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In this article, we give an alternative description of the 2–loop term of the $SU(2)$–Chern–Simons perturbation theory by using the technique developed in [11]. As an application, we give a Morse theoretic description of the 2–loop term of the $SU(2)$–Chern–Simons perturbation theory at a non-trivial connection.

1 Introduction

The G–Chern–Simons perturbation theory developed by M. Kontsevich in [5] and S. Axelrod and M. I. Singer in [1] gives a topological invariant of a closed oriented 3–manifold with a flat connection of the flat G-bundle over given 3–manifold, where G is an appropriate semi-simple Lie group. The Chern–Simons perturbation theory at the trivial connection was well studied by G. Kuperberg and D. Thurston in [6], C. Lescop in [7], D. Moussard in [10] and C. Taubes in [12]. In particular, Kuperberg and Thurston in [6] and Lescop in [7] showed that the Chern–Simons perturbation theory at the trivial connection gives a universal finite type invariant of integral homology 3–spheres. In 1999, R. Bott and A. S. Cattaneo gave a purely topological construction of the Chern-Simons perturbation theory at a non-trivial connection in [2].

In 1996, K. Fukaya constructed the Morse homotpy invariant by using Morse functions in [3]. The Morse homotopy invariant is an invariant of a closed oriented 3-manifold with two different flat connections of the trivial G bundle on given 3-manifold. Fukaya conjectured that the Morse homotopy invariant coincides with the 2–loop term of the Chern–Simons perturbation theory in some sense. M. Futaki in [4] pointed out that Fukaya's invariant depends on the choice of Morse functions by giving an explicit example.

In 2012, T. Watanabe constructed the Morse homotopy invariant for the trivial connection in [13]. This is an invariant of rational homology 3-spheres. We showed that Watanabe's invariant coincides with the Chern–Simons perturbation theory at the trivial connection in [11].

In this article, we give a generalized description of the 2–loop term of the $SU(2)$–Chern–Simons perturbation theory at a non-trivial flat connection by using the technique developed in [11]. A flat conncection gives a local system on given 3–manifold via the holonomy representation. The $SU(2)$–Chern–Simons perturbation theory gives an invariant of given 3–manifold with the local system. As an application of our generalized construction, we give the Morse homotopy for the 2–loop term of the $SU(2)$–Chern–Simons perturbation theory at a non-trivial connection.

A propagator plays an important role in the construction of the Chern-Simons perturbation theory. A propagator used in our construction is slightly different from that in [2]. There is an obstruction to existence of a propagator. We show that this obstruction is vanishing in $SU(2)$–Chern–Simons perturbation theory.

The invariant given by the Chern-Simons perturbation theory is the sum of the principal term and the correction term. Both terms are invariants of a flat connection on a 3–manifold with an extra information on the 3-manifold. In the Bott and Cattaneo's construction, they used a framing of given 3–manifold as an extra infromation. In this article, we use three 3–cycles of the unit sphere bundle of the tangent bundle of given 3-manifold. A framing gives a three linearly independent unit vector fields and each vector field gives a 3-cycle of the unit sphere bundle of the tangent bundle of the 3-manifold. In this meaning, our construction is a generalization of Bott and Cattaneo's construction.

Lescop defined an invariant of 3-manifold with Betti number 1 in [8]. In the construction of Lescop's invariant, she used a similar technique in the construction of the Chern–Simons perturbation theory. Watanabe constructed an invariant of 3-manifold with Betti number 1 by using his Morse homotopy technique in [14]. It is expected that the Chern-Simons perturbation theory at a non-trivial connection is related to Lescop' invariant and Watanabe's invariant in some sense.

The organization of this paper is as follows. In Section 2 we introduce some notations. In Section 3 we give a generalized construction of the 2–loop term of the Bott and Cattaneo's Chern–Simons perturbation theory. In Section 4 we give a Morse homotopy for the 2–loop term of the Chern–Simons perturbation theory as an application of the construction given in Section 3. In Section 5 we prove Theorem 4.7 stated in Section 4.

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2 Homology with local coefficients

In this section, we prepare some notations about homology with local coefficients.

2.1 Chains with local coefficients

We first prepare the notation about the chains or cycles with local coefficients. Let X be a manifold and E be a local system on M. Let $f : (A,a) \to (M,f(a))$ be a continuous map from a contractible compact k–dimensional manifold A with a base point $a \in A$ to X. Since A is contractible, the map f with $e \in E_{f(a)}$ gives a k–chain in $C_k(X;E)$ via an appropriate triangulation of A. We denote by $\langle A,a;e\rangle$ or $\langle f : A \to M,a;e\rangle$ this chain.

More generally, a continuous map $g : (B,b) \to (M,g(b))$ from a compact k dimensional manifold with a base point $b \in B$ to X with $e \in (E_{g(b)})^{g_*\pi_1(B,b)}$ gives a k–chain $\langle B,b;e\rangle$. Here $(E_{g(b)})^{g_*\pi_1(B,b)}$ is the invariant part of $E_{g(B)}$ under the action of $g_*\pi_1(B,b) < \pi_1(M,g(b))$.

For a chain $C = \langle f : A \to M,a;e\rangle$, *the support* $\mathrm{Supp}(C)$ of C is defined by $\mathrm{Supp}(C) = f(A)$.

Let $\pi : X \to M$ be a fiber bundle such that the typical fiber F is a compact oriented manifold. Let $c = \langle f : A \to M,a;e\rangle \in C_k(M;E)$ be a k–chain of M. Then

$$\pi^! c = \langle \widetilde{f} : f^*X \to X, v_a;e\rangle \in C_{k+\dim F}(X;\pi^*E)$$

is a $(k+\dim F)$–chain of X, where $\widetilde{f} : f^*X \to X$ is a bundle map induced by $f : A \to M$ and $v_a \in \pi^{-1}(a)$ is any point.

2.2 Intersection of chains

Let X be a compact n dimensional manifold with a boundary and E is an oriented local system on X. Let $c_1,\dots,c_k$ be singular chains of X such that:

(A) The degree of c_i is n_i: $c_i \in C_{n_i}(X;E)$ for each i,

(B) $\mathrm{Supp}(\partial c_i) \subset \partial X$ and there are open submanifolds $U_1, \dots, U_k$ of ∂X satisfying $\mathrm{Supp}(\partial c_i) \subset U_i$ and $\bigcap_i U_i = \emptyset$.

Here U is an *open submanifold* of ∂X if U is a $(\dim X - 1)$–dimensional submanifold and the boundary $\partial\overline{U}$ is a submanifold of ∂X, where $\overline{U}$ is a closure of U. Each c_i gives a homology class $[c_i] \in H_{n_i}(X, \overline{U_i}; E)$. We denote by $[c_i]_{P.D.} \in H^{n-n_i}(X, \partial X - U_i; E)$ the Poincaré dual of $[c_i]$. The cup product $[c_1]_{P.D.} \cup \ldots \cup [c_k]_{P.D.}$ is in

$$H^{\sum_i (n-n_i)}(X, \bigcup_i (\partial X - U_i); E^{\otimes k}).$$

Thanks to the assumption (B), $\bigcup_i (\partial X - U_i) = \partial X$. Then $[c_1]_{P.D.} \cup \ldots \cup [c_k]_{P.D.} \in H^{\sum_i (n-n_i)}(X, \partial X; E^{\otimes k})$.

Definition 2.1 The *intersection* $\bigcap_i c_i$ of $c_1, \dots, c_k$ is the homology class of degree $n - \sum_i (n - n_i)$ of X given by

$$\bigcap_i c_i = ([c_1]_{P.D.} \cup \ldots \cup [c_k]_{P.D.})_{P.D.} \in H_{n-\sum_i (n-n_i)}(X; E^{\otimes k}).$$

More generally, the *intersection* $\bigcap_i c_i$ is also defined for c_i with $\partial c_i \not\subset \partial X$. Let $c_1, \dots, c_k$ be the chains satisfying the following conditions:

(A) The degree of c_i is n_i: $c_i \in C_{n_i}(X; E)$ for any i,

(B) There are open submanifold $V_1, \dots, V_n \subset X$ and open submanifold $U_1, \dots, U_n \subset \partial X$ such that $\mathrm{Supp}(\partial c_i) \subset C_i \sqcup U_i$ and $\bigcap_i V_i = \bigcap_i U_i = \emptyset$.

In this case, each c_i gives a homology class in $H_{n_i}(M \setminus (\bigcup_i V_i), \overline{U_i} \cup \partial\overline{V_i}; E)$. Then we define $\bigcap_i c_i = ([c_1]_{P.D.} \cup \ldots \cup [c_k]_{P.D.})_{P.D.} \in H_{n-\sum_i (n-n_i)}(X \setminus (\cup_i V_i); E^{\otimes k})$.

Remark 2.2 We remark that if $c_1, \dots, c_k$ transversally intersect, the homology class $\bigcup_i c_i$ is represented by the geometric intersection of $c_1, \dots, c_k$.

3 An alternative description of the SU(2)–Chern–Simons perturbation theory

In this section we give a generalized construction of the 2–loop term of the Chern–Simons perturbation theory. This construction is a generalization of the construction given by Bott and Cattaneo in [2]. Let M be an oriented closed 3–manifold and G be a simply connected Lie group (After Theorem 3.13, we only consider the case of $G = SU(2)$). Let $\rho : \pi_1(M) \to G$ be a representation of $\pi_1(M)$. We consider a local

system on M as a covariant functor from the fundamental groupoid to a category of finite dimensional vector spaces. Let E be the local system on M corresponding to the representation $\mathrm{Ad}\circ\rho : \pi_1(M) \to \mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G and $Ad : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation of G. We denote by E_x the object corresponding to $x \in M$, γ_* the morphism corresponding to a path γ in M via the functor E. We assume that E is acyclic, namely $H_k(M;E) = 0$ for any $k \in \mathbb{Z}$.

We denote by $\mathbb{R}$ the local system corresponding to the rank one trivial representation. The natural transformation map

$$c : \mathbb{R} \to E \otimes E$$

is given by the following: Since E is corresponding to an orthonormal representation, there is a natural isomorphism $E_x \otimes E_x \cong E_x \otimes E_x^*$ for any $x \in M$ where E_x^* is the dual space of E_x. The evaluation map ev is in $E_x \otimes E_x^*$. Then we set $c(1) = \mathrm{ev}$.

Let us denote by $B\ell(M^2,\Delta)$ the manifold with corner obtained by the real blowing up of M^2 along $\Delta = \{(x,x) \mid x \in M\}$. We denote by $q : B\ell(M^2,\Delta) \to M^2$ the blow down map. Then $F = q^*(p_1^*E \otimes p_2^*E)$ is a local system on $B\ell(M^2,\Delta)$, where $p_1 : M^2 \to M$ and $p_2 : M^2 \to M$ are the projections. We remark that $F|_{q^{-1}(\Delta)} = q^*(E \otimes E)$. Let $T : B\ell(M^2,\Delta) \to B\ell(M^2,\Delta)$ be an involution induced by $T_0 : M^2 \to M^2, (x,y) \mapsto (y,x)$. We denote by $H_3^+(M^2;F), H_3^-(M^2;F)$ the $+1$ eigen space and -1 eigen space of the induced map $T_* : H_3(M^2;F) \to H_3(M^2;F)$ respectively.

There is a unique homology class $d(E) \in H_3^+(q^{-1}(\Delta);F)$ corresponding to $c_*([\Delta]) \in H_3(\Delta;F)$ under the following diagram, where $[\Delta] \in H_3(\Delta;\mathbb{R})$ is the fundamental homology class.

$$\begin{array}{ccc} H_4^+(B\ell(M^2,\Delta),\partial B\ell(M^2\Delta);F) & \xrightarrow{\cong} & H_4^+(M^2,\Delta;E\otimes E) \\ \downarrow\partial & \circlearrowleft & \cong\downarrow\partial \\ H_3^+(q^{-1}(\Delta);F) & \longrightarrow & H_3^+(\Delta;E\otimes E) \end{array}$$

The left vertical line is a part of the long exact sequence of the pair $(B\ell(M^2,\Delta),\partial B\ell(M^2,\Delta))$ and the right vertical line is a part of the long exact sequence of the pair (M^2,Δ). The top horizontal isomorphism is the excision isomorphism.

Let $s : \Delta \to q^{-1}(\Delta)$ be a section. Then $[s(\Delta) \cup T(s(\Delta))]$ is a homology class in $H_3^+(q^{-1}(\Delta);\mathbb{R})$.

Lemma 3.1 *The homology class $[s(\Delta) \cup T(s(\Delta))]$ is independent of the choice of a section s.*

Proof We show that $[s(\Delta) \cup T(s(\Delta))] \cap \alpha \in H_0(q^{-1}(\Delta);\mathbb{R})$ is independent of the choice of s for any $\alpha \in H_2(q^{-1}(\Delta);\mathbb{R})$. Thanks to the Poincaré duality, this implies that the homology class $[s(\Delta) \cup T(s(\Delta))]$ is independent of the choice of s.

Since $T : q^{-1}(\Delta) \to q^{-1}(\Delta)$ reverses the orientation, for any $a \in H_3^+(q^{-1}(\Delta);\mathbb{R})$ and $b \in H_2^+(q^{-1}(\Delta);\mathbb{R})$, $a \cap b = (a_{P.D.} \cup b_{P.D.})_{P.D.} \in H_0^-(q^{-1}(\Delta);\mathbb{R}) = 0$.

Since $H_2^-(\Delta;\mathbb{R}) = H_0^-(\Delta;\mathbb{R}) = 0$ and $H_0^-(S^2;\mathbb{R}) = 0$, we have $H_2^-(q^{-1}(\Delta);\mathbb{R}) \cong H_0^+(\Delta;\mathbb{R}) \times H_2^-(S^2;\mathbb{R})$. Then $H_2^-(q^{-1}(\Delta);\mathbb{R})$ is generated by $[q^{-1}(x)]$ for a point $x \in \Delta$. It is clear that $[s(\Delta) \cup T(s(\Delta))] \cap [q^{-1}(x)] = [\{s(x), T(s(x))\}] = 2 \in \mathbb{R} \cong H_0(q^{-1}(\Delta);\mathbb{R})$. This is obviously independent of the choice of s. □

Definition 3.2 $o(E) = q_*(d(E) \cap [s(\Delta) \cup T(s(\Delta))]) \in H_1(\Delta; E \otimes E)$.

Remark 3.3 We will show that $o(E)$ is an obstruction to the exsistence of a propagator in Lemma 3.7.

Lemma 3.4 *If $G = SU(2)$, then $H_1^-(\Delta; E \otimes E) = 0$. In particular $o(E) = 0$.*

Proof Since $T_0|_\Delta = \mathrm{id}$, $H_1^-(\Delta; E \otimes E) = H_1(\Delta; (E \otimes E)^-)$. The Lie bracket $[\cdot,\cdot]$ of $\mathfrak{su}_2$ induces a natural transformation map $b : E \otimes E \to E, f(x \otimes y) = [x, y]$. For each $(x, x) \in \Delta$, $b : E_x \otimes E_x \to E_x$ is subjective because $\mathfrak{su}_2$ is semi-simple. Then $\dim(\ker b) = 6$. Since $b(T(x \otimes y)) = -b(x \otimes y)$, $(E_x \otimes E_x)^-$ is a subspace of $\ker b$. On the other hand $\dim((E_x \otimes E_x)^-) = 6$. Therefore $(E \otimes E)^- \cong E$. Then $H_1(\Delta; (E \otimes E)^-) = H_1(\Delta; E) = 0$. □

We now define a propagator which plays an important role in the construction of the invariant.

Definition 3.5 (propagator) A $4-$cycle $\Sigma \in C_4(B\ell(M^2, \Delta), \partial B\ell(M^2, \Delta); F)$ is said to be a *propagator* if there is a cycle $\Sigma_{\mathbb{R}}^{\partial} \in C_3^+(q^{-1}(\Delta);\mathbb{R})$ such that the following conditons hold:

(1) $T_*\Sigma = \Sigma$,

(2) $q_*[\Sigma_{\mathbb{R}}^{\partial}] = [\Delta] \in H_3^+(\Delta;\mathbb{R})$ and

(3) $\partial\Sigma = c_*(\Sigma_{\mathbb{R}}^{\partial})$.

Remark 3.6 Since $H_1^+(\Delta;\mathbb{R}) = 0$, we have $H_3^+(q^{-1}(\Delta);\mathbb{R}) \cong H_3(\Delta;\mathbb{R}) \times H_0(S^2;\mathbb{R}) \cong H_3(\Delta;\mathbb{R})$. From the condition (2) of Definition 3.5, the homology class $[\Sigma_{\mathbb{R}}^{\partial}] \in H_3(q^{-1}(\Delta);\mathbb{R})$ is independent from $\Sigma_{\mathbb{R}}^{\partial}$ and Σ.

Lemma 3.7 *If $o(E) = 0$, there exist propagators.*

Proof Let $s : \Delta \to q^{-1}(\Delta)$ be a section. The cycle $\frac{1}{2}\{s(\Delta), T(s(\Delta))\} \in C_3^+(q^{-1}(\Delta); \mathbb{R})$ satisfies the condition (2) of Definition 3.5. Set $\Sigma_{\mathbb{R}}^{\partial} = \frac{1}{2}\{s(\Delta), T(s(\Delta))\}$. By the definition of $d(E)$,

$$d(E) - c_*[\Sigma_{\mathbb{R}}^{\partial}] \in \ker(q_* : H_3^+(\partial B\ell(M^2, \Delta); F) \to H_3^+(\Delta; E \otimes E)).$$

We fix a trivialization $q^{-1}(\Delta) \cong \Delta \times S^2$. Under the trivialization $q^{-1}(\Delta) \cong \Delta \times S^2$,

$$\ker q_* \cong H_1^-(\Delta; E \otimes E) \times H_2(S^2; \mathbb{R}) \overset{\varphi}{\cong} H_1^-(\Delta; E \otimes E).$$

For any $a \in H_1^-(\Delta; E \otimes E)$, $q_*((a \times [S^2]) \cap \Sigma_{\mathbb{R}}^{\partial}) = a$. Then the last isomorphism φ is given by $\varphi(\cdot) = q_*(\cdot \cap \Sigma_{\mathbb{R}}^{\partial})$. On the other hand $q_*(c_*[\Sigma_{\mathbb{R}}^{\partial}] \cap [\Sigma_{\mathbb{R}}^{\partial}]) = 0$. Therefore $0 = o(E) - q_*(c_*[\Sigma_{\mathbb{R}}^{\partial}] \cap [\Sigma_{\mathbb{R}}^{\partial}]) = \varphi^{-1}(d(E) - c_*[\Sigma_{\mathbb{R}}^{\partial}])$. Thus we have $d(E) = c_*[\Sigma_{\mathbb{R}}^{\partial}]$. This implies that

$$c_*[\Sigma_{\mathbb{R}}^{\partial}] \in \mathrm{Im}(H_4(B\ell(M^2, \Delta), \partial B\ell(M^2, \Delta); F) \to H_3(\partial B\ell(M^2, \Delta); F)).$$

So we can take $\Sigma_0 \in C_4(B\ell(M^2, \Delta); F)$ satisfying $\partial\Sigma_0 = c_*\Sigma_{\mathbb{R}}^{\partial}$. Then $\Sigma = \frac{1}{2}(\Sigma_0 + T(\Sigma_0)) \in C_4^+(B\ell(M^2, \Delta), \partial B\ell(M^2, \Delta); F)$ is a propagator. □

We take a triple of propagators $(\Sigma_1, \Sigma_2, \Sigma_3)$ satisfying the following conditions:

- There are submanifolds N_1, N_2 and N_3 of $q^{-1}(\Delta)$ such that $N_i \supset \mathrm{Supp}(\partial\Sigma_i)$ for $i = 1, 2, 3$,
- $N_1 \cap N_2 \cap N_3 = \emptyset$.

We will call such a triple *an admissible triple of propagators*. Under these conditions, we can apply the intersection theory to $\Sigma_1, \Sigma_2, \Sigma_3$ as in the usual homology theory. (More precisely, we consider the Poincaré dual of each propagator. See Section 2.2)

Lemma 3.8 *Let $(\Sigma_1, \Sigma_2, \Sigma_3)$ and $(\Sigma_1', \Sigma_2', \Sigma_3')$ be admissible triples of propagators such that $\partial\Sigma_i = \partial\Sigma_i'$ for any $i = 1, 2, 3$ and $[\Sigma_1 - \Sigma_1'] = 0 \in H_4^+(B\ell(M^2, \Delta); F)$. Then $\mathrm{Tr}^{\otimes 2}(\Sigma_1 \cap \Sigma_2 \cap \Sigma_3) = \mathrm{Tr}^{\otimes 2}(\Sigma_1' \cap \Sigma_2' \cap \Sigma_3') \in \mathbb{R} \cong H_0(B\ell(M^2, \Delta), \mathbb{R})$.*

Proof It is enough to show that $\mathrm{Tr}^{\otimes 2}(\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 - \Sigma_1' \cap \Sigma_2 \cap \Sigma_3) = 0$. Since $[\Sigma_2 \cap \Sigma_3] \in H_2(B\ell(M^2, \Delta), \partial B\ell(M^2, \Delta); F)$ and $[\Sigma_1 - \Sigma_1'] = 0 \in H_4(B\ell(M^2, \Delta); F)$, we have

$$\begin{aligned} & (\Sigma_1 \cap \Sigma_2 \cap \Sigma_3) - (\Sigma_1' \cap \Sigma_2 \cap \Sigma_3) \\ = \ & [\Sigma_1 - \Sigma_1'] \cap [\Sigma_2 \cap \Sigma_3] \\ = \ & 0 \cap [\Sigma_2 \cap \Sigma_3] = 0. \end{aligned}$$

□

Lemma 3.9 *If $G = SU(2)$, then $H_4^+(B\ell(M^2,\Delta);F) = 0$.*

Proof By the Poincaré duality and the excision isomorphism, $H_4^+(B\ell(M^2,\Delta);F) \cong H_-^2(B\ell(M^2,\Delta),\partial B\ell(M^2,\Delta);F) \cong H_-^2(M^2,\Delta;p_1^*E\otimes p_2^*E)$. Since $H^*(M;E)=0$, the Künneth formula shows that $H^*(M^2;p_1^*E\otimes p_2^*E)=0$. The cohomology exact sequence of pair (M^2,Δ) implies that $H_-^2(M^2,\Delta;p_1^*E\otimes p_2^*E) \cong H_-^1(\Delta;E\otimes E)$. By the same reason as in Lemma 3.4, $H_-^1(\Delta;E\otimes E) = H^1(\Delta;(E\otimes E)^-) = 0$. □

Corollary 3.10 *If $G = SU(2)$, $\mathrm{Tr}^{\otimes 2}(\Sigma_1,\Sigma_2,\Sigma_3) \in \mathbb{R}$ is independent of the choice of an admissible triple of propagators $(\Sigma_1,\Sigma_2,\Sigma_3)$.*

Let STM is the unit sphere bundle of TM. Let $F_{TM} \to STM$ be the tangent bundle along the fiber of $STM \to M$. We denote by $e(F_{TM}) \in H^2(STM;\mathbb{R})$ the Euler class of $F_{TM} \to M$. Then $e(F_{TM})_{P.D.} \in H_3(STM;\mathbb{R})$.

Let $c_1,c_2,c_3 \in C_3(STM;\mathbb{R})$ be 3−cycles such that $\bigcap_i \mathrm{Supp}(c_i) = \emptyset$ and $[c_1]=[c_2]=[c_3] = \frac{1}{2}e(F_{TM})_{P.D.}$. Let W be a compact oriented 4–manifold with $\partial W = M$ and $\chi(W) = 0$, where $\chi(W)$ is the Euler characteristic of W. Let $T^vW \subset TW$ be an oriented $\mathbb{R}^3$ bundle satisfying $T^vW|_M = TM$. The total space of the unit sphere bundle ST^vW is an oriented 6-dimensional manifold with $\partial ST^vW = STM$. Let $F_{T^vW} \to ST^vW$ be the tangent bundle along the fiber of $ST^vW \to W$ and let $e(F_{T^vW}) \in H^2(ST^vW;\mathbb{R})$ be the Euler class. Then $e(F_{T^vW})_{P.D.} \in H_4(ST^vW,\partial ST^vW;\mathbb{R})$. Take 4–cycles $C_1,C_2,C_3 \in C_4(ST^vW,\partial ST^vW;\mathbb{R})$ such that $\partial C_i = c_i$ and $[C_i,c_i] = \frac{1}{2}e(F_{T^vW})_{P.D.}$ for $i=1,2,3$. By the assumption of c_1,c_2,c_3, the intersection $(C_1\cap C_2\cap C_3) \in \mathbb{R} = H_0(ST^vW,\partial ST^vW;\mathbb{R})$ is well-defined.

Lemma 3.11 (Theorem4.1 in [11], see also Appendix of [11]) *$(C_1\cap C_2\cap C_3) - \frac{3}{4}\mathrm{Sign}W \in \mathbb{R}$ is independent of the choice of C_1,C_2,C_3 and W.*

Definition 3.12 $I(c_1,c_2,c_3) = (C_1\cap C_2\cap C_3) - \frac{3}{4}\mathrm{Sign}W \in \mathbb{R}$.

There is a natural bundle isomorphism $\psi : TM \xrightarrow{\cong} T(M^2)/T\Delta$ given by $\psi(x,v) = ((x,x),(v,-v))$, where $x \in M$ and $v \in T_xM$. ψ gives an bundle isomorphism between STM and $q^{-1}(\Delta)$. We identify STM with $q^{-1}(\Delta)$ via this bundle isomorphism. Under this identification, $\partial\Sigma$ is a 3–cycle in $C_3(STM;E\otimes E)$ for any propagator Σ. By the definition of propagator, there is a cycle $\Sigma_{\mathbb{R}}^{\partial} \in C_3(STM;\mathbb{R})$ such that $c_*(\Sigma_{\mathbb{R}}^{\partial}) = \partial\Sigma$. We recall that the homology class $[\Sigma_{\mathbb{R}}^{\partial}] \in H_3(STM;\mathbb{R})$ is independent of the choice of a propagator Σ (See Remark 3.6). Since $e(F_{TM})_{P.D.} \in H_3^+(STM;\mathbb{R}) \cong \mathbb{R}$, we have $[\Sigma_{\mathbb{R}}^{\partial}] = \frac{1}{2}e(F_{TM})_{P.D.}$.

From now we assume that $G = SU(2)$.

Theorem 3.13 *Let $(\Sigma_1,\Sigma_2,\Sigma_3)$ be an admissible triple of propagators. Let $\Sigma^{\partial}_{\mathbb{R},i}\in C_3(STM;\mathbb{R})$ be the cycle such that $c_*(\Sigma^{\partial}_{\mathbb{R},i})=\partial\Sigma_i$ for $i=1,2,3$. Then*

$$Z^{SU(2)}(M,E;\Sigma_1,\Sigma_2,\Sigma_3)=\mathrm{Tr}^{\otimes 2}(\Sigma_1\cap\Sigma_2\cap\Sigma_3)-6I(\Sigma^{\partial}_{\mathbb{R},1},\Sigma^{\partial}_{\mathbb{R},2},\Sigma^{\partial}_{\mathbb{R},3})\in\mathbb{R}$$

is independent of the choice of $(\Sigma_1,\Sigma_2,\Sigma_3)$.

Proof Let $\Sigma_i^\star$ be an alternative choice of Σ_i. Let $\Sigma^{\star,\partial}_{\mathbb{R},i}\in C_3(STM;\mathbb{R})$ be the cycle satisfying $c_*(\Sigma^{\star,\partial}_{\mathbb{R},i})=\partial\Sigma_i^\star$ for $i=1,2,3$. Let $C_i^\star, C_i$ be 4-cycles in $C_4(ST^vW;\mathbb{R})$ satisfying $\partial\Sigma_i^\star=\Sigma^{\star,\partial}_{\mathbb{R},i}, \partial\Sigma_i=\Sigma^{\partial}_{\mathbb{R},i}$ for $i=1,2,3$. Let $STM\times[0,1]\subset ST^vW, STM\times\{0\}=\partial ST^vX$ be a collar of STM in ST^vW. We identify $ST^vW\setminus(STM\times[0,1])$ with ST^vW by stretching the collar. Thanks to Lemma 3.11, without loss of generality, we may assume that $C_i^\star|_{ST^vW\setminus(STM\times[0,1])}=C_i$. Then

$$\begin{aligned}
& I(\Sigma^{\star,\partial}_{\mathbb{R},1},\Sigma^{\star,\partial}_{\mathbb{R},2},\Sigma^{\star,\partial}_{\mathbb{R},3})-I(\Sigma^{\partial}_{\mathbb{R},1},\Sigma^{\partial}_{\mathbb{R},2},\Sigma^{\partial}_{\mathbb{R},3})\\
=\;& C_1^\star\cap C_2^\star\cap C_3^\star\cap(STM\times[0,1])\\
=\;& C_1\cap C_2\cap C_3\cap(STM\times[0,1]).
\end{aligned}$$

Let $q^{-1}(\Delta)\times[0,1]\subset B\ell(M^2,\Delta), q^{-1}(\Delta)\times\{0\}=\partial B\ell(M^2,\Delta)$ be a collar of $q^{-1}(\Delta)$ in $B\ell(M^2,\Delta)$. Thanks to Lemma 3.9, without loss of generality, we may assume that $\Sigma_i|_{B\ell(M^2,\Delta)\setminus q^{-1}(\Delta)\times[0,1]}=\Sigma_i'$ and $\Sigma_i|_{q^{-1}(\Delta)\times[0,1]}=C_i|_{STM\times[0,1]}$. Then

$$\begin{aligned}
& \mathrm{Tr}^{\otimes 2}(\Sigma_1,\Sigma_2,\Sigma_3)-\mathrm{Tr}^{\otimes 2}(\Sigma_1^\star,\Sigma_2^\star,\Sigma_3^\star)\\
=\;& \mathrm{Tr}^{\otimes 2}(\Sigma_1\cap\Sigma_2\cap\Sigma_3\cap(q^{-1}(\Delta)\times[0,1])\\
=\;& \mathrm{Tr}^{\otimes 2}(c_*(1)\otimes c_*(1)\otimes c_*(1))(C_1\cap C_2\cap C_3\cap(STM\times[0,1]))\\
=\;& 6(C_1\cap C_2\cap C_3\cap(STM\times[0,1])).
\end{aligned}$$

Therefore $\mathrm{Tr}^{\otimes 2}(\Sigma_1\cap\Sigma_2\cap\Sigma_3)-6I(\Sigma^{\partial}_{\mathbb{R},1},\Sigma^{\partial}_{\mathbb{R},2},\Sigma^{\partial}_{\mathbb{R},3})=\mathrm{Tr}^{\otimes 2}(\Sigma_1^\star\cap\Sigma_2^\star\cap\Sigma_3^\star)-6I(\Sigma^{\star,\partial}_{\mathbb{R},1},\Sigma^{\star,\partial}_{\mathbb{R},2},\Sigma^{\star,\partial}_{\mathbb{R},3})$. □

Definition 3.14 $Z^{SU(2)}(M,E)=Z^{SU(2)}(M,E;\Sigma_1,\Sigma_2,\Sigma_3)$.

4 Morse homotopy for the $SU(2)$–Chern–Simons perturbation theory

In [11] we introduced the Morse theoretic description (Morse homotopy) of a propagator for the Chern–Simons perturbation theory at the trivial connection. This description

is deeply inspired by [3] and [13] and related to [9]. In this section, we give a Morse theoretic description of a propagator for $Z^{SU(2)}(M,E)$.

Let $f : M \to \mathbb{R}$ be a Morse function. We denote by $\mathrm{Crit}_j(f)$ the set of critical points of index j of f and set $\mathrm{Crit}(f) = \sqcup_{j=0,1,2,3}\mathrm{Crit}_j(f)$. We denote by $\mathrm{ind}(p)$ the index of $p \in \mathrm{Crit}(f)$. Let

$$(C_*(f;E),\partial) = \cdots 0 \overset{\partial_4}{\to} C_3(f;E) \overset{\partial_3}{\to} C_2(f;E) \overset{\partial_2}{\to} C_1(f;E) \overset{\partial_1}{\to} C_0(f;E) \overset{\partial_0}{\to} 0$$

be the Morse-Smale complex of the Morse function f with an acyclic local system E, namely $C_j(f;E) = \oplus_{p\in\mathrm{Crit}_j(f)}E_p$.

Let $(\Phi_f^t : M \overset{\cong}{\to} M)_{t\in\mathbb{R}}$ be the one–parameter family of diffeomorphisms associated to $\mathrm{grad}f$. Let $\mathcal{A}_p$ and $\mathcal{D}_p$ be the ascending manifold of p and descending manifold of p respectively for $p \in \mathrm{Crit}(f)$:

$$\mathcal{A}_p = \{x \in Y \mid \lim_{t\to-\infty} \Phi_f^t(x) = p\},$$

$$\mathcal{D}_p = \{x \in Y \mid \lim_{t\to\infty} \Phi_f^t(x) = p\}.$$

Let $\mathcal{M}(p,q)$ be the set of all trajectories connecting $q \in \mathrm{Crit}(f)$ and $p \in \mathrm{Crit}(f)$:

$$\mathcal{M}(p,q) = \{\gamma : \mathbb{R} \to M \mid d\gamma/dt = \mathrm{grad}f, \lim_{t\to-\infty} \gamma(t) = q, \lim_{t\to\infty} \gamma(t) = p\}.$$

The additive group $\mathbb{R}$ acts on $M(p,q)$ as shifting the parameter. We set

$$\mathcal{M}'(p,q) = \mathcal{M}(p,q)/\mathbb{R}.$$

We consider each $\gamma \in \mathcal{M}'(p,q)$ as a path from q to p. When $\mathrm{ind}(q) = \mathrm{ind}(p)+1$, we assign $\varepsilon(\gamma) \in \{+,-\}$ to each $\gamma \in \mathcal{M}'(p,q)$ as follows: $\varepsilon(\gamma) = +$ if and only if the orientation of γ is from q to p.

The orientation convention follows [11].

Definition 4.1 ([3],[13]) A family of homomorphisms $g = (g^k : C_k(f;E) \to C_{k+1}(f;E))_{k\in\mathbb{Z}}$ is said to be a *combinatorial propagator* if

$$g^{k-1} \circ \partial_k + \partial_{k+1} \circ g^k = id_{C_k(f;E)}$$

for all $k \in \mathbb{Z}$.

A combinatorial propagator g gives a morphism $g_{p,q} : E_p \to E_q$ for $p \in \mathrm{Crit}_k(f), q \in \mathrm{Crit}_{k+1}(f)$.

Let $f_\Delta^0 : M \times [0,\infty) \to M^2, f_\Delta(x,t) = (x, \Phi_t^f(x))$. Let$M_\to(f)$ be the closure of $f_\Delta^0(M \times [0,\infty))$ in M^2. $M_\to(f)$ is obtained from $f_\Delta(M \times [0,\infty))$ by adding all the

broken trajectories (See [9, Lemma 4.3] or [13, Proposition 3.4] for more details). The map f^0_Δ can be extended to the closure. We denote by $f_\Delta : M_\to(f) \to M^2$ the extended map. Then we get a 4–chain of M^2:

$$M_\to(f) = \langle M_\to(f), (x,x); c(1)\rangle \in C_4(M^2; p_1^*E \otimes p_2^*E).$$

The closure of $q^{-1}(M^0_\to(f) \setminus \Delta)$ in $B\ell(M^2,\Delta)$ gives a 4–chain in $B\ell(M^2,\Delta)$. We will denote by

$$M^{B\ell}_\to(f) \in C_4(B\ell(M^2;\Delta); F)$$

this chain.

We denote by $\overline{\mathcal{A}_p} \to M$ the extension of the embedding $\mathcal{A}_p \to M$ to the closure of $\mathcal{A}_p$. $\overline{\mathcal{A}_p}$ is obtained by adding all the broken trajectories (See [13, Proposition 3.17] for more detail). For $p \in \mathrm{Crit}_k(f)$ and $q \in \mathrm{Crit}_{k+1}(f)$, $(1 \otimes g_{p,q})$ is a morphism from $E_p \otimes E_p$ to $E_p \otimes E_q$. Therefore

$$\overline{\mathcal{A}_p} \times \overline{\mathcal{D}_q} = \langle \overline{\mathcal{A}_p} \times \overline{\mathcal{D}_q}, (p,q); (1 \otimes g_{p,q})c(1)\rangle$$

is a 4-chain in $C_4(M^2, p_1^*E \otimes p_2^*E)$. We denote by

$$(\overline{\mathcal{A}_p} \times \overline{\mathcal{D}_q})^{B\ell} \in C_4(B\ell(M^2,\Delta); F)$$

the 4–chain given from the closure of $q^{-1}(\overline{\mathcal{A}_p} \times \overline{\mathcal{D}_q} \setminus \Delta)$ in $B\ell(M^2,\Delta)$. Let

$$\mathcal{M}^0(f) = M^{B\ell}_\to(f) + \sum_{p,q\in\mathrm{Crit}(f),\mathrm{ind}p=\mathrm{ind}q-1} (\overline{\mathcal{A}_p} \times \overline{\mathcal{D}_q})^{B\ell},$$

$$\mathcal{M}(f) = \frac{1}{2}(\mathcal{M}^0(f) + T(\mathcal{M}^0(f))) \in C_4(B\ell(M^2,\Delta); F).$$

Lemma 4.2 *$\mathcal{M}(f)$ is a 4–cycle of $C_4(B\ell(M^2,\Delta), \partial B\ell(M^2,\Delta); F)$.*

Proof $M_\to(f) + \sum_{p,q} \overline{\mathcal{A}_p} \times \overline{\mathcal{D}_q}$ is a 4–chain of M^2. It is sufficient to show that

$$\mathrm{Supp}(\partial(M_\to(f) + \sum_{p,q} \overline{\mathcal{A}_p} \times \overline{\mathcal{D}_q})) \subset \Delta.$$

By Lemma 4.3 in [9] or Propositon 3.4 in [13] and Proposition 3.14 in [13],

$$\partial M_\to(f) \setminus \Delta = \sum_{p\in\mathrm{Crit}(f)} \langle \overline{\mathcal{A}_p} \times \overline{\mathcal{D}_p}, (p,p); c(1)\rangle,$$

$$\begin{aligned}
&\partial(\overline{\mathcal{A}_p} \times \overline{\mathcal{D}_q}) \\
&= \sum_{p'\in\mathrm{Crit}(f)} \bigcup_{\gamma_{p',q}\in\mathcal{M}'(p',q)} \langle \overline{\mathcal{A}_p} \times \overline{\mathcal{D}_{p'}}, (p,p'); (1 \otimes (\gamma_{p',q})_* \circ g_{p,q})c(1)\rangle \\
&\quad + \sum_{q'\in\mathrm{Crit}(f)} \bigcup_{\gamma_{p,q'}\in\mathcal{M}'(p,q')} \langle \overline{\mathcal{A}_{q'}} \times \overline{\mathcal{D}_q}, (q',q); ((\gamma_{p,q'})_*^{-1} \circ g_{p,q} \otimes 1)c(1)\rangle.
\end{aligned}$$

If $\mathrm{ind}(p') > \mathrm{ind}(p)$, then the image of $\bigcup_{\gamma_{p',q}\in\mathcal{M}'(p',q)}\langle\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_{p'}},(p,p');(1\otimes(\gamma_{p',q})_*\circ g_{p,q})c(1)\rangle$ is in at most 2–dimensional manifold $\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_{p'}}$. Then we can omit $\bigcup_{\gamma_{p',q}\in\mathcal{M}'(p',q)}\langle\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_{p'}},(p,p');(1\otimes(\gamma_{p',q})_*\circ g_{p,q})c(1)\rangle$ from the boundary of $\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_q}$. Therefore, we have

$$\begin{aligned}
&\partial(\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_q})\\
=&\sum_{p'\in\mathrm{Crit}_{\mathrm{ind}(p)}(f)}\Big\langle\sum_{\gamma_{p',q}\in\mathcal{M}'(p',q)}\varepsilon(\gamma_{p',q})\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_{p'}},(p,p');(1\otimes(\gamma_{p',q})_*\circ g_{p,q})c(1)\Big\rangle\\
&+\sum_{q'\in\mathrm{Crit}_{\mathrm{ind}(q)}(f)}\Big\langle\sum_{\gamma_{p,q'}\in\mathcal{M}'(p,q')}\varepsilon(\gamma_{p,q'})\overline{\mathcal{A}_{q'}}\times\overline{\mathcal{D}_q},(q',q);((\gamma_{p,q'})_*^{-1}\circ g_{p,q}\otimes 1)c(1)\Big\rangle\\
=&\sum_{p'\in\mathrm{Crit}_{\mathrm{ind}(p)}(f)}\Big\langle\sum_{\gamma_{p',q}\in\mathcal{M}'(p',q)}\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_{p'}},(p,p');(1\otimes\varepsilon(\gamma_{p',q})(\gamma_{p',q})_*\circ g_{p,q})c(1)\Big\rangle\\
&+\sum_{q'\in\mathrm{Crit}_{\mathrm{ind}(q)}(f)}\Big\langle\sum_{\gamma_{p,q'}\in\mathcal{M}'(p,q')}\overline{\mathcal{A}_{q'}}\times\overline{\mathcal{D}_q},(q',q);(1\otimes g_{p,q}\circ\varepsilon(\gamma_{p,q'})(\gamma_{p,q'})_*)c(1)\Big\rangle\\
=&\sum_{p'\in\mathrm{Crit}_{\mathrm{ind}(p)}(f)}\Big\langle\sum_{\gamma_{p',q}\in\mathcal{M}'(p',q)}\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_{p'}},(p,p');(1\otimes(\partial_{p',q})_\circ g_{p,q})c(1)\Big\rangle\\
&+\sum_{q'\in\mathrm{Crit}_{\mathrm{ind}(q)}(f)}\Big\langle\sum_{\gamma_{p,q'}\in\mathcal{M}'(p,q')}\overline{\mathcal{A}_{q'}}\times\overline{\mathcal{D}_q},(q',q),(1\otimes g_{p,q}\circ(\partial_{p,q'}))c(1)\Big\rangle.
\end{aligned}$$

Here $\partial_{p',q}:E_{p'}\to E_q$ is the boundary map. Thus

$$\begin{aligned}
&\partial(\sum_{p,q}\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_q})\\
=&\sum_k\sum_{p,p'\in\mathrm{Crit}_k(f)}(\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_{p'}},(p,p'),(1\otimes\sum_{q\in\mathrm{Crit}_{k+1}(f)}\partial_{p',q}\circ g_{p,q}+\sum_{r\in\mathrm{Crit}_{k-1}(f)}\partial_{r,p'}\circ g_{r,p}\otimes 1)c(1))\\
=&\sum_{p\in\mathrm{Crit}(f)}(\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_p},(p,p),c_*(1))\\
=&\partial M_\to(f)\setminus\Delta.
\end{aligned}$$

Therefore

$$\mathrm{Supp}(\partial(M_\to(f)+\sum_{p,q}\overline{\mathcal{A}_p}\times\overline{\mathcal{D}_q}))\subset\Delta.$$

□

We set

$$c(f)=\partial\mathcal{M}(f)\in C_3(q^{-1}(\Delta);F).$$

By the definition of $\mathcal{M}(f)$, $\partial\mathcal{M}(f) \in C_3(q^{-1}(\Delta); q^*(E\otimes E)) = C_3(STM; q^*(E\otimes E))$ is described as follows: For $v \in T_xM \setminus \{0\}$, we denote by $[v] \in ST_xM$ the image of v under the projection $T_xM \setminus \{0\} \to STM$.

$$\partial\mathcal{M}(f) = \langle \Delta(f), (x, [\mathrm{grad}_x f]); c(1)\rangle + q^! \left(\sum_{p,q} \langle \mathcal{M}(p,q), (p,p); (\mathrm{id}\otimes (g_{p,q} \circ \sum_{\gamma\in\mathcal{M}'(p,q)} \gamma_*))c(1)\rangle \right).$$

Here $\Delta(f) = \overline{\{[\mathrm{grad}_y f], -[\mathrm{grad}_y f] \in ST_yM \mid y \in M\setminus \mathrm{Crit}(f)\}} \hookrightarrow STM$ and $x \in M \setminus \mathrm{Crit}(f)$ is any point. We remark that $\Delta(f)$ is a 3–cycle of $C_3(STM;\mathbb{R})$ (See [11, Lemma] for details). Let us denote

$$c(f) = \sum_{p,q} \langle \mathcal{M}(p,q), (p,p); (\mathrm{id}\otimes (g_{p,q} \circ \sum_{\gamma\in\mathcal{M}'(p,q)} \gamma_*))c(1)\rangle.$$

We next introduce the linking number of 1–chains with local coefficients. Let $c_1, c_2 \in C_1^-(M, E\otimes E)$ be 1-cycles such that $\mathrm{Supp}(c_1)\cap\mathrm{Supp}(c_2) = \emptyset$. Since $H_1^-(M, E\otimes E) = 0$, there is a 2–chain $C_1 \in C_2^-(M; E\otimes E)$ such that $\partial C_1 = c_1$. Then we have a 0–cycle $C_1 \cap c_2 \in C_0(M; (E\otimes E)^{\otimes 2})$. Let $R : (E\otimes E)^{\otimes 2} \to \mathbb{R}, R((x_1\otimes x_2)\otimes(y_1\otimes y_2)) = \langle [x_1,y_1],[x_2,y_2]\rangle = Tr^{\otimes 2}((x_1\otimes x_2)\otimes(y_1\otimes y_2)\otimes c_*(1))$.

Lemma 4.3 *$R_*(C_1\cap c_2) \in \mathbb{R} = H_0(M;\mathbb{R})$ is independent of the choice of C_1.*

Proof Let C_1' be the alternative choice of C_1. $C_1 - C_1'$ is a 2–cycles of $C_2^-(M; E\otimes E)$. Since $H_2^-(M; E\otimes E) = 0$, $C_1\cap c_2 - C_1'\cap c_2 = [C_1 - C_1']\cap[c_2] = 0 \in H_0^-(M; (E\otimes E)^2)$. Therefore $R_*(C_1\cap c_2) - R_*(C_1'\cap c_2) = R_*(0) = 0$. □

Definition 4.4 $\mathrm{lk}_{(M,E)}(c_1, c_2) = R(C_1\cap c_2)$.

Definition 4.5 A triple of Morse functions (f_1, f_2, f_3) is said to be *generic* when the following conditions hold:

- $\mathrm{Supp}(c(f_i)) \cap \mathrm{Supp}(c(f_j)) = \emptyset$ for any $i \neq j$ and
- $\bigcap_i \Delta(f_i) = \emptyset$.

Remark 4.6 A triple of Morse functions which is generic in the products space $C^\infty(M,\mathbb{R})^3$ of a map space $C^\infty(M,\mathbb{R})$ is generic in the above sence.

Theorem 4.7 *For any generic triple of Morse functions (f_1, f_2, f_3),*

$$Z^{SU(2)}(M,E) = \mathrm{Tr}^2(\mathcal{M}(f_1)\cap\mathcal{M}(f_2)\cap\mathcal{M}(f_3)) - 6I(\Delta(f_1),\Delta(f_2),\Delta(f_3)) - \sum_{i<j\in\{1,2,3\}} \mathrm{lk}_{(M,E)}(c(f_i), c(f_j)).$$

5 Proof of Theorem 4.7

Since $H_1^-(\Delta; E\otimes E) = 0$, there are 2–chains $C(f_1), C(f_2), C(f_3) \in C_2^-(\Delta; E\otimes E)$ satisfying $\partial C(f_i) = c(f_i)$ for $i = 1,2,3$. Let $\partial B\ell(M^2,\Delta)\times[0,1]$ be a collar of $\partial B\ell(M^2,\Delta) \subset B\ell(M^2,\Delta)$ with $\partial B\ell(M^2,\Delta)\times\{0\} = \partial B\ell(M^2,\Delta)$. For $t\in[0,1]$,we denote by $q_t : \partial B\ell(M^2,\Delta)\times\{t\} \to \Delta$ the projection. We may assume that

- $\mathcal{M}(f_i)\cap \partial B\ell(M^2,\Delta)\times[0,1] = \partial\mathcal{M}(f_i)\times[0,1]$ for any $i = 1,2,3$.

We set

- $C(f_1)' = ((q_{1/4})^! C(f_1) + c(f_1)\times[0,1/4]) \in C_4(M^2,\Delta; E\otimes E)$,
- $C(f_2)' = ((q_{2/4})^! C(f_2) + c(f_2)\times[0,2/4]) \in C_4(M^2,\Delta; E\otimes E)$,
- $C(f_3)' = ((q_{3/4})^! C(f_3) + c(f_3)\times[0,3/4]) \in C_4(M^2,\Delta; E\otimes E)$,
- $\mathcal{M}(f_1)' = \mathcal{M}(f_1) - C(f_1)'$,
- $\mathcal{M}(f_2)' = \mathcal{M}(f_2) - C(f_2)'$ and
- $\mathcal{M}(f_3)' = \mathcal{M}(f_3) - C(f_3)'$.

Here $c(f_i)\times[0,t] = (\pi_t)^! c(f_i)$ for $t\in[0,1]$ and the projection $\pi_t : \partial B\ell(M^2,\Delta)\times[0,t] \to B\ell(M^2,\Delta)$. Then the triple $(\mathcal{M}(f_1)', \mathcal{M}(f_2)', \mathcal{M}(f_3)')$ is a triple of admissible propagators with $\partial\mathcal{M}(f_i)' = \langle \Delta(f_i), (x, [\mathrm{grad}_x f]); c(1)\rangle = c_*(\Delta(f_i))$. Then we have

$$Z_1^{SU(2)}(M,E) = \mathrm{Tr}^{\otimes 2}(\mathcal{M}(f_1)'\cap\mathcal{M}(f_2)'\cap\mathcal{M}(f_3)') - 6I(\Delta(f_1),\Delta(f_2),\Delta(f_3)).$$

Lemma 5.1 *For any $a_+, b_+, c_+ \in (E\otimes E)^+$, $a_-, b_-, c_- \in (E\otimes E)^-$ and any $\sigma_1,\sigma_2,\sigma_3 \in \{+,-\} \cong \mathbb{Z}/2$ with $\sigma_1\sigma_2\sigma_3 = -$, $\mathrm{Tr}^{\otimes 2}(a_{\sigma_1}\otimes b_{\sigma_2}\otimes c_{\sigma_3}) = 0$.*

Proof Since $\sigma_1\sigma_2\sigma_3 = -$,

$$T_0^{\otimes 3}(a_{\sigma_1}\otimes b_{\sigma_2}\otimes c_{\sigma_3}) = \sigma_1 a_{\sigma_1}\otimes\sigma_2 b_{\sigma_2}\otimes\sigma_3 c_{\sigma_3} = -a_{\sigma_1}\otimes b_{\sigma_2}\otimes c_{\sigma_3}.$$

On the other hand, $\mathrm{Tr}^{\otimes 2}\circ T_0^{\otimes 3} = \mathrm{Tr}^{\otimes 2}$. Then $\mathrm{Tr}^{\otimes 2}(a_{\sigma_1}\otimes b_{\sigma_2}\otimes c_{\sigma_3}) = 0$. □

Lemma 5.2 (1) $\mathrm{Tr}^{\otimes 2}(C(f_1)'\cap C(f_2)'\cap(\partial\mathcal{M}(f_3)\times[0,1])) = \mathrm{lk}_{(M,E)}(c(f_1),c(f_2))$,

(2) $\mathrm{Tr}^{\otimes 2}(C(f_1)'\cap(\partial\mathcal{M}(f_2)\times[0,1])\cap C(f_3)') = \mathrm{lk}_{(M,E)}(c(f_1),c(f_3))$,

(3) $\mathrm{Tr}^{\otimes 2}((\partial\mathcal{M}(f_1)\times[0,1])\cap C(f_2)'\cap C(f_3)') = \mathrm{lk}_{(M,E)}(c(f_2),c(f_3))$.

Proof We prove (1). For generic f_1, f_2, $\mathrm{Supp}(c(f_1)) \cap \mathrm{Supp}(c(f_2)) = \emptyset$. Therefore

$$\begin{aligned}
&\mathrm{Tr}^{\otimes 2}(C(f_1)' \cap C(f_2)' \cap \partial M(f_3) \times [0,1]) \\
=\ &\mathrm{Tr}^{\otimes 2}((q_{1/4})^! C(f_1) \cap (q_{1/4})^! c(f_2) \cap \partial \mathcal{M}(f_3) \times \{1/4\}) \\
=\ &\mathrm{Tr}^{\otimes 2}((q_{1/4})^! C(f_1) \cap (q_{1/4})^! c(f_2) \cap \langle \Delta(f_3), (x,x), c_*(1)) \rangle \\
&+ \mathrm{Tr}^{\otimes 2}((q_{1/4})^! C(f_1) \cap (q_{1/4})^! c(f_2) \cap (q_{1/4})^! c(f_3)).
\end{aligned}$$

Since $\mathrm{Supp}(c(f_2)) \cap \mathrm{Supp}(c(f_3)) = \emptyset$, the second term of the last equation is zero. Then

$$\begin{aligned}
&\mathrm{Tr}^{\otimes 2}(C(f_1)' \cap C(f_2)' \cap \partial M(f_3) \times [0,1]) \\
=\ &\mathrm{Tr}^{\otimes 2}((q_{1/4})^! C(f_1) \cap (q_{1/4})^! c(f_2) \cap \langle \Delta(f_3), (x,x), c_*(1)) \rangle \\
=\ &\mathrm{Tr}^{\otimes 2}((C(f_1) \cap c(f_2)) \otimes c_*(1)) \\
=\ &\mathrm{lk}_{(M,E)}(c(f_1), c(f_2)).
\end{aligned}$$

□

Proof of Theorem 4.7 For each $i = 1,2,3$, $\partial \mathcal{M}(f_i) \in C_3(q^{-1}(\Delta); q^*((E \otimes E)^+))$ and $C(f_i) \in C_2(\Delta; (E \otimes E)^-)$. Then, thanks to Lemma 5.1,

$$\begin{aligned}
&\mathrm{Tr}^{\otimes 2}(\mathcal{M}(f_1)' \cap \mathcal{M}(f_2)' \cap \mathcal{M}(f_3)') \\
=\ &\mathrm{Tr}^{\otimes 2}((\mathcal{M}(f_1) - C(f_1)') \cap (\mathcal{M}(f_2) - C(f_2)') \cap (\mathcal{M}(f_3) - C(f_3)')) \\
=\ &\mathrm{Tr}^{\otimes 2}(\mathcal{M}(f_1) \cap \mathcal{M}(f_2) \cap \mathcal{M}(f_3)) \\
&-\mathrm{Tr}^{\otimes 2}(C(f_1)' \cap C(f_2)' \cap (\partial \mathcal{M}(f_3) \times [0,1])) \\
&-\mathrm{Tr}^{\otimes 2}(C(f_1)' \cap (\partial \mathcal{M}(f_2) \times [0,1]) \cap C(f_3)') \\
&-\mathrm{Tr}^{\otimes 2}((\partial \mathcal{M}(f_1) \times [0,1]) \cap C(f_2)' \cap C(f_3)') \\
=\ &\mathrm{Tr}^{\otimes 2}(\mathcal{M}(f_1) \cap \mathcal{M}(f_2) \cap \mathcal{M}(f_3)) - \sum_{i<j} \mathrm{lk}_{(M,E)}(c(f_i), c(f_j)).
\end{aligned}$$

□

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