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Abstract

Problems of allocating indivisible goods to agents in an efficient and fair manner without money have long been investigated in the literature. The random assignment problem is one of them, where we are given a fixed feasible (available) set of indivisible goods and a profile of ordinal preferences over the goods, one for each agent, and we determine an assignment of goods to agents in a randomized way using lotteries. A seminal paper of Bogomolnaia and Moulin (2001) shows a probabilistic serial mechanism to give an efficient and envy-free solution to the assignment problem.

In this paper we consider an extension of the random assignment problem to the case where we are given a family $\mathcal{B}$ of feasible sets of indivisible goods. In particular we consider the case where $\mathcal{B}$ is a family of bases of a matroid. Under the agents' ordinal preferences over goods we show an extension of the probabilistic serial mechanism to give an efficient and envy-free solution that probabilistically makes a choice of a member (a base) of the family and its assignment to agents. The theory of submodular optimization plays a crucial rôle in the extension.

Keywords: Random assignment, probabilistic serial mechanism, ordinal preference, matchings, matroids, independent flows, submodular optimization

1. Introduction

Problems of allocating indivisible goods to agents in a fair and efficient manner without money have long been investigated in the literature (see, e.g., [19, 23, 1, 4, 13, 14, 3,

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10, 11, 2, 21]). Suppose that we are given a fixed feasible (available) set of indivisible goods and a profile of ordinal preferences over the goods, one for each agent. Then, using lotteries, we can guarantee fairness of a solution assigning available goods to agents probabilistically. Such a problem is called the *random assignment problem* [4]. A seminal paper of Bogomolnaia and Moulin [4] shows a probabilistic serial mechanism to give a solution to the random assignment problem.

In this paper we consider an extension of the random assignment problem to the case where we are given a family $\mathcal{B}$ of feasible sets of indivisible goods. In particular we consider the case where $\mathcal{B}$ is a family of bases of a matroid (see [22, 18]). Note that the random assignment problem considered in the literature treats only a single set of available indivisible goods. The random assignment problem with multiple indivisible goods was also investigated in [5, 11, 14], which is not treated in the present paper. Paper [5] also investigated extensions of the ordinary random assignment problem with additional constraints, which are diffrent from our matroidal ones.

Under the agents' ordinal preferences over goods we show an extension of the probabilistic serial mechanism introduced by Bogomolnaia and Moulin [4] to give a solution that makes a probabilistic choice of a member (a base) of the family and its assignment to agents. We show that our extended probabilistic serial mechanism gives a solution that is efficient and envy-free with respect to the partial order defined by the stochastic dominance relation introduced by Bogomolnaia and Moulin [4].

The well-known Birkhoff-von Neumann theorem on bi-stochastic matrices shows that every bi-stochastic matrix is expressed as a convex combination of permutation matrices, which plays a crucial rôle in designing the probabilistic serial mechanism developed by Bogomolnaia and Moulin [4]. On the other hand, our extended probabilistic serial mechanism heavily depends on the results of submodular optimization such as the integrality of the independent flow polyhedra ([6, 8]), which generalizes the Birkhoff-von Neumann theorem.

The present paper is organized as follows. Section 2 gives some definitions and preliminaries to be used later. In Section 3 we describe the random assignment problem and we first consider the base polytope of the given matroid as a set of *divisible* goods and show a procedure Random_Assignment to find an allocation of the divisible goods in an efficient and fair manner. In Section 4 we consider our original problem of allocating the *indivisible* goods and show how to design a lottery that realizes the probability distribution of getting goods for each agent which is obtained by procedure Random_Assignment. Section 5 gives concluding remarks.

2. Definitions and Preliminaries

Let N be a finite set of agents and E be that of goods. Each agent $i \in N$ has an ordinal preference $\succ_i$ over set E of goods, which is a linear ordering of E. Suppose that $|N| = n$ and $|E| = m$, where $|\cdot|$ denotes the cardinality. For any subset $X \subseteq E$ denote by χ_X the characteristic vector of X in $\mathbb{R}^E$, i.e., $\chi_X(e) = 1$ for $e \in X$ and $\chi_X(e) = 0$ for $e \in E \setminus X$. We also write χ_e instead of $\chi_{\{e\}}$ for $e \in E$.

For the ordinary random assignment problem considered in the literature we are given one available (feasible) set of goods to be allocated (e.g., [4, 13, 3]). In this paper we deal with a more general problem, where there exists a family $\mathcal{B} \subseteq 2^E$ of available (feasible) sets of goods that forms a family of bases of a matroid on E. We consider how to choose one set B from among $\mathcal{B}$ and at the same time how to allocate B to agents in an efficient and fair manner probabilistically.

A pair $(E, \mathcal{B})$ of a finite nonempty set E and a family $\mathcal{B} \subseteq 2^E$ is called a *matroid* with a family $\mathcal{B}$ of *bases* if the following two hold ([22, 18]).

- $\mathcal{B} \neq \emptyset$,
- for any $B_1, B_2 \in \mathcal{B}$ and any $e \in B_1 \setminus B_2$ there exists $e' \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{e'\} \in \mathcal{B}$.

The *rank function* $\rho : 2^E \to \mathbb{Z}_{\geq 0}$ associated with matroid $(E, \mathcal{B})$ is defined by

$$\rho(X) = \max\{|X \cap B| \mid B \in \mathcal{B}\} \qquad (\forall X \subseteq E). \tag{2.1}$$

Note that every base $B \in \mathcal{B}$ has the same cardinality equal to $\rho(E)$. In the sequel we assume without loss of generality that $|N| = \rho(E)$, unless otherwise stated. (If $|N| > \rho(E)$, we may introduce dummy $|N| - \rho(E)$ goods of coloops, while if $|N| < \rho(E)$, we may consider a truncation of $(E, \mathcal{B})$ to have bases of cardinality $|N|$.)[1]

Two simple examples of matroids are given as follows. They will be used to show the behavior of our solution in the next section.

Uniform matroids: For a positive integer $k \leq m(= |E|)$ every subset of cardinality k of E is feasible, i.e.,

$$\mathcal{B} = \{X \mid X \subseteq E,\ |X| = k\}.$$

When $k = m(= |E|)$, $\mathcal{B}$ consists of only one base E, which is the one available set of goods as is considered in the literature for the ordinary random assignment problem.

Graphic matroids: For a connected graph $G = (V, E)$ with a vertex set V and an edge set E every edge subset that forms a spanning tree is feasible, i.e., $(E, \mathcal{B})$ is the graphic

[1] We can also treat the case when $|N| > \rho(E)$ by imposing a matroidal constraint on N given by the uniform matroid on N of rank $\rho(E)$, without introducing dummy goods. See (4.7) for related arguments.

matroid represented by graph $G = (V, E)$ with $\mathcal{B}$ being the family of edge sets of spanning trees.

For a given matroid $(E, \mathcal{B})$ with its rank function $\rho : 2^E \to \mathbb{Z}_{\geq 0}$ let $\mathrm{B}(\rho)(\subset \mathbb{R}^E)$ be the *base polytope* of the matroid, which is given by the convex hull of all the characteristic vectors χ_B in $\mathbb{R}^E$ of bases $B \in \mathcal{B}$ and is represented by

$$\mathrm{B}(\rho) = \{x \in \mathbb{R}^E \mid \forall X \subset E : x(X) \leq \rho(X),\ x(E) = \rho(E)\}, \tag{2.2}$$

where for any $X \subseteq E$ we define $x(X) = \sum_{e \in X} x(e)$ (see, e.g., [8]). We assume that $\rho(\{e\}) = 1$ for all $e \in E$ without loss of generality. Also consider the lower hereditary closure of the base polytope $\mathrm{B}(\rho)$ given by

$$\mathrm{P}(\rho) = \{x \in \mathbb{R}^E \mid \forall X \subseteq E : x(X) \leq \rho(X)\}, \tag{2.3}$$

which is called the *submodular polyhedron* associated with ρ. Given a vector $x \in \mathrm{P}(\rho)$, a subset X of E is called *tight* for x if we have $x(X) = \rho(X)$, and there exists a unique maximal tight set, denoted by $\mathrm{sat}(x)$, for x, which is equal to the union of all tight sets for x. We also have

$$\mathrm{sat}(x) = \{e \in E \mid \forall \alpha > 0 : x + \alpha\chi_e \notin \mathrm{P}(\rho)\}, \tag{2.4}$$

which is the set of elements $e \in E$ for which we cannot increase $x(e)$ without leaving $\mathrm{P}(\rho)$. (See [8] for more details about these concepts and related facts.) Matroid $(E, \mathcal{B})$ is often denoted by (E, ρ) as well.

3. Random Assignment

In the seminal paper [4] by Bogomolnaia and Moulin they proposed a new solution through what is called the *probabilistic serial* (PS) mechanism from the point of view of stochastic dominance when there is only one feasible set of goods. We extend the PS mechanism so as to deal with the case where we are given a family $\mathcal{B}$ of feasible sets of goods that forms a base family of a matroid $(E, \mathcal{B})$ on E.

First, we consider the base polytope $\mathrm{B}(\rho)$ as a set of *divisible* goods and find an allocation of the divisible goods in an efficient and fair manner.

3.1. Random assignment as an allocation of divisible goods

Let P be an $N \times E$ matrix satisfying the following two:

1. $P(i, e) \geq 0$ for all $i \in N$ and $e \in E$.

2. Regarding each ith row $P_i = (P(i,e) \mid e \in E)$ of P as a vector in $\mathbb{R}^E_{\geq 0}$, we have

$$\sum_{i \in N} P_i \in \mathrm{B}(\rho). \tag{3.1}$$

Then we call P a *random assignment matrix* (or *random assignment* for short). Let us define the base x^*_P associated with P by

$$x^*_P = \sum_{i \in N} P_i. \tag{3.2}$$

For each $i \in N$ let agent i's preference be given by

$$L^i :\ e^i_1 \succ_i e^i_2 \succ_i \cdots \succ_i e^i_m, \tag{3.3}$$

where $\{e^i_1, e^i_2, \cdots, e^i_m\} = E$ and e^i_1 is the most favorite good for agent i. Let $\mathcal{L}$ be the profile of preferences L^i $(i \in N)$. Based on the collection (a multiset) of the first (most favorite) elements e^i_1 of all agents $i \in N$, define a nonnegative integral vector $b(\mathcal{L}) \in \mathbb{Z}^E_{\geq 0}$ by

$$b(\mathcal{L}) = \sum_{i \in N} \chi_{e^i_1}, \tag{3.4}$$

where note that we may have $e^i_1 = e^j_1$ for distinct $i, j \in N$.

Denote the random assignment problem by $\mathbf{P} = (N, E, \mathcal{L} = (L^i \mid i \in N), (E, \rho))$. Our random assignment algorithm by the extended PS mechanism is described as follows. During the execution of the following algorithm the current lists L^i may get shorter because of removal of exhausted goods.

Random_Assignment

Input: A preference profile $\mathcal{L} = (L^i \mid i \in N)$ and a matroid (E, ρ) with $\rho(E) \leq |N| (= n)$.

Output: A random assignment matrix $P \in \mathbb{R}^{N \times E}_{\geq 0}$.

Step 0: For each $i \in N$ put $x^i \leftarrow \mathbf{0} \in \mathbb{R}^E$ (the zero vector) and put $S_0 \leftarrow \emptyset$, $p \leftarrow 1$, and $x^* \leftarrow \mathbf{0}$.

Step 1: For current (updated) $\mathcal{L} = (L^i \mid i \in N)$, using $b(\mathcal{L})$ in (3.4), compute

$$\lambda^* = \max\{\lambda \geq 0 \mid x^* + \lambda b(\mathcal{L}) \in \mathrm{P}(\rho)\}. \tag{3.5}$$

For each $i \in N$ put $x^i \leftarrow x^i + \lambda^* \chi_{e^i_1}$.

Put $x^* \leftarrow x^* + \lambda^* b(\mathcal{L})$ and $S_p \leftarrow \mathrm{sat}(x^*)$.

Step 2: Put $T \leftarrow S_p \setminus S_{p-1}$.

Update L^i $(i \in N)$ by removing all elements of T from current L^i $(i \in N)$.

Step 3: If $\rho(S_p) < \rho(E)$, then put $p \leftarrow p+1$ and go to Step 1.
Otherwise ($\rho(S_p) = \rho(E)$) put $P(i,e) \leftarrow x^i(e)$ for all $i \in N$ and $e \in E$.
Return P.

For each agent $i \in N$ the ith row sum $\sum_{e \in E} P(i,e)$ of the output matrix P is equal to $\rho(E)/|N|$. Note that the procedure works for any matroid with $\rho(E) < |N|$ without introducing dummy goods as well.

It is worth mentioning that the procedure of determining λ^* in **Step 1** is the same as the monotone algorithm considered in [7] (also see [8, Sec. 9]), while a special case of the same problem for multi-terminal network flows was also considered earlier in [15] and later in [9].

To see the behavior of the procedure Random_Assignment let us consider two illustrative examples as follows.

Example 1: Consider $N = \{1,2,3,4\}$ and $E = \{a,b,c,d\}$. We are given a uniform matroid $\mathbf{M} = (E, \mathcal{B})$ of rank two, i.e.,

$$\mathcal{B} = \{X \mid X \subset E,\ |X| = 2\}. \tag{3.6}$$

Suppose that preferences of all agents are given as follows.

$i \in N$	preference L^i
1	$a \succ_1 b \succ_1 c \succ_1 d$
2	$a \succ_2 c \succ_2 b \succ_2 d$
3	$a \succ_3 c \succ_3 d \succ_3 b$
4	$b \succ_4 a \succ_4 d \succ_4 c$

(Here we consider the case when $|N| > \rho(E)$.)

Then by the procedure Random_Assignment we have

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1/3 & 1/6 & 0 & 0 \\ 1/3 & 0 & 1/6 & 0 \\ 1/3 & 0 & 1/6 & 0 \\ 0 & 1/3+1/6 & 0 & 0 \end{pmatrix} \end{array},$$

where

$$b(\mathcal{L}) = \overset{\begin{array}{cccc} a & b & c & d \end{array}}{\begin{pmatrix} 3, & 1, & 0, & 0 \end{pmatrix}}, \quad S_1 = \{a\}, \quad \lambda^* = 1/3 \ \text{ for } \ p = 1$$

and

$$b(\mathcal{L}) = (0,2,2,0), \quad S_2 = \{a,b,c,d\}, \quad \lambda^* = 1/6 \ \text{ for } \ p = 2$$

to get the random assignment matrix P. Also, vectors x_p^* on $T_p = S_p \setminus S_{p-1}$ for $p = 1, 2$ are given by

$$T_1 = \{a\}, \quad T_2 = \{b,c,d\},$$
$$x_1^*(a) = 1, \quad x_2^*(b) = 2/3(= 4/6), \ x_2^*(c) = 1/3(= 2/6), \ x_2^*(d) = 0.$$

Hence $x_P^* = (1, 2/3, 1/3, 0)$. Note that $\{a\}$, $\{a,b,c\}$, and $\{a,b,c,d\}(= \mathrm{sat}(x_P^*))$ are tight sets for x_P^*.

Note that if we add dummy goods e and f of coloops, we may have

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1/3 & 1/6 & 0 & 0 & 1/4 & 1/4 \\ 1/3 & 0 & 1/6 & 0 & 1/4 & 1/4 \\ 1/3 & 0 & 1/6 & 0 & 1/4 & 1/4 \\ 0 & 1/3+1/6 & 0 & 0 & 1/4 & 1/4 \end{pmatrix} \end{array},$$

whose every row sum is equal to one.

Example 2: Let us consider Example 1 except that set $\{a,b\}$ is excluded from the family $\mathcal{B}$ of feasible sets given by (3.6), i.e.,

$$\mathcal{B} = \{X \mid X \subset E, \ |X| = 2, \ X \neq \{a,b\}\}. \quad (3.7)$$

This is a graphic matroid, which can be represented by a graph shown in Figure 1.

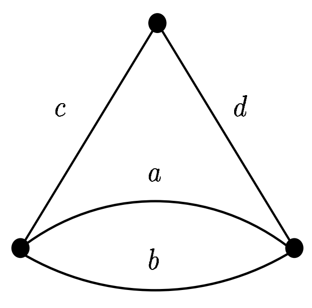

Figure 1: A graph with edge set $E = \{a,b,c,d\}$.

Under the same preferences given in Example 1 we get

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1/4 & 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 \end{pmatrix} \end{array},$$

where

$$b(\mathcal{L}) = (3,1,0,0), \quad S_1 = \{a,b\}, \quad \lambda^* = 1/4 \text{ for } p = 1,$$
$$b(\mathcal{L}) = (0,0,3,1), \quad S_2 = \{a,b,c,d\}, \quad \lambda^* = 1/4 \text{ for } p = 2,$$

and each row sum of P is equal to $1/4 + 1/4 = 1/2$. Also, vectors x_p^* on $T_p = S_p \setminus S_{p-1}$ for $p = 1, 2$ are given by

$$T_1 = \{a,b\}, \quad T_2 = \{c,d\},$$
$$x_1^*(a) = 3/4, \quad x_1^*(b) = 1/4,\ x_2^*(c) = 3/4,\ x_2^*(d) = 1/4.$$

Hence $x_P^* = (3/4, 1/4, 3/4, 1/4)$.

3.2. Ordinal efficiency and envy-freeness

We show that the random assignment matrix P obtained by Random_Assignment is an efficient and envy-free allocation of divisible goods in $\mathrm{B}(\rho)$, where precise definitions of efficiency and envy-freeness will be given below.

3.2.1. Ordinal efficiency

Let P and Q be random assignment matrices for Problem $\mathbf{P} = (N, E, \mathcal{L} = (L^i \mid i \in N), (E, \rho))$. For each agent $i \in N$ with preference relation $\succ_i$ given by $e_1^i \succ_i \cdots \succ_i e_m^i$, define a relation (*sd-dominance* relation[2]) $\succeq_i^{\mathrm{d}}$ between the ith rows P_i and Q_i of P and Q, respectively, as follows.

$$P_i \succeq_i^{\mathrm{d}} Q_i \quad \Longleftrightarrow \quad \forall \ell \in \{1, \cdots, m\}: \ \sum_{k=1}^{\ell} P(i, e_k^i) \geq \sum_{k=1}^{\ell} Q(i, e_k^i). \tag{3.8}$$

The random assignment matrix P is *sd-dominated* by Q if we have $Q_i \succeq_i^{\mathrm{d}} P_i$ for all $i \in N$ and $P \neq Q$. We say that P is *ordinally efficient* if P is not sd-dominated by any other random assignment ([4]).

[2]sd stands for stochastic dominance, which was introduced in [4].

Let us define a (directed) graph $H(P) = (V, A)$ with a vertex set V and an arc set A given by

$$V = \big(\bigcup_{i\in N} E^i\big) \cup E, \tag{3.9}$$

$$A = \big(\bigcup_{i\in N} A^i\big) \cup A_0 \cup A^*, \tag{3.10}$$

where each E^i is a disjoint copy of E, and A^i, A_0, and A^* are defined by

$$A^i = \{(e^i_{k+1}, e^i_k) \mid k = 1, \cdots, m-1\} \qquad (\forall i \in N), \tag{3.11}$$

$$A_0 = \{(e^i, e) \mid i \in N,\ e \in E,\ e^i \in E^i,\ e^i \text{ is a copy of } e\} \cup\{(e, e^i) \mid i \in N,\ e \in E,\ e^i \in E^i,\ e^i \text{ is a copy of } e,\ P(i, e) > 0\}, \tag{3.12}$$

$$A^* = \{(e, e') \mid e, e' \in E,\ e' \in \mathrm{dep}(x^*_P, e) \setminus \{e\}\}. \tag{3.13}$$

(See Figure 2 for a graph $H(P)$, where some of the broken arcs may not exist in A_0.) Here, we identify e^i_k $(k = 1, \cdots, m)$ appearing in (3.11) with the corresponding copies in E^i for each $i \in N$. Also note that dep appearing in (3.13) is what is called the *dependence function* ([8]) associated with the base polytope $\mathrm{B}(\rho)$, and $\mathrm{dep}(x, e)$ for any $x \in \mathrm{B}(\rho)$ and $e \in E$ is given by

$$\mathrm{dep}(x, e) = \{e' \in E \mid \exists \alpha > 0 : x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{B}(\rho)\}. \tag{3.14}$$

We now have the following theorem, which is a generalization of [4, Lemma 3].

Theorem 3.1: *A random assignment P is ordinally efficient if and only if there exists no directed cycle containing at least one arc of $\cup_{i\in N} A^i$ in $H(P) = (V, A)$ defined above.*

(Proof) "Only if' part: Let P be an ordinally efficient random assignment. Suppose to the contrary that there exists a directed cycle in $H(P) = (V, A)$ that contains at least one arc of $\cup_{i\in N} A^i$. Then there exists such a directed cycle C in $H(P)$ that has no short-cut arc taken from A^*. More precisely, let (e^i_{k+1}, e^i_k) be an arc in $C \cap A^i$ for some $i \in N$ and $k \in \{1, \cdots, m-1\}$ and suppose that going from e^i_k to e^i_{k+1} along the direction of C, we find arcs of A^* appearing in the order of $(e_1, e'_1), \cdots, (e_\ell, e'_\ell)$. Then C is chosen so that there is no short-cut arc $(e_p, e'_q) \in A^*$ with $1 \le p < q \le \ell$.

For a sufficiently small $\alpha > 0$ define for each $i \in N$ and $e \in E$

$$P'(i, e) = \begin{cases} P(i, e) + \alpha & \text{if } a = (e^i, e) \in C \cap A_0, \\ P(i, e) - \alpha & \text{if } a = (e, e^i) \in C \cap A_0, \\ P(i, e) & \text{otherwise.} \end{cases} \tag{3.15}$$

We can see from the choice of C that for a sufficiently small $\alpha > 0$, P' is nonnegative and $x^*_{P'}$ is a base in $\mathrm{B}(\rho)$ (see [8, Lemma 4.5]). Hence P' is a random assignment and sd-dominates P, a contradiction.

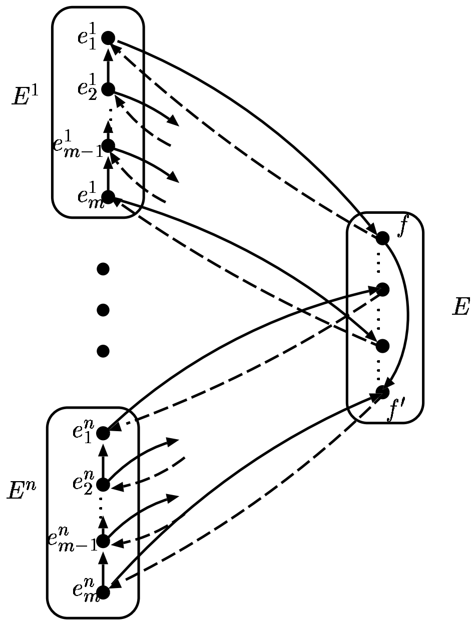

Figure 2: The graph $H(P)$ with $e_1^1 = f$, $e_m^n = f'$, and $f' \in \mathrm{dep}(x_P^*, f)$.

"If" part: Let P and Q be any random assignments such that P is sd-dominated by Q. Modify the graph $H(P) = (V, A)$ by replacing A^* by its subset

$$A^* \leftarrow \{(e, e') \in A^* \mid x_P^*(e) < x_Q^*(e),\ x_P^*(e') > x_Q^*(e')\}, \tag{3.16}$$

where recall that $x_P^*(e) = \sum_{i \in N} P(i, e)$ and $x_Q^*(e) = \sum_{i \in N} Q(i, e)$ for all $e \in E$. We can construct a nonnegative circulation $\varphi : A \to \mathbb{R}_{\geq 0}$ in $H(P)$ that satisfies

$$\varphi(e^i, e) = \max\{0, Q(i, e) - P(i, e)\}, \tag{3.17}$$

$$\varphi(e, e^i) = \max\{0, P(i, e) - Q(i, e)\} \tag{3.18}$$

for all arcs $(e^i, e), (e, e^i) \in A_0$ with $i \in N$ and $e \in E$ with $e^i \in E^i$ being a copy of e and

$$\partial\varphi(v) = 0 \qquad (\forall v \in V) \tag{3.19}$$

as follows. (Here $\partial\varphi(v)$ is the boundary of φ at v and $\partial\varphi(v) = 0$ means the flow conservation at v.) For each $i \in N$ put

$$\varphi(e_{k+1}^i, e_k^i) = \sum_{\ell=1}^{k} \left(Q(i, e_\ell^i) - P(i, e_\ell^i)\right) \geq 0 \qquad (k = 1, \cdots, m-1), \tag{3.20}$$

where the nonnegativity follows from the assumption that Q sd-dominates P. Equations (3.17), (3.18), and (3.20) imply the flow conservation at every $v \in E^i$ for all $i \in N$. Moreover, since x^*_P and x^*_Q are bases in $\mathrm{B}(\rho)$, there exist nonnegative flow values $\varphi(e, e')$ for all $(e, e') \in A^*$ such that for each $e \in E$ with $x^*_Q(e) > x^*_P(e)$

$$x^*_Q(e) - x^*_P(e) = \sum\{\varphi(e, e') \mid e' \in E \text{ with } (e, e') \in A^*\} \tag{3.21}$$

and for each $e' \in E$ with $x^*_Q(e') < x^*_P(e')$

$$x^*_P(e') - x^*_Q(e') = \sum\{\varphi(e, e') \mid e \in E \text{ with } (e, e') \in A^*\} \tag{3.22}$$

(see [8] and [6, Lemma 9]). The flow conservation at every $v \in E$ follows from (3.17), (3.18), (3.21), and (3.22).

Note that the nonnegative circulation φ can be decomposed into positive flows along elementary directed cycles C_k $(k \in K)$ and that at least one flow value $\varphi(a)$ for $a \in \cup_{i \in N} A^i$ defined by (3.20) is positive since $P \neq Q$. Hence there exists a directed cycle C_k in $H(P)$ for some $k \in K$ that contains at least one arc of $\cup_{i \in N} A^i$. □

Moreover, we show the following theorem. While we can show it based on Theorem 3.1, we give another direct proof of it, which may provide us further insight into the ordinal efficiency. We will examine the relationship between Theorem 3.1 and a certain minimum-cost flows in Section 3.4.

Theorem 3.2: *The random assignment matrix P computed by* Random_Assignment *is ordinally efficient.*

(Proof) By Procedure Random_Assignment we get a random assignment P together with a chain $S_0 = \emptyset \subset S_1 \subset \cdots \subset S_p = E$. Let Q be an arbitrary random assignment and suppose that $Q = P$ or Q sd-dominates P. It suffices to prove $Q = P$.

At the qth execution of Step 1 of Random_Assignment define

$$F_q = \{i \in N \mid e^i_1 \in T_q\}. \tag{3.23}$$

Let us denote e^i_1 (the top element in current L^i) at the qth execution of Step 1 by $e^i_1(q)$ and suppose that for some integer $q^* \geq 1$ we have

$$Q(i, e^i_1(q)) = P(i, e^i_1(q)) \qquad (\forall q = 1, \cdots, q^* - 1,\ \forall i \in F_q) \tag{3.24}$$

and we execute the q^*th Step 1. Then, because of Step 1 of Random_Assignment we have

$$\sum_{i \in F_q} P(i, e^i_1(q)) = \rho(S_q) - \rho(S_{q-1}) \qquad (q = 1, \cdots, q^*). \tag{3.25}$$

Since $Q = P$ or Q sd-dominates P, it follows from (3.24) that $Q(i, e_1^i(q^*)) \geq P(i, e_1^i(q^*))$ for all $i \in F_{q^*}$. Hence from (3.24) and (3.25) we must have

$$Q(i, e_1^i(q^*)) = P(i, e_1^i(q^*)) \qquad (\forall i \in F_{q^*}), \tag{3.26}$$

since we have $\sum_{i \in F_{q^*}} Q(i, e_1^i(q^*)) \leq \rho(S_{q^*}) - \rho(S_{q^*-1})$. (Here, $\sum_{q=1}^{q^*} \sum_{i \in F_q} Q(i, e_1^i(q)) \leq \rho(S_{q^*})$.)

Now, note that when $q^* = 1$, (3.24) is void (and thus holds). Hence, by induction on $q = 1, \cdots, p$, we have shown $Q = P$. □

3.2.2. Envy-freeness

We say a random assignment P is *envy-free* with respect to a profile of ordinal preferences $\succ_i$ for all $i \in N$ if for all $i, j \in N$ we have $P_i \succeq_i^{\mathrm{d}} P_j$.

We have the following theorem on envy-freeness. The proof is actually a direct adaptation of the one given by Bogomolnaia and Moulin [4] and Schulman and Vazirani [21] for a non-matroidal problem setting.

Theorem 3.3: *The random assignment matrix P computed by* Random_Assignment *is envy-free.*

(Proof) It suffices to show that for any $i \in N$ and $k \in \{1, \cdots, m\}$ we have

$$\sum_{\ell=1}^{k} P(i, e_\ell^i) \geq \sum_{\ell=1}^{k} P(j, e_\ell^i) \qquad (\forall j \in N). \tag{3.27}$$

Define

$$t_k^i = \sum_{\ell=1}^{k} P(i, e_\ell^i). \tag{3.28}$$

When good e_k^i is removed after an execution of Step 1, all goods e_ℓ^i ($\ell = 1, \cdots, k$) have been removed from E. It follows that for all $j \in N$ the sum of possible values $P(j, e_\ell^i)$ for goods e_ℓ^i ($\ell = 1, \cdots, k$) allocated to agent j is within t_k^i in total. Hence we must have

$$t_k^i \geq \sum_{\ell=1}^{k} P(j, e_\ell^i) \qquad (\forall j \in N). \tag{3.29}$$

□

3.3. A lexicographic characterization of the solution

Recently, Bogomolnaia [2] and Schulman and Vazirani [21] have shown a lexicographic characterization of the solution of the probabilistic serial mechanism for the ordinary random assignment problem. We show that their characterization extends to the random assignment problem considered in the present paper.

For each $i \in N$ and $j = 1, \cdots, m$ define

$$L^i(j) = \{e_1^i, \cdots, e_j^i\}, \tag{3.30}$$

which is the set of top j goods in the preference list L^i of agent i. Also let Q be any $N \times E$ random assignment matrix and define for each $j = 1, \cdots, m$

$$Q(i, L^i(j)) = Q(i, e_1^i) + \cdots + Q(i, e_j^i). \tag{3.31}$$

It should be noted that $(Q(i, L^i(j)) \mid i \in N, j \in \{1, \cdots, m\})$ is obtained by a unimodular transformation of $(Q(i, e_j^i) \mid i \in N, j \in \{1, \cdots, m\})$ defined by (3.31).

Moreover, let us denote by $\mathbf{T}(Q(i, L^i(j)) \mid i \in N, j \in \{1, \cdots, m\})$ the linear arrangement of values $Q(i, L^i(j))$ for all $i \in N$ and $j \in \{1, \cdots, m\}$ in nondecreasing order of magnitude. For simplicity we also write $\mathbf{T}(Q(i, L^i(j)))$ as $\mathbf{T}(Q(i, L^i(j)) \mid i \in N, j \in \{1, \cdots, m\})$ in the sequel.

The lexicographic order among all $\mathbf{T}(Q(i, L^i(j)))$ for all random assignment matrices Q is defined as usual. For any two random assignment matrices P and Q suppose that $\mathbf{T}(P(i, L^i(j))) = (p_1, p_2, \cdots, p_{nm})$ and $\mathbf{T}(Q(i, L^i(j))) = (q_1, q_2, \cdots, q_{nm})$. Then we say $\mathbf{T}(P(i, L^i(j)))$ is lexicographically greater than $\mathbf{T}(Q(i, L^i(j)))$ if there exists an integer $k \in \{1, \cdots, nm\}$ such that $p_i = q_i$ for all $i = 1, \cdots, k-1$ and $p_k > q_k$.

We have the following theorem, whose proof is a direct adaptation of the one for ordinary random assignment problem (with strict preferences) given by Bogomolnaia [2].

Theorem 3.4: *The random assignment matrix P obtained by* Random_Assignment *is the unique lexicographic maximizer of* $\mathbf{T}(Q(i, L^i(j)) \mid i \in N, j \in \{1, \cdots, m\})$ *among all random assignment matrices Q.*

(Proof) Let Q be an arbitrary $N \times E$ random assignment matrix such that $Q \neq P$. It suffices to show that $\mathbf{T}(Q(i, L^i(j)))$ is lexicographically smaller than $\mathbf{T}(P(i, L^i(j)))$.

Since $Q \neq P$, it follows from the procedure Random_Assignment that there exists a pair of $i \in N$ and $\ell \in \{1, \cdots, m\}$ such that $Q(i, L^i(\ell)) < P(i, L^i(\ell))$. Let (i^*, ℓ^*) be one such pair (i, ℓ) having the minimum value of $Q(i, L^i(\ell))$ and define for each $i \in N$

$$\ell^i = \min\{j \in \{1, \cdots, m\} \mid P(i, L^i(j-1)) \leq Q(i^*, L^{i^*}(\ell^*)) < P(i, L^i(j))\}, \tag{3.32}$$

where we define $P(i, L^i(0)) = 0$ and ℓ^i is well defined due to the definition of (i^*, ℓ^*).

Then it follows also from the procedure Random_Assignment that

$$Q(i, L^i(k)) = P(i, L^i(k)) \qquad (\forall i \in N,\ \forall k = 1, \cdots, \ell^i - 1), \tag{3.33}$$

$$Q(i^*, L^{i^*}(\ell^*)) < P(i, L^i(k)) \qquad (\forall i \in N,\ \forall k = \ell^i, \cdots, m). \tag{3.34}$$

This implies that $\mathbf{T}(Q(i, L^i(j)))$ is lexicographically smaller than $\mathbf{T}(P(i, L^i(j)))$. □

3.4. Relation to minimum-cost independent flows

Let us define a graph $\bar{H} = (V, \bar{A})$ with the vertex set V given by (3.9) and the arc set $\bar{A}$ given by

$$\bar{A} = \big(\bigcup_{i\in N} A^i\big) \cup \bar{A}_0, \tag{3.35}$$

where A^i is given by (3.11) and $\bar{A}_0$ is given by

$$\bar{A}_0 = \{(e^i, e) \mid i \in N,\ e \in E,\ e^i \in E^i,\ e^i \text{ is a copy of } e\}. \tag{3.36}$$

Let $\bar{S}^+ = \{e^i_m \mid i \in N\}$ be the set of entrances and $\bar{S}^- = E$ be the set of exits. Moreover, we consider a cost function $\gamma : \bar{A} \to \mathbb{R}$. Denote $\bar{\mathcal{N}} = (\bar{H} = (V, \bar{A}), \bar{S}^+, \bar{S}^-, (E, \rho), \gamma)$. We call a nonnegative flow $\varphi : \bar{A} \to \mathbb{R}_{\geq 0}$ an *independent flow* in $\bar{\mathcal{N}}$ if it satisfies

$$\partial^+\varphi(i) \leq 1 \quad (\forall i \in \bar{S}^+), \qquad \partial^+\varphi(\bar{S}^+) = \rho(E), \qquad \partial^-\varphi \in \mathrm{B}(\rho), \tag{3.37}$$

and

$$\partial\varphi(v) = 0 \qquad (\forall v \in \cup_{i\in N} E^i \setminus \bar{S}^+), \tag{3.38}$$

where $\partial^+\varphi$ is the vector of out-flow values on $\bar{S}^+$ and $\partial^-\varphi$ is the vector of in-flow values on $\bar{S}^-$ for φ. (See Figure 3.) The cost of φ is given by $\sum_{a\in\bar{A}} \gamma(a)\varphi(a)$. (See [6, 8] for independent flows defined in full generality.)

It should be noted that for any random assignment matrix P for Problem $\mathbf{P} = (N, E, \mathcal{L}{=}(L^i \mid i \in N), (E, \rho))$ there uniquely exists an independent flow φ in $\bar{\mathcal{N}}$ such that $\varphi(e^i, e) = P(i, e)$ and conversely, for any independent flow φ in $\bar{\mathcal{N}}$ the $N \times E$ matrix $P = (\varphi(e^i, e) \mid i \in N,\ e \in E)$ is a random assignment. Denote by φ_P the independent flow in $\bar{\mathcal{N}}$ determined by random assignment P. It should also be noted that $\varphi_P(e^i_{k+1}, e^i_k) = P(i, L^i(k))$ for all $i \in N$ and $k = 1, \cdots, m-1$ (see (3.31)).

As a consequence of Theorem 3.1 we have the following.

Theorem 3.5: *Suppose* $|N| = \rho(E)$. *Let* $\gamma : \bar{A} \to \mathbb{R}$ *be an arbitrary cost function satisfying*

$$\gamma(a) < 0 \quad (\forall a \in \bigcup_{i\in N} A^i), \qquad \gamma(a) = 0 \quad (\forall a \in \bar{A}_0). \tag{3.39}$$

If φ *is a minimum-cost independent flow in* $\bar{\mathcal{N}} = (\bar{H} = (V, \bar{A}), \bar{S}^+, \bar{S}^-, (E, \rho), \gamma)$*, then the random assignment matrix* P *with* $P(i, e) = \varphi(e^i, e)$ *for all* $i \in N$ *and* $e \in E$ *is ordinally efficient for Problem* $\mathbf{P} = (N, E, \mathcal{L} = (L^i \mid i \in N), (E, \rho))$.

Moreover, for the solution P *of* Random_Assignment*, the corresponding independent flow* φ_P *is of minimum cost for an appropriate cost function* γ *satisfying* (3.39).

(Proof) Suppose that the independent flow φ has the minimum cost with respect to the

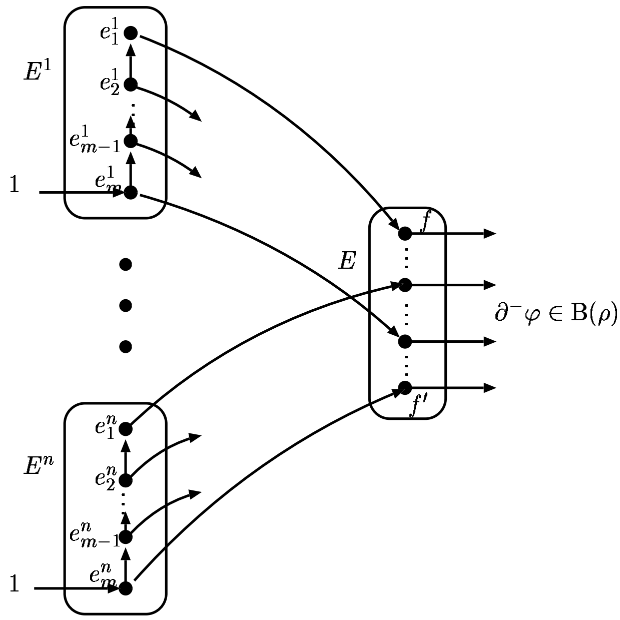

Figure 3: A network $\bar{\mathcal{N}} = (\bar{H}{=}(V, \bar{A}), \bar{S}^+, \bar{S}^-, (E, \rho), \gamma)$ when $|N| = \rho(E)$.

cost function γ. Consider an auxiliary graph $\bar{H}(\varphi) = (V, \bar{A}(\varphi))$ with the arc set

$$\begin{aligned} \bar{A}(\varphi) \;\; = \;\; & \bar{A} \cup \{(e^i_k, e^i_{k+1}) \mid i \in N,\ k \in \{1, \cdots, m-1\},\ \varphi(e^i_{k+1}, e^i_k) > 0\} \\ & \cup \{(e, e^i) \mid e \in E,\ i \in N,\ \varphi(e, e^i) > 0\} \\ & \cup \{(e, e') \mid e, e' \in E,\ e' \in \mathrm{dep}(\partial^- \varphi, e) \setminus \{e\}\}, \end{aligned} \tag{3.40}$$

where $\bar{A}$ is given by (3.35). Also consider a length function $\bar{\gamma}_\varphi : \bar{A}(\varphi) \to \mathbb{R}$ given by

$$\bar{\gamma}_\varphi(a) = \begin{cases} \gamma(a) & \text{if } \ a \in \cup_{i \in N} A^i \\ -\gamma(\tilde{a}) & \text{if } \ \tilde{a} \in \cup_{i \in N} A^i \\ 0 & \text{otherwise} \end{cases} \qquad (\forall a \in \bar{A}(\varphi)), \tag{3.41}$$

where $\tilde{a}$ denotes the reorientation of arc a.

It follows from the optimality of φ that there is no directed cycle of negative length in the auxiliary graph $\bar{H}(\varphi) = (V, \bar{A}(\varphi))$ with length function $\bar{\gamma}_\varphi$ ([8, 6]). Also note that graph $H(P) = (V, A)$ defined in Section 3.2.1 is a subgraph of $\bar{H}(\varphi)$ and has no arcs of positive length of $\bar{H}(\varphi)$ and the set of all arcs of negative length is exactly the set $\cup_{i \in N} A^i$ in $H(P) = (V, A)$. Hence there exists no directed cycle in $H(P) = (V, A)$

containing at least one arc of $\cup_{i\in N} A^i$, which implies that the random assignment matrix $P = (\varphi(e^i, e) \mid i \in N, e \in E)$ is ordinally efficient, due to Theorem 3.1.

Moreover, it should be noted that because of Theorem 3.4, the lexicographic optimality of the solution P of Random_Assignment, the corresponding independent flow φ_P is of minimum cost for an appropriate cost function γ satisfying (3.39). Here recall that $\varphi_P(e^i_{k+1}, e^i_k) = P(i, L^i(k))$. □

It should be noted that Theorem 3.5 does not assert that every ordinally efficient random assignment P gives a minimum-cost independent flow φ_P in $\bar{\mathcal{N}}$ for some cost function γ satisfying (3.39). The non-existence of a negative directed cycle in $\bar{H}(\varphi) = (V, \bar{A}(\varphi))$ with some length function $\bar{\gamma}_\varphi$ seems to be a slightly stronger requirement than the non-existence of a cycle in $H(P) = (V, A)$ containing at least one arc of $\cup_{i\in N} A^i$, for $\varphi = \varphi_P$.

4. Randomized Mechanism

Now consider our original problem of allocating the *indivisible* goods probabilistically. Given the sd-efficient and envy-free random assignment matrix P obtained as the output of the procedure Random_Assignment, we show how to design a lottery that realizes the probability distribution $P_i = (P(i, e) \mid e \in E)$ of getting goods $e \in E$ for each agent $i \in N$. Note that when $|N| > \rho(E)$, every agent i receives no good with probability $1 - \rho(E)/|N|$.

4.1. Random assignments and independent flows

Consider a complete bipartite graph $G = (S^+, S^-; A)$ with a vertex set $V = S^+ \cup S^-$ given by

$$S^+ = N, \qquad S^- = E \tag{4.1}$$

and an arc set A given by

$$A = N \times E. \tag{4.2}$$

For every arc $a \in A$ we consider its capacity $c(a) = +\infty$. The vertex set $S^+ = N$ is the set of entrances and $S^- = E$ is the set of exits. Denote by $\mathcal{N} = (G = (S^+, S^-, A), c, (E, \rho))$ the network with the matroidal constraints on the exit set $S^- = E$ defined as follows. (See Figure 4.)

Consider a nonnegative flow $\varphi : A \to \mathbb{R}_{\geq 0}$ in $\mathcal{N}$ and define $\partial^\pm \varphi : S^\pm \to \mathbb{R}_{\geq 0}$ by

$$\partial^+\varphi(i) = \sum\{\varphi(i, e) \mid e \in E\} \qquad (\forall i \in S^+ = N), \tag{4.3}$$

$$\partial^-\varphi(e) = \sum\{\varphi(i, e) \mid i \in N\} \qquad (\forall e \in S^- = E). \tag{4.4}$$

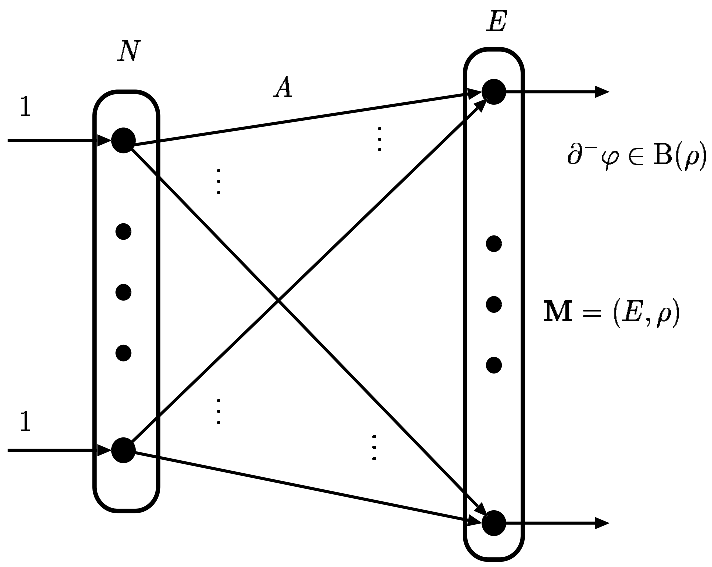

Figure 4: An independent-flow network $\mathcal{N}$.

Given the random assignment matrix P computed by Random_Assignment, put

$$\varphi_P(a) = P(i, e) \qquad (\forall a = (i, e) \in A). \tag{4.5}$$

Then the flow $\varphi_P : A \to \mathbb{R}_{\geq 0}$ is an independent flow in $\mathcal{N}$ and satisfies

$$\partial^- \varphi_P \in \mathrm{B}(\rho), \tag{4.6}$$

$$\partial^+ \varphi_P(i) \leq 1 \quad (\forall i \in N), \tag{4.7}$$

where note that $\partial^- \varphi_P = x^*_P$. Note that if $|N| = \rho(E)$, then (4.7) together with (4.6) implies

$$\partial^+ \varphi_P(i) = 1 \qquad (\forall i \in N). \tag{4.8}$$

Define a polytope P^* by

$$P^* = \{\varphi \mid \varphi \text{ is an independent flow in } \mathcal{N} \text{ satisfying } (4.7)\}. \tag{4.9}$$

It should be noted that by definition any independent flow φ in $\mathcal{N}$ satisfies (4.6) and that P^* is nonempty since $\varphi_P \in P^*$.

The following integrality property holds true for independent flows in $\mathcal{N}$ ([6, 8]), which plays a crucial rôle in our problem setting.

Proposition 4.1: *The polytope P^* defined by* (4.9) *is integral. Every integral independent flow in P^* is $\{0,1\}$-valued and is an extreme point of P^*.*

It should be noted that every integral independent flow in P^* corresponds to a $\{0,1\}$-valued random assignment matrix Q such that every row Q_i $(i \in N)$ of Q has at most one entry being equal to one and the set of columns $e \in E$ with $Q(i,e) = 1$ forms a base $B \in \mathcal{B}$. Denote such base B by B_Q. We call any $\{0,1\}$-valued random assignment matrix an *assignment matrix*.

Because of Proposition 4.1 and the remark given above we have

Theorem 4.2: *For the random assignment matrix P computed by* Random_Assignment *there exist assignment matrices $Q^{(k)}$ $(k \in K)$ and convex combination coefficients ν_k $(k \in K)$ such that*

$$P = \sum_{k \in K} \nu_k Q^{(k)} \tag{4.10}$$

and for each $k \in K$ the set $B_{Q^{(k)}} \subseteq E$ of non-zero column indices $e \in E$ of $Q^{(k)}$ is a base in $\mathcal{B}$.

It is worth mentioning that Theorem 4.2 is a generalization of the Birkhoff-von Neumann theorem on bi-stochastic matrices, even if the matroid $(E, \mathcal{B})$ is a uniform matroid. When $|N| = |E|$ and E is the unique base, i.e., $(E, \mathcal{B})$ is a free matroid, Theorem 4.2 becomes the Birkhoff-von Neumann theorem on bi-stochastic matrices.

The random assignment matrix P appearing in Example 1 in Section 3.1 can be expressed, for example, as

$$\begin{aligned} P \;=\; & \frac{1}{6}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ & + \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.11}$$

Here note that $|E| = 4 > 2 = \rho(E)$.

Based on the expression (4.10), we construct a randomized mechanism that gives P as probability distributions on the set E of goods assigned to agents.

Randomized_Mechanism

1. Let $B_{Q^{(k)}}$ be the base in $\mathcal{B}$ given by $Q^{(k)}$ for each $k \in K$ appearing in (4.10).
2. Generate a base from among bases $B_{Q^{(k)}}$ $(k \in K)$ according to the probability distribution ν_k $(k \in K)$ and let k^* be the generated index in K.
3. Assign goods in base $B_{Q^{(k^*)}}$ to agents in N according to assignment matrix $Q^{(k^*)}$.

Consequently, we have

Theorem 4.3: *For the random assignment matrix P computed by* Random_Assignment *each row P_i of P gives the probability distribution of receiving goods $e \in E$ for each agent $i \in N$ according to the randomized procedure* Randomized_Mechanism.

It should be noted that the expression (4.10) for P is not unique but this does not affect the probability distribution P_i of P for each $i \in N$ realized by Randomized_Mechanism.

In the following we show how to efficiently compute an expression (4.10).

4.2. Computing the probability distribution

For the random assignment matrix P (or independent flow φ_P) and base $x_P^* \in \mathrm{B}(\rho)$ computed by Random_Assignment we first consider the unique minimal face of P^* containing φ_P.

Denote by $\mathcal{D}(x_P^*)$ the set of all tight sets for x_P^* in $\mathrm{B}(\rho)$, where $\mathcal{D}(x_P^*)$ is closed with respect to the binary operations of set union and intersection and is a distributive lattice (see [8]). Let a maximal chain of $\mathcal{D}(x_P^*)$ be given by

$$\mathcal{C}(x_P^*) : \hat{S}_0 = \emptyset \subset \cdots \subset \hat{S}_p = E. \tag{4.12}$$

The chain of tight sets obtained during the execution of Random_Assignment is a subchain of (4.12). A maximal chain $\mathcal{C}(x_P^*)$ is determined by the dependence structure associated with $\mathrm{dep}(x_P^*, e)$ for all $e \in E$ and can be computed in strongly polynomial time ([8]). (Construct a directed graph $G(x_P^*)$ with vertex set E and the set of arcs (e, f) for all $e \in E$ and $f \in \mathrm{dep}(x_P^*, e) \setminus \{e\}$. The sets $\mathrm{dep}(x_P^*, e)$ $(e \in E)$ can be computed by using any strongly polynomial submodular function minimization algorithm such as [12, 20, 17]. Then $\mathcal{C}(x_P^*)$ is a maximal chain of subsets of E that have no leaving arcs in $G(x_P^*)$.)

For each $q = 1, \cdots, p$ consider the minor, denoted by $\mathbf{M}_q$, of matroid (E, ρ) obtained by its restriction to $\hat{S}_q$ followed by the contraction of $\hat{S}_{q-1}$. The minor $\mathbf{M}_q$ is the matroid on $T_q \equiv \hat{S}_q \setminus \hat{S}_{q-1}$ with the rank function ρ_q given by

$$\rho_q(X) = \rho(X \cup \hat{S}_{q-1}) - \rho(\hat{S}_{q-1}) \qquad (\forall X \subseteq T_q). \tag{4.13}$$

Also denote by x^*_q the restriction of x^*_P to $T_q(= \hat{S}_q \setminus \hat{S}_{q-1})$. Then x^*_q is a base of (T_q, ρ_q) regarded as a polymatroid. In other words, $x^*_q \in \mathrm{B}(\rho_q)$. Note that x^*_P is a base of the direct sum $\oplus_{q=1}^{p}\mathbf{M}_q$ of minors $\mathbf{M}_q$ $(q = 1, \cdots, p)$. Let $\hat{\rho}$ be the rank function of matroid $\oplus_{q=1}^{p}\mathbf{M}_q$. It should be noted that because of the maximality of chain $\mathcal{C}(x^*_P)$, for each $q = 1, \cdots, p$ the base polytope $\mathrm{B}(\rho_q)$ is full dimensional and base x^*_q is within the interior of $\mathrm{B}(\rho_q)$ and that x^*_P is within the relative interior of the base polytope $\mathrm{B}(\hat{\rho})$ of $\oplus_{q=1}^{p}\mathbf{M}_q$, which is the unique minimal face of $\mathrm{B}(\rho)$ containing x^*_P. (See [8, Chapter II].)

Put

$$\hat{A}^0 = \{a \in A \mid \varphi_P(a) = 0\}, \tag{4.14}$$

$$\hat{A}^+ = A \setminus \hat{A}^0, \tag{4.15}$$

$$\hat{I} = \{i \in N \mid \partial^+\varphi_P(i) = 1\}. \tag{4.16}$$

Then, define a face of P^* containing φ_P by

$$P^*(\varphi_P) = \{\varphi \in P^* \mid \forall i \in \hat{I} : \partial^+\varphi(i) = 1,\ \forall a \in \hat{A}^0 : \varphi(a) = 0,\ \partial^-\varphi \in \mathrm{B}(\hat{\rho})\}. \tag{4.17}$$

Lemma 4.4: *The polytope $P^*(\varphi_P)$ is the unique minimal face of P^* containing φ_P. Moreover, $P^*(\varphi_P)$ restricted in $\mathbb{R}^{\hat{A}^+}$ is the set of independent flows satisfying* (4.7) *in the network $\hat{\mathcal{N}} = (\hat{G} = (S^+, S^-; \hat{A}^+), c, (E, \hat{\rho}))$.*

(Proof) In the system of inequalities (and equations) that defines P^* of (4.9), the given φ_P satisfies $\partial^+\varphi_P(i) = 1$ for all $i \in \hat{I}$, $\varphi_P(a) = 0$ for all $a \in \hat{A}^0$, and

$$\partial^-\varphi_P(X) = \rho(X) \qquad (\forall X \in \mathcal{D}(x^*_P)), \tag{4.18}$$

which includes all the inequalities for P^* satisfied with equality by φ_P. Note that (4.18) is implied by

$$\partial^-\varphi_P(X) = \rho(X) \qquad (\forall X \in \mathcal{C}(x^*_P)), \tag{4.19}$$

since $\mathcal{D}(x^*_P)$ is a distributive lattice and ρ is modular on $\mathcal{D}(x^*_P)$. Also note that the system of equations (4.19) together with $\partial^-\varphi_P \in \mathrm{B}(\rho)$ is equivalent to $\partial^-\varphi_P \in \mathrm{B}(\hat{\rho})$. Hence (4.17) defines the unique minimal face of P^* containing φ_P.

Moreover, the latter statement holds true because of the definition of the network $\hat{\mathcal{N}}$.

□

We begin with base $x^*_1 \equiv x^*_P \in \mathrm{B}(\hat{\rho})$ and independent flow $\hat{\varphi}_1 \equiv$ (the restriction of φ_P on $\hat{A}^+$) in network $\hat{\mathcal{N}} = (\hat{G} = (S^+, S^-; \hat{A}^+), c, (E, \hat{\rho}))$ such that $x^*_1 = \partial^- \hat{\varphi}_1$. If φ_P is already $\{0,1\}$-valued, we are done. Hence we assume that φ_P is not $\{0,1\}$-valued. Perform the following procedure.

---

1. Put $t \leftarrow 1$.
2. Find a $\{0,1\}$-valued independent flow φ_t in $\hat{\mathcal{N}}$.
3. Compute
$$\beta^*_t = \max\{\beta > 0 \mid \hat{\varphi}_t + \beta(\hat{\varphi}_t - \varphi_t) \in P^*(\hat{\varphi}_t)\}. \tag{4.20}$$
4. Put $\hat{\varphi}_{t+1} \leftarrow \hat{\varphi}_t + \beta^*_t(\hat{\varphi}_t - \varphi_t)$ and $x^*_{t+1} \leftarrow x^*_t + \beta^*_t(x^*_t - \partial^- \varphi_t)$.
5. If flow $\hat{\varphi}_{t+1}$ is not $\{0,1\}$-valued, then put $t \leftarrow t+1$, update $\hat{\mathcal{N}}$ for the current base x^*_t and flow $\hat{\varphi}_t$, and go to Step 2.
   Otherwise put $\varphi_{t+1} \leftarrow \hat{\varphi}_{t+1}$.
   Return φ_s for all $s = 1, \cdots, t+1$ and β^*_s for all $s = 1, \cdots, t$.

During the execution of the above procedure $P^*(\hat{\varphi}_t)$ appearing in (4.20) is the unique minimal face of P^* containing $\hat{\varphi}_t$, due to Lemma 4.4. At the tth execution of Step 3 with current rank function $\hat{\rho}$ we have the unique minimal face $\mathrm{B}(\hat{\rho})$ of $\mathrm{B}(\rho)$ containing x^*_t. Then β^*_t in (4.20) is the maximum value of β that satisfies

$$\partial^+ \hat{\varphi}_t(i) \le 1 \qquad (\forall i \in N), \tag{4.21}$$
$$\hat{\varphi}_t(a) + \beta(\hat{\varphi}_t(a) - \varphi_t(a)) \ge 0 \qquad (\forall a \in \hat{A}^+_t), \tag{4.22}$$
$$x^*_t + \beta(x^*_t - \partial^- \varphi_t) \in \mathrm{B}(\hat{\rho}), \tag{4.23}$$

where $\hat{A}^+_t = \{a \in A \mid \hat{\varphi}_t(a) > 0\}$. Note that since $\hat{\varphi}_t$ is within the relative interior of $P^*(\hat{\varphi}_t)$, we get $\beta^*_t > 0$. We can compute β^*_t in strongly polynomial time (see [16]). Also note that the final value of t is $\mathrm{O}(|N||E|)$ since every execution of Step 3 and Step 4 makes at least one strict inequality in (4.21) or (4.22) hold with equality or makes the length of a maximal chain $\mathcal{C}(x^*_{t+1})$ greater than that of $\mathcal{C}(x^*_t)$.

We regard each φ_s $(s = 1, \cdots, t+1)$ as a flow in the original network $\mathcal{N}$ by putting $\varphi_s(a) = 0$ for all $a \in A \setminus \hat{A}^+_s$, and similarly for $\hat{\varphi}_s$ $(s = 1, \cdots, t+1)$. From the output φ_s for all $s = 1, \cdots, t+1$ and β^*_s for all $s = 1, \cdots, t$ we have

$$\hat{\varphi}_{s+1} = (1 + \beta^*_s)\hat{\varphi}_s - \beta^*_s \varphi_s \qquad (\forall s = 1, \cdots t), \tag{4.24}$$

or

$$\hat{\varphi}_s = (1 + \beta^*_s)^{-1}(\hat{\varphi}_{s+1} + \beta^*_s \varphi_s) \qquad (\forall s = 1, \cdots t). \tag{4.25}$$

Eliminating $\hat{\varphi}_s$ for $s = 1, \cdots, t$ and using $\hat{\varphi}_{t+1} = \varphi_{t+1}$, we can obtain the following expression.

$$\varphi_P (= \hat{\varphi}_1) = \sum_{s=1}^{t+1} \nu_s \varphi_s \tag{4.26}$$

for some convex combination coefficients ν_s $(s = 1, \cdots, t+1)$. Each $\{0, 1\}$-valued flow φ_s gives a desired assignment matrix $Q^{(s)}$, and ν_s $(s = 1, \cdots, t+1)$ the desired probability distribution on the set of assignment matrices $Q^{(s)}$ $(s = 1, \cdots, t+1)$. Note that (4.26) is equivalent to

$$P = \sum_{s=1}^{t+1} \nu_s Q^{(s)}, \tag{4.27}$$

which thus can be computed in strongly polynomial time.

5. Concluding Remarks

We have shown that the probabilistic serial mechanism of Bogomolnaia and Moulin [4] can be extended to the case where we are given a family of bases of a matroid as a family of available sets of goods. The present paper opens many research subjects on a new class of random assignment problems with additional constraints given by combinatorial submodularity structures such as matroids and polymatroids. We should further investigate extensions of those results which have been established till now for the ordinary random assignment problem in order to gain further insights into the random assignment problem in full generality.

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