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Abstract

We prove the Etingof conjecture stated that the quotient $Cact_n^{(0)}$ of the Cactus group $Cact_n$ [2],[5],[9] by the relation $(s_{12}s_{13})^6 = 1$ is isomorphic to the Gelfand–Tsetlin group, known also as the BK_n group, which has been introduced and studied in [1]. In particular, a subtraction free birational (as well as piece-wise linear one) action of the (reduced) Cactus group $Cact_n^{(0)}$ on the projective space $\mathbb{P}^{n^2}$ is described. This action is geometric/tropical lift of the combinatorial action of the local Schützenberger transformation on the set of semistandard Young tableaux, cf [1], [6].

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1 Introduction

The cactus group $Cact_n := \pi_1(\overline{M_{0,n+1}(\mathbb{R})})$ is the fundamental group of the real locus of the Deligne–Mumford compactification $\overline{M_{0,n+1}}$ of the moduli space of genus zero stable curves with $n+1$ marked points, see e.g., [2]. The action of (reduced) cactus group on the set of standard Young tableaux has been described in [5],[4]. On the other hand, a combinatorial action of the Gelfand–Tsetlin/Berenstein–Kirillov group GT_n/BK_n on the set of semistandard Young tableaux had been introduced by A. Lascoux and M.-P. Schützenberger, see e.g., [7]; see also [3], [6].

Based on results obtained in [1],[6], we construct birational/geometric representation of the (reduced) cactus group $Cact_n^{(0)}$, and identify it with the Gelfand–Tsetlin group GT_n. In fact, in [1] the affine extension of the Gelfand–Tsetlin group has been introduced and studied, as well as presented an action of the (affine) group GT_n on the set of semistandard Young tableaux and that of transportation matrices, cf [8]. It will be interesting task to find a geometric interpretations of affine Cactus and affine Gelfand–Tsetlin groups. Another interesting problem we are looking for, is to study of combinatorial and algebraic properties of the higher genus cactus group $Cact_{g,n} := \pi_1(\overline{M_{g,n+1}}(\mathbb{R}))$, work in progress.

2 Cactus group $Cact_n$, [2],[5], [4], [9]

Definition 2.1 *The Cactus group $Cact_n$ is a group (with a unit) generated by elements σ_{ij}, $1 \le i < j \le n$, subject to the set of relations*

- $\sigma_{ij}^2 = 1$, *if* $1 \le i < j \le n$,
- $\sigma_{ij}\ \sigma_{kl} = \sigma_{kl}\ \sigma_{ij}$, *if* $j < k$,
- $\sigma_{ij}\ \sigma_{kl}\ \sigma_{ij} = \sigma_{i+j-l,i+j-k}$, *if* $i \le k < l \le j$.

Let us set $\sigma_i := \sigma_{1,i+1}$, $1 \le i \le n-1$. It is clear that $\sigma_i^2 = 1$, and the elements $\sigma_1, \ldots, \sigma_{n-1}$ generate the Cactus group $Cact_n$. We denote by $Cact_n^{(0)}$ the quotient of the cactus group $Cact_n$ by the normal subgroup generated by the element $((\sigma_1\sigma_2)^6 - 1)$.

3 Gelfand–Tsetlin group, [1], [6]

Definition 3.1 *([1],[6]) The Gelfand–Tsetlin group GT_n, known also as the Berenstein–Kirillov group and denoted by BK_n, is a group (with a unit) generated by the elements $t_1, \ldots, t_{n-1}$ subject to the set of relations*

- $(t_1\ t_2)^6 = 1$,
- $t_i\ t_j = t_j\ t_i$, *if* $|i-j| \ge 2$,
- $(t_i\ q_j)^4 = 1$ *if* $j - i \ge 2$, *where*

$$q_i = t_1\ \underbrace{t_2\ t_1}\cdots\underbrace{t_i t_{i-1}\ldots t_1}, \quad 1 \le i < n.$$

Proposition 3.2 *([1],[6]) The following relations in the group BK_n are satisfied*

- $[t_i, q_k\ t_j\ q_k] = 0, \quad if \quad k \geq i+j+1,$
- $[q_i, q_k\ q_j\ q_k] = 0, \quad if \quad k \geq i+j+1,$

where $[a,b] := a\ b - b\ a$ denotes the commutator of elements a and b.

Theorem 3.3 *([1]) The elements of the Gelfand–Tsetlin group GT_n listed below*

$$s_i := q_i\ t_1\ q_i, \quad 1 \leq i < n$$

satisfy the following relations
- $s_i^2 = 1,$
- *(Coxeter relations)* $(s_i\ s_{i+1})^3 = 1,$ *if* $1 \leq i < n,$

(commutativity) $(s_i\ s_j)^2 = 1,$ *if* $|i-j| \geq 2.$ *We* **expect** *that the group generated by the elements $s_1, \ldots, s_{n-1}$ is isomorphic to the symmetric group $\mathbb{S}_n$.*

Theorem 3.4 *The maps $\sigma_i \longleftrightarrow q_i$ can be extended to the* **isomorphism** *of groups*

$$Cact_n^{(0)} \cong GT_n.$$

Indeed, it is clear that under the above correspondence one has

$$q_k\ q_j\ q_k = \sigma_{k+1-j,k+1}.$$

Therefore, if $k \geq i+j+1$, then $k-j+1 \geq i+2$, and therefore

$$[\sigma_{1,i+1}, \sigma_{k+1-j,k+1}] = [q_i, q_k\ q_j\ q_k] = 0.$$

Now let us check that the image of relations

$$\sigma_{in}\ \sigma_{jk} = \sigma_{i+n-k,i+n-j}\ \sigma_{in}, \quad 1 \leq j < k \leq n$$

are satisfied in the Gelfand–Tsetlin group. Indeed, we have to show that the relations

$$q_{n-1}\ q_{n-i}\ q_{n-1}\ q_{k-1}\ q_{k-j}\ q_{k-1} = q_{n+i-j-1}\ q_{k-j}\ q_{n+i-j-1}\ q_{n-1}\ q_{n-i}\ q_{n-1}$$

are valid in the group GT_n. By definition, $q_r = q_{r-1}\ p_r$, where $p_r := \underbrace{t_r\ t_{r-1} \cdots t_2\ t_1}$. Therefore,

$$q_{n-1} = q_{i+n-j-1}\ u_{n+i-j}^{(n-1)},$$

where we set $u_b^{(a)} := p_a \ldots p_b$, if $a > b$. Therefore one can rewrite relations we have to prove, in the following form

$$u_{i+n-j}^{(n-1)}\ q_{n-i}\ q_{n-1} q_{k-1}\ q_{k-ju} q_{k-1} = q_{k-j}\ u_{n+i-j}^{(n-1}\ q_{n-i}\ q_{n-1}.$$

Now we use the following deletion $u_{n-1} = q_{n-i}\ u^{(n-1)}_{n_i+1}$,so that we can rewrite the above relations in the form

$$u^{(n-1)}_{n+i-j}\ u^{(n-1)}_{n-i+1}\ q_{k-1} = q_{k-j}\ u^{(n-1)}_{n+i-j}\ u^{(n-1)}_{n-i+1}\ q_{k-1}.$$

Note that the number of terms in the product $u^{(n-1)}_{n+i-j}\ u^{(n-1)}_{n-i+1}$ is equal to $j-1$. Therefore,

$$u^{(n-1)}_{n+i-j}\ u^{(n-1)}_{n-i+1}\ q_{k-1} = q_{k-j} \times A_{j+2},$$

where A_{j+2} is a certain product of generators $t_n, \ldots, t_{j+2}$ only. This statement is clearly seen from the relations

$$p_a\ q_b = t_a\ t_{a-1} \cdots t_{b+1}\ q_{b-1}, \quad if \quad a > b.$$

The relations $s_{ij}\ s_{kl} = s_{kl}\ s_{ij},\ if\ j < k$, can be proved in the same fashion.

Clearly, one has the maps $GT_n \longleftrightarrow Cact^{(0)}_n$ given by

$$q_i \longleftrightarrow s_{1,i+1}$$

. Therefore we came to conclusion that the groups GT_n and $Cact^{(0)}_n$ are isomorphic.

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Remark After this note was written, we was informed about a preprint arXive:1609.2046 " The Berenstein-Kirillov group and cactus groups" by M. Chmutov, M. Glick, P. Pylyavskyy, which also contains another approach to identify the cactus group $Cactus)n^{(0)}$ and that BK_n.

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