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COHOMOLOGY OF THE GEOMETRIC FUNDAMENTAL GROUP OF HYPERBOLIC POLYCURVES

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Abstract. In the present paper, we study the cohomology groups of profinite groups obtained by successive extensions of families of (pro-Σ) surface groups. As an application, we show, among others, that the dimension of a hyperbolic polycurve (i.e., a successive extension of a family of hyperbolic curves) over a field can be reconstructed group-theoretically from its geometric fundamental group.

Introduction

Let p be a prime number, k a field of characteristic zero, $\overline{k}$ an algebraic closure of k, $G_k := \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k, and X a variety over k. Then the structure morphism $X \to \operatorname{Spec} k$ induces a natural (outer) surjection $\pi_1(X) \twoheadrightarrow G_k$. Write $\Delta_{X/k}$ for the kernel of the surjection $\pi_1(X) \twoheadrightarrow G_k$ and $\Delta^p_{X/k}$ for the maximal pro-p quotient of $\Delta_{X/k}$. A. Grothendieck proposed that, for certain types of k, if X is "an anabelian variety" over k, then the isomorphism class of X may be completely determined by $\pi_1(X) \twoheadrightarrow G_k$ (cf. [2],[3]). In [4], the Grothendieck conjecture for successive extensions of families of hyperbolic curves (hereinafter called "hyperbolic polycurves", cf. Definition 2.1(ii)) of dimension ≤ 4 was proved. Moreover, in [11], we studied the pro-p version of the Grothendieck conjecture, and obtained a similar result for hyperbolic polycurves satisfying condition $(*)_p$ (cf. Definition 2.5).

On the other hand, the Grothendieck conjecture for hyperbolic polycurves of dimension ≥ 5 is still open. However, even if the dimension is greater than 4, the étale fundamental group of a hyperbolic polycurve may have various geometric information. In the present paper, we discuss reconstruction of geometric invariants from the cohomology groups of $\Delta_{X/k}$ (resp. $\Delta^p_{X/k}$), where X is a hyperbolic polycurve over k (resp. a hyperbolic polycurve over k satisfying condition $(*)_p$). Note that in this case, $\Delta_{X/k}$ (resp. $\Delta^p_{X/k}$) is a successive extension of a family of (profinite (resp. pro-p)) surface groups. Here, we refer to a profinite group which is isomorphic to the maximal pro-Σ quotient (where Σ is a nonempty set of prime numbers) of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic zero as a (pro-Σ) surface group (cf. [7] Definition 1.2). So we consider successive extensions of families of surface groups. The following is the main result of the present paper.

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Theorem A (cf. Theorem 2.15)**.** *Let $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ be a successive extension of surface groups* (*cf. Definition 2.6*) *and m a nonnegative integer. Write $\Sigma := \bigcap_{j=1}^{n} \Sigma_j$. Suppose that $\Sigma \neq \emptyset$. Then the following conditions are equivalent:*

(1) $m = n$.

(2) *For any positive real number M, there exists an open subgroup $V \subset G$ of G such that, for any open subgroup $U \subset V$ of V, any nonzero finite Σ-torsion U-module A, and any nonnegative integer i such that $i \neq m$, it holds that $\log(\sharp H^m(U, A)) > M \log(\sharp H^j(U, A))$.*

The following result follows immediately from Theorem A.

Corollary B (cf. Corollary 2.16)**.** *Let X be a hyperbolic polycurve over k* (*resp. a hyperbolic polycurve over k satisfying condition $(*)_p$*)*. Then the dimension of X can be reconstructed group-theoretically from $\Delta_{X/k}$* (*resp. $\Delta^p_{X/k}$*)*.*

Remark B.1. In the special case that X/k is a configuration space of a hyperbolic curve over k (cf. e.g., [7] Definition 2.1), it has already been verified in [5] that the dimension can be reconstructed group-theoretically from $\Delta_{X/k}, \Delta^p_{X/k}$. (The reconstruction algorithm in [5] is different from our algorithm.)

The following generalization of Corollary B also follows from Theorem A.

Theorem C (cf. Corollary 2.20)**.** *Let p be a prime number, X, Y hyperbolic polycurves over k* (*resp. hyperbolic polycurves over k satisfying condition $(*)_p$*)*, and*

$$X = X_{\dim(X)} \to \cdots \to X_2 \to X_1 \to \operatorname{Spec} k = X_0,$$
$$Y = Y_{\dim(Y)} \to \cdots \to Y_2 \to Y_1 \to \operatorname{Spec} k = Y_0$$

sequences of parametrizing morphisms (*cf. Definition 2.1*(ii))*. Suppose that there exists an injective homomorphism $\Delta_{X/k} \hookrightarrow \Delta_{Y/k}$* (*resp. $\Delta^p_{X/k} \hookrightarrow \Delta^p_{Y/k}$*) *such that the image is normal in $\Delta_{Y/k}$* (*resp. $\Delta^p_{Y/k}$*)*. Then it holds that $\dim(X) \leq \dim(Y)$, $\sharp\{j \mid X_{j+1} \to X_j : \text{proper}\} \leq \sharp\{j \mid Y_{j+1} \to Y_j : \text{proper}\}$. Moreover, if $\dim(X) = \dim(Y)$, then $\Delta_{X/k} \hookrightarrow \Delta_{Y/k}$* (*resp. $\Delta^p_{X/k} \hookrightarrow \Delta^p_{Y/k}$*) *is open.*

On the other hand, by characterizing the condition that X is proper over k group-theoretically (by the method different from that of Theorem C), we obtain the following result.

Theorem D (cf. Corollary 2.24)**.** *Let p be a prime number and X, Y hyperbolic polycurves over k* (*resp. hyperbolic polycurves over k satisfying condition $(*)_p$*)*. Suppose that there exists an injective homomorphism $\Delta_{X/k} \hookrightarrow \Delta_{Y/k}$* (*resp. $\Delta^p_{X/k} \hookrightarrow \Delta^p_{Y/k}$*) *such that the image is normal in $\Delta_{Y/k}$* (*resp. $\Delta^p_{Y/k}$*)*. Suppose, moreover, that Y is proper over k. Then X is proper over k.*

1. Cohomology Groups of Profinite Groups

In the present §1, we study generalities on the cohomology groups of profinite groups. Let us fix a real number $q > 1$. Let $\mathfrak{Primes}$ be the set of all prime numbers.

Definition 1.1. Let G be a profinite group.

(i) A *G-module* A is a discrete abelian group A together with a continuous action of G on A.

(ii) Let A be a G-module. Then we shall write

$$H^n(G, A)$$

for the n-th cohomology group of G with coefficients in A. (For convenience, we set $H^n(G, A) := \{0\}$ for all $n < 0$.)

Theorem 1.2 (Hochschild-Serre spectral sequence (cf. e.g., [8] Theorem (2.4.1))). *Let G be a profinite group, $H \subset G$ a normal closed subgroup of G, and A a G-module. Then there exists a spectral sequence*

$$E_2^{ij} = H^i(G/H, H^j(H, A)) \Rightarrow H^{i+j}(G, A).$$

It is called the Hochshild-Serre spectral sequence.

Definition 1.3. Let G be a profinite group.

(i) Let A be a G-module. For each integer i, we shall write

$$h^i(G, A) := \log_q(\sharp H^i(G, A)).$$

(ii) Let A be a G-module. Suppose that $h^i(G, A) < \infty$ for any integer i, and that $h^i(G, A) = 0$ for all but finitely many integers i. Then we shall write

$$\chi(G, A) := \sum_{i=0}^{\infty} (-1)^i h^i(G, A).$$

In this case, we shall say that "$\chi(G, A)$ is defined".

(iii) Let $\Sigma \subset \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$. Suppose that there exists a (unique) constant $b \in \mathbb{R}$ such that, for any Σ-torsion G-module A (i.e., for any $a \in A$, there exists a positive integer n such that $na = 0$ and that every prime factor of n is contained in Σ), it holds that $\chi(G, A)$ is defined, and that $\chi(G, A) = b \log_q(\sharp A)$. Then we shall write

$$\chi_\Sigma(G) := b.$$

In this case, we shall say that "$\chi_\Sigma(G)$ is defined".

Remark 1.3.1.

(i) For the purpose of this paper, we can choose any $q > 1$. For this reason, we sometimes specify that $q = p$ in order to simplify calculations when we consider the case where A is a p-primary group (i.e., $\{p\}$-torsion group).

(ii) It is clear by definition that if $\chi_\Sigma(G)$ is defined, then $\chi_\Sigma(G)$ does not depend on q and $\chi_\Sigma(G) \in \mathbb{Z}$. Moreover, if $\chi_\Sigma(G)$ is defined, then, for any nonempty subset $\Sigma' \subset \Sigma$ of Σ, $\chi_{\Sigma'}(G)$ is also defined and it holds that $\chi_{\Sigma'}(G) = \chi_\Sigma(G)$.

(iii) If G is a pro-p group such that $\chi(G, \mathbb{F}_p)$ is defined, then it is well-known that $\chi_{\{p\}}(G)$ is defined. The value $\chi_{\{p\}}(G)$ is often called the Euler-Poincaré characteristic of G (cf. e.g., [12] §4.1).

Lemma 1.4. *Let $\Sigma \subset \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$. Then the following hold:*

(i) *Let G be a profinite group and U an open subgroup of G. Suppose that $\chi_\Sigma(G)$ is defined. Then $\chi_\Sigma(U)$ is also defined, and it holds that $\chi_\Sigma(U) = [G : U]\chi_\Sigma(G)$.*

(ii) *Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be a short exact sequence of profinite groups. Suppose that $\chi_\Sigma(G_3)$ is defined. Then for any finite Σ-torsion G_2-module A, if $\chi(G_1, A)$ is defined, then $\chi(G_2, A)$ is also defined, and it holds that $\chi(G_2, A) = \chi(G_1, A) \cdot \chi_\Sigma(G_3)$. In particular, if $\chi_\Sigma(G_1)$ is defined, then $\chi_\Sigma(G_2)$ is also defined, and it holds that $\chi_\Sigma(G_2) = \chi_\Sigma(G_1) \cdot \chi_\Sigma(G_3)$.*

Proof. First, we verify assertion (i). Let A be a finite Σ-torsion U-module. Then it holds that $\log_q(\sharp \operatorname{Ind}_G^U A) = [G : U] \log_q(\sharp A)$. Thus, since $H^i(U, A) = H^i(G, \operatorname{Ind}_G^U A)$ and $\operatorname{Ind}_G^U A$ is a finite Σ-torsion G-module, $\chi(U, A)$ is defined, and it holds that

$$\chi(U, A) = \chi(G, \operatorname{Ind}_G^U A) = \chi_\Sigma(G) \log_q(\sharp \operatorname{Ind}_G^U A) = [G : U]\chi_\Sigma(G) \log_q(\sharp A).$$

This implies that $\chi_\Sigma(U) = [G : U]\chi_\Sigma(G)$. This completes the proof of assertion (i). Next, we verify assertion (ii). Let A be a finite Σ-torsion G_2-module. Let us consider the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(G_3, H^j(G_1, A)) \Rightarrow H^{i+j}(G_2, A).$$

Note that for any nonnegative integer j, $H^j(G_1, A)$ is a finite Σ-torsion G_3-module. Since $\chi_\Sigma(G_3)$ and $\chi(G_1, A)$ are defined, $\chi(G_2, A)$ is also defined. Now it holds that

$$\begin{aligned}\sum_{i=0}^{\infty}(-1)^{i+j} \log_q(\sharp E_2^{ij}) &= \sum_{i=0}^{\infty}(-1)^{i+j} h^i(G_3, H^j(G_1, A)) \\ &= (-1)^j \chi(G_3, H^j(G_1, A)) \\ &= (-1)^j \chi_\Sigma(G_3) \cdot h^j(G_1, A).\end{aligned}$$

This implies that $\sum_{i,j=0}^{\infty}(-1)^{i+j} \log_q(\sharp E_2^{ij}) = \chi(G_1, A) \cdot \chi_\Sigma(G_3)$. On the other hand, one verifies easily that $\sum_{i,j=0}^{\infty}(-1)^{i+j} \log_q(\sharp E_r^{ij})$ does not depend on $r \geq 2$. Thus, it holds that $\chi(G_2, A) = \sum_{i,j=0}^{\infty}(-1)^{i+j} \log_q(\sharp E_\infty^{ij}) = \chi(G_1, A) \cdot \chi_\Sigma(G_3)$. If $\chi_\Sigma(G_1)$ is defined, then we have $\chi(G_2, A) = \chi(G_1, A) \cdot \chi_\Sigma(G_3) = \chi_\Sigma(G_1) \cdot \chi_\Sigma(G_3) \log_q(\sharp A)$, which implies that $\chi_\Sigma(G_2) = \chi_\Sigma(G_1) \cdot \chi_\Sigma(G_3)$. This completes the proof of assertion (ii). $\square$

Lemma 1.5 ([11] Lemma 2.17(i))**.** *Let G be a profinite group, $H \subset G$ a closed subgroup of G, and $V \subset H$ an open subgroup of H. Then there exists an open subgroup $U \subset G$ of G such that $V = H \cap U$.*

Lemma 1.6. *Let p be a prime number and G a profinite group. Suppose that $\operatorname{cd}_p G < \infty$. Then there exists an open subgroup $U \subset G$ of G such that $H^{\operatorname{cd}_p G}(U, \mathbb{F}_p) \neq \{0\}$.*

Proof. Let $G_p \subset G$ be a Sylow p-subgroup of G. Then it follows from [8] Corollary (3.3.6), Proposition (3.3.2) that $H^{\operatorname{cd}_p G}(G_p, \mathbb{F}_p) \neq \{0\}$. On the other hand, it follows from [10] Proposition 2.1.4(d), together with [8] Proposition (1.5.1), that $H^{\operatorname{cd}_p G}(G_p, \mathbb{F}_p) = \varinjlim_U H^{\operatorname{cd}_p G}(U, \mathbb{F}_p)$, where U runs over all open subgroups of G containing G_p. Thus, there exists an open subgroup $U \subset G$ of G such that $H^{\operatorname{cd}_p G}(U, \mathbb{F}_p) \neq \{0\}$. This completes the proof of Lemma 1.6. $\square$

Lemma 1.7 ([8] Proposition (3.3.8))**.** *Let p be a prime number, G a profinite group, and $H \subset G$ a normal closed subgroup of G. Then it holds that $\operatorname{cd}_p G \leq \operatorname{cd}_p(G/H) + \operatorname{cd}_p H$. If $\operatorname{cd}_p(G/H) < \infty$ and $\operatorname{cd}_p H < \infty$, and if $H^{\operatorname{cd}_p H}(U, \mathbb{F}_p)$ is finite for any open subgroup $U \subset H$ of H, then the equality holds.*

Lemma 1.8 ([8] Theorem (3.3.9))**.** *Let p be a prime number, G a profinite group, and $N \subset G$ a normal closed subgroup of G. Suppose that $\mathrm{cd}_p G < \infty$, and that $H^{\mathrm{cd}_p N}(N, \mathbb{F}_p)$ is finite and nonzero. Then it holds that $\mathrm{vcd}_p(G/N) = \mathrm{cd}_p G - \mathrm{cd}_p N$ (i.e., there exists an open subgroup $U \subset G/N$ of G/N such that $\mathrm{cd}_p U = \mathrm{cd}_p G - \mathrm{cd}_p N$).*

Corollary 1.9. *Let p be a prime number, G a profinite group, and $N \subset G$ a normal closed subgroup of G. Suppose that $\mathrm{cd}_p N < \infty$, and that $H^{\mathrm{cd}_p N}(U, \mathbb{F}_p)$ is finite for any open subgroup $U \subset N$ of N. Then it holds that $\mathrm{vcd}_p(G/N) = \mathrm{vcd}_p G - \mathrm{cd}_p N$.*

Proof. If $\mathrm{vcd}_p G = \infty$, then it follows from Lemma 1.7 that $\mathrm{vcd}_p(G/N) = \infty$. Now suppose that $\mathrm{vcd}_p G < \infty$. Let $W \subset G$ be an open subgroup of G such that $\mathrm{cd}_p W = \mathrm{vcd}_p G < \infty$. Then it follows from Lemma 1.6 that there exists an open subgroup $U \subset N \cap W$ of $N \cap W$ such that $H^{\mathrm{cd}_p N}(U, \mathbb{F}_p) \neq \{0\}$. Now it follows from Lemma 1.5 that there exists an open subgroup $V \subset W$ of W such that $V \cap N = U$. Then V/U is an open subgroup of G/N. Moreover, it follows from Lemma 1.8 that $\mathrm{vcd}_p(V/U) = \mathrm{cd}_p V - \mathrm{cd}_p U = \mathrm{cd}_p W - \mathrm{cd}_p N$. Thus, we conclude that $\mathrm{vcd}_p(G/N) = \mathrm{vcd}_p G - \mathrm{cd}_p N$. This completes the proof of Corollary 1.9. $\square$

2. Reconstruction of Invariants from Cohomology Groups

In the present §2, we consider reconstruction of invariants of certain profinite groups from their cohomology groups. As an application, we prove, among others, that the dimension of a hyperbolic polycurve (cf. Definition 2.1(ii)) over a field of characteristic zero can be reconstructed from its geometric fundamental group (cf. Corollary 2.16). Let us fix $q > 1$. Let k be a field of characteristic zero and $\mathfrak{Primes}$ the set of all prime numbers.

Definition 2.1 (cf. [4] Definition 2.1)**.** Let S be a scheme and X a scheme over S.

(i) We shall say that X is a *hyperbolic curve* (*of type* (g, r)) over S if there exist
  - a pair of nonnegative integers (g, r);
  - a scheme X^{cpt} which is smooth, proper, geometrically connected, and of relative dimension one over S;
  - a (possibly empty) closed subscheme $D \subset X^{\mathrm{cpt}}$ of X^{cpt} which is finite and étale over S

  such that
  - $2g - 2 + r > 0$;
  - any geometric fiber of $X^{\mathrm{cpt}} \to S$ is (a necessarily smooth proper curve) of genus g;
  - the finite étale covering $D \hookrightarrow X^{\mathrm{cpt}} \to S$ is of degree r;
  - X is isomorphic to $X^{\mathrm{cpt}} \setminus D$ over S.

(ii) We shall say that X is a *hyperbolic polycurve* (*of relative dimension* n) over S if there exist a positive integer n and a (not necessarily unique) factorization of the structure morphism $X \to S$

$$X = X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to S = X_0$$

such that, for each $i = 1, \dots, n$, $X_i \to X_{i-1}$ is a hyperbolic curve. We shall refer to the above morphism $X \to X_{n-1}$ as a *parametrizing morphism* for X and refer to the above factorization of $X \to S$ as a *sequence of parametrizing morphisms*.

Remark 2.1.1. A sequence of parametrizing morphisms of $X \to S$

$$X = X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to S = X_0$$

is not necessarily unique, but, when we call X/S a hyperbolic polycurve, we always fix a sequence of parametrizing morphisms of $X \to S$ unless otherwise specified.

Definition 2.2. Let X, Y be connected noetherian schemes and $f : X \to Y$ a morphism. Then we shall write

$$\Delta_f = \Delta_{X/Y} \subset \pi_1(X)$$

for the kernel of the outer homomorphism $\pi_1(X) \to \pi_1(Y)$ between étale fundamental groups induced by f.

Remark 2.2.1 (cf. [11] Remark 2.8). Let S be a connected noetherian separated normal scheme over k, and X a hyperbolic polycurve of relative dimension n over S. Then, for any triplet of integers (i, j, l) such that $0 \leq i < j < l \leq n$, we obtain a natural exact sequence of profinite groups

$$1 \to \Delta_{X_l/X_j} \to \Delta_{X_l/X_i} \to \Delta_{X_j/X_i} \to 1.$$

Definition 2.3. Let G be a group and $\Sigma \subset \mathfrak{Primes}$ a subset of $\mathfrak{Primes}$. Then we shall write

$$G^{\Sigma}$$

for the pro-Σ completion of G. Note that if G is a topologically finitely generated profinite group, then, since every homomorphism from G to any finite group is continuous (cf. [9] Theorem 1.1), G^{Σ} is the maximal pro-Σ quotient of G. Let p be a prime number. Then we shall write simply

$$G^p$$

for the pro-p group $G^{\{p\}}$. Moreover, we shall write simply

$$G^{\wedge}$$

for the profinite group $G^{\mathfrak{Primes}}$.

Definition 2.4. Let (g, r) be a pair of nonnegative integers. Then we shall write

$$\Pi_{g,r} := \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_r \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_r = 1 \rangle.$$

Note that if $r > 0$, then $\Pi_{g,r}$ is a free group of rank $2g + r - 1$.

Remark 2.4.1. Let S be a connected noetherian separated normal scheme over k, X a hyperbolic curve of type (g, r) over S, and $\Sigma \subset \mathfrak{Primes}$ a subset of $\mathfrak{Primes}$. Then it holds that $\Delta^{\Sigma}_{X/S} \cong \Pi^{\Sigma}_{g,r}$ (cf. e.g., [13] Proposition (1.1)(i), [4] Proposition 2.4(ii)). This profinite group is sometimes called a (pro-Σ) surface group (cf. [7] Definition 1.2). Note that any open subgroup of pro-Σ surface group is pro-Σ surface group.

Definition 2.5 (cf. [11] Definition 2.10)**.** Let p be a prime number, n a positive integer, S a connected noetherian separated normal scheme over k, and X a hyperbolic polycurve of relative dimension n over S. We shall say that X/S satisfies condition $(*)_p$ if for any triplet of integers (i, j, l) such that $0 \leq i < j < l \leq n$, the sequence of profinite groups

$$1 \to \Delta^p_{X_l/X_j} \to \Delta^p_{X_l/X_i} \to \Delta^p_{X_j/X_i} \to 1$$

is exact.

Remark 2.5.1. The validity of conditions $(*)_p$ depends on the sequence of parametrizing morphisms (at least by definition). So, precisely, we should say that

$$X = X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to S = X_0$$

satisfies condition $(*)_p$, but we shall say as in Definition 2.5 for simplicity.

Remark 2.5.2. If X/S satisfies condition $(*)_p$, then $\Delta_{X/S}$ admits various group-theoretic properties (cf. e.g., Corollary 2.8, [11] Proposition 2.16(iii)). However, it is unknown whether the validity of condition $(*)_p$ for X/S only depends on the profinite group $\Delta_{X/S}$ or not.

Definition 2.6. Let n be a positive integer. A *successive extension of surface groups* is data $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ consisting of

- a profinite group G;
- a sequence of profinite groups $(G_j)_{0\leq j\leq n}$;
- a sequence of nonempty sets of prime numbers $(\Sigma_j)_{1\leq j\leq n}$

such that

- $G_0 = G$, $G_n = \{1\}$;
- for any integer j such that $1 \leq j \leq n$, G_j is a normal closed subgroup of G_{j-1}, and, moreover, there exists a pair of nonnegative integers (g_j, r_j) such that $2g_j - 2 + r_j > 0$ and $G_{j-1}/G_j \cong \Pi^{\Sigma_j}_{g_j,r_j}$.

We shall refer to n as the *dimension* of $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$. For a prime number p, we shall refer to $n_p := \sharp\{j \mid p \in \Sigma_j\}$ as the *p-dimension* of $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$.

Example 2.6.1. Let n be a positive integer, S a connected noetherian separated normal scheme over k, and X a hyperbolic polycurve of relative dimension n over S. Then the data $(\Delta_{X/S}, (\Delta_{X/X_j})_{0\leq j\leq n}, (\mathfrak{Primes})_{1\leq j\leq n})$ is a successive extension of surface groups of dimension n. If, moreover, X/S satisfies condition $(*)_p$ (where p is a prime number), then $(\Delta^p_{X/S}, (\Delta^p_{X/X_j})_{0\leq j\leq n}, (\{p\})_{1\leq j\leq n})$ is also a successive extension of surface groups of dimension n.

Remark 2.6.2.

(i) If a profinite group G has two structures of successive extensions of surface groups $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$, $(G, (G'_j)_{0\leq j\leq n'}, (\Sigma'_j)_{1\leq j\leq n'})$, it is unknown (at least by definition) that they have the same dimension (or p-dimension).

(ii) Let $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ be a successive extension of surface groups and $U \subset G$ an open subgroup of G. Then U has a natural structure of a successive extension of surface groups $(U, (U\cap G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$. Moreover, for each integer i such that $0 \leq i < n$, $G_i \subset G$ has a natural structure of a successive extension of surface groups $(G_i, (G_{i+j})_{0\leq j\leq n-i}, (\Sigma_{i+j})_{1\leq j\leq n-i})$.

(iii) Since it holds that $\Sigma_j = \{p \in \mathfrak{Primes} \mid (G_{j-1}/G_j)^p \neq \{1\}\}$, the sequence $(\Sigma_j)_{1\leq j\leq n}$ of $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ is determined by $(G, (G_j)_{0\leq j\leq n})$.

Proposition 2.7. *Let $\Sigma \subset \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$ and (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$. Then the following hold:*

(i) *For any $p \in \Sigma$, if $r > 0$, then $\mathrm{cd}_p(\Pi^\Sigma_{g,r}) = 1$, and if $r = 0$, then $\mathrm{cd}_p(\Pi^\Sigma_{g,r}) = 2$.*

(ii) *For any Σ-torsion $\Pi^\Sigma_{g,r}$-module A and any nonnegative integer i, the natural homomorphism $H^i(\Pi^\Sigma_{g,r}, A) \to H^i(\Pi^\wedge_{g,r}, A)$ is an isomorphism.*

(iii) *For any finite Σ-torsion $\Pi^{\Sigma}_{g,r}$-module A, it holds that* $h^0(\Pi^{\Sigma}_{g,r}, A) \leq \log_q(\sharp A)$, $h^2(\Pi^{\Sigma}_{g,r}, A) \leq \log_q(\sharp A)$.

(iv) $\chi_\Sigma(\Pi^{\Sigma}_{g,r})$ *is defined, and it holds that* $\chi_\Sigma(\Pi^{\Sigma}_{g,r}) = 2 - 2g - r$.

Proof. Assertion (i) is well-known (cf. e.g., [12] §3.4, [1] Corollary 17). Assertion (ii) follows from assertion (i) and [1] Lemma 12 if $r > 0$, and follows from [1] Proposition 16 if $r = 0$. We verify assertions (iii) and (iv). Let A be a finite Σ-torsion $\Pi^{\Sigma}_{g,r}$-module. Let us consider the cohomology groups $H^i(\Pi_{g,r}, A)$ of the discrete group $\Pi_{g,r}$. Then it follows from [1] Lemma 13, Proposition 14 that for any nonnegative integer i, the natural homomorphism $H^i(\Pi^{\Sigma}_{g,r}, A) \to H^i(\Pi_{g,r}, A)$ is an isomorphism.

First, suppose that $r = 0$. Then it follows from [6] §11 that the $\mathbb{Z}[\Pi_{g,r}]$-module $\mathbb{Z}$ with trivial $\Pi_{g,r}$-action has a finite free resolution

$$0 \to \mathbb{Z}[\Pi_{g,r}] \to \mathbb{Z}[\Pi_{g,r}]^{\oplus 2g} \to \mathbb{Z}[\Pi_{g,r}] \to \mathbb{Z} \to 0.$$

Thus, $H^i(\Pi_{g,r}, A) \cong \mathrm{Ext}^i_{\mathbb{Z}[\Pi_{g,r}]}(\mathbb{Z}, A)$ is the i-th cohomology group of the complex

$$\mathrm{Hom}_{\mathbb{Z}[\Pi_{g,r}]}(\mathbb{Z}[\Pi_{g,r}], A) \to \mathrm{Hom}_{\mathbb{Z}[\Pi_{g,r}]}(\mathbb{Z}[\Pi_{g,r}]^{\oplus 2g}, A) \to \mathrm{Hom}_{\mathbb{Z}[\Pi_{g,r}]}(\mathbb{Z}[\Pi_{g,r}], A) \to 0.$$

This implies that

$$\sharp H^0(\Pi_{g,r}, A) \leq \sharp A,\ \sharp H^2(\Pi_{g,r}, A) \leq \sharp A,\ \frac{\sharp H^0(\Pi_{g,r}, A) \cdot \sharp H^2(\Pi_{g,r}, A)}{\sharp H^1(\Pi_{g,r}, A)} = \frac{\sharp A \cdot \sharp A}{\sharp A^{\oplus 2g}} = (\sharp A)^{2-2g}.$$

This completes the proof of assertions (iii) and (iv) in the case $r = 0$.

Next, suppose that $r > 0$. Then, since $\Pi_{g,r}$ is a free group of rank $2g + r - 1$, the $\mathbb{Z}[\Pi_{g,r}]$-module $\mathbb{Z}$ with trivial $\Pi_{g,r}$-action has a finite free resolution

$$0 \to \mathbb{Z}[\Pi_{g,r}]^{\oplus 2g+r-1} \to \mathbb{Z}[\Pi_{g,r}] \to \mathbb{Z} \to 0.$$

Thus, it follows from an argument similar to the above argument that

$$\sharp H^0(\Pi_{g,r}, A) \leq \sharp A,\ \sharp H^2(\Pi_{g,r}, A) = 1 \leq \sharp A,\ \frac{\sharp H^0(\Pi_{g,r}, A)}{\sharp H^1(\Pi_{g,r}, A)} = \frac{\sharp A}{\sharp A^{\oplus 2g+r-1}} = (\sharp A)^{2-2g-r}.$$

This completes the proof of assertions (iii) and (iv) in the case $r > 0$, hence also of Proposition 2.7. □

Corollary 2.8. *Let p be a prime number, S a connected noetherian separated normal scheme over k, X a hyperbolic polycurve over S satisfying condition $(*)_p$, and A a p-primary $\Delta^p_{X/S}$-module. Then, for any nonnegative integer i, the natural homomorphism $H^i(\Delta^p_{X/S}, A) \to H^i(\Delta_{X/S}, A)$ is an isomorphism.*

Proof. Write n for the relative dimension of the hyperbolic polycurve X/S. We verify Corollary 2.8 by induction on n. If $n = 1$, then Corollary 2.8 follows from Proposition 2.7(ii). Now suppose that $n \geq 2$, and that the induction hypothesis is in force. Then it follows from a construction of the Hochschild-Serre spectral sequence (cf. [8] Theorem (2.4.1)) that there exists a natural isomorphism between two Hochschild-Serre spectral sequences obtained by the exact sequences of profinite groups

$$1 \to \Delta_{X/X_1} \to \Delta_{X/S} \to \Delta_{X_1/S} \to 1,\ 1 \to \Delta^p_{X/X_1} \to \Delta^p_{X/S} \to \Delta^p_{X_1/S} \to 1.$$

This completes the proof of Corollary 2.8. □

Corollary 2.9. *Let* $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ *be a successive extension of surface groups. Suppose that* $\Sigma := \bigcap_{j=1}^{n} \Sigma_j \neq \emptyset$*. Then* $\chi_\Sigma(G)$ *is defined, and it holds that* $\chi_\Sigma(G) = \prod_{j=1}^{n} \chi_\Sigma(G_{j-1}/G_j)$*.*

Proof. This follows from Lemma 1.4(ii), Proposition 2.7(iv). □

Lemma 2.10. *Let* $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ *be a successive extension of surface groups and* A *a finite* G*-module. Then, for any nonnegative integer* i*,* $H^i(G, A)$ *is finite.*

Proof. We verify Lemma 2.10 by induction on n. If $n = 1$ and $p \in \Sigma_1$, then Lemma 2.10 follows from Proposition 2.7. If $n = 1$ and $p \notin \Sigma_1$, then since $\mathrm{cd}_p\, G = 0$, Lemma 2.10 follows from the fact that $H^0(G, A) = A^G$. Now suppose that $n \geq 2$, and that the induction hypothesis is in force. Then Lemma 2.10 follows from the Hochschild-Serre spectral sequence. This completes the proof of Lemma 2.10. □

Corollary 2.11. *Let* p *be a prime number and* $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ *a successive extension of surface groups. Then it holds that* $\mathrm{cd}_p\, G = \sum_{j=1}^{n} \mathrm{cd}_p(G_{j-1}/G_j)$*.*

Proof. This follows from Lemma 1.7, Lemma 2.10. □

Lemma 2.12. *Let* $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ *be a successive extension of surface groups,* M *a positive real number, and* A *a finite* G*-module. Then there exists an open subgroup* $U \subset G$ *of* G *such that, for any integers* i, j *such that* $i \geq 0$ *and* $0 < j \leq n$*,* U_{j-1}/U_j *acts trivially on* $H^i(U_j, A)$*, and, moreover, it holds that* $-\chi_{\Sigma_j}(U_{j-1}/U_j) > M$*, where we write* $U_j := U \cap G_j$ *for each integer* j *such that* $0 \leq j \leq n$*.*

Proof. We verify Lemma 2.12 by induction on n. If $n = 1$, then, since $-\chi_{\Sigma_1}(G) \geq 1$ (cf. Proposition 2.7(iv)), it follows from Lemma 1.4(i) that an open subgroup $U \subset G$ of G such that $[G : U] > M$ satisfies the conditions appearing in the statement of Lemma 2.12. Now suppose that $n \geq 2$, and that the induction hypothesis is in force. Then it follows from the induction hypothesis that there exists an open subgroup $V \subset G_1$ of G_1 such that, for any integers i, j such that $i \geq 0$ and $1 < j \leq n$, V_{j-1}/V_j acts trivially on $H^i(V_j, A)$, and, moreover, it holds that $-\chi_{\Sigma_j}(V_{j-1}/V_j) > M$, where we write $V_j := V \cap G_j$ for each integer j such that $1 \leq j \leq n$. Then it follows from Lemma 1.5 that there exists an open subgroup $W \subset G$ of G such that $W \cap G_1 = V$. On the other hand, it follows from Proposition 2.7(i) and Corollary 2.11 that $\mathrm{cd}\, V_1 \leq \mathrm{cd}\, G = \sup_p \mathrm{cd}_p\, G \leq 2n < \infty$, which implies that for all but finitely many integers i, it holds that $H^i(V_1, A) = \{0\}$. Moreover, it follows from Lemma 2.10 that $H^i(V_1, A)$ is finite. Thus, there exists an open subgroup $H \subset W/V$ of W/V such that for any nonnegative integer i, H acts trivially on $H^i(V_1, A)$. By replacing H by an open subgroup of W/V contained in H if necessary, we may assume that $[W/V : H] > M$. Then it follows from Lemma 1.4(i) and Proposition 2.7(iv) that $-\chi_{\Sigma_1}(H) > M$.

Write $U \subset W$ for the inverse image of $H \subset W/V$ by the natural surjection $W \twoheadrightarrow W/V$, and $U_j := U \cap G_j$ for each integer j such that $0 \leq j \leq n$. Then since $V \subset U \subset W$, it holds that $U_1 = U \cap G_1 = V$, which implies that, for each integer j such that $1 \leq j \leq n$, it holds that $U_j = V_j$. Moreover, it holds that $U_0/U_1 = U/V = H \subset W/V$. Thus, it follows from our choice of V and H that U satisfies the conditions appearing in the statement of Lemma 2.12. This completes the proof of Lemma 2.12. □

Proposition 2.13. *Let $(G, (G_j)_{0 \leq j \leq n}, (\Sigma_j)_{1 \leq j \leq n})$ be a successive extension of surface groups and $\Sigma \subset \mathfrak{Primes}$ a nonempty subset of $\mathfrak{Primes}$. Then the following conditions are equivalent:*

1. *$\Sigma \subset \bigcap_{j=1}^{n} \Sigma_j$.*
2. *$\chi_\Sigma(G)$ is defined.*
3. *For each $p \in \Sigma$ and for any open subgroup $U \subset G$ of G, it holds that $\chi(U, \mathbb{F}_p) = [G : U]\chi(G, \mathbb{F}_p)$, where the action of G on $\mathbb{F}_p$ is trivial.*

In particular, $\bigcap_{j=1}^{n} \Sigma_j$ is completely determined by the profinite group G.

Proof. The implication (1) $\Rightarrow$ (2) follows from Corollary 2.9, and the implication (2) $\Rightarrow$ (3) follows from Lemma 1.4(i). Now we verify the implication (3) $\Rightarrow$ (1). Suppose that $\Sigma \not\subset \bigcap_{j=1}^{n} \Sigma_j$. Let p be a prime number such that $p \in \Sigma \setminus \bigcap_{j=1}^{n} \Sigma_j$. Write l_0 for the minimum integer l such that $1 \leq l \leq n$ and $p \notin \Sigma_l$. Then it follows from Lemma 2.12 that there exists an open subgroup $V \subset G$ of G such that, for any integers i, j such that $i \geq 0$ and $0 < j \leq n$, V_{j-1}/V_j acts trivially on $H^i(V_j, A)$, where we write $V_j := V \cap G_j$ for each integer j such that $0 \leq j \leq n$.

Let $W \subsetneq V_{l_0-1}/V_{l_0}$ be a proper open subgroup of V_{l_0-1}/V_{l_0}. Then it follows from Lemma 1.5 that there exists an open subgroup $U \subset V$ of V such that, if we write $U_j := U \cap V_j$ for each integer j such that $0 \leq j \leq n$, then $U_{l_0-1} \subset V_{l_0-1}$ is the inverse image of $W \subsetneq V_{l_0-1}/V_{l_0}$ by the surjection $V_{l_0-1} \twoheadrightarrow V_{l_0-1}/V_{l_0}$. Note that since $U_{l_0-1} \supset V_{l_0}$, it holds that $U_j = V_j$ for each integer j such that $l_0 \leq j \leq n$. Now it follows from our choice of l_0, U, V that for each integer j such that $1 \leq j \leq n$ and $p \notin \Sigma_j$, V_{j-1}/V_j (resp. U_{j-1}/U_j) acts trivially on $H^i(V_j, \mathbb{F}_p)$ (resp. $H^i(U_j, \mathbb{F}_p)$). Thus, if $p \notin \Sigma_j$, then it holds that $H^i(V_{j-1}, \mathbb{F}_p) \cong H^i(V_j, \mathbb{F}_p)$, $H^i(U_{j-1}, \mathbb{F}_p) \cong H^i(U_j, \mathbb{F}_p)$. In particular, if $p \notin \Sigma_j$, then it holds that $\chi(V_{j-1}, \mathbb{F}_p) = \chi(V_j, \mathbb{F}_{|})$, $\chi(U_{j-1}, \mathbb{F}_p) = \chi(U_j, \mathbb{F}_{|})$. On the other hand, if $p \in \Sigma_j$, then it follows from Lemma 1.4(ii) that $\chi(V_{j-1}, \mathbb{F}_p) = \chi(V_j, \mathbb{F}_{|}) \cdot \chi_{\{p\}}(V_{j-1}/V_j)$, $\chi(U_{j-1}, \mathbb{F}_p) = \chi(U_j, \mathbb{F}_{|}) \cdot \chi_{\{p\}}(U_{j-1}/U_j)$. This implies that

$$\chi(V, \mathbb{F}_p) = \log_q p \cdot \prod_{\substack{1 \leq j \leq n \\ p \in \Sigma_j}} \chi_{\{p\}}(V_{j-1}/V_j) (\neq 0), \chi(U, \mathbb{F}_p) = \log_q p \cdot \prod_{\substack{1 \leq j \leq n \\ p \in \Sigma_j}} \chi_{\{p\}}(U_{j-1}/U_j).$$

Thus, it follows from Lemma 1.4(i) that

$$\begin{aligned} \frac{\chi(U, \mathbb{F}_p)}{\chi(V, \mathbb{F}_p)} &= \prod_{\substack{1 \leq j \leq n \\ p \in \Sigma_j}} \frac{\chi_{\{p\}}(U_{j-1}/U_j)}{\chi_{\{p\}}(V_{j-1}/V_j)} \\ &= \prod_{j=1}^{l_0-1} \frac{\chi_{\{p\}}(U_{j-1}/U_j)}{\chi_{\{p\}}(V_{j-1}/V_j)} \\ &= \prod_{j=1}^{l_0-1} [V_{j-1}/V_j : U_{j-1}/U_j] \\ &= \frac{[V : U]}{[V_{l_0-1}/V_{l_0} : U_{l_0-1}/U_{l_0}]} \neq [V : U]. \end{aligned}$$

On the other hand, if condition (3) holds, then we have

$$\chi(U, \mathbb{F}_p) = [G : U]\chi(G, \mathbb{F}_p) = [V : U] \cdot [G : V]\chi(G, \mathbb{F}_p) = [V : U]\chi(V, \mathbb{F}_p).$$

This completes the proof of the implication (3) $\Rightarrow$ (1), hence also of Proposition 2.13. $\square$

Lemma 2.14. *Let p be a prime number, $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ a successive extension of surface groups, A a nonzero finite p-primary G-module, and M a positive real number.*

(i) *Write $j_1 < j_2 < \cdots < j_{n_p}$ for integers such that $p \in \Sigma_{j_m}$ for each m. Suppose that for each integer m such that $1 \leq m \leq n_p$, it holds that*

$$-\chi_{\{p\}}(G_{j_m-1}/G_{j_m}) \geq \mathrm{cd}(G_{j_m-1}/G_{j_m}) \cdot 3^m M,$$

and, moreover, that for any integers i, j such that $i \geq 0$, $0 < j \leq n$ and $p \notin \Sigma_j$, G_{j-1}/G_j acts trivially on $H^i(G_j, A)$. Then for each nonnegative integer i such that $i \neq n_p$, it holds that $h^{n_p}(G, A) > Mh^i(G, A)$.

(ii) *There exists an open subgroup $U \subset G$ of G such that, for each nonnegative integer i such that $i \neq n_p$, it holds that $h^{n_p}(U, A) > Mh^i(U, A)$.*

Proof. We verify assertion (i) by induction on n. If $n = 1$ and $p \in \Sigma_1$, then $n_p = 1$, and for $i = 0, 2$, it follows from Proposition 2.7(iii) that

$$h^1(G, A) \geq -\chi_{\{p\}}(G) \log_q(\sharp A) + h^i(G, A) > -\chi_{\{p\}}(G)h^i(G, A) \geq Mh^i(G, A).$$

If $p \notin \Sigma_1$, then $n_p = 0, \mathrm{cd}_p\, G = 0$, and G acts trivially on $H^0(G_1, A) = A$. Thus, it holds that $H^0(G, A) = A$, which implies that $h^0(G, A) = \log_q(\sharp A) > 0$. This completes the proof of assertion (i) in the case $n = 1$.

Now suppose that $n \geq 2$, and that the induction hypothesis is in force. First, suppose that $p \notin \Sigma_1$. Then it follows from the induction hypothesis that, for each nonnegative integer i such that $i \neq n_p$, it holds that $h^{n_p}(G_1, A) > Mh^i(G_1, A)$. On the other hand, since $p \notin \Sigma_1$, it holds that $\mathrm{cd}_p(G/G_1) = 0$. Thus, since (we have assumed that) for each nonnegative integer i, G/G_1 acts trivially on $H^i(G_1, A)$, we obtain an isomorphism $H^i(G, A) \cong H^i(G_1, A)$, which implies that for each nonnegative integer i such that $i \neq n_p$, we have $h^{n_p}(G, A) > Mh^i(G, A)$.

Finally, suppose that $p \in \Sigma_1$. Then it follows from the induction hypothesis that, for each nonnegative integer i such that $i \neq n_p - 1$, it holds that $h^{n_p-1}(G_1, A) > 3Mh^i(G_1, A)$. Let us write $h^{i,j} := h^i(G/G_1, H^j(G_1, A))$ for each pair of integers (i, j). Then it follows from the Hochschild-Serre spectral sequence that

$$h^{n_p}(G, A) \geq h^{1,n_p-1} \geq -\chi_{\{p\}}(G/G_1)h^{n_p-1}(G, A).$$

Moreover, it follows from Proposition 2.7(iii),(iv) that, if $i \neq 1$ and $j \neq n_p - 1$, then we have

$$h^{i,j} \leq h^j(G_1, A) < \frac{1}{3M} h^{n_p-1}(G_1, A),$$

if $i \neq 1$ and $j = n_p - 1$, then we have

$$h^{i,j} \leq h^{n_p-1}(G_1, A),$$

and if $i = 1$ and $j \neq n_p - 1$, then we have

$$h^{i,j} \leq (-\chi_p(G/G_1) + \mathrm{cd}(G/G_1))h^j(G_1, A) < \frac{-\chi_p(G/G_1) + \mathrm{cd}(G/G_1)}{3M} h^{n_p-1}(G_1, A).$$

Now suppose that $\mathrm{cd}(G/G_1) = 1$. Then it follows from the Hochschild-Serre spectral sequence that for any nonnegative integer i, we have $h^i(G, A) \leq h^{0,i} +$

$h^{1,i-1}$. Thus, if $i \neq n_p$, then we have

$$
\begin{aligned}
Mh^i(G,A) &\leq M(h^{0,i} + h^{1,i-1}) \\
&< M\left(\max\left\{\frac{1}{3M}, 1\right\} + \frac{-\chi_{\{p\}}(G/G_1)+1}{3M}\right) h^{n_p-1}(G_1,A) \\
&= \frac{1}{3}(-\chi_{\{p\}}(G/G_1) + 1 + \max\{1, 3M\}) h^{n_p-1}(G_1,A) \\
&\leq \frac{-\chi_{\{p\}}(G/G_1) + 1 + \max\{1,3M\}}{-3\chi_{\{p\}}(G/G_1)} h^{n_p}(G,A).
\end{aligned}
$$

Since

$$
\frac{-\chi_{\{p\}}(G/G_1)+2}{-3\chi_{\{p\}}(G/G_1)} \leq 1, \ \frac{-\chi_{\{p\}}(G/G_1)+1+3M}{-3\chi_{\{p\}}(G/G_1)} \leq \frac{2}{3} + \frac{M}{3M} = 1,
$$

we conclude that $Mh^i(G,A) < h^{n_p}(G,A)$.

Next, suppose that $\mathrm{cd}(G/G_1) = 2$. In this case, it follows from Proposition 2.7(i), (iv) that $-\chi_{\{p\}}(G/G_1) \geq \max\{2, 6M\}$. Moreover, it follows from the Hochschild-Serre spectral sequence that for any nonnegative integer i, we have $h^i(G,A) \leq h^{0,i} + h^{1,i-1} + h^{2,i-2}$. Thus, if $i \neq n_p$, then we have

$$
\begin{aligned}
Mh^i(G,A) &\leq M(h^{0,i} + h^{1,i-1} + h^{2,i-2}) \\
&< M\left(\max\left\{\frac{1}{3M}, 1\right\} + \frac{1}{3M} + \frac{-\chi_{\{p\}}(G/G_1)+2}{3M}\right) h^{n_p-1}(G_1,A) \\
&= \frac{1}{3}(-\chi_{\{p\}}(G/G_1) + 3 + \max\{1, 3M\}) h^{n_p-1}(G_1,A) \\
&\leq \frac{-\chi_{\{p\}}(G/G_1) + 3 + \max\{1,3M\}}{-3\chi_{\{p\}}(G/G_1)} h^{n_p}(G,A).
\end{aligned}
$$

Since

$$
\frac{-\chi_{\{p\}}(G/G_1)+4}{-3\chi_{\{p\}}(G/G_1)} \leq 1, \ \frac{-\chi_{\{p\}}(G/G_1)+3+3M}{-3\chi_{\{p\}}(G/G_1)} \leq \frac{5}{6} + \frac{M}{6M} = 1,
$$

we conclude that $Mh^i(G,A) < h^{n_p}(G,A)$. This completes the proof of assertion (i). Assertion (ii) follows from assertion (i), together with Lemma 2.12. This completes the proof of Lemma 2.14. $\square$

Theorem 2.15. *Let $(G, (G_j)_{0 \leq j \leq n}, (\Sigma_j)_{1 \leq j \leq n})$ be a successive extension of surface groups and m a nonnegative integer. Write $\Sigma := \bigcap_{j=1}^n \Sigma_j$. Suppose that $\Sigma \neq \emptyset$. Then the following conditions are equivalent:*

(1) *$m = n$.*
(2) *For any positive real number M, there exists an open subgroup $V \subset G$ of G such that, for any open subgroup $U \subset V$ of V, any nonzero finite Σ-torsion U-module A, and any nonnegative integer i such that $i \neq m$, it holds that $h^m(U,A) > Mh^i(U,A)$.*
(3) *There exist a positive real number M and an open subgroup $V \subset G$ of G such that, for any open subgroup $U \subset V$, there exists a nonzero finite Σ-torsion U-module A such that, for any nonnegative integer i such that $i \neq m$, it holds that $h^m(U,A) > Mh^i(U,A)$.*

(4) *There exist a positive real number M and an open subgroup $U \subset G$ of G such that, for any nonzero finite Σ-torsion U-module A and any nonnegative integer i such that $i \neq m$, it holds that $h^m(U, A) > Mh^i(U, A)$.*

(5) *There exist an open subgroup $U \subset G$ of G and a nonzero finite Σ-torsion U-module A such that, for any nonnegative integer i such that $i \neq m$, it holds that $h^m(U, A) \geq 3^{\operatorname{cd} G} h^i(U, A)$.*

In particular, n is completely determined by the profinite group G.

Proof. The implications (2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4), and (2) $\Rightarrow$ (5) are immediate. First, we verify the implication (1) $\Rightarrow$ (2). It follows from Lemma 2.12 that there exists an open subgroup $V \subset G$ of G such that, for each integer j such that $0 < j \leq n$, it holds that $-\chi_\Sigma((V \cap G_{j-1})/(V \cap G_j)) \geq 2 \cdot 3^j M$. Then it follows from Lemma 2.14(i) that, for any $p \in \Sigma$, any open subgroup $U \subset V$ of V, any finite nonzero p-primary U-module A, and any nonnegative integer i such that $i \neq n$, it holds that $h^n(U, A) = h^n(V, \operatorname{Ind}_V^U A) > Mh^i(V, \operatorname{Ind}_V^U A) = Mh^i(U, A)$. Since any finite Σ-torsion U-module is isomorphic to a direct sum of finite p-primary U-modules (where $p \in \Sigma$), the same inequality holds for any nonzero finite Σ-torsion U-module. This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (3) $\Rightarrow$ (1). Suppose that condition (3) is satisfied. Let M be a positive real number and $V \subset G$ an open subgroup of G satisfying the condition appearing in (3). Then, since (we have already verified that) the implication (1) $\Rightarrow$ (2) holds, there exists an open subgroup $W \subset G$ of G satisfying the condition appearing in (2) for "V", where we take the data "(m, M)" to be (n, M^{-1}). Now it follows from our choice of V that there exists a nonzero finite Σ-torsion $V \cap W$-module A such that, for any nonnegative integer i such that $i \neq m$, it holds that $h^m(V \cap W, A) > Mh^i(V \cap W, A)$. On the other hand, it follows from our choice of W that, for any nonnegative integer i such that $i \neq n$, it holds that $h^n(V \cap W, A) > M^{-1} h^i(V \cap W, A)$. Thus, we conclude that $m = n$. This completes the proof of the implication (3) $\Rightarrow$ (1).

Next, we verify the implication (4) $\Rightarrow$ (1). Suppose that condition (4) is satisfied. Let M be a positive real number and $U \subset G$ an open subgroup of G satisfying the condition appearing in (4). Then, since (we have already verified that) the implication (1) $\Rightarrow$ (2) holds, there exists an open subgroup $V \subset G$ of G satisfying the condition appearing in (2), where we take the data "(m, M)" to be (n, M^{-1}). Let A be a nonzero finite Σ-torsion $U \cap V$-module. Then it follows from our choice of U that, for any nonnegative integer i such that $i \neq m$, it holds that $h^m(U \cap V, A) = h^m(U, \operatorname{Ind}_U^{U \cap V} A) > Mh^i(U, \operatorname{Ind}_U^{U \cap V} A) = Mh^i(U \cap V, A)$. Thus, by applying an argument similar to the argument used in the proof of the implication (3) $\Rightarrow$ (1), we conclude that $m = n$. This completes the proof of the implication (4) $\Rightarrow$ (1).

Finally, we verify the implication (5) $\Rightarrow$ (1). Let $p \in \Sigma$. Suppose that condition (5) is satisfied. Let us take an open subgroup $U \subset G$ of G and a nonzero finite Σ-torsion U-module A satisfying condition (5). Moreover, it follows from Proposition 2.7(i), Corollary 2.11 that $\operatorname{cd} G = \operatorname{cd}_p G = \sum_{j=1}^n \operatorname{cd}_p(G_{j-1}/G_j) \geq n$. Thus, by applying Lemma 2.14(i), where we take $M = 3^{-\operatorname{cd} G}$, for each nonnegative integer i such that $i \neq n$, we have $h^n(U, A) > 3^{-\operatorname{cd} G} h^i(U, A)$, which implies that $m = n$. This completes the proof of the implication (5) $\Rightarrow$ (1), hence also of Theorem 2.15. $\square$

Remark 2.15.1. It is unknown whether the p-dimension n_p of $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ is completely determined by the profinite group G or not (where p is a prime number such that $p \notin \bigcap_{j=1}^{n} \Sigma_j$).

Corollary 2.16. *Let p be a prime number, S a connected noetherian separated normal scheme over k, and X a hyperbolic polycurve over S (resp. a hyperbolic polycurve over S satisfying condition $(*)_p$). Then the relative dimension of X/S can be reconstructed group-theoretically from $\Delta_{X/S}$ (resp. $\Delta^p_{X/S}$).*

Proof. This follows from Theorem 2.15 (cf. Example 2.6.1). □

Corollary 2.17. *Let p be a prime number, S a connected noetherian separated normal scheme over k, and X a hyperbolic polycurve over S (resp. a hyperbolic polycurve over S satisfying condition $(*)_p$). Write n for the relative dimension of X/S. Then it holds that $\sharp\{j \mid X_{j+1} \to X_j : \mathit{proper}\} = \mathrm{cd}(\Delta_{X/S}) - n$ (resp. $\sharp\{j \mid X_{j+1} \to X_j : \mathit{proper}\} = \mathrm{cd}(\Delta^p_{X/S}) - n$). In particular, $\sharp\{j \mid X_{j+1} \to X_j : \mathit{proper}\}$ can be reconstructed group-theoretically from $\Delta_{X/S}$ (resp. $\Delta^p_{X/S}$).*

Proof. The first assertion follows from Proposition 2.7(i), Corollary 2.11. The second assertion follows from the first assertion, together with Corollary 2.16. □

Remark 2.17.1. Corollary 2.17 implies a purely algebrico-geometric fact that the number $\sharp\{j \mid X_{j+1} \to X_j : \text{proper}\}$ does not depend on the sequence of parametrizing morphisms of $X \to S$. The author does not know at the time of writing whether we can prove the above fact only by using a purely algebrico-geometric method or not.

Lemma 2.18. *Let p be a prime number and $1 \to N \to G \to H \to 1$ an exact sequence of profinite groups. Let us consider the H-module $\mathbb{F}_p$ with trivial action. Then the following hold:*

(i) *Suppose that for each nonnegative integer i, it holds that $H^i(N, \mathbb{F}_p)$ and $H^i(G, \mathbb{F}_p)$ are finite, and that the action of H on $H^i(N, \mathbb{F}_p)$ is trivial. Then, for each nonnegative integer i, $H^i(H, \mathbb{F}_p)$ is finite.*

(ii) *Let M be a positive real number and n_1, n_2 nonnegative integers. Suppose that the conditions appearing in* (i) *are satisfied. Suppose, moreover, that the following conditions are satisfied:*
   (1) $\mathrm{cd}_p H < \infty$.
   (2) *For each nonnegative integer i such that $i \neq n_1$, it holds that*
   $$h^{n_1}(G, \mathbb{F}_p) > (\mathrm{cd}_p H + 2M + 1)h^i(G, \mathbb{F}_p).$$
   (3) *For each nonnegative integer i such that $i \neq n_2$, it holds that*
   $$h^{n_2}(N, \mathbb{F}_p) > (\mathrm{cd}_p H + 2M + 1)h^i(N, \mathbb{F}_p).$$

   Then, for each nonnegative integer i such that $i \neq n_1 - n_2$, it holds that $h^{n_1-n_2}(H, \mathbb{F}_p) > Mh^i(H, \mathbb{F}_p)$. In particular, $0 \leq n_1 - n_2 \leq \mathrm{cd}_p H$. Moreover, if $M \geq 1$ and $n_1 = n_2$, then, for each nonnegative integer i, it holds that $H^i(G, \mathbb{F}_p) \cong H^i(N, \mathbb{F}_p)$.

Proof. For simplicity, we assume that $q = p$ (cf. Remark 1.3.1(i)). Note that we have the Hochschild-Serre spectral sequence
$$E_2^{ij} = H^i(H, H^j(N, \mathbb{F}_p)) \Rightarrow H^{i+j}(G, \mathbb{F}_p)$$

associated to the exact sequence $1 \to N \to G \to H \to 1$. First, we verify assertion (i) by induction on i. If $i = 0$, then assertion (i) is immediate. Now suppose that $i \geq 1$, and that for each nonnegative integer m such that $m < i$, $H^m(H, \mathbb{F}_p)$ is finite. Then since (we have assumed that) for each nonnegative integer j, H acts trivially on $H^j(N, \mathbb{F}_p)$, it follows from the fact that $H^j(N, \mathbb{F}_p) \cong \mathbb{F}_p^{h^j(N,\mathbb{F}_p)}$, together with the induction hypothesis, that for each nonnegative integer m such that $m < i$, it holds that $E_2^{mj} = H^m(H, H^j(N, \mathbb{F}_p)) \cong H^m(H, \mathbb{F}_p)^{h^j(N,\mathbb{F}_p)}$ is finite. Thus, for each integers j, m, r such that $j \geq 0, 0 \leq m < i, r \geq 2$, it holds that E_r^{mj} is finite. In particular, for each integer r such that $r \geq 2$, it holds that $E_r^{i-r,r-1}$, hence also $\mathrm{Im}(E_r^{i-r,r-1} \to E_r^{i,0})$, is finite. On the other hand, since $H^i(G, \mathbb{F}_p)$ is finite, it holds that $E_\infty^{i,0}$ is finite. Thus, it follows from the fact that $E_{r+1}^{i,0}$ is isomorphic to $E_r^{i,0}/\mathrm{Im}(E_r^{i-r,r-1} \to E_r^{i,0})$, together with the finiteness of $\mathrm{Im}(E_r^{i-r,r-1} \to E_r^{i,0})$, that for each integer r such that $r \geq 2$, $E_r^{i,0}$ is finite. In particular, $H^i(H, \mathbb{F}_p) = E_2^{i,0}$ is finite. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us choose a nonnegative integer i_0 such that $i_0 \neq n_1 - n_2$ and that $h^{i_0}(H, \mathbb{F}_p)$ is largest among $\{h^i(H, \mathbb{F}_p)\}_{i \neq n_1 - n_2}$. To verify the first assertion, it suffices to verify that $h^{n_1-n_2}(H, \mathbb{F}_p) > Mh^{i_0}(H, \mathbb{F}_p)$. Now, since for each pair of nonnegative integers (i, j), it holds that

$$H^i(H, H^j(N, \mathbb{F}_p)) \cong H^i(H, \mathbb{F}_p^{h^j(N,\mathbb{F}_p)}) \cong H^i(H, \mathbb{F}_p)^{h^j(N,\mathbb{F}_p)},$$

we have

$$\log_p(\sharp E_\infty^{ij}) \leq \log_p(\sharp E_2^{ij}) = h^i(H, H^j(N, \mathbb{F}_p)) = h^i(H, \mathbb{F}_p) \cdot h^j(N, \mathbb{F}_p).$$

Thus, it follows from our choice of i_0, together with assumption (3), that

$$\begin{aligned} h^{n_1}(G, \mathbb{F}_p) &= \sum_{i+j=n_1} \log_p(\sharp E_\infty^{ij}) \\ &\leq \sum_{i+j=n_1} h^i(H, \mathbb{F}_p) \cdot h^j(N, \mathbb{F}_p) \\ &= h^{n_1-n_2}(H, \mathbb{F}_p) \cdot h^{n_2}(N, \mathbb{F}_p) + \sum_{\substack{0 \leq i \leq \mathrm{cd}_p H \\ i \neq n_1 - n_2}} h^i(H, \mathbb{F}_p) \cdot h^{n_1-i}(N, \mathbb{F}_p) \\ &\leq h^{n_1-n_2}(H, \mathbb{F}_p) \cdot h^{n_2}(N, \mathbb{F}_p) + \frac{\mathrm{cd}_p H + 1}{\mathrm{cd}_p H + 2M + 1} h^{i_0}(H, \mathbb{F}_p) \cdot h^{n_2}(N, \mathbb{F}_p) \\ &\leq (h^{n_1-n_2}(H, \mathbb{F}_p) + h^{i_0}(H, \mathbb{F}_p)) h^{n_2}(N, \mathbb{F}_p). \end{aligned}$$

On the other hand, it holds that

$$\begin{aligned} \log_p(\sharp E_{r+1}^{i_0,n_2}) &\geq \log_p(\sharp E_r^{i_0,n_2}) - \log_p(\sharp E_r^{i_0+r,n_2-r+1}) - \log_p(\sharp E_r^{i_0-r,n_2+r-1}) \\ &\geq \log_p(\sharp E_r^{i_0,n_2}) - \log_p(\sharp E_2^{i_0+r,n_2-r+1}) - \log_p(\sharp E_2^{i_0-r,n_2+r-1}), \end{aligned}$$

which implies that

$$\begin{aligned} \log_p(\sharp E_\infty^{i_0,n_2}) &= \log_p(\sharp E_{\mathrm{cd}_p H+1}^{i_0,n_2}) \\ &\geq \log_p(\sharp E_2^{i_0,n_2}) - \sum_{r=2}^{\mathrm{cd}_p H} (\log_p(\sharp E_2^{i_0+r,n_2-r+1}) + \log_p(\sharp E_2^{i_0-r,n_2+r-1})) \\ &= h^{i_0}(H, \mathbb{F}_p) \cdot h^{n_2}(N, \mathbb{F}_p) - \sum_{r=2}^{\mathrm{cd}_p H} (\log_p(\sharp E_2^{i_0+r,n_2-r+1}) + \log_p(\sharp E_2^{i_0-r,n_2+r-1})). \end{aligned}$$

Moreover, it follows from our choice of i_0, together with assumption (3), that

$$\begin{aligned}
&\sum_{r=2}^{\mathrm{cd}_p H} (\log_p(\sharp E_2^{i_0+r, n_2-r+1}) + \log_p(\sharp E_2^{i_0-r, n_2+r-1})) \\
&\quad = \sum_{r=2}^{\mathrm{cd}_p H} (h^{i_0+r}(H, \mathbb{F}_p) \cdot h^{n_2-r+1}(N, \mathbb{F}_p) + h^{i_0-r}(H, \mathbb{F}_p) \cdot h^{n_2+r-1}(N, \mathbb{F}_p)) \\
&\quad \leq \frac{1}{\mathrm{cd}_p H + 2M + 1} h^{n_2}(N, \mathbb{F}_p) \sum_{r=2}^{\mathrm{cd}_p H} (h^{i_0+r}(H, \mathbb{F}_p) + h^{i_0-r}(H, \mathbb{F}_p)) \\
&\quad \leq \frac{1}{\mathrm{cd}_p H + 2M + 1} h^{n_2}(N, \mathbb{F}_p) \sum_{\substack{0 \leq r \leq \mathrm{cd}_p H \\ r \neq i_0}} h^r(H, \mathbb{F}_p) \\
&\quad \leq \frac{1}{\mathrm{cd}_p H + 2M + 1} h^{n_2}(N, \mathbb{F}_p)(\mathrm{cd}_p H \cdot h^{i_0}(H, \mathbb{F}_p) + h^{n_1-n_2}(H, \mathbb{F}_p)).
\end{aligned}$$

Thus, we conclude that

$$\log_p(\sharp E_\infty^{i_0, n_2}) \geq \frac{1}{\mathrm{cd}_p H + 2M + 1}((2M+1)h^{i_0}(H, \mathbb{F}_p) - h^{n_1-n_2}(H, \mathbb{F}_p))h^{n_2}(N, \mathbb{F}_p).$$

Now, since $i_0 \neq n_1 - n_2$, we have $(\mathrm{cd}_p H + 2M + 1)h^{i_0+n_2}(G, \mathbb{F}_p) < h^{n_1}(G, \mathbb{F}_p)$ (cf. assumption (2)). Moreover, it is clear that $h^{i_0+n_2}(G, \mathbb{F}_p) \geq \log_p(\sharp E_\infty^{i_0, n_2})$. Thus, we conclude that

$$(2M+1)h^{i_0}(H, \mathbb{F}_p) - h^{n_1-n_2}(H, \mathbb{F}_p) < h^{n_1-n_2}(H, \mathbb{F}_p) + h^{i_0}(H, \mathbb{F}_p),$$

which implies that $h^{n_1-n_2}(H, \mathbb{F}_p) > M h^{i_0}(H, \mathbb{F}_p)$. This completes the proof of the first assertion.

Finally, suppose that $M \geq 1$ and $n_1 = n_2$. Then, for each positive integer i, it holds that $\sharp H^i(H, \mathbb{F}_p) < \sharp H^0(H, \mathbb{F}_p) = p$, i.e., $H^i(H, \mathbb{F}_p) = \{0\}$. Thus, since (we have assumed that) the action of H on $H^j(N, \mathbb{F}_p)$ is trivial, we have

$$E_\infty^{ij} \cong E_2^{ij} \cong \begin{cases} H^j(N, \mathbb{F}_p) & (i = 0) \\ \{0\} & (i \neq 0), \end{cases}$$

which implies that $H^i(G, \mathbb{F}_p) \cong H^i(N, \mathbb{F}_p)$. This completes the proof of assertion (ii), hence also of Lemma 2.18. □

Theorem 2.19. *Let p be a prime number and $(G, (G_j)_{0 \leq j \leq n}, (\Sigma_j)_{1 \leq j \leq n})$, $(G', (G'_j)_{0 \leq j \leq n'}, (\Sigma'_j)_{1 \leq j \leq n'})$ successive extensions of surface groups of dimension n, n', respectively. Write n_p, n'_p for the p-dimension of $(G, (G_j)_{0 \leq j \leq n}, (\Sigma_j)_{1 \leq j \leq n})$, $(G', (G'_j)_{0 \leq j \leq n'}, (\Sigma'_j)_{1 \leq j \leq n'})$, respectively. Suppose that there exists an injective homomorphism $G \hookrightarrow G'$ such that the image is normal in G'. Suppose, moreover, that $p \in \bigcap_{j=1}^{n'} \Sigma'_j$. Then it holds that $n_p \leq n'_p$, $\mathrm{cd}_p G - n_p \leq \mathrm{cd}_p G' - n'_p$. Moreover, if $n_p = n'_p$, then $G \hookrightarrow G'$ is open, and $n = n' = n_p = n'_p$.*

Proof. Let us regard G as a normal closed subgroup of G' via $G \hookrightarrow G'$. It follows from Corollary 1.9, Lemma 2.10 that $\mathrm{vcd}_p(G'/G) = \mathrm{cd}_p G' - \mathrm{cd}_p G$. Thus, there exists an open subgroup $H \subset G'/G$ of G'/G such that $\mathrm{cd}_p H = \mathrm{cd}_p G' - \mathrm{cd}_p G$. Write $H' \subset G'$ for the inverse image of $H \subset G'/G$ by the natural surjection $G' \twoheadrightarrow G'/G$. Then it follows from the implication (1) ⇒ (2) of Theorem 2.15 that there exists an open subgroup $V' \subset H'$ of H' such that, for any open subgroup $U' \subset V'$ of V' and nonnegative integer i such that $i \neq n'_p = n'$, it holds that $h^{n'_p}(U', \mathbb{F}_p) > (\mathrm{cd}_p G' - \mathrm{cd}_p G + 3) h^i(U', \mathbb{F}_p)$. Moreover, it follows from Lemma 2.14(ii) that there exists an open subgroup $W \subset V' \cap G$ of $V' \cap G$ such that, for each nonnegative integer i such that $i \neq n_p$, it holds that $h^{n_p}(W, \mathbb{F}_p) > (\mathrm{cd}_p G' - \mathrm{cd}_p G + 3) h^i(W, \mathbb{F}_p)$. Now it follows from Lemma 1.5 that there exists an open subgroup $W' \subset V'$ such that $W' \cap G = W$. Then, since W'/W is an open subgroup of H, it follows from Lemma 2.10 that there exists an open subgroup $U \subset W'/W$ such that, for each nonnegative integer i, U acts trivially on $H^i(W, \mathbb{F}_p)$.

Write $U' \subset W'$ for the inverse image of $U \subset W'/W$ by the natural surjection $W' \twoheadrightarrow W'/W$. Then it follows from our choice of W, U', U that

- the sequence $1 \to W \to U' \to U \to 1$ is exact,
- W (resp. U', U) is an open subgroup of G (resp. G', H),
- for each nonnegative integer i, the action of U on $H^i(W, \mathbb{F}_p)$ is trivial,
- for each nonnegative integer i such that $i \neq n_p$, it holds that
$$h^{n_p}(W, \mathbb{F}_p) > (\mathrm{cd}_p G' - \mathrm{cd}_p G + 3) h^i(W, \mathbb{F}_p),$$
- for each nonnegative integer i such that $i \neq n'_p$, it holds that
$$h^{n'_p}(U', \mathbb{F}_p) > (\mathrm{cd}_p G' - \mathrm{cd}_p G + 3) h^i(U', \mathbb{F}_p).$$

Thus, it follows from Lemma 2.18(ii), together with Lemma 2.10, that $0 \leq n'_p - n_p \leq \mathrm{cd}_p U = \mathrm{cd}_p H = \mathrm{cd}_p G' - \mathrm{cd}_p G$, which implies that $n_p \leq n'_p, \mathrm{cd}_p G - n_p \leq \mathrm{cd}_p G' - n'_p$.

Next, assume that $n_p = n'_p$ and U is nontrivial. Then it follows from Lemma 2.18(ii) that, for each nonnegative integer i, it holds that $H^i(U', \mathbb{F}_p) \cong H^i(W, \mathbb{F}_p)$. In particular, we have $\chi(U', \mathbb{F}_p) = \chi(W, \mathbb{F}_p)$. On the other hand, since U is nontrivial, there exists a proper open subgroup $\widetilde{U} \subsetneq U$ of U. Write $\widetilde{U}' \subset U'$ for the inverse image of $\widetilde{U} \subsetneq U$ by the surjection $U' \twoheadrightarrow U$. Then, by applying an argument similar to the above argument, we have $\chi(\widetilde{U}', \mathbb{F}_p) = \chi(W, \mathbb{F}_p)$. Thus, we conclude that $\chi(U', \mathbb{F}_p) = \chi(\widetilde{U}', \mathbb{F}_p)$. However, since $\chi_{\{p\}}(U')$ and $\chi_{\{p\}}(\widetilde{U}')$ are defined and nonzero (cf. Proposition 2.7(iv), Corollary 2.9), it follows from Lemma 1.4(i) that $\chi_{\{p\}}(\widetilde{U}') = [U' : \widetilde{U}'] \chi_{\{p\}}(U') \neq \chi_{\{p\}}(U')$, which contradicts the equation above. Thus, if $n_p = n'_p$, then it holds that $U = \{1\}$, hence $U' = W$. This implies that $G \hookrightarrow G'$ is open. Then, since $(G, (G \cap G'_j)_{0 \leq j \leq n'}, (\Sigma'_j)_{1 \leq j \leq n'})$ is a successive extension of surface groups, it follows from Proposition 2.13 that $\bigcap_{j=1}^{n} \Sigma_j = \bigcap_{j=1}^{n'} \Sigma'_j \ni p$. Thus, it holds that $n = n_p = n'_p = n'$. This completes the proof of Theorem 2.19. $\square$

Corollary 2.20. *Let p be a prime number, k_1, k_2 fields of characteristic zero, S, T connected noetherian separated normal schemes over k_1, k_2, respectively, and X, Y hyperbolic polycurves over S, T (resp. hyperbolic polycurves over S, T satisfying condition $(*)_p$), respectively. Write n_X, n_Y for the relative dimension of $X/S, Y/T$, respectively. Suppose that there exists an injective homomorphism $\Delta_{X/S} \hookrightarrow \Delta_{Y/T}$ (resp. $\Delta^p_{X/S} \hookrightarrow \Delta^p_{Y/T}$) such that the image is normal in $\Delta_{Y/T}$ (resp. $\Delta^p_{Y/T}$). Then it holds that $n_X \leq n_Y$, $\sharp\{j \mid X_{j+1} \to X_j : proper\} \leq \sharp\{j \mid Y_{j+1} \to Y_j : proper\}$. Moreover, if $n_X = n_Y$, then $\Delta_{X/S} \hookrightarrow \Delta_{Y/T}$ (resp. $\Delta^p_{X/S} \hookrightarrow \Delta^p_{Y/T}$) is open.*

Proof. This follows from Corollary 2.17, Theorem 2.19. $\square$

Proposition 2.21. *Let p be a prime number and $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ a successive extension of surface groups of p-dimension n_p. Let us consider the G-module $\mathbb{F}_p$ with trivial action. Then the following conditions are equivalent:*

(1) $\mathrm{cd}_p G = 2n_p$.
(2) *For any open subgroup $U \subset G$ of G, it holds that $\sharp H^{\mathrm{cd}_p G}(U, \mathbb{F}_p) \leq p$.*

Proof. For simplicity, we assume that $q = p$ (cf. Remark 1.3.1(i)). For each integer j such that $1 \leq j \leq n$, let (g_i, r_i) be a pair of nonnegative integers such that $G_{j-1}/G_j \cong \Pi^{\Sigma_j}_{g_j,r_j}$. Since it holds that

$$\mathrm{cd}_p G = \sum_{j=1}^{n} \mathrm{cd}_p(G_{j-1}/G_j) = \sum_{j:\, p\in\Sigma_j} \mathrm{cd}_p(G_{j-1}/G_j)$$

(cf. Corollary 2.11), condition (1) holds if and only if $\mathrm{cd}_p(G_{j-1}/G_j) = 2$ for each j such that $p \in \Sigma_j$, i.e., $r_j = 0$ for each j such that $p \in \Sigma_j$ (cf. Proposition 2.7(i)).

Now we verify the implication (1) $\Rightarrow$ (2). Suppose that condition (1) is satisfied. We verify that condition (2) holds by induction on n. If $n = 1$ and $p \notin \Sigma_1$, then condition (2) immediately holds. If $n = 1$ and $p \in \Sigma_1$, then it follows from Proposition 2.7(iii) that condition (2) holds. Now suppose that $n \geq 2$, and that the induction hypothesis is in force. Let $U \subset G$ be an open subgroup of G. Write $U_1 := U \cap G_1$. Note that since $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ satisfies condition (1), $(G_1, (G_{j+1})_{0\leq j\leq n-1}, (\Sigma_{j+1})_{1\leq j\leq n-1})$ also satisfies condition (1). Thus, it follows from the induction hypothesis that $\sharp H^{\mathrm{cd}_p G_1}(U_1, \mathbb{F}_p) \leq p$. If $p \notin \Sigma_1$, then, since it holds that $H^{\mathrm{cd}_p G}(U, \mathbb{F}_p) = (H^{\mathrm{cd}_p G_1}(U_1, \mathbb{F}_p))^{U/U_1}$, we have $\sharp H^{\mathrm{cd}_p G}(U, \mathbb{F}_p) \leq \sharp H^{\mathrm{cd}_p G_1}(U_1, \mathbb{F}_p) \leq p$. If $p \in \Sigma_1$, then, since it holds that $H^{\mathrm{cd}_p G}(U, \mathbb{F}_p) \cong H^2(U/U_1, H^{\mathrm{cd}_p G_1}(U_1, \mathbb{F}_p))$, it follows from Proposition 2.7(iii) that $\sharp H^{\mathrm{cd}_p G}(U, \mathbb{F}_p) \leq \sharp H^{\mathrm{cd}_p G_1}(U_1, \mathbb{F}_p) \leq p$. This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that $\mathrm{cd}_p G \neq 2n_p$. We verify that there exists an open subgroup $U \subset G$ of G such that $\sharp H^{\mathrm{cd}_p G}(U, \mathbb{F}_p) > p$ (i.e., $h^{\mathrm{cd}_p G}(U, \mathbb{F}_p) > 1$) by induction on n. If $n = 1$, then, since $\mathrm{cd}_p G \neq 2n_p$, it holds that $p \in \Sigma_1$, $r_1 \neq 0$. Thus, it follows from Proposition 2.7(iv) that $h^1(G, \mathbb{F}_p) = h^0(G, \mathbb{F}_p) - \chi_{\{p\}}(G) > 1$. Now suppose that $n \geq 2$, and that the induction hypothesis is in force. Then it follows from Lemma 1.6 that there exists an open subgroup $V \subset G_1$ of G_1 such that $H^{\mathrm{cd}_p G_1}(V, \mathbb{F}_p) \neq \{0\}$. Note that, by the induction hypothesis, if $(G_1, (G_{j+1})_{0\leq j\leq n-1}, (\Sigma_{j+1})_{1\leq j\leq n-1})$ does not satisfy condition (1), then, by replacing V by an open subgroup of V if necessary, we may assume that $h^{\mathrm{cd}_p G_1}(V, \mathbb{F}_p) > 1$.

On the other hand, it follows from Lemma 1.5 that there exists an open subgroup $W \subset G$ of G such that $W \cap G_1 = V$. Moreover, since $H^{\mathrm{cd}_p G_1}(V, \mathbb{F}_p)$ is finite (cf. Lemma 2.10), there exists an open subgroup $H \subset W/V$ of W/V (hence also of G/G_1) such that H acts trivially on $H^{\mathrm{cd}_p G_1}(V, \mathbb{F}_p)$. By replacing H by an open subgroup of H if necessary, we may assume that $H^{\mathrm{cd}_p (G/G_1)}(H, \mathbb{F}_p) \neq \{0\}$ (cf. Lemma 1.6). Write $U \subset W$ for the inverse image of $H \subset W/V$ by the natural surjection $W \twoheadrightarrow W/V$. Then it follows from the Hochschild-Serre spectral sequence that

$$H^{\mathrm{cd}_p G}(U, \mathbb{F}_p) \cong H^{\mathrm{cd}_p (G/G_1)}(H, H^{\mathrm{cd}_p G_1}(V, \mathbb{F}_p)),$$

which implies that $h^{\mathrm{cd}_p G}(U, \mathbb{F}_p) = h^{\mathrm{cd}_p G_1}(V, \mathbb{F}_p) \cdot h^{\mathrm{cd}_p (G/G_1)}(H, \mathbb{F}_p)$. Note that at least one of $(G_1, (G_{j+1})_{0\leq j\leq n-1}, (\Sigma_{j+1})_{1\leq j\leq n-1})$ and $(G/G_1, (G_j/G_1)_{0\leq j\leq 1}, (\Sigma_1)_{1\leq j\leq 1})$

does not satisfy condition (1). If $(G_1, (G_{j+1})_{0\leq j\leq n-1}, (\Sigma_{j+1})_{1\leq j\leq n-1})$ does not satisfy condition (1), then, since we have assumed that $h^{\mathrm{cd}_p G_1}(V, \mathbb{F}_p) > 1$, we have $h^{\mathrm{cd}_p G}(U, \mathbb{F}_p) > 1$. If $(G/G_1, (G_j/G_1)_{0\leq j\leq 1}, (\Sigma_1)_{1\leq j\leq 1})$ does not satisfy condition (1), then it follows from an argument similar to the argument in the case of $n = 1$ that $h^{\mathrm{cd}_p(G/G_1)}(H, \mathbb{F}_p) > 1$, which implies that $h^{\mathrm{cd}_p G}(U, \mathbb{F}_p) > 1$. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Proposition 2.21. $\square$

Corollary 2.22. *Let p be a prime number, S a connected noetherian separated normal scheme over k, and X a hyperbolic polycurve over S (resp. a hyperbolic polycurve over S satisfying condition $(*)_p$). Let us consider the $\Delta_{X/S}$ (resp. $\Delta^p_{X/S}$)-module $\mathbb{F}_p$ with trivial action. Then the following conditions are equivalent:*

1. *$X \to S$ is proper.*
2. *For each integer j such that $1 \leq j \leq n$, $X_j \to X_{j-1}$ is proper.*
3. *For any open subgroup U of $\Delta_{X/S}$ (resp. $\Delta^p_{X/S}$), it holds that $\sharp H^{\mathrm{cd}(\Delta_{X/S})}(U, \mathbb{F}_p) \leq p$.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) follows from Corollary 2.17, Proposition 2.21. Moreover, the implication (2) $\Rightarrow$ (1) is trivial. We verify the implication (1) $\Rightarrow$ (2) by induction on n. If $n = 1$, then the implication (1) $\Rightarrow$ (2) is immediate. Now suppose that $n \geq 2$, and that the induction hypothesis is in force. Suppose that condition (1) is satisfied. Then, since $X_1 \to S$ is separated, $X \to X_1$ is proper. Thus, it follows from the induction hypothesis that for each integer j such that $2 \leq j \leq n$, $X_j \to X_{j-1}$ is proper. On the other hand, it is clear that $X_1 \to X_0 = S$ is separated and of finite type, $X \to X_1$ is surjective, and $X \to X_0$ is universally closed. Thus, since surjectivity is stable under base change, it holds that $X_1 \to X_0$ is universally closed, hence proper. This completes the proof of the implication (1) $\Rightarrow$ (2), hence also of Corollary 2.22. $\square$

Remark 2.22.1. In fact, by comparing the profinite cohomology groups with the étale cohomology groups, we can verify that the conditions appearing in Corollary 2.22 are equivalent to the following condition:

(3)′ For any open subgroup U of $\Delta_{X/S}$ (resp. $\Delta^p_{X/S}$), it holds that $H^{\mathrm{cd}(\Delta_{X/S})}(U, \mathbb{F}_p) \cong \mathbb{F}_p$.

Corollary 2.23. *Let p be a prime number and $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$, $(G', (G'_j)_{0\leq j\leq n'}, (\Sigma'_j)_{1\leq j\leq n'})$ successive extensions of surface groups of p-dimension n_p, n'_p, respectively. Suppose that there exists an injective homomorphism $G \hookrightarrow G'$ such that the image is normal in G'. Suppose, moreover, that $\mathrm{cd}_p G' = 2n'_p$. Then it holds that $\mathrm{cd}_p G = 2n_p$.*

Proof. For simplicity, we assume that $q = p$ (cf. Remark 1.3.1(i)). Let us regard G as a normal closed subgroup of G' via $G \hookrightarrow G'$. Suppose that $\mathrm{cd}_p G \neq 2n_p$. Then it follows from Proposition 2.21 that there exists an open subgroup $N \subset G$ of G such that $\sharp H^{\mathrm{cd}_p G}(N, \mathbb{F}_p) > p$. On the other hand, it follows from Lemma 1.5 that there exists an open subgroup $V \subset G'$ of G' such that $V \cap G = N$. Note that it follows from Lemma 2.10 that $H^{\mathrm{cd}_p G}(N, \mathbb{F}_p)$ is finite. Moreover, it follows from Corollary 1.9 that $\mathrm{vcd}_p(V/N) = \mathrm{cd}_p G' - \mathrm{cd}_p G$. Thus, there exists an open subgroup $H \subset V/N$ of V/N such that $\mathrm{cd}_p H = \mathrm{cd}_p G' - \mathrm{cd}_p G$, and the action of H on $H^{\mathrm{cd}_p G}(N, \mathbb{F}_p)$ is trivial. By replacing H by an open subgroup of H if necessary, we may assume that $H^{\mathrm{cd}_p G' - \mathrm{cd}_p G}(H, \mathbb{F}_p) \neq \{0\}$ (cf. Lemma 1.6).

Write $U \subset V$ for the inverse image of $H \subset V/N$ by the natural surjection $V \twoheadrightarrow V/N$. Then U is an open subgroup of G', and the sequence $1 \to N \to U \to H \to 1$ is exact. Thus, it follows from the Hochschild-Serre spectral sequence that

$$H^{\mathrm{cd}_p G'}(U, \mathbb{F}_p) \cong H^{\mathrm{cd}_p G' - \mathrm{cd}_p G}(H, H^{\mathrm{cd}_p G}(N, \mathbb{F}_p)),$$

which implies that $h^{\mathrm{cd}_p G'}(U, \mathbb{F}_p) = h^{\mathrm{cd}_p G}(N, \mathbb{F}_p) \cdot h^{\mathrm{cd}_p G' - \mathrm{cd}_p G}(H, \mathbb{F}_p) > 1$. However, since (we have assumed that) $\mathrm{cd}_p G' = 2n'_p$, we obtain a contradiction (cf. Proposition 2.21). This completes the proof of Corollary 2.23. □

Corollary 2.24. *Let p be a prime number, k_1, k_2 fields of characteristic zero, S, T connected noetherian separated normal schemes over k_1, k_2, respectively, and X, Y hyperbolic polycurves over S, T (resp. hyperbolic polycurves over S, T satisfying condition $(*)_p$), respectively. Suppose that there exists an injective homomorphism $\Delta_{X/S} \hookrightarrow \Delta_{Y/T}$ (resp. $\Delta^p_{X/S} \hookrightarrow \Delta^p_{Y/T}$) such that the image is normal in $\Delta_{Y/T}$ (resp. $\Delta^p_{Y/T}$). Suppose, moreover, that $Y \to T$ is proper. Then $X \to S$ is proper.*

Proof. This follows from Corollary 2.23, together with Corollary 2.17 and the equivalence (1) ⇔ (2) of Corollary 2.22. □

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