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Reconstruction of inertia groups associated to log divisors from a configuration space group equipped with log-full subgroups

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RECONSTRUCTION OF INERTIA GROUPS ASSOCIATED TO LOG DIVISORS FROM A CONFIGURATION SPACE GROUP EQUIPPED WITH LOG-FULL SUBGROUPS

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ABSTRACT. In the present paper, we study configuration space groups. The goal of this paper is to reconstruct group-theoretically various log divisors of a log configuration space of a smooth log curve from the associated configuration space group equipped with log-full subgroups.

0. Introduction

Let p, l be distinct prime numbers; k an algebraically closed field of characteristic zero or p; $S \overset{\text{def}}{=} \mathrm{Spec}(k)$; (g, r) a pair of nonnegative integers such that $2g-2+r > 0$; $X^{\log} \to S$ a smooth log curve of type (g, r) (cf. Notation 1.3, (iv)); $n \in \mathbb{Z}_{>1}$. In the present paper, we study the n-th log configuration space $X_n^{\log}$ associated to $X^{\log} \to S$ (cf. Definition 2.1). The log scheme $X_n^{\log}$ is a suitable compactification of the usual n-th configuration space U_{X_n} associated to the smooth curve determined by $X^{\log}$. Write $\Pi_n \overset{\text{def}}{=} \pi_1^{\text{pro-}l}(X_n^{\log})$ for the pro-l configuration space group determined by $X_n^{\log}$ (cf. [MzTa], Definition 2.3, (i)), i.e., the maximal pro-l quotient of the fundamental group of the log scheme $X_n^{\log}$. We shall refer to an irreducible divisor of the underlying scheme of $X_n^{\log}$ contained in the complement of U_{X_n} as a *log divisor* of $X_n^{\log}$. The log divisor V determines an inertia group $I_V(\simeq \mathbb{Z}_l) \subset \Pi_n$, which plays a central role in the present paper. Let $V_1, \ldots, V_n$ be distinct log divisors of $X_n^{\log}$ such that $V_1 \cap \cdots \cap V_n \neq \emptyset$. Then we shall refer to $P \overset{\text{def}}{=} V_1 \cap \cdots \cap V_n$ as a *log-full point* (cf. Definition 2.2, (ii), and Remark 2.3, (ii)). The log-full point $P = V_1 \cap \cdots \cap V_n$ determines a *log-full subgroup* $A(\simeq I_{V_1} \times \cdots \times I_{V_n} \simeq \mathbb{Z}_l^{\oplus n}) \subset \Pi_n$ (cf. Definition 2.2, (iii)). It is known that log-full subgroups of a configuration space group may be characterized group-theoretically whenever the configuration space group is equipped with a suitable action of a profinite group (cf. [HMM], Theorem 3.7). In the present paper, we reconstruct group-theoretically inertia groups associated to log divisors in a configuration space group from the configuration space group equipped with log-full subgroups. Moreover, we reconstruct group-theoretically inertia groups associated to tripodal divisors (cf. Definition 3.1, (ii)) and drift diagonals (cf. Definition 3.1, (v)), as well as drift collections (cf. Definition 8.14) and drift fiber subgroups (cf. Definition 9.1).

Our main result is as follows:

Theorem 0.1. *For $\Box \in \{\circ, \bullet\}$, let $p^\Box$, $l^\Box$ be distinct prime numbers; $k^\Box$ an algebraically closed field of characteristic zero or $p^\Box$; $S^\Box \overset{\text{def}}{=} \mathrm{Spec}(k^\Box)$; $(g^\Box, r^\Box)$ a pair* *of nonnegative integers such that* $2g^\square - 2 + r^\square > 0$*;*

$$X^{\log \square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$*;* $n^\square \in \mathbb{Z}_{>1}$*;* $X^{\log \square}_{n^\square}$ *the* $n^\square$*-th log configuration space associated to* $X^{\log \square} \to S^\square$*;* $\Pi^\square \stackrel{\text{def}}{=} \pi_1^{\text{pro-}l^\square}(X^{\log \square}_{n^\square})$*;*

$$\phi \colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups. We suppose that $r^\square > 0$*;* ϕ *induces a bijection between the set of log-full subgroups of* Π° *and the set of log-full subgroups of* $\Pi^\bullet$*. Then the following hold:*

(i) ϕ *induces a bijection between the set of inertia groups of* Π° *associated to log divisors of* $X^{\log \circ}_{n^\circ}$ *and the set of inertia groups of* $\Pi^\bullet$ *associated to log divisors of* $X^{\log \bullet}_{n^\bullet}$ *(cf. Theorem 5.3).*

(ii) ϕ *induces a bijection between the set of inertia groups of* Π° *associated to tripodal divisors of* $X^{\log \circ}_{n^\circ}$ *and the set of inertia groups of* $\Pi^\bullet$ *associated to tripodal divisors of* $X^{\log \bullet}_{n^\bullet}$ *(cf. Theorem 6.4).*

(iii) ϕ *induces a bijection between the set of inertia groups of* Π° *associated to drift diagonals of* $X^{\log \circ}_{n^\circ}$ *and the set of inertia groups of* $\Pi^\bullet$ *associated to drift diagonals of* $X^{\log \bullet}_{n^\bullet}$ *(cf. Theorem 7.3).*

(iv) ϕ *induces a bijection between the set of drift collections of* Π° *and the set of drift collections of* $\Pi^\bullet$ *(cf. Theorem 8.15).*

(v) ϕ *induces a bijection between the set of drift fiber subgroups of* Π° *and the set of drift fiber subgroups of* $\Pi^\bullet$ *(cf. Theorem 9.3).*

Note that one may define the notion of a log-full point even if $r = 0$ (cf. [HMM], Definition 1.1). Since there is no log-full point if $r = 0$, we however suppose that $r > 0$ in the present paper. Note also that, roughly speaking, Theorem 0.1, (i), asserts that we may extract group-theoretically a "geometric direct summand $\mathbb{Z}_l$" (i.e., a log divisor) from "$\mathbb{Z}_l^{\oplus n}$" (i.e., a log-full subgroup).

This paper is organized as follows: In §1, we explain some notations. In §2, we define log configuration spaces, log-full points, and log divisors. In §3, we define tripodal divisors and drift diagonals, and we study the geometry of various log divisors. In §4, we reconstruct scheme-theoretically non-degenerate elements (cf. Definition 4.5, (i)) of a log-full subgroup. In §5, we reconstruct log divisors. In §6, we reconstruct tripodal divisors. In §7, we reconstruct drift diagonals. In §8, we reconstruct drift collections. In §9, we reconstruct drift fiber subgroups.

1. **Notations**

Notation 1.1. (i) Let G be a group. If we apply the notation "e" to an element of G, then "$e \in G$" always denotes the identity element of G.

(ii) Let G be a group, $H \subseteq G$ a subgroup, and $\alpha \in G$. We write

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid gh = hg \text{ for any } h \in H\}$$

for the centralizer of H in G;

$$N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid gHg^{-1} = H\}$$

for the normalizer of H in G;

$$Z_G(\alpha) \stackrel{\text{def}}{=} Z_G(\langle\alpha\rangle) = \{g \in G \mid g\alpha = \alpha g\}.$$

Notation 1.2. Let $S^{\log}$ be an fs log scheme.

(i) Write S for the underlying scheme of $S^{\log}$.

(ii) Write $\mathcal{M}_S$ for the sheaf of monoids that defines the log structure of $S^{\log}$.

(iii) Let $\overline{s}$ be a geometric point of S. Then we shall denote by $I(\overline{s}, \mathcal{M}_S)$ the ideal of $\mathcal{O}_{S,\overline{s}}$ generated by the image of $\mathcal{M}_{S,\overline{s}} \setminus \mathcal{O}^{\times}_{S,\overline{s}}$ via the homomorphism of monoids $\mathcal{M}_{S,\overline{s}} \to \mathcal{O}_{S,\overline{s}}$ induced by $\mathcal{M}_S \to \mathcal{O}_S$ which defines the log structure of $S^{\log}$.

(iv) Let $s \in S$ and $\overline{s}$ a geometric point of S which lies over s. Write $(\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}})^{\mathrm{gp}}$ for the groupification of $\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}}$. Then we shall refer to the nonnegative integer $\mathrm{rank}(\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}})^{\mathrm{gp}}$ as the *log rank* at s. Note that one verifies easily that $\mathrm{rank}(\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}})^{\mathrm{gp}}$ is independent of the choice of $\overline{s}$, i.e., depends only on s.

(v) Let $m \in \mathbb{Z}$. Then write

$$S^{\log \le m} \stackrel{\text{def}}{=} \{s \in S \mid \text{the log rank at } s \text{ is } \le m\}.$$

Note that $S^{\log \le m}$ is open in S.

(vi) Write $U_S \stackrel{\text{def}}{=} S^{\log \le 0}$ and refer to U_S as the *interior* of $S^{\log}$.

Notation 1.3. Let (g, r) be a pair of nonnegative integers such that $2g-2+r > 0$.

(i) Write $\mathcal{M}_{g,r}$ for the moduli stack of smooth curves of type (g, r) over $\mathbb{Z}$ and $\overline{\mathcal{M}}_{g,r}$ for the moduli stack of pointed stable curves of type (g, r) over $\mathbb{Z}$. Here, we assume the marking sections to be ordered.

(ii) Write

$$\overline{\mathcal{C}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$$

for the tautological curve over $\overline{\mathcal{M}}_{g,r}$; $\overline{\mathcal{D}}_{g,r} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$ for the divisor at infinity.

(iii) Write $\overline{\mathcal{M}}^{\log}_{g,r}$ for the log stack obtained by equipping the moduli stack $\overline{\mathcal{M}}_{g,r}$ with the log structure determined by $\overline{\mathcal{D}}_{g,r}$.

(iv) The divisor given by the union of the divisor of $\overline{\mathcal{C}}_{g,r}$ corresponding to the marked points with the inverse image in $\overline{\mathcal{C}}_{g,r}$ of $\overline{\mathcal{D}}_{g,r}$ determines a log structure on $\overline{\mathcal{C}}_{g,r}$; we denote the resulting log stack by $\overline{\mathcal{C}}^{\log}_{g,r}$. Thus, we obtain a morphism of log stacks

$$\overline{\mathcal{C}}^{\log}_{g,r} \to \overline{\mathcal{M}}^{\log}_{g,r}$$

which we refer to as the *tautological log curve* over $\overline{\mathcal{M}}^{\log}_{g,r}$. If $S^{\log}$ is an arbitrary log scheme, then we shall refer to a morphism

$$C^{\log} \to S^{\log}$$

which is obtained as the pull-back of the tautological log curve via some morphism $S^{\log} \to \overline{\mathcal{M}}^{\log}_{g,r}$ as a *stable log curve* (of type (g, r)). If $C \to S$ is smooth, i.e., any geometric fiber of $C \to S$ has no nodes, then we shall refer to $C^{\log} \to S^{\log}$ as a *smooth log curve* (of type (g, r)).

(v) A smooth log curve of type $(0, 3)$ will be referred to as a *tripod.* A vertex of a semi-graph of anabelioids of pro-l PSC-type (cf. [CmbGC], Definition 1.1, (i)) of type $(0, 3)$ (cf. [CbTpI], Definition 2.3, (iii)) will be referred to as a *tripod.*

2. **Log configuration spaces and log divisors**

Let p, l be distinct prime numbers; k an algebraically closed field of characteristic zero or p; $S \overset{\mathrm{def}}{=} \mathrm{Spec}(k)$; (g,r) a pair of nonnegative integers such that $2g-2+r>0$;

$$X^{\log} \to S$$

a smooth log curve of type (g,r); $n \in \mathbb{Z}_{>0}$. We suppose that

$$r > 0.$$

In the present §2, we define log configuration spaces, log-full points, and log divisors.

Definition 2.1. The smooth log curve $X^{\log}$ over S determines a "classifying morphism" $S \to \overline{\mathcal{M}}^{\log}_{g,r}$. Thus, by pulling back the morphism $\overline{\mathcal{M}}^{\log}_{g,r+n} \to \overline{\mathcal{M}}^{\log}_{g,r}$ given by forgetting the last n marked points via this morphism $S \to \overline{\mathcal{M}}^{\log}_{g,r}$, we obtain a morphism of log schemes

$$X^{\log}_n \to S.$$

We shall refer to $X^{\log}_n$ as the *n-th log configuration space associated to* $X^{\log} \to S$. Note that $X^{\log}_1 = X^{\log}$. Write $X^{\log}_0 \overset{\mathrm{def}}{=} S$.

Definition 2.2. (i) Write

$$\Pi_n \overset{\mathrm{def}}{=} \pi_1^{\text{pro-}l}(X^{\log}_n)$$

for the maximal pro-l quotient of the fundamental group of the log scheme $X^{\log}_n$.

(ii) Let P be a closed point of X_n. We shall say that P is a *log-full point* of $X^{\log}_n$ if

$$\dim(\mathcal{O}_{X_n,P}/I(P,\mathcal{M}_{X_n})) = 0,$$

i.e., P is of maximal log rank (cf. Notation 1.2, (iv)).

(iii) Let P be a log-full point of $X^{\log}_n$ and $P^{\log}$ the log scheme obtained by restricting the log structure of $X^{\log}_n$ to the reduced closed subscheme of X_n determined by P. Then we obtain an outer homomorphism $\pi_1^{\text{pro-}l}(P^{\log}) \to \Pi_n$. We refer to $\mathrm{Im}(\pi_1^{\text{pro-}l}(P^{\log}) \to \Pi_n)$, well-defined up to conjugation, as a *log-full subgroup* at P.

(iv) Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-l PSC-type and $\mathbb{G}$ the underlying semi-graph of $\mathcal{G}$. Then we shall write

$$\mathrm{Cusp}(\mathcal{G})$$

for the set of cusps of $\mathcal{G}$ and

$$\mathrm{Cusp}(\mathbb{G})$$

for the set of cusps of $\mathbb{G}$. Thus, we have a natural bijection $\mathrm{Cusp}(\mathcal{G}) \overset{\sim}{\to} \mathrm{Cusp}(\mathbb{G})$.

(v) Let P be a point of $X^{\log}_n$. Then P parametrizes a pointed stable curve of type $(g, r+n)$ (cf. Definition 2.1), which thus determines a semi-graph of anabelioids of pro-l PSC-type. We shall write $\mathcal{G}_P$ for this semi-graph of anabelioids of pro-l PSC-type.

(vi) Let us fix an ordered set

$$C_{r,n} \stackrel{\mathrm{def}}{=} \{c_1, \ldots, c_r, x_1 \stackrel{\mathrm{def}}{=} c_{r+1}, \ldots x_n \stackrel{\mathrm{def}}{=} c_{r+n}\}.$$

Then, by definition, for each point P of $X_n^{\log}$, we have a natural bijection $C_{r,n} \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_P)$. In the following, let us identify the set $\mathrm{Cusp}(\mathcal{G}_P)$ with $C_{r,n}$.

(vii) We shall refer to an irreducible divisor of X_n contained in the complement $X_n \setminus U_{X_n}$ of the interior U_{X_n} of $X_n^{\log}$ as a *log divisor* of $X_n^{\log}$. That is to say, a log divisor of $X_n^{\log}$ is an irreducible divisor of X_n whose generic point parametrizes a pointed stable curve with precisely two irreducible components (cf. Definition 2.1).

(viii) Let V be a log divisor of $X_n^{\log}$. Then we shall write $\mathcal{G}_V$ for "$\mathcal{G}_P$" in the case where we take "P" to be the generic point of V.

(ix) For $1 \leq i \leq n$, write $p_i \colon X_n^{\log} \to X^{\log}$ for the projection morphism of co-profile $\{i\}$ (cf. [MzTa], Definition 2.1, (ii)). Let $\iota \stackrel{\mathrm{def}}{=} (p_i)_{1 \leq i \leq n} \colon X_n^{\log} \to X^{\log} \times_S \cdots \times_S X^{\log}$.

Remark 2.3. (i) By establishing a similar theory to the theory discussed in [Hsh2], §3, one verifies easily that, for each finite collection of log divisors $V_1, \ldots, V_m$, the intersection $V_1 \cap \cdots \cap V_m$ is isomorphic, over S, to

$$X_{i_1} \times_S (\overline{\mathcal{M}}_{0,i_2+3} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \overline{\mathcal{M}}_{0,i_j+3} \times_{\mathbb{Z}} S)$$

for some nonnegative integers $j, i_1, \ldots, i_j$. Thus, the intersection $V_1 \cap \cdots \cap V_m$ is irreducible (cf. also [Hsh2], Proposition 3.1, (i)).

(ii) By the definition, together with (i), for distinct log divisors $V_1, \ldots, V_n$, if $V_1 \cap \cdots \cap V_n \neq \emptyset$, then $P \stackrel{\mathrm{def}}{=} V_1 \cap \cdots \cap V_n$ is a log-full point.

3. **Various log divisors**

We continue with the notation of the preceding Section. We suppose that $n \in \mathbb{Z}_{>1}$. In the present §3, we define various log divisors and study the geometry of various log divisors.

Definition 3.1. (i) For positive integers $1 \leq i < j \leq n$, write

$$\pi_{i,j} \colon X \times_S \cdots \times_S X \to X \times_S X$$

for the projection of the fiber product of n copies of $X \to S$ to the i-th and j-th factors. Write $\delta'_{i,j}$ for the inverse image via $\pi_{i,j}$ of the image of the diagonal embedding $X \hookrightarrow X \times_S X$. Write $\delta_{i,j}$ for the uniquely determined log divisor of $X_n^{\log}$ whose generic point maps to the generic point of $\delta'_{i,j}$ via $X_n \to X \times_S \cdots \times_S X$ (cf. Definition 2.2, (ix)). We shall refer to the log divisor $\delta_{i,j}$ as a *naive diagonal* of $X_n^{\log}$.

(ii) Let V be a log divisor of $X_n^{\log}$. We shall say that V is a *tripodal divisor* if one of vertices of $\mathcal{G}_V$ is a tripod.

(iii) Let $y_1, y_2 \in C_{r,n}$ be distinct elements. We shall use the notation $V(y_1, y_2)$ to denote a tripodal divisor which satisfies the following condition (if it exists): Since $V(y_1, y_2)$ is a tripodal divisor of $X_n^{\log}$, $\mathcal{G}_{V(y_1,y_2)}$ has precisely two vertices v_1, v_2, one of which is a tripod. Let v_1 be a tripod. (Note that since $n > 1$ and (it is immediate that) v_2 is of type $(g, n+r-1)$, v_2 is not a tripod.) Then y_1, y_2 are cusps of $\mathcal{G}_{V(y_1,y_2)}|_{v_1}$ (cf. [CbTpI], Definition 2.1, (iii)).

(iv) Let V be a log divisor of $X_n^{\log}$. We shall say that V is a *(g,r)-divisor* if one of vertices of $\mathcal{G}_V$ is of type (g,r).

(v) Let V be a log divisor of $X_n^{\log}$. We shall say that V is a *drift diagonal* if there exist a naive diagonal δ and an automorphism α of $X_n^{\log}$ over S such that $V = \alpha(\delta)$.

Remark 3.2. (i) One verifies immediately that a tripodal divisor which satisfies the condition in Definition 3.1, (iii), (i.e., "$V(y_1, y_2)$ for fixed y_1, y_2") is unique (if it exists).

(ii) Let V be a tripodal divisor of $X_n^{\log}$. Then it follows immediately that there exist distinct elements $y_1, y_2 \in C_{r,n}$ such that $V = V(y_1, y_2)$.

Proposition 3.3. *The following hold.*

(i) It holds that

$$\{\textit{naive diagonals}\} = \{V(x_i, x_j) \mid 1 \leq i < j \leq n\}.$$

(ii) If $(g,r) \neq (0,3)$, then

$$\{\textit{tripodal divisors}\}$$

$$= \{V(y_1, y_2) \mid y_1, y_2 \in C_{r,n} \textit{ are distinct elements}, \{y_1, y_2\} \not\subseteq \{c_1, \ldots, c_r\}\}.$$

(iii) If $(g,r) = (0,3)$, then

$$\{\textit{tripodal divisors}\} = \{V(y_1, y_2) \mid y_1, y_2 \in C_{r,n} \textit{ are distinct elements}\}.$$

(iv) Let V be a tripodal divisor and α an automorphism of $X_n^{\log}$ over S. Then $\alpha(V)$ is a tripodal divisor.

Proof. First, assertion (i) follows immediately from the various definitions involved. Next, assertions (ii), (iii) follow immediately from Remark 3.2, (ii), together with the definition of tripodal divisors. Finally, assertion (iv) follows from the fact that α lifts to an automorphism of $X_{n+1}^{\log}$ relative to the natural morphism $X_{n+1}^{\log} \to X_n^{\log}$ (cf. [NaTa], Theorem D), which thus implies that $\mathcal{G}_V$ is isomorphic to $\mathcal{G}_{\alpha(V)}$. $\square$

Proposition 3.4. *The following hold.*

(i) It holds that

$$\{\textit{naive diagonals}\} \subseteq \{\textit{drift diagonals}\} \subseteq \{\textit{tripodal divisors}\} \subseteq \{\textit{log divisors}\}.$$

(ii) If $(g,r) \neq (0,3), (1,1)$, then

$$\{\textit{naive diagonals}\} = \{\textit{drift diagonals}\}.$$

(iii) If $(g,r) = (0,3)$ or $(1,1)$, then

$$\{\textit{drift diagonals}\} = \{\textit{tripodal divisors}\}.$$

Proof. First, we verify assertion (i). The first and third inclusions follow immediately from the various definitions involved. The second inclusion follows from Proposition 3.3, (i), (iv). This completes the proof of assertion (i). Next, assertion (ii) follows from [CbTpII], Lemma 2.7, (iii). Finally, we consider assertion (iii). Let us first suppose that $(g,r) = (0,3)$. Then it follows immediately that $X_n^{\log}$ is isomorphic to $(\overline{\mathcal{M}}_{0,n+3}^{\log})_k \overset{\mathrm{def}}{=} \overline{\mathcal{M}}_{0,n+3}^{\log} \times_{\mathbb{Z}} S$ over S, on which the symmetric group on $n+3$ letters naturally acts. Thus, by considering the automorphism of $C_{r,n} = C_{3,n}$ which permutes the third (resp. first; second; fourth) marked point to the $(n+3)$-rd (resp. fourth; third; $(n+3)$-rd) marked point, we obtain an automorphism α_1 (resp. α_2; α_3; α_4) of $X_n^{\log}$ over S. Then it holds that

$$\alpha_1(V(x_i,x_n)) = V(x_i,c_3)\ (1 \le i \le n-1),\ \alpha_4\alpha_1(V(x_1,x_n)) = V(x_n,c_3),$$
$$\alpha_2(V(x_i,x_1)) = V(x_i,c_1)\ (2 \le i \le n),\ \alpha_4\alpha_2(V(x_1,x_n)) = V(x_1,c_1),$$
$$\alpha_3\alpha_1(V(x_i,x_n)) = V(x_i,c_2)\ (1 \le i \le n-1),\ \alpha_4\alpha_3\alpha_1(V(x_1,x_n)) = V(x_n,c_2),$$
$$\alpha_3\alpha_1\alpha_2(V(x_1,x_n)) = V(c_1,c_2),\ \alpha_1\alpha_4\alpha_3\alpha_1(V(x_1,x_n)) = V(c_2,c_3),$$
$$\alpha_2\alpha_1(V(x_1,x_n)) = V(c_1,c_3).$$

Thus, it follows from Proposition 3.3, (i), (iii), that every tripodal divisor is a drift diagonal. This completes the proof of assertion (iii) in the case where $(g,r) = (0,3)$.

Next, suppose that $(g,r) = (1,1)$. Thus, the underlying scheme X of $X^{\log} = X_1^{\log}$ is naturally equipped with a structure of elliptic curve over S. (The group operation of this elliptic curve will be written additively.) Now we have two automorphisms of U_{X_n} over S

$$\alpha\colon U_{X_n} \xrightarrow{\sim} U_{X_n} : (z_1,\dots,z_n) \mapsto (z_n - z_1,\dots,z_n - z_{n-1}, z_n),$$
$$\beta\colon U_{X_n} \xrightarrow{\sim} U_{X_n} : (z_1,\dots,z_n) \mapsto (z_1, z_1 - z_2,\dots,z_1 - z_n)$$

which thus induce the automorphisms α, β of $X_n^{\log}$ over S. Then

$$\alpha(V(x_i,x_n)) = V(x_i,c_1)\ (1 \le i \le n-1),\ \beta(V(x_n,x_1)) = V(x_n,c_1).$$

Thus, it follows from Proposition 3.3, (i), (ii), that every tripodal divisor is a drift diagonal. This completes the proof of assertion (iii) in the case where $(g,r) = (1,1)$. $\square$

Definition 3.5. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-l PSC-type.

(i) We shall say that a vertex of $\mathcal{G}$ is a *terminal vertex* if precisely one node abuts to it.

(ii) We shall say that a node of $\mathcal{G}$ is a *terminal node* if it abuts to a terminal vertex.

(iii) Write
$$\mathrm{Node}(\mathcal{G})$$
for the set of nodes of $\mathcal{G}$.

(iv) Write
$$\mathrm{TerNode}(\mathcal{G}) \subseteq \mathrm{Node}(\mathcal{G})$$
for the set of terminal nodes of $\mathcal{G}$.

(v) Write
$$\mathrm{Vert}(\mathcal{G})$$
for the set of vertices of $\mathcal{G}$.

(vi) Write
$$\mathrm{Edge}(\mathcal{G})$$
for the set of edges of $\mathcal{G}$.

Proposition 3.6. *Let P be a log-full point of $X_n^{\log}$ and A a log-full subgroup at P. The following hold.*

(i) It holds that $\sharp\mathrm{Node}(\mathcal{G}_P) = n$ and $\mathcal{G}_P$ has a precisely $n+1$ vertices, one of which is of type (g,r) and other vertices are tripods. Moreover, the underlying semi-graph of $\mathcal{G}_P$ is a tree.

(ii) *Write* $\mathrm{Node}(\mathcal{G}_P) = \{e_1, \ldots, e_n\}$ *(cf. (i)). Then for each* $1 \leq i \leq n$*, there exists a unique log divisor* V_i *such that there exists a natural isomorphism of* $\mathcal{G}_{V_i}$ *with* $(\mathcal{G}_P)_{\rightsquigarrow \mathrm{Node}(\mathcal{G}_P)\setminus\{e_i\}}$ *(cf.* [CbTpI]*, Definition 2.8) which preserves ordering of the sets of cusps. In this situation, let us identify* $\mathcal{G}_{V_i}$ *with* $(\mathcal{G}_P)_{\rightsquigarrow \mathrm{Node}(\mathcal{G}_P)\setminus\{e_i\}}$*. Moreover, these* V_i*'s satisfy that* $P = V_1 \cap \cdots \cap V_n$ *and* $A = I_{V_1} \times \cdots \times I_{V_n}$*, where* $I_{V_i} \subseteq \Pi_n$ *is a suitable inertia group associated to* V_i *contained in* A*.*

Proof. Assertion (i) and the first assertion of assertion (ii) follow immediately from the various definitions involved. The final assertion of assertion (ii) follows from [CbTpI], Lemma 5.4, (ii). $\square$

Definition 3.7. Let P be a log-full point of $X_n^{\log}$ and $V_1, \ldots, V_n$ the log divisors such that $P = V_1 \cap \cdots \cap V_n$ (cf. Proposition 3.6, (ii)). We shall say that V_i is a *terminal divisor* of P if there exists a terminal node $e \in \mathrm{TerNode}(\mathcal{G}_P)$ such that $\mathcal{G}_{V_i} = (\mathcal{G}_P)_{\rightsquigarrow \mathrm{Node}(\mathcal{G}_P)\setminus\{e\}}$ (cf. Proposition 3.6, (ii)).

Lemma 3.8. *Let* P *be a log-full point of* $X_n^{\log}$ *and* $V_1, \ldots, V_n$ *the log divisors such that* $P = V_1 \cap \cdots \cap V_n$ *(cf. Proposition 3.6, (ii)). The following hold.*

(i) If V_i *is a terminal divisor of* P*, then* V_i *is a tripodal divisor or a* (g, r)*-divisor.*

(ii) If V_i *is a tripodal divisor, then* V_i *is a terminal divisor of* P*.*

Proof. Assertion (i) follows from Proposition 3.6, (i). Assertion (ii) follows immediately from the various definitions involved. $\square$

Theorem 3.9. *For* $\square \in \{\circ, \bullet\}$*, let* $p^\square$*,* $l^\square$ *be distinct prime numbers;* $k^\square$ *an algebraically closed field of characteristic zero or* $p^\square$*;* $S^\square \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k^\square)$*;* $(g^\square, r^\square)$ *a pair of nonnegative integers such that* $2g^\square - 2 + r^\square > 0$*;*

$$X^{\log\square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$*;* $n^\square \in \mathbb{Z}_{>1}$*;* $X_{n^\square}^{\log\square}$ *the* $n^\square$*-th log configuration space associated to* $X^{\log\square} \to S^\square$*;* $\Pi^\square \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l^\square}(X_{n^\square}^{\log\square})$*;*

$$\phi\colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups. Then the following hold:

(i) $(g^\circ, r^\circ, n^\circ) = (g^\bullet, r^\bullet, n^\bullet)$*.*

(ii) If $(g^\square, r^\square) \neq (0,3), (1,1)$*, then* ϕ *induces a bijection between the set of fiber subgroups (cf.* [MzTa]*, Definition 2.3, (iii)) of* Π° *and the set of fiber subgroups of* $\Pi^\bullet$*.*

(iii) We suppose that $(g^\square, r^\square) \neq (0,3), (1,1)$*. Write* $\iota^\square\colon \Pi^\square \to \Pi_1^\square \times \cdots \times \Pi_1^\square$ *for the outer homomorphism induced by* $X_{n^\square}^{\log\square} \to X^{\log\square} \times_{S^\square} \cdots \times_{S^\square} X^{\log\square}$ *(cf. Definition 2.2, (ix)), where* $\Pi_1^\square \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l}(X^{\log\square})$*. Then* ϕ *induces a commutative diagram*

$$\begin{tikzcd} \Pi^\circ \arrow[r, "\phi"] \arrow[d, "\iota^\circ"'] & \Pi^\bullet \arrow[d, "\iota^\bullet"'] \\ \Pi_1^\circ \times \cdots \times \Pi_1^\circ \arrow[r, dotted, "\sim"] & \Pi_1^\bullet \times \cdots \times \Pi_1^\bullet. \end{tikzcd}$$

Proof. Assertion (i) follows from [HMM], Theorem 2.4, (i). Assertion (ii) follows from [MzTa], Corollary 6.3. Assertion (iii) follows from assertion (ii). $\square$

4. **Reconstruction of non-degenerate elements of log-full subgroups**

We continue with the notation of the preceding Section. In the present §4, we reconstruct scheme-theoretically non-degenerate elements (cf. Definition 4.5, (i), below) of a log-full subgroup (cf. Theorem 4.14, below).

Proposition 4.1. *Let P be a log-full point of $X_n^{\log}$, $V_1, \ldots, V_n$ the log divisors such that $P = V_1 \cap \cdots \cap V_n$, and $A = I_{V_1} \times \cdots \times I_{V_n}$ the log-full subgroup at P (cf. Proposition 3.6, (ii)). The following hold.*

(i) There exists a tripodal divisor in $\{V_i\}_{1 \le i \le n}$. Suppose that V_1 is a tripodal divisor. Thus, $\mathcal{G}_{V_1}$ has precisely two vertices v_1, v_1', one of which is a tripod. Suppose that v_1 is a tripod.

(ii) If $r = 1$, then there exists a (g,r)-divisor in $\{V_i\}_{1 \le i \le n}$. Suppose that V_n is a (g,r)-divisor.

(iii) In the situation of (i), if $(g,r) \neq (0,3)$, then there exists $i_0 \in \{1, \ldots, n\}$ such that x_{i_0} is a cusp of $\mathcal{G}_{V_1}|_{v_1}$ (cf. [CbTpI], *Definition 2.1, (iii)). In this case, write $p\colon X_n^{\log} \to X_{n-1}^{\log}$ for the projection morphism of profile $\{i_0\}$ (cf.* [MzTa], *Definition 2.1, (ii)).*

(iv) In the situation of (i), if $(g,r) = (0,3)$, then there exists $i_0 \in \{1, \ldots, 3+n\}$ such that c_{i_0} is a cusp of $\mathcal{G}_{V_1}|_{v_1}$. In this case, write $p\colon X_n^{\log} \to X_{n-1}^{\log}$ for the morphism determined by the morphism $(\overline{\mathcal{M}}_{0,n+3}^{\log})_k \to (\overline{\mathcal{M}}_{0,n+2}^{\log})_k$ obtained by forgetting the i_0-th marked point (cf. the proof of Proposition 3.4, (iii)).

(v) In the situation of (iii) or (iv), it holds that $V_1' \stackrel{\text{def}}{=} p(V_1) = X_{n-1}$ and $V_i' \stackrel{\text{def}}{=} p(V_i)$ is a log divisor of $X_{n-1}^{\log}$ $(2 \le i \le n)$.

(vi) In the situation of (v), it holds that $V_i' \neq V_j'$ $(1 \le i < j \le n)$.

(vii) In the situation of (v), it holds that $p(P)$ is a log-full point of $X_{n-1}^{\log}$.

(viii) In the situation of (iii) or (iv), for each (g,r), by abuse of notation, we write $p\colon \Pi_n \to \Pi_{n-1}$ for the outer homomorphism induced by p. Then $A' \stackrel{\text{def}}{=} p(A)$ is a log-full subgroup of Π_{n-1} and we obtain exact sequences

$$1 \longrightarrow \Pi_{n/n-1} \stackrel{\text{def}}{=} \mathrm{Ker}(p) \longrightarrow \Pi_n \stackrel{p}{\longrightarrow} \Pi_{n-1} \longrightarrow 1$$

$$1 \longrightarrow I_{V_1} \longrightarrow A \stackrel{p}{\longrightarrow} A' \longrightarrow 1.$$

Proof. Assertions (i), (ii) follow from Proposition 3.6, (i), and Lemma 3.8, (i). Assertion (iii) follows from Proposition 3.3, (ii). Assertion (iv) is immediate. Assertion (v) follows from our choice of $p\colon X_n^{\log} \to X_{n-1}^{\log}$. We verify assertion (vi). By assertion (v), it holds that $V_1' \neq V_i'$ $(1 < i \le n)$. Thus, we may assume without lose of generality that $\mathcal{G}_{V_i}$ has precisely two vertices v_i, v_i' such that x_{i_0} is a cusp of $\mathcal{G}_{V_i}|_{v_i'}$. Let us recall that we have identified $\mathrm{Cusp}(\mathcal{G}_{V_i})$, $\mathrm{Cusp}(\mathcal{G}_{V_j})$ with $C_{r,n}$ (cf. Definition 2.2, (vi)). We assume that $V_i' = V_j'$. Then one verifies easily that $\mathcal{G}_{V_j}$ has precisely two vertices v_j, v_j' such that

$$(\mathrm{Cusp}(\mathcal{G}_{V_i}|_{v_i}) \cap \mathrm{Cusp}(\mathcal{G}_{V_i})) \cup \{x_{i_0}\} = \mathrm{Cusp}(\mathcal{G}_{V_j}|_{v_j}) \cap \mathrm{Cusp}(\mathcal{G}_{V_j});$$

$$\sharp\mathrm{Cusp}(\mathcal{G}_{V_i}|_{v_i}) + 1 = \sharp\mathrm{Cusp}(\mathcal{G}_{V_j}|_{v_j});$$

$$(\mathrm{Cusp}(\mathcal{G}_{V_j}|_{v_j'}) \cap \mathrm{Cusp}(\mathcal{G}_{V_j})) \cup \{x_{i_0}\} = \mathrm{Cusp}(\mathcal{G}_{V_i}|_{v_i'}) \cap \mathrm{Cusp}(\mathcal{G}_{V_i});$$

$$\sharp\mathrm{Cusp}(\mathcal{G}_{V_j}|_{v'_j}) + 1 = \sharp\mathrm{Cusp}(\mathcal{G}_{V_i}|_{v'_i});$$

$$g(v_i) = g(v_j),\ g(v'_i) = g(v'_j),$$

where we write $g(v_{(-)}), g(v'_{(-)})$ for the "genus" of $\mathcal{G}_{V_{(-)}}|_{v_{(-)}}, \mathcal{G}_{V_{(-)}}|_{v'_{(-)}}$ (cf. [CbTpI], Definition 2.3, (ii)). Thus, we obtain a contradiction (cf. our choice of $p\colon X_n^{\log} \to X_{n-1}^{\log}$). In particular, we conclude that $V'_i \neq V'_j$. Assertion (vii) is immediate. Assertion (viii) follows from assertion (v), (vii). □

Proposition 4.2. *Let P be a log-full point of $X_n^{\log}$; $V, V_1, \ldots, V_n$ log divisors such that $P = V_1 \cap \cdots \cap V_n$; I_V an inertia group associated to V. Then it holds that*

$$P \in V \iff \textit{there exists a log-full subgroup } A \textit{ at } P \textit{ such that } I_V \subset A.$$

Proof. $\Longrightarrow$ is immediate. We consider $\Longleftarrow$. We suppose that $I_V \subset A = I_{V_1} \times \cdots \times I_{V_n}$. We apply induction on n.

First, we suppose that $n = 2$. Write $p_i\colon X_2^{\log} \to X_1^{\log}$ for the projection morphism of profile $\{i\}$ $(i = 1, 2)$ and, by abuse of notation, $p_i\colon \Pi_2 \to \Pi_1$ for the outer homomorphism induced by p_i $(i = 1, 2)$. Then we obtain exact sequences

$$1 \longrightarrow \mathrm{Ker}(p_1) \longrightarrow \Pi_2 \xrightarrow{p_1} \Pi_1 \longrightarrow 1,$$

$$1 \longrightarrow \mathrm{Ker}(p_2) \longrightarrow \Pi_2 \xrightarrow{p_2} \Pi_1 \longrightarrow 1.$$

Suppose that $p_1(I_V) = \{e\}$, which thus implies that $I_V \subset \mathrm{Ker}(p_1)$. Then it follows that I_V may be regarded as an inertia subgroup of $\mathrm{Ker}(p_1)$ associated to a cusp of the fiber of p_1. Now let us observe that one verifies easily that $p_1(P)$ is a log-full point. In particular, $\mathrm{Ker}(p_1|_A)$ is isomorphic to $\mathbb{Z}_l$. Moreover, one also verifies easily that $\mathrm{Ker}(p_1|_A)$ may be regarded as an inertia subgroup of $\mathrm{Ker}(p_1)$ associated to a cusp or node of the fiber, at $p_1(P)$, of p_1. Thus, since $I_V \subset A$, by [CmbGC], Proposition 1.2, (i), it holds that $I_V = \mathrm{Ker}(p_1|_A)$, which thus implies that $\mathrm{Ker}(p_1|_A)$ is an inertia subgroup of (not a node but) a cusp. In particular, it follows immediately that $\mathrm{Ker}(p_1|_A) = I_{V_j}$ for some $j = 1, 2$. Thus, by again [CmbGC], Proposition 1.2, (i), we conclude that $V = V_j$. In particular, $P \in V$.

Suppose that $p_i(I_V) \neq \{e\}$ $(i = 1, 2)$. Then one verifies easily that $p_i(I_V), p_i(A)$ are log-full subgroups of Π_1 $(i = 1, 2)$. By [CmbGC], Proposition 1.2, (i), it holds that $p_i(I_V) = p_i(A)$ $(i = 1, 2)$ and $p_i(V) = p_i(P)$ $(i = 1, 2)$. Then one verify easily that there exists $j = 1, 2$ such that $V = V_j$. In particular, $P \in V$.

Next, we suppose that $n \geq 3$, and that the induction hypothesis is in force. Write $p_i\colon X_n^{\log} \to X_{n-1}^{\log}$ for the projection morphism of profile $\{i\}$ $(i = 1, 2)$ and, by abuse of notation, $p_i\colon \Pi_n \to \Pi_{n-1}$ for the outer homomorphism induced by p_i $(i = 1, 2)$. Then we obtain exact sequences

$$1 \longrightarrow \mathrm{Ker}(p_1) \longrightarrow \Pi_n \xrightarrow{p_1} \Pi_{n-1} \longrightarrow 1$$

$$1 \longrightarrow \mathrm{Ker}(p_2) \longrightarrow \Pi_n \xrightarrow{p_2} \Pi_{n-1} \longrightarrow 1.$$

If $p_1(I_V) = \{e\}$, then it follows immediately from a similar argument to the argument applied in the proof in the case of $n = 2$ and "$p_1(I_V) = \{e\}$" that there exists $1 \leq j \leq n$ such that $I_V = I_{V_j}$ and $V = V_j$. In particular, $P \in V$.

If $p_i(I_V) \neq \{e\}$ $(i = 1, 2)$, one verifies easily that $p_i(A)$ is a log-full subgroup of Π_{n-1} $(i = 1, 2)$ and $p_i(I_V)$ is a inertia group associated to $p_i(V)$ $(i = 1, 2)$. Since

$p_i(I_V) \subset p_i(A)$ $(i = 1, 2)$, by the induction hypothesis, $p_i(P) \in p_i(V)$ $(i = 1, 2)$. Then one verify easily that there exists $1 \leq j \leq n$ such that $V = V_j$. In particular, $P \in V$. □

Proposition 4.3. *Let V, W be log divisors and I_V, I_W inertia groups associated to V, W, respectively. Then it holds that*

$$V = W \iff \textit{there exists } g \in \Pi_n \textit{ such that } I_V = gI_Wg^{-1}.$$

Proof. It follows from a similar argument to the argument applied in the proof of Proposition 4.2. □

Proposition 4.4. *Let P_1, P_2 be log-full points of $X_n^{\log}$, A_1 a log-full subgroup at P_1, and A_2 a log-full subgroup at P_2. Then it holds that*

$$P_1 = P_2 \iff \textit{there exists } g \in \Pi_n \textit{ such that } A_1 = gA_2g^{-1}.$$

In particular,

$$\sharp\{\textit{log-full points}\} = \sharp\{\textit{conjugacy classes of log-full subgroups}\}.$$

Proof. The final assertion follows from the first assertion. Let us prove the first assertion. $\Longrightarrow$ is immediate. We consider $\Longleftarrow$. We suppose that $A_1 = A_2$. Let $V_1, \ldots, V_n$ be log divisors such that $P_1 = V_1 \cap \cdots \cap V_n$. Thus, we obtain that $A_1 = I_{V_1} \times \cdots \times I_{V_n}$. In particular, for each $1 \leq j \leq n$, $I_{V_j} \subset A_1 = A_2$. In particular, it follows from Proposition 4.2 that $P_2 \in V_j$. Thus, $P_2 \in V_1 \cap \cdots \cap V_n = P_1$. □

In the remainder of the present §4, we shall apply the notational convention introduced in the statement of Proposition 4.1.

Definition 4.5. Let $\alpha \in A$ and

$$A = I_{V_1} \times \cdots \times I_{V_n} : \alpha \mapsto (a_1, \ldots, a_n).$$

(i) We shall say that α is *scheme-theoretically non-degenerate* if $a_i \neq e$ for any i.
(ii) We shall say that α is *group-theoretically non-degenerate* if $Z_{\Pi_n}(\alpha)$ is an abelian group.

Theorem 4.6. *It holds that*

$$\{\textit{scheme-theoretically non-degenerate elements of } A\}$$
$$= \{\textit{group-theoretically non-degenerate elements of } A\}.$$

Proof. If $r \neq 1$, this follows from Claim 4.8 and Claim 4.11, below.
If $r = 1$, this follows from Claim 4.8, Claim 4.11, and Claim 4.13, below. □

Lemma 4.7. *It holds that*

$$N_{\Pi_n}(A) = A,$$

i.e., a log-full subgroup is normally terminal in Π_n.

Proof. We apply induction on n. By the definition, $N_{\Pi_n}(A) \supset A$. Let $\alpha \in N_{\Pi_n}(A)$. Since $\alpha A \alpha^{-1} = A$, it follows that $p(\alpha)A'p(\alpha)^{-1} = A'$, where $A' \stackrel{\mathrm{def}}{=} p(A)$. Note that it follows immediately from Proposition 4.1, (viii), that A' is a log-full subgroup of Π_{n-1}. Since A' is normally terminal (by the induction hypothesis and [CmbGC], Proposition 1.2, (ii)), it follows that $p(\alpha) \in A'$. Thus, $p(N_{\Pi_n}(A)) \subseteq A'$. Since $p(N_{\Pi_n}(A)) \supset p(A) = A'$, it follows that $p(N_{\Pi_n}(A)) = A'$.

By Proposition 4.1, (viii), $N_{\Pi_n}(A) \cap \Pi_{n/n-1} \supset I_{V_1}$. Let $\alpha \in N_{\Pi_n}(A) \cap \Pi_{n/n-1}$. Since $\alpha A \alpha^{-1} = A$, it follows that $\alpha I_{V_1} \alpha^{-1} \subseteq A$. Thus, since $\alpha \in \Pi_{n/n-1}$, it follows from Proposition 4.1, (viii), that $\alpha I_{V_1} \alpha^{-1} \subseteq A \cap \Pi_{n/n-1} = I_{V_1}$. By replacing α by α^{-1}, it follows that $\alpha I_{V_1} \alpha^{-1} = I_{V_1}$, i.e., that $\alpha \in N_{\Pi_{n/n-1}}(I_{V_1}) = I_{V_1}$ (cf. [CmbGC], Proposition 1.2, (ii)). Thus, we conclude that $N_{\Pi_n}(A) \cap \Pi_{n/n-1} = I_{V_1}$.

It follows from the above discussion that we have an exact sequence

$$1 \longrightarrow I_{V_1} \longrightarrow N_{\Pi_n}(A) \xrightarrow{p} A' \longrightarrow 1.$$

By the five lemma (cf. Proposition 4.1, (viii)), it follows that $N_{\Pi_n}(A) = A$. $\square$

Claim 4.8. *Let* $(a_1, \ldots, a_n) \in I_{V_1} \times \cdots \times I_{V_n} = A$. *If* $a_1, \ldots, a_n \neq e$, *then* $Z_{\Pi_n}(a_1 \cdots a_n)$ *is an abelian group.*

Proof. Let $X_{n+1}^{\log} \to X_n^{\log}$ be the projection morphism of profile $\{n+1\}$. This projection induces an exact sequence

$$1 \longrightarrow \mathrm{Ker}(\Pi_{n+1} \to \Pi_n) \longrightarrow \Pi_{n+1} \longrightarrow \Pi_n \longrightarrow 1,$$

which gives rise to an outer representation $\rho\colon \Pi_n \to \mathrm{Out}(\mathrm{Ker}(\Pi_{n+1} \to \Pi_n))$. It follows that ρ is injective (cf. [Asd], Remark of Theorem 1). Then there exists an isomorphism $\Pi_{\mathcal{G}_P} \xrightarrow{\sim} \mathrm{Ker}(\Pi_{n+1} \to \Pi_n)$ such that ρ determines an isomorphism

$$A \xrightarrow{\sim} \mathrm{Dehn}(\mathcal{G}_P)$$

(cf. [CbTpI], Definition 4.4; [CbTpI], Proposition 5.6, (ii)), and, moreover, it holds that

$$\mathrm{Aut}(\mathcal{G}_P) = N_{\mathrm{Out}^{\mathrm{C}}(\mathrm{Ker}(\Pi_{n+1} \to \Pi_n))}(\mathrm{Dehn}(\mathcal{G}_P))$$

(cf. [CbTpI], Theorem 5.14, (iii)).

Since $A \simeq \mathbb{Z}_l^{\oplus n}$ is an abelian group, to verify that $Z_{\Pi_n}(a_1 \cdots a_n)$ is an abelian group, it suffices to verify that $Z_{\Pi_n}(a_1 \cdots a_n) = A$. Since A is an abelian group and $a_1 \cdots a_n \in A \subseteq \Pi_n$, it follows that $Z_{\Pi_n}(a_1 \cdots a_n) \supset A$. By [NodNon], Theorem A, and [CbTpI], Corollary 5.9, (ii), it follows that $\rho(Z_{\Pi_n}(a_1 \cdots a_n)) \subseteq \mathrm{Aut}(\mathcal{G}_P)$. Thus, it follows that

$$\begin{aligned}\rho(Z_{\Pi_n}(a_1 \cdots a_n)) &\subseteq \mathrm{Aut}(\mathcal{G}_P) \cap \rho(\Pi_n) = N_{\mathrm{Out}^{\mathrm{C}}(\mathrm{Ker}(\Pi_{n+1} \to \Pi_n))}(\mathrm{Dehn}(\mathcal{G}_P)) \cap \rho(\Pi_n) \\ &= N_{\rho(\Pi_n)}(\mathrm{Dehn}(\mathcal{G}_P)) = N_{\rho(\Pi_n)}(\rho(A)) = \rho(N_{\Pi_n}(A)).\end{aligned}$$

In particular, $Z_{\Pi_n}(a_1 \cdots a_n) \subseteq N_{\Pi_n}(A)$. By Lemma 4.7, it follows that

$$Z_{\Pi_n}(a_1 \cdots a_n) = A.$$

$\square$

Definition 4.9. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-l PSC-type and $\mathbb{G}$ the underlying semi-graph of $\mathcal{G}$. Suppose that $\mathbb{G}$ is a tree.

(i) Let $e \in \mathrm{Edge}(\mathcal{G})$; $v \in \mathrm{Vert}(\mathcal{G})$ such that e abuts to v; b a branch of e that abuts to v. By replacing e by open edges e_1, e_2 such that e_1 abuts to v and e_2 abuts to the vertex $\neq v$ to which e abuts (resp. e_1 abuts to v and e_2 is an edge which abuts to no vertex) if $e \in \mathrm{Node}(\mathcal{G})$ (resp. $e \in \mathrm{Cusp}(\mathcal{G})$), we obtain two connected semi-graphs. Write $\mathbb{G}_{\not\ni b}$ for the semi-graph (among these two connected semi-graphs) that does not contain b. Write $\mathbb{G}_{\ni b}$ for the semi-graph (among these two connected semi-graphs) that contains b.

(ii) Let $e_1, e_2 \in \mathrm{Edge}(\mathcal{G})$; b_1, b_1' the two branches of e_1; b_2, b_2' the two branches of e_2. We suppose that $\mathbb{G}_{\not\ni b_1} \cap \mathbb{G}_{\not\ni b_2} = \emptyset$. Write $\mathbb{H}$ for the semi-graph obtained by considering the "intersection" of $\mathbb{G}_{\ni b_1}$ and $\mathbb{G}_{\ni b_2}$. Then we define the semi-graph of anabelioids of pro-l PSC-type

$$\mathcal{G}_{b_1 \curlyvee b_2}$$

as follows: We take the underlying semi-graph of $\mathcal{G}_{b_1 \curlyvee b_2}$ to be the semi-graph obtained by "gluing" $\mathbb{H}$, $\mathbb{G}_{\not\ni b_1}$, and $\mathbb{G}_{\not\ni b_2}$ by the correspondence "the branch of $\mathbb{H}$ corresponding to $b_1 \leftrightarrow$ the branch of $\mathbb{G}_{\not\ni b_2}$ corresponding to b_2'", "the branch of $\mathbb{H}$ corresponding to $b_2 \leftrightarrow$ the branch of $\mathbb{G}_{\not\ni b_1}$ corresponding to b_1'". Then the various connected anabelioids in $\mathcal{G}$ naturally determine a semi-graph of anabelioids of pro-l PSC-type $\mathcal{G}_{b_1 \curlyvee b_2}$ whose underlying semi-graph is the above resulting semi-graph.

Proposition 4.10. *Suppose that $r \neq 1$ (resp. $r = 1$). Let $1 \leq i \leq n$ (resp. $1 \leq i \leq n-1$). Then there exists a log divisor $H \neq V_i$ such that*

$$V_1 \cap \cdots \cap V_{i-1} \cap H \cap V_{i+1} \cap \cdots \cap V_n$$

is a log-full point.

Proof. It follows from Proposition 3.6, (ii), that there exists $e \in \mathrm{Node}(\mathcal{G}_P)$ such that $\mathcal{G}_{V_i} = (\mathcal{G}_P)_{\rightsquigarrow \mathrm{Node}(\mathcal{G}_P) \setminus \{e\}}$. Let w_1, w_2 be distinct vertices of $\mathcal{G}_P$ such that e abuts to w_1, w_2.

First, let us suppose that w_1, w_2 are tripods. Let e, y_1, y_2 be cusps of $\mathcal{G}_P|_{w_1}$ and e, y_3, y_4 cusps of $\mathcal{G}_P|_{w_2}$, where $y_1, y_2, y_3, y_4 \in (C_{r,n} \coprod \mathrm{Node}(\mathcal{G}_P)) \setminus \{e\}$ are distinct elements.

Let b_1 be a branch of y_1 that abuts to w_1; b_2 a branch of y_3 that abuts to w_2; $\mathcal{G}' \stackrel{\mathrm{def}}{=} (\mathcal{G}_P)_{b_1 \curlyvee b_2}$ (cf. Definition 4.9, (ii)). Then it follows immediately from the definition that there exists a log divisor $H \neq V_i$ such that $\mathcal{G}_H$ is naturally isomorphic to $\mathcal{G}'_{\rightsquigarrow \mathrm{Node}(\mathcal{G}') \setminus \{e\}}$ and $V_1 \cap \cdots \cap V_{i-1} \cap H \cap V_{i+1} \cap \cdots \cap V_n$ is a log-full point. This completes the proof of Proposition 4.10 in the case where w_1, w_2 are tripods.

Thus, we may assume without loss of generality that w_2 is not a tripod. Then it follows from Proposition 3.6, (i), that w_1 is a tripod and w_2 is of type $(g, r) \neq (0, 3)$. Next, let us observe that $r \neq 1$. Indeed, if $r = 1$, then it follows immediately from the fact that w_2 is of type $(g, r) \neq (0, 3)$, together with the definition of V_n (cf. Proposition 4.1, (ii)), that $V_i = V_n$. Thus, we obtain a contradiction (cf. our assumption that $i \leq n-1$ if $r = 1$). Thus, in summary, we are in the situation that w_1 is a tripod, w_2 is of type $(g, r) \neq (0, 3)$, and $r \neq 1$.

Let e, y_1, y_2 be cusps of $\mathcal{G}_P|_{w_1}$, where $y_1, y_2 \in (C_{r,n} \coprod \mathrm{Node}(\mathcal{G}_P)) \setminus \{e\}$ are distinct elements. Since $r \neq 0, 1$, it follows that $r + 1 \geq 3$. Let $e, y_3, \ldots, y_{r+1}$ be cusps of $\mathcal{G}_P|_{w_2}$, where $y_3, \ldots, y_{r+1} \in (C_{r,n} \coprod \mathrm{Node}(\mathcal{G}_P)) \setminus \{e, y_1, y_2\}$ are distinct elements.

Let b_1 be a branch of y_1 that abuts to w_1; b_2 a branch of y_2 that abuts to w_1; b_3 a branch of y_3 that abuts to w_2; $\mathbb{G}_P$ the underlying semi-graph of $\mathcal{G}_P$. Then it holds that $\mathrm{Cusp}((\mathbb{G}_P)_{\not\ni b_1}) \cap \{c_1, \ldots, c_r\} = \emptyset$ or $\mathrm{Cusp}((\mathbb{G}_P)_{\not\ni b_2}) \cap \{c_1, \ldots, c_r\} = \emptyset$. We suppose that $\mathrm{Cusp}((\mathbb{G}_P)_{\not\ni b_2}) \cap \{c_1, \ldots, c_r\} = \emptyset$. Let $\mathcal{G}' \stackrel{\mathrm{def}}{=} (\mathcal{G}_P)_{b_1 \curlyvee b_3}$. Then it follows immediately from the definition that there exists a log divisor $H \neq V_i$ such that $\mathcal{G}_H$ is naturally isomorphic to $\mathcal{G}'_{\rightsquigarrow \mathrm{Node}(\mathcal{G}') \setminus \{e\}}$ and $V_1 \cap \cdots \cap V_{i-1} \cap H \cap V_{i+1} \cap \cdots \cap V_n$ is a log-full point. $\square$

Claim 4.11. *Suppose that $r \neq 1$ (resp. $r = 1$). Let $1 \leq i \leq n$ (resp. $1 \leq i \leq n-1$) and $(a_1, \ldots, a_n) \in I_{V_1} \times \cdots \times I_{V_n} = A$. Then $Z_{\Pi_n}(a_1 \cdots a_{i-1}a_{i+1} \cdots a_n)$ is a non-abelian group.*

Proof. By Proposition 4.10, there exists a log divisor $H \neq V_i$ such that $V_1 \cap \cdots \cap V_{i-1} \cap H \cap V_{i+1} \cap \cdots \cap V_n$ is a log-full point. Since

$$a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in I_{V_1} \times \cdots \times I_{V_n}, I_{V_1} \times \cdots \times I_{V_{i-1}} \times I_H \times I_{V_{i+1}} \times \cdots \times I_{V_n}$$

and $I_{V_1} \times \cdots \times I_{V_n}, I_{V_1} \times \cdots \times I_{V_{i-1}} \times I_H \times I_{V_{i+1}} \times \cdots \times I_{V_n}$ are abelian groups, it follows that

$$I_{V_1} \times \cdots \times I_{V_n}, I_{V_1} \times \cdots \times I_{V_{i-1}} \times I_H \times I_{V_{i+1}} \times \cdots \times I_{V_n} \subseteq Z_{\Pi_n}(a_1 \cdots a_{i-1}a_{i+1} \cdots a_n).$$

Since $I_{V_1} \times \cdots \times I_{V_n}, I_{V_1} \times \cdots \times I_{V_{i-1}} \times I_H \times I_{V_{i+1}} \times \cdots \times I_{V_n}$ are distinct log-full subgroups (cf. Proposition 4.4) and contained in $Z_{\Pi_n}(a_1 \cdots a_{i-1}a_{i+1} \cdots a_n)$, by Lemma 4.7, it follows that $Z_{\Pi_n}(a_1 \cdots a_{i-1}a_{i+1} \cdots a_n)$ is a non-abelian group. □

Proposition 4.12. *If $r = 1$, then there exists $1 \leq i \leq n$ such that q induces an isomorphism $V_1 \cap \cdots \cap V_{n-1} \xrightarrow{\sim} X$, where $q\colon X_n^{\log} \to X^{\log}$ is the projection morphism of co-profile $\{i\}$ (cf.* [MzTa]*, Definition 2.1, (ii)).*

Proof. Let w_1 be the unique vertex of $\mathcal{G}_P$ of genus g. (Note that since $r = 1$, it holds that $g \neq 0$.) Then it follows immediately from Proposition 3.6, (i), together with our assumption that $r = 1$, that there exist a unique vertex w_2 of $\mathcal{G}_P$ and a unique node $e \in \mathrm{Node}(\mathcal{G}_P)$ such that e abuts to w_1, w_2 and, moreover, w_2 is a tripod. Let e, y_1, y_2 be cusps of $\mathcal{G}_P|_{w_2}$, where $y_1, y_2 \in (C_{r,n} \coprod \mathrm{Node}(\mathcal{G}_P)) \setminus \{e\}$ are distinct elements; b_1 a branch of y_1 that abuts to w_2; b_2 a branch of y_2 that abuts to w_2; $\mathbb{G}_P$ the underlying semi-graph of $\mathcal{G}_P$. Then it holds that $\mathrm{Cusp}((\mathbb{G}_P)_{\not\ni b_1}) \cap \{c_1\} = \emptyset$ or $\mathrm{Cusp}((\mathbb{G}_P)_{\not\ni b_2}) \cap \{c_1\} = \emptyset$. We suppose that $\mathrm{Cusp}((\mathbb{G}_P)_{\not\ni b_2}) \cap \{c_1\} = \emptyset$. Now let us observe that it follows immediately from the definition that there exists $x_i \in \{x_1, \ldots, x_n\}$ such that x_i be a cusp of $(\mathbb{G}_P)_{\not\ni b_2}$. Then it follows immediately from our choice of i that the projection morphism q of co-profile $\{i\}$ satisfies that $q\colon V_1 \cap \cdots \cap V_{n-1} \xrightarrow{\sim} X$. □

Claim 4.13. *Let $(a_1, \ldots, a_n) \in I_{V_1} \times \cdots \times I_{V_n} = A$. If $r = 1$, then $Z_{\Pi_n}(a_1 \cdots a_{n-1})$ is a non-abelian group.*

Proof. By Proposition 4.12, there exists $1 \leq i \leq n$ such that $q\colon V_1 \cap \cdots \cap V_{n-1} \xrightarrow{\sim} X$, where $q\colon X_n^{\log} \to X^{\log}$ is the projection morphism of co-profile $\{i\}$. By abuse of notation, we write $q\colon \Pi_n \to \Pi_1$ for the outer homomorphism induced by q. Let $V_1^{\log} \cap \cdots \cap V_{n-1}^{\log}$ be the log scheme obtained by restricting the log structure of $X_n^{\log}$ to the reduced closed subscheme of X_n determined by $V_1 \cap \cdots \cap V_{n-1}$. Then it follows immediately that the morphism $V_1^{\log} \cap \cdots \cap V_{n-1}^{\log} \to X^{\log}$ induced by q determines a sequence of profinite groups

$$\pi_1^{\text{pro-}l}(V_1^{\log} \cap \cdots \cap V_{n-1}^{\log}) \to D_{V_1} \cap \cdots \cap D_{V_{n-1}} \hookrightarrow \Pi_n \to \Pi_1,$$

where $D_{V_j} \stackrel{\text{def}}{=} Z_{\Pi_n}(I_{V_j})$ is the decomposition group associated to V_j determined by I_{V_j}. It follows from a consideration of objects parametrized by the various schemes that $V_1^{\log} \cap \cdots \cap V_{n-1}^{\log} \to X^{\log}$ is of type $\mathbb{N}^{\oplus n-1}$ (cf. [Hsh], Definition 6; the statement of [Hsh2], Proposition 3.2). Since $V_1^{\log} \cap \cdots \cap V_{n-1}^{\log} \to X^{\log}$ is of type $\mathbb{N}^{\oplus n-1}$, one verifies immediately that for any connected ket covering (i.e., connected finite Kummer log étale morphism (cf. [Kato], (3.3), and [Naka], Definition (2.1.2), (ii))) $Z^{\log} \to X^{\log}$, $(V_1^{\log} \cap \cdots \cap V_{n-1}^{\log}) \times_{X^{\log}} Z^{\log} \to V_1^{\log} \cap \cdots \cap V_{n-1}^{\log}$ is a connected ket covering, i.e., $\pi_1^{\text{pro-}l}(V_1^{\log} \cap \cdots \cap V_{n-1}^{\log}) \to \Pi_1$ is a surjection. In particular, the composite $D_{V_1} \cap \cdots \cap D_{V_{n-1}} \hookrightarrow \Pi_n \to \Pi_1$ is a surjection, i.e., $q(D_{V_1} \cap \cdots \cap D_{V_n}) = \Pi_1$. Thus, it follows immediately from the definitions that

$$\begin{aligned}\Pi_1 &= q(D_{V_1} \cap \cdots \cap D_{V_{n-1}}) = q(Z_{\Pi_n}(I_{V_1}) \cap \cdots \cap Z_{\Pi_n}(I_{V_{n-1}}))\\ &\subseteq q(Z_{\Pi_n}(a_1) \cap \cdots \cap Z_{\Pi_n}(a_{n-1})) \subseteq q(Z_{\Pi_n}(a_1 \cdots a_{n-1})) \subseteq q(\Pi_n) = \Pi_1.\end{aligned}$$

In particular, $Z_{\Pi_n}(a_1 \cdots a_{n-1})$ is a non-abelian group. $\square$

Theorem 4.14. *For $\square \in \{\circ, \bullet\}$, let $p^\square$, $l^\square$ be distinct prime numbers; $k^\square$ an algebraically closed field of characteristic zero or $p^\square$; $S^\square \overset{\text{def}}{=} \operatorname{Spec}(k^\square)$; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$;*

$$X^{\log \square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$; $n^\square \in \mathbb{Z}_{>1}$; $X_{n^\square}^{\log \square}$ the $n^\square$-th log configuration space associated to $X^{\log \square} \to S^\square$; $\Pi^\square \overset{\text{def}}{=} \pi_1^{\text{pro-}l^\square}(X_{n^\square}^{\log \square})$;

$$\phi \colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups; A° a log-full subgroup of Π°. We suppose that $r^\square > 0$ and $A^\bullet \overset{\text{def}}{=} \phi(A^\circ)$ is a log-full subgroup of $\Pi^\bullet$. Then ϕ induces a bijection between the set of scheme-theoretically non-degenerate elements (cf. Definition 4.5, (i)) of A° and the set of scheme-theoretically non-degenerate elements of $A^\bullet$.

Proof. This follows from Theorem 4.6. $\square$

5. Reconstruction of log divisors

We continue with the notation of the preceding Section. In the present §5, we reconstruct the set of inertia groups associated to log divisors (cf. Theorem 5.3, below).

Definition 5.1. Let A be a log-full subgroup of Π_n and $a \in A$. Write

$$I_a \overset{\text{def}}{=} \{b \in A \mid \langle a \rangle \subseteq \overline{\langle b \rangle} \text{ or } \langle b \rangle \subseteq \overline{\langle a \rangle}\},$$

where we write $\overline{(-)}$ for the closed subgroup generated by $(-)$.

Lemma 5.2. *The following hold.*

- *(i) There exist subgroups $B_0, \ldots, B_n \subseteq A$ and elements $b_{i,j} \in A$ ($0 \leq i \leq n-1$, $1 \leq j \leq n-1$) such that the following hold:*
  - *(a) $B_0 = \{e\}$.*
  - *(b) $b_{i,j} \notin B_0 \cup \cdots \cup B_i$ ($0 \leq i \leq n-1$, $1 \leq j \leq n-1$).*
  - *(c) $I_{b_{i,1}} \subsetneq \langle I_{b_{i,1}}, I_{b_{i,2}} \rangle \subsetneq \cdots \subsetneq \langle I_{b_{i,1}}, \ldots, I_{b_{i,n-1}} \rangle$ ($0 \leq i \leq n-1$).*
  - *(d) $B_{i+1} = \langle I_{b_{i,1}}, \ldots, I_{b_{i,n-1}} \rangle$ ($0 \leq i \leq n-1$).*
  - *(e) Every element of B_i is not (group-theoretically) non-degenerate ($0 \leq i \leq n$).*
- *(ii) In the situation of (i), $\{B_i \mid 1 \leq i \leq n\} = \{\prod_{1 \leq i \leq n, i \neq i_0} I_{V_i} \mid 1 \leq i_0 \leq n\}$.*
- *(iii) In the situation of (i), $\{I_{V_1}, \ldots, I_{V_n}\} = \{\bigcap_{1 \leq i \leq n, i \neq i_0} B_i \mid 1 \leq i_0 \leq n\}$.*

Proof. Assertions (i), (ii) follow immediately from a straightforward consideration. Assertion (iii) follows immediately from assertion (ii). $\square$

Theorem 5.3. *For $\square \in \{\circ, \bullet\}$, let $p^\square$, $l^\square$ be distinct prime numbers; $k^\square$ an algebraically closed field of characteristic zero or $p^\square$; $S^\square \stackrel{\text{def}}{=} \mathrm{Spec}(k^\square)$; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$;*

$$X^{\log \square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$; $n^\square \in \mathbb{Z}_{>1}$; $X^{\log \square}_{n^\square}$ the $n^\square$-th log configuration space associated to $X^{\log \square} \to S^\square$; $\Pi^\square \stackrel{\text{def}}{=} \pi_1^{\text{pro-}l^\square}(X^{\log \square}_{n^\square})$;

$$\phi \colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups. We suppose that $r^\square > 0$; ϕ induces a bijection between the set of log-full subgroups of Π° and the set of log-full subgroups of $\Pi^\bullet$. Then ϕ induces a bijection between the set of inertia groups of Π° associated to log divisors of $X^{\log \circ}_{n^\circ}$ and the set of inertia groups of $\Pi^\bullet$ associated to log divisors of $X^{\log \bullet}_{n^\bullet}$.

Proof. This follows from Theorem 4.14 and Lemma 5.2. $\square$

6. **Reconstruction of tripodal divisors**

We continue with the notation of the preceding Section. In the present §6, we reconstruct the set of inertia groups associated to tripodal divisors (cf. Theorem 6.4, below).

Lemma 6.1. *Let V be a log divisor of $X^{\log}_n$. Write $V^{\log}$ for the log scheme obtained by equipping V with the log structure induced by the log structure of $X^{\log}_n$. Let $Y^{\log} \to S$ be a smooth log curve of type $(0, 3)$ and, for any $m \in \mathbb{Z}_{>0}$, $Y^{\log}_m$ the m-th log configuration space associated to $Y^{\log} \to S$.*

- *(i) If V is a tripodal divisor, then $V^{\log \le 1}$ is isomorphic to $U_{X_{n-1}}$.*
- *(ii) If V is a (g, r)-divisor, then $V^{\log \le 1}$ is isomorphic to $U_{Y_{n-1}}$.*
- *(iii) If V is neither a tripodal divisor nor a (g, r)-divisor, then there exists $1 \le m \le n - 2$ such that $V^{\log \le 1}$ is isomorphic to $U_{Y_m} \times_S U_{X_{n-1-m}}$.*

Proof. This follows immediately from a consideration of objects parametrized by the various schemes which appear in the statements. $\square$

Definition 6.2. We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \simeq G_1 \times G_2$, where G_1, G_2 are profinite groups, it follows that either G_1 or G_2 is the trivial group. We shall say that a profinite group G is *decomposable* if G is not indecomposable.

Lemma 6.3. *Let V be a log divisor of $X^{\log}_n$ and I_V an inertia group associated to V. The following holds.*

- *(i) $Z_{\Pi_n}(I_V)/I_V$ is either decomposable, isomorphic to Π_{n-1} (cf. Definition 2.2, (i)), or isomorphic to $\Pi^{\text{tripod}}_{n-1} \stackrel{\text{def}}{=} \pi_1^{\text{pro-}l}(Y^{\log}_{n-1})$ (cf. Lemma 6.1).*
- *(ii) If $(g, r) \ne (1, 1)$ or $n \ge 3$, then it holds that V is a tripodal divisor if and only if $Z_{\Pi_n}(I_V)/I_V$ is isomorphic to Π_{n-1}.*
- *(iii) If $(g, r) = (1, 1)$ and $n = 2$, then $\sharp\{$log divisors$\} = 4$, $\sharp\{$tripodal divisors$\} = 3$, and $\sharp\{$log-full points$\} = 3$.*

(iv) If $(g, r) = (1, 1)$ and $n = 2$, then it holds that V is not a tripodal divisor if and only if for any log-full subgroup A, there exists an inertia group associated to V which is contained in A.

Proof. Assertions (i), (ii) follow from Lemma 6.1, and [Hsh], Corollary 2; Remark B.2. Assertion (iii) follows immediately from the various definitions involved. Assertion (iv) follows from assertion (iii) and Proposition 4.2. $\square$

Theorem 6.4. *For $\square \in \{\circ, \bullet\}$, let $p^{\square}$, $l^{\square}$ be distinct prime numbers; $k^{\square}$ an algebraically closed field of characteristic zero or $p^{\square}$; $S^{\square} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k^{\square})$; $(g^{\square}, r^{\square})$ a pair of nonnegative integers such that $2g^{\square} - 2 + r^{\square} > 0$;*

$$X^{\log \square} \to S^{\square}$$

a smooth log curve of type $(g^{\square}, r^{\square})$; $n^{\square} \in \mathbb{Z}_{>1}$; $X_{n^{\square}}^{\log \square}$ the $n^{\square}$-th log configuration space associated to $X^{\log \square} \to S^{\square}$; $\Pi^{\square} \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l^{\square}}(X_{n^{\square}}^{\log \square})$;

$$\phi \colon \Pi^{\circ} \xrightarrow{\sim} \Pi^{\bullet}$$

an isomorphism of profinite groups. We suppose that $r^{\square} > 0$; ϕ induces a bijection between the set of log-full subgroups of Π° and the set of log-full subgroups of $\Pi^{\bullet}$. Then ϕ induces a bijection between the set of inertia groups of Π° associated to tripodal divisors of $X_{n^{\circ}}^{\log \circ}$ and the set of inertia groups of $\Pi^{\bullet}$ associated to tripodal divisors of $X_{n^{\bullet}}^{\log \bullet}$.

Proof. Note that it follows from a well-known structure of the fundamental group of a smooth log curve of type (g, r) over an algebraically closed field of characteristic zero that $\pi_1^{\text{pro-}l^{\circ}}(X^{\log \circ})$ is isomorphic to $\pi_1^{\text{pro-}l^{\bullet}}(X^{\log \bullet})$ if and only if $l^{\circ} = l^{\bullet}$ and $2g^{\circ} - 2 + r^{\circ} = 2g^{\bullet} - 2 + r^{\bullet}$; it follows from [Ind], Theorem 3.5, that $\Pi^{\square}$ is indecomposable. Thus, Theorem 6.4 follows from Theorem 5.3, Lemma 6.3, and Theorem 3.9, (i). $\square$

7. Reconstruction of drift diagonals

We continue with the notation of the preceding Section. In the present §7, we reconstruct the set of inertia groups associated to drift diagonals (cf. Theorem 7.3, below).

Lemma 7.1. *The outer homomorphism $\iota \colon \Pi_n \to \Pi_1 \times \cdots \times \Pi_1$ induced by $\iota \colon X_n^{\log} \to X^{\log} \times_S \cdots \times_S X^{\log}$ (cf. Definition 2.2, (ix)) is surjective whose kernel is the closure of*

$$\langle I \mid I \text{ is an inertia group associated to a naive diagonal} \rangle.$$

Proof. It follows from [Hsh], Remark B.2, that in the commutative diagram

$$\begin{tikzcd} \pi_1^{\text{pro-}l}(U_{X_n}) \arrow[r] \arrow[d] & \pi_1^{\text{pro-}l}(U_{X_1}) \times \cdots \times \pi_1^{\text{pro-}l}(U_{X_1}) \arrow[d] \\ \Pi_n \arrow[r, "\iota"'] & \Pi_1 \times \cdots \times \Pi_1, \end{tikzcd}$$

where $\pi_1^{\text{pro-}l}(U_{X_n}) \to \pi_1^{\text{pro-}l}(U_{X_1}) \times \cdots \times \pi_1^{\text{pro-}l}(U_{X_1})$ is the outer surjective homomorphism induced by the open immersion $U_{X_n} \hookrightarrow U_{X_1} \times_S \cdots \times_S U_{X_1}$, the two vertical arrows are isomorphisms. Thus, $\iota\colon \Pi_n \to \Pi_1 \times \cdots \times \Pi_1$ is surjective. By [SGA1], Exposé X, Théorème 3.1,

$$\mathrm{Ker}(\iota) = \overline{\langle \alpha I \alpha^{-1} \mid \alpha \in \Pi_n, I \text{ is an inertia group associated to a naive diagonal } \rangle}.$$

This completes the proof of Lemma 7.1. $\square$

Lemma 7.2. *Let V be a tripodal divisor and I_V an inertia group associated to V. Write $\iota\colon \Pi_n \to \Pi_1 \times \cdots \times \Pi_1$ for the outer homomorphism induced by $\iota\colon X_n^{\log} \to X^{\log} \times_S \cdots \times_S X^{\log}$ (cf. Definition 2.2, (ix)). The following hold.*

(i) If V is a naive diagonal, then $\iota(I_V) = \{e\}$.
(ii) If V is not a naive diagonal, then $\iota(I_V) \neq \{e\}$.

Proof. Assertion (i) follows from Lemma 7.1. Assertion (ii) follows immediately from Proposition 3.3, (i), (ii), (iii). $\square$

Theorem 7.3. *For $\square \in \{\circ, \bullet\}$, let $p^\square$, $l^\square$ be distinct prime numbers; $k^\square$ an algebraically closed field of characteristic zero or $p^\square$; $S^\square \overset{\mathrm{def}}{=} \mathrm{Spec}(k^\square)$; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$;*

$$X^{\log\,\square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$; $n^\square \in \mathbb{Z}_{>1}$; $X_{n^\square}^{\log\,\square}$ the $n^\square$-th log configuration space associated to $X^{\log\,\square} \to S^\square$; $\Pi^\square \overset{\mathrm{def}}{=} \pi_1^{\text{pro-}l^\square}(X_{n^\square}^{\log\,\square})$;

$$\phi\colon \Pi^\circ \overset{\sim}{\to} \Pi^\bullet$$

an isomorphism of profinite groups. We suppose that $r^\square > 0$; ϕ induces a bijection between the set of log-full subgroups of Π° and the set of log-full subgroups of $\Pi^\bullet$. Then ϕ induces a bijection between the set of inertia groups of Π° associated to drift diagonals of $X_{n^\circ}^{\log\,\circ}$ and the set of inertia groups of $\Pi^\bullet$ associated to drift diagonals of $X_{n^\bullet}^{\log\,\bullet}$.

Proof. Let us suppose that $(g^\square, r^\square) = (0,3)$ or $(1,1)$. Then it follows from Theorem 6.4 and Proposition 3.4, (iii), that ϕ induces a bijection between the set of inertia groups of Π° associated to drift diagonals of $X_{n^\circ}^{\log\,\circ}$ and the set of inertia groups of $\Pi^\bullet$ associated to drift diagonals of $X_{n^\bullet}^{\log\,\bullet}$. This completes the proof of Theorem 7.3 in the case where $(g^\square, r^\square) = (0,3)$ or $(1,1)$.

Let us suppose that $(g^\square, r^\square) \neq (0,3), (1,1)$. Write $\Pi_1^\square \overset{\mathrm{def}}{=} \pi_1^{\text{pro-}l^\square}(X^{\log\,\square})$. Then it follows from Theorem 3.9, (iii), that ϕ induces a commutative diagram

$$\begin{tikzcd}
\Pi^\circ \arrow[r, "\phi"] \arrow[d, "\iota^\circ"'] & \Pi^\bullet \arrow[d, "\iota^\bullet"'] \\
\Pi_1^\circ \times \cdots \times \Pi_1^\circ \arrow[r, dotted, "\sim"] & \Pi_1^\bullet \times \cdots \times \Pi_1^\bullet,
\end{tikzcd}$$

where $\iota^\square\colon \Pi^\square \to \Pi_1^\square \times \cdots \times \Pi_1^\square$ is the outer homomorphism induced by $\iota^\square\colon X_{n^\square}^{\log\,\square} \to X^{\log\,\square} \times_{S^\square} \cdots \times_{S^\square} X^{\log\,\square}$ (cf. Definition 2.2, (ix)). Thus, it follows from Theorem 6.4, Lemma 7.2, and Proposition 3.4, (ii), that ϕ induces a bijection between the set of inertia groups of Π° associated to drift diagonals of $X_{n^\circ}^{\log\,\circ}$ and the set of inertia groups of $\Pi^\bullet$ associated to drift diagonals of $X_{n^\bullet}^{\log\,\bullet}$. This completes the proof of Theorem 7.3 in the case where $(g^\square, r^\square) \neq (0,3), (1,1)$. $\square$

8. **Reconstruction of drift collections**

We continue with the notation of the preceding Section. In the present §8, we reconstruct drift collections (cf. Definition 8.14, below, and Theorem 8.15, below).

Definition 8.1. Let Λ be a set of drift diagonals. We shall say that Λ is a *scheme-theoretic drift collection* if there exists an automorphism α of $X_n^{\log}$ over S such that $\Lambda = \{\alpha(V) \mid V \text{ is a naive diagonal}\}$.

Definition 8.2. Let V_1, V_2 be distinct drift diagonals and I_{V_1}, I_{V_2} inertia groups associated to V_1, V_2, respectively.

(i) Since V_1, V_2 are tripodal divisors (cf. Proposition 3.4, (i)), there exists a unique vertex v_1 (resp. v_2) of $\mathcal{G}_{V_1}$ (resp. $\mathcal{G}_{V_2}$) such that v_1, v_2 are tripods (cf. Definition 3.1, (iii)). We shall say that $\{V_1, V_2\}$ is a *scheme-theoretically co-cuspidal pair* if there exists a cusp $y \in C_{r,n}$ that is a cusp of $\mathcal{G}_{V_1}|_{v_1}$, $\mathcal{G}_{V_2}|_{v_2}$.

(ii) We shall say that $\{V_1, V_2\}$ is a *group-theoretically co-cuspidal pair* if there is no log-full subgroup A such that a conjugate of A contains I_{V_1} and a conjugate of A contains I_{V_2}.

Lemma 8.3. *Let V_1, V_2 be distinct drift diagonals. Then it holds that*

$$\begin{aligned} &\{V_1, V_2\} \text{ is a group-theoretically co-cuspidal pair} \\ \iff\ &\text{there is no log-full point contained in } V_1 \cap V_2. \end{aligned}$$

Proof. This follows from Proposition 4.2. □

Lemma 8.4. *A scheme-theoretically co-cuspidal pair is a group-theoretically co-cuspidal pair.*

Proof. Let $\{V_1, V_2\}$ be a scheme-theoretically co-cuspidal pair, v_1 the unique vertex of $\mathcal{G}_{V_1}$ which is a tripod, and y_1, y_2 cusps of $\mathcal{G}_{V_1}|_{v_1}$, where $y_1, y_2 \in C_{r,n}$ are distinct elements. We assume that there exists a log-full point P contained in $V_1 \cap V_2$. Then one verifies easily that for any generization $\mathcal{G}'$ of $\mathcal{G}_P$, there exists a vertex v of $\mathcal{G}'$ such that y_1, y_2 are cusps of $\mathcal{G}'|_v$. Thus, it follows immediately from the assumption that $\{V_1, V_2\}$ is a scheme-theoretically co-cuspidal pair that $P \notin V_2$, which thus implies a contradiction. □

Lemma 8.5. *A group-theoretically co-cuspidal pair is a scheme-theoretically co-cuspidal pair.*

Proof. Let $\{V_1, V_2\}$ be a pair of distinct drift diagonals which is not a scheme-theoretically co-cuspidal pair. There exists a unique vertex v_1 (resp. v_2) of $\mathcal{G}_{V_1}$ (resp. $\mathcal{G}_{V_2}$) such that v_1, v_2 are tripods. Let y_1, y_2 be cusps of $\mathcal{G}_{V_1}|_{v_1}$ and y_3, y_4 cusps of $\mathcal{G}_{V_2}|_{v_2}$, where $y_1, y_2, y_3, y_4 \in C_{r,n}$ are distinct elements. Then one verifies easily that there exist a log-full point P and terminal vertices t_1, t_2 of $\mathcal{G}_P$ such that t_1, t_2 are tripods, y_1, y_2 are cusps of $\mathcal{G}_P|_{t_1}$, and y_3, y_4 are cusps of $\mathcal{G}_P|_{t_2}$. In particular, $P \in V_1 \cap V_2$. Thus, $\{V_1, V_2\}$ is not a group-theoretically co-cuspidal pair (cf. Lemma 8.3). □

Definition 8.6. Let V_1, V_2, V_3 be distinct drift diagonals.

(i) Since V_1, V_2, V_3 are tripodal divisors, there exists a unique vertex v_1 (resp. v_2, v_3) of $\mathcal{G}_{V_1}$ (resp. $\mathcal{G}_{V_2}$, $\mathcal{G}_{V_3}$) such that v_1, v_2, v_3 are tripods. We shall say that $\{V_1, V_2, V_3\}$ is a *scheme-theoretically co-cuspidal triple* if there exist cusps

$y_1, y_2, y_3 \in C_{r,n}$ such that y_1, y_2 are cusps of $\mathcal{G}_{V_1}|_{v_1}$; y_2, y_3 are cusps of $\mathcal{G}_{V_2}|_{v_2}$; y_1, y_3 are cusps of $\mathcal{G}_{V_3}|_{v_3}$.

(ii) We shall say that $\{V_1, V_2, V_3\}$ is a *group-theoretically co-cuspidal triple* if there exist log divisors $W_2, \ldots, W_n$ such that $I_{V_1} \times I_{W_2} \times \cdots \times I_{W_n}$, $I_{V_2} \times I'_{W_2} \times \cdots \times I'_{W_n}$, $I_{V_3} \times I''_{W_2} \times \cdots \times I''_{W_n}$ are log-full subgroups, where $I_{(-)}$, $I'_{(-)}$, $I''_{(-)}$ are inertia groups of $(-)$.

Lemma 8.7. *Let V_1, V_2, V_3 be distinct drift diagonals. Then it holds that $\{V_1, V_2, V_3\}$ is a group-theoretically co-cuspidal triple if and only if there exist log divisors $W_2, \ldots, W_n$ such that $V_1 \cap W_2 \cap \cdots \cap W_n$, $V_2 \cap W_2 \cap \cdots \cap W_n$, $V_3 \cap W_2 \cap \cdots \cap W_n$ are log-full points.*

Proof. This follows from Proposition 4.4 and Proposition 4.2. □

Lemma 8.8. *A scheme-theoretically co-cuspidal triple is a group-theoretically co-cuspidal triple*

Proof. Let $\{V_1, V_2, V_3\}$ be a scheme-theoretically co-cuspidal triple. Since V_1, V_2, V_3 are tripodal divisors, there exists a unique vertex v_1 (resp. v_2, v_3) of $\mathcal{G}_{V_1}$ (resp. $\mathcal{G}_{V_2}$, $\mathcal{G}_{V_3}$) such that v_1, v_2, v_3 are tripods. Then there exist cusps $y_1, y_2, y_3 \in C_{r,n}$ such that y_1, y_2 are cusps of $\mathcal{G}_{V_1}|_{v_1}$; y_2, y_3 are cusps of $\mathcal{G}_{V_2}|_{v_2}$; y_1, y_3 are cusps of $\mathcal{G}_{V_3}|_{v_3}$. Thus, there exists a log divisor W_2 such that $\mathcal{G}_{W_2}$ has a vertex w which satisfies the conditions that w is a vertex of type $(0,4)$ (cf. [CbTpI], Definition 2.3, (iii)) and y_1, y_2, y_3 are cusps of $\mathcal{G}_{W_2}|_w$. In particular, one verifies easily that $\{V_1, V_2, V_3\}$ is a group-theoretically co-cuspidal triple (cf. Lemma 8.7). □

Lemma 8.9. *A group-theoretically co-cuspidal triple is a scheme-theoretically co-cuspidal triple*

Proof. Let $\{V_1, V_2, V_3\}$ be a group-theoretically co-cuspidal triple. There exist $y_1, y_2 \in C_{r,n}$ such that $V_1 = V(y_1, y_2)$. By lemma 8.7, there exist log divisors $W_2, \ldots, W_n$ such that $V_1 \cap W_2 \cap \cdots \cap W_n$, $V_2 \cap W_2 \cap \cdots \cap W_n$, $V_3 \cap W_2 \cap \cdots \cap W_n$ are log-full points. Let Q be a generic point of $W_2 \cap \cdots \cap W_n$. Then there exists a unique vertex v of $\mathcal{G}_Q$ such that y_1 is a cusp of $\mathcal{G}_Q|_v$. Since $V_1 \cap W_2 \cap \cdots \cap W_n$ is a log-full point, v is a vertex of type $(0,4)$ and there exists $y_3 \in C_{r,n}$ such that y_1, y_2, y_3 are cusps of $\mathcal{G}_Q|_v$. Then it follows immediately from the definitions that $\{V_1, V_2, V_3\} = \{V(y_1, y_2), V(y_2, y_3), V(y_1, y_3)\}$. Thus, $\{V_1, V_2, V_3\}$ is a scheme-theoretically co-cuspidal triple. □

Definition 8.10. Let Λ be a set of drift diagonals such that $\sharp\Lambda = \frac{n(n-1)}{2}$. We shall say that Λ is a *group-theoretic drift collection* if there exist distinct drift diagonals $V_{i,j}$ $(1 \le i < j \le n)$ such that $\Lambda = \{V_{i,j} \mid 1 \le i < j \le n\}$, and, moreover, the following hold:

(a) For any $1 \le i \le n-2$, $\{V_{i,i+1}, V_{i+1,i+2}\}$ is a (group-theoretically) co-cuspidal pair.

(b) For any $1 \le i < j \le n-1$, if $j \ne i+1$, then $\{V_{i,i+1}, V_{j,j+1}\}$ is not a (group-theoretically) co-cuspidal pair.

(c) For any $1 \le i < j \le n$, if $j \ne i+1$, $\{V_{i,j}, V_{i,i+1}, V_{i+1,j}\}$ is a (group-theoretically) co-cuspidal triple.

Theorem 8.11. *Let Λ be a set of drift diagonals. Then Λ is group-theoretic drift collection if and only if Λ is scheme-theoretic drift collection (cf. Definition 8.1).*

Proof. This follows immediately from Claim 8.12 and Claim 8.13, below. □

Claim 8.12. *A scheme-theoretic drift collection is a group-theoretic drift collection.*

Proof. Let Λ be a scheme-theoretic drift collection. Then it follows from Proposition 3.3, (i), that there exist distinct elements $y_1, \ldots, y_n \in C_{r,n}$ such that

$$\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}.$$

Then one verifies easily that if we write $V_{i,j} \overset{\text{def}}{=} V(y_i, y_j)$, then $V_{i,j}$'s satisfy the condition of Definition 8.10, which thus implies that Λ is a group-theoretic drift collection. □

Claim 8.13. *A group-theoretic drift collection is a scheme-theoretic drift collection.*

Proof. Let Λ be a group-theoretic drift collection. By Remark 3.2, (ii), and Definition 8.10, (a), there exist $y_1, y_2, y_3 \in C_{r,n}$ such that $V_{1,2} = V(y_1, y_2), V_{2,3} = V(y_2, y_3)$. By Remark 3.2, (ii), and Definition 8.10, (a), (b), there exist $y_4, \ldots, y_n \in C_{r,n}$ such that $V_{i,i+1} = V(y_i, y_{i+1})$. By Remark 3.2, (ii), and Definition 8.10, (c), it holds that $V_{i,j} = V(y_i, y_j)$. Thus, if $(g, r) = (0, 3)$ or $(1, 1)$, then it follows from the proof of Proposition 3.4, (iii), that $\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}$ is a scheme-theoretic drift collection. Moreover, if $(g, r) \neq (0, 3), (1, 1)$, then it follows from Proposition 3.4, (ii), that $\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}$ is a scheme-theoretic drift collection. □

Definition 8.14. We shall refer to $\{I_V \mid V \in \Lambda\}$ as a *drift collection* of Π_n, where Λ is a (group-theoretic) drift collection and I_V is an inertia group of Π_n associated to $V \in \Lambda$.

Theorem 8.15. *For $\square \in \{\circ, \bullet\}$, let $p^\square$, $l^\square$ be distinct prime numbers; $k^\square$ an algebraically closed field of characteristic zero or $p^\square$; $S^\square \overset{\text{def}}{=} \mathrm{Spec}(k^\square)$; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$;*

$$X^{\log \square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$; $n^\square \in \mathbb{Z}_{>1}$; $X^{\log \square}_{n^\square}$ the $n^\square$-th log configuration space associated to $X^{\log \square} \to S^\square$; $\Pi^\square \overset{\text{def}}{=} \pi_1^{\text{pro-}l^\square}(X^{\log \square}_{n^\square})$;

$$\phi \colon \Pi^\circ \overset{\sim}{\to} \Pi^\bullet$$

an isomorphism of profinite groups. We suppose that $r^\square > 0$; ϕ induces a bijection between the set of log-full subgroups of Π° and the set of log-full subgroups of $\Pi^\bullet$. Then ϕ induces a bijection between the set of drift collections of Π° and the set of drift collections of $\Pi^\bullet$.

Proof. This follows from Theorem 7.3 and Theorem 8.11. □

9. **Reconstruction of drift fiber subgroups**

We continue with the notation of the preceding Section. In the present §9, we reconstruct drift fiber subgroups (cf. Definition 9.1, below, and Theorem 9.3, below).

Definition 9.1. Let H be a closed subgroup of Π_n. We shall say that H is a *drift fiber subgroup* if there exist an automorphism α of $X_n^{\log}$ over S and a fiber subgroup $F \subseteq \Pi_n$ (cf. [MzTa], Definition 2.3, (iii)) such that $H = \beta(F)$, where β is an automorphism of Π_n which arises from α.

Proposition 9.2. *If* $(g, r) \neq (0, 3), (1, 1)$*, then*

$$\{\textit{drift fiber subgroups}\} = \{\textit{fiber subgroups}\}.$$

Proof. This follows immediately from [CbTpII], Lemma 2.7, (iii). $\square$

Theorem 9.3. *For* $\square \in \{\circ, \bullet\}$*, let* $p^\square$*,* $l^\square$ *be distinct prime numbers;* $k^\square$ *an algebraically closed field of characteristic zero or* $p^\square$*;* $S^\square \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k^\square)$*;* $(g^\square, r^\square)$ *a pair of nonnegative integers such that* $2g^\square - 2 + r^\square > 0$*;*

$$X^{\log \square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$*;* $n^\square \in \mathbb{Z}_{>1}$*;* $X_{n^\square}^{\log \square}$ *the* $n^\square$*-th log configuration space associated to* $X^{\log \square} \to S^\square$*;* $\Pi^\square \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l^\square}(X_{n^\square}^{\log \square})$*;*

$$\phi \colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups. We suppose that $r^\square > 0$*;* ϕ *induces a bijection between the set of log-full subgroups of* Π° *and the set of log-full subgroups of* $\Pi^\bullet$*. Then* ϕ *induces a bijection between the set of drift fiber subgroups of* Π° *and the set of drift fiber subgroups of* $\Pi^\bullet$*.*

Proof. For each j, write $\Pi_j^\square \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l^\square}(X_j^{\log \square})$. Let $F^\circ \subset \Pi^\circ$ be a drift fiber subgroup of Π°. Then there exists a drift collection Λ° of Π° such that the following holds:

> Write $\iota^\circ \colon \Pi^\circ \to Q^\circ$ for the surjection obtained by taking the quotient by the normal closed subgroup generated by the elements of a drift collection Λ° of Π°. Now it follows from Lemma 7.1 and [MzTa], Corollary 3.4, that there exist n° surjections $Q^\circ \to \Pi_1^\circ$ which determine an isomorphism $Q^\circ \xrightarrow{\sim} \Pi_1^\circ \times \cdots \times \Pi_1^\circ$. Then there exists a surjection p° among these n° surjections whose kernel contains F°.

Next, let us observe that we have a commutative diagram

$$\begin{tikzcd} \Pi^\circ \arrow[r, "\phi"] \arrow[d, "p^\circ"'] & \Pi^\bullet \arrow[d, "p^\bullet"'] \\ \Pi_1^\circ \arrow[r, dotted] & \Pi_1^\bullet, \end{tikzcd}$$

where $p^\bullet$ is the surjection corresponding to p° via ϕ. It follows immediately from Theorem 8.15, together with the definition p°, that $\mathrm{Ker}(p^\square)$ has a natural structure of configuration space group and F° is a fiber subgroup of $\mathrm{Ker}(p^\circ)$. By [MzTa], Corollary 6.3, $F^\bullet \stackrel{\mathrm{def}}{=} \phi(F^\circ)$ is a fiber subgroup of $\mathrm{Ker}(p^\bullet)$. Thus, again by the definition of p°, together with the various definitions involved, $F^\bullet$ is a drift fiber subgroup of $\Pi^\bullet$. $\square$

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