F_j\right)\\ =& \int \left.[z^{j+1}\varphi_{k,B}(z,t)]\right|_{t=y+\mu\log(x-z)} (x-z)^{-1}dz + \tilde{F}_1 + (x-\mu\partial_y\partial_x^{-1})F_j,\end{aligned}$$

where $\tilde{F}_1$ is such that $\partial_x\tilde{F}_1 = 0$. Since

$$\begin{aligned}\partial_x^{j+1}(\tilde{F}_1 + (x-\mu\partial_y\partial_x^{-1})F_j) =& \partial_x^{j+1}(xF_j) - \mu\partial_y\partial_x^jF_j\\ =& x\partial_x^{j+1}F_j + (j+1)\partial_x^jF_j = 0\end{aligned}$$

holds, (3.10) is also true for $j+1$. This completes the proof of Lemma 3.2. □

Using Lemma 3.2, we now prove that $\chi(x,y)$ satisfies some partial differential equation.

Proposition 3.2. *Let $\tilde{P}_B$ be the following partial differential operator:*

$$\tilde{P}_B = \partial_x^l \sum_{j=0}^{n} \sum_{k=0}^{\deg a_j} a_{jk}(x - \mu\partial_y\partial_x^{-1})^k \partial_y^{n-j}\partial_x^j.$$

Then $\tilde{P}_B\chi(x,y) = 0$ holds.

Proof. It follows from (3.8) that

$$\begin{aligned}
\tilde{P}_B\chi(x,y) =& \partial_x^l \sum_{j=0}^{n} \sum_{k=0}^{\deg a_j} a_{jk}(x - \mu\partial_y\partial_x^{-1})^k \partial_y^{n-j}\partial_x^j \chi(x,y) \\
=& \partial_x^l \sum_{j=0}^{n} \sum_{k=0}^{\deg a_j} a_{jk}(x - \mu\partial_y\partial_x^{-1})^{k-j} (x - \mu\partial_y\partial_x^{-1})^j \partial_x^j \\
& \times \int \left[\partial_t^{n-j}\varphi_{k,B}(z,t)\right]\Big|_{t=y+\mu\log(x-z)} (x-z)^{-1} dz.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&(x\partial_x - \mu\partial_y) \int \varphi_{k,B}(z, y + \mu\log(x-z))(x-z)^{-1}dz \\
=& \int \left[x\partial_z\varphi_{k,B}(z,t) - \mu\partial_t\varphi_{k,B}(z,t)\right]|_{t=y+\mu\log(x-z)} (x-z)^{-1}dz \\
=& \int \left[\left(1 + \frac{z}{x-z}\right)\partial_z\varphi_{k,B}(z,t) - \frac{\mu}{x-z}\partial_t\varphi_{k,B}(z,t)\right]\Big|_{t=y+\mu\log(x-z)} dz \\
=& \int \left[z\partial_z\varphi_{k,B}(z,t)\right]|_{t=y+\mu\log(x-z)} (x-z)^{-1}dz \\
& \quad + \int \left[\left(\partial_z - \frac{\mu}{x-z}\partial_t\right)\varphi_{k,B}(z,t)\right]\Big|_{t=y+\mu\log(x-z)} dz \\
=& \int \left[z\partial_z\varphi_{k,B}(z,t)\right]|_{t=y+\mu\log(x-z)} (x-z)^{-1}dz \\
& \quad + \int \partial_z(\varphi_{k,B}(z, y + \mu\log(x-z)))dz \\
=& \int \left[z\partial_z\varphi_{k,B}(z,t)\right]|_{t=y+\mu\log(x-z)} (x-z)^{-1}dz.
\end{aligned} \tag{3.11}$$

Hence, using (3.11) and the following formulas of differential operators

$$(x - \mu\partial_y\partial_x^{-1})^j\partial_x^j = (x\partial_x - \mu\partial_y)(x\partial_x - \mu\partial_y - 1)\cdots(x\partial_x - \mu\partial_y - (j-1)),$$

$$x\partial_x(x\partial_x - 1)\cdots(x\partial_x - (j-1)) = x^j\partial_x^j,$$

we can calculate $\tilde{P}_B\chi(x,y)$ as follows:

$$\begin{aligned}
&\tilde{P}_B\chi(x,y)\\
=&\partial_x^l\sum_{j=0}^{n}\sum_{k=0}^{\deg a_j} a_{jk}(x-\mu\partial_y\partial_x^{-1})^{k-j}(x\partial_x-\mu\partial_y)(x\partial_x-\mu\partial_y-1)\cdots\\
&\cdots(x\partial_x-\mu\partial_y-(j-1))\int\left[\partial_t^{n-j}\varphi_{k,B}(z,t)\right]\Big|_{t=y+\mu\log(x-z)}(x-z)^{-1}dz\\
=&\partial_x^l\sum_{j=0}^{n}\sum_{k=0}^{\deg a_j} a_{jk}(x-\mu\partial_y\partial_x^{-1})^{k-j}\\
&\times\int\left[z\partial_z\cdots(z\partial_z-(j-1))\partial_t^{n-j}\varphi_{k,B}(z,t)\right]\Big|_{t=y+\mu\log(x-z)}(x-z)^{-1}dz\\
=&\partial_x^l\sum_{j=0}^{n}\sum_{k=0}^{\deg a_j} a_{jk}(x-\mu\partial_y\partial_x^{-1})^{k-j}\\
&\times\int\left[z^j\partial_z^j\partial_t^{n-j}\varphi_{k,B}(z,t)\right]\Big|_{t=y+\mu\log(x-z)}(x-z)^{-1}dz.
\end{aligned}$$

Finally, by using (3.10), we thus obtain

$$\begin{aligned}
&\tilde{P}_B\chi(x,y)\\
=&\partial_x^l\sum_{j=0}^{n}\sum_{k=0}^{\deg a_j} a_{jk}\left(\int\left[z^{k-j}z^j\partial_z^j\partial_t^{n-j}\varphi_{k,B}(z,t)\right]\Big|_{t=y+\mu\log(x-z)}(x-z)^{-1}dz+F_{k-j}\right)\\
=&\partial_x^l\left(\int\left[P_B\varphi_{k,B}(z,t)\right]\big|_{t=y+\mu\log(x-z)}(x-z)^{-1}dz+\sum_{j=0}^{n}\sum_{k=0}^{\deg a_j}a_{jk}F_{k-j}\right)\\
=&0.
\end{aligned}$$

□

As ψ_j satisfies

$$\tilde{P}\psi_j=\eta^{-l-n}\partial_x^l\sum_{j=0}^{n}\sum_{k=0}^{\deg a_j}a_{jk}(x-\mu\eta\partial_x^{-1})^k\eta^{n-j}\partial_x^j\psi_j=0$$

due to (1.5), its Borel transform $\psi_{j,B}(x,y)$ also satisfies $\tilde{P}_B\psi_{j,B}=0$. That is, $\chi(x,y)$ and the Borel transform of ψ_j satisfy the same partial differential equation. Now we will prove that $\chi(x,y)$ is a constant multiple of the Borel transform of $\eta^{-1/2}\psi_j$.

Substituting the definition of the Borel transform into the integrand of

$\chi(x,y)$, we have

$$\begin{aligned}
&\chi(x,y)\\
=&\int\sum_{n\geq 0}\frac{\varphi_{k,n}(z)}{\Gamma(n+1/2)}\left(y+\mu\log(x-z)+\int_a^z\zeta_k(z)dz\right)^{n-1/2}(x-z)^{-1}dz\\
=&\int_{-\sqrt{w}}^{\sqrt{w}}\sum_{n\geq 0}\frac{\varphi_{k,n}(z(u))}{\Gamma(n+1/2)}\left(y+f_{k,0}-u^2\right)^{n-1/2}(x-z(u))^{-1}\frac{\partial z}{\partial u}du\\
=&\int_{-\sqrt{w}}^{\sqrt{w}}\sum_{n\geq 0}\frac{\varphi_{k,n}(z(u))}{\Gamma(n+1/2)}(x-z(u))^{-1}\frac{\partial z}{\partial u}\left(w-u^2\right)^{n-1/2}du. \qquad (3.12)
\end{aligned}$$

Here $z(u)$ denotes an implicit function from u to z defined by $f_k(z)=f_{k,0}-u^2$ and we put $w=y+f_{k,0}$ in the last equality.

We will prove that

$$\Phi_{k,n}(u)=\frac{\varphi_{k,n}(z(u))}{\Gamma(n+1/2)}(x-z(u))^{-1}\frac{\partial z}{\partial u}$$

is holomorphic at $u=0$. By the Taylor expansion of f_k, we have

$$\begin{aligned}
&f_{k,0}-u^2=f_k\\
=&f_{k,0}+\left.\frac{\partial f_k}{\partial z}\right|_{z=z_j(x)}(z-z_j(x))+\frac{1}{2}\left.\frac{\partial^2 f_k}{\partial z^2}\right|_{z=z_j(x)}(z-z_j(x))^2+\cdots\\
=&f_{k,0}+\frac{1}{2}\left.\frac{\partial^2 f_k}{\partial z^2}\right|_{z=z_j(x)}(z-z_j(x))^2+\frac{1}{3!}\left.\frac{\partial^3 f_k}{\partial z^3}\right|_{z=z_j(x)}(z-z_j(x))^3+\cdots,
\end{aligned}$$

which implies

$$u^2=-\left(\frac{1}{2}\left.\frac{\partial^2 f_k}{\partial z^2}\right|_{z=z_j(x)}(z-z_j(x))^2+\frac{1}{3!}\left.\frac{\partial^3 f_k}{\partial z^3}\right|_{z=z_j(x)}(z-z_j(x))^3+\cdots\right).$$

On the other hand,

$$\left.\frac{\partial^2 f_k}{\partial z^2}\right|_{z=z_j(x)}\neq 0$$

holds since $z_j(x)$ is a simple zero of $\partial f_k/\partial z$. Therefore we have

$$u=\sqrt{-\frac{1}{2}\left.\frac{\partial^2 f_k}{\partial z^2}\right|_{z=z_j(x)}}(z-z_j(x))g(z)$$

where $g(z)$ is holomorphic at $z=z_j(x)$ and satisfies $g(z_j(x))=1$. Hence we have

$$\left.\frac{\partial u}{\partial z}\right|_{z=z_j(x)}=\sqrt{-\frac{1}{2}\left.\frac{\partial^2 f_k}{\partial z^2}\right|_{z=z_j(x)}}\neq 0,$$

which means $\partial z/\partial u$ is holomorphic at $u=0$. Thus $\Phi_{k,n}(u)$ is also holomorphic at $u=0$.

Next we will prove that the integrand of (3.12) converges absolutely and uniformly on $[-\sqrt{w}, \sqrt{w}]$ for sufficiently small w. We take sufficiently small R so that

$$\begin{aligned} U &:= \{u \in \mathbb{C}\,;\, |u| < R\} \\ &\subset (\text{domain of holomorphy for } \Phi_{k,n}(u) \text{ including } u=0) \end{aligned}$$

holds. We also take sufficiently small w so that $\{u \in \mathbb{C}\,;\, |u| \le \sqrt{|w|}\} \subset U$ holds. Then there exists a constant $M > 0$ independent of n such that

$$\left| (x - z(u))^{-1} \frac{\partial z}{\partial u} \right| \le M \quad (|u| \le \sqrt{|w|}).$$

Note that $z_j(x) \ne x$ because $z(0) = z_j(x)$ is a zero of $\partial f_k/\partial z$ and x is a pole of $\partial f_k/\partial z$. Let $\bar{U}$ be the closure of U and put $K = z(\bar{U})$. Since K is a compact set of $\mathbb{C} \setminus \{\text{turning point of } P\}$, there exist constants $A, C > 0$ such that

$$\sup_{u \in U} |\varphi_{k,n}(z(u))| = \sup_{z \in K} |\varphi_{k,n}(z)| \le AC^n n! \quad (n = 0, 1, 2, \dots).$$

On the other hand, we have

$$\lim_{n \to \infty} \frac{n^{-\frac{1}{2}} n!}{\Gamma(n + 1/2)} = 1$$

by Stirling's formula. Hence, if we take a constant C' such that $n^{1/2} < (C'/C)^n$ for all $n = 0, 1, 2, \dots$, we have

$$|\Phi_{k,n}(u)| \le \frac{AC^n n!}{n^{-1/2} n!} M \le AMC'^n$$

on $|u| \le \sqrt{|w|}$ for all $n = 0, 1, 2, \dots$. We rewrite AM and C' as A and C, respectively. Then the integrand of (3.12) converges uniformly on $|w| \le 1/4C$ since the following estimate holds on $|w| \le 1/4C$:

$$|\Phi_{k,n}(u)(w - u^2)^n| \le AC^n (2|w|)^n \le \frac{A}{2^n}.$$

Therefore we can write $\chi(x, y)$ as follows:

$$\chi(x, y) = \sum_{n \ge 0} \int_{-\sqrt{w}}^{\sqrt{w}} \Phi_{k,n}(u)(w - u^2)^{n-1/2} du$$

where $w = y + f_{k,0}$. Let $\Phi_{k,n}(u) = \sum_{l \ge 0} a_{n,l} u^l$ be the Taylor expansion of

$\Phi_{k,n}(u)$ at $u = 0$. Then we have

$$\begin{aligned}
\chi(x,y) =& \sum_{n\geq 0} \int_{-\sqrt{w}}^{\sqrt{w}} \sum_{l\geq 0} a_{n,l} u^l (w-u^2)^{n-1/2} du \\
=& \sum_{n\geq 0}\sum_{l\geq 0} a_{n,l} \int_{-\sqrt{w}}^{\sqrt{w}} u^l (w-u^2)^{n-1/2} du \\
=& \sum_{n\geq 0}\sum_{l\geq 0} a_{n,l} \int_{-1}^{1} (\sqrt{w}t)^l (w-wt^2)^{n-1/2} \sqrt{w} dt \\
=& \sum_{n\geq 0}\sum_{l\geq 0} a_{n,l} \int_{-1}^{1} t^l (1-t^2)^{n-1/2} dt w^{n+\frac{1}{2}l} \\
=& \sum_{n\geq 0}\sum_{l\geq 0} a_{n,2l} \int_{-1}^{1} t^{2l} (1-t^2)^{n-1/2} dt w^{n+l}.
\end{aligned}$$

Here we have used a change of variables $u = \sqrt{w}t$ in the third equality and the fact that the integral is equal to 0 when l is odd in the last equality. The constant term of this series is

$$\begin{aligned}
a_{0,0} \int_{-1}^{1} t^0 (1-t^2)^{0-\frac{1}{2}} dt =& \pi \Phi_{k,0}(0) \\
=& \pi \frac{\varphi_{k,0}(z_j(x))}{\Gamma(1/2)} (x - z_j(x))^{-1} \left.\frac{\partial z}{\partial u}\right|_{u=0}.
\end{aligned}$$

Thanks to the assumption that a saddle point $z_j(x)$ of f_k is not a turning point of P, $\varphi_{k,0}(z_j(x)) \neq 0$ holds. Furthermore, as we have already dicussed above, $(\partial z/\partial u)|_{u=0} \neq 0$ holds. Therefore, if we put $w = y + \int_{\tilde{a}}^{x} \frac{\mu dx}{x - z_j(x)}$, $\chi(x,y)$ is a holomorphic function of w in a neighborhood of $w = 0$ and satisfies $\chi(x,y)|_{w=0} \neq 0$.

Having these facts in mind, we will prove Theorem 3.1. We rewrite $\chi(x,y)$ as follows:

$$\begin{aligned}
\chi(x,y) =& \sum_{n\geq 0} \chi_n w^n \\
=& \sum_{n\geq 0} \chi_n \left(y + \int_{\tilde{a}}^{x} \frac{\mu dx}{x - z_j(x)} \right)^n \\
=& \mathcal{B}\left[\exp\left(\eta \int_{\tilde{a}}^{x} \frac{\mu dx}{x - z_j(x)} \right) \sum_{n\geq 0} \chi_n \Gamma(n+1) \eta^{-n-1} \right].
\end{aligned}$$

Define T_l by

$$\exp\left(\sum_{l\geq 0} T_l \eta^{-l} \right) = \sum_{n\geq 0} \chi_n \Gamma(n+1) \eta^{-n},$$

then we can write

$$\chi(x,y)=\mathcal{B}\left[\eta^{-1}\exp\left(\eta\int_{\tilde{a}}^{x}\frac{\mu dx}{x-z_j(x)}+\sum_{l\geq 0}T_l\eta^{-l}\right)\right].$$

It follows from Proposition 3.2 that

$$\begin{aligned}&\mathcal{B}\left[\eta^{n+l}\tilde{P}\left(\eta^{-1}\exp\left(\eta\int_{\tilde{a}}^{x}\frac{\mu dx}{x-z_j(x)}+\sum_{l\geq 0}T_l\eta^{-l}\right)\right)\right]\\ =&\tilde{P}_B\left(\mathcal{B}\left[\eta^{-1}\exp\left(\eta\int_{\tilde{a}}^{x}\frac{\mu dx}{x-z_j(x)}+\sum_{l\geq 0}T_l\eta^{-l}\right)\right]\right)\\ =&\tilde{P}_B\chi(x,y)=0.\end{aligned}$$

Therefore we have

$$\tilde{P}\left(\eta^{-1/2}\exp\left(\eta\int_{\tilde{a}}^{x}\frac{\mu dx}{x-z_j(x)}+\sum_{l\geq 0}T_l\eta^{-l}\right)\right)=0.$$

Thus both $\eta^{1/2}\mathcal{B}^{-1}\chi(x,y)$ and ψ_j are infinite series of the same form with the same (leading part of the) exponential factor and satisfy $\tilde{P}\psi=0$. Hence, thanks to the uniqueness of WKB solutions, we can conclude they coincide (up to constant multiple). That is, we have

$$\psi_j=\eta^{-1/2}\exp\left(\eta\int_{\tilde{a}}^{x}\frac{\mu dx}{x-z_j(x)}+\sum_{l\geq 0}T_l\eta^{-l}\right). \tag{3.13}$$

(To be more precise, we determine the normalization of ψ_j by the right-hand side of (3.13).) Thus $\chi(x,y)=\mathcal{B}[\eta^{-1/2}\psi_j]$, which completes the proof of Theorem 3.1.

4 Proof of Main Theorem, II — Global aspect and exact steepest descent paths

In the previous section we studied the behavior of $\chi(x,y)$ near $y=-\int_{\tilde{a}}^{x}\frac{\mu dx}{x-z_j(x)}$ by making use of properties of the integral (3.3) near saddle points. In this section, to complete the proof of Main Theorem, we investigate the global behavior of $\chi(x,y)$ by taking account of the geometry of the integration path of (3.3), i.e., the steepest descent path $C^{(k)}_{z_j(x)}$.

In what follows, as in the preceding section, we assume that P is of second order and $\zeta_k(x)$, $\zeta_{k'}(x)$ denote the roots of its characteristic equation. We further assume $f_{k,0} = f_k(z_j(x))$ and $C^{(k)}_{z_j(x)}$ is parametrized by $f_k = f_{k,0} - u^2$ $(u \in \mathbb{R})$. We first consider the case where $C^{(k)}_{z_j(x)}$ does not cross any Stokes curve of $P\varphi = 0$. In this case, as variables y and $w \geq 0$ are related by

$$y = -\int_{\tilde{a}}^{x} \frac{\mu dx}{x - z_j(x)} + w = -f_{k,0} + w = -f_k(z_j(x)) + w$$

in the integral (3.6), the second variable of the integrand $\varphi_{k,B}$ of the definition (3.7) of $\chi(x,y)$ can be written in terms of w as follows:

$$\begin{aligned} y + \mu \log(x - z) &= - f_k(z_j(x)) + w + \mu \log(x - z) \\ &= - f_k(z) - u^2 + w + \mu \log(x - z) \\ &= - \int_a^z \zeta_k(z) dz - u^2 + w. \end{aligned} \tag{4.1}$$

Since $C^{(k)}_{z_j(x)}$ does not cross a Stokes curve by the assumption, it follows from Theorem 2.1 that $\varphi_k(z,\eta)$ is Borel summable when z is on $C^{(k)}_{z_j(x)}$, that is, $\varphi_{k,B}(z,y)$ is analytically continuable along $y = -\int_a^z \zeta_k dz + v$, $v \geq 0$. Hence, in this case, the integration path of (3.7) does not hit a singularity of $\varphi_{k,B}(z, y + \mu \log(x-z))$ when u satisfies $-\sqrt{w} \leq u \leq \sqrt{w}$ in view of (4.1). This implies that $\chi(x,y)$ is analytically continuable along $y = -\int_{\tilde{a}}^{x} \frac{\mu dx}{x - z_j(x)} + w$, $w \geq 0$. Furthermore, the Borel summability of $\varphi_k(z,\eta)$ implies the following exponential estimate:

$$\left| \varphi_{k,B}\left(z, -\int_a^z \zeta_k dz + v\right) \right| \leq C_1(|z|+1)^{-\alpha} \exp(C_2|v|) \quad (v \geq 0),$$

where α, C_1, C_2 are positive constants. (The constant α is determined by the coefficients $a_j(x)$ of $P\varphi = 0$. See [15] for the details. In this paper, however, the value of α is irrelevant.) Therefore we have

$$\begin{aligned} &\left| \chi\left(x, -\int_{\tilde{a}}^{x} \frac{\mu dx}{x - z_j(x)} + w\right) \right| = |\chi(x,y)| \\ &= \left| \int_{\substack{f_k(z) = f_{k,0} - u^2 \\ -\sqrt{w} \leq u \leq \sqrt{w}}} \varphi_{k,B}\left(z, -\int_a^z \zeta_k dz + w - u^2\right) (x-z)^{-1} dz \right| \\ &\leq \int_{\substack{f_k(z) = f_{k,0} - u^2 \\ -\sqrt{w} \leq u \leq \sqrt{w}}} C_1(|z|+1)^{-\alpha} \exp(C_2(w - u^2)) |x-z|^{-1} |dz| \\ &\leq C_1 \exp(C_2|w|) \int_{\substack{f_k(z) = f_{k,0} - u^2 \\ -\sqrt{w} \leq u \leq \sqrt{w}}} \exp(-C_2 u^2)(|z|+1)^{-\alpha} |x-z|^{-1} |dz| \end{aligned}$$

for $w \geq 0$. Since the integral in the last equality is bounded by a positive constant M, we obtain the following estimate:

$$\left| \chi\left(x, -\int_{\tilde{a}}^{x} \frac{\mu dx}{x - z_j(x)} + w\right) \right| \leq C_1 M \exp(C_2|w|).$$

Summarizing the above, we have verified the following.

Theorem 4.1. *Under the assumptions of Main Theorem, if the steepest descent path $C^{(k)}_{z_j(x)}$ of $\Re f_k$ passing through a saddle point $z_j(x)$ does not cross any Stokes curve of $P\varphi = 0$ and, further, if it does not flow into any other saddle point of f_k, then $\eta^{-1/2}\psi_j$ is Borel summable.*

We next consider the case where the steepest descent path $C^{(k)}_{z_j(x)}$ crosses a Stokes curve of $P\varphi = 0$ once. In this case, a singularity appears in the integration path of (3.7). We will study the effects of this singularity to (the analytic continuation of) $\chi(x, y)$.

Assume that the steepest descent path $C^{(k)}_{z_j(x)}$ crosses a Stokes curve of type $k > k'$, which emanates from a turning point $z = a_0$ of P, at $z = z_0$ in an anti-clockwise manner. We let u_0 be a parameter corresponding to z_0, i.e., $f_k(z_0) = f_{k,0} - u_0^2$, and let $v_0 := \int_{a_0}^{z_0}(\zeta_k - \zeta_{k'})dz > 0$ and $w_0 := u_0^2 + v_0$. We denote by $\varphi_*^{(0)}(z, \eta)$ $(* = k, k')$ a WKB solution of $P\varphi = 0$ normalized at a_0, that is, normalized at the turning point where the Stokes curve in question emanates. Then the following relation holds between φ_* and $\varphi_*^{(0)}$:

$$\varphi_*(z, \eta) = W_* \exp\left(\eta \int_a^{a_0} \zeta_*(z)dz\right) \varphi_*^{(0)}(z, \eta) \quad (* = k, k'), \tag{4.2}$$

where W_* $(= W_+$ or $W_-)$ is defined by

$$W_\pm = \exp\left(\pm \int_a^{a_0} (S_{\mathrm{odd},0} + \eta^{-1}S_{\mathrm{odd},1} + \cdots)\, dz\right).$$

Here $S_{\mathrm{odd}} = \eta S_{\mathrm{odd},-1} + S_{\mathrm{odd},0} + \eta^{-1}S_{\mathrm{odd},1} + \cdots$ denotes the odd part of $(\partial/\partial x)\log\varphi_\pm$. Note that W_* is independent of z and Borel summable under our current assumptions. As a matter of fact, evaluating both sides of (4.2) at some point z outside Stokes curves, we find both $\varphi_*(z, \eta)$ and $\varphi_*^{(0)}(z, \eta)$ are Borel summable thanks to Theorem 2.1 and so is W_*.

Using the relation (4.2) and Theorem 2.2, we now verify that a singularity of the integrand of (3.7) appears on its integration path when $z = z_0$. It follows from Theorem 2.2 that $\varphi_{k,B}^{(0)}(z, y)$ has singularities at

$$y = -\int_{a_0}^{z} \zeta_k(z)dz, \ -\int_{a_0}^{z} \zeta_{k'}(z)dz$$

when $z = z_0$. Then, since (4.2) implies

$$\varphi_{k,B}(z, y) = (W_{k,B} * \varphi_{k,B}^{(0)})\left(z, y + \int_a^{a_0} \zeta_k(z)\, dz\right) \tag{4.3}$$

and $W_{k,B}$ is holomorphic near the positive real axis by its Borel summability noted above, $\varphi_{k,B}(z, y)$ has singularities at

$$y = -\int_a^{z} \zeta_k(z)dz, \ -\int_{a_0}^{z} \zeta_{k'}(z)dz - \int_a^{a_0} \zeta_k(z)dz$$

when $z = z_0$. Using (4.1) at $z = z_0$, i.e.,

$$y + \mu \log(x - z_0) = -\int_a^{z_0} \zeta_k(z)dz - u_0^2 + w$$

for the second variable of the integrand of (3.7), we thus find that the integrand has a singularity at

$$w = u_0^2 + \int_{a_0}^{z_0} (\zeta_k - \zeta_{k'})\, dz = u_0^2 + v_0 =: w_0.$$

That is, when $z = z_0$ and $y = -f_{k,0} + w_0 = -f_{k,0} + u_0^2 + v_0$, a singularity of the integrand hits the integration path of (3.7). Therefore, to consider the analytic continuation of $\chi(x, y)$ for $w \geq w_0$, we need to deform the integration path so that we may avoid this singularity.

We denote this singularity by z_*. The above discussion imples that z_* is determined by

$$y + \mu \log(x - z_*) = -\int_{a_0}^{z_*} \zeta_{k'}(z)dz - \int_a^{a_0} \zeta_k(z)dz \tag{4.4}$$

for a given x, y. Then we can verify the following:

Lemma 4.1. *The singularity z_* is located on a steepest descent path*

$$C_{z_0}^{(k')} : \ f_{k'}(z) = f_{k'}(z_0) - \tilde{u} \ (\tilde{u} \geq 0)$$

of $\Re f_{k'}$ emanating from the crossing point z_0. Here $f_{k'}$ is defined by $f_{k'} = \mu \log(x - z) + \int_a^z \zeta_{k'}(z)dz$.

Proof. By (4.4) we have

$$\begin{aligned}
f_{k'}(z_*) &= \mu \log(x - z_*) + \int_a^{z_*} \zeta_{k'}(z)dz \\
&= -y - \int_a^{a_0} (\zeta_k(z) - \zeta_{k'}(z))dz \\
&= f_{k,0} - w - \int_a^{a_0} (\zeta_k(z) - \zeta_{k'}(z))dz,
\end{aligned} \tag{4.5}$$

while

$$\begin{aligned}
f_{k'}(z_0) &= \mu \log(x - z_0) + \int_a^{z_0} \zeta_{k'}(z)dz \\
&= \mu \log(x - z_0) + \int_a^{z_0} \zeta_k(z)dz - \int_a^{z_0} (\zeta_k(z) - \zeta_{k'}(z))dz \\
&= f_k(z_0) - v_0 - \int_a^{a_0} (\zeta_k(z) - \zeta_{k'}(z))dz \\
&= f_{k,0} - w_0 - \int_a^{a_0} (\zeta_k(z) - \zeta_{k'}(z))dz.
\end{aligned} \tag{4.6}$$

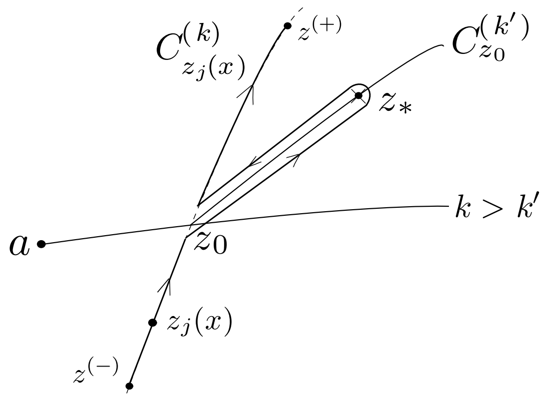


Figure 2: The integration path on z-plane when $w \geq w_0$.

Hence

$$f_{k'}(z_*) - f_{k'}(z_0) = w_0 - w \leq 0 \tag{4.7}$$

holds, that is, z_* is a point on $C^{(k')}_{z_0}$ satisfying $f_{k'}(z_*) = f_{k'}(z_0) - \tilde{u}_*$ with $\tilde{u}_* = w - w_0$. This completes the proof of Lemma 4.1. □

Lemma 4.1 neatly explains why the bifurcated steepest descent path $C^{(k')}_{z_0}$ enters into the theory. Taking account of Lemma 4.1, we decompose (the analytic continuation of) $\chi(x, y)$ for $w \geq w_0$ as follows:

$$\begin{aligned}\chi(x,y) = &\int_{C^{(k)}_{z_j(x)}:-\sqrt{w}\leq u \leq u_0} \varphi_{k,B}(z, y + \mu \log(x-z))(x-z)^{-1} dz \\ &+ \int_{C^{(k)}_{z_j(x)}:u_0\leq u \leq \sqrt{w}} \varphi_{k,B}(z, y + \mu \log(x-z))(x-z)^{-1} dz \\ &+ \int_\gamma \varphi_{k,B}(z, y + \mu \log(x-z))(x-z)^{-1} dz. \end{aligned} \tag{4.8}$$

Here γ designates a path that starts from z_0, goes to z_* along the steepest descent path $C^{(k')}_{z_0}$ of $\Re f_{k'}$ and returns to z_0 after encircling z_*. (See Figure 2.) Since $C^{(k)}_{z_j(x)}$ has no further crossing points with Stokes curves other than z_0 in the current situation, the reasoning employed in the proof of Theorem 4.1 entails that the first two terms of the right-hand side of (4.8) can be analytically continued to all $w \geq w_0$ and satisfy exponential estimates there. Thus what remains to be proved is to verify the analytic continuability of the third term of the right-hand side of (4.8).

Let I denote the third term of (4.8):

$$I := \int_\gamma \varphi_{k,B}(z, y + \mu \log(x-z))(x-z)^{-1}\, dz. \tag{4.9}$$

To consider the continuability of (4.8), we first note that the Stokes phenomenon for φ_k on the Stokes curve in question emanating from a_0 is described by

$$\begin{aligned}\varphi_k &= W_k \exp\left(\eta \int_a^{a_0} \zeta_k\,dz\right)\varphi_k^{(0)} \\ &\to W_k \exp\left(\eta \int_a^{a_0} \zeta_k\,dz\right)(\varphi_k^{(0)} + i\varphi_{k'}^{(0)}) \\ &= \varphi_k + iW_k W_{k'}^{-1} \exp\left(\eta \int_a^{a_0} (\zeta_k - \zeta_{k'})\,dz\right)\varphi_{k'} \end{aligned} \tag{4.10}$$

in view of Theorem 2.3. Making use of this connection formula (4.10), we prove the continuability of I by the following argument. It follows from (4.3) that

$$I = \int_\gamma (W_{k,B} * \varphi_{k,B}^{(0)})\left(z, y + \mu\log(x-z) + \int_a^{a_0} \zeta_k\,dz\right)(x-z)^{-1}\,dz. \tag{4.11}$$

Then, when z lies on $C_{z_0}^{(k')}$, that is, when $f_{k'}(z) = f_{k'}(z_0) - \tilde{u}$ ($\tilde{u} \geq 0$) holds, the second variable of the integrand of (4.11) can be rewritten as

$$\begin{aligned} & y + \mu\log(x-z) + \int_a^{a_0} \zeta_k dz \\ &= -f_{k,0} + w + \mu\log(x-z) + \int_a^{a_0} \zeta_k dz \\ &= -f_{k,0} + w + f_{k'}(z) - \int_a^z \zeta_{k'} dz + \int_a^{a_0} \zeta_k dz \\ &= -f_{k,0} + w + f_{k'}(z) + \int_a^{a_0} (\zeta_k - \zeta_{k'}) dz - \int_{a_0}^z \zeta_{k'} dz \\ &= f_{k'}(z) - f_{k'}(z_*) - \int_{a_0}^z \zeta_{k'} dz & (4.12) \\ &= f_{k'}(z_0) - f_{k'}(z_*) - \tilde{u} - \int_{a_0}^z \zeta_{k'} dz \\ &= w - w_0 - \tilde{u} - \int_{a_0}^z \zeta_{k'} dz. & (4.13) \end{aligned}$$

Here we have used (4.5) and (4.7) in the fourth and sixth equalities, respectively. Note that, since z_0 and z_* correspond respectively to $\tilde{u} = 0$ and $\tilde{u} = w - w_0$ in view of (4.7), (4.13) is located on a line emanating from $-\int_{a_0}^z \zeta_{k'} dz$ and running parallel to the positive real axis when z is on the integration path γ of I. By the assumption that the exact steepest descent path passing through $z_j(x)$ does not flow into any other saddle point, we find z_* is not a saddle point of $f_{k'}$. Hence it follows from (4.12) that the second variable of the integrand of (4.11) has a non-zero derivative at $z = z_*$ and consequently encircles the singularity in an anti-clockwise manner when z encircles z_* in an anti-clockwise manner. Thus the integral I can be represented as

$$\int_{z_0}^{z_*} \Big[(W_{k,B} * \varphi_{k,B}^{(0)})\left(z, -\int_{a_0}^{z} \zeta_{k'}dz + w - w_0 - \tilde{u} + i0\right)$$
$$- (W_{k,B} * \varphi_{k,B}^{(0)})\left(z, -\int_{a_0}^{z} \zeta_{k'}dz + w - w_0 - \tilde{u} - i0\right)\Big] (x-z)^{-1}dz$$
$$= \int_{z_0}^{z_*} \Delta_{y=-\int_{a_0}^{z}\zeta_{k'}dz}(W_{k,B} * \varphi_{k,B}^{(0)})\left(z, -\int_{a_0}^{z} \zeta_{k'}dz + w - w_0 - \tilde{u}\right)(x-z)^{-1}dz,$$

where $-\int_{a_0}^{z}\zeta_{k'}dz + w - w_0 - \tilde{u} \pm i0$ indicates we are taking the boundary value of the multi-valued function $W_{k,B} * \varphi_{k,B}^{(0)}$ from the upper-side or lower-side of the cut. (In the current situation we place the cut along the integration path γ.)

We now employ the following lemma.

Lemma 4.2. *Let $\hat{v}$ denote $-\int_{a_0}^{z}\zeta_{k'}dz$. Then, when z lies on $C_{z_0}^{(k')}$ and $|z-z_0|$ is sufficiently small, the following relation holds:*

$$\Delta_{y=\hat{v}}(W_{k,B} * \varphi_{k,B}^{(0)})(z, \hat{v}+v) = (W_{k,B} * (\Delta_{y=\hat{v}}\varphi_{k,B}^{(0)}))(z, \hat{v}+v).$$

Proof. It follows from Theorem 2.2 that $\varphi_{k,B}^{(0)}(z,y)$ has singularities at $y = -\int_{a_0}^{z}\zeta_k dz, -\int_{a_0}^{z}\zeta_{k'}dz$ and except at these two points it is analytic in a neighborhood of $E := \{y = -\int_{a_0}^{z}\zeta_k dz + v,\ v \geq 0\}$ when $|z-z_0|$ is sufficiently small. Furthermore, as already mentioned above, W_k is Borel summable and, in particular, $W_{k,B}$ is holomorphic near the positive real axis. Hence we find that $(W_{k,B} * \varphi_{k,B}^{(0)})(z,y)$ is also analytic in a neighborhood of E except at the above two singularities.

Once we confirm the location of singularities of $W_{k,B} * \varphi_{k,B}^{(0)}$, we can readily compute its discontinuity as follows:

$$\begin{aligned}
&\Delta_{y=\hat{v}}(W_{k,B} * \varphi_{k,B}^{(0)})(z, \hat{v}+v) \\
&\quad = (W_{k,B} * \varphi_{k,B}^{(0)})(z, \hat{v}+v+i0) - (W_{k,B} * \varphi_{k,B}^{(0)})(z, \hat{v}+v-i0) \\
&\quad = \int_0^{\hat{v}+v} W_{k,B}(t)\, \varphi_{k,B}^{(0)}(z, \hat{v}+v-t+i0)\, dt \\
&\qquad - \int_0^{\hat{v}+v} W_{k,B}(t)\, \varphi_{k,B}^{(0)}(z, \hat{v}+v-t-i0)\, dt \\
&\quad = \int_0^{\hat{v}+v} W_{k,B}(t)\, (\Delta_{y=\hat{v}}\varphi_{k,B}^{(0)})(z, \hat{v}+v-t)\, dt \\
&\quad = (W_{k,B} * (\Delta_{y=\hat{v}}\varphi_{k,B}^{(0)}))(z, \hat{v}+v).
\end{aligned}$$

□

Making use of Lemma 4.2, we can write I as

$$\int_{z_0}^{z_*} (W_{k,B} * (\Delta_{y=-\int_{a_0}^{z} \zeta_{k'} dz} \varphi_{k,B}^{(0)})) \left(z, -\int_{a_0}^{z} \zeta_{k'} dz + w - w_0 - \tilde{u} \right) (x-z)^{-1} dz$$

when $w - w_0$ (> 0) is sufficiently small. Thanks to Theorem 2.3, this coincides with

$$i \int_{z_0}^{z_*} (W_{k,B} * \varphi_{k',B}^{(0)}) \left(z, -\int_{a_0}^{z} \zeta_{k'} dz + w - w_0 - \tilde{u} \right) (x-z)^{-1} dz.$$

Using (4.2) again, we have

$$\varphi_{k',B}^{(0)}(z,y) = (W_{k',B}^{-1} * \varphi_{k',B}) \left(z, y - \int_{a}^{a_0} \zeta_{k'}(z)\, dz \right).$$

Hence we obtain the following expression for the integral I:

$$\begin{aligned} I &= i \int_{z_0}^{z_*} (W_{k,B} * W_{k',B}^{-1} * \varphi_{k',B}) \left(z, -\int_{a}^{z} \zeta_{k'} dz + w - w_0 - \tilde{u} \right) (x-z)^{-1}\, dz \\ &= i \int_{z_0}^{z_*} (W_{k,B} * W_{k',B}^{-1} * \varphi_{k',B}) \left(z, y + \mu \log(x-z) + \int_{a}^{a_0} (\zeta_k - \zeta_{k'}) dz \right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times (x-z)^{-1}\, dz. \end{aligned}$$

(We have used (4.13) in deriving the last equality.) In view of this expression, if we assume that $C_{z_0}^{(k')}$ has no further crossing points with Stokes curves of $P\varphi = 0$, we find that I is analytically continuable to all $w \geq w_0$ since both W_k and $W_{k'}$ are Borel summable. Furthermore, the same reasoning as in the proof of Theorem 4.1 also deduces an exponential estimate for I.

Thus we have verified the following:

Theorem 4.2. *Assume that the steepest descent path $C_{z_j(x)}^{(k)}$ passing through a saddle point $z_j(x)$ crosses a Stokes curve of $P\varphi = 0$ of type $k > k'$ once at z_0. Let $C_{z_0}^{(k')}$ denote a steepest descent path of $\Re f_{k'}$ emanating from z_0. We also assume that $C_{z_j(x)}^{(k)}$ and $C_{z_0}^{(k')}$ have no further crossing points with Stokes curves other than z_0. Then under the assumptions of Main Theorem, if neither $C_{z_j(x)}^{(k)}$ nor $C_{z_0}^{(k')}$ flows into any other saddle point, $\eta^{-1/2}\psi_j$ is Borel summable. Furthermore, the analytic continuation of its Borel transform $\chi(x,y)$ for $w \geq w_0$ is given by the sum of the three integrals as follows:*

$$\begin{aligned} \chi(x,y) &= \int_{C_{z_j(x)}^{(k)}: -\sqrt{w} \leq u \leq u_0} \varphi_{k,B}(z, y + \mu \log(x-z))(x-z)^{-1} dz \\ &\quad + \int_{C_{z_j(x)}^{(k)}: u_0 \leq u \leq \sqrt{w}} \varphi_{k,B}(z, y + \mu \log(x-z))(x-z)^{-1} dz \\ &\quad + I, \end{aligned} \tag{4.14}$$

where

$$I = i\int_{z_0}^{z_*} (W_{k,B} * W_{k',B}^{-1} * \varphi_{k',B}) \left(z, y + \mu \log(x-z) + \int_a^{a_0} (\zeta_k - \zeta_{k'}) dz \right) \times (x-z)^{-1}\, dz. \qquad (4.15)$$

Remark 4.1. If the starting point a_0 of the Stokes curve that crosses with $C^{(k)}_{z_j(x)}$ at z_0 is the same as the turning point a used for the normalization of $\varphi_*(z,\eta)$, the explicit form of I is simplified as follows:

$$I = i\int_{z_0}^{z_*} \varphi_{k',B}(z, y + \mu \log(x-z))(x-z)^{-1} dz.$$

Remark 4.2. The function $iW_{k,B} * W_{k',B}^{-1} = i(W_k W_{k'}^{-1})_B$ that appears in the expression (4.15) of I is nothing but the Borel transform of the connection coefficient (or the Stokes coefficient) for the connection formula (4.10).

We now compute the Borel sum of $\eta^{-1/2}\psi_j$ under the current situation discussed in Theorem 4.2. We compute the contribution of each term of (4.14) to the Borel sum separately.

The contribution of the first term to the Borel sum is easily computed as follows:

$$\begin{aligned}
&\int_{\substack{y=-f_{k,0}+w \\ 0\le w\le u_0^2}} e^{-\eta y} \int_{C^{(k)}_{z_j(x)}: -\sqrt{w}\le u\le\sqrt{w}} \varphi_{k,B}(z, y+\mu\log(x-z))(x-z)^{-1} dz dy \\
&\quad + \int_{\substack{y=-f_{k,0}+w \\ u_0^2\le w\le w_0}} e^{-\eta y} \int_{C^{(k)}_{z_j(x)}: -\sqrt{w}\le u\le u_0} \varphi_{k,B}(z, y+\mu\log(x-z))(x-z)^{-1} dz dy \\
&\quad + \int_{\substack{y=-f_{k,0}+w \\ w_0\le w}} e^{-\eta y} \int_{C^{(k)}_{z_j(x)}: -\sqrt{w}\le u\le u_0} \varphi_{k,B}(z, y+\mu\log(x-z))(x-z)^{-1} dz dy \\
&= \int_{C^{(k)}_{z_j(x)}: u\le u_0} \int_{\substack{y=-f_{k,0}+w \\ u^2\le w}} e^{-\eta y}\varphi_{k,B}(z, y+\mu\log(x-z))(x-z)^{-1} dy dz \\
&= \int_{C^{(k)}_{z_j(x)}: u\le u_0} (x-z)^{\mu\eta-1} \int_{\substack{\tilde{y}=-\int_a^z \zeta_k(z)dz+v \\ v\ge 0}} e^{-\eta\tilde{y}}\varphi_{k,B}(z,\tilde{y}) d\tilde{y} dz \\
&= \int_{C^{(k)}_{z_j(x)}: u\le u_0} (x-z)^{\mu\eta-1}\Phi_k(z,\eta) dz.
\end{aligned}$$

Here $\Phi_k(z,\eta)$ denotes the Borel sum of $\varphi_k(z,\eta)$ and we put $\tilde{y} = y + \mu\log(x-z)$ and $v = w - u^2$ in obtaining the second equality.

Similarly, the contribution of the second term is given by

$$\int_{\substack{y=-f_{k,0}+w\\ u_0^2\le w\le w_0}} e^{-\eta y}\int_{C^{(k)}_{z_j(x)}:u_0\le u\le\sqrt{w}}\varphi_{k,B}(z,y+\mu\log(x-z))(x-z)^{-1}dzdy$$
$$+\int_{\substack{y=-f_{k,0}+w\\ w_0\le w}} e^{-\eta y}\int_{C^{(k)}_{z_j(x)}:u_0\le u\le\sqrt{w}}\varphi_{k,B}(z,y+\mu\log(x-z))(x-z)^{-1}dzdy$$
$$=\int_{C^{(k)}_{z_j(x)}:u_0\le u\le\sqrt{w_0}}(x-z)^{\mu\eta-1}\int_{\substack{\tilde{y}=-\int_a^z\zeta_k(z)dz+v\\ 0\le v\le w_0-u^2}} e^{-\eta\tilde{y}}\varphi_{k,B}(z,\tilde{y})d\tilde{y}dz$$
$$+\int_{C^{(k)}_{z_j(x)}:u_0\le u\le\sqrt{w_0}}(x-z)^{\mu\eta-1}\int_{\substack{\tilde{y}=-\int_a^z\zeta_k(z)dz+v\\ w_0-u^2\le v}} e^{-\eta\tilde{y}}\varphi_{k,B}(z,\tilde{y})d\tilde{y}dz$$
$$+\int_{C^{(k)}_{z_j(x)}:\sqrt{w_0}\le u}(x-z)^{\mu\eta-1}\int_{\substack{\tilde{y}=-\int_a^z\zeta_k(z)dz+v\\ v\ge 0}} e^{-\eta\tilde{y}}\varphi_{k,B}(z,\tilde{y})d\tilde{y}dz.$$

Note that the integration path of the inner integral of the first term and that of the second term on the left-hand side lie on the same side viewed from the singular point z_* of the integrand. Hence, also in the right-hand side, the branch of the integrand of the first term and that of the second term are the same. Thus the contribution of the second term is expressed also by one integral as follows:

$$\int_{C^{(k)}_{z_j(x)}:u_0\le u}(x-z)^{\mu\eta-1}\Phi_k(z,\eta)dz.$$

Finally we compute the contribution of the third term, that is,

$$\int_{\substack{y=-f_{k,0}+w\\ w_0\le w}} e^{-\eta y}I$$
$$=i\int_{\substack{y=-f_{k,0}+w\\ w_0\le w}} e^{-\eta y}\left[\int_{z_0}^{z_*}(W_{k,B}*W^{-1}_{k',B}*\varphi_{k',B})\right.$$
$$\left.\left(z,y+\mu\log(x-z)+\int_a^{a_0}(\zeta_k-\zeta_{k'})dz\right)(x-z)^{-1}dz\right]dy.$$

The inner integral (i.e., I) is taken along the steepest descent path $C^{(k')}_{z_0}$. In terms of the parametrization $f_{k'}(z)=f_{k'}(z_0)-\tilde{u}$ of $C^{(k')}_{z_0}$ it is done on the interval $\tilde{u}\in[0,w-w_0]$ as the points z_0 and z_* correspond to $\tilde{u}=0$ and $w-w_0$, respectively (cf. (4.7)). Hence, by changing the order of integration, we can rewrite the above integral as

$$i\int_{C^{(k')}_{z_0};0\le\tilde{u}}\left[\int_{\substack{y=-f_{k,0}+w\\ w_0+\tilde{u}\le w}} e^{-\eta y}(W_{k,B}*W^{-1}_{k',B}*\varphi_{k',B})\right.$$
$$\left.\left(z,y+\mu\log(x-z)+\int_a^{a_0}(\zeta_k-\zeta_{k'})dz\right)(x-z)^{-1}dy\right]dz$$

$$= i \int_{C_{z_0}^{(k')};0\le \tilde{u}} (x-z)^{\mu\eta-1} \exp\left(\eta \int_a^{a_0} (\zeta_k - \zeta_{k'})dz\right)$$
$$\left[\int_{\substack{\tilde{y}=-\int_a^z \zeta_{k'}dz+v \\ 0\le v}} e^{-\eta\tilde{y}} (W_{k,B} * W_{k',B}^{-1} * \varphi_{k',B})(z,y)dy\right] dz.$$

Here we put $\tilde{y} = y + \mu \log(x-z) + \int_a^{a_0}(\zeta_k - \zeta_{k'})dz$ and $v = w - w_0 - \tilde{u}$. Note that $\tilde{y} = -\int_a^z \zeta_{k'}dz + w - w_0 - \tilde{u}$ holds in view of (4.13). Hence the contribution of the third term is expressed as

$$i \exp\left(\eta \int_a^{a_0} (\zeta_k - \zeta_{k'})dz\right) \mathcal{W}_k \mathcal{W}_{k'}^{-1} \int_{C_{z_0}^{(k')}} (x-z)^{\mu\eta-1} \Phi_{k'}(z,\eta)dz,$$

where $\mathcal{W}_k$, $\mathcal{W}_{k'}$ and $\Phi_{k'}$ denote the Borel sum of W_k, $W_{k'}$ and $\varphi_{k'}$, respectively.

Summing up, we obtain

Theorem 4.3. *Under the same assumptions as in Theorem 4.2, the Borel sum of $\eta^{-1/2}\psi_j$ is expressed by the following:*

$$\int_{C_{z_j(x)}^{(k)};u\le u_0} (x-z)^{\mu\eta-1} \Phi_k(z,\eta)dz + \int_{C_{z_j(x)}^{(k)};u_0\le u} (x-z)^{\mu\eta-1} \Phi_k(z,\eta)dz$$
$$+ i \exp\left(\eta \int_a^{a_0} (\zeta_k - \zeta_{k'})dz\right) \mathcal{W}_k \mathcal{W}_{k'}^{-1} \int_{C_{z_0}^{(k')}} (x-z)^{\mu\eta-1} \Phi_{k'}(z,\eta)dz,$$

or, more simply,

$$\int_{C_{z_j(x)}^{(k)}} (x-z)^{\mu\eta-1} \Phi_k(z,\eta)dz$$
$$+ i \exp\left(\eta \int_a^{a_0} (\zeta_k - \zeta_{k'})dz\right) \mathcal{W}_k \mathcal{W}_{k'}^{-1} \int_{C_{z_0}^{(k')}} (x-z)^{\mu\eta-1} \Phi_{k'}(z,\eta)dz.$$

The case where $C_{z_j(x)}^{(k)}$ or $C_{z_0}^{(k')}$ has another crossing point with a Stokes curve of $P\varphi = 0$ can be handled in a similar manner. As a matter of fact, in such a case a singularity appears in the integration path of one of the three integrals in the right-hand side of (4.14). Then, since all integrals in the right-hand side of (4.14) have the same form as the integral (3.7) (except for the multiplicative factor $iW_{k,B} * W_{k',B}^{-1}$ and the shift by $\int_a^{a_0}(\zeta_k - \zeta_{k'})dz$ in the second variable of (4.15), which induce no problem at all), the reasoning employed so far is again applicable in the completely same manner as above. Further, we can repeat this procedure in an inductive way as far as the number of crossing points of steepest descent paths with Stokes curves is finite, in other words, under the assumption that the bifurcation process of steepest descent paths terminates in finite steps. As a consequence we obtain the following theorem, which completes the proof of Main Theorem.

Theorem 4.4. *Suppose that the same assumptions as in Main Theorem are satisfied. We label the (finitely many) crossing points of an exact steepest descent path passing through a saddle point $z_j(x)$ with Stokes curves of $P\varphi = 0$ as $z_1, \ldots, z_N$. To each steepest descent path C contained in the exact steepest descent path we can assign a sequence $(z_j(x), z_{\tau_1}, \ldots, z_{\tau_l})$ of crossing points, a sequence $(k, k_{\tau_1}, \ldots, k_{\tau_l})$ of types, and a sequence $(C_0, C_{\tau_1}, \ldots, C_{\tau_l})$ of steepest descent paths in the following way:*

> *Starting with the steepest descent path $C_0 = C_j^{(k)}$ passing through $z_j(x)$, we bifurcate a steepest descent path C_{τ_1} from a crossing point z_{τ_1} of C_0 with a Stokes curve of type $k > k_{\tau_1}$, and repeat this procedure. That is, when $C_{\tau_{\mu-1}}$ crosses a Stokes curve of type $k_{\tau_{\mu-1}} > k_{\tau_\mu}$ at z_{τ_μ}, then we bifurcate a steepest descent path C_{τ_μ} from z_{τ_μ} $(2 \leq \mu \leq l)$. The steepest descent path $C = C_{\tau_l}$ in question is then obtained at the final step of this procedure.*

Using these sequences, we denote C by $C = C^{(k,k_{\tau_1},\cdots,k_{\tau_l})}_{z_j(x),z_{\tau_1},\cdots,z_{\tau_l}}$. (Similarly, C_{τ_μ} is denoted by $C_{\tau_\mu} = C^{(k,k_{\tau_1},\cdots,k_{\tau_\mu})}_{z_j(x),z_{\tau_1},\cdots,z_{\tau_\mu}}$.) We also denote the exact steepest descent path passing through $z_j(x)$ by $\bigcup_{(\tau_1,\cdots,\tau_l)\in\Lambda} C^{(k,k_{\tau_1},\cdots,k_{\tau_l})}_{z_j(x),z_{\tau_1},\cdots,z_{\tau_l}}$ (where Λ is a finite set of indices), and assume that it does not pass through any other saddle point. Then $\eta^{-1/2}\psi_j(x,\eta)$ is Borel summable. Furthermore, its Borel sum is given by the following integral along the exact steepest descent path $\bigcup_{(\tau_1,\ldots,\tau_l)\in\Lambda} C^{(k,k_{\tau_1},\cdots,k_{\tau_l})}_{z_j(x),z_{\tau_1},\cdots,z_{\tau_l}}$:

$$\int_{C^{(k)}_{z_j(x)}} (x-z)^{\mu\eta-1}\Phi_k(z,\eta)dz$$
$$+ \sum_{(\tau_1,\cdots,\tau_l)\in\Lambda, l\geq 1} A^{(k,k_{\tau_1},\cdots,k_{\tau_l})}_{z_j(x),z_{\tau_1},\cdots,z_{\tau_l}} \int_{C^{(k,k_{\tau_1},\cdots,k_{\tau_l})}_{z_j(x),z_{\tau_1},\cdots,z_{\tau_l}}} (x-z)^{\mu\eta-1}\Phi_{k_{\tau_l}}(z,\eta)dz,$$

where Φ_k, $\Phi_{k_{\tau_l}}$ designate the Borel sum of φ_k, $\varphi_{k_{\tau_l}}$ and

$$A^{(k,k_{\tau_1},\cdots,k_{\tau_l})}_{z_j(x),z_{\tau_1},\cdots,z_{\tau_l}} = \prod_{1\leq\mu\leq l} c^{(k_{\tau_\mu})}_{z_{\tau_\mu}}.$$

Here $c^{(k_{\tau_\mu})}_{z_{\tau_\mu}}$ denotes the Borel sum of the Stokes coefficient for the Stokes phenomenon

$$\varphi^{(0)}_{k_{\tau_{\mu-1}}} \to \varphi^{(0)}_{k_{\tau_{\mu-1}}} + c^{(k_{\tau_\mu})}_{z_{\tau_\mu}} \varphi^{(0)}_{k_{\tau_\mu}}$$

observed at $z = z_{\tau_\mu}$.

Remark 4.3. As we will see in the subsequent section, it sometimes happens that several steepest descent paths with different notations introduced in Theorem 4.4 may overlap.

5 Examples — Verification via numerical analysis

In this section, to check Main Theorem numerically and to show its practical usefulness, we discuss some concrete examples with the aid of a computer. We mainly study the case where a steepest descent path crosses Stokes curves of P more than twice.

Example 5.1. Let us consider

$$P_1 = 3(\eta^{-1}\partial)^2 + 2c(\eta^{-1}\partial) + x.$$

Note that a differential equation $P_1\varphi = 0$ is equivalent to the Airy equation $((\eta^{-1}\partial_{\tilde{x}})^2 - \tilde{x})\tilde{\varphi} = 0$ via a change of variables $\varphi(x) = \exp(-\eta cx/3)\tilde{\varphi}(x)$, $-3^{1/3}\tilde{x} = x - c^2/3$. After applying the middle convolution, we have

$$\tilde{P}_1 = 3(\eta^{-1}\partial)^3 + 2c(\eta^{-1}\partial)^2 + x(\eta^{-1}\partial) - \mu + \eta^{-1},$$

which is the restriction to $x_1 = x$, $x_2 = c$ of the so-called $(1,4)$ hypergeometirc system in two variables (x_1, x_2) (cf. [11]). We take parameters as $c = -3+3i, \mu = 1-6i$ here. Following the recipe given in [2], we obtain the Stokes geometry of $\tilde{P}_1$ as shown in Figure 3. In Figure 3 red points, green small points, and blue small points designate (ordinary and virtual) turning points, ordered crossing points, and non-ordered crossing points of $\tilde{P}_1$, respectively. Dotted curves indicate parts of new Stokes curves on which no Stokes phenomenon occurs. (We will also use these notations in Example 5.2 below.)

We let x_{1A} denote a turning point situated near the bottom of Figure 3. Figures 9, 10 and 11 show the exact steepest descent paths of $\Re f_\pm$ passing through saddle points of

$$f_\pm = \mu \log(x - z) - \frac{1}{3}cz \pm \int^z \sqrt{-\frac{1}{3}\left(z - \frac{1}{3}c^2\right)}dz$$

when $x = x_{1A} + 0.1\exp(k\pi i/9)\,(0 \le k \le 17)$. In these Figures blue points and blue lines designate turning points and Stokes curves of $P\varphi = 0$, respectively. Red big points are saddle points of $f_\pm$, red small points are crossing points of a steepest descent path of $\Re f_\pm$ and a Stokes curve of type $+ \gtrless -$ of $P\varphi = 0$, and red solid lines are exact steepest descent paths of $\Re f_\pm$. (These notations will be also used in Example 5.2.) We observe that an exact steepest descent path flows into a saddle point between $k = 4$ and $k = 5$, between $k = 10$ and $k = 11$, and between $k = 16$ and $k = 17$. In view of Figure 3 we find that these directions are the same as the directions of Stokes curves emanating from x_{1A}.

Similarly we let x_{1B} denote an ordered crossing point near (slightly lower of) the center of Figure 3. Figures 12, 13 and 14 show the exact steepest descent paths of $\Re f_\pm$ when $x = x_{1B} + 0.1\exp(k\pi i/9)\,(0 \le k \le 17)$. We observe that an exact steepest descent path flows into a saddle point six times, that is, between $k = 2$ and $k = 3$, between $k = 5$ and $k = 6$, between $k = 8$ and $k = 9$, between

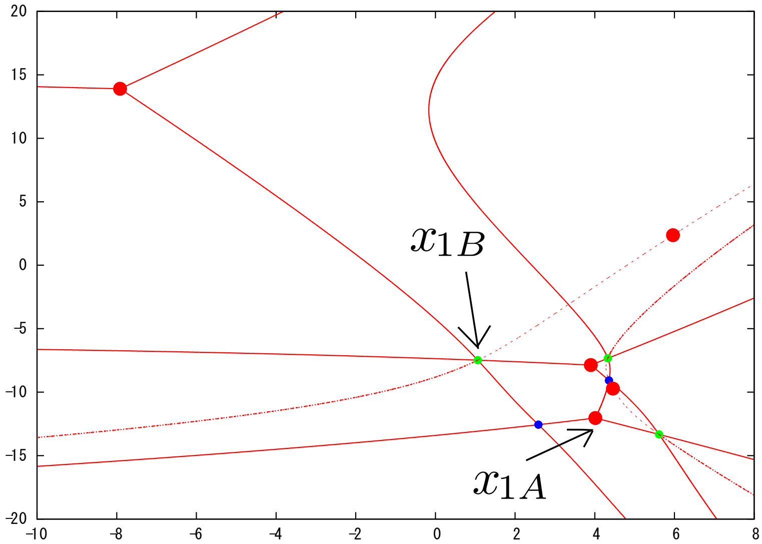


Figure 3: Stokes geometry of $\tilde{P}_1$.

$k = 11$ and $k = 12$, between $k = 14$ and $k = 15$, and between $k = 17$ and $k = 0$. Among these six directions, a Stokes phenomenon does not occur between $k = 2$ and $k = 3$. Indeed, the following holds.

Proposition 5.1. *Let P be a second order differential operator of the form* (1.1)*. Let $\zeta_\pm(x)$ be roots of the characteristic equation and $x = a$ be a turning point of P. Put*

$$f_\pm = \mu \log(x - z) + \int_a^z \zeta_\pm(z)dz$$

and let $z_j(x)$ be a saddle point of $f_\pm$. Let $C^{(\pm)}_{z_j(x)}$ be a steepest descent path of $\Re f_\pm$ passing through $z_j(x)$. As shown in Figure 4, we label types of three Stokes curves emanating from a. (A wavy line designates a cut to determine the branch of $\zeta_\pm$.) Under these notations we assume that $C^{(+)}_{z_j(x)}$ crosses a Stokes curve of type $+ > -$ at $z = z_0$ and a Stokes curve of type $- > +$ at $z = z_1$. Then a steepest descent path $C^{(-)}_{z_0}$ emanating from the crossing point z_0 also passes through z_1.

Proof. The defining equations of $C^{(+)}_{z_j(x)}$ and $C^{(-)}_{z_0}$ are given by

$$C^{(+)}_{z_j(x)} : \Im(f_+ - f_+|_{z=z_j(x)}) = 0,$$
$$C^{(-)}_{z_0} : \Im(f_- - f_-|_{z=z_0}) = 0,$$

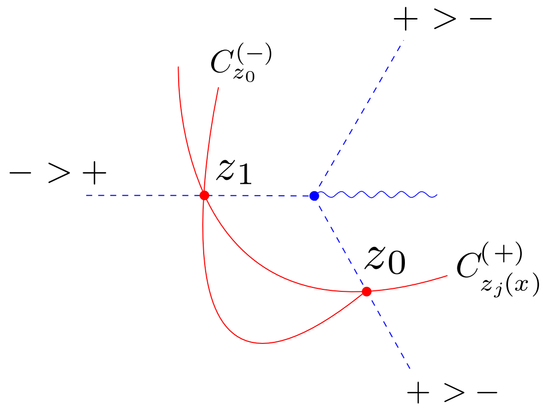


Figure 4: Steepest descent path which crosses two Stokes curves emanating from the same turning point.

respectively. Since $C_{z_j(x)}^{(+)}$ passes through z_0 and z_1, we have

$$\Im(f_+|_{z=z_0} - f_+|_{z=z_j(x)}) = 0,$$
$$\Im(f_+|_{z=z_1} - f_+|_{z=z_j(x)}) = 0.$$

Hence

$$\Im(f_+|_{z=z_0} - f_+|_{z=z_1}) = 0 \tag{5.1}$$

holds. On the other hand, we have

$$\Im(f_+|_{z=z_k} - f_-|_{z=z_k}) = \Im \int_a^{z_k} (\zeta_+(z) - \zeta_-(z))dz = 0 \tag{5.2}$$

for $k = 0, 1$ because z_0 and z_1 lie on Stokes curves of P emanating from a. Combining (5.1) and (5.2), we have $\Im(f_-|_{z=z_0} - f_-|_{z=z_1}) = 0$, which means $C_{z_0}^{(-)}$ passes through z_1. □

It follows from Proposition 5.1 that a steepest descent path bifurcated from $C_{z_0}^{(-)}$ at z_1 overlaps with $C_{z_j(x)}^{(+)}$. We parametrize $C_{z_j(x)}^{(+)}$ by $f_+ = f_+(z_j(x)) - u^2$ $(u \in \mathbb{R})$ and let $u = u_1$ be a value of the parameter corresponding to z_1. We may assume $u_1 > 0$ without loss of generality. Then, if we use WKB solutions $\varphi_\pm$ normalized at the unique turning point of P, the integral along the exact steepest descent path becomes

$$\begin{aligned}
&\int_{C^{(+)}_{z_j(x)}} (x-z)^{\mu\eta-1}\Phi_+(z)dz \\
&\quad - i\left(\int_{C^{(-)}_{z_0}} (x-z)^{\mu\eta-1}\Phi_-(z)dz - i\int_{C^{(+)}_{z_j(x)};u\geq u_1} (x-z)^{\mu\eta-1}\Phi_+(z)dz\right) \\
=&\int_{C^{(+)}_{z_j(x)};u\leq u_1} (x-z)^{\mu\eta-1}\Phi_+(z)dz - i\int_{C^{(-)}_{z_0}} (x-z)^{\mu\eta-1}\Phi_-(z)dz,
\end{aligned}$$

which shows that the integral along the exact steepest descent path has no contribution from the portion of the steepest descent path $C^{(+)}_{z_j(x)}$ passing over z_1, i.e., the overlapping portion of $C^{(+)}_{z_j(x)}$. Hence a Stokes phenomenon does not occur between $k=2$ and $k=3$ since only the overlapping portion of $C^{(+)}_{z_j(x)}$ flows into a saddle point there.

In conclusion, Stokes phenomena occur five times around x_{1B}. This is consistent with the configuration of Stokes curves around x_{1B} given in Figure 3.

Example 5.2. Let us consider

$$P_2 = (\eta^{-1}\partial)^2 + x^2 + c.$$

After applying the middle convolution, we have

$$\tilde{P}_2 = (\eta^{-1}\partial)^4 + (x^2+c)(\eta^{-1}\partial)^2 + (-2\mu x + 4x\eta^{-1})(\eta^{-1}\partial) + \mu^2 - 3\mu\eta^{-1} + 2\eta^{-2}.$$

We take parameters as $c = 1 + 0.1i$, $\mu = 1 - 6i$. The Stokes geometry of $\tilde{P}_2$ is shown in Figure 5. Enlarging it near the center, we have Figure 6. Figure 7 is a more enlarged version (enlarged in $[0,2]\times[0,2]$).

We let x_{2A} denote an (ordinary) turning point situated near the top of Figure 6. Figures 15, 16 and 17 show the exact steepest descent paths of $\Re f_\pm$ passing through saddle points of

$$f_\pm = \mu\log(x-z) \pm \int^z \sqrt{-(z^2+c)}dz$$

when $x = x_{2A} + 0.1\exp(k\pi i/9)\,(0 \leq k \leq 17)$. An exact steepest descent path flows into a saddle point between $k=1$ and $k=2$, between $k=7$ and $k=8$, and between $k=13$ and $k=14$. This is consistent with the Stokes geometry around x_{2A} given in Figure 6.

We next let x_{2B} denote an ordered crossing point situated near the bottom of Figure 7. Figures 18, 19 and 20 show the exact steepest descent paths of $\Re f_\pm$ when $x = x_{2B} + 0.01\exp(k\pi i/9)\,(0 \leq k \leq 17)$. An exact steepest descent path flows into a saddle point six times, that is, between $k=1$ and $k=2$, between $k=4$ and $k=5$, between $k=6$ and $k=7$, between $k=10$ and $k=11$, between $k=13$ and $k=14$, and between $k=14$ and $k=15$. However, Figure 7 indicates

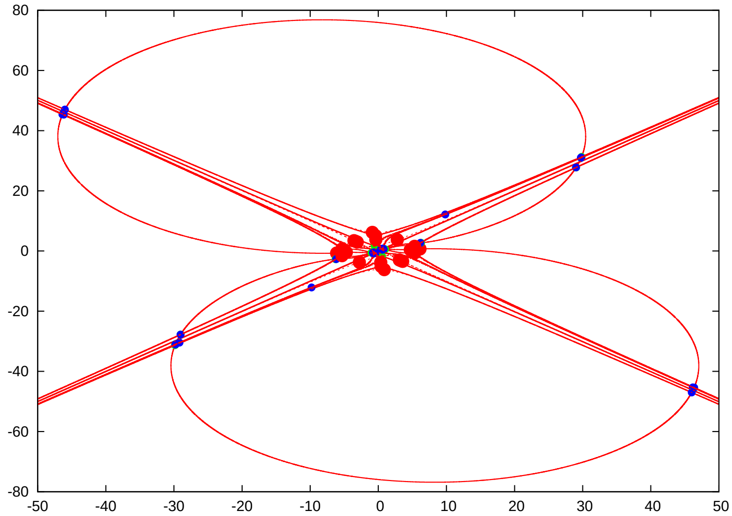


Figure 5: Stokes geometry of $\tilde{P}_2$.

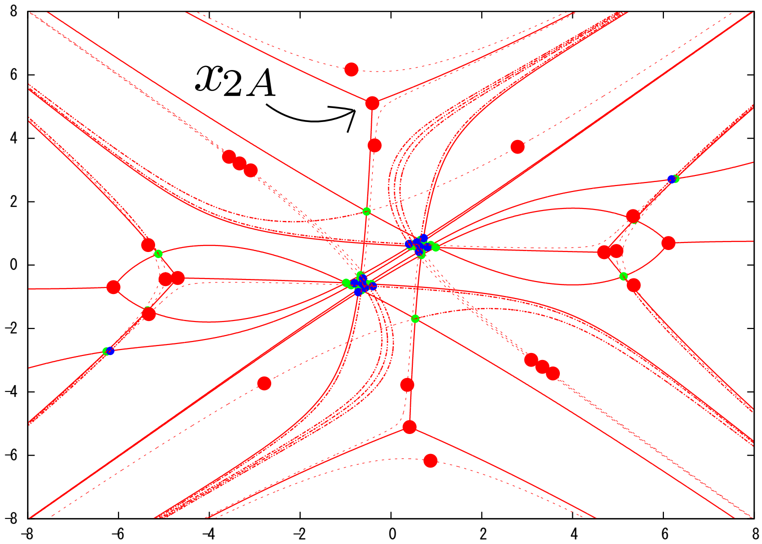


Figure 6: Stokes geometry of $\tilde{P}_2$; enlarged near the center.

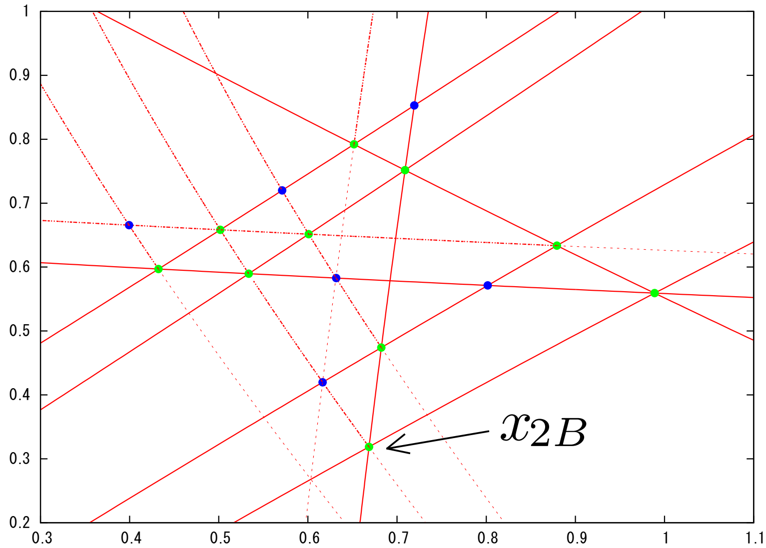


Figure 7: Stokes geometry of $\tilde{P}_2$; more enlarged in $[0,2] \times [0,2]$.

that no Stokes phenomenon occurs between $k = 14$ and $k = 15$. In the following, we discuss this inconsistency in more details.

Figure 8 shows a part relevant to the change between $k = 14$ and $k = 15$ of Figure 20. Dotted lines and wavy lines are the Stokes curves of P_2 and cuts to determine the branch of $\zeta_\pm = \pm\sqrt{-(z^2+c)}$, respectively. Solid lines are relevant portions of the exact steepest descent path of $\Re f_\pm$ passing through $z_j(x)$ and a heavy line designates the most important portion which changes between $k = 14$ and $k = 15$. We label turning points of P and steepest descent paths as in Figure 8. Here we denote the crossing point of $C^{(-)}_{z_1} = C^{(+,-)}_{z_j(x),z_1}$ and a Stokes curve of type $+ > -$ of P by z_5. We will prove that $C^{(+)}_{z_2} = C^{(+,-,+)}_{z_j(x),z_0,z_2}$ also passes through z_5. Let

$$f^{(+)}_\pm = \mu \log(x-z) + \int_{a_+}^{z} \zeta_\pm(z)dz,$$

where $\zeta_\pm(z)$ are the characteristic roots of P_2, i.e., $\zeta_\pm(z) = \pm\sqrt{-(z^2+c)}$. Further we denote

$$g(z) = \mu \log(x-z), \quad h_\pm(z) = \int_{a_+}^{z} \zeta_\pm(z)dz.$$

Then we have

$$\begin{aligned} & f^{(+)}_+|_{z=z_5} - f^{(+)}_+|_{z=z_2} \\ =& g(z_5) + h_+(z_5) - g(z_2) - h_+(z_2) \end{aligned}$$

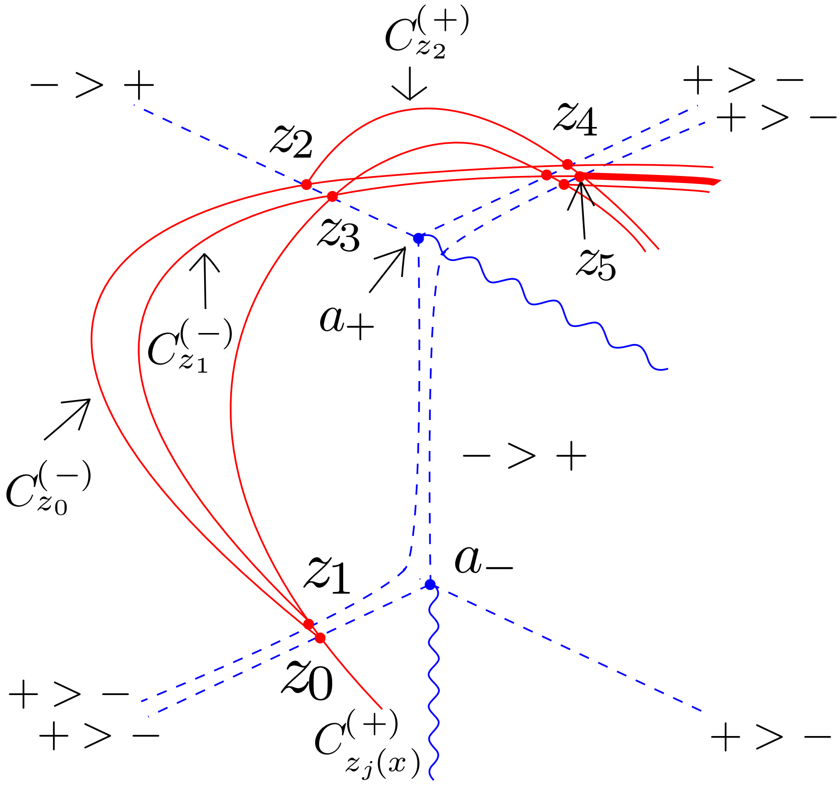


Figure 8: Steepest descent paths in question.

$$
\begin{aligned}
=&(h_+(z_5) - h_-(z_5)) + (g(z_5) + h_-(z_5) - g(z_1) - h_-(z_1)) \\
&+ (h_-(z_1) - h_+(z_1)) + (g(z_1) + h_+(z_1) - g(z_0) - h_+(z_0)) \\
&+ (h_+(z_0) - h_-(z_0)) + (g(z_0) + h_-(z_0) - g(z_2) - h_-(z_2)) \\
&+ (h_-(z_2) - h_+(z_2)) \\
=&(h_+(z_5) - h_-(z_5)) + (f_-^{(+)}(z_5) - f_-^{(+)}(z_1)) + (h_-(z_1) - h_+(z_1)) \\
&+ (f_+^{(+)}(z_1) - f_+^{(+)}(z_0)) + (h_+(z_0) - h_-(z_0)) \\
&+ (f_-^{(+)}(z_0) - f_-^{(+)}(z_2)) + (h_-(z_2) - h_+(z_2)).
\end{aligned}
$$

Since z_5 is on $C_{z_1}^{(-)}$ and z_1 is on a Stokes curve emanating from a_+ etc., we thus obtain

$$
\begin{aligned}
&\Im(f_+^{(+)}|_{z=z_5} - f_+^{(+)}|_{z=z_2}) \\
=&\Im(h_+(z_5) - h_-(z_5) + h_+(z_0) - h_-(z_0))
\end{aligned}
$$

$$
\begin{aligned}
&=\Im\left(\int_{a_+}^{z_5}(\zeta_+(z)-\zeta_-(z))dz+\int_{a_+}^{z_0}(\zeta_+(z)-\zeta_-(z))dz\right)\\
&=\Im\left(\int_{a_-}^{z_5}(\zeta_+(z)-\zeta_-(z))dz+\int_{a_+}^{a_-}(\zeta_-(z)-\zeta_+(z))dz\right.\\
&\qquad\left.+\int_{a_-}^{z_0}(\zeta_+(z)-\zeta_-(z))dz+\int_{a_+}^{a_-}(\zeta_+(z)-\zeta_-(z))dz\right)\\
&=0.
\end{aligned}
$$

Here, in obtaining the second term in the fourth line, we have used the fact that $\zeta_+(x)$ and $\zeta_-(x)$ are exchanged after crossing the cut emanating from a_+. It follows from this equality that $C_{z_2}^{(+)}$ also passes through z_5. Hence several steepest descent paths overlap on the relevant portion designated by heavy line in Figure 8. Among them, thanks to Proposition 5.1 and computations done just after that, $C_{z_j(x)}^{(+)}$ and $C_{z_3}^{(+)}=C_{z_j(x),z_1,z_3}^{(+,-,+)}$ have no contribution to the integral along the portion in question. Thus, to discuss the Stokes phenomenon between $k=14$ and $k=15$, it suffices to compute the contribution to the integral from $C_{z_1}^{(-)}$ and $C_{z_5}^{(-)}=C_{z_j(x),z_0,z_2,z_5}^{(+,-,+,-)}$, a steepest descent path bifurcated from $C_{z_2}^{(+)}$ at z_5.

We denote WKB solutions of $P_2\varphi=0$ normalized at a_+ and a_- by $\psi_\pm$ and $\varphi_\pm$, respectively:

$$
\psi_\pm=\frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_{a_+}^{x}S_{\mathrm{odd}}dx\right),\quad \varphi_\pm=\frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_{a_-}^{x}S_{\mathrm{odd}}dx\right),
$$

where $S_\pm=\pm S_{\mathrm{odd}}+S_{\mathrm{even}}$ denotes the logarithmic derivative of WKB solutions of $P_2\varphi=0$ satisfying $S_{-1}=\zeta_\pm(x)$ (cf. (2.4)). We have the following relations between these solutions:

$$
\psi_+=\exp\left(-\int_{a_-}^{a_+}S_{\mathrm{odd}}dx\right)\varphi_+,\quad \psi_-=\exp\left(\int_{a_-}^{a_+}S_{\mathrm{odd}}dx\right)\varphi_-.
$$

We first compute the contribution from $C_{z_1}^{(-)}$. A Stokes phenomenon observed when passing through z_1 is

$$
\begin{aligned}
\varphi_+&=\exp\left(\int_{a_-}^{a_+}S_{\mathrm{odd}}dx\right)\psi_+\\
&\to\exp\left(\int_{a_-}^{a_+}S_{\mathrm{odd}}dx\right)(\psi_+-i\psi_-)\\
&=\varphi_+-i\exp\left(2\int_{a_-}^{a_+}S_{\mathrm{odd}}dx\right)\varphi_-.
\end{aligned}
$$

Hence the coefficient of the integral (in terms of $\varphi_\pm$) along $C^{(-)}_{z_1}$ is

$$-i\exp\left(2\int_{a_-}^{a_+} S_{\text{odd}}dx\right).$$

We next compute the contribution from $C^{(-)}_{z_5}$. Since a Stokes phenomenon observed when passing through z_0 is $\varphi_+ \to \varphi_+ - i\varphi_-$, the coefficient of the integral along $C^{(-)}_{z_0} = C^{(+,-)}_{z_j(x),z_0}$ is $-i$. Furthermore, a Stokes phenomenon observed when passing through z_2 along $C^{(-)}_{z_0}$ is

$$\begin{aligned}
\varphi_- &= \exp\left(-\int_{a_-}^{a_+} S_{\text{odd}}dx\right)\psi_- \\
&\to \exp\left(-\int_{a_-}^{a_+} S_{\text{odd}}dx\right)(\psi_- - i\psi_+) \\
&= \varphi_- - i\exp\left(-2\int_{a_-}^{a_+} S_{\text{odd}}dx\right)\varphi_+.
\end{aligned}$$

Hence, taking the coefficient of the integral along $C^{(-)}_{z_0}$ computed above into account, we find that the coefficient of the integral (in terms of $\varphi_\pm$) along $C^{(+)}_{z_2}$ is given by

$$-i\left(-i\exp\left(-2\int_{a_-}^{a_+} S_{\text{odd}}dx\right)\right) = -\exp\left(-2\int_{a_-}^{a_+} S_{\text{odd}}dx\right).$$

Define $\tilde{\varphi}_\pm$ by

$$\tilde{\varphi}_\pm = \frac{1}{\sqrt{S_{\text{odd}}}}\exp\left(\pm\int_{a_-\to x} S_{\text{odd}}dx\right),$$

where the integral is taken along a path starting from a_-, crossing the cut emanating from a_+, and ending at x. The branch of the integrand stands for the one at the starting point a_-. We also let γ_{+-} denote a path starting from a_-, going to a_+, then returning to a_- after changing the branch at a_+ (that is, after crossing the cut emanating from a_+). In the following we assume the branch of the integrand stands for the one at the starting point a_- for the integral whose integration path is γ_{+-}. Then the relations between $\varphi_\pm$ and $\tilde{\varphi}_\pm$ are

$$\varphi_+ = \exp\left(\int_{\gamma_{+-}} S_{\text{odd}}dx\right)\tilde{\varphi}_-, \quad \varphi_- = \exp\left(-\int_{\gamma_{+-}} S_{\text{odd}}dx\right)\tilde{\varphi}_+.$$

Under these notations we find that a Stokes phenomenon observed when passing through z_5 along $C_{z_2}^{(+)}$ is

$$\begin{aligned}\varphi_+ &= \exp\left(\int_{\gamma_{+-}} S_{\mathrm{odd}}dx\right)\tilde{\varphi}_- \\ &\to \exp\left(\int_{\gamma_{+-}} S_{\mathrm{odd}}dx\right)(\tilde{\varphi}_- - i\tilde{\varphi}_+) \\ &= \varphi_+ - i\exp\left(2\int_{\gamma_{+-}} S_{\mathrm{odd}}dx\right)\varphi_- \\ &= \varphi_+ - i\exp\left(4\int_{a_-}^{a_+} S_{\mathrm{odd}}dx\right)\varphi_-.\end{aligned}$$

Therefore the coefficient of the integral along $C_{z_5}^{(-)}$ (which overlaps with $C_{z_1}^{(-)}$ as discussed above) is

$$-\exp\left(-2\int_{a_-}^{a_+} S_{\mathrm{odd}}dx\right)\left(-i\exp\left(4\int_{a_-}^{a_+} S_{\mathrm{odd}}dx\right)\right) = i\exp\left(2\int_{a_-}^{a_+} S_{\mathrm{odd}}dx\right).$$

Since this is -1 multiple of the coefficient of the integral along $C_{z_1}^{(-)}$, the contributions from $C_{z_1}^{(-)}$ and $C_{z_5}^{(-)}$ are also cancelled out. Hence the relevant overlapping portion of the steepest descent paths in question (i.e., the portion designated by heavy line in Figure 8) has no contribution to the integral along the exact steepest descent path.

Thus no Stokes phenomenon occurs between $k = 14$ and $k = 15$. In conclusion, the behavior of the exact steepest descent paths around x_{2B} is consistent with the Stokes geometry given in Figure 7.

References

[1] T. Aoki, T. Kawai and Y. Takei, The Bender-Wu analysis and the Voros theory, Special Functions, Springer-Verlag, 1991, pp. 1–29.

[2] ———, New turning points in the exact WKB analysis for higher-order ordinary differential equations, Analyse algébrique des perturbations singulières, I, Méthodes résurgentes, Hermann, 1994, pp. 69–84.

[3] ———, On the exact steepest descent method: A new method for the description of Stokes curves, J. Math. Phys., **42**(2001), 3691–3713.

[4] ———, The exact steepest descent method – a new steepest descent method based on the exact WKB analysis, Adv. Stud. in Pure Math., Vol. 42, Math. Soc. Japan, 2004, pp. 45–61.

[5] W. Balser, From Divergent Power Series to Analytic Functions, Lecture Notes in Math., Vol. 1582, Springer-Verlag, 1994.

[6] H.L. Berk, W.M. Nevins and K.V. Roberts, New Stokes' line in WKB theory, J. Math. Phys, **23**(1982), 988–1002.

[7] O. Costin, Asymptotics and Borel Summability, Monographs and Surveys in Pure and Applied Mathematics, Vol. 141, CRC Press, 2009.

[8] E. Delabaere and F. Pham, Resurgent methods in semi-classical asymptotics, Ann. Inst. H. Poincaré, **71**(1999), 1–94.

[9] M. Dettweiler and S. Reiter, Middle convolution of Fuchsian systems and the construction of rigid differential systems, J. Algebra, **318**(2007), 1–24.

[10] N. Honda, T. Kawai and Y. Takei, Virtual Turning Points, SpringerBriefs in Mathematical Physics, Vol. 4, Springer-Verlag, 2015.

[11] K. Iwaki and T. Koike, On the computation of Voros coefficients via middle convolutions, RIMS Kôkyûroku Bessatsu, **B52**(2014), 55–70.

[12] S. Kamimoto and T. Koike, On the Borel summability of WKB theoretic transformation series, to appear in RIMS Kôkyûroku Bessatsu.

[13] N. Katz, Rigid Local Systems, Ann. of Math. Stud., Vol. 139, Princeton University Press, 1996.

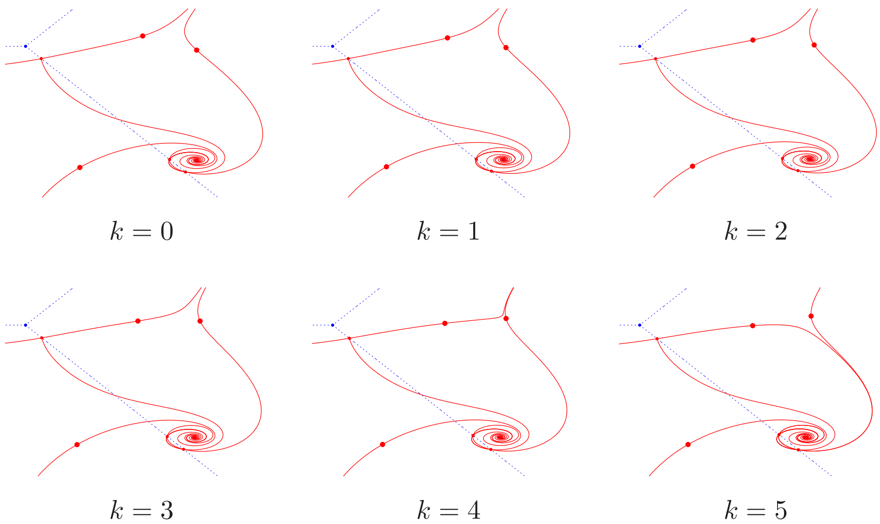
[14] T. Kawai and Y. Takei, Algebraic Analysis of Singular Perturbation Theory, Translations of Mathematical Monographs, Vol. 227, Amer. Math. Soc., 2005.

[15] T. Koike and R. Schäfke, On the Borel summability of WKB solutions of Schrödinger equations with polynomial potentials and its application, in preparation.

[16] T. Koike and Y. Takei, Exact WKB analysis of second-order non-homogeneous linear ordinary differential equations, RIMS Kôkyûroku Bessatsu, **B40**(2013), 293–312.

[17] T. Oshima, Fractional Calculus of Weyl Algebra and Fuchsian Differential Equations, MSJ Memoirs, Vol. 28, Math. Soc. Japan, 2012.

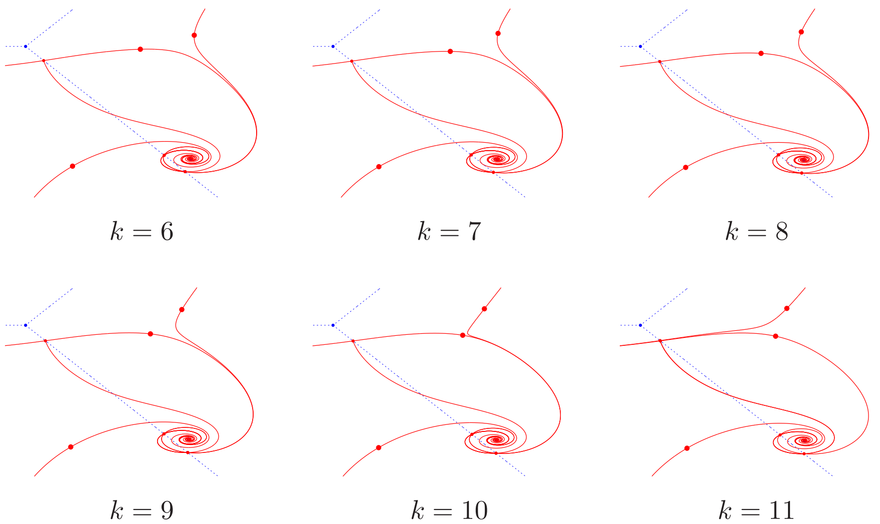
[18] A. Voros, The return of the quartic oscillator. The complex WKB method, Ann. Inst. H. Poincaré, **39**(1983), 211–338.



$k = 0$ $k = 1$ $k = 2$

$k = 3$ $k = 4$ $k = 5$

Figure 9: The exact steepest descent paths around x_{1A} $(0 \leq k \leq 5)$.



$k = 6$ $k = 7$ $k = 8$

$k = 9$ $k = 10$ $k = 11$

Figure 10: The exact steepest descent paths around x_{1A} $(6 \leq k \leq 11)$.

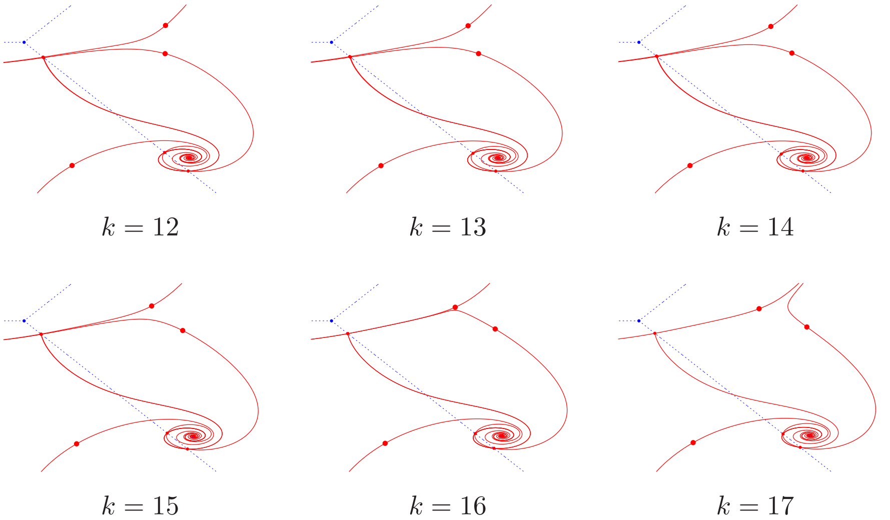


Figure 11: The exact steepest descent paths around x_{1A} $(12 \leq k \leq 17)$.

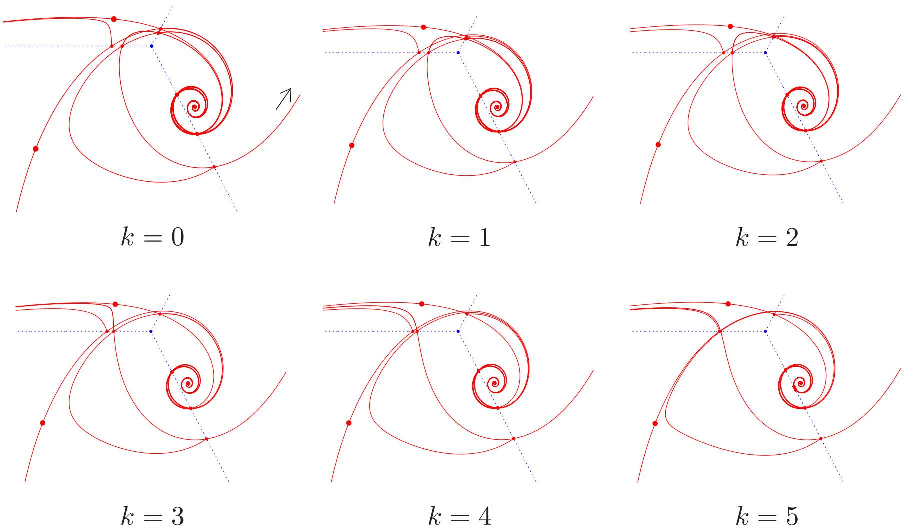


Figure 12: The exact steepest descent paths around x_{1B} $(0 \leq k \leq 5)$. The third saddle point is not within the scope of these figures, but located on the extension (in the direction indicated by an arrow in the figure for $k = 0$) of a steepest descent path.

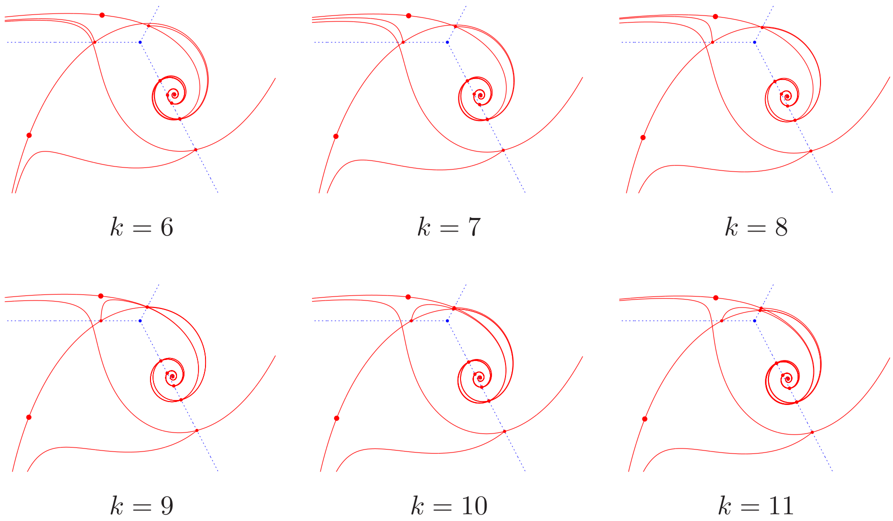


Figure 13: The exact steepest descent paths around x_{1B} ($6 \leq k \leq 11$).

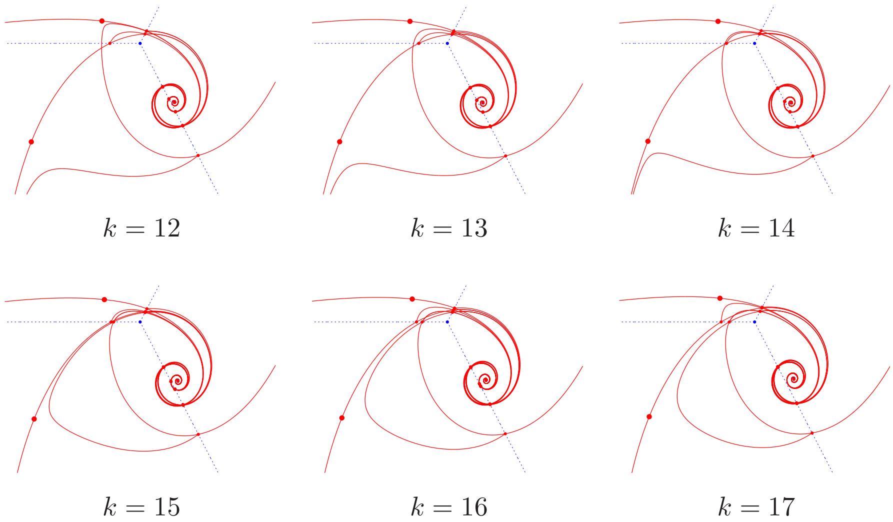


Figure 14: The exact steepest descent paths around x_{1B} ($12 \leq k \leq 17$).

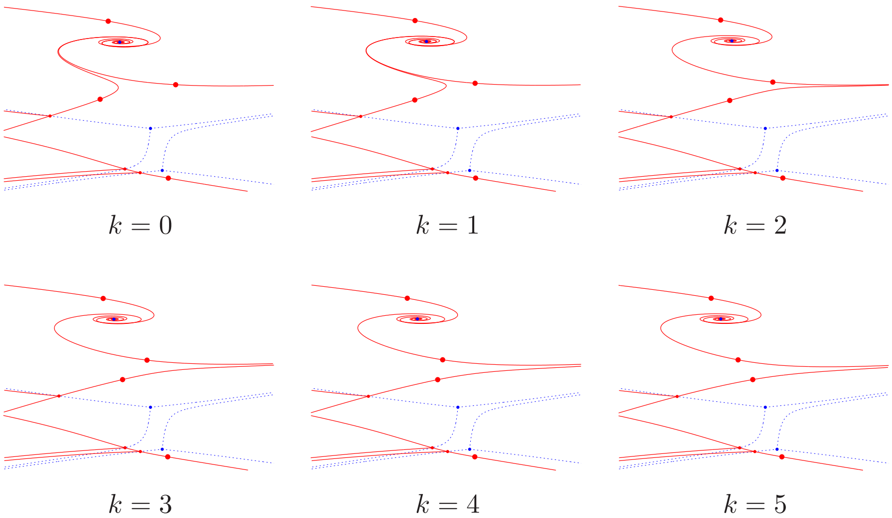


Figure 15: The exact steepest descent paths around x_{2A} $(0 \leq k \leq 5)$.

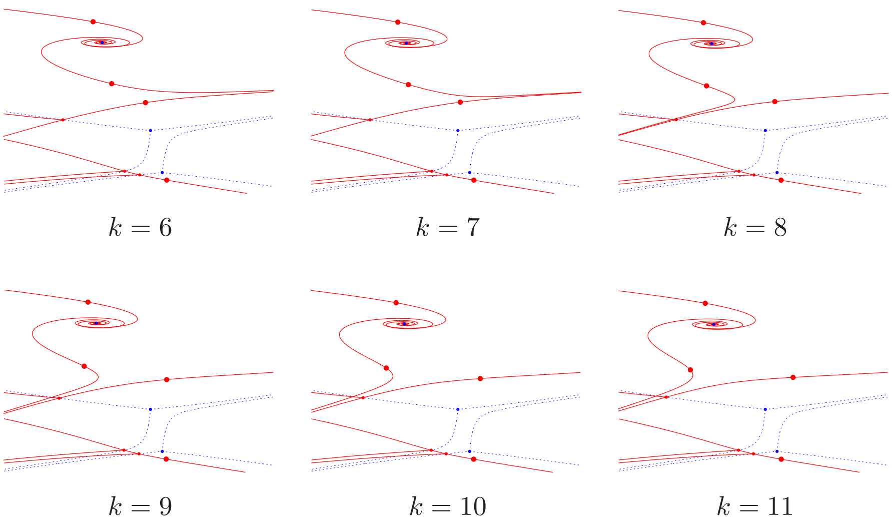


Figure 16: The exact steepest descent paths around x_{2A} $(6 \leq k \leq 11)$.

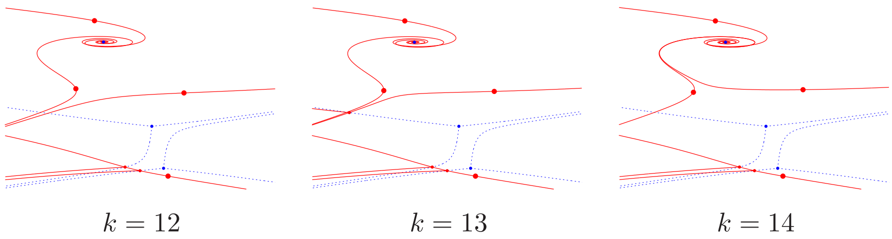


Figure 17: The exact steepest descent paths around x_{2A} $(12 \leq k \leq 17)$.

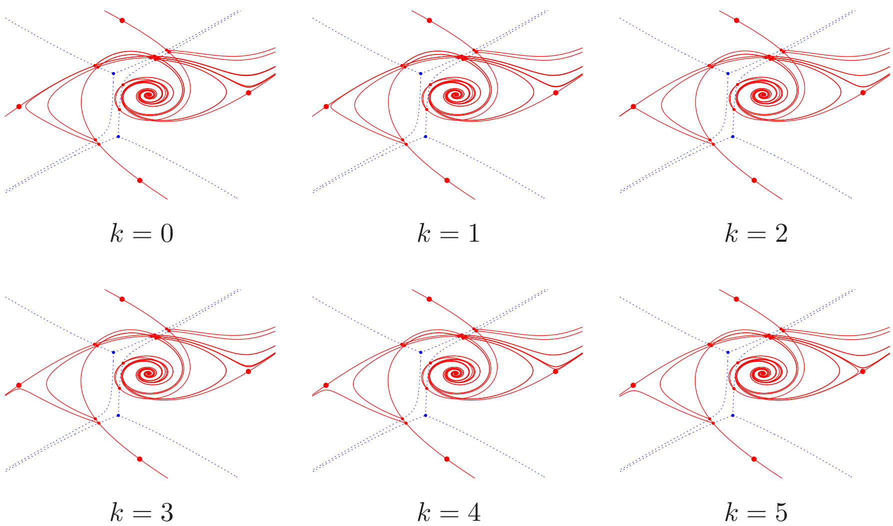
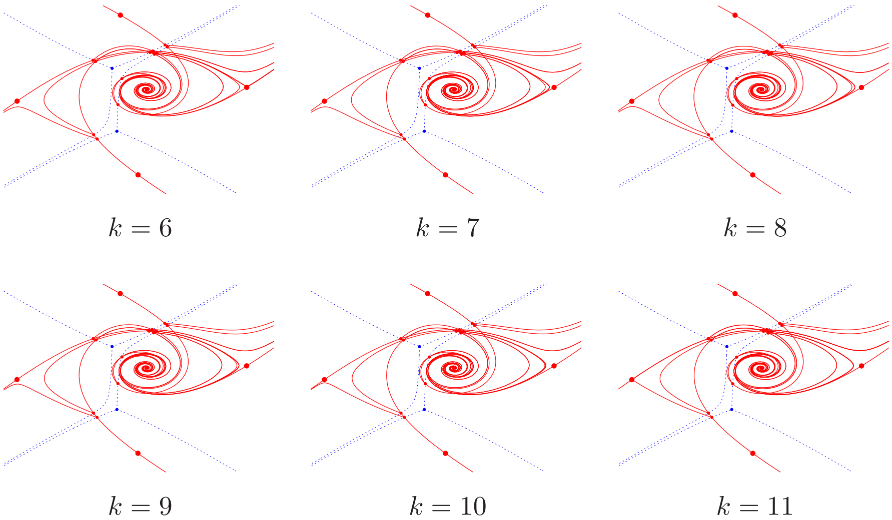


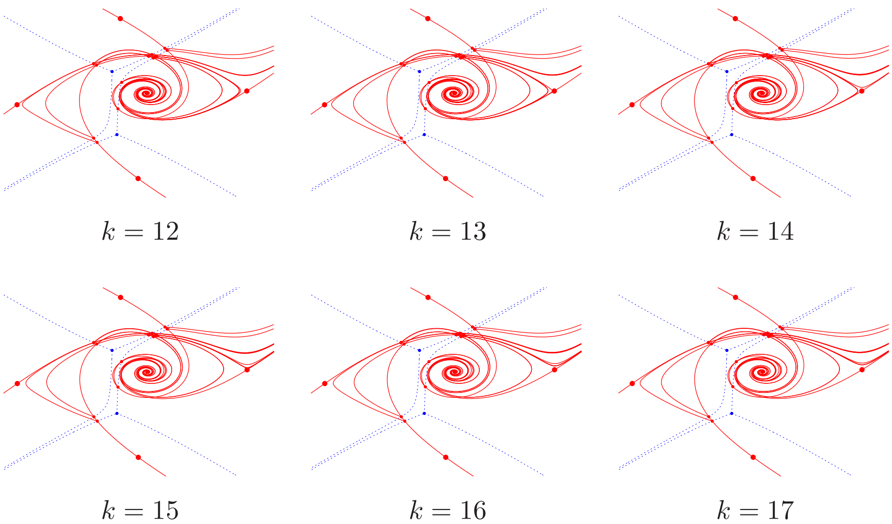
Figure 18: The exact steepest descent paths around x_{2B} $(0 \leq k \leq 5)$.



$k = 6$ $k = 7$ $k = 8$

$k = 9$ $k = 10$ $k = 11$

Figure 19: The exact steepest descent paths around x_{2B} $(6 \leq k \leq 11)$.



$k = 12$ $k = 13$ $k = 14$

$k = 15$ $k = 16$ $k = 17$

Figure 20: The exact steepest descent paths around x_{2B} $(12 \leq k \leq 17)$.