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# Formulas for Local and Global $p$-Ranks of Coverings of Curves

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**Abstract**

In the present paper, we investigate the $p$-ranks of coverings of curves. Let $G$ be a finite $p$-group, $f : \mathscr{Y} \longrightarrow \mathscr{X}$ a morphism of pointed semi-stable curves over a complete discrete valuation ring $R$ with algebraically closed residue field of characteristic $p > 0$, and $x$ a closed point of $\mathscr{X}$. Write $\eta$ for the generic point of $S := \operatorname{Spec} R$, and $s$ for the closed point of $S$. Suppose that the generic fiber $\mathscr{X}_\eta$ of $\mathscr{X}$ is a smooth pointed stable curve over $\eta$, and that the morphism $f_\eta : \mathscr{Y}_\eta \longrightarrow \mathscr{X}_\eta$ induced by $f$ on generic fibers is a Galois covering whose Galois group is isomorphic to $G$, and whose branch locus is contained in the set of marked points of $\mathscr{X}_\eta$. If $f^{-1}(x)$ is not a finite set, we shall call $x$ a vertical point associated to $f$ and $f^{-1}(x)$ the vertical fiber associated to $x$. We give an explicit formula for the $p$-rank $\sigma(\mathscr{Y}_s)$ of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ and an explicit formula for the $p$-rank $\sigma(f^{-1}(x))$ of the vertical fiber $f^{-1}(x)$ associated to $x$. The formula for the $p$-rank $\sigma(\mathscr{Y}_s)$ can be regarded as a relative version of the Deuring-Shafarevich formula, and the formula for the $p$-rank $\sigma(f^{-1}(x))$ generalizes a result of M. Saïdi concerning the formula for the $p$-rank $\sigma(f^{-1}(x))$ to the case where $G$ is an arbitrary $p$-group.

Keywords: $p$-rank, pointed semi-stable curve, pointed semi-stable covering, semi-graph with $p$-rank, Deuring-Shafarevich formula.

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# 1 Introduction

Let $C$ be a smooth projective curve over an algebraically closed field of characteristic $p > 0$. There are two natural invariants associated to $C$: the genus $g(C)$ and the $p$-rank $\sigma(C) := \dim_{\mathbb{F}_p} \mathrm{H}^1_{\text{ét}}(C, \mathbb{F}_p)$. These two invariants determine, respectively, the isomorphism classes (as profinite groups) of the maximal pro-$\Sigma$ and pro-$p$ quotients of the étale fundamental group $\pi_1(C)$ of $C$, for $\Sigma$ a set of prime numbers which does not contain $p$. The genus and $p$-rank have some similar properties. Let $G$ be a finite group and $h : C' \longrightarrow C$ a $G$-Galois covering of smooth projective curves (i.e., the extension of function fields $K(C')/K(C)$ induced by $h$ is a Galois extension with Galois group $G$). The genus $g(C')$ of $C'$ can be calculated by the Riemann-Hurwitz formula. In particular, if $(\sharp G, p) = 1$, then the Riemann-Hurwitz formula has the following form:

$$g(C') = \sharp G \cdot (g(C) - 1) + \sum_{c' \in (C')^{\mathrm{cl}}} (e_{c'} - 1)/2 + 1,$$

where $(C')^{\mathrm{cl}}$ denotes the set of the closed points of $C'$, $e_{c'}$ denotes the ramification index at $c'$, and $\sharp G$ denotes the order of $G$. If $G$ is a $p$-group, then we have the Deuring-Shafarevich formula (cf. [C]) for the $p$-rank $\sigma(C')$, as follows:

$$\sigma(C') = \sharp G \cdot (\sigma(C) - 1) + \sum_{c' \in (C')^{\mathrm{cl}}} (e_{c'} - 1) + 1,$$

where $(C')^{\mathrm{cl}}$ denotes the set of the closed points of $C'$, $e_{c'}$ denotes the ramification index at $c'$, and $\sharp G$ denotes the order of $G$. In the present paper, we study the geometry of coverings of curves over a complete discrete valuation ring and prove a **relative version of the Deuring-Shafarevich formula** (cf. Theorem 4.5).

Let $R$ be a complete valuation ring with algebraically closed residue field $k$ of characteristic $p > 0$. Write $K$ for the quotient field of $R$, $S := \operatorname{Spec} R$, $\eta : \operatorname{Spec} K \longrightarrow S$ and $s : \operatorname{Spec} k \longrightarrow S$ for the natural morphisms. Let $G$ be a finite group and $\mathscr{X} = (X, D_X)$ a pointed semi-stable curve of genus $g_X$ over $S$, where $X$ denotes the underlying curve of $\mathscr{X}$ and $D_X$ denotes the set of marked points (each of which is a section $S \longrightarrow X$ of $X \longrightarrow S$) of $\mathscr{X}$. Write $\mathscr{X}_\eta = (X_\eta, D_{X_\eta})$ and $\mathscr{X}_s = (X_s, D_{X_s})$ for the result of base-changing $\mathscr{X}$ by $\eta$ and $s$, respectively. Moreover, we suppose that $\mathscr{X}_\eta$ is a smooth pointed stable curve over $\eta$.

Let $\mathscr{Y}_\eta = (Y_\eta, D_{Y_\eta})$ be a smooth pointed stable curve over $\eta$ and $f_\eta : \mathscr{Y}_\eta \longrightarrow \mathscr{X}_\eta$ a morphism of pointed stable curves over $\eta$. Write $\mathrm{Im}_{D_{Y_\eta}}$ and $\mathrm{Im}_{D_{X_\eta}}$ for the sets of the images of the elements of $D_{Y_\eta}$ and $D_{X_\eta}$, respectively. Suppose that $f_\eta$ is a Galois covering whose Galois group is isomorphic to $G$ such that $f_\eta^{-1}(\mathrm{Im}_{D_{X_\eta}}) = \mathrm{Im}_{D_{Y_\eta}}$, and that the branch locus of $f_\eta$ is contained in $\mathrm{Im}_{D_{X_\eta}}$. By replacing $S$ by a finite extension of $S$

(i.e., the spectrum of the normalization of $R$ in a finite extension of $K$), we may assume that $f_\eta$ extends uniquely to a $G$-**pointed semi-stable covering** (cf. Definition 3.3) $f : \mathscr{Y} = (Y, D_Y) \longrightarrow \mathscr{X}$ over $S$ (cf. Proposition 3.4). We are interested in understanding the structure of the special fiber $\mathscr{Y}_s = (Y_s, D_{Y_s})$ of $\mathscr{Y}$. If the order $\sharp G$ of $G$ is prime to $p$, then by the specialization theorem for log étale fundamental groups, the morphism $f_s : \mathscr{Y}_s \longrightarrow \mathscr{X}_s$ on special fibers induced by $f$ is an admissible covering (cf. [V], [Y1]); thus, $\mathscr{Y}_s$ may be obtained by gluing together tame coverings of the irreducible components of $\mathscr{X}_s$. On the other hand, if $p|\sharp G$, then $f_s$ is not a finite morphism in general. For example, if $\mathrm{char}(K) = 0, \mathrm{char}(k) = p > 0$, and $\mathscr{X}$ is a stable curve (i.e., $D_X = \emptyset$), then there exists a Zariski dense subset $Z$ of the set of closed points $X^{\mathrm{cl}}$ of $X$, which may in fact be taken to be $X^{\mathrm{cl}}$ when $k$ is an algebraic closure of $\mathbb{F}_p$, such that for any $x \in Z$, after possibly replacing $S$ by a finite extension of $S$, there exist a finite group $H$ and an $H$-pointed semi-stable covering $f_{\mathscr{W}} : \mathscr{W} \longrightarrow \mathscr{X}$ over $S$ such that the fiber $(f_{\mathscr{W}})^{-1}(x)$ is not finite (cf. [T], [Y2]). If $f^{-1}(x)$ is not finite, then we shall call $x$ a **vertical point associated to** $f$ and call $f^{-1}(x)$ the **vertical fiber associated to** $x$ (cf. Definition 3.6).

In order to investigate the properties of $\mathscr{Y}_s$, we focus on the $p$-rank $\sigma(\mathscr{Y}_s)$ of $\mathscr{Y}_s$ (cf. Definition 3.1 (b)). By the definition of the $p$-rank of a pointed semi-stable curve, to calculate $\sigma(\mathscr{Y}_s)$, it suffices to calculate $\dim_{\mathbb{C}} \mathrm{H}^1(\Gamma_{\mathscr{Y}_s}, \mathbb{C})$ (where $\Gamma_{\mathscr{Y}_s}$ denotes the dual semi-graph of $\mathscr{Y}_s$ (cf. Definition 3.1 (a))), the $p$-ranks of the irreducible components of $\mathscr{Y}_s$ which are finite over $\mathscr{X}_s$, and the $p$-ranks of the vertical fibers of $f$. In the present paper, we study the $p$-ranks of vertical fibers and special fibers of $G$-pointed semi-stable coverings and consider the following Question:

**Question 1.1.** *Let $G$ be a finite $p$-group and $f : \mathscr{Y} \longrightarrow \mathscr{X}$ a $G$-pointed semi-stable covering over $S$.*

*(Global Version): Does there exist an explicit formula for the $p$-rank $\sigma(\mathscr{Y}_s)$ of $\mathscr{Y}_s$ in terms of the dual semi-graph of $\mathscr{X}_s$ and the inertia subgroups of the irreducible components and marked points of $\mathscr{Y}_s$?*

*(Local Version): Let $x$ be a vertical point associated to $f$. Then does there exist an explicit formula for the $p$-rank $\sigma(f^{-1}(x))$ of $f^{-1}(x)$ in terms of the inertia subgroups of the irreducible components and marked points of $f^{-1}(x)$?*

If the vertical point $x$ is not contained in $D_{X_s}$, Question 1.1 (Local Version) had been studied by M. Raynaud and M. Saïdi. If $x$ is a smooth point of $\mathscr{X}_s$ which is not contained in $D_{X_s}$, then Raynaud proved that $\sigma(f^{-1}(x)) = 0$ (cf. [R, Théorème and Proposition 2 (ii)]). If $G$ is a cyclic $p$-group, and $x$ is a singular point of $\mathscr{X}_s$, then an explicit formula has been obtained by M. Saïdi (cf. [S, Proposition 1] and Corollary 4.9 of the present paper).

The main theorems of the present paper give an answer to Question 1.1 (cf. Theorem 4.5 for the global version and Theorem 4.6 for the local version). Theorem 4.5 can be regarded as a certain analogue of the Deuring-Shafarevich formula for $G$-pointed semi-stable coverings over $S$. On the other hand, if $x$ is a singular point of $\mathscr{X}_s$, then the explicit formula for $\sigma(f^{-1}(x))$ assumes a simple form (cf. Theorem 4.7), which generalizes Saïdi's result to the case where $G$ is an arbitrary $p$-group.

The present paper is organized as follows. In Section 2, we introduce a kind of purely combinatorial object called a **semi-graph with $p$-rank**. We define $p$-ranks, coverings, and $G$-coverings for semi-graphs with $p$-rank; then we calculate the $p$-ranks of $G$-coverings of semi-graphs with $p$-rank. In Section 3, by using the theory of semi-stable curves, we prove that for any $G$-pointed semi-stable covering, one may construct a $G$-covering of semi-graphs with $p$-rank associated to the $G$-pointed semi-stable covering (resp. the vertical fibers of the $G$-pointed semi-stable covering) in a natural way. In Section 4, we study the relationship between the inertia groups of irreducible components and the inertia groups of nodes of the special fiber of a $G$-pointed semi-stable covering; then together with the results obtained in Section 2 and Section 3, we obtain our main theorems.

Finally, we would like to mention that by applying the formulas for global and local $p$-ranks associated to a $G$**-semi-stable covering** $f$ (cf. Definition 3.3 and Theorem 4.7), in [Y3], we answer an open problem posed by M. Saïdi concerning the boundedness of $p$-ranks of vertical fibers (cf. [S, Question]) in the case where $G$ is an arbitrary abelian $p$-group. Moreover, we prove that the global and local $p$-ranks associated to $f$ are determined by a kind of **combinatorial data** associated to $f$.

# 2 Semi-graphs with $p$-rank

In this section, we develop the theory of semi-graphs with $p$-rank. We always assume that $G$ is a finite $p$-group with order $p^r$.

## 2.1 Definitions

We begin with some general remarks concerning semi-graphs (cf. [M]). A **semi-graph** $\mathbb{G}$ consists of the following data: (i) A set $\mathscr{V}_{\mathbb{G}}$ whose elements we refer to as vertices; (ii) A set $\mathscr{E}^{\mathbb{G}}$ whose elements we refer to as edges. Any element $e \in \mathscr{E}^{\mathbb{G}}$ is a set of cardinality 2 satisfying the following property: For each $e \neq e' \in \mathscr{E}^{\mathbb{G}}$, we have $e \cap e' = \emptyset$; (iii) A set of maps $\{\zeta_e^{\mathbb{G}}\}_{e \in \mathscr{E}^{\mathbb{G}}}$ such that $\zeta_e : e \longrightarrow \mathscr{V} \cup \{\mathscr{V}\}$ is a map from the set $e$ to the set $\mathscr{V} \cup \{\mathscr{V}\}$. For an edge $e \in \mathscr{E}^{\mathbb{G}}$, we shall refer to an element $b \in e$ as a branch of the edge $e$. An edge $e \in \mathscr{E}^{\mathbb{G}}$ is called closed (resp. open) if $\zeta_e^{-1}(\{\mathscr{V}^{\mathbb{G}}\}) = \emptyset$ (resp. $\zeta_e^{-1}(\{\mathscr{V}^{\mathbb{G}}\}) \neq \emptyset$). A semi-graph will be called finite if both its set of vertices and its set of edges are finite. In the present paper, we only consider finite semi-graphs. Since a semi-graph can be regarded as a topological space, we shall call $\mathbb{G}$ a connected semi-graph if $\mathbb{G}$ is connected as a topological space.

Let $\mathbb{G}$ be a semi-graph. Write $v(\mathbb{G})$ for the set of vertices of $\mathbb{G}$, $e^{\mathrm{cl}}(\mathbb{G})$ for the set of closed edges of $\mathbb{G}$, and $e^{\mathrm{op}}(\mathbb{G})$ for the set of open edges of $\mathbb{G}$. For each $v \in v(\mathbb{G})$, write $b(v)$ for the set of branches $\cup_{e \in e^{\mathrm{cl}}(\mathbb{G}) \cup e^{\mathrm{op}}(\mathbb{G})} \zeta_e^{-1}(v)$ and $e(v)$ for the set of edges which abuts to $v$. For each $e \in e^{\mathrm{cl}}(\mathbb{G}) \cup e^{\mathrm{op}}(\mathbb{G}))$, write $v(e)$ for the set which consists of the elements of $v(\mathbb{G})$ which are abutted by $e$. An edge $e \in e^{\mathrm{cl}}(\mathbb{G})$ is called a loop if $\sharp v(e) = 1$, and we use the notation $e^{\mathrm{lp}}(v)$ to denotes the set of loops which abut to $v$ for each $v \in v(\mathbb{G})$.

A morphism between semi-graphs $\mathbb{G} \longrightarrow \mathbb{H}$ is a collection of maps $v(\mathbb{G}) \longrightarrow v(\mathbb{H})$; $e^{\mathrm{cl}}(\mathbb{G}) \cup e^{\mathrm{op}}(\mathbb{G}) \longrightarrow e^{\mathrm{cl}}(\mathbb{H}) \cup e^{\mathrm{op}}(\mathbb{H})$ such for each $e_{\mathbb{G}} \in e^{\mathrm{cl}}(\mathbb{G})$ (resp. $e_{\mathbb{G}} \in e^{\mathrm{op}}(\mathbb{G})$) mapping to $e_{\mathbb{H}} \in e^{\mathrm{cl}}(\mathbb{H})$ (resp. $e_{\mathbb{H}} \in e^{\mathrm{op}}(\mathbb{H})$) is a bijection $e_{\mathbb{G}} \xrightarrow{\sim} e_{\mathbb{H}}$, and all of which are compatible with the $\{\zeta_e^{\mathbb{G}}\}_{e \in e^{\mathrm{cl}}(\mathbb{G}) \cup e^{\mathrm{op}}(\mathbb{G})}$ and $\{\zeta_e^{\mathbb{H}}\}_{e \in e^{\mathrm{cl}}(\mathbb{H}) \cup e^{\mathrm{op}}(\mathbb{H})}$.

A sub-semi-graph $\mathbb{G}'$ of $\mathbb{G}$ is a semi-graph satisfying the following properties: (i) $v(\mathbb{G}')$

(resp. $e^{\mathrm{cl}}(\mathbb{G}') \cup e^{\mathrm{op}}(\mathbb{G}')$) is a subset of $v(\mathbb{G})$ (resp. $e^{\mathrm{cl}}(\mathbb{G}) \cup e^{\mathrm{op}}(\mathbb{G})$); (ii) if $e \in e^{\mathrm{cl}}(\mathbb{G}')$, $\zeta_e^{\mathbb{G}'}(e) = \zeta_e^{\mathbb{G}}(e)$; (iii) if $e = \{b_1, b_2\} \in e^{\mathrm{op}}(\mathbb{G}')$ such that $\zeta_e^{\mathbb{G}}(b_1) \in v(\mathbb{G}')$ and $\zeta_e^{\mathbb{G}}(b_2) \notin v(\mathbb{G}')$, $\zeta_e^{\mathbb{G}'}(b_1) = \zeta_e^{\mathbb{G}}(b_1)$ and $\zeta_e^{\mathbb{G}'}(b_2) = \{v(\mathbb{G}')\}$.

**Definition 2.1.** Let $\mathbb{G}'$ be a sub-semi-graph of a semi-graph $\mathbb{G}$. We define a semi-graph $\mathbb{G}\setminus\mathbb{G}'$ as follows: (i) $v(\mathbb{G}\setminus\mathbb{G}') := v(\mathbb{G})\setminus v(\mathbb{G}')$; (ii) $e^{\mathrm{cl}}(\mathbb{G}\setminus\mathbb{G}') := \{e \in e^{\mathrm{cl}}(\mathbb{G}) \mid v(e) \cap v(\mathbb{G}') = \emptyset \text{ in } \mathbb{G}\}$; (iii) $e^{\mathrm{op}}(\mathbb{G} \setminus \mathbb{G}') := \{e \in e^{\mathrm{cl}}(\mathbb{G}) \mid v(e) \cap v(\mathbb{G}') \neq \emptyset \text{ in } \mathbb{G} \text{ and } v(e) \cap v(\mathbb{G} \setminus \mathbb{G}') \neq \emptyset \text{ in } \mathbb{G}\} \cup \{e \in e^{\mathrm{op}}(\mathbb{G}) \mid v(e) \cap v(\mathbb{G} \setminus \mathbb{G}') \neq \emptyset \text{ in } \mathbb{G}\}$; (iv) For each $e = \{b_i\}_{i=\{1,2\}} \in e^{\mathrm{cl}}(\mathbb{G} \setminus \mathbb{G}') \cup e^{\mathrm{op}}(\mathbb{G} \setminus \mathbb{G}')$, we set $\zeta_e^{\mathbb{G}\setminus\mathbb{G}'}(b_i) = \zeta_e^{\mathbb{G}}(b_i)$ (resp. $\zeta_e^{\mathbb{G}\setminus\mathbb{G}'}(b_i) = \{v(\mathbb{G} \setminus \mathbb{G}')\}$) if $\zeta_e^{\mathbb{G}}(b_i) \notin v(\mathbb{G}')$ (resp. $\zeta_e^{\mathbb{G}}(b_i) \in v(\mathbb{G}')$).

**Definition 2.2.** (a) A pair $\mathfrak{G} := (\mathbb{G}, \sigma_{\mathfrak{G}})$ which consists of a semi-graph $\mathbb{G}$ and a map $\sigma_{\mathfrak{G}} : v(\mathbb{G}) \longrightarrow \mathbb{Z}$. We shall call $\mathfrak{G}$ a semi-graph with $p$-rank. We shall refer to $\mathbb{G}$ as the underlying semi-graph of $\mathfrak{G}$ and $\sigma_{\mathfrak{G}}$ as the $p$-rank map of $\mathfrak{G}$, respectively.

(b) We define the ***p*-rank** $\sigma_{\mathfrak{G}}(\mathfrak{G})$ of $\mathfrak{G}$ as follows:

$$\sigma_{\mathfrak{G}}(\mathfrak{G}) := \sum_{v \in v(\mathbb{G})} \sigma_{\mathfrak{G}}(v) + \sum_{\mathbb{G}_i \in \pi_0(\mathbb{G})} \dim_{\mathbb{C}} \mathrm{H}^1(\mathbb{G}_i, \mathbb{C}),$$

where $\pi_0(-)$ denotes the set of the connected components of $(-)$, and $\mathbb{C}$ denotes the field of complex number.

(c) A semi-graph with $p$-rank is called connected if the underlying semi-graph $\mathbb{G}$ is a connected semi-graph.

(d) A morphism between semi-graphs with $p$-rank $\mathfrak{b} : \mathfrak{G}^1 := (\mathbb{G}^1, \sigma_{\mathfrak{G}^1}) \longrightarrow \mathfrak{G}^2 := (\mathbb{G}^2, \sigma_{\mathfrak{G}^2})$ is defined by a morphism of the underlying semi-graphs $\beta : \mathbb{G}^1 \longrightarrow \mathbb{G}^2$; we shall refer to the morphism $\beta$ as the underlying morphism of $\mathfrak{b}$.

From now on, we assume that all the semi-graphs with $p$-rank are connected.

**Definition 2.3.** Let $\mathfrak{b} : \mathfrak{G}^1 := (\mathbb{G}^1, \sigma_{\mathfrak{G}^1}) \longrightarrow \mathfrak{G}^2 := (\mathbb{G}^2, \sigma_{\mathfrak{G}^2})$ be a morphism of semi-graphs with $p$-rank.

(a) We shall call $\mathfrak{b}$ ***p*-étale** (resp. ***p*-purely inseparable**) at an edge $e \in e^{\mathrm{cl}}(\mathbb{G}^1) \cup e^{\mathrm{op}}(\mathbb{G}^1)$ if $\sharp\beta^{-1}(\beta(e)) = p$ (resp. $\sharp\beta^{-1}(\beta(e)) = 1$), where $\sharp(-)$ denotes the cardinality of $(-)$. We shall call $\mathfrak{b}$ ***p*-generically étale** at $v \in v(\mathbb{G}^1)$ if one of the following étale types holds:

(Type-I) If $\sharp\beta^{-1}(\beta(v)) = p$, then we have $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^2}(\beta(v))$;

(Type-II) If $\sharp\beta^{-1}(\beta(v)) = 1$, then we have the following Deuring-Shafarevich type formula:

$$\sigma_{\mathfrak{G}^1}(v) - 1 = p(\sigma_{\mathfrak{G}^2}(\beta(v)) - 1) + \sum_{e \in e(v)} (r_e - 1),$$

where $r_e$ is equal to $p$ if $\sharp\beta^{-1}(\beta(e)) = 1$, and $r_e$ is equal to 1 if $\sharp\beta^{-1}(\beta(e)) = p$.

(b) We shall call $\mathfrak{b}$ **purely inseparable** at $v \in v(\mathbb{G}^1)$ if $\sharp\beta^{-1}(\beta(v)) = 1$ and $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^2}(\beta(v))$ hold.

(c) We shall call $\mathfrak{b}$ a ***p*-covering** if the following conditions hold: (i) there exists a $\mathbb{Z}/p\mathbb{Z}$-action (which may be trivial) on $\mathbb{G}^1$ (resp. a trivial $\mathbb{Z}/p\mathbb{Z}$-action on $\mathbb{G}^2$), and the underlying morphism $\beta$ of $\mathfrak{b}$ is compatible with the $\mathbb{Z}/p\mathbb{Z}$-actions; then the natural morphism $\mathbb{G}^1/\mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}^2$ induced by $\mathfrak{b}$ is an isomorphism; (ii) for each $v \in v(\mathbb{G}^1)$, $\mathfrak{b}$ is either $p$-generically étale or purely inseparable at $v$; (iii) let $e \in e^{\mathrm{cl}}(\mathbb{G}^1)$ and $v(e) = \{v, v'\}$ ($v$ and $v'$ may be equal if $e$ is a loop); if $\mathfrak{b}$ is $p$-generically étale at $v$ and $v'$, then $\mathfrak{b}$ is $p$-étale at $e$; (iv) For each $v \in v(\mathbb{G}^1)$, then $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^1}(\tau(v))$ holds for each $\tau \in \mathbb{Z}/p\mathbb{Z}$.

Note that by the definition of $p$-covering, the identity morphism of a semi-graph with $p$-rank is a $p$-covering.

(d) We shall call $\mathfrak{b}$ a **covering** if $\mathfrak{b}$ is a composite of $p$-coverings.

(e) We shall call

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

an **maximal normal filtration** of $G$ if $G_j$ is a normal subgroup of $G$ and $G_j/G_{j+1} \cong \mathbb{Z}/p\mathbb{Z}$ for each $j = 0, \ldots, r-1$. Suppose that $\mathbb{G}^1$ (resp. $\mathbb{G}^2$) admits a (resp. trivial) $G$-action (which may be trivial). Then a maximal normal filtration $\Phi$ of $G$ induces a sequence of semi-graphs:

$$\Phi : \mathbb{G}^1 = \mathbb{G}_r \xrightarrow{\beta_r} \mathbb{G}_{r-1} \xrightarrow{\beta_{r-1}} \ldots \xrightarrow{\beta_1} \mathbb{G}_0,$$

where $\mathbb{G}_j, j = 0, \ldots, r$, denotes the quotient of $\mathbb{G}^1$ by $G_j$. We shall call $\mathfrak{b}$ a **$G$-covering** if there exist a maximal normal filtration $\Phi$ of $G$ and a set of $p$-coverings $\{\mathfrak{b}_j : \mathfrak{G}_j \longrightarrow \mathfrak{G}_{j-1},\ j = 1, \ldots, r\}$ such that the following conditions hold: (i) the underlying morphism $\beta$ of $\mathfrak{b}$ is compatible with the $G$-actions, and the natural morphism $\mathbb{G}_0 = \mathbb{G}^1/G \longrightarrow \mathbb{G}^2$ induced by $\beta$ is an isomorphism; (ii) the underlying graph of $\mathfrak{G}_j$ is equal to $\mathbb{G}_j$ for each $j = 0, \ldots, r$; (iii) the underlying morphism $\mathbb{G}_j \longrightarrow \mathbb{G}_{j-1}$ of $\mathfrak{b}_j$ is equal to $\beta_j$ for each $j = 1, \ldots, r$; (iv) The composite morphism $\mathfrak{b}_1 \circ \cdots \circ \mathfrak{b}_r$ is equal to $\mathfrak{b}$.

If $\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2$ is a $G$-covering, then we have a maximal normal filtration $\Phi$ of $G$ and a sequence of $p$-coverings:

$$\Phi_{\mathfrak{G}^1/\mathfrak{G}^2} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \ldots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{G}^2.$$

We shall call $\Phi_{\mathfrak{G}^1/\mathfrak{G}^2}$ **a sequence of $p$-coverings induced by** $\Phi$.

**Definition 2.4.** Let $\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2$ be a $G$-covering, $\beta : \mathbb{G}^1 \longrightarrow \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$, $v^1 \in v(\mathbb{G}^1)$, and $e^1 \in e^{\mathrm{cl}}(\mathbb{G}^1) \cup e^{\mathrm{op}}(\mathbb{G}^1)$. By the definition of $G$-coverings, we have a maximal normal filtration $\Phi$ of $G$ and a sequence of $p$-coverings induced by $\Phi$:

$$\Phi_{\mathfrak{G}^1/\mathfrak{G}^2} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \ldots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{G}^2.$$

Write $\beta_j : \mathbb{G}_j \longrightarrow \mathbb{G}_{j-1}, j = 1, \ldots, r$, for the underlying morphism $\mathfrak{b}_j$. Write $v_j$ (resp. $e_j$) for the image $\beta_{j+1} \circ ... \circ \beta_r(v)$ (resp. $\beta_{j+1} \circ ... \circ \beta_r(e)$), $j = 0, \ldots, r-1$, and $v_r$ for $v^1$. Then we set

$$\log_p(\sharp I_{v^1}) := \sharp\{j \in \{1, \ldots, r\} \mid \mathfrak{b}_j \text{ is purely inseparable at } v_j\},$$

$$\log_p(\sharp I_{e^1}) := \sharp\{j \in \{1, \ldots, r\} \mid \mathfrak{b}_j \text{ is purely inseparable at } e_j\},$$

$$D_{v^1} := \{\tau \in G \mid \tau(v^1) = v^1\}.$$

Note that if $e^1 \in e(v^1)$, then we have $\sharp I_{v^1} | \sharp I_{e^1}$. In particular, if $e^1$ is a loop, then we have $\sharp I_{v^1} = \sharp I_{e^1}$ by using Definition 2.3 (c, iii). Moreover, Definition 2.3 (c, iii) implies that $\sharp I_{e^1} | \sharp D_{v^1}$. Write $v^2$ (resp. $e^2$) for $v_0 = \beta(v^1)$ (resp. $e_0 = \beta(e^1)$). Let $(v^1)'$ (resp. $(e^1)'$) be an arbitrary element of $\beta^{-1}(v^2)$ (resp. $\beta^{-1}(e^2)$). By the action of $G$ on $\mathbb{G}^1$, we have $\sharp I_{v^1} = \sharp I_{(v^1)'}$, $\sharp I_{e^1} = \sharp I_{(e^1)'}$, and $\sharp D_{v^1} = \sharp D_{(v^1)'}$. Thus, we use the notation $\sharp I_{v^2}$ (resp. $\sharp I_{e^2}$, $\sharp D_{v^2}$) to denote $\sharp I_{v^1}$ (resp. $\sharp I_{e^1}$, $\sharp D_{v^1}$). Then we obtain $\sharp I_{v^2} | \sharp I_{e^2} | \sharp D_{v^2}$.

**Remark 2.4.1.** We follow the notations in Definition 2.4. It is easy to compute the $p$-rank $\sigma_{\mathfrak{G}^1}(v^1)$ by using Definition 2.3 (a, Type-II). Then we have the following Deuring-Shafarevich type formula (cf. Proposition 3.2 for the Deuring-Shafarevich formula for curves)

$$\sigma_{\mathfrak{G}^1}(v^1) - 1 = \sharp D_{v^2}/\sharp I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e^2 \in e(v^2)} (\sharp D_{v^2}/\sharp I_{e^2})(\sharp I_{e^2}/\sharp I_{v^2} - 1)$$

$$= \sharp D_{v^2}/\sharp I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e^2 \in e(v^2) \setminus e^{\mathrm{lp}}(v^2)} (\sharp D_{v^2}/\sharp I_{e^2})(\sharp I_{e^2}/\sharp I_{v^2} - 1).$$

The second equality follows from Definition 2.3 (c, iii).

## 2.2 An operator on $G$-coverings of semi-graphs with $p$-rank

Let $\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2$ be a $G$-covering and $\beta : \mathbb{G}^1 \longrightarrow \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$. In this subsection, we introduce an operator for $\mathfrak{b}$. Let $v^2 \in v(\mathbb{G}^2)$ and $v^1 \in \beta^{-1}(v^2)$. First, let us define a new semi-graph $(\mathbb{G}^1)^*[v^2]$.

If $\beta^{-1}(v^2) = \{v^1\}$ (i.e., $D_{v^1} = G$), then we define $(\mathbb{G}^1)^*[v^2]$ to be $\mathbb{G}^1$.

If $\beta^{-1}(v^2) \neq \{v^1\}$, we define a new semi-graph $(\mathbb{G}^1)^*[v^2]$ as follows. Define $v((\mathbb{G}^1)^*[v^2])$ (resp. $e^{\mathrm{cl}}((\mathbb{G}^1)^*[v^2]) \cup e^{\mathrm{op}}((\mathbb{G}^1)^*[v^2])$) to be the disjoint union $(v(\mathbb{G}^1) \setminus \beta^{-1}(v^2)) \coprod \{v^*[v^2]\}$ (resp. $e^{\mathrm{cl}}(\mathbb{G}^1) \cup e^{\mathrm{op}}(\mathbb{G}^1)$).

The collection of maps $\{\zeta_e^{(\mathbb{G}^1)^*[v^2]}\}_e$ is as follows: (i) for each branch $b \notin \cup_{v \in \beta^{-1}(v^2)} b(v)$, $\zeta_e^{(\mathbb{G}^1)^*[v^2]}(b) = \zeta_e^{\mathbb{G}^1}(b)$ if $b \in e$ and $\zeta_e^{(\mathbb{G}^1)^*[v^2]}(b) = \emptyset$ if $b \notin e$; (ii) for each $v \in \beta^{-1}(v^2)$ and each branch $b \in b(v)$, $\zeta_e^{(\mathbb{G}^1)^*[v^2]}(b) = v^*[v^2]$ if $b \in e$ and $\zeta_e^{(\mathbb{G}^1)^*[v^2]}(b) = \emptyset$ if $b \notin e$.

Second, we define a map $\sigma_{(\mathfrak{G}^1)^*[v^2]} : v((\mathbb{G}^1)^*[v^2]) \longrightarrow \mathbb{Z}$ as follows: (i) if $v^*[v^2] \neq v \in v((\mathbb{G}^1)^*[v^2])$, we set $\sigma_{(\mathfrak{G}^1)^*[v^2]}(v) := \sigma_{\mathfrak{G}^1}(v)$; (ii) if $v = v^*[v^2]$, we set

$$\sigma_{(\mathfrak{G}^1)^*[v^2]}(v^*[v^2]) := \sharp G/\sharp I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e \in e(v^2)} \sharp G/\sharp I_e(\sharp I_e/\sharp I_{v^2} - 1) + 1.$$

We define $(\mathfrak{G}^1)^*[v^2]$ to be the pair $((\mathbb{G}^1)^*[v^2], \sigma_{(\mathfrak{G}^1)^*[v^2]})$ which is a semi-graph with $p$-rank.

We define a morphism of semi-graphs $\beta^*[v^2] : (\mathbb{G}^1)^*[v^2] \longrightarrow \mathbb{G}^2$ as follows: (i) for each $v \in v((\mathbb{G}^1)^*[v^2])$, $\beta^*[v^2](v) := v^2$ if $v = v^*[v^2]$ and $\beta^*[v^2](v) := \beta(v)$ if $v \neq v^*[v^2]$; (ii) if $e \in e^{\mathrm{cl}}((\mathbb{G}^1)^*[v^2]) \cup e^{\mathrm{op}}((\mathbb{G}^1)^*[v^2])$, we set $\beta^*[v^2](e) := \beta(e)$. Then we obtain a morphism of semi-graphs with $p$-rank $\mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \longrightarrow \mathfrak{G}^2$ induced by $\beta^*[v^2]$.

Moreover, $(\mathbb{G}^1)^*[v^2]$ admits a natural $G$-action as follows: (i) the action of $G$ on $v(\mathbb{G}^1)^*[v^2]) \setminus \{v^*[v^2]\}$ (resp. $e^{\mathrm{cl}}((\mathbb{G}^1)^*[v^2]) \cup e^{\mathrm{op}}((\mathbb{G}^1)^*[v^2]))$ is the action of $G$ on $v(\mathbb{G}) \setminus \beta_{\mathfrak{G}}^{-1}(p_i)$ (resp. $e^{\mathrm{cl}}(\mathbb{G}) \cup e^{\mathrm{op}}(\mathbb{G})$); (ii) the action of $G$ on $v^*[v^2]$ is a trivial action.

Let us explain that with the $G$-action defined above, $\mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \longrightarrow \mathfrak{G}^2$ is a $G$-covering. Since $\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2$ is a $G$-covering, there exist a maximal normal filtration

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

of $G$ and a sequence of $p$-coverings of semi-graphs with $p$-rank

$$\Phi_{\mathfrak{G}^1/\mathfrak{G}^2} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{G}^2.$$

Note that for each $j = 1, \dots, r$, $\mathfrak{b}_1 \circ ... \circ \mathfrak{b}_j : \mathfrak{G}_j \longrightarrow \mathfrak{G}_0$ is a $G/G_j$-covering. Then we obtain a sequence of morphisms of semi-graphs with $p$-rank

$$\Phi_{(\mathfrak{G}^1)^*[v^2]/\mathfrak{G}^2} : (\mathfrak{G}^1)^*[v^2] = \mathfrak{G}_r^*[v^2] \xrightarrow{\mathfrak{b}_r^*[v^2]} \mathfrak{G}_{r-1}^*[v^2] \xrightarrow{\mathfrak{b}_{r-1}^*[v^2]} \dots \xrightarrow{\mathfrak{b}_1^*[v^2]} \mathfrak{G}_0 = \mathfrak{G}^2.$$

By the construction of $\mathfrak{G}_j^*[v^2]$, it is easy to see that $\mathfrak{b}_j^*[v^2]$ is a $p$-covering for each $j = 1, \dots, r$. Thus, $\mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \longrightarrow \mathfrak{G}^2$ can be regarded as a $G$-covering.

**Definition 2.5.** Let $\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2$ be a $G$-covering and $v^2 \in v(\mathbb{G}^2)$. We define an operator $\rightleftharpoons_{II}^{I} [v^2]$ from a $G$-covering to a $G$-covering to be

$$\rightleftharpoons_{II}^{I} [v^2](\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2) := \mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \longrightarrow \mathfrak{G}^2.$$

## 2.3 Formula for $p$-ranks of $G$-coverings of semi-graphs with $p$-rank

In this subsection, we give an explicit formula for the $p$-rank of $G$-coverings of semi-graphs with $p$-rank.

**Lemma 2.6.** *Let $\mathbb{G}$ be a connected semi-graph, $\{\mathbb{G}_i\}_{i=1,\dots,n}$ a set of connected sub-semi-graph of $\mathbb{G}$, and $v_i \in v(\mathbb{G}_i), i = 1, \dots, n$. Suppose that $\mathbb{G}_s \cap \mathbb{G}_t = \emptyset$ for each $s, t \in \{1, \dots, n\}$. Let $\mathbb{G}^{\mathrm{c}}$ be a semi-graph defined as follows: (i) $v(\mathbb{G}^{\mathrm{c}}) = v(\mathbb{G}) \coprod \{v^{\mathrm{c}}\}$; (ii) $e^{\mathrm{op}}(\mathbb{G}^{\mathrm{c}}) = e^{\mathrm{op}}(\mathbb{G})$; (iii) $e^{\mathrm{cl}}(\mathbb{G}^{\mathrm{c}}) = e^{\mathrm{cl}}(\mathbb{G}) \coprod \{e_i^{\mathrm{c}}\}_{i=1,\dots,n}$; (iv) if $e \notin \{e_i^{\mathrm{c}}\}_{i=1,\dots,n}$, we set $\zeta_e^{\mathbb{G}^{\mathrm{c}}}(b) = \zeta_e^{\mathbb{G}}(b)$ if $b \in e$ and $\zeta_e^{\mathbb{G}^{\mathrm{c}}}(b) =$ if $b \notin e$; (v) let $e_i^{\mathrm{c}} = \{b_{e_i^{\mathrm{c}}}^1, b_{e_i^{\mathrm{c}}}^2\}$, $i = 1, \dots, n$; we set $\zeta_{e_i^{\mathrm{c}}}^{\mathbb{G}^{\mathrm{c}}}(b_{e_i^{\mathrm{c}}}^1) = v_i$ and $\zeta_{e_i^{\mathrm{c}}}^{\mathbb{G}^{\mathrm{c}}}(b_{e_i^{\mathrm{c}}}^2) = v^{\mathrm{c}}$ for each $i = 1, \dots, n$. Then we have*

$$\dim_{\mathbb{C}} \mathrm{H}^1(\mathbb{G}, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{H}^1(\mathbb{G}^{\mathrm{c}}, \mathbb{C}) - n + 1.$$

*Proof.* The lemma follows from the construction of $\mathbb{G}^{\mathrm{c}}$. $\square$

The following proposition is a key for calculating the $p$-ranks of $G$-coverings.

**Proposition 2.7.** *Let $\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2$ be a $G$-covering of semi-graphs with $p$-rank, $v^2 \in v(\mathbb{G}^2)$, and $\rightleftharpoons_{II}^{I} [v^2](\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2) := \mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \longrightarrow \mathfrak{G}^2$. Then we have*

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1) = \sigma_{(\mathfrak{G}^1)^*[v^2]}((\mathfrak{G}^1)^*[v^2]).$$

*Proof.* Write $\beta$ (resp. $\beta^*[v^2]$) for the underlying morphism of $\mathfrak{b}$ (resp. $\mathfrak{b}^*[v^2]$). Write $r$ (resp. $r_{\backslash\{v^2\}}, r^*, r^*_{\backslash\{v^2\}}$) for $\dim_{\mathbb{C}}\mathrm{H}^1(\mathbb{G}^1,\mathbb{C})$ (resp. $\dim_{\mathbb{C}}\mathrm{H}^1(\mathbb{G}^1\backslash\beta^{-1}(v^2),\mathbb{C}), \dim_{\mathbb{C}}\mathrm{H}^1((\mathbb{G}^1)^*[v^2],\mathbb{C})$, $\dim_{\mathbb{C}}\mathrm{H}^1((\mathbb{G}^1)^*[v^2]\setminus(\beta^*[v^2])^{-1}(v^2),\mathbb{C})$). Then we have

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1)=\sum_{v\in v(\mathbb{G}^1\backslash\beta^{-1}(v^2))}\sigma_{\mathfrak{G}^1}(v)+\sum_{v\in\beta^{-1}(v^2)}\sigma_{\mathfrak{G}^1}(v)+r_{\backslash\{v^2\}}+r-r_{\backslash\{v^2\}}$$

and

$$\sigma_{(\mathfrak{G}^1)^*[v^2]}((\mathfrak{G}^1)^*[v^2])=\sum_{v\in v((\mathbb{G}^1)^*[v^2]\backslash(\beta^*[v^2])^{-1}(v^2))}\sigma_{(\mathfrak{G}^1)^*[v^2]}(v)+\sigma_{(\mathfrak{G}^1)^*[v^2]}(v^*[v^2])+r^*_{\backslash\{v^2\}}+r^*-r^*_{\backslash\{v^2\}}.$$

Note that by the construction of $(\mathfrak{G}^1)^*[v^2]$, we have

$$A:=\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1)=\sum_{v\in v(\mathbb{G}^1\backslash\beta^{-1}(v^2))}\sigma_{\mathfrak{G}^1}(v)=\sum_{v\in v((\mathbb{G}^1)^*[v^2]\backslash(\beta^*[v^2])^{-1}(v^2))}\sigma_{(\mathfrak{G}^1)^*[v^2]}(v)$$

and

$$B:=r_{\backslash\{v^2\}}=r^*_{\backslash\{v^2\}}.$$

Let us calculate $r-r_{\backslash\{v^2\}}$ and $r^*-r^*_{\backslash\{v^2\}}$. Follows from Lemma 2.6, it is sufficient to treat the case where $\mathbb{G}^1\setminus\beta^{-1}(v^2)=(\mathbb{G}^1)^*[v^2]\setminus(\beta^*[v^2])^{-1}(v^2)$ is connected. Then we obtain

$$r-r_{\backslash\{v^2\}}=\sharp G/\sharp D_{v^2}((\sum_{e\in e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2)}\sharp D_{v^2}/\sharp I_e)-1)+\sharp e^{\mathrm{lp}}(v^2)(\sharp G/\sharp I_{v^2})$$

and

$$r^*-r^*_{\backslash\{v^2\}}=(\sum_{e\in e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2)}\sharp G/\sharp I_e)-1+\sharp e^{\mathrm{lp}}(v^2)(\sharp G/\sharp I_{v^2}).$$

The Remark 2.4.1 implies that for each $v\in\beta^{-1}(v^2)$, we have

$$\sigma_{\mathfrak{G}^1}(v)=\sharp D_{v^2}/\sharp I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2)-1)+\sum_{e\in e(v^2)}(\sharp D_{v^2}/\sharp I_e)(\sharp I_e/\sharp I_{v^2}-1)+1.$$

On the other hand, the construction of $(\mathfrak{G}^1)^*[v^2]$ implies that

$$\sigma_{(\mathfrak{G}^1)^*[v^2]}(v^*[v^2])=\sharp G/\sharp I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2)-1)+\sum_{e\in e(v^2)}\sharp G/\sharp I_e(\sharp I_e/\sharp I_{v^2}-1)+1.$$

Thus, we obtain

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1)=A+B+\sum_{v\in\beta^{-1}(v^2)}(\sharp D_{v^2}/\sharp I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2)-1)+\sum_{e\in e(v^2)}(\sharp D_{v^2}/\sharp I_e)(\sharp I_e/\sharp I_{v^2}-1)+1)$$

$$+\sharp G/\sharp D_{v^2}((\sum_{e\in e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2)}\sharp D_{v^2}/\sharp I_e)-1)+\sharp e^{\mathrm{lp}}(v^2)(\sharp G/\sharp I_{v^2})$$

$$= A + B + \sharp G/\sharp D_{v^2}(\sharp D_{v^2}/\sharp I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e \in e(v^2)} (\sharp D_{v^2}/\sharp I_e)(\sharp I_e/\sharp I_{v^2} - 1) + 1)$$

$$+\sharp G/\sharp D_{v^2}((\sum_{e \in e(v^2) \cap e^{\mathrm{cl}}(\mathbb{G}^2)} \sharp D_{v^2}/\sharp I_e) - 1) + \sharp e^{\mathrm{lp}}(v^2)(\sharp G/\sharp I_{v^2})$$

$$= A + B + (\sharp G/\sharp I_{v^2})\sigma_{\mathfrak{G}^2}(v^2) - \sharp G/\sharp I_{v^2} + \sum_{e \in e(v^2)} \sharp G/\sharp I_{v^2} - \sum_{e \in e(v^2)} \sharp G/\sharp I_e$$

$$+ \sum_{e \in e(v^2) \cap e^{\mathrm{cl}}(\mathbb{G}^2)} \sharp G/\sharp I_e + \sharp e^{\mathrm{lp}}(v^2)(\sharp G/\sharp I_{v^2})$$

$$= A+B+(\sharp G/\sharp I_{v^2})\sigma_{\mathfrak{G}^2}(v^2) - \sharp G/\sharp I_{v^2} + \sum_{e \in e(v^2)} \sharp G/\sharp I_{v^2} - \sum_{e \in e(v^2) \cap e^{\mathrm{op}}(\mathbb{G}^2)} \sharp G/\sharp I_e + \sharp e^{\mathrm{lp}}(v^2)(\sharp G/\sharp I_{v^2})$$

and

$$\sigma_{(\mathfrak{G}^1)^*[v^2]}((\mathfrak{G}^1)^*[v^2]) = A + B + \sharp G/\sharp I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e \in e(v^2)} \sharp G/\sharp I_e(\sharp I_e/\sharp I_{v^2} - 1) + 1$$

$$+(\sum_{e \in e(v^2) \cap e^{\mathrm{cl}}(\mathbb{G}^2)} \sharp G/\sharp I_e) - 1 + \sharp e^{\mathrm{lp}}(v^2)(\sharp G/\sharp I_{v^2})$$

$$= A+B+(\sharp G/\sharp I_{v^2})\sigma_{\mathfrak{G}^2}(v^2) - \sharp G/\sharp I_{v^2} + \sum_{e \in e(v^2)} \sharp G/\sharp I_{v^2} - \sum_{e \in e(v^2) \cap e^{\mathrm{op}}(\mathbb{G}^2)} \sharp G/\sharp I_e \sharp e^{\mathrm{lp}}(v^2)(\sharp G/\sharp I_{v^2}).$$

This means that

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1) = \sigma_{(\mathfrak{G}^1)^*[v^2]}((\mathfrak{G}^1)^*[v^2]).$$

We complete the proof of the proposition. □

**Theorem 2.8.** *Let $\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2$ be a $G$-covering of semi-graphs with $p$-rank and $\beta$ the underlying morphism of $\mathfrak{b}$. Then we have*

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1) = \sum_{v \in v(\mathbb{G}^2)} (\sharp G/\sharp I_v(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e \in e(v) \setminus e^{\mathrm{lp}}(v)} \sharp G/\sharp I_e(\sharp I_e/\sharp I_v - 1) + 1)$$

$$+ \sum_{e \in e^{\mathrm{cl}}(\mathbb{G}^2) \setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\sharp G/\sharp I_e - 1) + \sum_{v \in v(\mathbb{G}^2)} \sharp e^{\mathrm{lp}}(v)(\sharp G/\sharp I_v - 1) + \dim_{\mathbb{C}} \mathrm{H}^1(\mathbb{G}^2, \mathbb{C}).$$

*Proof.* By applying Proposition 2.7, to calculate the $p$-rank of $\mathfrak{G}^1$, it is sufficient to assume that $\sharp\beta^{-1}(v) = 1$ for each $v \in v(\mathbb{G}^2)$. Then for each $v \in v(\mathbb{G}^2)$, we have

$$\sigma_{\mathfrak{G}^1}(\beta^{-1}(v)) = \sharp G/\sharp I_v(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e \in e(v)} \sharp G/\sharp I_e(\sharp I_e/\sharp I_v - 1) + 1$$

$$= \sharp G/\sharp I_v(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e \in e(v) \setminus e^{\mathrm{lp}}(v)} \sharp G/\sharp I_e(\sharp I_e/\sharp I_v - 1) + 1.$$

On the other hand, it is easy to see that

$$\dim_{\mathbb{C}}\mathrm{H}^1(\mathbb{G}^1,\mathbb{C}) = \sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)}(\sharp G/\sharp I_e - 1) + \dim_{\mathbb{C}}\mathrm{H}^1(\mathbb{G}^2,\mathbb{C}) - \sum_{v\in v(\mathbb{G}^2)}\sharp e^{\mathrm{lp}}(v)$$

$$+ \sum_{v\in v(\mathbb{G}^2)}\sharp e^{\mathrm{lp}}(v)(\sharp G/\sharp I_v)$$

$$= \sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)}(\sharp G/\sharp I_e - 1) + \dim_{\mathbb{C}}\mathrm{H}^1(\mathbb{G}^2,\mathbb{C}) + \sum_{v\in v(\mathbb{G}^2)}\sharp e^{\mathrm{lp}}(v)(\sharp G/\sharp I_v - 1).$$

Thus, we obtain

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1) = \sum_{v\in v(\mathbb{G}^2)}(\sharp G/\sharp I_v(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e\in e(v)\setminus e^{\mathrm{lp}}(v)}\sharp G/\sharp I_e(\sharp I_e/\sharp I_v - 1) + 1)$$

$$+ \sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)}(\sharp G/\sharp I_e - 1) + \sum_{v\in v(\mathbb{G}^2)}\sharp e^{\mathrm{lp}}(v)(\sharp G/\sharp I_v - 1) + \dim_{\mathbb{C}}\mathrm{H}^1(\mathbb{G}^2,\mathbb{C}).$$

This completes the proof of the theorem. □

**Definition 2.9.** Let $n$ be a positive natural number and $\mathbb{P}_n$ a semi-graph such that the following conditions hold: (i) $v(\mathbb{P}_n) = \{p_1,\dots,p_n\}$; (ii) $e^{\mathrm{cl}}(\mathbb{P}_n) = \{e_{1,2},\dots,e_{n,n-1}\}$ and $e^{\mathrm{op}}(\mathbb{P}_n) = \{e_{0,1}, e_{n,n+1}\}$; (iii)$v(e_{0,1}) = \{p_1\}$ and $v(e_{n,n+1}) = \{p_n\}$; (iv) $v(e_{i,i+1}) = \{p_i, p_{i+1}\}$. We define a semi-graph with $p$-rank $\mathfrak{P}_n$ to be $(\mathbb{P}_n, \sigma_{\mathfrak{P}_n})$, where $\sigma_{\mathfrak{P}_n}(p_i)$ is equal to 0 for each $i = 1,\dots,n$. We shall call $\mathfrak{P}_n$ a $n$-**chain**.

We have the following corollary.

**Corollary 2.10.** *Let $\mathfrak{b} : \mathfrak{G} \longrightarrow \mathfrak{P}_n$ be a $G$-covering of semi-graphs with $p$-rank. Then we have*

$$\sigma_{\mathfrak{G}}(\mathfrak{G}) = \sum_{i=1}^{n}\sharp G/\sharp I_{p_i} - \sum_{i=1}^{n+1}\sharp G/\sharp I_{e_{i-1,i}} + 1.$$

*Proof.* Since $\sum_{v\in v(\mathbb{P}_n)}\sharp e^{\mathrm{lp}}(v)(\sharp G/\sharp I_v - 1)$ and $\dim_{\mathbb{C}}\mathrm{H}^1(\mathbb{P}_n,\mathbb{C})$ are equal to 0, the corollary follows from Theorem 2.8. □

In the next section, we will use Theorem 2.8 and Corollary 2.10 to calculate the global and local $p$-ranks of coverings of curves over a complete discrete valuation ring.

# 3 Semi-graphs with $p$-rank associated to pointed semi-stable coverings

## 3.1 $p$-ranks and pointed semi-stable coverings

**Definition 3.1.** (a) Let $\mathscr{C} := (C, D_C)$ be a pointed semi-stable curve over a scheme $A$. We shall call $C$ the underlying curve of $\mathscr{C}$ and $D_C$ the set of marked points of $\mathscr{C}$. Write $\mathrm{Im}_{D_C}$ for the scheme theoretic images of the elements of $D_C$; we identify $D_C$ with $\mathrm{Im}_{D_C}$.

If $A$ is a field, we write $\mathrm{Irr}(C)$ for the set of the irreducible components of $C$ and $C^{\mathrm{sing}}$ for the set the singular points of $C$; the dual semi-graph $\Gamma_{\mathscr{C}}$ of the pointed semi-stable curve $\mathscr{C}$ is a semi-graph defined as follows: (i) $v(\Gamma_{\mathscr{C}}) := \{v_E\}_{E\in \mathrm{Irr}(C)}$; (ii) $e^{\mathrm{cl}}(\Gamma_{\mathscr{C}}) := \{e_s\}_{s\in C^{\mathrm{sing}}}$; (iii) $e^{\mathrm{op}}(\Gamma_{\mathscr{C}}) := \{e_m\}_{m\in D_C}$; (iv) for each $e_s = \{b_s^1, b_s^2\} \in e^{\mathrm{cl}}(\Gamma_{\mathscr{C}})$, $\zeta^{\Gamma_{\mathscr{C}}}_{e_s}(b_s^1) \neq \emptyset$, $\zeta^{\Gamma_{\mathscr{C}}}_{e_s}(b_s^2) \neq \emptyset$, and $\zeta^{\Gamma_{\mathscr{C}}}_{e_s}(e_s) := \{v_E \in v(\Gamma_{\mathscr{C}}) \mid a \in E\}$; (v) for each $e_m = \{b_m^1, b_m^2\} \in e^{\mathrm{op}}(\Gamma_{\mathscr{C}})$, $\zeta^{\Gamma_{\mathscr{C}}}_{e_m}(b_m^1) := \{v_E \in v(\Gamma_{\mathscr{C}}) \mid a \in E\}$, $\zeta^{\Gamma_{\mathscr{C}}}_{e_m}(b_m^2) := \{v(\Gamma_{\mathscr{C}})\}$.

(b) Let $C$ be a disjoint union of projective curves over an algebraically closed field of characteristic $p > 0$. We define the **$p$-rank of $C$** as follows:

$$\sigma(C) := \dim_{\mathbb{F}_p} \mathrm{H}^1_{\text{ét}}(C, \mathbb{F}_p).$$

Suppose that $C$ is a semi-stable curve over an algebraically closed field of characteristic $p > 0$. Write $\Gamma_C$ for the dual semi-graph of $C$ and $v(\Gamma_C)$ for the set of vertices of $\Gamma_C$. Then we have

$$\sigma(C) = \sum_{v\in v(\Gamma_C)} \sigma(\widetilde{C}_v) + \dim_{\mathbb{C}} \mathrm{H}^1(\Gamma_C, \mathbb{C}),$$

where for $v \in v(\Gamma)$, $\widetilde{C}_v$ denotes the normalization of the irreducible component $C_v$ of $C$ corresponding to $v$. Let $\mathscr{C} := (C, D_C)$ be a pointed semi-stable curve over an algebraically closed field of characteristic $p > 0$, where $C$ denotes the underlying curve of $\mathscr{C}$, and $D_C$ denotes the set of marked points of $\mathscr{C}$. We define the $p$-rank $\sigma(\mathscr{C})$ of $\mathscr{C}$ to be the $p$-rank $\sigma(C)$.

Let $G$ be a finite group. The $p$-rank of a $G$-Galois covering (i.e., Galois covering whose Galois group is isomorphic to $G$) of a smooth projective curve is difficult to calculate in general. If $G$ is a $p$-group, then the $p$-rank of a $G$-Galois covering can be calculated by the Deuring-Shafarevich formula as follows (cf. [C]):

**Proposition 3.2.** *Let $h : C' \longrightarrow C$ be a Galois covering (possibly ramified) of smooth projective curves over an algebraically closed field of characteristic $p > 0$, whose Galois group is a finite $p$-group $G$. Then we have*

$$\sigma(C') - 1 = \sharp G(\sigma(C) - 1) + \sum_{c'\in (C')^{\mathrm{cl}}} (e_{c'} - 1),$$

*where $(C')^{\mathrm{cl}}$ denotes the set of closed points of $C'$, $e_{c'}$ denotes the ramification index at $c'$, and $\sharp G$ denotes the order of $G$.*

From now on, let $R$ be a complete discrete valuation ring with algebraically closed residue field $k$ of characteristic $p > 0$ and $K$ the quotient field. We use the notation $S$ to denote the spectrum of $R$. Write $\eta$ and $s$ for the generic point and the closed point corresponding to the natural morphisms $\mathrm{Spec}\, K \longrightarrow S$ and $\mathrm{Spec}\, k \longrightarrow S$, respectively. Let $\mathscr{X} := (X, D_X)$ be a pointed semi-stable curve over $S$. Write $\mathscr{X}_\eta := (X_\eta, D_{X_\eta})$ and $\mathscr{X}_s := (X_s, D_{X_s})$ for the generic fiber and the special fiber, respectively. Write $\Gamma_{\mathscr{X}_s}$ for the dual semi-graph of $\mathscr{X}$. Moreover, we suppose that $\mathscr{X}_\eta$ is a smooth pointed stable curve over $\eta$.

**Definition 3.3.** Let $f : \mathscr{Y} := (Y, D_Y) \longrightarrow \mathscr{X}$ be a morphism of pointed semi-stable curves over $S$ and $G$ a finite group. The morphism $f$ is called a **pointed semi-stable covering** (resp. **$G$-pointed semi-stable covering**) over $S$ if the morphism $f_\eta : \mathscr{Y}_\eta = (Y_\eta, D_{Y_\eta}) \longrightarrow \mathscr{X}_\eta = (X_\eta, D_{X_\eta})$ over $\eta$ induced by $f$ on generic fibers is a finite generically étale morphism (resp. a Galois covering whose Galois group is isomorphic to $G$) such that the following conditions are satisfied: (i) the branch locus of $f_\eta$ is contained in $D_{X_\eta}$; (ii) $f_\eta^{-1}(D_{X_\eta}) = D_{Y_\eta}$; (iii) the following universal property holds: if $g : \mathscr{Z} \longrightarrow \mathscr{X}$ is a morphism of pointed semi-stable curves over $S$ such that the generic fiber $\mathscr{Z}_\eta$ of $\mathscr{Z}$ and the morphism $g_\eta : \mathscr{Z}_\eta \longrightarrow \mathscr{X}_\eta$ induced by $g$ on generic fibers are equal to $\mathscr{Y}_\eta$ and $f_\eta$, respectively, then there exists a unique morphism $h : \mathscr{Z} \longrightarrow \mathscr{Y}$ such that $f = g \circ h$. We shall call $f$ a **pointed stable covering** (resp. **$G$-pointed stable covering**) over $S$ if $f$ is a pointed semi-stable covering (resp. $G$-pointed semi-stable covering) over $S$, and $\mathscr{X}$ is a pointed stable curve. We shall call $f$ a **semi-stable covering** (resp. **stable covering, $G$-semi-stable covering, $G$-stable covering**) over $S$ if $f$ is a pointed semi-stable covering (resp. pointed stable covering, $G$-pointed semi-stable covering, $G$-pointed stable covering) over $S$, and $D_X$ is empty.

**Proposition 3.4.** *Let $f_\eta : \mathscr{Y}_\eta := (Y_\eta, D_{Y_\eta}) \longrightarrow \mathscr{X}_\eta$ be a morphism of pointed smooth curves over $\eta$. Suppose that the branch locus of $f_\eta$ is contained in $D_{X_\eta}$ and $f_\eta^{-1}(D_{X_\eta}) = D_{Y_\eta}$. Then by replacing $S$ by a finite extension of $S$, $f_\eta$ extends uniquely to a pointed semi-stable covering $f : \mathscr{Y} = (Y, D_Y) \longrightarrow \mathscr{X}$ over $S$ such that the restriction of $f$ to $\eta$ is $f_\eta$.*

*Proof.* Write $Y'$ for the normalization of $X$ in the function field $K(Y_\eta)$ induced by the natural injection $K(X_\eta) \hookrightarrow K(Y_\eta)$ induced by $f$.

Let $D_{X_\eta}^{\mathrm{add}}$ be a set of closed points of $X_\eta$, $D_X^{\mathrm{add}}$ the set

$$\{\text{the closure of } x_\eta \text{ in } X\}_{x_\eta \in D_{X_\eta}^{\mathrm{add}}},$$

and $D_{Y_\eta}^{\mathrm{add}}$ the set inverse images $\{f_\eta^{-1}(x_\eta)\}_{x_\eta \in D_{X_\eta}^{\mathrm{add}}}$ such that the following conditions hold: (1) $D_X^{\mathrm{add}} \cap D_X = \emptyset$; (2) for any $x_\eta \in D_{X_\eta}^{\mathrm{add}}$, the reduction $x_s$ of $x_\eta$ in $X$ is a smooth point of $X_s$; (3) for any irreducible component $E_X \subseteq X_s$, there exists a unique point $x_\eta \in D_{X_\eta}^{\mathrm{add}} \cup D_{X_\eta}$ such that the reduction $x_s$ of $x_\eta$ in $X$ is contained in $E_X \setminus (X_s)^{\mathrm{sing}}$, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$; (4) for any $y_\eta \in D_{Y_\eta}^{\mathrm{add}}$, the reduction $y_s$ of $y_\eta$ in $Y'$ is a smooth point of the special fiber $Y'_s$ of $Y'$. It is easy to see the existence of $D_{X_\eta}^{\mathrm{add}}$. Furthermore, by replacing $S$ by a finite extension of $S$, we may assume that all the closed points which are contained in $D_{X_\eta}^{\mathrm{add}}$ and $D_{Y_\eta}^{\mathrm{add}}$ are $K$-rational points, and $(Y_\eta, D_{Y_\eta}^{\mathrm{add}} \cup D_{Y_\eta})$ admits a pointed stable model $\mathscr{Y}^{\mathrm{add}}$ over $S$. Write $Y$ for the underlying curve of $\mathscr{Y}^{\mathrm{add}}$, $D_Y$ for the set

$$\{\text{the closure of } y_\eta \text{ in } Y\}_{y_\eta \in D_{Y_\eta}},$$

and $D_Y^{\mathrm{add}}$ for the set

$$\{\text{the closure of } y_\eta \text{ in } Y\}_{y_\eta \in D_{Y_\eta}^{\mathrm{add}}}.$$

Note that the set of the marked points of $\mathscr{Y}^{\mathrm{add}}$ is $D_Y^{\mathrm{add}} \cup D_Y$. Then we obtain a morphism of pointed stable curves $f^{\mathrm{add}} : \mathscr{Y}^{\mathrm{add}} \longrightarrow \mathscr{X}^{\mathrm{add}} = (X, D_X^{\mathrm{add}} \cup D_X)$ over $S$. We define $\mathscr{Y}$ to be the pointed semi-stable curve $(Y, D_Y)$. Then by forgetting $D_Y^{\mathrm{add}}$ and $D_X^{\mathrm{add}}$, we obtain a natural morphism $f : \mathscr{Y} \longrightarrow \mathscr{X}$ such that the restricting morphism $f|_\eta$ is $f_\eta$. Note that $\mathscr{Y}$ and $f$ do not depend on the choices of $D_{X_\eta}^{\mathrm{add}}$.

Next, let us prove that $\mathscr{Y}$ satisfies the universal property defined in Definition 3.3. Let $\mathscr{Z} = (Z, D_Z)$ be a pointed semi-stable curve over $S$ and $g : \mathscr{Z} \longrightarrow \mathscr{X}$ is a morphism of pointed semi-stable curves over $S$ such that the generic fiber $\mathscr{Z}_\eta$ of $\mathscr{Z}$ and the morphism of generic fibers $g_\eta$ induced by $g$ are equal to $\mathscr{Y}_\eta$ and $f_\eta$, respectively. We may choose $D_{X_\eta}^{\mathrm{add}}$ such that the following conditions: for any $z_\eta \in D_{Y_\eta}^{\mathrm{add}} = \{f_\eta^{-1}(x_\eta)\}_{x_\eta \in D_{X_\eta}^{\mathrm{add}}} \subseteq Z_\eta = Y_\eta$, the reduction $z_s$ of $z_\eta$ in $Z$ is a smooth point of the special fiber $Z_s$ of $Z$. Write $D_Z^{\mathrm{add}}$ for the set

$$\{\text{the closure of } z_\eta \text{ in } Z\}_{z_\eta \in D_{Y_\eta}^{\mathrm{add}}}.$$

Then we obtain a pointed semi-stable curve $\mathscr{Z}^{\mathrm{add}} = (Z, D_Z^{\mathrm{add}} \cup D_Z)$ and a morphism of pointed semi-stable curves $g^{\mathrm{add}} : \mathscr{Z}^{\mathrm{add}} \longrightarrow \mathscr{X}^{\mathrm{add}}$ over $S$. Note that the generic fiber of $\mathscr{Z}^{\mathrm{add}}$ is equal to the generic fiber of $\mathscr{Y}^{\mathrm{add}}$. Since $\mathscr{Y}^{\mathrm{add}}$ is a pointed stable curve over $S$, we obtain a natural morphism $h^{\mathrm{add}} : \mathscr{Z}^{\mathrm{add}} \longrightarrow \mathscr{Y}^{\mathrm{add}}$ over $S$ such that $f^{\mathrm{add}} = g^{\mathrm{add}} \circ h^{\mathrm{add}}$. By forgetting $D_Z^{\mathrm{add}}$, $D_Y^{\mathrm{add}}$, and $D_X^{\mathrm{add}}$, we obtain a morphism $h : \mathscr{Z} \longrightarrow \mathscr{Y}$ such that $f = h \circ g$. This completes the proof of the proposition. □

**Proposition 3.5.** *Let $G$ be a finite group, $f : \mathscr{Y} = (Y, D_Y) \longrightarrow \mathscr{X}$ a finite $G$-pointed semi-stable covering over $S$, and $\Gamma_{\mathscr{Y}_s}$ the dual semi-graph of $\mathscr{Y}_s$. Then the images of nodes (resp. smooth points) of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ are nodes (resp. smooth points) of $\mathscr{X}_s$. In particular, the map of dual semi-graphs $\Gamma_{\mathscr{Y}_s} \longrightarrow \Gamma_{\mathscr{X}_s}$ induced by the morphism of the special fiber $f_s : \mathscr{Y}_s \longrightarrow \mathscr{X}_s$ over $s$ induced by $f$ is a morphism of semi-graphs.*

*Proof.* Let $y$ be a closed point of $Y$. Write $I_y \subseteq G$ for the inertia subgroup of $y$. Thus, the natural morphism $Y/I_y \longrightarrow X$ induced by $f$ is étale at the image of $y$ under the quotient morphism $Y \longrightarrow Y/I_y$. Then to verify the lemma, we may assume that $G = I_y$.

If $y$ is a smooth point, then $x$ is a smooth point (cf. [R, Proposition 5]). If $y$ is a node, let $Y_1$ and $Y_2$ be the irreducible components (which may be equal) of $Y$ which contain $y$. Write $D_1 \subseteq G$ and $D_2 \subseteq G$ for the decomposition subgroups of $Y_1$ and $Y_2$, respectively. The proof of [R, Proposition 5] implies that (i) if $D_1$ and $D_2$ are not equal to $I_y = G$, then $x$ is a smooth point; (ii) if $D_{v_1} = D_{v_2} = G$, then $x$ is a node.

Let us prove that the case (i) does not happen. If $D_1$ and $D_2$ are not equal to $I_y = G$, then there exists an element $\tau \in G$ such that $\tau(Y_1) = Y_2$. Thus, we have $D := D_1 = D_2$ is a normal subgroup of $G$ of index 2. By replacing $I_y$ by $I_y/D$ and $Y$ by $Y/D$ and applying the case (ii), we may assume that $D$ is trivial. Then the morphism of the special fibers $f_s : \mathscr{Y}_s \longrightarrow \mathscr{X}_s$ induced by $f$ is étale at $\eta_{Y_1}$ and $\eta_{Y_2}$, where $\eta_{Y_1}$ and $\eta_{Y_2}$ denote the generic points of $Y_1$ and $Y_2$, respectively. Consider the local morphism $f_y : \mathrm{Spec}\, \mathscr{O}_{Y,y} \longrightarrow \mathrm{Spec}\, \mathscr{O}_{X,x}$ induced by $f$. Since the restricting morphism $f_y|_\eta$ is étale, and the restricting morphism $f_y|_s$ is étale at $\eta_{Y_1}$ and $\eta_{Y_2}$, $f_y$ is étale at all the points of heights 1. By applying the Zariski-Nagata purity theorem, we obtain that $f_y$ is étale. Thus, $y$ is a smooth point. This is a contradiction. We complete the proof of the proposition. □

**Definition 3.6.** Let $f : \mathscr{Y} \longrightarrow \mathscr{X}$ be a pointed semi-stable covering over $S$. A closed point $x \in \mathscr{X}$ is called a **vertical point associated to** $f$, or for simplicity, a **vertical**

**point** when there is no fear of confusion, if $f^{-1}(x)$ is not a finite set. The inverse image $f^{-1}(x)$ is called the **vertical fiber associated to** $x$.

If a vertical point $x$ is a smooth point of $\mathscr{X}_s$ and $x \notin D_{X_s}$, the following result was proved by Raynaud (cf. [R, Théorème 1, Proposition 1, and Proposition 2]).

**Proposition 3.7.** *Let $G$ be a finite $p$-group, $f : \mathscr{Y} \longrightarrow \mathscr{X}$ a $G$-pointed semi-stable covering over $S$, and $x$ a vertical point associated to $f$. If $x$ is a smooth point of $\mathscr{X}_s$ and $x \notin D_{X_s}$, then the $p$-rank of each connected component of the vertical fiber $f^{-1}(x)$ associated to $x$ is equal to* 0. *On the other hand, by contracting the vertical fibers $f^{-1}(x)$, we obtain a curve $\mathscr{Y}^{\mathrm{c}}$ over $S$. Write $c : \mathscr{Y} \longrightarrow \mathscr{Y}^{\mathrm{c}}$ for the contracting morphism. Then the closed points $c(f^{-1}(x))$ are geometrically unibranch.*

## 3.2 Global cases

From now on, we always assume that $G$ is a finite $p$-group with order $p^r$. Let $f : \mathscr{Y} = (Y, D_Y) \longrightarrow \mathscr{X} = (X, D_X)$ be a $G$-pointed semi-stable covering over $S$ and

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

a maximal normal filtration of $G$. It follows from [R, Appendice, Corollaire], $\mathscr{Y}_j := \mathscr{Y}/G_j$, $j = 0, \dots, r$, is a pointed semi-stable curve over $S$. Write $\mathscr{X}^{\mathrm{sst}} := (X^{\mathrm{sst}}, D_{X^{\mathrm{sst}}})$ for $\mathscr{Y}_0$. We obtain two natural morphisms of pointed semi-stable curves $h : \mathscr{Y} \longrightarrow \mathscr{X}^{\mathrm{sst}}$ and $g : \mathscr{X}^{\mathrm{sst}} \longrightarrow \mathscr{X}$ induced by $f$ such that $g \circ h = f$. The maximal normal filtration $\Phi$ of $G$ induces a sequence of morphisms of pointed semi-stable curves over $S$

$$\Phi_{\mathscr{Y}/\mathscr{X}^{\mathrm{sst}}} : \mathscr{Y} = \mathscr{Y}_r \xrightarrow{\phi_r} \mathscr{Y}_{r-1} \xrightarrow{\phi_{r-1}} \dots \xrightarrow{\phi_1} \mathscr{Y}_0 = \mathscr{X}^{\mathrm{sst}}$$

such that $\phi_1 \circ ... \circ \phi_r = h$. Note that $\phi_j, j = 1, \dots, r$, is a finite $\mathbb{Z}/p\mathbb{Z}$-pointed semi-stable covering over $S$. For each $j \in \{0, \dots, r\}$, write $\Gamma_{\mathscr{Y}_j}$ for the dual semi-graph of the special fiber $(\mathscr{Y}_j)_s$ of $\mathscr{Y}_j$. Then for each $j = 1, \dots, r$, the morphism of the special fibers $(\phi_j)_s : (\mathscr{Y}_j)_s \longrightarrow (\mathscr{Y}_{j-1})_s$ induces a map of semi-graphs $\beta_j : \Gamma_{\mathscr{Y}_j} \longrightarrow \Gamma_{\mathscr{Y}_{j-1}}$. Moreover, Proposition 3.5 implies that $\beta_j, j = 1, \dots, r$, is a morphism of semi-graphs.

For each $v \in v(\Gamma_{\mathscr{Y}_j})$, write $\widetilde{Y}_v^j$ for the normalization of the irreducible component $Y_v^j \subseteq (\mathscr{Y}_j)_s$ corresponding to $v$. We define a semi-graph with $p$-rank $\mathfrak{G}_{\mathscr{Y}_j} := (\mathbb{G}_{\mathscr{Y}_j}, \sigma_{\mathfrak{G}_{\mathscr{Y}_j}}), j = 0, \dots, r$, associated to $(\mathscr{Y}_j)_s$ as follows: (i) $\mathbb{G}_{\mathscr{Y}_j} = \Gamma_{\mathscr{Y}_j}$; (ii) for each $v \in v(\mathbb{G}_{\mathscr{Y}_j})$, $\sigma(v) := \sigma(\widetilde{Y}_v^j)$. Then $\Phi_{\mathscr{Y}/\mathscr{X}^{\mathrm{sst}}}$ induces a sequence of morphisms of semi-graphs with $p$-rank

$$\Phi_{\mathfrak{G}_{\mathscr{Y}}/\mathfrak{G}_{\mathscr{X}^{\mathrm{sst}}}} : \mathfrak{G}_{\mathscr{Y}} := \mathfrak{G}_{\mathscr{Y}_r} \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{\mathscr{Y}_{r-1}} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_{\mathscr{X}^{\mathrm{sst}}} := \mathfrak{G}_{\mathscr{Y}_0},$$

where $\mathfrak{b}_j : \mathfrak{G}_{\mathscr{Y}_j} \longrightarrow \mathfrak{G}_{\mathscr{Y}_{j-1}}, j = 1, \dots, r$, is induced by $\beta_j : \Gamma_{\mathscr{Y}_j} \longrightarrow \Gamma_{\mathscr{Y}_{j-1}}, j = 1, \dots, r$. By using the Deuring-Shafarevich formula and Zariski-Nagata purity, it is easy to see that $\mathfrak{b}_j, j = 1, \dots, r$, is a $p$-covering, moreover, $\mathfrak{b} := \mathfrak{b}_1 \circ \cdots \circ \mathfrak{b}_r$ is a $G$-covering. Then we have $\sigma_{\mathfrak{G}_{\mathscr{Y}}}(\mathfrak{G}_{\mathscr{Y}}) = \sigma(\mathscr{Y}_s)$. Summarizing the discussion above, we obtain the following proposition.

**Proposition 3.8.** *Let $f : \mathscr{Y} \longrightarrow \mathscr{X}$ be a $G$-pointed semi-stable covering over $S$ and $\mathscr{Y}_s$ the special fiber of $\mathscr{Y}$. Then there exists a $G$-covering of semi-graphs with $p$-rank $\mathfrak{b} : \mathfrak{G}_{\mathscr{Y}} \longrightarrow \mathfrak{G}_{\mathscr{X}}$ associated to $f$ which is constructed above such that $\sigma(\mathscr{Y}_s) = \sigma_{\mathfrak{G}_{\mathscr{Y}}}(\mathfrak{G}_{\mathscr{Y}})$.*

### 3.3 Local cases

We maintain the notations introduced in Section 3.2. Let $x$ be a vertical point associated to $f$. Write $Y'$ for the normalization of $X$ in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by $f$ and $\psi$ for the normalization morphism $Y' \longrightarrow X$. Then $Y'$ admits a natural action of $G$ induced by the action of $G$ on the generic fiber of $Y$. Let $y' \in \psi^{-1}(x)$. Write $I_{y'} \subseteq G$ for the inertia group of $y'$. Proposition 3.4 implies that the morphism of pointed smooth curves $(Y_\eta/I_{y'}, D_Y/I_{y'}) \longrightarrow \mathscr{X}_\eta$ over $\eta$ induced by $f$ extends to a pointed semi-stable covering $\mathscr{Y}_{I_{y'}} \longrightarrow \mathscr{X}$ over $S$. In order to calculate the $p$-rank of $f^{-1}(x)$, since the morphism $\mathscr{Y}_{I_{y'}} \longrightarrow \mathscr{X}$ is finite étale over $x$, by replacing $\mathscr{X}$ by $\mathscr{Y}_{I_{y'}}$, we may assume that $G$ is equal to $I_{y'}$. In the remainder of this subsection, we shall assume that $G = I_{y'}$. Then $f^{-1}(x)$ is connected.

Write $\mathscr{X}_s^{\mathrm{sst}} = (X_s^{\mathrm{sst}}, D_{X_s^{\mathrm{sst}}})$ (resp. $\mathscr{Y}_s = (Y_s, D_{Y_s})$) for the special fiber of $\mathscr{X}$ (resp. $\mathscr{Y}$), and $(-)_{\mathrm{red}}$ for the reduced induced closed subscheme of $(-)$. By the general theory of semi-stable curves, $g^{-1}(x)_{\mathrm{red}} \subset \mathscr{X}_s^{\mathrm{sst}}$ (resp. $f^{-1}(x)_{\mathrm{red}} = h^{-1}(g^{-1}(x))_{\mathrm{red}} \subset \mathscr{Y}_s$) is a pointed semi-stable curve over $s$. In particular, the irreducible components of $\underline{g^{-1}(x)_{\mathrm{red}}}$ are isomorphic to $\mathbb{P}^1$. Write $V_X$ for the set of closed points $g^{-1}(x)_{\mathrm{red}} \cap \overline{\mathscr{X}_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}}$, where $\overline{\mathscr{X}_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}}$ denotes the closure of $\mathscr{X}_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}$ in $\mathscr{X}_s^{\mathrm{sst}}$, and $V_Y \subset \mathscr{Y}_s$ for the set of closed points $\{h^{-1}(p)_{\mathrm{red}}\}_{p \in V_X}$. Note that $V_X$ consists of a closed point (resp. two closed points) of $X_s^{\mathrm{sst}}$ if $x$ is a smooth point (resp. a node) of $X_s$. Write $\underline{g^{-1}(x)_{\mathrm{red}}}$ (resp. $\underline{f^{-1}(x)_{\mathrm{red}}}$) for the underlying curve of $g^{-1}(x)_{\mathrm{red}}$ (resp. $f^{-1}(x)_{\mathrm{red}}$).

We define two pointed semi-stable curves over $s$ to be $\mathscr{E}_X := (\underline{g^{-1}(x)_{\mathrm{red}}}, (D_{X^{\mathrm{sst}}} \cap \underline{g^{-1}(x)_{\mathrm{red}}}) \cup V_X)$ and $\mathscr{E}_Y := (\underline{f^{-1}(x)_{\mathrm{red}}}, (D_Y \cap \underline{f^{-1}(x)_{\mathrm{red}}}) \cup V_Y)$. Then we obtain a morphism of pointed semi-stable curves $\rho_{\mathscr{E}_Y/\mathscr{E}_X} : \mathscr{E}_Y \longrightarrow \mathscr{E}_X$ induced by $h$. Moreover, since $f^{-1}(x)$ is connected, $\mathscr{E}_Y$ admits a natural action of $G$ induced by the action of $G$ on the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$. Write $\Gamma_{\mathscr{E}_Y}$ (resp. $\Gamma_{\mathscr{E}_X}$) for the dual semi-graph of $\mathscr{E}_Y$ (resp. $\mathscr{E}_X$). We obtain a map of semi-graphs $\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \longrightarrow \Gamma_{\mathscr{E}_X}$ induced by $\rho_{\mathscr{E}_Y/\mathscr{E}_X}$. Proposition 3.5 implies that the map $\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \longrightarrow \Gamma_{\mathscr{E}_X}$ is a morphism of semi-graphs. Note that $\Gamma_{\mathscr{E}_X}$ is a tree.

For each $v \in v(\mathbb{E}_Y)$, write $\widetilde{Y}_v$ for the normalization of the irreducible component $Y_v \subseteq Y_s$ corresponding to $v$. We define a semi-graph with $p$-rank $\mathfrak{E}_Y := (\mathbb{E}_Y, \sigma_{\mathfrak{E}_Y})$ (resp. $\mathfrak{E}_X := (\mathbb{E}_X, \sigma_{\mathfrak{E}_X})$) associated to $\mathscr{E}_Y$ (resp. $\mathscr{E}_X$) as follows: (i) $\mathbb{E}_Y = \Gamma_{\mathscr{E}_Y}$ (resp. $\mathbb{E}_X = \Gamma_{\mathscr{E}_X}$); (ii) for each $v \in v(\mathbb{E}_Y)$ (resp. $v \in v(\mathbb{E}_X)$), we set $\sigma_{\mathfrak{E}_Y}(v) := \sigma(\widetilde{Y}_v)$ (resp. $\sigma_{\mathfrak{E}_X}(v) := 0$). The morphism of dual semi-graphs $\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \longrightarrow \Gamma_{\mathscr{E}_X}$ induces a morphism of semi-graphs with $p$-rank $\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X} : \mathfrak{E}_Y \longrightarrow \mathfrak{E}_X$. Moreover, $\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X}$ is a $G$-covering. Then we have $\sigma_{\mathfrak{E}_Y}(\mathfrak{E}_Y) = \sigma(f^{-1}(x)_{\mathrm{red}}) = \sigma(f^{-1}(x))$. Summarizing the discussion above, we obtain the following proposition.

**Proposition 3.9.** *Let $f : \mathscr{Y} \longrightarrow \mathscr{X}$ be a $G$-pointed semi-stable covering over $S$ and $x$ a vertical point associated to $f$. Suppose that $f^{-1}(x)$ is connected. Then there exists a $G$-covering of semi-graphs with $p$-rank $\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X} : \mathfrak{E}_Y \longrightarrow \mathfrak{E}_X$ associated to $f$ and $x$ which is constructed above such that $\sigma_{\mathfrak{E}_Y}(\mathfrak{E}_Y) = \sigma(f^{-1}(x))$.*

In the remainder of this subsection, we suppose that the vertical point $x$ is a node of $\mathscr{X}_s$. Write $X_1'$ and $X_2'$ (which may be equal) for the irreducible components of $\mathscr{X}_s$ which contain $x$. Write $X_1$ and $X_2$ for the strict transforms of $X_1'$ and $X_2'$ under the birational morphism $g : \mathscr{X}^{\mathrm{sst}} \longrightarrow \mathscr{X}$, respectively. By the general theory of semi-stable curves,

$g^{-1}(x)_{\rm red} \subseteq \mathscr{X}_s^{\rm sst}$ is a semi-stable curve (i.e., $g^{-1}(x)_{\rm red} \cap D_{X_s^{\rm sst}} = \emptyset$) over $s$ whose irreducible components are isomorphic to $\mathbb{P}_k^1$. Write $C$ for the semi-stable subcurve of $g^{-1}(x)_{\rm red}$ which is a chain of projective lines $\cup_{i=1}^n P_i$ such that the following conditions hold: (i) for any $s, t = 1, \dots, n$, $P_s \cap P_t = \emptyset$ if $|s-t| \geq 2$ and $P_s \cap P_t$ is reduced to a point if $|s-t| = 1$; (ii) $P_1 \cap X_1$ (resp. $P_n \cap X_2$) is reduced to a point; (iii) $C \cap \overline{\{X^{\rm sst} \setminus C\}} = (P_1 \cap X_1) \cup (P_n \cap X_2)$, where $\overline{\{X^{\rm sst} \setminus C\}}$ denotes the closure of $X^{\rm sst} \setminus C$ in $X^{\rm sst}$. Then we have

$$g^{-1}(x)_{\rm red} = C \cup B,$$

where $B$ denotes the topological closure of $g^{-1}(x)_{\rm red} \setminus C$ in $g^{-1}(x)_{\rm red}$. Note that $B \cap C$ are smooth points of $C$. Then it follows from Proposition 3.7, the $p$-ranks of the connected components of $B$ are equal to 0. Thus, we have $\sigma(f^{-1}(x)) = \sigma(h^{-1}(C))$.

Let $\{V_i\}_{i=1}^n$ be a set of irreducible components of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ such that the following conditions hold: (i) $h(V_i) = P_i$ for $i = 1, \dots, n$; (ii) the union $\cup_{i=1}^n V_i \subseteq \mathscr{Y}_s$ is a connected semi-stable curve (i.e., $(\cup_{i=1}^n V_i) \cap D_{Y_s} = \emptyset$ ) over $s$; write $I_{V_i} \subseteq G, i = 1, \dots, n$ for the inertia group of $V_i$ and for any closed point $y_i \in V_i$, $I_{y_i} \subseteq G$ for the inertia group of $y_i$. Then we have the following lemma.

**Lemma 3.10.** *Write* $\mathrm{Ray}_{V_i}, i = 1, \dots, n$, *for the set of the closed points* $h^{-1}(C \cap B)_{\rm red} \cap V_i$. *Then for any* $y_i \in \mathrm{Ray}_{V_i}$, *we have* $I_{y_i} = I_{V_i}$.

*Proof.* Since $I_{y_i} \supseteq I_{V_i}$, we only need to prove that $I_{y_i} \subseteq I_{V_i}$. Note that $I_{V_i}$ is a normal subgroup of $I_{y_i}$. By replacing $G$ and $\mathscr{X}^{\rm sst}$ by $I_{y'}$ and $\mathscr{Y}/I_{y_i}$, respectively, we may assume that $G = I_{y_i}$. Then we have $\sharp h^{-1}(h(y_i))_{\rm red} = 1$.

Consider the quotient curve $\mathscr{Y}/I_{V_i}$. By [R, Appendice Corollaire], $\mathscr{Y}/I_{V_i}$ is a pointed semi-stable curve over $S$. Write $h_{I_{V_i}}$ for the quotient morphism $\mathscr{Y} \longrightarrow \mathscr{Y}/I_{V_i}$ and $g_{I_{V_i}}$ for the $\mathscr{Y}/I_{V_i} \longrightarrow \mathscr{X}^{\rm sst}$ induced by $h$ such that $h = g_{I_{V_i}} \circ h_{I_{V_i}}$. Write $E_{y_i}$ for the connected component of $h^{-1}(B)_{\rm red}$ which contains $y_i$. Contracting $h_{I_{V_i}}(E_{y_i})$ (resp. $h(E_{y_i})$) which is contained in the special fiber of $\mathscr{Y}/I_{V_i}$ (resp. $\mathscr{X}^{\rm sst}$), we obtain a fiber surface $(\mathscr{Y}/I_{V_i})^{\rm c}$ and a semi-stable curve $(\mathscr{X}^{\rm sst})^{\rm c}$ over $S$. Moreover, we obtain three morphisms of fiber surfaces $c_{h_{I_{V_i}}(E_{y_i})} : \mathscr{Y}/I_{V_i} \longrightarrow (\mathscr{Y}/I_{V_i})^{\rm c}$, $c_{h(E_{y_i})} : \mathscr{X}^{\rm sst} \longrightarrow (\mathscr{X}^{\rm sst})^{\rm c}$, and $g_{I_{V_i}}^{\rm c} : (\mathscr{Y}/I_{V_i})^{\rm c} \longrightarrow (\mathscr{X}^{\rm sst})^{\rm c}$ such that $c_{h(E_{y_i})} \circ g_{I_{V_i}} = g_{I_{V_i}}^{\rm c} \circ c_{h_{I_{V_i}}(E_{y_i})}$. Note that $c_{h(E_{y_i})} \circ h(y_i)$ is a smooth point of the special fiber of $(\mathscr{X}^{\rm sst})^{\rm c}$, and $g_{I_{V_i}}^{\rm c}$ is étale at the generic point of $c_{h_{I_{V_i}}(E_{y_i})} \circ h_{I_{V_i}}(V_i)$.

Write $y_i^{\rm c} \in (\mathscr{Y}/I_{V_i})^{\rm c}$ and $x_i^{\rm c} \in (\mathscr{X}^{\rm sst})^{\rm c}$ for $c_{h_{I_{V_i}}(E_{y_i})} \circ h_{I_{V_i}}(y_i)$ and $c_{h(E_{y_i})} \circ h(y_i)$, respectively. Consider the morphism $g_{y_i^{\rm c}} : \mathrm{Spec}\, \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\rm c}, y_i^{\rm c}} \longrightarrow \mathrm{Spec}\, \mathcal{O}_{(\mathscr{X}^{\rm sst})^{\rm c}, x_i^{\rm c}}$ induced by $g_{I_{V_i}}^{\rm c}$. Proposition 3.7 implies that the special fiber of $\mathrm{Spec}\, \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\rm c}, y_i^{\rm c}}$ is irreducible. Then $g_{y_i^{\rm c}}$ is generically étale at the generic point of the special fiber of $\mathrm{Spec}\, \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\rm c}, y_i^{\rm c}}$. Thus, by applying Zariski-Nagata purity, $g_{y_i^{\rm c}}$ is étale.

If $I_{V_i} \neq I_{y_i}$, then we obtain that $g_{y_i^{\rm c}}$ is not an identity. Thus, we have $\sharp h^{-1}(h(y_i))_{\rm red} \neq 1$. This is a contradiction. Then we have $I_{V_i} = I_{y_i}$. □

Let $\mathscr{C}_Y := (h^{-1}(C)_{\rm red}, h^{-1}((C \cap X_1) \cup (C \cap X_2)))$ and $\mathscr{C}_X := (C, (C \cap X_1) \cup (C \cap X_2))$ be two pointed semi-stable curves and $\rho_{\mathscr{C}_Y/\mathscr{C}_X} : \mathscr{C}_Y \longrightarrow \mathscr{C}_X$ the natural morphism over $s$ induced by $h : \mathscr{Y} \longrightarrow \mathscr{X}^{\rm sst}$. Moreover, since $f^{-1}(x)_{\rm red}$ is connected, $\mathscr{C}_Y$ admits a natural action of $G$ induced by the action of $G$ on $f^{-1}(x)_{\rm red}$. Write $\Gamma_{\mathscr{C}_Y}$ (resp. $\Gamma_{\mathscr{C}_X}$) for the dual semi-graph of $\mathscr{C}_Y$ (resp. $\mathscr{C}_X$). Proposition 3.5 implies that the map of semi-graphs $\delta_{\mathscr{C}_Y/\mathscr{C}_X} : \Gamma_{\mathscr{C}_Y} \longrightarrow \Gamma_{\mathscr{C}_X}$ induced by $\rho_{\mathscr{C}_Y/\mathscr{C}_X}$ is a morphism of semi-graphs.

For each $v \in v(\Gamma_{\mathscr{C}_Y})$, write $\widetilde{Y}_v$ for the normalization of the irreducible component $Y_v \subseteq Y_s$ corresponding to $v$. We define a semi-graph with $p$-rank $\mathfrak{C}_Y := (\mathbb{C}_Y, \sigma_{\mathfrak{C}_Y})$ (resp. $\mathfrak{C}_X := (\mathbb{C}_X, \sigma_{\mathfrak{C}_X})$) associated to $\mathscr{C}_Y$ (resp. $\mathscr{C}_X$) as follows: (i) $\mathbb{C}_Y = \Gamma_{\mathscr{C}_Y}$ (resp. $\mathbb{C}_X = \Gamma_{\mathscr{C}_X}$); (ii) For each $v \in v(\mathbb{C}_Y)$ (resp. $v \in v(\mathbb{C}_X)$), we set $\sigma_{\mathfrak{C}_Y}(v) := \sigma(\widetilde{Y}_v)$ (resp. $\sigma_{\mathfrak{C}_X}(v) := 0$).

The morphism of dual semi-graphs $\delta_{\mathscr{C}_Y/\mathscr{C}_X} : \Gamma_{\mathscr{C}_Y} \longrightarrow \Gamma_{\mathscr{C}_X}$ induces a morphism of semi-graphs with $p$-rank $\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X} : \mathfrak{C}_Y \longrightarrow \mathfrak{C}_X$. Moreover, $\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X} : \mathfrak{C}_Y \longrightarrow \mathfrak{C}_X$ is a $G$-covering. Note that by the construction, $\mathfrak{C}_X$ is a $n$-chain (cf. Definition 2.9). Furthermore, Lemma 3.10 implies that $\sigma_{\mathfrak{C}_Y}(\mathfrak{C}_Y) = \sigma(h^{-1}(C)) = \sigma(f^{-1}(x))$. Summarizing the discussion above, we obtain the following proposition.

**Proposition 3.11.** *Let $f : \mathscr{Y} \longrightarrow \mathscr{X}$ be a $G$-pointed semi-stable covering over $S$ and $x \in \mathscr{X}_s$ a vertical point associated to $f$ such that $x$ is a node of $\mathscr{X}_s$. Suppose that $f^{-1}(x)$ is connected. Then there exists a $G$-covering of semi-graphs with $p$-rank $\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X} : \mathfrak{C}_Y \longrightarrow \mathfrak{C}_X$ associated to $f$ and $x$ which is constructed above such that $\mathfrak{C}_X$ is a $n$-chain and $\sigma_{\mathfrak{C}_Y}(\mathfrak{C}_Y) = \sigma(f^{-1}(x))$.*

# 4 Formulas for local and global $p$-ranks of coverings of curves

## 4.1 Inertia groups and a criterion for the existence of vertical fibers

In this subsection, we study the relationship between the inertia groups of nodes and the inertia groups of irreducible components of special fibers of $G$-pointed semi-stable coverings.

**Lemma 4.1.** *Let $f : \mathscr{Y} = (Y, D_Y) \longrightarrow \mathscr{X}$ be a finite $G$-semi-stable covering over $S$, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$, $y \in Y_s$ a node, and $Y_1$ and $Y_2$ the irreducible components of $Y_s$ which contain $y$ (which may be equal). Write $I_y \subseteq G$ (resp. $I_{Y_1} \subseteq G$, $I_{Y_2} \subseteq G$) for the inertia group of $y$ (resp. $Y_1$, $Y_2$). If $G$ is a $p$-group, then inertia group $I_y$ is generated by $I_{Y_1}$ and $I_{Y_2}$.*

*Proof.* Write $I$ for the group generated by $I_{Y_1}$ and $I_{Y_2}$. Then we have $I \subseteq I_y$. Consider the quotient $\mathscr{Y}/I$. We obtain two morphism of pointed semi-stable curves $\mu_1 : \mathscr{Y} \longrightarrow \mathscr{Y}/I$ and $\mu_2 : \mathscr{Y}/I \longrightarrow \mathscr{X}$ over $S$ such that $\mu_2 \circ \mu_1 = f$. Note that $\mathscr{Y}/I$ is a pointed semi-stable curve over $S$, and $\mu_1(y)$ is node of the special fiber $(\mathscr{Y}/I)_s$ of $\mathscr{Y}/I$ (cf. [R, Appendice, Corollaire] and the proof). Moreover, $\mu_2$ is generically étale at the generic points of $\mu_1(Y_1)$ and $\mu_1(Y_2)$. Then applying [T, Lemma 2.1 (iii)] to $\operatorname{Spec} \mathcal{O}_{\mathscr{Y}/I,\mu_1(y)} \longrightarrow \operatorname{Spec} \mathcal{O}_{\mathscr{X},f(y)}$, we obtain that $\mu_2$ is tamely ramified at $\mu_1(y)$. Moreover, since $G$ is a $p$-group, $\mu_2$ is étale at $\mu_1(y)$. This means that $I_y \subseteq I$. Thus, we obtain $I_y = I$. $\square$

The following criterion for the existence of vertical fibers due to A. Tamagawa (cf. [T, Propoisiton 4.3 (ii)]).

**Proposition 4.2.** *Let $f : \mathscr{Y} \longrightarrow \mathscr{X}$ be a $G$-semi-stable covering over $S$, and $x$ a node of $\mathscr{X}_s$. Suppose that for each irreducible component $Z := \overline{\{z\}}$ of $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{X}_s,x}$ and each point $w$ of the fiber $\mathscr{Y} \times_{\mathscr{X}} z$, the natural morphism from the integral closure $W^s$ of $Z$ in $k(w)^s$ to $Z$ is wildly ramified, where $k(w)^s$ denotes the maximal separable subextension of $k(w)$ in $k(z)$. Then $x$ is a vertical point associated to $f$.*

We prove a criterion of existence of vertical fibers over nodes as follows:

**Proposition 4.3.** *Let $f : \mathscr{Y} = (Y, D_Y) \longrightarrow \mathscr{X}$ be a $G$-semi-stable covering over $S$, $\mathscr{Y}_\eta = (Y_\eta, D_{Y_\eta})$ the generic fiber of $\mathscr{Y}$, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$, and $x$ a node of $\mathscr{X}_s$. Write $\mathscr{Y}'$ for the normalization of $\mathscr{X}$ in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by $f$, and $\psi_2$ for the resulting normalization morphism $\mathscr{Y}' \longrightarrow \mathscr{X}$. There is a natural morphism of fiber surfaces $\psi_1 : \mathscr{Y} \longrightarrow \mathscr{Y}'$ induced by $f$ such that $\psi_2 \circ \psi_1 = f$. Write $X_1$ and $X_2$ for the irreducible components of $\mathscr{X}_s$ which contain $x$ (which may be equal). Let $y' \in \psi_2^{-1}(x)$, $Y_1$ and $Y_2$ the irreducible components of $\mathscr{Y}_s$ such that $y' \in \psi_1(Y_1) \cap \psi_1(Y_2)$, and $I_{Y_1} \subseteq G$ and $I_{Y_2} \subseteq G$ the inertia group of $Y_1$ and $Y_2$, respectively. If neither $I_{Y_1} \subseteq I_{Y_2}$ nor $I_{Y_1} \supseteq I_{Y_2}$ holds, then $x$ is a vertical point associated to $f$.*

*Proof.* To verify the proposition, we assume that $x$ is not a vertical point associated to $f$. Then $f^{-1}(x)$ is a finite set, and $\psi_1$ and $\psi_2$ coincide with $f$ over $x$. Write $y$ for $y' \in \psi_1^{-1}(x) = f^{-1}(x)$. By replacing $\mathscr{X}$ by the quotient $\mathscr{Y}/D_y$ and $G$ by $D_y \subseteq G$, where $D_y$ denotes the decomposition group of $y$, we may assume that $f^{-1}(x) = \{y\} \subseteq Y_1 \cap Y_2$. Consider the quotient $\mathscr{Y}/I_{Y_1}$ (resp. $\mathscr{Y}/I_{Y_2}$) which is a semi-stable curve over $S$. We obtain two morphism of semi-stable curves $\lambda_1^1 : \mathscr{Y} \longrightarrow \mathscr{Y}/I_{Y_1}$ and $\lambda_2^1 : \mathscr{Y}/I_{Y_1} \longrightarrow \mathscr{X}$ over $S$ such that $\lambda_2^1 \circ \lambda_1^1 = f$ (resp. $\lambda_1^2 : \mathscr{Y} \longrightarrow \mathscr{Y}/I_{Y_2}$ and $\lambda_2^2 : \mathscr{Y}/I_{Y_2} \longrightarrow \mathscr{X}$ over $S$ such that $\lambda_2^2 \circ \lambda_1^2 = f$). Note that $\lambda_2^1$ (resp. $\lambda_2^2$) is étlae at the generic point of $\lambda_1^1(Y_1)$ (resp. $\lambda_1^2(Y_2)$) with degree $\sharp G/\sharp I_{Y_1}$ (resp. $\sharp G/\sharp I_{Y_2}$).

If $\lambda_2^1$ (resp. $\lambda_2^2$) is generically étale at the generic point of $\lambda_1^1(Y_2)$ (resp. $\lambda_1^2(Y_1)$), then by applying [T, Lemma 2.1 (iii)] to $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{Y}/I_{Y_1},\lambda_1^1(y)} \longrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{X},x}$ (resp. $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{Y}/I_{Y_2},\lambda_1^2(y)} \longrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{X},x}$), we obtain $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_1^1(Y_1),\lambda_1^1(y)} \longrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{X_1,x}$ (resp. $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_1^2(Y_2),\lambda_1^2(y)} \longrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{X_2,x}$) induced by $\lambda_2^1$ (resp. $\lambda_2^2$) is tamely ramified with ramification index $t_1$ (resp. $t_2$). Thus, we have $(t_1, p) = 1$ (resp. $(t_2, p) = 1$. On the other hand, since $I_{Y_1}$ (resp. $I_{Y_2}$) does not contain $I_{Y_2}$ (resp. $I_{Y_1}$), and $I_{Y_1}$ (resp. $I_{Y_2}$) is a $p$-group, we have $p|t^1$ (resp. $p|t^2$). This is a contradiction. Thus, $\lambda_2^1$ (resp. $\lambda_2^2$) is not generically étale at the generic point of $\lambda_1^1(Y_2)$ (resp. $\lambda_1^2(Y_1)$).

Moreover, the morphism $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_1^1(Y_1),\lambda_1^1(y)} \longrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{X_1,x}$ (resp. $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_1^2(Y_2),\lambda_1^2(y)} \longrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{X_2,x}$) induced by $\lambda_2^1$ (resp. $\lambda_2^2$) is wildly ramified. Thus, Proposition 2.3 implies that $x$ is a vertical point associated to $f$. This is a contradiction. We complete the proof of the proposition. □

**Corollary 4.4.** *Let $f : \mathscr{Y} = (Y, D_Y) \longrightarrow \mathscr{X}$ be a $G$-semi-stable covering over $S$, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$, $y \in Y_s$ a node, and $Y_1$ and $Y_2$ the irreducible components of $\mathscr{Y}_s$ which contain $y$ (which may be equal). Write $I_y \subseteq G$ (resp. $I_{Y_1} \subseteq G$, $I_{Y_2} \subseteq G$) for the inertia group of $y$ (resp. $Y_1$, $Y_2$). Suppose that $G$ is a $p$-group, and $f$ is a finite morphism. Then either $I_{Y_1} \subseteq I_{Y_2}$ or $I_{Y_1} \supseteq I_{Y_2}$ holds, moreover, the inertia group $I_y$ is equal to either $I_{Y_1}$ or $I_{Y_2}$.*

*Proof.* The corollary follows from Lemma 4.1 and Proposition 4.3. $\square$

## 4.2 Global version

Let $f : \mathscr{Y} = (Y, D_Y) \longrightarrow \mathscr{X} = (X, D_X)$ be a $G$-pointed semi-stable covering over $S$, $h : \mathscr{Y} \longrightarrow \mathscr{Y}/G := \mathscr{X}^{\mathrm{sst}} = (X^{\mathrm{sst}}, D_{X^{\mathrm{sst}}})$ the quotient morphism, and $\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$ the dual semi-graph of the special fiber $\mathscr{X}_s^{\mathrm{sst}} = (X_s^{\mathrm{sst}}, D_{X_s^{\mathrm{sst}}})$ of $\mathscr{X}^{\mathrm{sst}}$. For each $v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$ (resp. $e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}) \cup e^{\mathrm{op}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$), write $X_v$ (resp. $x_e$) for the irreducible component of $\mathscr{X}_s^{\mathrm{sst}}$ corresponding to $v$ (resp. for the node of $\mathscr{X}_s^{\mathrm{sst}}$ corresponding to $e$ if $e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$ or a marked point of $\mathscr{X}_s^{\mathrm{sst}}$ corresponding to $e$ if $e \in e^{\mathrm{op}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$), $\widetilde{X}_v$ for the normalization of $X_v$. For each $v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$ (resp. $e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}) \cup e^{\mathrm{op}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$), let $Y_v$ (resp. $y_e$) be an connected component of $h^{-1}(X_v)_{\mathrm{red}}$ (resp. a point of $h^{-1}(x_e)_{\mathrm{red}}$). Write $I_{Y_v} \subseteq G$ (resp. $I_{y_e} \subseteq G$) for the inertia group of $Y_v$ (resp. $y_e$). Since $\sharp I_{Y_v}$ (resp. $\sharp I_{y_e}$) does not depend on the choices of $Y_v$ (resp. $y_e$), we may use the notation $\sharp I_v$ (resp. $\sharp I_e$) to denote $\sharp I_{Y_v}$ (resp. $\sharp I_{y_e}$). For any $e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$, write $\sharp I_e^{\mathrm{m}}$ for $\max_{v \in e(v)}\{I_v\}$. By Corollary 4.4, we have $\sharp I_e = \sharp I_e^{\mathrm{m}}$. Then Theorem 2.8 and Proposition 3.8 imply the following theorem.

**Theorem 4.5.** *We maintain the notations introduced above. Then we have*

$$\sigma(\mathscr{Y}_s) = \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} (\sharp G/\sharp I_v(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v) \setminus e^{\mathrm{lp}}(v)} \sharp G/\sharp I_e(\sharp I_e/\sharp I_v - 1) + 1)$$

$$+ \sum_{e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}) \setminus e^{\mathrm{lp}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} (\sharp G/\sharp I_e - 1) + \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} \sharp e^{\mathrm{lp}}(v)(\sharp G/\sharp I_v - 1) + \dim_{\mathbb{C}} \mathrm{H}^1(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}, \mathbb{C}).$$

*In particular, if $f : \mathscr{Y} \longrightarrow \mathscr{X}$ is a $G$-semi-stable covering, then we have*

$$\sigma(\mathscr{Y}_s) = \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} (\sharp G/\sharp I_v(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v) \setminus e^{\mathrm{lp}}(v)} \sharp G/\sharp I_e^{\mathrm{m}}(\sharp I_e^{\mathrm{m}}/\sharp I_v - 1) + 1)$$

$$+ \sum_{e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}) \setminus e^{\mathrm{lp}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} (\sharp G/\sharp I_e^{\mathrm{m}} - 1) + \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} \sharp e^{\mathrm{lp}}(v)(\sharp G/\sharp I_v - 1) + \dim_{\mathbb{C}} \mathrm{H}^1(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}, \mathbb{C}).$$

## 4.3 Local version

We follow the notations of Section 4.2. Let $x$ be a vertical point associated to $f$. Suppose that $f^{-1}(x)$ is connected. Write $g$ for the natural morphism $\mathscr{X}^{\mathrm{sst}} \longrightarrow \mathscr{X}$ over $S$ induced by $f$ such that $f = g \circ h$. Write

$$\underline{g^{-1}(x)_{\mathrm{red}}}$$

for the underlying curve of $g^{-1}(x)_{\mathrm{red}}$ and $V_X$ for the set of closed points

$$\underline{g^{-1}(x)_{\mathrm{red}}} \cap \overline{X_s^{\mathrm{sst}} \setminus \underline{g^{-1}(x)_{\mathrm{red}}}},$$

where $\overline{X_s^{\mathrm{sst}} \setminus \underline{g^{-1}(x)_{\mathrm{red}}}}$ denotes the closure of $X_s^{\mathrm{sst}} \setminus \underline{g^{-1}(x)_{\mathrm{red}}}$ in $X_s^{\mathrm{sst}}$. Let

$$\mathscr{E}_X := (\underline{g^{-1}(x)_{\mathrm{red}}}, (D_{X_s^{\mathrm{sst}}} \cap \underline{g^{-1}(x)_{\mathrm{red}}}) \cup V_X)$$

and $\Gamma_{\mathscr{E}_X}$ the dual semi-graph of $\mathscr{E}_X$. Note that $\Gamma_{\mathscr{E}_X}$ is a tree. Then Theorem 2.8 and Proposition 3.9 imply the following theorem.

**Theorem 4.6.** *We maintain the notations introduced above. Then we have*

$$\sigma(f^{-1}(x)) = \sum_{v\in v(\Gamma_{\mathscr{E}_X})} (-\sharp G/\sharp I_v + \sum_{e\in e(v)} \sharp G/\sharp I_e(\sharp I_e/\sharp I_v - 1) + 1)$$

$$+ \sum_{e\in e^{\mathrm{cl}}(\Gamma_{\mathscr{E}_X})} (\sharp G/\sharp I_e - 1).$$

In the remainder of this section, we suppose that $x$ is a node of $\mathscr{X}_s$. Write $X_1'$ and $X_2'$ (which may be equal) for the irreducible components of $\mathscr{X}_s$ which contain $x$. Write $X_1$ and $X_2$ for the strict transforms of $X_1'$ and $X_2'$ under the birational morphism $g : \mathscr{X}^{\mathrm{sst}} \longrightarrow \mathscr{X}$, respectively. By the general theory of semi-stable curves, $g^{-1}(x)_{\mathrm{red}} \subseteq \mathscr{X}_s^{\mathrm{sst}}$ is a semi-stable curve over $s$ whose irreducible components are isomorphic to $\mathbb{P}_k^1$. Write $C$ for the semi-stable subcurve of $g^{-1}(x)_{\mathrm{red}}$ which is a chain of projective lines $\cup_{i=1}^n P_i$ such that the following conditions hold: (i) for any $s, t = 1, \dots, n, P_s \cap P_t = \emptyset$ if $|s-t| \geq 2$ and $P_s \cap P_t$ is reduced to a point if $|s-t| = 1$; (ii) $P_1 \cap X_1$ (resp. $P_n \cap X_2$) is reduced to a point; (iii) $C \cap \overline{\{X^{\mathrm{sst}} \setminus C\}} = (P_1 \cap X_1) \cup (P_n \cap X_2)$, where $\overline{\{X^{\mathrm{sst}} \setminus C\}}$ denotes the closure of $X^{\mathrm{sst}} \setminus C$ in $X^{\mathrm{sst}}$. Then we have

$$g^{-1}(x)_{\mathrm{red}} = C \cup B,$$

where $B$ denotes the topological closure of $g^{-1}(x)_{\mathrm{red}} \backslash C$ in $g^{-1}(x)_{\mathrm{red}}$. Write $B_i, i = 1, \dots, n$, for the set of the connected components of $B$ which intersect with $V_i$ are not empty.

Let $\{V_i\}_{i=0}^{n+1}$ be a set of irreducible components of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ such that the following conditions hold: (i) $h(V_i) = P_i$ for $i = 1, \dots, n$; (ii) $h(V_0) = X_1$ and $h(V_{n+1}) = X_2$; (iii) the union $\cup_{i=0}^{n+1} V_i \subseteq Y_s$ is a connected semi-stable curve over $s$. Write $I_{V_i} \subseteq G, i = 0, \dots, n+1$ for the inertia group of $V_i$.

**Lemma 4.7.** *We have $G = \langle I_{V_1}, I_{V_{n+1}} \rangle$, where $\langle I_{V_1}, I_{V_{n+1}} \rangle$ denotes the subgroup of $G$ generated by $I_{V_1}$ and $I_{V_{n+1}}$.*

*Proof.* If $G \neq \langle I_{V_1}, I_{V_{n+1}} \rangle$, since $G$ is a $p$-group, then there exists a normal subgroup $H \subseteq G$ of index $p$ such that $\langle I_{V_1}, I_{V_{n+1}} \rangle \subseteq H$. Write $\mathscr{Y}'$ for the normalization of $\mathscr{X}$ in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by $f$. The normalization $\mathscr{Y}'$ admits an action of $G$ induced by the action of $G$ on $\mathscr{Y}$. Consider the quotient $\mathscr{Y}'/H$. Then we obtain a morphism of fiber surfaces $f_H : \mathscr{Y}'/H \longrightarrow X$ over $S$ induced by $f$. Moreover, $\mathscr{Y}'/H$ admits an action of $G/H \cong \mathbb{Z}/p\mathbb{Z}$ induced by the action of $G$ on $\mathscr{Y}'$. Then $f_H$ is generically étale above $X_1'$ and $X_2'$. Thus, [T, Lemma 2.1 (iii)] implies that $f_H$ is étale above $x$. Then $f^{-1}(x)$ is not connected. This is a contradiction. We complete the proof of the lemma. □

Let $(u, w) \in \{0, \dots, n+1\} \times \{0, \dots, n+1\}$ be a pair such that $u \leq w$. We shall call a group $I_{u,w}^{\min}$ a minimal element associated to $\{I_{V_i}\}_{i=0}^{n+1}$ if one of the following conditions hold: (i) $(u, w) = (0, n+1)$ and for any $I_{V_i}, i = 0, \dots, n+1$, $I_{0,n+1}^{\min} = I_{V_i}$; (ii) $(u, w) = (0, w) \neq (0, n+1)$, $I_{0,w}^{\min} = I_{V_0} = I_{V_1} = \cdots = I_{V_w} \subset I_{V_{w+1}}$; (iii) $(u, w) = (u, n+1) \neq (0, n+1)$, $I_{V_{u-1}} \supset I_{V_u} = I_{V_{u+1}} \cdots = I_{V_{n+1}} = I_{u,n+1}^{\min}$; (iv) $u \neq 0$, $w \neq n+1$, and $I_{V_{u-1}} \supset I_{u,w}^{\min} = I_{V_u} = I_{V_{u+1}} \cdots = I_{V_w} \subset I_{V_{w+1}}$. We shall call a group $J_{u,w}^{\max}$ a maximal element associated to $\{I_{V_i}\}_{i=0}^{n+1}$ if one of the following conditions holds: (i) $(u, w) = (0, n+1)$ and for any

$I_{V_i}, i=0,\ldots,n+1$, $J^{\max}_{0,n+1}=I_{V_i}$; (ii) $(u,w)=(0,w)\neq(0,n+1)$, $J^{\max}_{0,w}=I_{V_0}=I_{V_1}=\cdots=I_{V_w}\supset I_{V_{w+1}}$; (iii) $(u,w)=(u,n+1)\neq(0,n+1)$, $I_{V_{u-1}}\subset I_{V_u}=I_{V_{u+1}}\cdots=I_{V_{n+1}}=J^{\max}_{u,n+1}$; (iv) $u\neq 0$, $w\neq n+1$, and $I_{V_{u-1}}\subset J^{\max}_{u,w}=I_{V_u}=I_{V_{u+1}}\cdots=I_{V_w}\supset I_{V_{w+1}}$. We define Min to be the set

$$\{I^{\min}_{u,w}\}_{(u,w)\in\{1,\ldots,n\}\times\{1,\ldots,n+1\}} \text{ or } \{I^{\min}_{0,n+1}\},$$

and Max to be the set

$$\{I^{\max}_{u,w}\}_{(u,w)\in\{0,\ldots,n+1\}\times\{0,\ldots,n+1\}}.$$

Note that Min may be an empty set. Then Corollary 2.10, Proposition 3.11, Lemma 4.1, Corollary 4.4, and Lemma 4.7 imply the following theorem.

**Theorem 4.8.** *We maintain the notations introduced above. Then we have*

$$\sigma(f^{-1}(x))=\sum_{i=1}^{n}\sharp G/\sharp I_{V_i}-\sum_{i=1}^{n+1}\sharp G/\sharp\langle I_{V_{i-1}},I_{V_i}\rangle+1$$

$$=\sum_{i=1}^{n}\sharp G/\sharp I_{V_i}-\sum_{i=1}^{n+1}\sharp G/\sharp I_{i-1,i}+1,$$

*where for each $i=1,\ldots,n+1$, $\langle I_{V_{i-1}},I_{V_i}\rangle$ denotes the subgroup of $G$ generated by $I_{V_{i-1}}$ and $I_{V_i}$, and $\sharp I_{i-1,i}$ denotes $\max\{\sharp I_{V_{i-1}},\sharp I_{V_i}\}$. Note that $\sharp I_{V_i}, i=0,\ldots,n+1$, does not depend on the choices of $V_i$. Moreover, we have*

$$\sigma(f^{-1}(x))=\sum_{I\in\mathrm{Min}}\sharp G/\sharp I-\sum_{J\in\mathrm{Max}}\sharp G/\sharp J+1,\ \textit{if}\ \mathrm{Min}\neq\{I^{\min}_{0,n+1}\},$$

*and*

$$\sigma(f^{-1}(x))=0\ \textit{if}\ \mathrm{Min}=\{I^{\min}_{0,n+1}\}.$$

**Remark 4.8.1.** The formulas

$$\sigma(f^{-1}(x))=\sum_{I\in\mathrm{Min}}\sharp G/\sharp I-\sum_{J\in\mathrm{Max}}\sharp G/\sharp J+1,\ \text{if}\ \mathrm{Min}\neq\{I^{\min}_{0,n+1}\},$$

and

$$\sigma(f^{-1}(x))=0\ \text{if}\ \mathrm{Min}=\{I^{\min}_{0,n+1}\}$$

are the key in the calculation of bounds of vertical fibers (cf. [Y3]).

If $G$ is an abelian $p$-group, then $I_{V_i}, i=0,\ldots,n+1$, does not depend on the choices of $V_i$. Then if $G$ is abelian, we use the notation $I_{P_i}, i=0,\ldots,n+1$, to denote $I_{V_i}$.

**Lemma 4.9.** *We maintain the notations introduced above. If $G$ is a cyclic $p$-group, then there exists $0\leq u\leq n+1$ such that*

$$I_{P_0}\supseteq I_{P_1}\supseteq I_{P_2}\supseteq\cdots\supseteq I_{P_u}\subseteq\cdots\subseteq I_{P_{n-1}}\subseteq I_{P_n}\subseteq I_{P_{n+1}}.$$

*Proof.* If the lemma is not true, there exist $s, t$ and $v$ such that $I_{P_v} \neq I_{P_s}$, $I_{P_v} \neq I_{P_t}$ and $I_{P_s} \subset I_{P_{s+1}} = \cdots = I_{P_v} = \cdots = I_{P_{t-1}} \supset I_{P_t}$. Since $G$ is a cyclic group, we may assume $I_{P_s} \supseteq I_{P_t}$.

Considering the quotient of $\mathscr{Y}$ by $I_{P_s}$, we obtain a natural morphism of pointed semi-stable curves $h_s : \mathscr{Y}/I_{P_s} \longrightarrow \mathscr{X}^{\mathrm{sst}}$ over $S$. By contacting $P_{s+1}, P_{s+2}, \ldots, P_{t-1}, B_{s+1}, \ldots, B_{t-1}$ (resp. $(h_s)^{-1}(P_{s+1})_{\mathrm{red}}, (h_s)^{-1}(P_{s+2})_{\mathrm{red}}, \ldots, (h_s)^{-1}(P_{t-1})_{\mathrm{red}}, (h_s)^{-1}(B_{s+1})_{\mathrm{red}}, \ldots, (h_s)^{-1}(B_{t-1})_{\mathrm{red}}$), we obtain a pointed semi-stable curve $(\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ (resp. a fiber surface $(\mathscr{Y}/I_{P_s})^{\mathrm{c}}$) and a contacting morphism $c_{\mathscr{X}^{\mathrm{sst}}} : \mathscr{X}^{\mathrm{sst}} \longrightarrow (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ (resp. $c_{\mathscr{Y}/I_{P_s}} : \mathscr{Y}/I_{P_s} \longrightarrow (\mathscr{Y}/I_{P_s})$). The morphism $h_s$ induces a morphism of fiber surfaces $h_s^{\mathrm{c}} : (\mathscr{Y}/I_{P_s})^{\mathrm{c}} \longrightarrow (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$. Then we have the following commutative diagram as follows:

$$\begin{array}{ccc} \mathscr{Y}/I_{P_s} & \xrightarrow{c_{\mathscr{Y}/I_{P_s}}} & (\mathscr{Y}/I_{P_s})^{\mathrm{c}} \\ {\scriptstyle h_s}\downarrow & & \downarrow{\scriptstyle h_s^{\mathrm{c}}} \\ \mathscr{X}^{\mathrm{sst}} & \xrightarrow{c_{\mathscr{X}^{\mathrm{sst}}}} & (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}. \end{array}$$

Write $P_s^{\mathrm{c}}$ and $P_t^{\mathrm{c}}$ for the images $c_{\mathscr{X}^{\mathrm{sst}}}(P_s)$ and $c_{\mathscr{X}^{\mathrm{sst}}}(P_t)$, respectively, and $x_{st}^{\mathrm{c}}$ for the closed point $P_s^{\mathrm{c}} \cap P_t^{\mathrm{c}}$. Since $h_s^{\mathrm{c}}$ is generically étale above $P_s^{\mathrm{c}}$ and $P_t^{\mathrm{c}}$, [T, Lemma 2.1 (iii)] implies that $(h_s^{\mathrm{c}})^{-1}(x_{st}^{\mathrm{c}})_{\mathrm{red}}$ are nodes. Thus, $(\mathscr{Y}/I_{P_s})^{\mathrm{c}}$ is a semi-stable curve over $S$, moreover, we have $h_s^{\mathrm{c}}$ is étale over $x_{st}^{\mathrm{c}}$. Then the inertia groups of the closed points $(h_s^{\mathrm{c}})^{-1}(x_{st}^{\mathrm{c}})_{\mathrm{red}}$ of the special fiber $(\mathscr{Y}/I_{P_s})_s^{\mathrm{c}}$ of $(\mathscr{Y}/I_{P_s})^{\mathrm{c}}$ are trivial.

On the other hand, since $I_{P_s}$ is a proper subgroup of $I_{P_v}$, we obtain the natural action of $G/I_{P_s}$ on the irreducible components of $h_s^{-1}(\cup_{j=s+1}^{t-1} P_j)_{\mathrm{red}}$ is trivial. Thus, the inertia groups of the closed points $c_{\mathscr{Y}/I_{P_s}}(h_s^{-1}(\cup_{j=s+1}^{t-1} P_j)_{\mathrm{red}}) = (h_s^{\mathrm{c}})^{-1}(x_{st}^{\mathrm{c}})_{\mathrm{red}}$ of the special fiber $(\mathscr{Y}/I_{P_s})_s^{\mathrm{c}}$ of $(\mathscr{Y}/I_{P_s})^{\mathrm{c}}$ are not trivial. This is a contradiction. Then we complete the proof of the lemma. □

Then Theorem 4.8 and Lemma 4.9 imply the following corollary.

**Corollary 4.10.** *Suppose that $G$ is a cyclic $p$-group, and $I_{P_0}$ is equal to $G$. Then we have*

$$\sigma(f^{-1}(x)) = \sharp G/\sharp I_{\min} - \sharp G/\sharp I_{P_{n+1}},$$

*where $I_{\min}$ denotes the group $\cap_{i=0}^{n+1} I_{P_i}$.*

**Remark 4.10.1.** The formula in Corollary 4.10 had been obtained by M. Saïdi (cf. [S, Proposition 1]).

## References


[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models,* Ergeb. Math. Grenz., 21. Springer, New York-Heidelberg-Berlin. 1990.

[C] R. Crew, *Étale p-covers in characteristic p,* Compositio Math. 52 (1984), 31-45.

[L] Q. Liu, *Algebraic geometry and arithmetic curves,* Oxford Graduate Texts in Mathematics, 6. Oxford University Press, Oxford, 2002.

[M] S. Mochizuki, *Semi-graphs of anabelioids,* Publ. Res. Inst. Math. Sci. 42 (2006), 221-322.

[R] M. Raynaud, *p-groupes et réduction semi-stable des courbes,* The Grothendieck Festschrift, Vol. III, 179197, Progr. Math., 88, Birkhäuser Boston, Boston, MA, 1990.

[S] M. Saïdi, *p-rank and semi-stable reduction of curves,* C. R. Acad. Sci. Paris, t. 326, Série I, 63-68, 1998.

[T] A. Tamagawa, *Resolution of nonsingularities of families of curves,* Publ. Res. Inst. Math. Sci. 40 (2004), 1291-1336.

[V] I. Vidal, *Contributions à la cohomologie étale des schémas et des log-schémas,* Thèse, U. Paris-Sud (2001).

[Y1] Y. Yang, *Degeneration of period matrices of stable curves,* RIMS Preprint 1835. http://www.kurims.kyoto-u.ac.jp/~yuyang/papersandpreprints/DPM.pdf

[Y2] Y. Yang, *On the existence of vertical fibers of coverings of curves over a complete discrete valuation ring,* RIMS Preprint 1843. http://www.kurims.kyoto-u.ac.jp/~yuyang/papersandpreprints/WRN-I.pdf

[Y3] Y. Yang, *On the boundedness and graph-theoreticity of p-ranks of coverings of curves.* http://www.kurims.kyoto-u.ac.jp/~yuyang/papersandpreprints/BG.pdf



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