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On the Asymptotic Expansions of the Kashaev Invariant of the Knots with 6 Crossings

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Abstract

We give presentations of the asymptotic expansions of the Kashaev invariant of the knots with 6 crossings. In particular, we show the volume conjecture for these knots, which states that the leading terms of the expansions present the hyperbolic volume and the Chern-Simons invariant of the complements of the knots. As higher coefficients of the expansions, we obtain a new series of invariants of these knots.

A non-trivial part of the proof is to apply the saddle point method to calculate the asymptotic expansion of an integral which presents the Kashaev invariant. A key step of this part is to give a concrete homotopy of the (real 3-dimensional) domain of the integral in $\mathbb{C}^3$ in such a way that the boundary of the domain always stays in a certain domain in $\mathbb{C}^3$ given by the potential function of the hyperbolic structure.

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1 Introduction

In [12, 13] Kashaev defined the Kashaev invariant $\langle L \rangle_N \in \mathbb{C}$ of a link L for $N = 2, 3, \cdots$ by using the quantum dilogarithm. In [14] he conjectured that, for any hyperbolic link L, $\frac{2\pi}{N} \log \langle L \rangle_N$ goes to the hyperbolic volume of $S^3 - L$ as $N \to \infty$, and verified the conjecture for some simple knots, by formal calculations. In 1999, H. Murakami and J. Murakami [18] proved that the Kashaev invariant $\langle L \rangle_N$ of any link L is equal to the N-colored Jones polynomial $J_N(L; e^{2\pi\sqrt{-1}/N})$ of L evaluated at $e^{2\pi\sqrt{-1}/N}$, where $J_N(L;q)$ denotes the invariant obtained as the quantum invariant of links associated with the N-dimensional irreducible representation of the quantum group $U_q(sl_2)$. Further, as an extension of Kashaev's conjecture, they conjectured that, for any knot K, $\frac{2\pi}{N} \log |J_N(K; e^{2\pi\sqrt{-1}/N})|$ goes to the (normalized) simplicial volume of $S^3 - K$. This is called *the volume conjecture*. As a complexification of the volume conjecture, it is conjectured in [19] that, for a hyperbolic link L,

$$J_N(L; e^{2\pi\sqrt{-1}/N}) \underset{N\to\infty}{\sim} e^{N\varsigma(L)},$$

where we put

$$\varsigma(L) = \frac{1}{2\pi\sqrt{-1}} \big(\mathrm{cs}(S^3 - L) + \sqrt{-1}\,\mathrm{vol}(S^3 - L) \big),$$

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and "cs" and "vol" denote the Chern-Simons invariant and the hyperbolic volume. Furthermore, it is conjectured [9] (see also [3, 10, 35]) from the viewpoint of the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory that the asymptotic expansion of $J_N(K; e^{2\pi\sqrt{-1}/k})$ of a hyperbolic knot K as $N, k \to \infty$ fixing $u = N/k$ is presented by the following form,

$$J_N(K; e^{2\pi\sqrt{-1}/k}) \underset{\substack{N,k\to\infty \\ u=N/k:\ \text{fixed}}}{\sim} e^{N\varsigma} N^{3/2}\, \omega \cdot \Big(1 + \sum_{i=1}^{\infty} \kappa_i \cdot \big(\frac{2\pi\sqrt{-1}}{N}\big)^i\Big) \tag{1}$$

for some scalars $\varsigma, \omega, \kappa_i$ depending on K and u, though they do not discuss the Jones polynomial in the Chern-Simons theory in the case of vanishing quantum dimension, which is discussed in [29]. Further, the first author showed in [20] that, when K is the 5_2 knot, the asymptotic expansions of the Kashaev invariant is presented by the following form,

$$\langle\, K \,\rangle_N \;=\; e^{N\varsigma(K)} N^{3/2}\, \omega(K) \cdot \Big(1 + \sum_{i=1}^{d} \kappa_i(K) \cdot \big(\frac{2\pi\sqrt{-1}}{N}\big)^i \; + O\big(\frac{1}{N^{d+1}}\big)\Big), \tag{2}$$

for any d, where $\omega(K)$ and $\kappa_i(K)$'s are some scalars.

The volume conjecture has been rigorously proved for some particular knots and links such as torus knots [15] (see also [4][1]), the figure-eight knot (by Ekholm, see also [1][2]), Whitehead doubles of $(2,p)$-torus knots [36], positive iterated torus knots [27], the 5_2 knot [16, 20], and some links [8, 11, 26, 27, 28, 36]; for details see *e.g.* [17].

The aim of this paper is to extend the formula (2) to the knots with 6 crossings, that is, we show the following theorem. In particular, this means that the volume conjecture holds for these knots.

Theorem 1.1. *The asymptotic expansions of the Kashaev invariant $\langle\, K \,\rangle_N$ of the knots K with 6 crossings are presented by the form* (2) *for any d, where $\omega(K)$ and $\kappa_i(K)$'s are some constants depending on K.*

The knots with 6 crossings are the 6_1, 6_2 and 6_3 knots, and all of them are hyperbolic. Their $\varsigma(K)$ are presented by

$$\begin{aligned} \varsigma(6_1) &= \overline{\varsigma(6_1)} \;=\; 0.5035603... + \sqrt{-1}\cdot 1.08078... \,, \\ \varsigma(6_2) &= 0.700414... - \sqrt{-1}\cdot 0.934648... \,, \\ \varsigma(6_3) &= 0.906072... \,. \end{aligned}$$

We note that their $\omega(K)$ and $\kappa_i(K)$'s are also their invariants; in particular, we will show that their $\omega(K)$ are presented by

$$\begin{aligned} \omega(6_1) &= \overline{\omega(6_1)} \;=\; -0.52139... - \sqrt{-1}\cdot 0.071732... \,, \\ \omega(6_2) &= -0.42920... + \sqrt{-1}\cdot 0.20337... \,, \end{aligned}$$

[1]A detailed asymptotic expansion of the colored Jones polynomial for torus knots is given in [4].

[2]A detailed proof of the volume conjecture for the figure-eight knot was given in [1] and the term $N^{3/2}$ in (1) and (2) was also verified there.

$$\omega(6_3) \;=\; 0.416927... \;.$$

We note that the values of $\varsigma(6_3)$ and $\omega(6_3)$ are real, since the 6_3 knot is amphicheiral. It is shown [22] that $2\sqrt{-1}\,\omega^2(K)$ for these knots is equal to the twisted Reidemeister torsion associated with the action on $\mathfrak{sl}_2$ of the holonomy representation of the hyperbolic structure. We also remark that Dimofte and Garoufalidis [2] define a formal power series from an ideal tetrahedral decomposition of a knot complement, which is expected to be equal to the asymptotic expansion of the Kashaev invariant of the knot.

We show proofs of the theorem for the 6_1, 6_2, 6_3 knots in Sections 3.1, 4.1, 5.1 respectively. An outline of the proofs is as follows. From the definition of the Kashaev invariant, the Kashaev invariant of K is presented by a sum. We rewrite the sum by an integral by the Poisson summation formula (Proposition 2.3). When we apply the Poisson summation formula, the right-hand side of the Poisson summation formula consists of infinitely many summands, and we show that we can ignore them except for the one at 0 in the sense that they are of sufficiently small order at $N \to \infty$. Further, by the saddle point method (Proposition 2.6), we calculate the asymptotic expansion of the integral, and obtain the presentation of the theorem.

A non-trivial part of the proof is to apply the saddle point method, whose concrete procedure is quite different from the proof for the 5_2 knot in [20]. In this part, we need to calculate the asymptotic behavior of an integral of the following form as $N \to \infty$,

$$\int_{\Delta'} \exp\Big(N\big(V(t,s,u) - \varsigma\big)\Big)\, dt\, ds\, du,$$

for a certain function $V(t,s,u)$ which depends on N, where $V(t,s,u)$ converges to $\hat{V}(t,s,u)$ as $N \to \infty$, and ς is a critical value of $\hat{V}(t,s,u)$. Here, $\hat{V}(t,s,u)$ is the potential function of the hyperbolic structure of the knot complement. As we mention in Remark 2.7, the above calculation is reduced to the calculation of the following case,

$$\int_{\Delta'} \exp\Big(N\big(\hat{V}(t,s,u) - \varsigma\big)\Big)\, dt\, ds\, du.$$

The domain Δ' of the integral is a compact domain in $\mathbb{R}^3$, and its boundary is included in the following domain

$$\big\{(t,s,u) \in \mathbb{C}^3 \;\big|\; \mathrm{Re}\,\big(\hat{V}(t,s,u) - \varsigma\big) < 0\big\}. \tag{3}$$

The critical value ς is given by a critical point (t_0, s_0, u_0), and it is located near Δ' in $\mathbb{C}^3$. In order to apply the saddle point method, we need to show that we can move Δ' into imaginary direction by a homotopy in such a way that the new domain Δ'_1 contains (t_0, s_0, u_0), and $\Delta'_1 - \{(t_0, s_0, u_0)\}$ is included in (3), and the boundary of Δ' always stays in (3) when we apply the homotopy. We note that, when we restrict the domain (3) to a sufficiently small neighborhood of (t_0, s_0, u_0), the resulting space is homotopy equivalent to a 2-sphere. The existence of the above homotopy means that the boundary of Δ' is homotopic to this 2-sphere in the domain (3). It is a non-trivial task to see that they are homotopic in the domain (3), since it is not easy to see the topological type of the domain

(3) directly. We give such a homotopy concretely in Sections 3.5, 4.5, 5.5 for the 6_1, 6_2, 6_3 knots respectively. This part of the proof is quite non-trivial; in fact, we note that we can not make such a homotopy for the 7_2 knot as shown in [21].

By the method of this paper, the asymptotic behavior of the Kashaev invariant is discussed for the hyperbolic knots with 7 crossings in [21] and for some hyperbolic knots with 8 crossings in [24].

The paper is organized as follows. In Section 2, we review definitions and basic properties of the notation used in this paper. In Sections 3, 4, 5, we show proofs of Theorem 1.1 for the 6_1, 6_2, 6_3 knots respectively. In Appendices A, B, C and D, we show proofs of some lemmas used in Sections 3, 4, 5.

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2 Preliminaries

In this section, we review definitions and basic properties of the notation used in this paper.

2.1 Integral presentation of $(q)_n$

In this section, we review the integral expression of q-factorials $(q)_n$ and their basic properties.

Let N be an integer ≥ 2. We put $q = \exp(2\pi\sqrt{-1}/N)$, and put

$$(x)_n = (1-x)(1-x^2)\cdots(1-x^n)$$

for $n \geq 0$. It is known [18] (see also [20]) that for any n, m with $n \leq m$,

$$(q)_n(\overline{q})_{N-n-1} = N, \tag{4}$$

$$\sum_{n\leq k\leq m} \frac{1}{(q)_{m-k}(\overline{q})_{k-n}} = 1. \tag{5}$$

Following Faddeev [5], we define a holomorphic function $\varphi(t)$ on $\{t \in \mathbb{C} \mid 0 < \operatorname{Re} t < 1\}$ by

$$\varphi(t) = \int_{-\infty}^{\infty} \frac{e^{(2t-1)x}dx}{4x \sinh x \sinh(x/N)},$$

noting that this integrand has poles at $n\pi\sqrt{-1}$ $(n \in \mathbb{Z})$, where, to avoid the pole at 0, we choose the following contour of the integral,

$$\gamma = (-\infty, -1] \cup \big\{z \in \mathbb{C} \bigm| |z| = 1, \ \operatorname{Im} z \geq 0\big\} \cup [1, \infty).$$

It is known [7, 31] that

$$\begin{aligned}(q)_n &= \exp\Big(\varphi\big(\frac{1}{2N}\big) - \varphi\big(\frac{2n+1}{2N}\big)\Big),\\ (\overline{q})_n &= \exp\Big(\varphi\big(1-\frac{2n+1}{2N}\big) - \varphi\big(1-\frac{1}{2N}\big)\Big).\end{aligned} \tag{6}$$

We put $\hbar = 2\pi\sqrt{-1}/N$, and put

$$\Phi_d(z) \;=\; \mathrm{Li}_2\big(z\big) + \sum_{1\le k\le d} \hbar^{2k} c_{2k} \cdot \Big(z\frac{d}{dz}\Big)^{2k-2} \frac{z}{1-z}\,,$$

where we define c_{2k} by

$$\frac{t/2}{\sinh(t/2)} \;=\; \sum_{k\ge 0} c_{2k}\, t^{2k}.$$

Then, it is known [7, 31] (see also [20]) that, for any $d \ge 0$,

$$\varphi(t) \;=\; \frac{N}{2\pi\sqrt{-1}}\,\Phi_d(e^{2\pi\sqrt{-1}\,t}) \;+ O(\hbar^{2d+1}), \tag{7}$$

$$\varphi^{(k)}(t) \;=\; \frac{N}{2\pi\sqrt{-1}}\Big(\frac{d}{dt}\Big)^k \Phi_d(e^{2\pi\sqrt{-1}\,t}) \;+ O(\hbar^{2d+1}), \tag{8}$$

for each $k > 0$. More precisely, as for the convergence of $\frac{1}{N}\varphi(t)$ as $N \to \infty$, we recall the following proposition.

Proposition 2.1 (See [20]). *We fix any sufficiently small $\delta > 0$ and any $M > 0$. Let d be any non-negative integer. Then, in the domain*

$$\big\{ t \in \mathbb{C} \;\big|\; \delta \le \mathrm{Re}\, t \le 1-\delta, \;\; |\mathrm{Im}\, t| \le M \big\}, \tag{9}$$

$\varphi(t)$ is presented by

$$\varphi(t) \;=\; \frac{N}{2\pi\sqrt{-1}}\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t}) \;+ O\big(\frac{1}{N}\big),$$

where $O(1/N)$ means the error term whose absolute value is bounded by C/N for some $C > 0$, which is independent of t (but possibly dependent on δ). In particular, $\frac{1}{N}\varphi(t)$ uniformly converges to $\frac{1}{2\pi\sqrt{-1}}\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t})$ in the domain (9).

As for properties of $\varphi(t)$, it is a consequence of (4) and (6) (see [20]) that, for any $t \in \mathbb{C}$ with $0 < \mathrm{Re}\ t < 1$,

$$\varphi(t) + \varphi(1-t) \;=\; 2\pi\sqrt{-1}\,\Big(-\frac{N}{2}\big(t^2 - t + \frac{1}{6}\big) + \frac{1}{24N}\Big). \tag{10}$$

Further, the following formulas are known (due to Kashaev, see [20]),

$$\begin{aligned} \varphi\big(\frac{1}{2N}\big) \;&=\; \frac{N}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6} + \frac{1}{2}\log N + \frac{\pi\sqrt{-1}}{4} - \frac{\pi\sqrt{-1}}{12N}\,, \\ \varphi\big(1-\frac{1}{2N}\big) \;&=\; \frac{N}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6} - \frac{1}{2}\log N + \frac{\pi\sqrt{-1}}{4} - \frac{\pi\sqrt{-1}}{12N}\,. \end{aligned} \tag{11}$$

2.2 Some behaviors of the dilogarithm function

In this section, we show some behaviors of the dilogarithm function.

We put

$$\Lambda(t) \;=\; \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t})\Big).$$

Since

$$\Lambda'(t) \;=\; -\log 2\sin\pi t, \qquad \Lambda''(t) \;=\; -\pi\cot\pi t,$$

the behavior of $\Lambda(t)$ is as follows.

t	0	$\cdots$	$\frac{1}{6}$	$\cdots$	$\frac{1}{2}$	$\cdots$	$\frac{5}{6}$	$\cdots$	1
$\Lambda(t)$	0	$\nearrow$	$\Lambda(\frac{1}{6})$	$\searrow$	0	$\searrow$	$-\Lambda(\frac{1}{6})$	$\nearrow$	0
$\Lambda'(t)$		$+$	0	$-$	$-$	$-$	0	$+$	
$\Lambda''(t)$		$-$	$-$	$-$	0	$+$	$+$	$+$	

Here, $\Lambda(\frac{1}{6}) = 0.161533...$. For the graph of $\Lambda(t)$, see Figure 1.

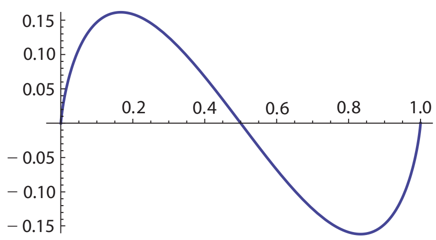


Figure 1: The graph of $\Lambda(t)$ for $0 \le t \le 1$

Further, the behavior of $\operatorname{Li}_2\big(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}\big)$ fixing t is presented by the following lemma.

Lemma 2.2. *Let t be a real number with $0 < t < 1$. Then, there exists $C > 0$ such that*

$$\begin{aligned}\left(\begin{cases}0 & \text{if } X \ge 0\\ 2\pi\big(t-\frac{1}{2}\big)X & \text{if } X < 0\end{cases}\right) - C \;&<\; \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2\big(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}\big)\Big)\\ &<\; \left(\begin{cases}0 & \text{if } X \ge 0\\ 2\pi\big(t-\frac{1}{2}\big)X & \text{if } X < 0\end{cases}\right) + C\end{aligned}$$

for any $X \in \mathbb{R}$.

Proof. Since

$$\lim_{X\to\infty}\operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2\big(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}\big)\Big) \;=\; \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2(1)\Big) \;=\; 0,$$

the "$X \ge 0$" part of the lemma is satisfied.

Further, by (7) and (10),

$$\mathrm{Li}_2\big(e^{2\pi\sqrt{-1}\,\tilde{t}}\big) + \mathrm{Li}_2\big(e^{-2\pi\sqrt{-1}\,\tilde{t}}\big) \;=\; 2\pi^2\big(\tilde{t}^2 - \tilde{t} + \frac{1}{6}\big).$$

Hence,

$$\mathrm{Re}\,\frac{1}{2\pi\sqrt{-1}}\Big(\mathrm{Li}_2\big(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}\big) + \mathrm{Li}_2\big(e^{-2\pi\sqrt{-1}\,(t+X\sqrt{-1})}\big)\Big) \;=\; 2\pi\big(t-\frac{1}{2}\big)X.$$

Therefore, we obtain the "$X<0$" part of the lemma from the "$X \ge 0$" part. □

2.3 Definition of the Kashaev invariant

In this section, we review the definition of the Kashaev invariant of oriented knots.

Following Yokota [33],[3] we review the definition of the Kashaev invariant. We put

$$\mathcal{N} \;=\; \{0, 1, \cdots, N-1\}.$$

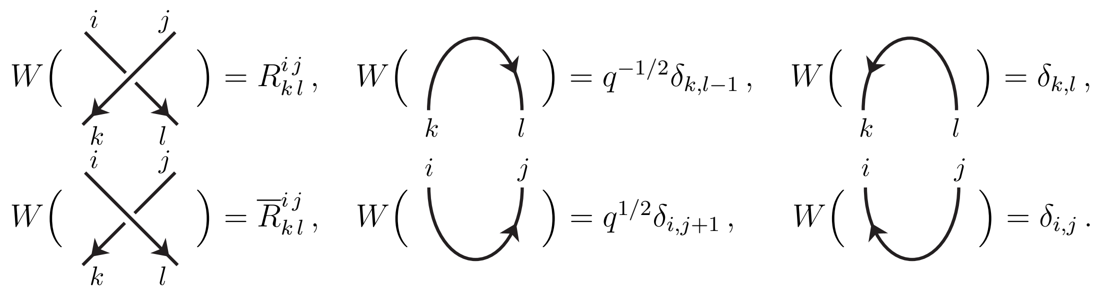
For $i, j, k, l \in \mathcal{N}$, we put

$$R^{i\,j}_{k\,l} \;=\; \frac{N\,q^{-\frac{1}{2}+i-k}\,\theta^{i\,j}_{k\,l}}{(q)_{[i-j]}(\overline{q})_{[j-l]}(q)_{[l-k-1]}(\overline{q})_{[k-i]}}\,, \qquad \overline{R}^{i\,j}_{k\,l} \;=\; \frac{N\,q^{\frac{1}{2}+j-l}\,\theta^{i\,j}_{k\,l}}{(\overline{q})_{[i-j]}(q)_{[j-l]}(\overline{q})_{[l-k-1]}(q)_{[k-i]}}\,,$$

where $[m] \in \mathcal{N}$ denotes the residue of m modulo N, and we put

$$\theta^{i\,j}_{k\,l} \;=\; \begin{cases} 1 & \text{if } \ [i-j]+[j-l]+[l-k-1]+[k-i] = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Let K be an oriented knot. We consider a 1-tangle whose closure is isotopic to K such that its string is oriented downward at its end points. Let D be a diagram of the 1-tangle. We present D by a union of elementary tangle diagrams shown in (12). We decompose the string of D into edges by cutting it at crossings and critical points with respect to the height function of $\mathbb{R}^2$. A *labeling* is an assignment of an element of $\mathcal{N}$ to each edge. Here, we assign 0 to the two edges adjacent to the end points of D. For example, see (28). We define the *weights* of labeled elementary tangle diagrams by



$$W(\text{crossing}) = R^{i\,j}_{k\,l}\,,\quad W(\text{cap}) = q^{-1/2}\delta_{k,l-1}\,,\quad W(\text{cap}) = \delta_{k,l}\,,$$
$$W(\text{crossing}) = \overline{R}^{i\,j}_{k\,l}\,,\quad W(\text{cup}) = q^{1/2}\delta_{i,j+1}\,,\quad W(\text{cup}) = \delta_{i,j}\,. \qquad (12)$$

[3]We make a minor modification of the definition of weights of critical points from the definition in [33], in order to make $\langle K \rangle_N$ invariant under Reidemeister moves.

Then, *the Kashaev invariant* $\langle K \rangle_N$ of K is defined by

$$\langle K \rangle_N = \sum_{\text{labelings}} \prod_{\substack{\text{crossings} \\ \text{of } D}} W(\text{crossings}) \prod_{\substack{\text{critical} \\ \text{points of } D}} W(\text{critical points}) \in \mathbb{C}.$$

2.4 The Poisson summation formula

In this section, we review the Poisson summation formula and a proposition obtained from it.

Recall (see *e.g.* [23]) that the Poisson summation formula states that

$$\sum_{\mathbf{m} \in \mathbb{Z}^n} f(\mathbf{m}) = \sum_{\mathbf{m} \in \mathbb{Z}^n} \hat{f}(\mathbf{m}) \tag{13}$$

for a continuous integrable function f on $\mathbb{R}^n$ which satisfies that

$$|f(\mathbf{z})| \leq C\big(1+|\mathbf{z}|\big)^{-n-\delta}, \qquad |\hat{f}(\mathbf{z})| \leq C\big(1+|\mathbf{z}|\big)^{-n-\delta} \tag{14}$$

for some $C, \delta > 0$, where $\hat{f}$ is the Fourier transform of f defined by

$$\hat{f}(\mathbf{w}) = \int_{\mathbb{R}^n} f(\mathbf{z})\, e^{-2\pi\sqrt{-1}\,\mathbf{w}^T\mathbf{z}} d\mathbf{z}.$$

The following proposition is obtained from the Poisson summation formula.

Proposition 2.3 (see [20]). *For $(c_1, c_2, c_3) \in \mathbb{C}^3$ and an oriented 3-ball D' in $\mathbb{R}^3$, we put*

$$\Lambda = \Big\{ \big(\frac{i}{N} + c_1,\, \frac{j}{N} + c_2,\, \frac{k}{N} + c_3\big) \in \mathbb{C}^3 \;\Big|\; i, j, k \in \mathbb{Z},\ \big(\frac{i}{N}, \frac{j}{N}, \frac{k}{N}\big) \in D' \Big\},$$
$$D = \big\{ (t + c_1,\, s + c_2,\, u + c_3) \in \mathbb{C}^3 \;\big|\; (t, s, u) \in D' \subset \mathbb{R}^3 \big\}.$$

Let $\psi(t, s, u)$ be a holomorphic function defined in a neighborhood of $\mathbf{0} \in \mathbb{C}^3$ including D. We assume that ∂D is included in the domain

$$\big\{ (t, s, u) \in \mathbb{C}^3 \;\big|\; \operatorname{Re}\psi(t, s, u) < -\varepsilon_0 \big\}$$

for some $\varepsilon_0 > 0$. Further, we assume that ∂D is null-homotopic in each of the following domains,

$$\big\{ (t + \delta\sqrt{-1},\, s,\, u) \in \mathbb{C}^3 \;\big|\; (t, s, u) \in D',\ \delta \geq 0,\ \operatorname{Re}\psi(t + \delta\sqrt{-1},\, s,\, u) < 2\pi\delta \big\}, \tag{15}$$
$$\big\{ (t - \delta\sqrt{-1},\, s,\, u) \in \mathbb{C}^3 \;\big|\; (t, s, u) \in D',\ \delta \geq 0,\ \operatorname{Re}\psi(t - \delta\sqrt{-1},\, s,\, u) < 2\pi\delta \big\}, \tag{16}$$
$$\big\{ (t,\, s + \delta\sqrt{-1},\, u) \in \mathbb{C}^3 \;\big|\; (t, s, u) \in D',\ \delta \geq 0,\ \operatorname{Re}\psi(t,\, s + \delta\sqrt{-1},\, u) < 2\pi\delta \big\}, \tag{17}$$
$$\big\{ (t,\, s - \delta\sqrt{-1},\, u) \in \mathbb{C}^3 \;\big|\; (t, s, u) \in D',\ \delta \geq 0,\ \operatorname{Re}\psi(t,\, s - \delta\sqrt{-1},\, u) < 2\pi\delta \big\}, \tag{18}$$
$$\big\{ (t,\, s,\, u + \delta\sqrt{-1}) \in \mathbb{C}^3 \;\big|\; (t, s, u) \in D',\ \delta \geq 0,\ \operatorname{Re}\psi(t,\, s,\, u + \delta\sqrt{-1}) < 2\pi\delta \big\}, \tag{19}$$
$$\big\{ (t,\, s,\, u - \delta\sqrt{-1}) \in \mathbb{C}^3 \;\big|\; (t, s, u) \in D',\ \delta \geq 0,\ \operatorname{Re}\psi(t,\, s,\, u - \delta\sqrt{-1}) < 2\pi\delta \big\}. \tag{20}$$

Then,

$$\frac{1}{N^3}\sum_{(t,s,u)\,\in\,\Lambda} e^{N\,\psi(t,s,u)} \;=\; \int_D e^{N\,\psi(t,s,u)} dt\,ds\,du \;+O(e^{-N\varepsilon}),$$

for some $\varepsilon > 0$.

Proof. We briefly review the proof; for details, see [20].

The sum of the left-hand side of the required formula is rewritten,

$$\sum_{i,j,k} \exp\Big(N\cdot\psi\big(\frac{i}{N}+c_1,\ \frac{j}{N}+c_2,\ \frac{k}{N}+c_3\big)\Big). \tag{21}$$

In order to apply the Poisson summation formula, we put

$$f(t,s,u) \;=\; g\big(\frac{t}{N}+c_1,\ \frac{s}{N}+c_2,\ \frac{u}{N}+c_3\big)\ \exp\Big(N\cdot\psi\big(\frac{t}{N}+c_1,\ \frac{s}{N}+c_2,\ \frac{u}{N}+c_3\big)\Big),$$

where g is a differentiable function on $\mathbb{R}^n+\mathbf{c}$ satisfying that

$$g(x,y,z) \;=\; \begin{cases} 1 & \text{if } (x,y,z)\in D,\\ 0 & \text{if } (x,y,z)\notin N(D),\end{cases}$$
$$0 \le g(x,y,z) \le 1 \quad \text{if } (x,y,z)\in N(D)-D.$$

Here, $N(D)$ is a neighborhood of D in $\mathbb{R}^3+(c_1,c_2,c_3)$ such that $N(D)-D$ is included in the domain $\big\{(t,s,u)\in\mathbb{C}^3 \mid \operatorname{Re}\psi(t,s,u) < -\varepsilon_0/2\big\}$. Then, the Fourier transform of f is given by

$$\begin{aligned}\hat f(m_1,m_2,m_3) \;&=\; \int_{\mathbb{R}^3} g\big(\frac{t}{N}+c_1,\ \frac{s}{N}+c_2,\ \frac{u}{N}+c_3\big)\\ &\qquad \times \exp\Big(N\cdot\psi\big(\frac{t}{N}+c_1,\ \frac{s}{N}+c_2,\ \frac{u}{N}+c_3\big)\Big)\, e^{-2\pi\sqrt{-1}\,(m_1t+m_2s+m_3u)} dt\,ds\,du\\ &=\; N^3\int_{\mathbb{R}^3+(c_1,c_2,c_3)} g(x,y,z)\, e^{N\big(\psi(x,y,z)-2\pi\sqrt{-1}\,(m_1(x-c_1)+m_2(y-c_2)+m_3(z-c_3))\big)} dx\,dy\,dz,\end{aligned}$$

where we put $x=t/N+c_1$, $y=s/N+c_2$ and $z=u/N+c_3$. Further,

$$\begin{aligned}&(\zeta_1^2+\zeta_2^2+\zeta_3^2)^2\,\hat f(\zeta_1,\zeta_2,\zeta_3)\\ &=\; N^3\big(\frac{-1}{4\pi^2N}\big)^2\int_{\mathbb{R}^3+(c_1,c_2,c_3)} g(x,y,z)\,e^{N\,\psi(x,y,z)}\\ &\qquad\qquad \times\Big(\big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\big)^2 e^{-2\pi\sqrt{-1}\,N\big(\zeta_1(x-c_1)+\zeta_2(y-c_2)+\zeta_3(z-c_3)\big)}\Big) dx\,dy\,dz\\ &=\; N^3\big(\frac{-1}{4\pi^2N}\big)^2\int_{\mathbb{R}^3+(c_1,c_2,c_3)} \Big(\big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\big)^2 g(x,y,z)\,e^{N\,\psi(x,y,z)}\Big)\\ &\qquad\qquad \times e^{-2\pi\sqrt{-1}\,N\big(\zeta_1(x-c_1)+\zeta_2(y-c_2)+\zeta_3(z-c_3)\big)} dx\,dy\,dz\\ &=\; N^3\big(\frac{-1}{4\pi^2N}\big)^2\int_{\mathbb{R}^3+(c_1,c_2,c_3)} h(x,y,z)\,e^{N\,\psi(x,y,z)}\,e^{-2\pi\sqrt{-1}\,N\big(\zeta_1(x-c_1)+\zeta_2(y-c_2)+\zeta_3(z-c_3)\big)} dx\,dy\,dz,\end{aligned}$$

where $h(x,y,z)$ is some polynomial in derivatives of $g(x,y,z)$ and $\psi(x,y,z)$. Here, we obtain the second equality by repeatedly using the fact that $\int_b^a F'(w)G(w)dw + \int_b^a F(w)G'(w)dw = \int_b^a \big(F(w)G(w)\big)' dw = 0$ if $F(w)G(w) = 0$ for $w \in \mathbb{R} - (a,b)$. Since the above integral is bounded independently of (x,y,z), $\hat{f}(x,y,z)$ satisfies the assumption (14) of the Poisson summation formula. Further, $f(t,s,u)$ also satisfies (14). Therefore, by the Poisson summation formula (13),

$$(21) \;=\; \sum_{m_1,m_2,m_3 \,\in \mathbb{Z}} \hat{f}(m_1,m_2,m_3).$$

When $(m_1,m_2,m_3) \neq (0,0,0)$, we have that

$$\begin{aligned}
\hat{f}(m_1,m_2,m_3) \;&=\; N^2\Big(\frac{-1}{4\pi^2 N}\Big)^2 \cdot \frac{1}{(m_1^2+m_2^2+m_3^2)^2} \\
&\qquad \times \int_{\mathbb{R}^3+(c_1,c_2,c_3)} h(x,y,z)\, e^{N\big(\psi(x,y,z) - 2\pi\sqrt{-1}\,(m_1(x-c_1)+m_2(y-c_2)+m_3(z-c_3))\big)}\, dx\,dy\,dz \\
&= N^2\Big(\frac{-1}{4\pi^2 N}\Big)^2 \cdot \frac{1}{(m_1^2+m_2^2+m_3^2)^2} \\
&\qquad \times \int_D \Psi(x,y,z)\, e^{N\big(\psi(x,y,z) - 2\pi\sqrt{-1}\,(m_1(x-c_1)+m_2(y-c_2)+m_3(z-c_3))\big)}\, dx\,dy\,dz \qquad (22) \\
&\quad + N^2\Big(\frac{-1}{4\pi^2 N}\Big)^2 \cdot \frac{1}{(m_1^2+m_2^2+m_3^2)^2} \\
&\qquad \times \int_{N(D)-D} h(x,y,z)\, e^{N\big(\psi(x,y,z) - 2\pi\sqrt{-1}\,(m_1(x-c_1)+m_2(y-c_2)+m_3(z-c_3))\big)}\, dx\,dy\,dz, \qquad (23)
\end{aligned}$$

where $\Psi(x,y,z)$ is some polynomial in (at most the 4th) derivatives of $\psi(x,y,z)$. Further, since $\operatorname{Re}\psi(x,y,z) < \varepsilon_0/2$ for $(x,y,z) \in N(D) - D$,

$$\sum_{(m_1,m_2,m_3)\neq(0,0,0)} (23) \;=\; O(e^{-N\varepsilon_1})$$

for some $\varepsilon_1 > 0$. Furthermore, when $m_1 > 0$, pushing the contour D into the domain (16), we obtain

$$\sum_{\substack{m_1,m_2,m_3 \,\in \mathbb{Z} \\ m_1>0}} (22) \;=\; O(e^{-N\varepsilon_2})$$

for some $\varepsilon_2 > 0$. Similarly we obtain

$$\sum_{\substack{m_1,m_2,m_3 \,\in \mathbb{Z} \\ m_1<0}} (22) \;=\; O(e^{-N\varepsilon_3})$$

for some $\varepsilon_3 > 0$, by pushing D into the domain (15). Hence,

$$\sum_{(m_1,m_2,m_3)\neq(0,0,0)} (22) \;=\; \sum_{\substack{(m_1,m_2,m_3)\neq(0,0,0) \\ m_1=0}} (22) \;+ O(e^{-N\varepsilon_4})$$

for some $\varepsilon_4 > 0$. By repeating this argument for m_2 and m_3, we obtain

$$\sum_{(m_1,m_2,m_3)\neq(0,0,0)} (22) \;=\; O(e^{-N\varepsilon_5})$$

for some $\varepsilon_5 > 0$. Therefore,

$$(21) \;=\; \hat{f}(0,0,0) + O(e^{-N\varepsilon_6}) \;=\; N^3 \int_D e^{N\,\psi(x,y,z)} dx\,dy\,dz \;+ O(e^{-N\varepsilon_6})$$

for some $\varepsilon_6 > 0$, and this implies the required formula. □

Remark 2.4. By modifying the proof of Proposition 2.3, we can show that, instead of the domains (15)–(20), we can use the domains (17)–(20) and

$$\begin{aligned} &\big\{(t+\delta\sqrt{-1},\, s-\delta\sqrt{-1},\, u) \in \mathbb{C}^3 \;\big|\; \\ &\qquad (t,s,u) \in \Delta',\; \delta \geq 0,\; \operatorname{Re}\psi(t+\delta\sqrt{-1},\, s-\delta\sqrt{-1},\, u) < 2\pi\delta\big\}, \end{aligned} \tag{24}$$

$$\begin{aligned} &\big\{(t-\delta\sqrt{-1},\, s+\delta\sqrt{-1},\, u) \in \mathbb{C}^3 \;\big|\; \\ &\qquad (t,s,u) \in \Delta',\; \delta \geq 0,\; \operatorname{Re}\psi(t-\delta\sqrt{-1},\, s+\delta\sqrt{-1},\, u) < 2\pi\delta\big\}. \end{aligned} \tag{25}$$

In this case, we can prove the proposition by considering the cases where $m_1 \neq m_2$, $m_2 \neq 0$ or $m_3 \neq 0$, instead of the cases where $m_1 \neq 0$, $m_2 \neq 0$ or $m_3 \neq 0$.

Remark 2.5. Similarly as in [20, Remark 4.8], Proposition 2.3 can naturally be extended to the case where the holomorphic function $\psi(t,s,u)$ depends on N, if $\psi(t,s,u)$ uniformly converges to $\psi_0(t,s,u)$ as $N \to \infty$, and $\psi_0(t,s,u)$ satisfies the assumption of the proposition, and $|\Psi(t,s,u)|$ is bounded by a constant which is independent of N. We note that we can choose ε of the proposition independently of N in this case.

2.5 The saddle point method

In this section, we review a proposition obtained from the saddle point method.

Proposition 2.6 (see [20])**.** *Let A be a non-singular symmetric complex 3×3 matrix, and let $\psi(z_1,z_2,z_3)$ and $r(z_1,z_2,z_3)$ be holomorphic functions of the forms,*

$$\begin{aligned} \psi(z_1,z_2,z_3) &= \mathbf{z}^T A\,\mathbf{z} + r(z_1,z_2,z_3), \\ r(z_1,z_2,z_3) &= \textstyle\sum_{i,j,k} b_{ijk} z_i z_j z_k + \sum_{i,j,k,l} c_{ijkl} z_i z_j z_k z_l + \cdots\,, \end{aligned} \tag{26}$$

defined in a neighborhood of $\mathbf{0} \in \mathbb{C}^3$. The restriction of the domain

$$\big\{(z_1,z_2,z_3) \in \mathbb{C}^3 \;\big|\; \operatorname{Re}\psi(z_1,z_2,z_3) < 0\big\} \tag{27}$$

to a neighborhood of $\mathbf{0} \in \mathbb{C}^3$ is homotopy equivalent to S^2. Let D be an oriented 3-ball embedded in $\mathbb{C}^3$ such that ∂D is included in the domain (27) whose inclusion is homotopic to a homotopy equivalence to the above S^2 in the domain (27). Then,

$$\int_D e^{N\,\psi(z_1,z_2,z_3)} dz_1\,dz_2\,dz_3 \;=\; \frac{\pi^{3/2}}{N^{3/2}\sqrt{\det(-A)}}\Big(1 + \sum_{i=1}^{d} \frac{\lambda_i}{N^i} \;+ O\big(\frac{1}{N^{d+1}}\big)\Big),$$

for any d, where we choose the sign of $\sqrt{\det(-A)}$ as explained in [20]*, and λ_i's are constants presented by using coefficients of the expansion of $\psi(z_1, z_2, z_3)$; such presentations are obtained by formally expanding the following formula,*

$$1 + \sum_{i=1}^{\infty} \frac{\lambda_i}{N^i} \;=\; \exp\Big(N\, r\big(\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, \frac{\partial}{\partial w_3}\big)\Big) \exp\Bigg(-\frac{1}{4N} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}^T A^{-1} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \Bigg) \Bigg|_{w_1=w_2=w_3=0}.$$

For a proof of the proposition, see [20].

Remark 2.7. As mentioned in [20, Remark 3.6], we can extend Proposition 2.6 to the case where $\psi(z_1, z_2, z_3)$ depends on N in such a way that $\psi(z_1, z_2, z_3)$ is of the form

$$\begin{aligned} \psi(z_1, z_2, z_3) \;=\; & \psi_0(z_1, z_2, z_3) + \psi_1(z_1, z_2, z_3)\frac{1}{N} + \psi_2(z_1, z_2, z_3)\frac{1}{N^2} \\ & + \cdots + \psi_m(z_1, z_2, z_3)\frac{1}{N^m} + r_m(z_1, z_2, z_3)\frac{1}{N^{m+1}}, \end{aligned}$$

where $\psi_i(z_1, z_2, z_3)$'s are holomorphic functions independent of N, and we assume that $\psi_0(z_1, z_2, z_3)$ satisfies the assumption of the proposition and $|r_m(z_1, z_2, z_3)|$ is bounded by a constant which is independent of N.

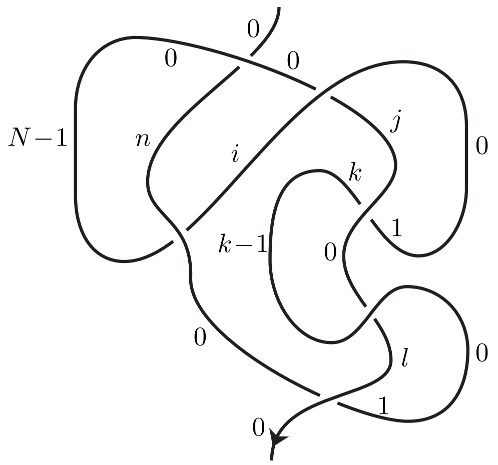
3 The 6_1 knot

In this section, we show Theorem 1.1 for the 6_1 knot. We give a proof of the theorem in Section 3.1, using lemmas shown in Section 3.2–3.5.

3.1 Proof of Theorem 1.1 for the 6_1 knot

In this section, we show a proof of Theorem 1.1 for the 6_1 knot.

Since the Kashaev invariant of the mirror image of a knot is equal to the complex conjugate of the Kashaev invariant of the original knot, it is sufficient to show the theorem for the mirror image $\overline{6_1}$ of the 6_1 knot. *The $\overline{6_1}$ knot* is the closure of the following tangle.



(28)

As shown in [33], we can put the labelings of edges adjacent to the unbounded regions as shown above. Hence, from the definition of the Kashaev invariant, the Kashaev invariant of the $\overline{6_1}$ knot is presented by

$$\begin{aligned}
\langle\, \overline{6_1}\,\rangle_N &= \sum \frac{N\,q^{\frac{1}{2}}}{(\overline{q})_{N-n-1}(q)_n} \times \frac{N\,q^{\frac{1}{2}+i}}{(\overline{q})_{n-i}(q)_i(q)_{N-n-1}} \times \frac{N\,q^{-\frac{1}{2}-i}}{(\overline{q})_{N-j}(q)_{j-i-1}(\overline{q})_i} \\
&\qquad \times \frac{N\,q^{-\frac{1}{2}+k}}{(q)_{k-j}(\overline{q})_{j-1}(\overline{q})_{N-k}} \times \frac{N\,q^{-\frac{1}{2}-k+1}}{(\overline{q})_{N-l}(q)_{l-k}(\overline{q})_{k-1}} \times \frac{N\,q^{-\frac{1}{2}}}{(q)_{N-l}(\overline{q})_{l-1}} \\
&= \sum_{0\le i<j\le k\le N} \frac{N^4\,q^{-1}}{(q)_i(\overline{q})_i(q)_{j-i-1}(\overline{q})_{j-1}(\overline{q})_{N-j}(q)_{k-j}(\overline{q})_{k-1}(\overline{q})_{N-k}} \\
&= \sum_{0\le i\le j\le k<N} \frac{N^4\,q^{-1}}{(q)_i(\overline{q})_i(q)_{j-i}(\overline{q})_j(\overline{q})_{N-j-1}(q)_{k-j}(\overline{q})_k(\overline{q})_{N-k-1}} \\
&= \sum_{\substack{0\le i,j,k \\ i+j+k<N}} \frac{N^4\,q^{-1}}{(q)_i(\overline{q})_i(q)_j(\overline{q})_{i+j}(\overline{q})_{N-i-j-1}(q)_k(\overline{q})_{i+j+k}(\overline{q})_{N-i-j-k-1}}\,, \qquad (29)
\end{aligned}$$

where the second equality is obtained by (4) and (5), the third equality is obtained by replacing j and k with $j+1$ and $k+1$ respectively, and the last equality is obtained by replacing j and k with $i+j$ and $i+j+k$ respectively.

Proof of Theorem 1.1 for the $\overline{6_1}$ knot. By (6), the above presentation of $\langle\, \overline{6_1}\,\rangle_N$ is rewritten

$$\langle\, \overline{6_1}\,\rangle_N = N^4 q^{-1} \sum_{\substack{0\le i,j,k \\ i+j+k<N}} \exp\Big(N\,\tilde{V}\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}\big)\Big),$$

where we put

$$\begin{aligned}
\tilde{V}(t,s,u) &= \frac{1}{N}\Big(\varphi(t) - \varphi(1-t) + \varphi(s) - \varphi\big(t+s-\frac{1}{2N}\big) - \varphi\big(1-t-s+\frac{1}{2N}\big) \\
&\qquad + \varphi(u) - \varphi\big(t+s+u-\frac{1}{N}\big) - \varphi(1-t-s-u+\frac{1}{N}\big) \\
&\qquad - 3\,\varphi\big(\frac{1}{2N}\big) + 5\,\varphi\big(1-\frac{1}{2N}\big)\Big) \\
&= \frac{1}{N}\Big(2\,\varphi(t) + \varphi(s) + \varphi(u)\Big) + \frac{1}{2\pi\sqrt{-1}}\cdot\frac{\pi^2}{3} - \frac{4}{N}\,\log N + \frac{\pi\sqrt{-1}}{2N} - \frac{\pi\sqrt{-1}}{6N^2} \\
&\quad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\Big(\big(t+s+u-\frac{1}{N}\big)^2 + \big(t+s-\frac{1}{2N}\big)^2 + t^2 - 3t - 2s - u + \frac{1}{2} + \frac{3}{2N} - \frac{1}{4N^2}\Big).
\end{aligned}$$

Here, we obtain the last equality by (10) and (11). Hence, by putting

$$V(t,s,u) = \tilde{V}(t,s,u) + \frac{4}{N}\log N$$

$$= \frac{1}{N}\Big(2\,\varphi(t)+\varphi(s)+\varphi(u)\Big) + \frac{1}{2\pi\sqrt{-1}}\cdot\frac{\pi^2}{3} + \frac{\pi\sqrt{-1}}{2N} - \frac{\pi\sqrt{-1}}{6N^2}$$
$$+\,2\pi\sqrt{-1}\cdot\frac{1}{2}\Big(\big(t+s+u-\frac{1}{N}\big)^2 + \big(t+s-\frac{1}{2N}\big)^2 + t^2 - 3t - 2s - u + \frac{1}{2} + \frac{3}{2N} - \frac{1}{4N^2}\Big),$$

the presentation of $\langle\, \overline{6_1}\,\rangle_{N}$ is rewritten

$$\langle\, \overline{6_1}\,\rangle_{N} \;=\; q^{-1}\sum_{\substack{0\,\leq\, i,j,k\\ i+j+k<N}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big)\Big)$$
$$=\; q^{-1}\sum_{\substack{i,j,k\,\in\,\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big)\Big),$$

where the range Δ of $(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})$ of the above sum is given by the following domain,

$$\Delta \;=\; \big\{(t,s,u)\in\mathbb{R}^3 \;\big|\; 0\leq t,s,u\leq 1,\quad t+s+u\leq 1+\frac{1}{N}\big\}.$$

By Proposition 2.1, as $N\to\infty$, $V(t,s,u)$ converges to the following $\hat{V}(t,s,u)$ in the interior of Δ,

$$\hat{V}(t,s,u) \;=\; \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,s}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,u}) + \frac{\pi^2}{3}\Big)$$
$$+\,2\pi\sqrt{-1}\cdot\frac{1}{2}\big((t+s+u)^2 + (t+s)^2 + t^2 - 3t - 2s - u + \frac{1}{2}\big).$$

By concrete computation, we can check that the boundary of Δ is included in the domain

$$\big\{(t,s,u)\in\Delta \;\big|\; \mathrm{Re}\,\hat{V}(t,s,u) < \varsigma_{R} - \varepsilon\big\} \tag{30}$$

for some sufficiently small $\varepsilon>0$, where we put $\varsigma_{R} = 0.5035603...$ as in (41); we will know later that this value is equal to the real part of the critical value of $\hat{V}$ at the critical point of Lemma 3.3. Since we will know later that the sum of the problem is of the order $O(e^{N\varsigma_R})$, we can ignore the sum of the problem restricted in the above domain, and hence, we can remove this domain from Δ; see Appendix D for concrete procedure of this argument. Therefore, we can choose a new domain Δ' in the interior of Δ such that $\Delta-\Delta'\subset(30)$; more concretely, we can choose Δ' as

$$\Delta' \;=\; \left\{(t,s,u)\in\Delta \;\middle|\; \begin{array}{l} 0.03\leq t\leq 0.4,\quad 0.001\leq s\leq 0.5,\\ 0.001\leq u\leq 0.5,\quad t+s+u\leq 0.94 \end{array}\right\}, \tag{31}$$

where we calculate the concrete values of the bounds of these inequalities in Section 3.2.

Hence, since $\Delta - \Delta' \subset$ (30), we obtain the second equality of the following formula,

$$\begin{aligned}
\langle\, \overline{6_1}\, \rangle_{N} \;&=\; e^{N\varsigma} q^{-1} \sum_{\substack{i,j,k\,\in\mathbb{Z} \\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big) - N\varsigma \Big) \\
&=\; e^{N\varsigma}\Bigg(q^{-1} \sum_{\substack{i,j,k\,\in\mathbb{Z} \\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta'}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big) - N\varsigma \Big) \; + O(e^{-N\varepsilon}) \Bigg), \qquad (32)
\end{aligned}$$

for some $\varepsilon > 0$. To be precise, in order to obtain the second equality, we need to estimate of the summand of the above sum in terms of $\hat{V}(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N})$; see Appendix D for the proof of the equality of (32).

Further, by Proposition 2.3 (Poisson summation formula, see also Remark 2.5), the above sum is presented by

$$\langle\, \overline{6_1}\, \rangle_{N} \;=\; e^{N\varsigma}\Bigg(N^3 q^{-1} \int_{\Delta'} \exp\big(N\cdot V(t,s,u) - N\varsigma \big)\, dt\, ds\, du \;\; + O(e^{-N\varepsilon}) \Bigg), \qquad (33)$$

noting that we verify the assumption of Proposition 2.3 in Lemma 3.4. Furthermore, by Proposition 2.6 (saddle point method, see also Remark 2.7), there exist some κ_i''s such that

$$\langle\, \overline{6_1}\, \rangle_{N} \;=\; N^3 q^{-1} \exp\big(N\cdot V(t_c,s_c,u_c) \big) \cdot \frac{(2\pi)^{3/2}}{N^{3/2}} \big(\det(-H)\big)^{-1/2} \Big(1 + \sum_{i=1}^{d} \kappa_i' \hbar^i + O(\hbar^{d+1}) \Big),$$

for any $d > 0$, noting that we verify the assumption of Proposition 2.6 in Lemma 3.9. Here, (t_c, s_c, u_c) is the critical point of V which corresponds to the critical point (t_0, s_0, u_0) of $\hat{V}$ of Lemma 3.3, and H is the Hesse matrix of V at (t_c, s_c, u_c).

We calculate the right-hand side of the above formula. Since $t_c = t_0 + O(\hbar)$, $s_c = s_0 + O(\hbar)$ and $u_c = u_0 + O(\hbar)$, we have that $V(t_c,s_c,u_c) = V(t_0,s_0,u_0) + O(\hbar^2)$. Further, by comparing $V(t_0,s_0,u_0)$ and $\hat{V}(t_0,s_0,u_0) = \varsigma$, we have that

$$V(t_0,s_0,u_0) \;=\; \varsigma + \frac{\pi\sqrt{-1}}{2N} - \frac{2\pi\sqrt{-1}}{N}\big(t_0+s_0+u_0-\frac{1}{2}\big) - \frac{2\pi\sqrt{-1}}{2N}\big(t_0+s_0-\frac{1}{2}\big) + O(\hbar^2).$$

Therefore, there exist some κ_i's such that

$$\langle\, \overline{6_1}\, \rangle_{N} \;=\; e^{N\varsigma} N^{3/2}\, \omega \cdot \Big(1 + \sum_{i=1}^{d} \kappa_i \hbar^i + O(\hbar^{d+1}) \Big),$$

for any $d > 0$. Here, we put

$$\begin{aligned}
\omega \;&=\; (2\pi)^{3/2}\sqrt{-1}\,\big(-x_0y_0z_0\big)^{-1}\big(-x_0y_0\big)^{-1/2}\big(\det(-H_0)\big)^{-1/2} \\
&=\; -0.52139... + \sqrt{-1}\cdot 0.071732...\;,
\end{aligned}$$

where we put $x_0 = e^{2\pi\sqrt{-1}\,t_0}$, $y_0 = e^{2\pi\sqrt{-1}\,s_0}$, $z_0 = e^{2\pi\sqrt{-1}\,u_0}$, and H_0 is the Hesse matrix of $\hat{V}$ at (t_0,s_0,u_0) whose concrete presentation is given in (42); see also [22] for a relation of this value and the twisted Reidemeister torsion. Hence, we obtain the theorem for the $\overline{6_1}$ knot. $\square$

3.2 Estimate of the range of Δ'

In this section, we calculate the concrete values of the bounds of the inequalities in (31) so that they satisfy that $\Delta - \Delta' \subset$ (30).

Putting Λ as in Section 2.2, we have that

$$\operatorname{Re} \hat{V}(t,s,u) \;=\; 2\,\Lambda(t) + \Lambda(s) + \Lambda(u).$$

We consider the domain

$$\big\{(t,s,u) \in \Delta \;\big|\; 2\,\Lambda(t) + \Lambda(s) + \Lambda(u) \;\geq\; \varsigma_{R} \big\}, \tag{34}$$

where we put $\varsigma_{R} = 0.5035603...$ as in (41). We note that this domain is symmetric with respect to the exchange of s and u. The aim of this section is to show that this domain is included in the interior of the domain Δ' of (31). For this purpose, we estimate ranges of t, s, u and $t{+}s{+}u$ in the domain (34).

We calculate the minimal value $t_{\min}$ and the maximal value $t_{\max}$ of t. Since $\Lambda(\,\cdot\,) \leq \Lambda(\frac{1}{6})$,

$$2\,\Lambda(t) \;\geq\; \varsigma_{R} - 2\,\Lambda\big(\frac{1}{6}\big) \;=\; 0.180494...\;.$$

The minimal and maximal values of t are solutions of the following equation,

$$2\,\Lambda(t) \;=\; \varsigma_{R} - 2\,\Lambda\big(\frac{1}{6}\big). \tag{35}$$

Noting that the behavior of $\Lambda(t)$ is as shown in Section 2.2, this equation has exactly two solutions in $0 < t < 0.5$. By calculating a solution of this equation by Newton's method from $t = 0.01$, we obtain $t_{\min} = 0.0364809...$, and from $t = 0.4$, we obtain $t_{\max} = 0.363674...$. Therefore, we obtain an estimate of t in Δ' as

$$0.03 \;\leq\; t \;\leq\; 0.4.$$

Remark 3.1. To be precise, the above argument is not partially rigorous, since we do not estimate the error terms of the numerical solutions of Newton's method, though the above argument is practically useful, since we can guess that such error terms would be sufficiently small for the above purpose. In order to complete the above argument, we give a rigorous proof of estimates of solutions of (35) in Lemma A.1.

We calculate the minimal value $s_{\min}$ and the maximal value $s_{\max}$ of s. Since $\Lambda(\,\cdot\,) \leq \Lambda(\frac{1}{6})$,

$$\Lambda(s) \;\geq\; \varsigma_{R} - 3\,\Lambda\big(\frac{1}{6}\big) \;=\; 0.0189614...\;.$$

The minimal and maximal values of s are solutions of the following equation,

$$\Lambda(s) \;=\; \varsigma_{R} - 3\,\Lambda\big(\frac{1}{6}\big).$$

Similarly as above, we note that this equation has exactly two solutions in $0 < s < 0.5$. By calculating a solution of the equation by Newton's method from $s = 0.001$, we obtain

$s_{\min} = 0.00406176...$, and from $s = 0.5$, we obtain $s_{\max} = 0.472596...$. Therefore, we obtain an estimate of s in Δ' as

$$0.001 \le s \le 0.5.$$

To be precise (see Remark 3.1), we can rigorously verify the above estimate of the solutions of the above equation in a similar way as in Section A.1.

We obtain the estimate of u in Δ' from the above formula by the symmetry with respect to the exchange of s and u.

Before calculating $t+s+u$, we show that the domain (34) is convex, as follows. As mentioned in Section 2.2, the function $\Lambda(t)$ is a concave function for $0 < t < 0.5$, whose second derivative is negative. Hence, the function $2\Lambda(t) + \Lambda(s) + \Lambda(u)$ is concave on $\{(t, s, u) \mid 0 < t, s, u < 0.5\}$, whose Hesse matrix is negative definite. Therefore, the domain (34) is convex. Further, we note that its boundary is a smooth closed surface in $\{(t, s, u) \mid 0 < t, s, u < 0.5\}$, whose Gaussian curvature is positive everywhere (see Lemma B.2).

We calculate the maximal value $(t+s+u)_{\max}$ of $t+s+u$. We consider the plane $t+s+u = c$ for a constant c. The range of $t+s+u$ is given as the range of c such that this plane and the domain (34) has non-empty intersection. Since the domain (34) is a compact convex domain whose boundary is a smooth closed surface, the maximal and minimal values are given by the planes tangent to this domain. Putting $w = t+s+u$, the tangent points of such planes are given by solutions of the following equations,

$$\begin{cases} 2\,\Lambda(w-s-u) + \Lambda(s) + \Lambda(u) \;=\; \varsigma_R\,, \\ \dfrac{\partial}{\partial s}\big(2\,\Lambda(w-s-u) + \Lambda(s) + \Lambda(u)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(w-s-u) + \Lambda(s) + \Lambda(u)\big) \;=\; 0. \end{cases}$$

These equations are rewritten

$$\begin{cases} 2\,\Lambda(w-s-u) + \Lambda(s) + \Lambda(u) \;=\; \varsigma_R\,, \\ -2\,\Lambda'(w-s-u) + \Lambda'(s) \;=\; 0, \\ -2\,\Lambda'(w-s-u) + \Lambda'(u) \;=\; 0. \end{cases}$$

Since the boundary of the domain (34) is a smooth closed surface whose Gaussian curvature is positive everywhere (see Lemma B.2), there are exactly two such tangent points, and the above system of equations has exactly two solutions, corresponding to the maximal and minimal values of $t+s+u$; we put the maximal value to be $w_{\max}$. Since the above system of equations is symmetric with respect to the exchange of s and u, the (unique) solution of the form $(w_{\max}, s, u)$ satisfies that $s = u$. Hence, we can rewrite the above system of equations as

$$\begin{cases} 2\,\Lambda(w-2s) + 2\,\Lambda(s) \;=\; \varsigma_R\,, \\ -2\,\Lambda'(w-2s) + \Lambda'(s) \;=\; 0. \end{cases} \tag{36}$$

By calculating a solution of these equations by Newton's method from $(w,s)=(1,0.3)$, we obtain $(t{+}s{+}u)_{\max}=0.925048...$. Therefore, we obtain an estimate of $t+s+u$ in Δ' as

$$t+s+u \;\le\; 0.94.$$

Remark 3.2. To be precise, the above argument is not partially rigorous, since we do not estimate the error term of the numerical solution of Newton's method, though the above argument is practically useful, since we can guess that such an error term would be sufficiently small for the above purpose. In order to complete the above argument, we give a rigorous proof of an estimate of a solution of (36) in Lemma A.2.

3.3 Calculation of the critical value

In this section, we calculate the concrete values of a critical point and the Hesse matrix of $\hat{V}$.

The differentials of $\hat{V}$ are presented by

$$\frac{\partial}{\partial t}\hat{V}(t,s,u) \;=\; -2\,\log(1-x)+2\pi\sqrt{-1}\,\big(3t+2s+u-\frac{3}{2}\big), \tag{37}$$

$$\frac{\partial}{\partial s}\hat{V}(t,s,u) \;=\; -\log(1-y)+2\pi\sqrt{-1}\,(2t+2s+u-1), \tag{38}$$

$$\frac{\partial}{\partial u}\hat{V}(t,s,u) \;=\; -\log(1-z)+2\pi\sqrt{-1}\,\big(t+s+u-\frac{1}{2}\big), \tag{39}$$

where $x=e^{2\pi\sqrt{-1}\,t}$, $y=e^{2\pi\sqrt{-1}\,s}$ and $z=e^{2\pi\sqrt{-1}\,u}$.

Lemma 3.3. *$\hat{V}$ has a unique critical point (t_0,s_0,u_0) in $P^{-1}(\Delta')$, where $P:\mathbb{C}^3\to\mathbb{R}^3$ is the projection to the real parts of the entries.*

Proof. Any critical point of $\hat{V}$ is given by a solution of $\frac{\partial}{\partial t}\hat{V}=\frac{\partial}{\partial s}\hat{V}=\frac{\partial}{\partial u}\hat{V}=0$, and these equations are rewritten,

$$(1-x)^2 \;=\; -x^3\,y^2\,z, \qquad 1-y \;=\; x^2\,y^2\,z, \qquad 1-z \;=\; -x\,y\,z.$$

Putting $y'=xy$ and $z'=xyz$, they are rewritten,

$$(1-x)^2 \;=\; -x\,y'z', \qquad 1-\frac{y'}{x} \;=\; y'z', \qquad 1-\frac{z'}{y'} \;=\; -z'.$$

From the third formula, we have that $y'=z'/(1+z')$. Hence, from the second formula, we have that $x=-z'/(z'^2-z'-1)$. By substituting them into the first formula, we obtain that

$$2z'^4-z'^3-2z'^2+z'+1 \;=\; 0.$$

Its solutions are given by

$$z' \;=\; -0.677958...\pm\sqrt{-1}\cdot 0.15778...\,,\quad 0.927958...\pm\sqrt{-1}\cdot 0.413327...\,.$$

Further, their corresponding values of (t, s) are given by

$$\begin{aligned}(t,s) \;&= (0.16676... - \sqrt{-1}\cdot 0.0928453...\,,\quad 0.224343... - \sqrt{-1}\cdot 0.0127069...),\\ &\quad (0.83324... - \sqrt{-1}\cdot 0.0928453...\,,\quad 0.775657... - \sqrt{-1}\cdot 0.0127069...),\\ &\quad (0.111002... + \sqrt{-1}\cdot 0.0376865...\,,\quad 0.922078... + \sqrt{-1}\cdot 0.0678661...),\\ &\quad (0.888998... + \sqrt{-1}\cdot 0.0376865...\,,\quad 0.0779221... + \sqrt{-1}\cdot 0.0678661...).\end{aligned}$$

Among these, the first solution is in Δ', from which we have that

$$\begin{aligned} x_0 &= 0.895123... + \sqrt{-1}\cdot 1.55249...\,, & t_0 &= 0.16676... - \sqrt{-1}\cdot 0.0928453...\,,\\ y_0 &= 0.17385... + \sqrt{-1}\cdot 1.06907...\,, & s_0 &= 0.224343... - \sqrt{-1}\cdot 0.0127069...\,,\\ z_0 &= 0.322042... + \sqrt{-1}\cdot 0.15778...\,, & u_0 &= 0.0725053... + \sqrt{-1}\cdot 0.163214...\,,\end{aligned}$$

where $x_0 = e^{2\pi\sqrt{-1}\,t_0}$, $y_0 = e^{2\pi\sqrt{-1}\,s_0}$ and $z_0 = e^{2\pi\sqrt{-1}\,u_0}$. These give a unique critical point of $\hat{V}$ in $P^{-1}(\Delta')$. □

The critical value of $\hat{V}$ at the critical point of Lemma 3.3 is presented by

$$\begin{aligned}\varsigma \;&=\; \hat{V}(t_0, s_0, u_0)\\ &=\; \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(x_0) + \mathrm{Li}_2(y_0) + \mathrm{Li}_2(z_0) + \frac{\pi^2}{3}\Big)\\ &\qquad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big((t_0+s_0+u_0)^2 + (t_0+s_0)^2 + t_0^2 - 3t_0 - 2s_0 - u_0 + \frac{1}{2}\big) \qquad (40)\\ &=\; 0.5035603... - \sqrt{-1}\cdot 1.08078...\,.\end{aligned}$$

Further, we put its real part to be ς_R,

$$\varsigma_R \;=\; \mathrm{Re}\,\varsigma \;=\; 0.5035603...\,. \qquad (41)$$

We calculate the Hesse matrix of $\hat{V}$. Since $x = e^{2\pi\sqrt{-1}\,t}$, $\frac{d}{dt} = 2\pi\sqrt{-1}\,x\frac{d}{dx}$. Hence, from (37), we have that

$$\frac{\partial^2}{\partial t^2}\hat{V} \;=\; 2\pi\sqrt{-1}\,x\frac{\partial}{\partial x}\big(-2\log(1-x)\big) + 2\pi\sqrt{-1}\cdot 3 \;=\; 2\pi\sqrt{-1}\cdot\frac{3-x}{1-x}\,.$$

Similarly, we have that

$$\begin{aligned}&\frac{\partial^2}{\partial t\partial s}\hat{V} \;=\; 2\pi\sqrt{-1}\cdot 2, \qquad \frac{\partial^2}{\partial t\partial u}\hat{V} \;=\; 2\pi\sqrt{-1}, \qquad \frac{\partial^2}{\partial s^2}\hat{V} \;=\; \frac{2-y}{1-y},\\ &\frac{\partial^2}{\partial s\partial u}\hat{V} \;=\; 2\pi\sqrt{-1}, \qquad \frac{\partial^2}{\partial u^2}\hat{V} \;=\; \frac{1}{1-z}\,.\end{aligned}$$

Hence, the Hesse matrix of $\hat{V}$ at (t_0, s_0, u_0) is presented by

$$H_0 \;=\; 2\pi\sqrt{-1}\begin{pmatrix}\dfrac{3-x_0}{1-x_0} & 2 & 1\\ 2 & \dfrac{2-y_0}{1-y_0} & 1\\ 1 & 1 & \dfrac{1}{1-z_0}\end{pmatrix}. \qquad (42)$$

3.4 Verifying the assumption of the Poisson summation formula

In this section, we verify the assumption of the Poisson summation formula in Lemma 3.4, which is used in the proof of Theorem 1.1 for the $\overline{6_1}$ knot in Section 3.1. As we mentioned in Remark 2.5, we verify the assumption for $\hat{V}(t,s,u)$ instead of $V(t,s,u)$, since $V(t,s,u)$ converges uniformly to $\hat{V}(t,s,u)$ on Δ' in the form mentioned in Remark 2.5.

$\operatorname{Re}\hat{V}(t,s,u)$ has a unique maximal point in Δ' at

$$x_0 = y_0 = y_0 = e^{\pi\sqrt{-1}/3}, \qquad t_0 = s_0 = u_0 = \frac{1}{6},$$

and its maximal value is 0.646131... . Hence,

$$\operatorname{Re}\hat{V}(t,s,u) - \varsigma_R \ \leq\ 0.142571... \tag{43}$$

for any $(t,s,u)\in\Delta'$. Therefore, in the proof of Lemma 3.4, it is sufficient to decrease, say, $\operatorname{Re}\hat{V}(t,\ s,\ u+\delta\sqrt{-1}) - 2\pi\delta$ by this value, by moving δ (though we do not use this value in the proof of the lemma).

Lemma 3.4. $\hat{V}(t,s,u)-\varsigma_R$ *satisfies the assumption of Proposition 2.3.*

Proof. We show that $\partial\Delta'$ is null-homotopic in each of (17)–(20), (24) and (25).

As for (19), we show that we can move Δ' into the following domain,

$$\big\{(t,\ s,\ u+\delta\sqrt{-1})\in\mathbb{C}^3 \ \big|\ (t,s,u)\in\Delta',\ \delta\geq 0,\ \operatorname{Re}\hat{V}(t,\ s,\ u+\delta\sqrt{-1})-\varsigma_R-2\pi\delta<0\big\}.$$

Hence, putting

$$F(\delta) \ = \ \operatorname{Re}\hat{V}(t,\ s,\ u+\delta\sqrt{-1})-\varsigma_R-2\pi\delta,$$

it is sufficient to show that there exists $\delta_0>0$ such that

$$\begin{aligned} &F(\delta_0) \ < \ 0 \quad \text{for any } (t,s,u)\in\Delta', \text{ and} \\ &F(\delta) \ < \ 0 \quad \text{for any } (t,s,u)\in\partial\Delta' \text{ and } \delta\in[0,\delta_0]. \end{aligned} \tag{44}$$

Therefore, it is sufficient to show that

$$\frac{d}{d\delta}F(\delta) \ = \ \frac{\partial}{\partial\delta}\operatorname{Re}\hat{V}(t,\ s,\ u+\delta\sqrt{-1})-\varsigma_R-2\pi \ < \ -\varepsilon',$$

for some $\varepsilon'>0$ (because, if the above formula holds, then (44) holds for a sufficiently large δ_0). Hence, it is sufficient to show that

$$\operatorname{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t,\ s,\ u+\delta\sqrt{-1})\Big) \ < \ 2\pi-\varepsilon'.$$

Further, as for (20), similarly as above, it is sufficient to show that

$$\operatorname{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t,\ s,\ u-\delta\sqrt{-1})\Big) \ < \ 2\pi-\varepsilon'$$

for some $\varepsilon'>0$.

Hence, as for (19) and (20), it is sufficient to show that

$$-(2\pi-\varepsilon') \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat V(t,\,s,\,u+\delta\sqrt{-1})\Big) \;<\; 2\pi-\varepsilon' \tag{45}$$

for some $\varepsilon'>0$. The middle term is calculated as

$$\begin{aligned}\mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat V(t,\,s,\,u+\delta\sqrt{-1})\Big) &= \mathrm{Re}\Big(\sqrt{-1}\cdot\frac{\partial}{\partial u}\hat V(t,\,s,\,u+\delta\sqrt{-1})\Big)\\ &= -\mathrm{Im}\Big(-\log(1-z)+2\pi\sqrt{-1}\,\big(t+s+u-\frac{1}{2})\Big)\\ &= \mathrm{Arg}\,(1-z)-2\pi\big(t+s+u-\frac{1}{2}\big),\end{aligned}$$

where $z=e^{2\pi\sqrt{-1}\,u}$. Since $0<u\le\frac{1}{2}$,

$$-2\pi\big(\frac{1}{2}-u\big) \;<\; \mathrm{Arg}\,(1-z) \;<\; 0.$$

Hence,

$$-2\pi(t+s) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat V(t,\,s,\,u+\delta\sqrt{-1})\Big) \;<\; 2\pi\big(\frac{1}{2}-t-s-u\big).$$

Therefore, since $t+s\le 0.4+0.5=0.9$ and $t,s,u\ge 0$,

$$-2\pi\cdot 0.9 \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat V(t,\,s,\,u+\delta\sqrt{-1})\Big) \;<\; 2\pi\cdot 0.5,$$

and hence, (45) is satisfied.

As for (17) and (18), similarly as above, it is sufficient to show that

$$-(2\pi-\varepsilon') \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat V(t,\,s+\delta\sqrt{-1},u)\Big) \;<\; 2\pi-\varepsilon' \tag{46}$$

for some $\varepsilon'>0$. The middle term is calculated as

$$\mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat V(t,\,s+\delta\sqrt{-1},u)\Big) \;=\; \mathrm{Arg}\,(1-y)-2\pi\big(2t+2s+u-1\big),$$

where $y=e^{2\pi\sqrt{-1}\,s}$. Since $0<s\le\frac{1}{2}$,

$$-2\pi\big(\frac{1}{2}-s\big) \;<\; \mathrm{Arg}\,(1-y) \;<\; 0.$$

Hence,

$$-2\pi\big(2t+s+u-\frac{1}{2}\big) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat V(t,\,s+\delta\sqrt{-1},u)\Big) \;<\; 2\pi(1-2t-2s-u).$$

Therefore, since $t+(t+s+u)\le 0.4+0.94=1.34$ and $t\ge 0.03,\ s,u\ge 0$,

$$-2\pi\cdot 0.84 \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat V(t,\,s+\delta\sqrt{-1},u)\Big) \;<\; 2\pi\cdot 0.94,$$

and hence, (46) is satisfied.

As for (24) and (25), similarly as above, it is sufficient to show that

$$-(2\pi - \varepsilon') \ < \ \mathrm{Re}\Big(\frac{\partial}{\partial \delta}\hat{V}(t+\delta\sqrt{-1}, s-\delta\sqrt{-1}, u)\Big) \ < \ 2\pi - \varepsilon' \qquad (47)$$

for some $\varepsilon' > 0$. The middle term is calculated as

$$\mathrm{Re}\Big(\frac{\partial}{\partial \delta}\hat{V}(t+\delta\sqrt{-1}, s-\delta\sqrt{-1}, u)\Big) \ = \ 2\,\mathrm{Arg}\,(1-x) - \mathrm{Arg}\,(1-y) - 2\pi\big(t-\frac{1}{2}\big),$$

where $x = e^{2\pi\sqrt{-1}\,t}$. Since $0 < t \le \frac{1}{2}$,

$$-2\pi\big(\frac{1}{2}-t\big) \ < \ \mathrm{Arg}\,(1-x) \ < \ 0.$$

Further, since $\mathrm{Arg}\,(1-y)$ is in the range mentioned above,

$$-2\pi\big(\frac{1}{2}-t\big) \ < \ \mathrm{Re}\Big(\frac{\partial}{\partial \delta}\hat{V}(t+\delta\sqrt{-1}, s-\delta\sqrt{-1}, u)\Big) \ < \ 2\pi\big(1-t-s\big).$$

Therefore, since $t \ge 0.03$ and $s \ge 0$,

$$-2\pi\cdot 0.5 \ < \ \mathrm{Re}\Big(\frac{\partial}{\partial \delta}\hat{V}(t+\delta\sqrt{-1}, s-\delta\sqrt{-1}, u)\Big) \ < \ 2\pi\cdot 0.97,$$

and hence, (47) is satisfied. □

3.5 Verifying the assumption of the saddle point method

In this section, we verify the assumption of the saddle point method in Lemma 3.9. In order to show this lemma, we show Lemmas 3.5–3.8 in advance. As we mentioned in Remark 2.7, we verify the assumption for $\hat{V}(t,s,u)$ instead of $V(t,s,u)$, since $V(t,s,u)$ converges uniformly to $\hat{V}(t,s,u)$ on Δ' in the form mentioned in Remark 2.7.

In the proof of Lemma 3.9, by (43), it is sufficient to decrease $\mathrm{Re}\,\hat{V}(t,s,u)$ by 0.142571... by pushing t,s,u into the imaginary directions. In order to calculate this concretely, putting

$$f(X,Y,Z) \ = \ \mathrm{Re}\,\hat{V}(t+X\sqrt{-1}, s+Y\sqrt{-1}, u+Z\sqrt{-1}) - \varsigma_R\,,$$

we consider the behavior of f at each fiber of the projection $\mathbb{C}^3 \to \mathbb{R}^3$.

Lemma 3.5 ([34]). *Fixing X and Y, we regard f as a function of Z.*
(1) If $t+s+u \ge \frac{1}{2}$, then f is monotonically decreasing.
(2) If $t+s+u < \frac{1}{2}$, then f has a unique minimal point at $Z = g_3(t,s,u)$, where

$$g_3(t,s,u) \ = \ \frac{1}{2\pi}\,\log\frac{\sin 2\pi(t+s)}{\sin 2\pi(\frac{1}{2}-t-s-u)}\,.$$

In particular, $g_3(t,s,u)$ goes to ∞ as $t+s+u \to \frac{1}{2}-0$.

Proof. As a function of Z, the differential of f is presented by

$$\frac{\partial f}{\partial Z} = \mathrm{Arg}\,(1-z) - 2\pi\big(t+s+u-\frac{1}{2}\big),$$

where $z = e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})}$. Since $0 < u < \frac{1}{2}$,

$$-2\pi\big(\frac{1}{2}-u\big) \ < \ \mathrm{Arg}\,(1-z) \ < \ 0,$$

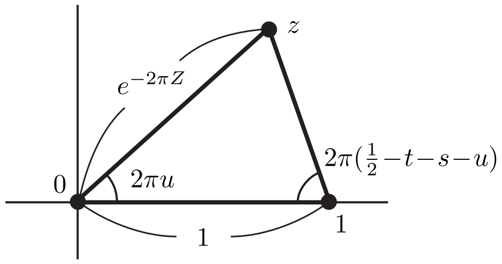
and $\mathrm{Arg}\,(1-z)$ is monotonically increasing as a function of Z. Further,

$$\frac{\partial f}{\partial Z}\Big|_{Z\to\infty} = 2\pi\big(\frac{1}{2}-t-s-u\big),$$

$$\frac{\partial f}{\partial Z}\Big|_{Z\to-\infty} = -2\pi\big(\frac{1}{2}-u\big) - 2\pi\big(t+s+u-\frac{1}{2}\big) = -2\pi(t+s) \ < \ 0.$$

If $t+s+u \geq \frac{1}{2}$, then $\frac{\partial f}{\partial Z}$ is always negative, and (1) holds.

If $t+s+u < \frac{1}{2}$, then there is a unique zero of $\frac{\partial f}{\partial Z}$, which gives a unique minimal point of f. When $\frac{\partial f}{\partial Z} = 0$, we have that $-\mathrm{Arg}\,(1-z) = 2\pi(\frac{1}{2}-t-s-u)$, and t, s, u and Z are related as shown in the following picture.



Hence,

$$\frac{e^{-2\pi Z}}{\sin 2\pi(\frac{1}{2}-t-s-u)} = \frac{1}{\sin 2\pi(t+s)}.$$

Therefore,

$$Z = \frac{1}{2\pi}\log\frac{\sin 2\pi(t+s)}{\sin 2\pi(\frac{1}{2}-t-s-u)},$$

and this gives the minimal point of the lemma. Hence, (2) holds. □

Lemma 3.6 ([34]). *Fixing X and Z, we regard f as a function of Y.*
(1) *If $2t+2s+u \geq 1$, then f is monotonically decreasing.*
(2) *If $2t+2s+u < 1$ and $2t+s+u > \frac{1}{2}$, then f has a unique minimal point at $Y = g_2(t,s,u)$, where*

$$g_2(t,s,u) = \frac{1}{2\pi}\log\frac{\sin 2\pi(2t+s+u-\frac{1}{2})}{\sin 2\pi(1-2t-2s-u)}.$$

In particular, $g_2(t,s,u)$ goes to ∞ as $2t+2s+u \to 1-0$, and goes to $-\infty$ as $2t+s+u \to \frac{1}{2}+0$.
(3) *If $2t+s+u \leq \frac{1}{2}$, then f is monotonically increasing.*

Proof. As a function of Y, the differential of f is presented by

$$\frac{\partial f}{\partial Y} \;=\; \mathrm{Arg}\,(1-y) - 2\pi\big(2t+2s+u-1\big),$$

where $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$. Since $0 < s < \frac{1}{2}$,

$$-2\pi\big(\frac{1}{2}-s\big) \;<\; \mathrm{Arg}\,(1-y) \;<\; 0,$$

and $\mathrm{Arg}\,(1-y)$ is monotonically increasing as a function of Y. Further,

$$\begin{aligned}
\frac{\partial f}{\partial Y}\Big|_{Y\to\infty} &\;=\; 2\pi\big(1-2t-2s-u\big),\\
\frac{\partial f}{\partial Y}\Big|_{Y\to-\infty} &\;=\; -2\pi\big(\frac{1}{2}-s\big) - 2\pi\big(2t+2s+u-1\big) \;=\; -2\pi\big(2t+s+u-\frac{1}{2}\big).
\end{aligned}$$

If $2t+2s+u \ge 1$, then $\frac{\partial f}{\partial Y}$ is always negative, and (1) holds.

If $2t+s+u \le \frac{1}{2}$, then $\frac{\partial f}{\partial Y}$ is always positive, and (3) holds.

If $2t+2s+u < 1$ and $2t+s+u > \frac{1}{2}$, then there is a unique zero of $\frac{\partial f}{\partial Y}$, which gives a unique minimal point of f. We can obtain the value of the minimal point in the same way as in the proof of Lemma 3.5. □

Lemma 3.7 ([34])**.** *Fixing Y and Z, we regard f as a function of X.*
(1) *If $3t+2s+u \ge \frac{3}{2}$, then f is monotonically decreasing.*
(2) *If $3t+2s+u < \frac{3}{2}$ and $t+2s+u > \frac{1}{2}$, then f has a unique minimal point at $X = g_1(t,s,u)$, where*

$$g_1(t,s,u) \;=\; \frac{1}{2\pi}\,\log\frac{\sin 2\pi(t+2s+u-\frac{1}{2})}{\sin 2\pi(\frac{3}{2}-3t-2s-u)}\,.$$

In particular, $g_1(t,s,u)$ goes to ∞ as $3t+2s+u \to \frac{3}{2}-0$, and goes to $-\infty$ as $t+2s+u \to \frac{1}{2}+0$.
(3) *If $t+2s+u \le \frac{1}{2}$, then f is monotonically increasing.*

Proof. As a function of X, the differential of f is presented by

$$\frac{\partial f}{\partial X} \;=\; 2\,\mathrm{Arg}\,(1-x) - 2\pi\big(3t+2s+u-\frac{3}{2}\big),$$

where $x = e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$. Since $0 < t < \frac{1}{2}$,

$$-2\pi\big(\frac{1}{2}-t\big) \;<\; \mathrm{Arg}\,(1-x) \;<\; 0,$$

and $\mathrm{Arg}\,(1-x)$ is monotonically increasing as a function of X. Further,

$$\frac{\partial f}{\partial X}\Big|_{X\to\infty} \;=\; 2\pi\big(\frac{3}{2}-3t-2s-u\big),$$

$$\frac{\partial f}{\partial X}\Big|_{X\to-\infty} = -2\pi\big(1-2t\big) - 2\pi\big(3t+2s+u-\frac{3}{2}\big) = -2\pi\big(t+2s+u-\frac{1}{2}\big).$$

If $3t+2s+u \geq \frac{3}{2}$, then $\frac{\partial f}{\partial X}$ is always negative, and (1) holds.

If $t+2s+u \leq \frac{1}{2}$, then $\frac{\partial f}{\partial X}$ is always positive, and (3) holds.

If $3t+2s+u < \frac{3}{2}$ and $t+2s+u > \frac{1}{2}$, then there is a unique zero of $\frac{\partial f}{\partial X}$, which gives a unique minimal point of f. We can obtain the value of the minimal point in the same way as in the proof of Lemma 3.5. □

Lemma 3.8 ([34]). *For each $(t,s,u) \in \Delta'$ satisfying that $2t+2s+u<1,\ \ t+s+u<\frac{1}{2}$, $t+2s+u > \frac{1}{2}$ and $2t+s+u>\frac{1}{2}$, f has a unique minimal point and no other critical points.*

Proof. From the definition of f,

$$\begin{aligned} f(X,Y,Z) &= \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\cdot 2\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})})\Big) + 2\pi\big(\frac{3}{2}-3t-2s-u\big)X \\ &\quad + \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\cdot \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})})\Big) + 2\pi\big(1-2t-2s-u\big)Y \\ &\quad + \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\cdot \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})})\Big) + 2\pi\big(\frac{1}{2}-t-s-u\big)Z. \end{aligned}$$

By Lemma 3.7, fixing Y and Z, f has a unique minimal point at $X = g_1(t,s,u)$. By Lemma 3.6, fixing X and Z, f has a unique minimal point at $Y = g_2(t,s,u)$. By Lemma 3.5, fixing X and Y, f has a unique minimal point at $Z = g_3(t,s,u)$. Since the contribution to f from X, Y and Z are independent, f has a unique minimal point at $(X,Y,Z) = \big(g_1(t,s,u), g_2(t,s,u), g_3(t,s,u)\big)$. □

Lemma 3.9 ([34]). *When we apply Proposition 2.6 to* (33)*, the assumption of Proposition 2.6 holds.*

Proof. We show that there exists a homotopy Δ'_δ $(0 \leq \delta \leq 1)$ between $\Delta'_0 = \Delta'$ and Δ'_1 such that

$$(t_0,s_0,u_0) \in \Delta'_1, \tag{48}$$

$$\Delta'_1 - \{(t_0,s_0,u_0)\} \ \subset\ \big\{(t,s,u)\in\mathbb{C}^3 \ \big|\ \operatorname{Re}\hat{V}(t,s,u) < \varsigma_R\big\}, \tag{49}$$

$$\partial\Delta'_\delta \ \subset\ \big\{(t,s,u)\in\mathbb{C}^3 \ \big|\ \operatorname{Re}\hat{V}(t,s,u) < \varsigma_R\big\}. \tag{50}$$

For a sufficiently large $R>0$, we put

$$\hat{g}_1(t,s,u) = \begin{cases} R & \text{if } \ 3t+2s+u \geq \frac{3}{2}, \\ \max\big\{-R,\ \min\{R,\ g_1(t,s,u)\}\big\} & \text{if } \ 3t+2s+u<\frac{3}{2} \text{ and } t+2s+u>\frac{1}{2}, \\ -R & \text{if } \ t+2s+u\leq\frac{1}{2}, \end{cases}$$

$$\hat{g}_2(t,s,u) = \begin{cases} R & \text{if } \ 2t+2s+u \geq 1, \\ \max\big\{-R,\ \min\{R,\ g_2(t,s,u)\}\big\} & \text{if } \ 2t+2s+u<1 \text{ and } 2t+s+u>\frac{1}{2}, \\ -R & \text{if } \ 2t+s+u\leq\frac{1}{2}, \end{cases}$$

$$\hat{g}_3(t,s,u) \;=\; \begin{cases} R & \text{if } \; t+s+u \geq \frac{1}{2}, \\ \min\{R,\; g_3(t,s,u)\} & \text{if } \; t+s+u < \frac{1}{2}. \end{cases}$$

We note that, since $g_3(t,s,u) \to \infty$ as $t+s+u \to \frac{1}{2}$, $\hat{g}_3(t,s,u)$ is continuous, and similarly, we can check that $\hat{g}_1(t,s,u)$ and $\hat{g}_2(t,s,u)$ are also continuous. We set the ending of the homotopy by

$$\Delta'_1 \;=\; \big\{ \big(t+\hat{g}_1(t,s,u)\sqrt{-1},\; s+\hat{g}_2(t,s,u)\sqrt{-1}, u+\hat{g}_3(t,s,u)\sqrt{-1}\big) \in \mathbb{C}^3 \;\big|\; (t,s,u) \in \Delta' \big\}.$$

Further, we define the internal part Δ'_δ $(0<\delta<1)$ of the homotopy by setting it along the flow from $\big(t,\,s,\,u\big)$ determined by the vector field $\big(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}\big)$.

We show (50), as follows. From the definition of Δ',

$$\partial\Delta' \;\subset\; \big\{(t,s,u) \in \mathbb{C}^3 \;\big|\; \operatorname{Re} \hat{V}(t,s,u) < \varsigma_R \big\}.$$

Further, by the construction of the homotopy, $\operatorname{Re} \hat{V}$ monotonically decreases by the homotopy. Hence, (50) holds.

We show (48) and (49), as follows. Consider the following functions

$$\begin{aligned} F(t,s,u,X,Y,Z) \;&=\; \operatorname{Re} \hat{V}\big(t+X\sqrt{-1},\; s+Y\sqrt{-1},\; u+Z\sqrt{-1}\big), \\ h(t,s,u) \;&=\; F\big(t,\; s,\; u,\; \hat{g}_1(t,s,u),\; \hat{g}_2(t,s,u),\; \hat{g}_3(t,s,u)\big). \end{aligned}$$

When $3t+2s+u \geq \frac{3}{2}$, $-h(t,s,u)$ is sufficiently large (because we let R be sufficiently large), and (49) holds in this case. Similarly, we can check that (49) holds when $\hat{g}_i(t,s,u) = \pm R$. The remaining case is the case where $\hat{g}_i(t,s,u) = g_i(t,s,u)$ for $i=1,2,3$. In this case, we show (49), as follows. It is shown from the definitions of $g_i(t,s,u)$ that $\frac{\partial F}{\partial X} = 0$ at $X = g_1(t,s,u)$ and $\frac{\partial F}{\partial Y} = 0$ at $Y = g_2(t,s,u)$ and $\frac{\partial F}{\partial Z} = 0$ at $Z = g_3(t,s,u)$. Hence, $\operatorname{Im} \frac{\partial \hat{V}}{\partial t} = \operatorname{Im} \frac{\partial \hat{V}}{\partial s} = \operatorname{Im} \frac{\partial \hat{V}}{\partial u} = 0$ at $\big(t+g_1(t,s,u)\sqrt{-1},\; s+g_2(t,s,u)\sqrt{-1},\; u+g_3(t,s,u)\sqrt{-1}\big)$. Further, $\frac{\partial h}{\partial t} = \operatorname{Re} \frac{\partial \hat{V}}{\partial t}$ and $\frac{\partial h}{\partial s} = \operatorname{Re} \frac{\partial \hat{V}}{\partial s}$ and $\frac{\partial h}{\partial u} = \operatorname{Re} \frac{\partial \hat{V}}{\partial u}$ at $\big(t+g_1(t,s,u)\sqrt{-1},\; s+g_2(t,s,u)\sqrt{-1},\; u+g_3(t,s,u)\sqrt{-1}\big)$. Therefore, when (t,s,u) is a critial point of $h(t,s,u)$, $\big(t+g_1(t,s,u)\sqrt{-1},\; s+g_2(t,s,u)\sqrt{-1},\; u+g_3(t,s,u)\sqrt{-1}\big)$ is a critical point of $\hat{V}$. Hence, by Lemma 3.3, $h(t,s,u)$ has a unique maximal point at $(t,s,u) = (\operatorname{Re} t_0, \operatorname{Re} s_0, \operatorname{Re} u_0)$. Therefore, (48) and (49) hold. □

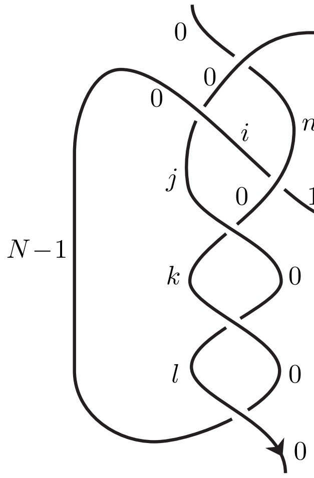
4 The 6_2 knot

In this section, we show Theorem 1.1 for the 6_2 knot. We give a proof of the theorem in Section 4.1, using lemmas shown in Section 4.2–4.5.

4.1 Proof of Theorem 1.1 for the 6_2 knot

In this section, we show a proof of Theorem 1.1 for the 6_2 knot.

The 6_2 *knot* is the closure of the following tangle.



As shown in [33], we can put the labelings of edges adjacent to the unbounded regions as shown above. Hence, from the definition of the Kashaev invariant, the Kashaev invariant of the 6_2 knot is presented by

$$\begin{aligned}
\langle\, 6_2 \,\rangle_{N} \;&=\; \sum \frac{N\,q^{-\frac{1}{2}}}{(\overline{q})_{N-n}(q)_{n-1}} \times \frac{N\,q^{\frac{1}{2}-i}}{(q)_{N-i}(\overline{q})_{i-j-1}(q)_j} \times \frac{N\,q^{-\frac{1}{2}+i}}{(q)_{i-n}(\overline{q})_{n-1}(\overline{q})_{N-i}} \\
&\qquad \times \frac{N\,q^{\frac{1}{2}}}{(\overline{q})_j(\overline{q})_{N-k-1}(q)_{k-j}} \times \frac{N\,q^{\frac{1}{2}}}{(\overline{q})_k(\overline{q})_{N-l-1}(q)_{l-k}} \times \frac{N\,q^{\frac{1}{2}}}{(\overline{q})_l(q)_{N-l-1}} \\
&=\; \sum_{\substack{0\le j<i\le N\\ j\le k<N}} \frac{N^4\,q}{(q)_{N-i}(\overline{q})_{N-i}(\overline{q})_{i-j-1}(q)_j(\overline{q})_j(q)_{k-j}(\overline{q})_k(\overline{q})_{N-k-1}} \\
&=\; \sum_{\substack{0\le j\le i<N\\ 0\le k<N-j}} \frac{N^4\,q}{(q)_{N-i-1}(\overline{q})_{N-i-1}(\overline{q})_{i-j}(q)_j(\overline{q})_j(q)_k(\overline{q})_{j+k}(\overline{q})_{N-j-k-1}}
\end{aligned} \tag{51}$$

where the second equality is obtained by (4) and (5), and the last equality is obtained by replacing i with $i+1$ and replacing k with $j+k$.

Proof of Theorem 1.1 for the 6_2 *knot.* By (6), the above presentation of $\langle\, 6_2 \,\rangle_{N}$ is rewritten

$$\langle\, 6_2 \,\rangle_{N} \;=\; N^4\,q \sum_{\substack{0\le j\le i<N\\ 0\le k<N-j}} \exp\Big(N\,\tilde{V}\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}\big)\Big),$$

where we put

$$\begin{aligned}
\tilde{V}(t,s,u) \;&=\; \frac{1}{N}\Big(\varphi(1-t)-\varphi(t)-\varphi\big(1-t+s-\frac{1}{2N}\big)+\varphi(s)-\varphi(1-s)+\varphi(u) \\
&\qquad -\varphi\big(1-s-u+\frac{1}{2N}\big)-\varphi\big(s+u-\frac{1}{2N}\big)-3\,\varphi\big(\frac{1}{2N}\big)+5\,\varphi\big(1-\frac{1}{2N}\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N}\Big(-2\,\varphi(t)-\varphi\big(1-t+s-\frac{1}{2N}\big)+2\,\varphi(s)+\varphi(u)\Big)-\frac{4}{N}\,\log N\\
&\quad+\frac{1}{2\pi\sqrt{-1}}\cdot\frac{\pi^2}{3}+\frac{\pi\sqrt{-1}}{2N}-\frac{\pi\sqrt{-1}}{6N^2}\\
&\quad+2\pi\sqrt{-1}\cdot\frac{1}{2}\Big(-t^2+s^2+\big(s+u-\frac{1}{2N}\big)^2+t-2s-u+\frac{1}{6}+\frac{1}{2N}-\frac{1}{12N^2}\Big).
\end{aligned}
$$

Here, we obtain the last equality by (10) and (11). Hence, by putting

$$
\begin{aligned}
V(t,s,u) &= \tilde{V}(t,s,u)+\frac{4}{N}\log N\\
&= \frac{1}{N}\Big(-2\,\varphi(t)-\varphi\big(1-t+s-\frac{1}{2N}\big)+2\,\varphi(s)+\varphi(u)\Big)\\
&\quad+\frac{1}{2\pi\sqrt{-1}}\cdot\frac{\pi^2}{3}+\frac{\pi\sqrt{-1}}{2N}-\frac{\pi\sqrt{-1}}{6N^2}\\
&\quad+2\pi\sqrt{-1}\cdot\frac{1}{2}\Big(-t^2+s^2+\big(s+u-\frac{1}{2N}\big)^2+t-2s-u+\frac{1}{6}+\frac{1}{2N}-\frac{1}{12N^2}\Big),
\end{aligned}
$$

the presentation of $\langle\, 6_2 \,\rangle_N$ is rewritten

$$
\begin{aligned}
\langle\, 6_2 \,\rangle_N &= q\sum_{\substack{0\le j\le i<N\\ 0\le k<N-j}}\exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big)\Big)\\
&= q\sum_{\substack{i,j,k\,\in\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta}}\exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big)\Big),
\end{aligned}
$$

where the range Δ of $(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})$ of the above sum is given by the following domain,

$$
\Delta = \big\{(t,s,u)\in\mathbb{R}^3 \;\big|\; 0\le s\le t\le 1,\quad 0\le u\le 1-s\big\}.
$$

By Proposition 2.1, as $N\to\infty$, $V(t,s,u)$ converges to the following $\hat{V}(t,s,u)$ in the interior of Δ,

$$
\begin{aligned}
\hat{V}(t,s,u) &= \frac{1}{2\pi\sqrt{-1}}\Big(-2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t})-\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s-t)})\\
&\qquad\qquad+2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,s})+\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,u})+\frac{\pi^2}{3}\Big)\\
&\quad+2\pi\sqrt{-1}\cdot\frac{1}{2}\big(-t^2+s^2+(s+u)^2+t-s-(s+u)+\frac{1}{6}\big).
\end{aligned}
$$

By concrete computation, we can check that the boundary of Δ is included in the domain

$$
\big\{(t,s,u)\in\Delta \;\big|\; \mathrm{Re}\,\hat{V}(t,s,u)<\varsigma_R-\varepsilon\big\} \tag{52}
$$

for some sufficiently small $\varepsilon>0$, where we put $\varsigma_R=0.700414...$ as in (63); we will know later that this value is equal to the real part of the critical value of $\hat{V}$ at the critical point of Lemma 4.2. Hence, similarly as in Section 3.1, we choose a new domain Δ', which satisfies that $\Delta - \Delta' \subset$ (52), as

$$\Delta' \;=\; \left\{ (t,s,u) \in \Delta \;\middle|\; \begin{array}{l} 0.6 \le t \le 0.86, \quad 0.14 \le s \le 0.4, \quad 0.05 \le u \le 0.4, \\ 0.3 \le t-s \le 0.67, \quad 2s+u \le 0.97 \end{array} \right\}, \tag{53}$$

where we calculate the concrete values of the bounds of these inequalities in Section 4.2. Then, similarly as in Section 3.1, we can restrict Δ to Δ' as

$$\langle\, 6_2 \,\rangle_N \;=\; e^{N\varsigma}\Bigg(q \sum_{\substack{i,j,k\,\in\,\mathbb{Z} \\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta'}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big) - N\varsigma\Big) \;+ O(e^{-N\varepsilon})\Bigg), \tag{54}$$

for some $\varepsilon > 0$.

Further, by Proposition 2.3 (Poisson summation formula, see also Remark 2.5), the above sum is presented by

$$\langle\, 6_2 \,\rangle_N \;=\; e^{N\varsigma}\Bigg(N^3 q \int_{\Delta'} \exp\big(N\cdot V(t,s,u) - N\varsigma\big)\, dt\, ds\, du \;+ O(e^{-N\varepsilon})\Bigg), \tag{55}$$

noting that we verify the assumption of Proposition 2.3 in Lemma 4.3. Furthermore, by Proposition 2.6 (saddle point method, see also Remark 2.7), there exist some κ_i''s such that

$$\langle\, 6_2 \,\rangle_N \;=\; N^3 q \,\exp\big(N\cdot V(t_c,s_c,u_c)\big)\cdot \frac{(2\pi)^{3/2}}{N^{3/2}}\big(\det(-H)\big)^{-1/2}\Big(1 + \sum_{i=1}^{d} \kappa_i' \hbar^i + O(\hbar^{d+1})\Big),$$

for any $d > 0$, noting that we verify the assumption of Proposition 2.6 in Lemma 4.9. Here, (t_c,s_c,u_c) is the critical point of V which corresponds to the critical point (t_0,s_0,u_0) of $\hat{V}$ of Lemma 4.2, and H is the Hesse matrix of V at (t_c,s_c,u_c).

We calculate the right-hand side of the above formula. Similarly as in Section 3.1, we have that $V(t_c,s_c,u_c) = V(t_0,s_0,u_0) + O(\hbar^2)$. Further,

$$\begin{aligned} \varphi\big(1 - t_0 + s_0 - \frac{1}{2N}\big) \;&=\; \varphi\big(1-t_0+s_0\big) - \varphi'\big(1-t_0+s_0\big)\cdot\frac{1}{2N} + O(\hbar^2) \\ &=\; \varphi\big(1-t_0+s_0\big) + \frac{1}{2}\log\big(1 - \frac{y_0}{x_0}\big) + O(\hbar^2), \end{aligned}$$

where we put $x_0 = e^{2\pi\sqrt{-1}\,t_0}$ and $y_0 = e^{2\pi\sqrt{-1}\,s_0}$. Hence, by comparing $V(t_0,s_0,u_0)$ and $\hat{V}(t_0,s_0,u_0) = \varsigma$, we have that

$$V(t_0,s_0,u_0) \;=\; \varsigma + \frac{\pi\sqrt{-1}}{2N} - \frac{1}{2N}\log\big(1-\frac{y_0}{x_0}\big) - \frac{2\pi\sqrt{-1}}{2N}\big(s_0+u_0-\frac{1}{2}\big) + O(\hbar^2).$$

Therefore, there exist some κ_i's such that

$$\langle\, 6_2 \,\rangle_N \;=\; e^{N\varsigma} N^{3/2}\,\omega\cdot\Big(1 + \sum_{i=1}^{d}\kappa_i\hbar^i + O(\hbar^{d+1})\Big),$$

for any $d > 0$. Here, we put

$$\omega = (2\pi)^{3/2}\sqrt{-1}\,\big(1-\frac{y_0}{x_0}\big)^{-1/2}\big(-y_0z_0\big)^{-1/2}\big(\det(-H_0)\big)^{-1/2}$$
$$= -0.42920... + \sqrt{-1}\cdot 0.20337...\ ,$$

where we put $z_0 = e^{2\pi\sqrt{-1}\,u_0}$, and H_0 is the Hesse matrix of $\hat{V}$ at (t_0, s_0, u_0) whose concrete presentation is given in (64); see also [22] for a relation of this value and the twisted Reidemeister torsion. Hence, we obtain the theorem for the 6_2 knot. □

4.2 Estimate of the range of Δ'

In this section, we calculate the concrete values of the bounds of the inequalities in (53) so that they satisfy that $\Delta - \Delta' \subset$ (52).

Putting Λ as in Section 2.2, we have that

$$\operatorname{Re}\hat{V}(t,s,u) = -2\,\Lambda(t) + \Lambda(t-s) + 2\,\Lambda(s) + \Lambda(u).$$

We consider the domain

$$\big\{(t,s,u)\in\Delta \;\big|\; -2\,\Lambda(t)+\Lambda(t-s)+2\,\Lambda(s)+\Lambda(u) \ \ge\ \varsigma_R\big\}. \tag{56}$$

We note that this domain is symmetric with respect to the exchange of (t, s, u) and $(1-s, 1-t, u)$. The aim of this section is to show that this domain is included in the interior of the domain Δ' of (53).

In order to estimate t and s, we consider the area of (t, s) such that (t, s, u) belongs to the domain (56) for some u. Since $\Lambda(\,\cdot\,) \le \Lambda(\frac{1}{6})$,

$$-2\,\Lambda(t) \ \ge\ \varsigma_R - 4\,\Lambda\big(\frac{1}{6}\big) \ =\ 0.054282... \ >\ 0,$$

and hence, $\frac{1}{2} < t < 1$. Similarly, we have that $0 < s < \frac{1}{2}$. Further, since $\Lambda(u) \le \Lambda(\frac{1}{6})$,

$$-2\,\Lambda(t)+\Lambda(t-s)+2\,\Lambda(s) \ \ge\ \varsigma_R - \Lambda\big(\frac{1}{6}\big) \ =\ 0.538881...\ .$$

We consider the following domain

$$\big\{(t,s)\mid 0.5<t<1,\ \ 0<s<0.5,\ \ -2\,\Lambda(t)+\Lambda(t-s)+2\,\Lambda(s) \ \ge\ \varsigma_R - \Lambda\big(\frac{1}{6}\big)\big\}. \tag{57}$$

We graphically show this domain in Figure 2.

Before calculating t and s, we show that the domain (57) is convex. We put $F(t,s) = -2\,\Lambda(t)+\Lambda(t-s)+2\,\Lambda(s)$. Its differentials are given by

$$\frac{\partial F}{\partial t} = -2\,\Lambda'(t)+\Lambda'(t-s), \qquad \frac{\partial F}{\partial s} = 2\,\Lambda'(s)-\Lambda'(t-s).$$

Hence,

$$\frac{\partial^2 F}{\partial t^2} = -2\,\Lambda''(t)+\Lambda''(t-s) = 2\pi\cot\pi t - \pi\cot\pi(t-s),$$

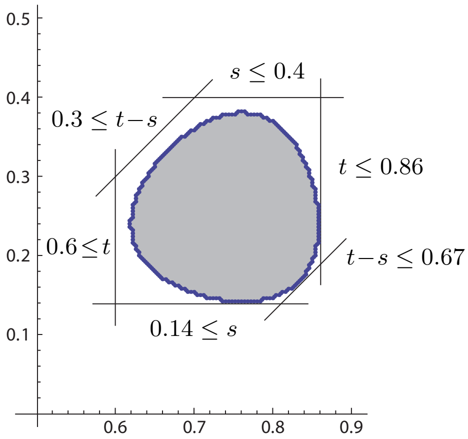


Figure 2: The domain (57)

$$\frac{\partial^2 F}{\partial t\,\partial s} = -\Lambda''(t-s) = \pi \cot \pi (t-s),$$
$$\frac{\partial^2 F}{\partial s^2} = 2\,\Lambda''(s) + \Lambda''(t-s) = -2\pi \cot \pi s - \pi \cot \pi (t-s).$$

We put $a = -\cot \pi t$ and $b = \cot \pi s$, noting that they are positive since $0.5 < t < 1$ and $0 < s < 0.5$. Further, noting that $\cot(\alpha+\beta) = (\cot\alpha \cot\beta - 1)/(\cot\alpha + \cot\beta)$, we have that

$$\frac{1}{\pi} \cdot \frac{\partial^2 F}{\partial t^2} = -2a + \frac{ab-1}{a+b}, \qquad \frac{1}{\pi} \cdot \frac{\partial^2 F}{\partial t\,\partial s} = -\frac{ab-1}{a+b}, \qquad \frac{1}{\pi} \cdot \frac{\partial^2 F}{\partial s^2} = -2b + \frac{ab-1}{a+b}.$$

We put the Hesse matrix of F to be H. Then,

$$\frac{1}{\pi} \cdot \operatorname{trace} H = -2a - 2b + 2 \cdot \frac{ab-1}{a+b} = -\frac{2\big((a+b)^2 - ab + 1\big)}{a+b} = -\frac{2\big(a^2 + b^2 + ab + 1\big)}{a+b} < 0.$$

Further,

$$\frac{1}{\pi^2} \cdot \det H = 4ab - (2a+2b) \cdot \frac{ab-1}{a+b} = 2(ab+1) > 0.$$

Hence, the two eigenvalues of H are nagative, and H is negative definite. Therefore, F is a concave function on $\big\{(t,s) \bigm| 0.5 < t < 1,\ 0 < s < 0.5\big\}$, whose Hesse matrix is negative definite. Hence, the domain (57) is a compact convex domain, and its boundary is a smooth closed curve whose curvature is non-zero everywhere (see Lemma B.1).

We calculate the minimal value $t_{\min}$ and the maximal value $t_{\max}$ of t in the domain (57). We consider the line $t = c$ in the (t,s) plane for a constant c. The values $t_{\min}$ and $t_{\max}$ are given when this line is tangent to the domain (57). The tangent points of such lines are given by the following equations,

$$\begin{cases} -2\,\Lambda(t) + \Lambda(t-s) + 2\,\Lambda(s) = \varsigma_R - \Lambda\big(\frac{1}{6}\big), \\ \dfrac{\partial}{\partial s}\big(-2\,\Lambda(t) + \Lambda(t-s) + 2\,\Lambda(s)\big) = 0. \end{cases}$$

Since the curvature of the boundary curve of the domain (57) is non-zero everywhere (see Lemma B.1), there are exactly two such tangent points, and the above system of equations has exactly two solutions $t_{\min}$ and $t_{\max}$. By calculating a solution by Newton's method from $(t, s) = (0.6, 0.25)$, we obtain $t_{\min} = 0.619717...$, and from $(t, s) = (0.85, 0.25)$, we obtain $t_{\max} = 0.857766...$. Therefore, we obtain an estimate of t in Δ' as

$$0.6 \;\leq\; t \;\leq\; 0.86.$$

To be precise (see Remark 3.2), we can rigorously verify the above estimate of the solutions of the above system of equations in a similar way as in Section A.2.

We obtain an estimate of s in Δ' from the above formula by using the symmetry exchanging (t, s) and $(1-s, 1-t)$, and hence, we obtain that

$$0.14 \;\leq\; s \;\leq\; 0.4.$$

We calculate the minimal value $u_{\min}$ and the maximal value $u_{\max}$ of u. The maximal point of $-2\,\Lambda(t) + \Lambda(t-s) + 2\,\Lambda(s)$ is given by a solution of the following equations,

$$\begin{cases} \dfrac{\partial}{\partial t}\big(-2\,\Lambda(t) + \Lambda(t-s) + 2\,\Lambda(s)\big) \;=\; 0, \\ \dfrac{\partial}{\partial s}\big(-2\,\Lambda(t) + \Lambda(t-s) + 2\,\Lambda(s)\big) \;=\; 0. \end{cases}$$

Their solution is $(t, s) = (\frac{3}{4}, \frac{1}{4})$. Hence,

$$\Lambda(u) \;\geq\; \varsigma_R - 4\,\Lambda\big(\frac{1}{4}\big) \;=\; 0.117292... \,.$$

By calculating solutions of the equality of the above formula by Newton's method, we obtain $u_{\min} = 0.0586318...$ and $u_{\max} = 0.315289...$. Therefore, we obtain an estimate of u in Δ' as

$$0.05 \;\leq\; u \;\leq\; 0.4.$$

To be precise (see Remark 3.1), we can rigorously verify the above estimate in a similar way as in Section A.1.

We calculate the minimal value $(t-s)_{\min}$ and the maximal value $(t-s)_{\max}$ of $t-s$. Putting $w = t - s$, they satisfy the following equations,

$$\begin{cases} -2\,\Lambda(t) + \Lambda(w) + 2\,\Lambda(t-w) \;=\; \varsigma_R - \Lambda\big(\frac{1}{6}\big), \\ \dfrac{\partial}{\partial t}\big(-2\,\Lambda(t) + \Lambda(w) + 2\,\Lambda(t-w)\big) \;=\; 0. \end{cases}$$

We note that this system of equations has exactly two solutions. By calculating a solution of them by Newton's method from $(t, w) = (0.65, 0.3)$, we obtain $(t-s)_{\min} = 0.334132...$, and from $(t, w) = (0.85, 0.7)$, we obtain $(t-s)_{\max} = 0.665868...$. Therefore, we obtain an estimate of $t-s$ in Δ' as

$$0.3 \;\leq\; t-s \;\leq\; 0.67.$$

To be precise (see Remark 3.2), we can rigorously verify the above estimate of the solutions of the above system of equations in a similar way as in Section A.2.

Before calculating $2s+u$, we show that the domain (56) is convex. Since the function $-2\Lambda(t)+\Lambda(t-s)+2\Lambda(s)$ and $\Lambda(u)$ are concave functions whose Hesse matrices are negative definite, the function $-2\Lambda(t)+\Lambda(t-s)+2\Lambda(s)+\Lambda(u)$ is also such a function on $\big\{(t,s,u)\ \big|\ 0.5<t<1,\ 0<s<0.5,\ 0<u<0.5\big\}$. Hence, the domain is a convex domain and its boundary is a smooth closed surface whose Gaussian curvature is positive everywhere (see Lemma B.2).

We calculate the maximal value $(2s+u)_{\max}$ of $2s+u$. We consider the plane $2s+u=c$ for a constant c. The maximal value $(2s+u)_{\max}$ is obtained when this plane is tangent to the domain (56); we note that there are exactly two such tangent points. Putting $2w'=2s+u$, such tangent points are given by the following equations,

$$\begin{cases} -2\,\Lambda(t)+\Lambda\big(t-w'+\frac{1}{2}u\big)+2\,\Lambda\big(w'-\frac{1}{2}u\big)+\Lambda(u) \ = \ \varsigma_R\,, \\ \dfrac{\partial}{\partial t}\big(-2\,\Lambda(t)+\Lambda\big(t-w'+\dfrac{1}{2}u\big)+2\,\Lambda\big(w'-\dfrac{1}{2}u\big)+\Lambda(u)\big) \ = \ 0, \\ \dfrac{\partial}{\partial u}\big(-2\,\Lambda(t)+\Lambda\big(t-w'+\dfrac{1}{2}u\big)+2\,\Lambda\big(w'-\dfrac{1}{2}u\big)+\Lambda(u)\big) \ = \ 0. \end{cases} \tag{58}$$

We note that this system of equations has exactly two solutions. By calculating a solution of them by Newton's method from $(t,w',u)=(0.75,0.5,0.25)$, we obtain $(2s+u)_{\max}=0.958506...$. Therefore, we obtain an estimate of $2s+u$ in Δ' as

$$2s+u \ \le \ 0.97.$$

Remark 4.1. To be precise, the above argument is not partially rigorous, since we do not estimate the error term of the numerical solution of Newton's method, though the above argument is practically useful, since we can guess that such an error term would be sufficiently small for the above purpose. In order to complete the above argument, we give a rigorous proof of an estimate of a solution of (58) in Lemma A.3.

4.3 Calculation of the critical value

In this section, we calculate the concrete values of a critical point and the Hesse matrix of $\hat{V}$.

The differentials of $\hat{V}$ are presented by

$$\frac{\partial}{\partial t}\hat{V}(t,s,u) \ = \ 2\,\log(1-x)-\log\big(1-\frac{y}{x}\big)-2\pi\sqrt{-1}\,\big(t-\frac{1}{2}\big), \tag{59}$$

$$\frac{\partial}{\partial s}\hat{V}(t,s,u) \ = \ -2\,\log(1-y)+\log\big(1-\frac{y}{x}\big)+2\pi\sqrt{-1}\,\big(2s+u-1\big), \tag{60}$$

$$\frac{\partial}{\partial u}\hat{V}(t,s,u) \ = \ -\log(1-z)+2\pi\sqrt{-1}\,\big(s+u-\frac{1}{2}\big), \tag{61}$$

where $x=e^{2\pi\sqrt{-1}\,t}$, $y=e^{2\pi\sqrt{-1}\,s}$ and $z=e^{2\pi\sqrt{-1}\,u}$.

Lemma 4.2. *$\hat{V}$ has a unique critical point (t_0,s_0,u_0) in $P^{-1}(\Delta')$, where $P:\mathbb{C}^3\to\mathbb{R}^3$ is the projection to the real parts of the entries.*

Proof. Any critical point of $\hat{V}$ is given by a solution of $\frac{\partial}{\partial t}\hat{V} = \frac{\partial}{\partial s}\hat{V} = \frac{\partial}{\partial u}\hat{V} = 0$, and these equations are rewritten,

$$(1-x)^2 \;=\; -x\big(1-\frac{y}{x}\big), \qquad (1-y)^2 \;=\; y^2\, z\big(1-\frac{y}{x}\big), \qquad 1-z \;=\; -y\, z.$$

By using the first and third formula, we remove y and z from the second formula, to obtain

$$x^5 - 2x^4 + 2x^3 - 3x^2 + 2x - 1 \;=\; 0.$$

Its solutions are given by

$$x \;=\; -0.232705... \pm \sqrt{-1}\cdot 1.09381... \;, \quad 0.438694... \pm \sqrt{-1}\cdot 0.557752... \;, \quad 1.58802... \;.$$

Since $x = e^{2\pi\sqrt{-1}\, t}$,

$$\begin{aligned} t \;=\; & 0.283362... - \sqrt{-1}\cdot 0.0177936... \;, \quad 0.716638... - \sqrt{-1}\cdot 0.0177936... \;, \\ & 0.143927... + \sqrt{-1}\cdot 0.0545975... \;, \quad 0.856073... + \sqrt{-1}\cdot 0.0545975... \;, \\ & -\sqrt{-1}\cdot 0.0736072... \;. \end{aligned}$$

Among these, the second and fourth solutions are in the range of t in Δ'. Further, as for the fourth solution, $s = 0.0243944... + \sqrt{-1}\cdot 0.127823...$, and this is not in Δ'. From the second solution, we have that

$$\begin{aligned} x_0 &= -0.232705... - \sqrt{-1}\cdot 1.09381... \;, & t_0 &= 0.716638... - \sqrt{-1}\cdot 0.0177936... \;, \\ y_0 &= 0.0904327... + \sqrt{-1}\cdot 1.60288... \;, & s_0 &= 0.24103... - \sqrt{-1}\cdot 0.0753425... \;, \\ z_0 &= 0.267792... + \sqrt{-1}\cdot 0.471915... \;, & u_0 &= 0.167853... + \sqrt{-1}\cdot 0.0973042... \;, \end{aligned}$$

where $x_0 = e^{2\pi\sqrt{-1}\, t_0}$, $y_0 = e^{2\pi\sqrt{-1}\, s_0}$ and $z_0 = e^{2\pi\sqrt{-1}\, u_0}$. These give a unique critical point in $P^{-1}(\Delta')$. □

The critical value of $\hat{V}$ at the critical point of Lemma 4.2 is presented by

$$\begin{aligned} \varsigma \;&=\; \hat{V}(t_0, s_0, u_0) \\ &=\; \frac{1}{2\pi\sqrt{-1}}\Big(-2\,\mathrm{Li}_2(x_0) - \mathrm{Li}_2\big(\frac{y_0}{x_0}\big) + 2\,\mathrm{Li}_2(y_0) + \mathrm{Li}_2(z_0) + \frac{\pi^2}{3}\Big) \\ &\quad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(-t_0^2 + s_0^2 + (s_0+u_0)^2 + t_0 - s_0 - (s_0+u_0) + \frac{1}{6}\big) \\ &=\; 0.700414... - \sqrt{-1}\cdot 0.934648... \;. \end{aligned} \tag{62}$$

Further, we put its real part to be ς_R ,

$$\varsigma_R \;=\; \mathrm{Re}\,\varsigma \;=\; 0.700414... \;. \tag{63}$$

We calculate the Hesse matrix of $\hat{V}$. Since $x = e^{2\pi\sqrt{-1}\, t}$, $\frac{d}{dt} = 2\pi\sqrt{-1}\, x\frac{d}{dx}$. Hence, from (59), we have that

$$\frac{\partial^2}{\partial t^2}\hat{V} \;=\; 2\pi\sqrt{-1}\, x\frac{\partial}{\partial x}\Big(2\log(1-x) - \log\big(1-\frac{y}{x}\big)\Big) - 2\pi\sqrt{-1} \;=\; 2\pi\sqrt{-1}\,\Big(-\frac{1+x}{1-x} - \frac{\frac{y}{x}}{1-\frac{y}{x}}\Big).$$

By calculating other entries similarly, the Hesse matrix of $\hat{V}$ at (t_0, s_0, u_0) is presented by

$$H_0 \;=\; 2\pi\sqrt{-1}\begin{pmatrix} -\dfrac{1+x_0}{1-x_0} - \dfrac{\frac{y_0}{x_0}}{1-\frac{y_0}{x_0}} & \dfrac{\frac{y_0}{x_0}}{1-\frac{y_0}{x_0}} & 0 \\ \dfrac{\frac{y_0}{x_0}}{1-\frac{y_0}{x_0}} & \dfrac{2}{1-y_0} - \dfrac{\frac{y_0}{x_0}}{1-\frac{y_0}{x_0}} & 1 \\ 0 & 1 & \dfrac{1}{1-z_0} \end{pmatrix}. \tag{64}$$

4.4 Verifying the assumption of the Poisson summation formula

In this section, we verify the assumption of the Poisson summation formula in Lemma 4.3, which is used in the proof of Theorem 1.1 for the 6_2 knot in Section 4.1. As we mentioned in Remark 2.5, we verify the assumption for $\hat{V}(t,s,u)$ instead of $V(t,s,u)$, since $V(t,s,u)$ converges uniformly to $\hat{V}(t,s,u)$ on Δ' in the form mentioned in Remark 2.5.

By computer calculation, we can see that the maximal value of $\mathrm{Re}\,\hat{V} - \varsigma_R$ is about 0.06. Therefore, in the proof of Lemma 4.3, it is sufficient to decrease, say, $\mathrm{Re}\,\hat{V}(t,\ s,\ u+\delta\sqrt{-1}) - 2\pi\delta$ by 0.06, by moving δ (though we do not use this value in the proof of the lemma).

Lemma 4.3. $\hat{V}(t,s,u) - \varsigma_R$ *satisfies the assumption of Proposition 2.3.*

Proof. We show that $\partial\Delta'$ is null-homotopic in each of (15)–(20).

As for (15) and (16), similarly as the proof of Lemma 3.4, it is sufficient to show that

$$-(2\pi - \varepsilon') \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t+\delta\sqrt{-1},\ s,\ u)\Big) \;<\; 2\pi - \varepsilon' \tag{65}$$

for some $\varepsilon' > 0$. The middle term is calculated as

$$\mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t+\delta\sqrt{-1},\ s,\ u)\Big) \;=\; -2\,\mathrm{Arg}\,(1-x) + \mathrm{Arg}\,\big(1-\frac{y}{x}\big) + 2\pi\big(t-\frac{1}{2}\big),$$

where $x = e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$ and $y/x = e^{-2\pi\sqrt{-1}\,(t-s)}e^{2\pi(X-Y)}$. Since $0.6 \le t \le 0.9$,

$$0 \;<\; \mathrm{Arg}\,(1-x) \;<\; 2\pi\big(t-\frac{1}{2}\big).$$

Hence,

$$-2\pi\big(t-\frac{1}{2}\big) \;<\; -2\,\mathrm{Arg}\,(1-x) + 2\pi\big(t-\frac{1}{2}\big) \;<\; 2\pi\big(t-\frac{1}{2}\big).$$

Further,

$$\min\big\{0,\ -2\pi\big(t-s-\frac{1}{2}\big)\big\} \;<\; \mathrm{Arg}\,\big(1-\frac{y}{x}\big) \;<\; \max\big\{2\pi\big(\frac{1}{2}-t+s\big),\ 0\big\}.$$

Therefore,

$$\min\big\{-2\pi\big(t-\frac{1}{2}\big),\ -2\pi\big(2t-s-1\big)\big\} \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t+\delta\sqrt{-1},\ s,\ u)\Big) \;<\; \max\big\{2\pi\cdot s,\ 2\pi\big(t-\frac{1}{2}\big)\big\}.$$

Since $t \le 0.9,\ t + (t-s) \le 0.9 + 0.67 = 1.57$ and $s \le 0.4$,

$$-2\pi \cdot 0.6 \ < \ \mathrm{Re}\Big(\frac{\partial}{\partial \delta} \hat{V}(t+\delta\sqrt{-1},\, s,\, u) \Big) \ < \ 2\pi \cdot 0.4,$$

and hence, (65) is satisfied.

As for (17) and (18), similarly as above, it is sufficient to show that

$$-(2\pi - \varepsilon') \ < \ \mathrm{Re}\Big(\frac{\partial}{\partial \delta} \hat{V}(t,\, s+\delta\sqrt{-1},\, u) \Big) \ < \ 2\pi - \varepsilon' \tag{66}$$

for some $\varepsilon' > 0$. The middle term is calculated as

$$\mathrm{Re}\Big(\frac{\partial}{\partial \delta} \hat{V}(t,\, s+\delta\sqrt{-1},\, u) \Big) \ = \ 2\,\mathrm{Arg}\,(1-y) - \mathrm{Arg}\,\big(1 - \frac{y}{x}\big) - 2\pi(2s+u-1),$$

where $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$ and $y/x = e^{-2\pi\sqrt{-1}\,(t-s)} e^{2\pi(X-Y)}$. Since $0.14 \le s \le 0.4$,

$$-2\pi\big(\frac{1}{2} - s\big) \ < \ \mathrm{Arg}\,(1-y) \ < \ 0.$$

Hence,

$$-2\pi \cdot u \ < \ 2\,\mathrm{Arg}\,(1-y) - 2\pi(2s+u-1) \ < \ 2\pi(1-2s-u).$$

Since $u \le 0.4$ and $2s+u \ge 0.33$,

$$-2\pi \cdot 0.4 \ < \ 2\,\mathrm{Arg}\,(1-y) - 2\pi(2s+u-1) \ < \ 2\pi \cdot 0.67.$$

Further,

$$\min\big\{0,\ -2\pi\big(t-s-\frac{1}{2}\big)\big\} \ < \ \mathrm{Arg}\,\big(1-\frac{y}{x}\big) \ < \ \max\big\{2\pi\big(\frac{1}{2} - t + s\big),\ 0\big\}.$$

Since $0.3 \le t-s \le 0.67$,

$$-2\pi \cdot 0.17 \ < \ \mathrm{Arg}\,\big(1-\frac{y}{x}\big) \ < \ 2\pi \cdot 0.2.$$

Therefore,

$$-2\pi \cdot 0.57 \ < \ \mathrm{Re}\Big(\frac{\partial}{\partial \delta} \hat{V}(t,\, s+\delta\sqrt{-1},\, u) \Big) \ < \ 2\pi \cdot 0.87,$$

and hence, (66) is satisfied.

As for (19) and (20), similarly as above, it is sufficient to show that

$$-(2\pi - \varepsilon') \ < \ \mathrm{Re}\Big(\frac{\partial}{\partial \delta} \hat{V}(t,\, s,\, u+\delta\sqrt{-1}) \Big) \ < \ 2\pi - \varepsilon' \tag{67}$$

for some $\varepsilon' > 0$. The middle term is calculated as

$$\mathrm{Re}\Big(\frac{\partial}{\partial \delta} \hat{V}(t,\, s,\, u+\delta\sqrt{-1}) \Big) \ = \ \mathrm{Arg}\,(1-z) - 2\pi\big(s+u-\frac{1}{2}\big),$$

where $z = e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})}$. Since $0.05 \le u \le 0.4$,

$$-2\pi\big(\frac{1}{2} - u\big) \;<\; \mathrm{Arg}\,(1-z) \;<\; 0.$$

Therefore,

$$-2\pi \cdot s \;<\; \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat{V}(t,\,s,\,u+\delta\sqrt{-1})\Big) \;<\; 2\pi\big(\frac{1}{2} - s - u\big).$$

Since $s \le 0.4$ and $s+u \ge 0.19$,

$$-2\pi \cdot 0.4 \;<\; \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat{V}(t,\,s,\,u+\delta\sqrt{-1})\Big) \;<\; 2\pi \cdot 0.31,$$

and hence, (67) is satisfied. □

4.5 Verifying the assumption of the saddle point method

In this section, we verify the assumption of the saddle point method in Lemma 4.9. In order to show this lemma, we show Lemmas 4.4–4.8 in advance. As we mentioned in Remark 2.7, we verify the assumption for $\hat{V}(t,s,u)$ instead of $V(t,s,u)$, since $V(t,s,u)$ converges uniformly to $\hat{V}(t,s,u)$ on Δ' in the form mentioned in Remark 2.7.

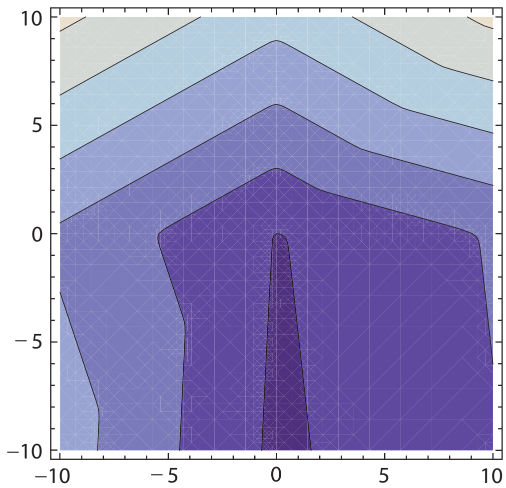


Figure 3: Contour lines of $\mathrm{Re}\,\hat{V}(0.8 + X\sqrt{-1},\, 0.18 + Y\sqrt{-1},\, 0.1) - \varsigma_R$

In the proof of Lemma 4.9, as mentioned at the beginning of Section 4.4, it is sufficient to decrease $\mathrm{Re}\,\hat{V}(t,s,u)$ by 0.06, by pushing t,s,u into the imaginary directions. In order to calculate this concretely, putting

$$f(X,Y,Z) \;=\; \mathrm{Re}\,\hat{V}(t+X\sqrt{-1},\, s+Y\sqrt{-1},\, u+Z\sqrt{-1}) - \varsigma_R$$

as in Section 3.5, we consider the behavior of f at each fiber of the projection $\mathbb{C}^3 \to \mathbb{R}^3$. Then, unlike Section 3.5, there are some $(t,s,u) \in \mathbb{R}^3$ such that, fixing X and Z, f has a maximal point as a function of Y; for example, see Figure 3. We also note that the

Hessian of f is not positive where the contour lines are not convex, though we like to show that the Hesse matrix of f is positive definite at any critical point of f.

Lemma 4.4. *Fixing X and Y, we regard f as a function of Z.*
(1) *If $s+u \geq \frac{1}{2}$, then f is monotonically decreasing.*
(2) *If $s+u < \frac{1}{2}$, then f has a unique minimal point at $Z = g_3(t,s,u)$, where*

$$g_3(t,s,u) \;=\; \frac{1}{2\pi} \log \frac{\sin 2\pi s}{\sin 2\pi\big(\frac{1}{2} - s - u\big)}\,.$$

In particular, this minimal point goes to ∞ as $s+u \to \frac{1}{2} - 0$.

Proof. As a function of Z, the differential of f is presented by

$$\frac{\partial f}{\partial Z} \;=\; \operatorname{Arg}(1-z) - 2\pi\,\big(s+u-\frac{1}{2}\big),$$

where $z = e^{2\pi\sqrt{-1}\,(u+\mathbb{Z}\sqrt{-1})}$. Hence, we can show the lemma in a similar way as the proof of Lemma 3.5. □

When we regard f as a function of Z fixing X and Y, by Lemma 4.4, f has a unique minimal point at $Z = g_3(t,s,u)$; we note that this minimal point does not depend on X and Y. In order to consider the behavior of f, putting

$$\check{f}(X,Y) \;=\; f\big(X,Y,g_3(t,s,u)\big), \tag{68}$$

we consider the behavior of $\check{f}$ in each fiber of the projection $\mathbb{C}^3 \to \mathbb{R}^3$ at $(t,s,u) \in \Delta' \subset \mathbb{R}^3$.

Lemma 4.5. *For each $(t,s,u) \in \Delta'$ satisfying that $t-\frac{1}{2} < s+u < \frac{1}{2}$, $\check{f}$ has a unique minimal point and no other critical points.*

Proof. The differentials of $\check{f}$ are presented by

$$\begin{aligned}
\frac{\partial \check{f}}{\partial X} \;&=\; \frac{\partial f}{\partial X} \;=\; \operatorname{Im}\Big(-2\log(1-x) + \log\big(1-\frac{y}{x}\big)\Big) + 2\pi\big(t-\frac{1}{2}\big),\\
\frac{\partial \check{f}}{\partial Y} \;&=\; \frac{\partial f}{\partial Y} \;=\; \operatorname{Im}\Big(2\log(1-y) - \log\big(1-\frac{y}{x}\big)\Big) - 2\pi\big(2s+u-1\big),
\end{aligned}$$

where $x = e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$ and $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$. Hence, since $\frac{dx}{dX} = -2\pi x$,

$$\begin{aligned}
\frac{\partial^2 \check{f}}{\partial X^2} \;&=\; \operatorname{Im}\Big(\big(\frac{2}{1-x} + \frac{\frac{y}{x^2}}{1-\frac{y}{x}}\big)\,\frac{dx}{dX}\Big) \;=\; 2\pi \operatorname{Im}\Big(-\frac{2x}{1-x} - \frac{\frac{y}{x}}{1-\frac{y}{x}}\Big)\\
&=\; 2\pi\operatorname{Im}\Big(-\frac{2}{1-x} - \frac{1}{1-\frac{y}{x}}\Big),
\end{aligned}$$

where we obtain the last equality by changing the content of the parentheses by a real number, which does not change its imaginary part. Similarly, we have that

$$\frac{\partial^2 \check{f}}{\partial X \partial Y} \;=\; 2\pi \operatorname{Im}\frac{1}{1-\frac{y}{x}}\,, \qquad \frac{\partial^2 \check{f}}{\partial Y^2} \;=\; 2\pi\operatorname{Im}\Big(\frac{2}{1-y} - \frac{1}{1-\frac{y}{x}}\Big).$$

Therefore, the Hesse matrix of $\check{f}$ is presented by

$$2\pi \begin{pmatrix} 2a_1 + b & -b \\ -b & 2a_2 + b \end{pmatrix}, \tag{69}$$

where we put

$$a_1 \;=\; -\operatorname{Im}\frac{1}{1-x}, \quad a_2 \;=\; \operatorname{Im}\frac{1}{1-y}, \quad b \;=\; -\operatorname{Im}\frac{1}{1-\frac{y}{x}}.$$

Since $0.6 \le t \le 0.9$, $\operatorname{Im}(1-x) > 0$, and hence, $a_1 > 0$. Similarly, since $0.14 \le s \le 0.4$, we have that $a_2 > 0$.

When $t - s \le \frac{1}{2}$, we have that $\operatorname{Im}\big(1 - \frac{y}{x}\big) \ge 0$, and hence, $b \ge 0$. Then, we can verify that the trace and the determinant of the Hesse matrix (69) are positive. Hence, the Hesse matrix (69) is positive definite. Therefore, by Lemma 4.7, $\check{f}$ has a unique minimal point and no other critical points, as required.

When $t - s > \frac{1}{2}$, putting $b' = -b$, we have that $b' > 0$. Since any critical point of $\check{f}$ is in the domain (72) by Lemma 4.6 below, it is sufficient (by Lemma 4.7) to show that the Hesse matrix (69) is positive definite in the domain (72). Hence, it is sufficient to show that

$$\big(\text{the trace of (69)}\big) \;=\; 4\pi(a_1 + a_2 - b') \;>\; 0, \tag{70}$$

$$\begin{aligned}\big(\text{the determinant of (69)}\big) \;&=\; 4\pi^2\big((2a_1 - b')(2a_2 - b') - b'^2\big) \\ &=\; 8\pi^2 a_1 a_2 b' \big(\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}\big) \;>\; 0.\end{aligned} \tag{71}$$

We show that (71) $\Rightarrow$ (70), as follows. Suppose that (71) holds. Then, $\frac{1}{b'} > \frac{1}{2}\big(\frac{1}{a_1} + \frac{1}{a_2}\big)$. Since a_1, a_2 and b' are positive, $\frac{1}{b'} > \frac{1}{a_1}$ or $\frac{1}{b'} > \frac{1}{a_2}$. Hence, $b' < a_1$ or $b' < a_2$. Therefore, (70) holds.

We show (71), as follows. Since $x = e^{2\pi\sqrt{-1}\,(t + X\sqrt{-1})}$, a_1 is presented by

$$\begin{aligned} a_1 \;&=\; -\operatorname{Im}\frac{1}{1-x} \;=\; -\operatorname{Im}\frac{1-\overline{x}}{\big|1-x\big|^2} \;=\; -\frac{e^{-2\pi X}\sin 2\pi t}{(1 - e^{2\pi\sqrt{-1}\,t}e^{-2\pi X})(1 - e^{-2\pi\sqrt{-1}\,t}e^{-2\pi X})} \\ &=\; -\frac{\sin 2\pi t}{e^{2\pi X} + e^{-2\pi X} - 2\cos 2\pi t} \;=\; \frac{\sin 2\pi(1-t)}{e^{2\pi X} + e^{-2\pi X} - 2\cos 2\pi(1-t)}.\end{aligned}$$

Hence,

$$\frac{1}{a_1} \;=\; \frac{e^{2\pi X} + e^{-2\pi X} - 2\cos 2\pi(1-t)}{\sin 2\pi(1-t)}.$$

Similarly, we have that

$$\frac{1}{a_2} \;=\; \frac{e^{2\pi Y} + e^{-2\pi Y} - 2\cos 2\pi s}{\sin 2\pi s}, \qquad \frac{1}{b'} \;=\; \frac{e^{2\pi(X-Y)} + e^{2\pi(Y-X)} - 2\cos 2\pi(1-t+s)}{\sin 2\pi(1-t+s)}.$$

Therefore, the differential of $\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}$ with respect to X is given by

$$\frac{1}{2\pi} \cdot \frac{\partial}{\partial X}\Big(\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}\Big) \ = \ 2 \cdot \frac{e^{2\pi(X-Y)} - e^{2\pi(Y-X)}}{\sin 2\pi(1-t+s)} - \frac{e^{2\pi X} - e^{-2\pi X}}{\sin 2\pi(1-t)} \,.$$

Since $X > 0$, $Y < 0$, $0.33 \leq 1-t+s < 0.5$ and $0.1 \leq 1-t \leq 0.4$,

$$\begin{aligned}
&\frac{1}{2\pi} \cdot \frac{\partial}{\partial X}\Big(\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}\Big) \ > \ \big(e^{2\pi X} - e^{-2\pi X}\big)\Big(\frac{2}{\sin 2\pi(1-t+s)} - \frac{1}{\sin 2\pi(1-t)}\Big) \\
&> \ \big(e^{2\pi X} - e^{-2\pi X}\big)\Big(\frac{2}{\sin 2\pi \cdot 0.33} - \max\Big\{\frac{1}{\sin 2\pi \cdot 0.1}\,,\ \frac{1}{\sin 2\pi \cdot 0.4}\Big\}\Big) \\
&> \ \big(e^{2\pi X} - e^{-2\pi X}\big)\Big(\frac{2}{0.876307...} - \max\Big\{\frac{1}{0.587785...}\,,\ \frac{1}{0.587785...}\Big\}\Big) \\
&> \ \big(e^{2\pi X} - e^{-2\pi X}\big) \cdot 0.581004... \ > \ 0.
\end{aligned}$$

Hence, it is sufficient to show the required inequality when $X = 0$. Similarly, since $X > 0$, $Y < 0$, $0.33 \leq 1-t+s < 0.5$ and $0.14 \leq s \leq 0.4$,

$$\begin{aligned}
&-\frac{1}{2\pi} \cdot \frac{\partial}{\partial Y}\Big(\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}\Big) \ = \ 2 \cdot \frac{e^{2\pi(X-Y)} - e^{2\pi(Y-X)}}{\sin 2\pi(1-t+s)} - \frac{e^{-2\pi Y} - e^{2\pi Y}}{\sin 2\pi s} \\
&> \ \big(e^{-2\pi Y} - e^{2\pi Y}\big)\Big(\frac{2}{\sin 2\pi(1-t+s)} - \frac{1}{\sin 2\pi s}\Big) \\
&> \ \big(e^{-2\pi Y} - e^{2\pi Y}\big)\Big(\frac{2}{\sin 2\pi \cdot 0.33} - \max\Big\{\frac{1}{\sin 2\pi \cdot 0.14}\,,\ \frac{1}{\sin 2\pi \cdot 0.4}\Big\}\Big) \\
&> \ \big(e^{-2\pi Y} - e^{2\pi Y}\big)\Big(\frac{2}{0.876307...} - \max\Big\{\frac{1}{0.770513...}\,,\ \frac{1}{0.587785...}\Big\}\Big) \\
&> \ \big(e^{-2\pi Y} - e^{2\pi Y}\big) \cdot 0.581004... \ > \ 0.
\end{aligned}$$

Hence, it is sufficient to show the required inequality when $Y = 0$. When $X = Y = 0$,

$$\frac{1}{2}\Big(\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}\Big) \ = \ 2 \cdot \frac{1 - \cos 2\pi(1-t+s)}{\sin 2\pi(1-t+s)} - \frac{1 - \cos 2\pi(1-t)}{\sin 2\pi(1-t)} - \frac{1 - \cos 2\pi s}{\sin 2\pi s} \,.$$

Further, since $\frac{1-\cos 2\pi\alpha}{\sin 2\pi\alpha} = \frac{2\,\sin^2 \pi\alpha}{2\,\sin \pi\alpha\,\cos \pi\alpha} = \tan \pi\alpha$,

$$\frac{1}{2}\Big(\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}\Big) \ = \ 2\,\tan \pi(1-t+s) - \tan \pi(1-t) - \tan \pi s \ > \ 0.$$

Hence, we obtain (71), as required. □

The following two lemmas are used in the proof of Lemma 4.5.

Lemma 4.6. *Let $\check{f}$ be as defined in* (68)*. If $s < t - \frac{1}{2} < s + u < \frac{1}{2}$, then any critical point of $\check{f}$ is in the following domain,*

$$\big\{(X, Y) \in \mathbb{R}^2 \;\big|\; X > 0, \quad Y < 0\big\}. \tag{72}$$

Proof. Let (X, Y) be a critical point of $\check{f}$.

We show that $X > 0$, as follows. Since $\frac{\partial \check{f}}{\partial X} = 0$,

$$-2\,\mathrm{Arg}\,(1-x) + 2\pi\big(t - \frac{1}{2}\big) \;=\; -\mathrm{Arg}\,\big(1 - \frac{y}{x}\big),$$

where $x = e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$ and $y/x = e^{-2\pi\sqrt{-1}\,(t-s)} e^{2\pi(X-Y)}$. Since $\frac{1}{2} < t - s \le 0.67$, the right-hand side of the above formula is positive. Further, since $0.6 \le t \le 0.9$, the left-hand side is monotonically increasing with respect to X, and is equal to 0 at $X = 0$. Therefore, $X > 0$.

We show that $Y < 0$, as follows. Since $\frac{\partial \check{f}}{\partial Y} = 0$,

$$2\,\mathrm{Arg}\,(1-y) \;=\; \mathrm{Arg}\,\big(1 - \frac{y}{x}\big) + 2\pi\big(2s + u - 1\big),$$

where $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$ and $y/x = e^{-2\pi\sqrt{-1}\,(t-s)} e^{2\pi(X-Y)}$. Since $\mathrm{Arg}\,\big(1 - \frac{y}{x}\big) < 0$ as mentioned above,

$$2\,\mathrm{Arg}\,(1-y) \;<\; 2\pi\big(2s + u - 1\big).$$

Hence,

$$2\,\mathrm{Arg}\,(1-y) + 2\pi\big(\frac{1}{2} - s\big) \;<\; 2\pi\big(s + u - \frac{1}{2}\big).$$

From the assumption of the lemma, the right-hand side of the above formula is negative. Further, since $0.14 \le s \le 0.4$, the left-hand side is monotonically increasing with respect to Y, and is equal to 0 at $Y = 0$. Therefore, $Y < 0$. □

Lemma 4.7. *For each $(t, s, u) \in \Delta'$ satisfying that $t - \frac{1}{2} < s + u$, $\check{f}(X, Y) \to \infty$ as $X^2 + Y^2 \to \infty$.*

Proof. From the definition of $\hat{V}$, we have that

$$\begin{aligned}
&\mathrm{Re}\,\hat{V}\big(t + X\sqrt{-1}, s + Y\sqrt{-1}, u + Z\sqrt{-1}\big) \\
&= \;\mathrm{Re}\;\frac{1}{2\pi\sqrt{-1}}\Big(-2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}) - \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s-t+Y\sqrt{-1}-X\sqrt{-1})}) \\
&\qquad\qquad + 2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})})\Big) \\
&\quad + 2\pi\big(t - \frac{1}{2}\big)X - 2\pi\big(s - \frac{1}{2}\big)Y - 2\pi\big(s + u - \frac{1}{2}\big)(Y + Z).
\end{aligned}$$

Since $\check{f}$ is defined from the above formula by fixing Z, it is sufficient to show that

$$\begin{aligned}
&\mathrm{Re}\;\frac{1}{2\pi\sqrt{-1}}\Big(-2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}) + 2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}) \\
&\quad - \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s-t+Y\sqrt{-1}-X\sqrt{-1})})\Big) \;+ 2\pi\big(t - \frac{1}{2}\big)X + 2\pi\big(1 - 2s - u\big)Y
\end{aligned}$$

goes to ∞ as $X^2+Y^2\to\infty$. Hence, putting

$$F(X,Y) = \left(\begin{cases}\big(t-\frac{1}{2}\big)X & \text{if } X\geq 0\\ -\big(t-\frac{1}{2}\big)X & \text{if } X<0\end{cases}\right) + \left(\begin{cases}(1-2s-u)Y & \text{if } Y\geq 0\\ -uY & \text{if } Y<0\end{cases}\right) + \left(\begin{cases}\big(t-s+\frac{1}{2}\big)(X-Y) & \text{if } X\geq Y\\ 0 & \text{if } X<Y\end{cases}\right), \tag{73}$$

by Lemma 2.2, it is sufficient to show that $F(X,Y)\to\infty$ as $X^2+Y^2\to\infty$.

We show that $F(X,Y)\to\infty$ as $X^2+Y^2\to\infty$, as follows. As for the first summand of (73), since $t>0.5$, this summand goes to ∞ as $|X|\to\infty$. As for the second summand of (73), since $1-2s-u>0$ and $u>0$, this summand goes to ∞ as $|Y|\to\infty$. As for the third summand of (73), since $t-s+\frac{1}{2}>0$, this summand is non-negative. Hence, $F(X,Y)\to\infty$ as $X^2+Y^2\to\infty$, as required. □

Lemma 4.8. *Suppose that* $t-\frac{1}{2}\geq s+u$, *and consider the flow from* $(X,Y)=(0,0)$ *along the vector field* $\big(-\frac{\partial\check{f}}{\partial X},-\frac{\partial\check{f}}{\partial Y}\big)$. *Then,* $Y\to-\infty$.

Proof. The differential of $\check{f}$ with respect to X is presented by

$$\frac{\partial\check{f}}{\partial X} = \frac{\partial f}{\partial X} = -2\,\mathrm{Arg}\,(1-x)+\mathrm{Arg}\,\big(1-\frac{y}{x}\big)+2\pi\big(t-\frac{1}{2}\big),$$

where $x=e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$ and $\frac{y}{x}=e^{-2\pi\sqrt{-1}\,(t-s)}e^{2\pi(X-Y)}$. In particular, when $X=0$, noting that $\mathrm{Arg}\,(1-x)=\pi\big(t-\frac{1}{2}\big)$,

$$\frac{\partial\check{f}}{\partial X}\Big|_{X=0} = \mathrm{Arg}\,\big(1-\frac{y}{x}\big) < 0,$$

since $\frac{1}{2}<t-s\leq 0.7$. Hence, the flow of the lemma goes in the domain $\{X>0\}$. We suppose that $X>0$ in the following of this proof.

The differential of $\check{f}$ with respect to Y is presented by

$$\frac{\partial\check{f}}{\partial Y} = \frac{\partial f}{\partial Y} = 2\,\mathrm{Arg}\,(1-y)-\mathrm{Arg}\,\big(1-\frac{y}{x}\big)-2\pi(2s+u-1),$$

where $y=e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$ and $\frac{y}{x}=e^{-2\pi\sqrt{-1}\,(t-s)}e^{2\pi(X-Y)}$. Hence,

$$\frac{\partial\check{f}}{\partial Y}\Big|_{Y\to-\infty} = -2\cdot 2\pi\big(\frac{1}{2}-s\big)+2\pi\big(t-s-\frac{1}{2}\big)-2\pi(2s+u-1) = 2\pi\big(t-s-u-\frac{1}{2}\big) \geq 0.$$

Further, when $Y\leq 0$, similarly as in the proof of Lemma 4.5, we have that

$$\frac{1}{2\pi}\cdot\frac{\partial^2\check{f}}{\partial Y^2} = \mathrm{Im}\Big(\frac{2}{1-y}-\frac{1}{1-\frac{y}{x}}\Big)$$
$$= \frac{2\sin 2\pi s}{e^{2\pi Y}+e^{-2\pi Y}-\cos 2\pi s}-\frac{\sin 2\pi(1-t+s)}{e^{2\pi(X-Y)}+e^{2\pi(Y-X)}-\cos 2\pi(1-t+s)}$$

$$> \frac{2\sin 2\pi s - \sin 2\pi(1-t+s)}{e^{2\pi Y} + e^{-2\pi Y} - \cos 2\pi s},$$

where we obtain the last inequality since $0.14 \le s \le 1-t+s < \frac{1}{2}$. Further, since

$$\begin{aligned} \sin 2\pi s \ &\ge\ \max\bigl\{ \sin 2\pi \cdot 0.14,\ \sin 2\pi(1-t+s)\bigr\} \\ &=\ \max\bigl\{0.770513... \ ,\ \sin 2\pi(1-t+s)\bigr\}\ >\ \frac{1}{2}\sin 2\pi(1-t+s), \end{aligned}$$

we have that $\frac{\partial^2 \check{f}}{\partial Y^2} > 0$. Hence, $\frac{\partial \check{f}}{\partial Y}$ is monotonically increasing. Therefore, since $\frac{\partial \check{f}}{\partial Y}\Big|_{Y\to-\infty} \ge 0$ as shown above, we have that $\frac{\partial \check{f}}{\partial Y} \ge 0$. Hence, by the flow of the lemma, $Y \to -\infty$. $\square$

Lemma 4.9. *When we apply Proposition 2.6 to* (55)*, the assumption of Proposition 2.6 holds.*

Proof. We show that there exists a homotopy Δ'_δ $(0 \le \delta \le 2)$ between $\Delta'_0 = \Delta'$ and Δ'_2 such that

$$(t_0, s_0, u_0) \in \Delta'_2, \tag{74}$$

$$\Delta'_2 - \{(t_0, s_0, u_0)\}\ \subset\ \bigl\{(t,s,u) \in \mathbb{C}^3 \bigm| \operatorname{Re}\hat{V}(t,s,u) < \varsigma_R\bigr\}, \tag{75}$$

$$\partial\Delta'_\delta\ \subset\ \bigl\{(t,s,u) \in \mathbb{C}^3 \bigm| \operatorname{Re}\hat{V}(t,s,u) < \varsigma_R\bigr\}. \tag{76}$$

For a sufficiently large $R > 0$, we put

$$\hat{g}_3(t,s,u)\ =\ \begin{cases} \min\{R,\ g_3(t,s,u)\} & \text{if } s+u < \frac{1}{2}, \\ R & \text{if } s+u \ge \frac{1}{2}, \end{cases}$$

where $g_3(t,s,u)$ is given in Lemma 4.4. We note that, since $g_3(t,s,u) \to \infty$ as $s+u \to \frac{1}{2}-0$, $\hat{g}_3(t,s,u)$ is continuous. We set the middle part Δ'_1 of the homotopy by

$$\Delta'_1\ =\ \bigl\{\bigl(t,\ s,\ u + \hat{g}_3(t,s,u)\sqrt{-1}\bigr) \in \mathbb{C}^3 \bigm| (t,s,u) \in \Delta'\bigr\}.$$

Further, we define the internal part Δ'_δ $(0 < \delta < 1)$ of the homotopy by setting it along the flow from (t,s,u) determined by the vector field $\bigl(0, 0, -\frac{\partial f}{\partial Z}\bigr)$. By Lemma 4.4, such a flow goes to Δ'_1. We note that, when $s+u \ge \frac{1}{2}$, $\operatorname{Re}\hat{V}$ is sufficiently small on Δ'_1. It is a problem how to move Δ'_1 further to make $\operatorname{Re}\hat{V}$ smaller, when $s+u < \frac{1}{2}$.

We consider the behavior of the flow from each point of Δ'_1 determined by the vector field $\bigl(-\frac{\partial \check{f}}{\partial X}, -\frac{\partial \check{f}}{\partial Y}, 0\bigr)$. When $t - \frac{1}{2} < s+u$, by Lemma 4.5, $\check{f}$ has a unique minimal point, and the flow goes to it; we put this minimal point to be $\bigl(g_1(t,s,u), g_2(t,s,u)\bigr)$. When $t - \frac{1}{2} \ge s+u$, by Lemma 4.8, Y goes to $-\infty$ by the flow. We put

$$\hat{g}_2(t,s,u)\ =\ \begin{cases} \max\{-R,\ g_2(t,s,u)\} & \text{if } t-\frac{1}{2} < s+u, \\ -R & \text{if } t-\frac{1}{2} \ge s+u. \end{cases}$$

We note that, since $g_2(t,s,u) \to -\infty$ as $t-s-u \to \frac{1}{2}-0$, $\hat{g}_2(t,s,u)$ is continuous. We put $\hat{g}_1(t,s,u)$ so that $\big(\hat{g}_1(t,s,u), \hat{g}_2(t,s,u)\big)$ is on the flow. We set the ending of the homotopy by

$$\Delta'_2 \;=\; \big\{ \big(t+\hat{g}_1(t,s,u)\sqrt{-1},\; s+\hat{g}_2(t,s,u)\sqrt{-1},\; u+\hat{g}_3(t,s,u)\sqrt{-1}\big) \in \mathbb{C}^3 \;\big|\; (t,s,u) \in \Delta' \big\}.$$

Further, we define the internal part Δ'_δ $(1 < \delta < 2)$ of the homotopy by setting it along the flow from $\big(t,\, s,\, u+\hat{g}_3(t,s,u)\sqrt{-1}\big)$ determined by the vector field $\big(-\frac{\partial \check{f}}{\partial X}, -\frac{\partial \check{f}}{\partial Y}, 0\big)$.

We show (76), as follows. From the definition of Δ',

$$\partial \Delta' \;\subset\; \big\{ (t,s,u) \in \mathbb{C}^3 \;\big|\; \operatorname{Re} \hat{V}(t,s,u) < \varsigma_{R} \big\}.$$

Further, by the construction of the homotopy, $\operatorname{Re}\hat{V}$ monotonically decreases by the homotopy. Hence, (76) holds.

We show (74) and (75), as follows. Consider the following functions

$$\begin{aligned} F(t,s,u,X,Y,Z) &\;=\; \operatorname{Re}\hat{V}\big(t+X\sqrt{-1},\, s+Y\sqrt{-1},\, u+Z\sqrt{-1}\big), \\ h(t,s,u) &\;=\; F\big(t,\, s,\, u,\, \hat{g}_1(t,s,u),\, \hat{g}_2(t,s,u),\, \hat{g}_3(t,s,u)\big). \end{aligned}$$

When $t-\frac{1}{2} \geq s+u$ or $s+u \geq \frac{1}{2}$, $-h(t,s,u)$ is sufficiently large (because we let R be sufficiently large), and (75) holds in this case. When $t-\frac{1}{2} < s+u < \frac{1}{2}$, similarly as in the proof of Lemma 3.9, we can show that, if (t,s,u) is a critical point of $h(t,s,u)$, $\big(t+g_1(t,s,u)\sqrt{-1},\, s+g_2(t,s,u)\sqrt{-1},\, u+g_3(t,s,u)\sqrt{-1}\big)$ is a critical point of $\hat{V}$. Hence, by Lemma 4.2, $h(t,s,u)$ has a unique maximal point at $(t,s,u) = (\operatorname{Re} t_0, \operatorname{Re} s_0, \operatorname{Re} u_0)$. Therefore, (74) and (75) hold. □

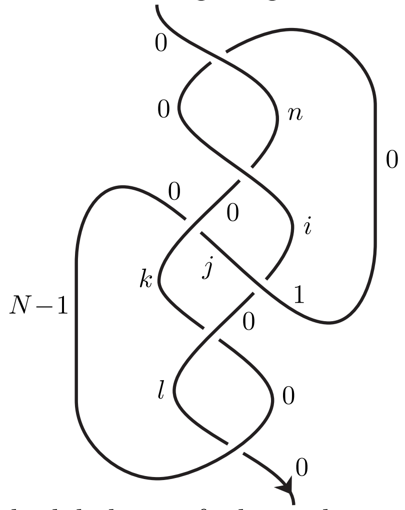
5 The 6_3 knot

In this section, we show Theorem 1.1 for the 6_3 knot. We give a proof of the theorem in Section 5.1, using lemmas shown in Section 5.2–5.5.

5.1 Proof of Theorem 1.1 for the 6_3 knot

In this section, we show a proof of Theorem 1.1 for the 6_3 knot.

The 6_3 *knot* is the closure of the following tangle.



As shown in [33], we can put the labelings of edges adjacent to the unbounded regions as shown above. Hence, from the definition of the Kashaev invariant, the Kashaev invariant of the 6_3 knot is presented by

$$\begin{aligned}\langle 6_3 \rangle_N &= \sum \frac{N\, q^{\frac{1}{2}-n}}{(q)_{N-n}(\overline{q})_{n-1}} \times \frac{N\, q^{\frac{1}{2}+n-i}}{(\overline{q})_{N-n}(q)_{n-i}(\overline{q})_{i-1}} \times \frac{N\, q^{-\frac{1}{2}-k}}{(\overline{q})_{N-j}(q)_{j-k-1}(\overline{q})_{k}} \\ &\qquad \times \frac{N\, q^{\frac{1}{2}+i-1}}{(\overline{q})_{j-i}(q)_{i-1}(q)_{N-j}} \times \frac{N\, q^{-\frac{1}{2}+k-l}}{(q)_{k}(q)_{N-l-1}(\overline{q})_{l-k}} \times \frac{N\, q^{-\frac{1}{2}+l+1}}{(q)_{l}(\overline{q})_{N-l-1}} \\ &= \sum_{\substack{0<i\leq j\leq N \\ 0\leq k<j}} \frac{N^4}{(q)_{i-1}(\overline{q})_{i-1}(\overline{q})_{j-i}(q)_{N-j}(\overline{q})_{N-j}(q)_{j-k-1}(q)_{k}(\overline{q})_{k}} \\ &= \sum_{\substack{0\leq i\leq j<N \\ 0\leq k\leq j}} \frac{N^4}{(q)_{i}(\overline{q})_{i}(\overline{q})_{j-i}(q)_{N-j-1}(\overline{q})_{N-j-1}(q)_{j-k}(q)_{k}(\overline{q})_{k}} \end{aligned} \tag{77}$$

where the second equality is obtained by (4) and (5), and the last equality is obtained by replacing i with $i+1$ and replacing j with $j+1$.

Proof of Theorem 1.1 for the 6_3 *knot.* By (6), the above presentation of $\langle 6_3 \rangle_N$ is rewritten

$$\langle 6_3 \rangle_N = N^4 \sum_{\substack{0\leq i\leq j<N \\ 0\leq k\leq j}} \exp\Big(N\,\tilde{V}\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}\big)\Big),$$

where we put

$$\begin{aligned}\tilde{V}(t,s,u) &= \frac{1}{N}\Big(\varphi(t) - \varphi(1-t) - \varphi\big(1-s+t-\frac{1}{2N}\big) - \varphi(s) + \varphi(1-s) \\ &\qquad + \varphi\big(s-u+\frac{1}{2N}\big) + \varphi(u) - \varphi(1-u) - 4\,\varphi\big(\frac{1}{2N}\big) + 4\,\varphi\big(1-\frac{1}{2N}\big)\Big)\end{aligned}$$

$$= \frac{1}{N}\Big(2\,\varphi(t) - \varphi\big(1-s+t-\frac{1}{2N}\big) - 2\,\varphi(s) + \varphi\big(s-u+\frac{1}{2N}\big) + 2\,\varphi(u)\Big)$$
$$+ 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t^2-s^2+u^2-t+s-u+\frac{1}{6}\big) - \frac{4}{N}\log N.$$

Here, we obtain the last equality by (10) and (11). Hence, by putting

$$\begin{aligned} V(t,s,u) &= \tilde V(t,s,u) + \frac{4}{N}\log N \\ &= \frac{1}{N}\Big(2\,\varphi(t) - \varphi\big(1-s+t-\frac{1}{2N}\big) - 2\,\varphi(s) + \varphi\big(s-u+\frac{1}{2N}\big) + 2\,\varphi(u)\Big) \\ &\quad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t^2-s^2+u^2-t+s-u+\frac{1}{6}\big), \end{aligned}$$

the presentation of $\langle\, 6_3 \,\rangle_{N}$ is rewritten

$$\begin{aligned} \langle\, 6_3 \,\rangle_{N} &= \sum_{\substack{0\le i\le j<N \\ 0\le k\le j}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big)\Big) \\ &= \sum_{\substack{i,j,k\,\in\mathbb{Z} \\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big)\Big), \end{aligned}$$

where the range Δ of $(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})$ of the above sum is given by the following domain,

$$\Delta \;=\; \big\{(t,s,u)\in\mathbb{R}^3 \;\big|\; 0\le t\le s\le 1, \quad 0\le u\le s\big\}.$$

By Proposition 2.1, as $N\to\infty$, $V(t,s,u)$ converges to the following $\hat V(t,s,u)$ in the interior of Δ,

$$\begin{aligned} \hat V(t,s,u) &= \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t}) - \mathrm{Li}_2(e^{-2\pi\sqrt{-1}\,(s-t)}) - 2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,s}) \\ &\quad + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s-u)}) + 2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,u})\Big) + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t^2-s^2+u^2-t+s-u+\frac{1}{6}\big). \end{aligned}$$

By concrete computation, we can check that the boundary of Δ is included in the domain

$$\big\{(t,s,u)\in\Delta \;\big|\; \mathrm{Re}\,\hat V(t,s,u) \;<\; \varsigma-\varepsilon\big\} \tag{78}$$

for some sufficiently small $\varepsilon>0$, where ς is given by (86), noting that ς is real unlike the cases of the 6_1 and 6_2 knots. Hence, similarly as in Section 3.1, we choose a new domain Δ', which satisfies that $\Delta-\Delta'\subset(78)$, as

$$\Delta' \;=\; \left\{(t,s,u)\in\Delta \;\middle|\; \begin{array}{l} 0.18\le t\le 0.3, \quad 0.54\le s\le 0.7, \quad 0.18\le u\le 0.3 \\ 0.3\le s-t\le 0.47, \quad 0.3\le s-u\le 0.47 \end{array}\right\}, \tag{79}$$

where we calculate the concrete values of the bounds of these inequalities in Section 5.2. Then, similarly as in Section 3.1, we can restrict Δ to Δ' as

$$\langle\, 6_3 \,\rangle_{N} \;=\; e^{N\varsigma}\Bigg(\sum_{\substack{i,j,k\,\in\mathbb{Z} \\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta'}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big) - N\varsigma\Big) \;+O(e^{-N\varepsilon})\Bigg), \tag{80}$$

for some $\varepsilon > 0$.

Further, by Proposition 2.3 (Poisson summation formula, see also Remark 2.5), the above sum is presented by

$$\langle 6_3 \rangle_N \;=\; e^{N\varsigma}\bigg(N^3\int_{\Delta'}\exp\big(N\cdot V(t,s,u)-N\varsigma\big)\,dt\,ds\,du\;+O(e^{-N\varepsilon})\bigg), \tag{81}$$

noting that we verify the assumption of Proposition 2.3 in Lemma 5.2. Furthermore, by Proposition 2.6 (saddle point method, see also Remark 2.7), there exist some κ_i''s such that

$$\langle 6_3 \rangle_N \;=\; N^3\exp\big(N\cdot V(t_c,s_c,u_c)\big)\cdot\frac{(2\pi)^{3/2}}{N^{3/2}}\big(\det(-H)\big)^{-1/2}\Big(1+\sum_{i=1}^{d}\kappa_i'\hbar^i+O(\hbar^{d+1})\Big),$$

for any $d > 0$, noting that we verify the assumption of Proposition 2.6 in Lemma 5.5. Here, (t_c,s_c,u_c) is the critical point of V which corresponds to the critical point (t_0,s_0,u_0) of $\hat{V}$ of Lemma 5.1, and H is the Hesse matrix of V at (t_c,s_c,u_c).

We calculate the right-hand side of the above formula. Similarly as in Section 3.1, we have that $V(t_c,s_c,u_c) = V(t_0,s_0,u_0)+O(\hbar^2)$. Further, similarly as in Section 4.1, we have that

$$\begin{aligned}\varphi\big(1-s_0+t_0-\frac{1}{2N}\big) \;&=\; \varphi\big(1-s_0+t_0\big)+\frac{1}{2}\log\big(1-\frac{x_0}{y_0}\big)+O(\hbar^2),\\ \varphi\big(s_0-u_0+\frac{1}{2N}\big) \;&=\; \varphi\big(s_0-u_0\big)-\frac{1}{2}\log\big(1-\frac{y_0}{z_0}\big)+O(\hbar^2),\end{aligned}$$

where we put $x_0 = e^{2\pi\sqrt{-1}\,t_0}$, $y_0 = e^{2\pi\sqrt{-1}\,s_0}$ and $z_0 = e^{2\pi\sqrt{-1}\,u_0}$. Hence, by comparing $V(t_0,s_0,u_0)$ and $\hat{V}(t_0,s_0,u_0)=\varsigma$, we have that

$$V(t_0,s_0,u_0) \;=\; \varsigma-\frac{1}{2N}\log\big(1-\frac{x_0}{y_0}\big)-\frac{1}{2N}\log\big(1-\frac{y_0}{z_0}\big)+O(\hbar^2).$$

Therefore, there exist some κ_i's such that

$$\langle 6_3 \rangle_N \;=\; e^{N\varsigma}N^{3/2}\,\omega\cdot\Big(1+\sum_{i=1}^{d}\kappa_i\hbar^i+O(\hbar^{d+1})\Big)$$

for any $d > 0$. Here, we put

$$\omega \;=\; (2\pi)^{3/2}\big(1-\frac{x_0}{y_0}\big)^{-1/2}\big(1-\frac{y_0}{z_0}\big)^{-1/2}\big(\det(-H_0)\big)^{-1/2} \;=\; 0.416927...$$

where H_0 is the Hesse matrix of $\hat{V}$ at (t_0,s_0,u_0) whose concrete presentation is given in (88); see also [22] for a relation of this value and the twisted Reidemeister torsion. Hence, we obtain the theorem for the 6_3 knot. □

5.2 Estimate of the range of Δ'

In this section, we calculate the concrete values of the bounds of the inequalities in (79) so that they satisfy that $\Delta - \Delta' \subset$ (78).

Putting Λ as in Section 2.2, we have that

$$\mathrm{Re}\,\hat{V}(t,s,u) \;=\; 2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) + 2\,\Lambda(u).$$

We consider the domain

$$\big\{(t,s,u) \in \Delta \;\big|\; 2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) + 2\,\Lambda(u) \;\geq\; \varsigma\big\}, \tag{82}$$

where we put $\varsigma = 0.906072...$ as in (86). We note that this domain is symmetric with respect to the exchange of t and u. The aim of this section is to show that this domain is included in the interior of the domain Δ' of (79). We graphically show the set of (t,s) satisfying that $\mathrm{Re}\,\hat{V}(t,s,u) \geq \varsigma$ for some u in Figure 4.

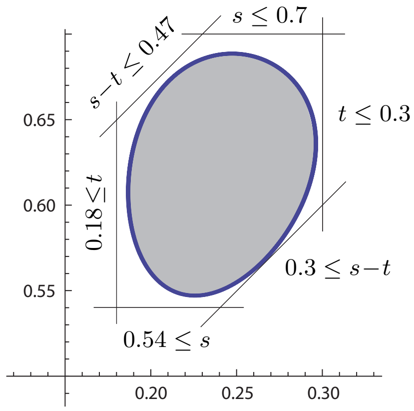


Figure 4: The set of (t,s) satisfying that $\mathrm{Re}\,\hat{V}(t,s,u) \geq \varsigma$ for some u

We can show (see Lemma C.1) that the domain (82) is a compact convex domain and its boundary is a smooth closed surface whose Gaussian curvature is positive everywhere.

We calculate the minimal value $t_{\min}$ and the maximal value $t_{\max}$ of t. We consider the plane $t = c$ for a constant c. The range of t is given as the range of c such that this plane and the domain (82) has non-empty intersection. Since the domain (82) is a compact convex domain whose boundary is a smooth closed surface, the maximal and minimal values are given by the planes tangent to this domain. The tangent points of such planes are given by solutions of the following equations,

$$\begin{cases} 2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) + 2\,\Lambda(u) \;=\; \varsigma, \\ \dfrac{\partial}{\partial s}\big(2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) + 2\,\Lambda(u)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) + 2\,\Lambda(u)\big) \;=\; 0. \end{cases}$$

Since the boundary of the domain (82) is a smooth closed surface whose Gaussian curvature is positive everywhere, there are exactly two such tangent points, and the above system of equations has exactly two solutions, corresponding to the minimal and maximal values of t. By calculating a solution by Newton's method from $(t, s, u) = (0.2,\ 0.6,\ 0.2)$, we obtain $t_{\min} = 0.186629...$, and from $(t, s, u) = (0.3,\ 0.6,\ 0.3)$, we obtain $t_{\max} = 0.296109...$. Therefore, we obtain an estimate of t in Δ' as

$$0.18 \ \le\ t\ \le\ 0.3.$$

To be precise (see Remark 4.1), we can rigorously verify the above estimate of the solutions of the above system of equations in a similar way as in Section A.3.

We calculate the minimal value $s_{\min}$ and the maximal value $s_{\max}$ of s. They satisfy the following equations,

$$\begin{cases} 2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) + 2\,\Lambda(u) \ =\ \varsigma, \\ \dfrac{\partial}{\partial t}\big(2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) + 2\,\Lambda(u)\big) \ =\ 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) + 2\,\Lambda(u)\big) \ =\ 0. \end{cases}$$

They are rewritten

$$\begin{cases} 2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) + 2\,\Lambda(u) \ =\ \varsigma, \\ 2\,\Lambda'(t) \ =\ \Lambda'(s-t), \\ 2\,\Lambda'(u) \ =\ \Lambda'(s-u). \end{cases}$$

We note that this system of equations has exactly two solutions. Since the above system of equations is symmetric with respect to the exchange of t and u, the solutions of the form $(t, s_{\min}, u)$ and $(t, s_{\max}, u)$ satisfy that $t = u$. Hence, the above system of equations is rewritten

$$\begin{cases} 2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s) \ =\ \tfrac{1}{2}\varsigma, \\ 2\,\Lambda'(t) \ =\ \Lambda'(s-t). \end{cases}$$

We note again that this system of equations has exactly two solutions. By calculating a solution of them by Newton's method from $(t, s) = (0.25,\ 0.55)$, we obtain $s_{\min} = 0.547094...$, and from $(t, s) = (0.25,\ 0.7)$, we obtain $s_{\max} = 0.688624...$. Therefore, we obtain an estimate of s in Δ' as

$$0.54 \ \le\ s\ \le\ 0.7.$$

To be precise (see Remark 3.2), we can rigorously verify the above estimate of the solutions of the above system of equations in a similar way as in Section A.2.

We calculate the minimal value $(s-t)_{\min}$ and the maximal value $(s-t)_{\max}$ of $s-t$. Putting $w = s - t$, they satisfy the following equations,

$$\begin{cases} 2\,\Lambda(t) + \Lambda(w) - 2\,\Lambda(t+w) + \Lambda(t+w-u) + 2\,\Lambda(u) \ =\ \varsigma, \\ \dfrac{\partial}{\partial t}\big(2\,\Lambda(t) + \Lambda(w) - 2\,\Lambda(t+w) + \Lambda(t+w-u) + 2\,\Lambda(u)\big) \ =\ 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t) + \Lambda(w) - 2\,\Lambda(t+w) + \Lambda(t+w-u) + 2\,\Lambda(u)\big) \ =\ 0. \end{cases}$$

We note that this system of equations has exactly two solutions. By calculating a solution by Newton's method from $(w,t,u) = (0.3,\ 0.25,\ 0.25)$, we obtain $(s-t)_{\min} = 0.301964...$, and from $(w,t,u) = (0.5,\ 0.2,\ 0.2)$, we obtain $(s-t)_{\max} = 0.462284...$. Therefore, we obtain an estimate of $s-t$ in Δ' as

$$0.3 \ \le\ s-t\ \le\ 0.47.$$

To be precise (see Remark 4.1), we can rigorously verify the above estimate of the solutions of the above system of equations in a similar way as in Section A.3.

We obtain the other bounds of Δ' from the above bounds by the symmetry with respect to the exchange of t and u.

5.3 Calculation of the critical value

In this section, we calculate the concrete values of a critical point and the Hesse matrix of $\hat{V}$.

The differentials of $\hat{V}$ are presented by

$$\frac{\partial}{\partial t}\hat{V}(t,s,u) \ =\ -2\,\log(1-x) + \log\big(1-\frac{x}{y}\big) + 2\pi\sqrt{-1}\,\big(t-\frac{1}{2}\big), \tag{83}$$

$$\frac{\partial}{\partial s}\hat{V}(t,s,u) \ =\ 2\,\log(1-y) - \log\big(1-\frac{x}{y}\big) - \log\big(1-\frac{y}{z}\big) - 2\pi\sqrt{-1}\,\big(s-\frac{1}{2}\big), \tag{84}$$

$$\frac{\partial}{\partial u}\hat{V}(t,s,u) \ =\ -2\,\log(1-z) + \log\big(1-\frac{y}{z}\big) + 2\pi\sqrt{-1}\,\big(u-\frac{1}{2}\big), \tag{85}$$

where $x = e^{2\pi\sqrt{-1}\,t}$, $y = e^{2\pi\sqrt{-1}\,s}$ and $z = e^{2\pi\sqrt{-1}\,u}$.

Lemma 5.1. *$\hat{V}$ has a unique critical point (t_0,s_0,u_0) in $P^{-1}(\Delta')$, where $P:\mathbb{C}^3\to\mathbb{R}^3$ is the projection to the real parts of the entries.*

Proof. Any critical point of $\hat{V}$ is given by a solution of $\frac{\partial}{\partial t}\hat{V} = \frac{\partial}{\partial s}\hat{V} = \frac{\partial}{\partial u}\hat{V} = 0$, and these equations are rewritten,

$$\begin{aligned}(1-x)^2 &= -x\big(1-\frac{x}{y}\big),\\ (1-y)^2 &= -y\big(1-\frac{x}{y}\big)\big(1-\frac{y}{z}\big),\\ (1-z)^2 &= -z\big(1-\frac{y}{z}\big).\end{aligned}$$

From the first formula, we have that $y = x^2/(x^2-x+1)$. Hence, from the second formula, we have that $z = x^3/\big((x-1)(x^2+1)\big)$. By substituting these into the third formula, we obtain

$$x^6 - 2x^5 + 5x^4 - 6x^3 + 5x^2 - 3x + 1 \ =\ 0.$$

Its solutions are given by

$$\begin{aligned}x \ &= 0.108378... \pm \sqrt{-1}\cdot 0.818891...\ ,\quad 0.23185... \pm \sqrt{-1}\cdot 1.65564...\ ,\\ &\quad 0.659772... \pm \sqrt{-1}\cdot 0.298454...\ .\end{aligned}$$

Since $x = e^{2\pi\sqrt{-1}\,t}$,

$$\begin{aligned} t \;&= 0.229058... + \sqrt{-1}\cdot 0.030418...\;, \quad 0.770942... + \sqrt{-1}\cdot 0.030418...\;, \\ &\;\;\; 0.227856... - \sqrt{-1}\cdot 0.081789...\;, \quad 0.772144... - \sqrt{-1}\cdot 0.081789...\;, \\ &\;\;\; 0.067611... + \sqrt{-1}\cdot 0.051371...\;, \quad 0.932389... + \sqrt{-1}\cdot 0.051371...\;. \end{aligned}$$

Among these, the first and third solutions are in the range of t in Δ'. Further, as for the third solution, $u = 0.932389... - \sqrt{-1}\cdot 0.0513713...$, and this is not in Δ'. From the first solution, we have that

$$\begin{aligned} x_0 &= 0.108378... + \sqrt{-1}\cdot 0.818891...\;, & t_0 &= 0.229058... + \sqrt{-1}\cdot 0.030418...\;, \\ y_0 &= -0.57395... - \sqrt{-1}\cdot 0.818891...\;, & s_0 &= 0.652705...\;, \\ z_0 &= 0.158836... + \sqrt{-1}\cdot 1.20014...\;, & u_0 &= 0.229058... - \sqrt{-1}\cdot 0.030418...\;, \end{aligned}$$

where $x_0 = e^{2\pi\sqrt{-1}\,t_0}$, $y_0 = e^{2\pi\sqrt{-1}\,s_0}$ and $z_0 = e^{2\pi\sqrt{-1}\,u_0}$. These give a unique critical point of $\hat{V}$ in $P^{-1}(\Delta')$. □

The critical value of $\hat{V}$ at the critical point of Lemma 5.1 is presented by

$$\begin{aligned} \varsigma \;&=\; \hat{V}(t_0, s_0, u_0) \\ &=\; \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(x_0) - \mathrm{Li}_2\big(\frac{x_0}{y_0}\big) - 2\,\mathrm{Li}_2(y_0) + \mathrm{Li}_2\big(\frac{y_0}{z_0}\big) + 2\,\mathrm{Li}_2(z_0)\Big) \\ &\qquad\qquad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t_0^2 - s_0^2 + u_0^2 - t_0 + s_0 - u_0 + \frac{1}{6}\big) \qquad (86) \\ &=\; 0.906072...\;, \end{aligned}$$

noting that this value is real, since the 6_3 knot is amphicheiral and the above value is invariant under the symmetry

$$(t,\,s,\,u) \longmapsto (\overline{u},\,\overline{s},\,\overline{t}), \qquad (x,\,y,\,z) \longmapsto (1/\overline{z},\,1/\overline{y},\,1/\overline{x}). \qquad (87)$$

We calculate the Hesse matrix of $\hat{V}$. Since $x = e^{2\pi\sqrt{-1}\,t}$, $\frac{d}{dt} = 2\pi\sqrt{-1}\,x\frac{d}{dx}$. Hence, from (83), we have that

$$\frac{\partial^2}{\partial t^2}\hat{V} \;=\; 2\pi\sqrt{-1}\,\Big(x\,\big(\frac{2}{1-x} - \frac{\frac{1}{y}}{1-\frac{x}{y}}\big) + 1\Big) \;=\; 2\pi\sqrt{-1}\,\Big(\frac{1+x}{1-x} - \frac{\frac{x}{y}}{1-\frac{x}{y}}\Big).$$

By calculating other entries similarly, the Hesse matrix of $\hat{V}$ at (t_0, s_0, u_0) is presented by

$$H_0 \;=\; 2\pi\sqrt{-1}\begin{pmatrix} \frac{1+x_0}{1-x_0} - \frac{\frac{x_0}{y_0}}{1-\frac{x_0}{y_0}} & \frac{\frac{x_0}{y_0}}{1-\frac{x_0}{y_0}} & 0 \\ \frac{\frac{x_0}{y_0}}{1-\frac{x_0}{y_0}} & -\frac{1+y_0}{1-y_0} - \frac{\frac{x_0}{y_0}}{1-\frac{x_0}{y_0}} + \frac{\frac{y_0}{z_0}}{1-\frac{y_0}{z_0}} & -\frac{\frac{y_0}{z_0}}{1-\frac{y_0}{z_0}} \\ 0 & -\frac{\frac{y_0}{z_0}}{1-\frac{y_0}{z_0}} & \frac{1+z_0}{1-z_0} + \frac{\frac{y_0}{z_0}}{1-\frac{y_0}{z_0}} \end{pmatrix}. \qquad (88)$$

5.4 Verifying the assumption of the Poisson summation formula

In this section, we verify the assumption of the Poisson summation formula in Lemma 5.2, which is used in the proof of Theorem 1.1 for the 6_3 knot in Section 5.1. As we mentioned in Remark 2.5, we verify the assumption for $\hat{V}(t,s,u)$ instead of $V(t,s,u)$, since $V(t,s,u)$ converges uniformly to $\hat{V}(t,s,u)$ on Δ' in the form mentioned in Remark 2.5.

By computer calculation, we can see that the maximal value of $\operatorname{Re}\hat{V}-\varsigma$ is about 0.02. Therefore, in the proof of Lemma 5.2, it is sufficient to decrease, say, $\operatorname{Re}\hat{V}(t,\ s,\ u+\delta\sqrt{-1})-2\pi\delta$ by 0.02, by moving δ (though we do not use this value in the proof of the lemma).

Lemma 5.2. *$\hat{V}(t,s,u)-\varsigma$ satisfies the assumption of Proposition 2.3.*

Proof. We show that $\partial\Delta'$ is null-homotopic in each of (15)–(20).

As for (15) and (16), similarly as the proof of Lemma 3.4, it is sufficient to show that

$$-(2\pi-\varepsilon') \ < \ \operatorname{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t+\delta\sqrt{-1},\ s,\ u)\Big) \ < \ 2\pi-\varepsilon' \tag{89}$$

for some $\varepsilon'>0$. The middle term is calculated as

$$\operatorname{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t+\delta\sqrt{-1},\ s,\ u)\Big) \ = \ 2\operatorname{Arg}(1-x)-\operatorname{Arg}\big(1-\frac{x}{y}\big)-2\pi\big(t-\frac{1}{2}\big),$$

where $x=e^{2\pi\sqrt{-1}(t+\delta\sqrt{-1})}$. Since $0.18\le t\le 0.3$,

$$-2\pi\big(\frac{1}{2}-t\big) \ < \ \operatorname{Arg}(1-x) \ < \ 0.$$

Further, since $0.3\le s-t\le 0.47$,

$$0 \ < \ \operatorname{Arg}\big(1-\frac{x}{y}\big) \ < \ 2\pi\big(\frac{1}{2}-s+t\big).$$

Therefore,

$$-2\pi(1-s) \ < \ \operatorname{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t+\delta\sqrt{-1},\ s,\ u)\Big) \ < \ 2\pi\big(\frac{1}{2}-t\big).$$

Since $s\ge 0.54$ and $t\ge 0.18$,

$$-2\pi\cdot 0.46 \ < \ \operatorname{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t+\delta\sqrt{-1},\ s,\ u)\Big) \ < \ 2\pi\cdot 0.32,$$

and hence, (89) is satisfied.

As for (17) and (18), similarly as above, it is sufficient to show that

$$-(2\pi-\varepsilon') \ < \ \operatorname{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t,\ s+\delta\sqrt{-1},\ u)\Big) \ < \ 2\pi-\varepsilon' \tag{90}$$

for some $\varepsilon'>0$. The middle term is calculated as

$$\operatorname{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t,\ s+\delta\sqrt{-1},\ u)\Big) \ = \ -2\operatorname{Arg}(1-y)+\operatorname{Arg}\big(1-\frac{x}{y}\big)+\operatorname{Arg}\big(1-\frac{y}{z}\big)+2\pi\big(s-\frac{1}{2}\big)$$

where $y = e^{2\pi\sqrt{-1}(s+\delta\sqrt{-1})}$. Since $0.54 \le s \le 0.7$,

$$0 \;<\; \mathrm{Arg}\,(1-y) \;<\; 2\pi\big(s-\frac{1}{2}\big).$$

Further, since $0.3 \le s-t \le 0.47$,

$$0 \;<\; \mathrm{Arg}\,\big(1-\frac{x}{y}\big) \;<\; 2\pi\big(\frac{1}{2}-s+t\big).$$

Furthermore, since $0.3 \le s-u \le 0.47$,

$$-2\pi\big(\frac{1}{2}-s+u\big) \;<\; \mathrm{Arg}\,\big(1-\frac{y}{z}\big) \;<\; 0.$$

Therefore,

$$-2\pi\cdot u \;<\; \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat{V}(t,\, s+\delta\sqrt{-1},\, u)\Big) \;<\; 2\pi\cdot t.$$

Since $u \le 0.3$ and $t \le 0.3$,

$$-2\pi\cdot 0.3 \;<\; \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat{V}(t,\, s+\delta\sqrt{-1},\, u)\Big) \;<\; 2\pi\cdot 0.3,$$

and hence, (90) is satisfied.

As for (19) and (20), similarly as above, it is sufficient to show that

$$-(2\pi-\varepsilon') \;<\; \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat{V}(t,\, s,\, u+\delta\sqrt{-1})\Big) \;<\; 2\pi-\varepsilon'$$

for some $\varepsilon' > 0$. We obtain this formula from (89) by the symmetry (87). □

5.5 Verifying the assumption of the saddle point method

In this section, we verify the assumption of the saddle point method in Lemma 5.5. In order to show this lemma, we show Lemmas 5.3 and 5.4 in advance. As we mentioned in Remark 2.7, we verify the assumption for $\hat{V}(t,s,u)$ instead of $V(t,s,u)$, since $V(t,s,u)$ converges uniformly to $\hat{V}(t,s,u)$ on Δ' in the form mentioned in Remark 2.7.

In the proof of Lemma 5.5, as mentioned at the beginning of Section 5.4, it is sufficient to decrease $\mathrm{Re}\,\hat{V}(t,s,u)$ by 0.02, by pushing t,s,u into the imaginary directions. In order to calculate this concretely, putting

$$f(X,Y,Z) = \mathrm{Re}\,\hat{V}(t+X\sqrt{-1},\, s+Y\sqrt{-1},\, u+Z\sqrt{-1}) - \varsigma,$$

as in Section 3.5, we consider the behavior of f at each fiber of the projection $\mathbb{C}^3 \to \mathbb{R}^3$.

Lemma 5.3. *For each $(t,s,u) \in \Delta'$, f has a unique minimal point and no other critical points.*

Proof. By Lemma 5.4, it is sufficient to show that the Hesse matrix of f is positive definite at any $(X,Y,Z)\in\mathbb{R}^3$. Similarly as in the proof of Lemma 4.5, we have that

$$\begin{aligned}
\frac{\partial^2 f}{\partial X^2} &= 2\pi\,\mathrm{Im}\Big(\frac{2}{1-x}-\frac{1}{1-\frac{x}{y}}\Big), & \frac{\partial^2 f}{\partial X\,\partial Y} &= 2\pi\,\mathrm{Im}\,\frac{1}{1-\frac{x}{y}}\,,\\
\frac{\partial^2 f}{\partial Y^2} &= 2\pi\,\mathrm{Im}\Big(-\frac{2}{1-y}-\frac{1}{1-\frac{x}{y}}+\frac{1}{1-\frac{y}{z}}\Big), & \frac{\partial^2 f}{\partial Y\,\partial Z} &= -2\pi\,\mathrm{Im}\,\frac{1}{1-\frac{y}{z}}\,,\\
\frac{\partial^2 f}{\partial Z^2} &= 2\pi\,\mathrm{Im}\Big(\frac{2}{1-z}+\frac{1}{1-\frac{y}{z}}\Big),
\end{aligned}$$

where $x=e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$, $y=e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$ and $z=e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})}$. Hence, the Hesse matrix is presented by

$$2\pi\begin{pmatrix} a_1+b_1 & -b_1 & 0\\ -b_1 & a_2+b_1+b_2 & -b_2\\ 0 & -b_2 & a_3+b_2\end{pmatrix},$$

where we put

$$a_1 = \mathrm{Im}\,\frac{2}{1-x}\,,\quad a_2 = \mathrm{Im}\,\frac{-2}{1-y}\,,\quad a_3 = \mathrm{Im}\,\frac{2}{1-z}\,,\quad b_1 = \mathrm{Im}\,\frac{-1}{1-\frac{x}{y}}\,,\quad b_2 = \mathrm{Im}\,\frac{1}{1-\frac{y}{z}}\,,$$

noting that these numbers are positive since $(t,s,u)\in\Delta'$. Since the above matrix is equivalent as a quadratic form to the following matrix,

$$2\pi\begin{pmatrix} a_1+b_1 & 0 & 0\\ 0 & a_2+\frac{a_1b_1}{a_1+b_1}+\frac{a_3b_2}{a_3+b_2} & 0\\ 0 & 0 & a_3+b_2\end{pmatrix},$$

the Hesse matrix of f is positive definite, as required. □

The following lemma is used in the proof of Lemma 5.3.

Lemma 5.4. *For each $(t,s,u)\in\Delta'$, $f(X,Y,Z)\to\infty$ as $X^2+Y^2+Z^2\to\infty$.*

Proof. From the definition of f, we have that

$$\begin{aligned}
f(X,Y,Z) \;=\; \mathrm{Re}\,\frac{1}{2\pi\sqrt{-1}}\Big(&2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})})-\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(1+t-s)+2\pi(Y-X)})\\
&-2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})})+\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s-u)+2\pi(Z-Y)})+2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})})\Big)\\
&-2\pi\big(t-\tfrac{1}{2}\big)X+2\pi\big(s-\tfrac{1}{2}\big)Y-2\pi\big(u-\tfrac{1}{2}\big)Z-\varsigma.
\end{aligned}$$

Hence, by Lemma 2.2, putting

$$\begin{aligned} F(X,Y,Z) \;=\; & \left(\begin{cases} \big(\frac{1}{2}-t\big)X & \text{if } X \ge 0 \\ -\big(\frac{1}{2}-t\big)X & \text{if } X < 0 \end{cases} \right) + \left(\begin{cases} 0 & \text{if } X \ge Y \\ \big(\frac{1}{2}-s+t\big)(Y-X) & \text{if } X < Y \end{cases} \right) \\ & + \left(\begin{cases} \big(s-\frac{1}{2}\big)Y & \text{if } Y \ge 0 \\ -\big(s-\frac{1}{2}\big)Y & \text{if } Y < 0 \end{cases} \right) + \left(\begin{cases} 0 & \text{if } Y \ge Z \\ \big(\frac{1}{2}-s+u\big)(Z-Y) & \text{if } Y < Z \end{cases} \right) \\ & + \left(\begin{cases} \big(\frac{1}{2}-u\big)Z & \text{if } X \ge 0 \\ -\big(\frac{1}{2}-u\big)Z & \text{if } X < 0 \end{cases} \right), \end{aligned}$$

it is sufficient to show that $F(X,Y,Z) \to \infty$ as $X^2+Y^2+Z^2 \to \infty$. Since $F(c\,X, c\,Y, c\,Z) = c\,F(X,Y,Z)$ for any $c > 0$, it is sufficient to show that $F(X,Y,Z)$ is positive on the unit sphere $\{(X,Y,Z) \in \mathbb{R}^3 \mid X^2+Y^2+Z^2 = 1\}$.

Note that all of the five summands in the right-hand side of the formula of $F(X,Y,Z)$ are non-negative. Further, the first summand is positive if $X \ne 0$, the third summand is positive if $Y \ne 0$, and the last summand is positive if $Z \ne 0$. Hence, $F(X,Y,Z)$ is positive on the unit sphere, as required. □

Lemma 5.5. *When we apply Proposition 2.6 to* (81)*, the assumption of Proposition 2.6 holds.*

Proof. We show that there exists a homotopy Δ'_δ $(0 \le \delta \le 1)$ between $\Delta'_0 = \Delta'$ and Δ'_1 such that

$$(t_0, s_0, u_0) \in \Delta'_1, \tag{91}$$

$$\Delta'_1 - \{(t_0,s_0,u_0)\} \;\subset\; \big\{(t,s,u) \in \mathbb{C}^3 \;\big|\; \operatorname{Re}\hat{V}(t,s,u) < \varsigma\big\}, \tag{92}$$

$$\partial\Delta'_\delta \;\subset\; \big\{(t,s,u) \in \mathbb{C}^3 \;\big|\; \operatorname{Re}\hat{V}(t,s,u) < \varsigma\big\}. \tag{93}$$

At each fiber of the projection $\mathbb{C}^3 \to \mathbb{R}^3$, we consider the behavior of $f(X,Y,Z) = \operatorname{Re}\hat{V}(t+X\sqrt{-1},\, s+Y\sqrt{-1},\, u+Z\sqrt{-1}) - \varsigma$. By Lemma 5.3, f has a unique minimal point. We put this minimal point to be $(X,Y,Z) = \big(g_1(t,s,u), g_2(t,s,u), g_3(t,s,u)\big)$. We define the ending of the homotopy to be the set of these minimal points,

$$\Delta'_1 \;=\; \big\{\big(t+g_1(t,s,u)\sqrt{-1},\, s+g_2(t,s,u)\sqrt{-1},\, u+g_3(t,s,u)\sqrt{-1}\big) \in \mathbb{C}^3 \;\big|\; (t,s,u) \in \Delta'\big\}.$$

Further, we define the internal part of the homotopy by setting it along the flow from (t,s,u) determined by the vector field $\big(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}\big)$.

We can show (91), (92) and (93) by Lemma 5.1 in a similar way as the proof of Lemma 3.9. □

A Estimate of solutions of systems of equations

In Sections 3.2, 4.2 and 5.2, we estimate solutions of some systems of equations by using numerical solutions obtained by Newton's method. As mentioned in Remarks 3.1, 3.2 and

4.1, this argument is not partially rigorous, since we do not estimate the error terms of such solutions. In order to complete their proofs, we show how to estimate such solutions rigorously in this appendix. We explain 1-, 2-, 3-variable cases in Sections A.1, A.2, A.3 respectively.

A.1 Estimate of solutions of (35)

In this section, we explain how to estimate solutions in 1-variable case. We explain it for solutions of (35); we can estimate solutions in other 1-variable cases in similar ways.

As mentioned in Section 3.2, there are exactly two solutions of (35) in $0 < t < 0.5$. We put them to be $t_{\min}$ and $t_{\max}$ (such that $t_{\min} < t_{\max}$). As we mentioned in Remark 3.1, we give a rigorous proof of estimates of them in the following lemma.

Lemma A.1. $0.03 \leq t_{\min}$ *and* $t_{\max} \leq 0.4$.

Proof. We put

$$P(t) \;=\; 2\,\Lambda(t) - \varsigma_R + 2\,\Lambda\Big(\frac{1}{6}\Big).$$

Then, (35) is rewritten as $P(t) = 0$. Since

$$P(0.03) \;=\; -0.020344... \;<\; 0, \quad P\Big(\frac{1}{6}\Big) \;=\; 0.142571... \;>\; 0, \quad P(0.4) \;=\; -0.045188... \;<\; 0,$$

we have that

$$0.03 \;<\; t_{\min} \;<\; \frac{1}{6} \;<\; t_{\max} \;<\; 0.4,$$

as required. □

A.2 Estimate of a solution of (36)

In this section, we explain how to estimate a solution in 2-variable case. We explain it for a solution of (36); we can estimate solutions in other 2-variable cases in similar ways.

As mentioned in Section 3.2, there are exactly two solutions of (36). We put $w_{\max}$ to be w of the solution whose w is the larger one. As we mentioned in Remark 3.2, we give a rigorous proof of an estimate of $w_{\max}$ in the following lemma.

Lemma A.2. $w_{\max} \leq 0.94$.

Proof. The system of equations (36) is rewritten

$$\begin{cases} \Lambda(w-2s) + \Lambda(s) - \frac{1}{2}\varsigma_R \;=\; 0, \\ 2\,\Lambda'(w-2s) \;=\; \Lambda'(s). \end{cases}$$

Since $\Lambda'(t) = -\log 2\sin \pi t$, the second equation is rewritten

$$2\sin^2 \pi(w-2s) \;=\; \sin \pi s.$$

We put $t = w - s - u = w - 2s$. Since $0.03 \leq t \leq 0.4$ as shown in Section 3.2, $\sin \pi(w-2s) > 0$. Therefore,

$$\sin \pi(w-2s) \;=\; \sqrt{\frac{1}{2}\sin \pi s}\,,$$

and

$$w \;=\; 2s + \frac{1}{\pi}\arcsin\sqrt{\frac{1}{2}\sin \pi s}\,.$$

We put the right-hand side of this formula to be $f(s)$, that is, $w = f(s)$.

In order to transform a range of s to a range of w later, we show that $f(s)$ is monotonically increasing in an area where we consider a solution, as follows. We put

$$F(w,s) \;=\; 2\sin^2 \pi(w-2s) - \sin \pi s.$$

Since $F(w,s) = 0$,

$$\frac{d}{ds}F\big(f(s),s\big) \;=\; \frac{\partial F}{\partial w}\cdot f'(s) + \frac{\partial F}{\partial s} \;=\; 0.$$

Hence, since

$$\frac{\partial F}{\partial w} \;=\; 4\pi \sin \pi(w-2s)\cos \pi(w-2s) \;=\; 2\pi \sin 2\pi(w-2s),$$

$$\frac{\partial F}{\partial s} \;=\; -4\pi \sin 2\pi(w-2s) - \pi \sin \pi s,$$

we have that

$$f'(s) \;=\; -\frac{\frac{\partial F}{\partial w}}{\frac{\partial F}{\partial s}} \;=\; 2 + \frac{\cos \pi s}{2\sin 2\pi(w-2s)} \;=\; 2 + \frac{\cos \pi s}{2\sin 2\pi t} \;>\; 0,$$

where we obtain the last inequality since $0.001 \leq s \leq 0.5$ and $0.03 \leq t \leq 0.4$ as shown in Section 3.2. Therefore, $f(s)$ is monotonically increasing in the area of such s and t.

We estimate a solution of (36) by rewriting it as a 1-variable equation, as follows. We put

$$P(s) \;=\; \Lambda\big(f(s)-2s\big) + \Lambda(s) - \frac{1}{2}\varsigma_R\,.$$

Then, (36) is rewritten as $P(s) = 0$. Since

$$P(0.346) \;=\; 0.0003872... \;>\; 0, \qquad P(0.347) \;=\; -0.0002516... \;<\; 0,$$

a solution of $P(s) = 0$ is in the following range,

$$0.346 \;<\; s \;<\; 0.347.$$

Further, since $f(s)$ is monotonically increasing, w of a solution of (36) is in the following range

$$f(0.346) \;<\; w_{\max} \;<\; f(0.347).$$

Therefore,

$$w_{\max} \;<\; f(0.347) \;=\; 0.925927... \;<\; 0.94,$$

as required. □

A.3 Estimate of a solution of (58)

In this section, we explain how to estimate a solution in 3-variable case. We explain it for a solution of (58); we can estimate solutions in other 3-variable cases in similar ways.

As mentioned in Section 4.2, there are exactly two solutions of (58). These solutions give the maximal and minimal values of $2s+u$. We put the solution for the maximal value to be (t_1, s_1, u_1). Then, the maximal value of $2s+u$ is given by $(2s+u)_{\max} = 2s_1 + u_1$. As we mentioned in Remark 4.1, we give a rigorous proof of an estimate of $(2s+u)_{\max}$ in the following lemma.

Lemma A.3. $(2s+u)_{\max} \ \le\ 0.97.$

Proof. The system of equations (58) is rewritten

$$\begin{cases} -2\,\Lambda(t) + \Lambda\big(t - w' + \tfrac{1}{2}u\big) + 2\,\Lambda\big(w' - \tfrac{1}{2}u\big) + \Lambda(u) - \varsigma_{R} \ = \ 0, \\ -2\,\Lambda'(t) + \Lambda'\big(t - w' + \tfrac{1}{2}u\big) \ = \ 0, \\ \tfrac{1}{2}\Lambda\big(t - w' + \tfrac{1}{2}u\big) - \Lambda\big(w' - \tfrac{1}{2}u\big) + \Lambda'(u) \ = \ 0, \end{cases}$$

where $\varsigma_{R} = 0.700414248\ldots$. Putting $s = w' - \frac{1}{2}$, they are rewritten

$$\begin{cases} -2\,\Lambda(t) + \Lambda(t-s) + 2\,\Lambda(s) + \Lambda(u) - \varsigma_{R} \ = \ 0, \\ 2\,\Lambda'(t) \ = \ \Lambda'(t-s), \\ \tfrac{1}{2}\,\Lambda'(t-s) - \Lambda'(s) + \Lambda'(u) \ = \ 0. \end{cases} \tag{94}$$

The second equation is rewritten

$$2\sin^2 \pi t \ = \ \sin \pi(t-s).$$

Hence,[4]

$$s \ = \ t - \frac{1}{\pi}\arcsin\big(2\sin^2 \pi t\big).$$

We put the right-hand side of this formula to be $f(t)$, that is, $s = f(t)$. By the second and third equations of (94), we obtain that

$$\Lambda'(u) \ = \ \Lambda'(s) - \Lambda'(t).$$

Hence,

$$u \ = \ \frac{1}{\pi}\arcsin\Big(\frac{\sin \pi s}{2\sin \pi t}\Big).$$

Putting $s = f(t)$, we put the right-hand side of this formula to be $g(t)$, that is, $u = g(t)$. Further, putting

$$P(t) \ = \ -2\,\Lambda(t) + \Lambda\big(t - f(t)\big) + 2\,\Lambda\big(f(t)\big) + \Lambda\big(g(t)\big) - \varsigma_{R}\,,$$

the system of equations (94) is rewritten as a single equation $P(t) = 0$.

[4]Here, we choose a branch of arcsin as $-\pi/2 \le \arcsin(\cdot) \le \pi/2$. By choosing another branch, we obtain an estimate of the minimal value of $2s+u$.

We estimate the solution (t_1, s_1, u_1), as follows. Since

$$P(0.7586) \;=\; 0.000307098... \;>\; 0, \qquad P(0.7587) \;=\; -0.000200034... \;<\; 0,$$

we have that

$$0.7586 \;<\; t_1 \;<\; 0.7587.$$

In this range, $\arcsin\big(2\sin^2 \pi t\big)$ is monotonically decreasing. Hence,

$$f(0.7586) \;<\; s_1 \;<\; f(0.7587).$$

Further, since

$$f(0.7586) \;=\; 0.363692... \;>\; 0.3636, \qquad f(0.7587) \;=\; 0.364407... \;<\; 0.3645,$$

we have that

$$0.3636 \;<\; s_1 \;<\; 0.3645.$$

In these ranges, $\frac{\sin \pi s}{\sin \pi t}$ is monotonically increasing with respect to t and s respectively. Hence,

$$\frac{1}{\pi}\arcsin\Big(\frac{\sin \pi\, 0.3636}{2\sin \pi\, 0.7586}\Big) \;<\; u_1 \;<\; \frac{1}{\pi}\arcsin\Big(\frac{\sin \pi\, 0.3645}{2\sin \pi\, 0.7587}\Big).$$

Therefore, we obtain that

$$(2s+u)_{\max} \;=\; 2s_1 + u_1 \;<\; 2\cdot 0.3645 + \frac{1}{\pi}\arcsin\Big(\frac{\sin \pi\, 0.3645}{2\sin \pi\, 0.7587}\Big) \;=\; 0.959440... \leq 0.97,$$

as required. □

B Curvature of the boundary of the domain $\{\operatorname{Re}\hat{V} \geq \varsigma_R\}$

In Sections 3.2, 4.2 and 5.2, we estimated the maximal and minimal values of some linear function $L(t,s,u)$ on the domain $\big\{(t,s,u) \mid \operatorname{Re}\hat{V}(t,s,u) \geq \varsigma_R\big\}$. This domain is a convex domain whose boundary is a smooth closed surface. These maximal and minimal values are obtained when the plane $L(t,s,u) = c$ (where c is a constant) is tangent to this domain. In order to show that there are exactly two such planes,[5] we show that Gaussian curvature of the boundary surface of this domain is positive in this section. We explain 2-, 3-variable cases in Sections B.1 and B.2 respectively.

B.1 Curvature of the boundary of a convex domain in a plane

In this section, we consider 2-variable case.

Let $F(x,z)$ be a smooth concave function whose maximal value is positive. Then, the domain

$$\big\{(x,z) \in \mathbb{R}^2 \;\big|\; F(x,z) \geq 0\big\} \tag{95}$$

is a convex domain and its boundary is a smooth curve. The aim of this section is to show the following lemma.

[5]We use this fact, when we show that the system of equations presenting such tangent points has exactly two solutions in Sections 3.2, 4.2, 5.2 and Appendix A.

Lemma B.1. *Let $F(x,z)$ be a smooth concave function whose maximal value is positive and Hesse matrix is negative definite. Then, the domain* (95) *is a convex domain and its boundary is a smooth curve whose curvature is non-zero everywhere.*

Proof. We choose any point on the boundary of the domain (95). It is sufficient to show that the curvature of the boundary curve at this point is non-zero. By changing the coordinate (x,z) by an affine transformation appropriately, we can assume that this point is the origin $(0,0)$, and the domain (95) is in the upper half plane and it is tangent to the x-axis at the origin. Then, in a neighborhood of the origin, we can present the boundary curve as $z = f(x)$ with some smooth function f defined in a neighborhood of 0. It is sufficient to show that $f''(0) \neq 0$.

We show that $f''(0) \neq 0$, as follows. Since the boundary curve is given by $F(x,z) = 0$, we have that $F\big(x, f(x)\big) = 0$. Its differential is given by

$$F_x(x,z) + F_z(x,z)\, f'(x) \;=\; 0,$$

where the subscripts of x and z means $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$ respectively. Since the domain (95) is tangent to the x-axis at the origin, $F_x(0,0) = f'(0) = 0$. Further, the differential of the above formula is given by

$$F_{xx}(x,z) + 2\,F_{xz}(x,z)\, f'(x) + F_{zz}(x,z)\big(f'(x)\big)^2 + F_z(x,z)\, f''(x) \;=\; 0.$$

By putting $x = z = 0$, we have that

$$F_{xx}(0,0) + F_z(0,0)\, f''(0) \;=\; 0.$$

We obtain $\big(F_{xx}\big)$ from the Hesse matrix

$$\begin{pmatrix} F_{xx} & F_{xz} \\ F_{xz} & F_{zz} \end{pmatrix}$$

by restricting $\mathbb{R}^2$ to $\mathbb{R}\times\{0\}$. Since the Hesse matrix is negative definite, we obtain that $F_{xx}(x,z) < 0$ for any (x,z). Further, we consider the function $g(z) = F(0,z)$. This is a concave function such that $g''(z) < 0$ and the maximal point z_0 of $g(z)$ is positive. Hence, the function $g'(z)$ is monotonically decreasing, and $g'(z_0) = 0$ for $z_0 > 0$. Therefore, $F_z(0,0) = g'(0) > 0$. Hence, we obtain that

$$f''(0) \;=\; -\frac{F_{xx}(0,0)}{F_z(0,0)} \;>\; 0,$$

as required. □

B.2 Curvature of the boundary of the domain $\{\operatorname{Re}\hat{V}(t,s,u) \geq \varsigma_R\}$

In this section, we consider 3-variable case. Instead of the function $\operatorname{Re}\hat{V}(t,s,u) - \varsigma_R$, we consider a function $F(x,y,z)$ in general.

Let $F(x,y,z)$ be a smooth concave function whose maximal value is positive. Then, the domain

$$\big\{(x,y,z) \in \mathbb{R}^3 \;\big|\; F(x,y,z) \geq 0\big\} \tag{96}$$

is a convex domain and its boundary is a smooth surface. The aim of this section is to show the following lemma.

Lemma B.2. *Let $F(x,y,z)$ be a smooth concave function whose maximal value is positive and Hesse matrix is negative definite. Then, the domain* (96) *is a convex domain and its boundary is a smooth surface whose Gaussian curvature is positive everywhere.*

Proof. We choose any point on the boundary of the domain (96). Similarly as in the proof of Lemma B.1, we can assume that this point is the origin $(0,0,0)$, and the domain (96) is in the upper half space and it is tangent to the (x,y)-plane at the origin. Then, in a neighborhood of the origin, we can present the boundary curve as $z=f(x,y)$ with some smooth function f defined in a neighborhood of 0. It is sufficient to show that the Hesse matrix of f is positive definite at the origin.

We show that the Hesse matrix of f is positive definite at the origin, as follows. Since the boundary curve is given by $F(x,y,z)=0$, we have that $F\big(x,y,f(x,y)\big)=0$. Its differentials are given by

$$\begin{aligned} F_x(x,y,z)+F_z(x,y,z)\,f_x(x,y) &= 0,\\ F_y(x,y,z)+F_z(x,y,z)\,f_y(x,y) &= 0. \end{aligned}$$

Since the domain (95) is tangent to the (x,y)-plane at the origin, $f_x(0,0)=f_y(0,0)=0$. Further, the differentials of the above formulas are given by

$$\begin{aligned} &F_{xx}(x,y,z)+2\,F_{xz}(x,y,z)\,f_x(x,y)+F_{zz}(x,y,z)\big(f_x(x,y)\big)^2+F_z(x,y,z)\,f_{xx}(x,y) = 0,\\ &F_{xy}(x,y,z)+F_{xz}(x,y,z)\,f_y(x,y)+F_{yz}(x,y,z)\,f_x(x,y)\\ &\qquad +F_{zz}(x,y,z)\,f_x(x,y)\,f_y(x,y)+F_z(x,y,z)\,f_{xy}(x,y) = 0,\\ &F_{yy}(x,y,z)+2\,F_{yz}(x,y,z)\,f_y(x,y)+F_{zz}(x,y,z)\big(f_x(x,y)\big)^2+F_z(x,y,z)\,f_{yy}(x,y) = 0. \end{aligned}$$

By putting $x=y=z=0$, we have that

$$\begin{aligned} F_{xx}(0,0,0)+F_z(0,0,0)\,f_{xx}(0,0) &= 0,\\ F_{xy}(0,0,0)+F_z(0,0,0)\,f_{xy}(0,0) &= 0,\\ F_{yy}(0,0,0)+F_z(0,0,0)\,f_{yy}(0,0) &= 0. \end{aligned}$$

Further, by considering the function $g(z)=F(0,0,z)$, we can show that $F_z(0,0,0)=g'(0)>0$, similarly as in the proof of Lemma B.1. Furthermore, we obtain the following matrix

$$-F_z(0,0,0)\begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0)\\ f_{xy}(0,0) & f_{yy}(0,0)\end{pmatrix} = \begin{pmatrix} F_{xx}(0,0,0) & F_{xy}(0,0,0)\\ F_{xy}(0,0,0) & F_{yy}(0,0,0)\end{pmatrix}$$

from the Hesse matrix of F by restricting $\mathbb{R}^3$ to $\mathbb{R}^2\times\{0\}$. Therefore, the above matrix is negative definite. Hence, the Hesse matrix of f is positive definite at the origin, as required. □

C On the domain $\{\operatorname{Re}\hat{V}(t,s,u)\geq\varsigma\}$ for the 6_3 knot

In Section 5.1, we put

$$\Delta = \big\{(t,s,u)\in\mathbb{R}^3 \bigm| 0\leq t\leq s<1,\quad 0\leq u\leq s\big\},$$

and consider the following domain in (82), which we put to be $\hat{\Delta}$ in this section,

$$\hat{\Delta} \;=\; \big\{(t,s,u)\in\Delta \;\big|\; 2\,\Lambda(t)+\Lambda(s-t)-2\,\Lambda(s)+\Lambda(s-u)+2\,\Lambda(u)-\varsigma \;\geq\; 0\big\}.$$

We note that this domain is symmetric with respect to the exchange of t and u. The aim of this section is to show the following lemma.

Lemma C.1. *$\hat{\Delta}$ is a compact convex domain in the interior of Δ, and the boundary of $\hat{\Delta}$ is a smooth closed surface whose Gaussian curvature is positive everywhere.*

We show a proof of the lemma later in this section. Before showing it, we show some lemmas.

Lemma C.2. *The boundary of Δ is included in the complement of $\hat{\Delta}$ in Δ. That is, $\hat{\Delta}$ is included in the interior of Δ.*

Proof. We consider the defining inequality of $\hat{\Delta}$,

$$2\,\Lambda(t)+\Lambda(s-t)-2\,\Lambda(s)+\Lambda(s-u)+2\,\Lambda(u)-\varsigma \;\geq\; 0. \tag{97}$$

It is sufficient to show that any boundary point of Δ does not satisfy this inequality. The boundary of Δ consists of five faces: $\{t=0\}$, $\{t=s\}$, $\{s=1\}$, $\{u=0\}$ and $\{u=s\}$.

When $t=0$, the left-hand side of (97) is rewritten

$$-\Lambda(s)+\Lambda(s-u)+2\,\Lambda(u)-\varsigma \;\leq\; 4\,\Lambda\big(\frac{1}{6}\big)-\varsigma \;=\; -0.25994... \;<\; 0.$$

Hence, the face $\{t=0\}$ is included in the complement of $\hat{\Delta}$.

When $t=s$, the left-hand side of (97) is rewritten

$$\Lambda(t-u)+2\,\Lambda(u)-\varsigma \;\leq\; 3\,\Lambda\big(\frac{1}{6}\big)-\varsigma \;=\; -0.421473... \;<\; 0.$$

Hence, the face $\{t=s\}$ is included in the complement of $\hat{\Delta}$.

When $s=1$, the left-hand side of (97) is rewritten

$$\Lambda(t)+\Lambda(u)-\varsigma \;\leq\; 2\,\Lambda\big(\frac{1}{6}\big)-\varsigma \;=\; -0.583006... \;<\; 0.$$

Hence, the face $\{s=1\}$ is included in the complement of $\hat{\Delta}$.

Further, it is shown that the other faces are included in the complement of $\hat{\Delta}$ from the above results by the symmetry with respect to the exchange of t and u.

Therefore, we obtain the lemma. □

Lemma C.3. *Let $(t,s,u)\in\hat{\Delta}$. Then, $s>0.5$.*

Proof. We suppose that s was less than or equal to 0.5 (and show a contradiction). The function of the defining inequality of $\hat{\Delta}$ splits into the following two summands,

$$\begin{aligned}&2\,\Lambda(t)+\Lambda(s-t)-2\,\Lambda(s)+\Lambda(s-u)+2\,\Lambda(u)-\varsigma\\ &=\; \big(2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s)-\frac{1}{2}\varsigma\big)+\big(2\,\Lambda(u)+\Lambda(s-u)-\Lambda(s)-\frac{1}{2}\varsigma\big).\end{aligned} \tag{98}$$

We show that, for such an arbitrarily fixed s, the first summand is negative,

$$2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s) - \frac{1}{2}\varsigma \;<\; 0 \qquad \text{for}\;\; 0 \le t \le s. \tag{99}$$

(Then, it is shown similarly that the second summand of (98) is also negative, and this contradicts the defining inequality of $\hat{\Delta}$, which implies the assumption of the lemma.) Hence, it is sufficient to show (99) for $s \le 0.5$.

We show (99), as follows. For a fixed $s \le 0.5$, we put

$$f(t) \;=\; 2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s) - \frac{1}{2}\varsigma$$

for $0 \le t \le s$. Then,

$$f'(t) \;=\; 2\,\Lambda'(t) - \Lambda'(s-t).$$

Since $\Lambda(t)$ and $\Lambda(s-t)$ are concave functions in $0 \le t \le s$ whose second derivatives are negative, $f(t)$ has a unique maximal point in $0 \le t \le s$, noting that $\lim_{t\to+0} f'(t) = \infty$ and $\lim_{t\to s-0} f'(t) = -\infty$. We put such a unique maximal point to be $\hat{t}$, that is, $\hat{t}$ is a unique solution of $f'(t) = 0$ in $0 \le t \le s$. Since $f'(\hat{t}) = 0$,

$$2\,\Lambda'(\hat{t}) \;=\; \Lambda'(s-\hat{t}).$$

Further, since $\Lambda'(t) = -\log 2\sin \pi t$,

$$\sin \pi(s-\hat{t}) \;=\; 2\sin^2 \pi \hat{t}.$$

Since $2\sin^2 \pi\hat{t} \le 1$ and $0 \le \hat{t} \le 0.5$, we have that $0 \le \hat{t} \le 0.25$. Hence,

$$s \;=\; \hat{t} + \frac{1}{\pi}\arcsin\big(2\sin^2 \pi\hat{t}\,\big).$$

We put the right-hand side of this formula to be $g(\hat{t})$, that is, $s = g(\hat{t})$. Since $\sin^2 \pi\hat{t}$ is monotonically increasing in $0 \le \hat{t} \le 0.25$, $g(\hat{t})$ is also monotonically increasing. We note that $g(0) = 0$ and $g(0.25) = 0.75$. So, in fact, we can define $g(\hat{t})$ for $0 \le \hat{t} \le 0.25$, but we substantially consider the case where

$$0 \;\le\; \hat{t} \;\le\; g^{-1}(0.5) \;=\; 0.2148... \;\le\; 0.22.$$

In order to show (99), it is sufficient to show that

$$2\,\Lambda(\hat{t}) + \Lambda\big(g(\hat{t}) - \hat{t}\big) - \Lambda\big(g(\hat{t})\big) - \frac{1}{2}\varsigma \;<\; 0$$

for $0 \le \hat{t} \le g^{-1}(0.5)$. We put the left-hand side of the above formula to be $P(\hat{t})$. We have that

$$\begin{aligned} P'(\hat{t}) \;&=\; 2\,\Lambda'(\hat{t}) + \Lambda'\big(g(\hat{t})-\hat{t}\big)\big(g'(\hat{t})-1\big) - \Lambda'\big(g(\hat{t})\big)\,g'(\hat{t}) \\ &=\; \Lambda'(s-\hat{t}) + \Lambda'(s-\hat{t})\big(g'(\hat{t})-1\big) - \Lambda'(s)\,g'(\hat{t}) \\ &=\; \big(\Lambda'(s-\hat{t}) - \Lambda'(s)\big)\,g'(\hat{t}), \end{aligned}$$

where we obtain the second equality by $2\,\Lambda'(\hat{t}) = \Lambda'(s-\hat{t})$. Since $g(\hat{t})$ is monotonically increasing, $g'(\hat{t}) > 0$. Further, since $\Lambda'(v)$ is monotonically decreasing in $0 < v \leq 0.5$ and $0 \leq s-\hat{t} \leq s \leq 0.5$, we have that $\Lambda'(s-\hat{t}) - \Lambda'(s) > 0$ for $0 < \hat{t} \leq g^{-1}(0.5)$. Therefore, $P'(\hat{t}) > 0$, and $P(\hat{t})$ is monotonically increasing. Since

$$P(0) \;=\; -0.453036... \;<\; 0 \qquad \text{and} \qquad P(0.22) \;=\; -0.00453122... \;<\; 0,$$

we obtain that $P(\hat{t}) < 0$ for $0 \leq \hat{t} \leq g^{-1}(0.5)$. This implies (99), completing the proof of the lemma. □

Lemma C.4. *Let* $(t, s, u) \in \hat{\Delta}$*. Then,* $t < 0.5$ *or* $u < 0.5$*.*

Proof. We suppose that $0.5 \leq t \leq 1$ and $0.5 \leq u \leq 1$. Then, the function of the defining inequality of $\hat{\Delta}$ is calculated as

$$\begin{aligned} &2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) + 2\,\Lambda(u) - \varsigma \\ &\leq\; \Lambda(s-t) - 2\,\Lambda(s) + \Lambda(s-u) - \varsigma \;\leq\; 4\,\Lambda\Big(\frac{1}{6}\Big) - \varsigma \;=\; -0.25994... \;<\; 0, \end{aligned}$$

and this contradicts the defining inequality of $\hat{\Delta}$, which implies the assumption of the lemma.

Hence, $t < 0.5$ or $u < 0.5$, as required. □

Lemma C.5. *Let* $(t, s, u) \in \hat{\Delta}$*. Then,* $t < 0.5$ *and* $u < 0.5$*.*

Proof. By Lemma C.4, we have that $t < 0.5$ or $u < 0.5$. We consider each of these two cases in the following of this proof.

We suppose that $t < 0.5$. Then, we show that $u < 0.5$, as follows. We consider the maximal value of $2\Lambda(t) + \Lambda(s-t) - 2\Lambda(s)$ for $0 \leq t < 0.5$ and $0.5 < s \leq 1$. As we showed in Section 4.2 (exchanging t and s), when $(t, s) = (\frac{1}{4}, \frac{3}{4})$, this maximal value is given by $4\Lambda(\frac{1}{4})$. Hence, by the defining inequality of $\hat{\Delta}$,

$$\begin{aligned} 0 \;&\leq\; \big(2\,\Lambda(t) + \Lambda(s-t) - 2\,\Lambda(s)\big) + \Lambda(s-u) + 2\,\Lambda(u) - \varsigma \\ &\leq\; 4\,\Lambda\Big(\frac{1}{4}\Big) + \Lambda(s-u) + 2\,\Lambda(u) - \varsigma\,. \end{aligned}$$

Therefore,

$$2\,\Lambda(u) \;\geq\; \varsigma - 4\,\Lambda\Big(\frac{1}{4}\Big) - \Lambda(s-u) \;\geq\; \varsigma - 4\,\Lambda\Big(\frac{1}{4}\Big) - \Lambda\Big(\frac{1}{6}\Big) \;=\; 0.161417... \;>\; 0.$$

Hence, $u < 0.5$, and we obtain the conclusion of the lemma in this case.

We suppose that $u < 0.5$. Then, it is shown that $t < 0.5$ from the above argument by exchanging t and u. Hence, we also obtain the conclusion of the lemma in this case. □

We now show a proof of Lemma C.1 by using above lemmas.

Proof of Lemma C.1. By Lemmas C.2, C.3 and C.5, the domain $\hat{\Delta}$ is included in

$$\big\{(t,s,u) \;\big|\; 0<t<0.5,\quad 0.5<s<1,\quad 0<u<0.5\big\}.$$

We put H to be the Hesse matrix of the function

$$2\,\Lambda(t)+\Lambda(s-t)-2\,\Lambda(s)+\Lambda(s-u)+2\,\Lambda(u)-\varsigma,$$

which defines $\hat{\Delta}$. By Lemma B.2, it is sufficient to show that H is negative definite.

We show that H is negative definite, as follows. We have that

$$\begin{aligned} H \;&=\; \begin{pmatrix} 2\,\Lambda''(t)+\Lambda''(s-t) & -\Lambda''(s-t) & 0 \\ -\Lambda''(s-t) & -2\,\Lambda''(s)+\Lambda''(s-t)+\Lambda''(s-u) & -\Lambda''(s-u) \\ 0 & -\Lambda''(s-u) & 2\,\Lambda''(u)+\Lambda''(s-u) \end{pmatrix} \\ &=\; -\pi \begin{pmatrix} 2\cot\pi t+\cot\pi(s{-}t) & -\cot\pi(s{-}t) & 0 \\ -\cot\pi(s{-}t) & -2\cot\pi s+\cot\pi(s{-}t)+\cot\pi(s{-}u) & -\cot\pi(s{-}u) \\ 0 & -\cot\pi(s{-}u) & 2\cot\pi u+\cot\pi(s{-}u) \end{pmatrix}. \end{aligned}$$

We put $a=\cot\pi t$, $b=-\cot\pi s$ and $c=\cot\pi u$, noting that they are positive. Further, noting that $\cot(\alpha{+}\beta)=(\cot\alpha\cot\beta-1)/(\cot\alpha+\cot\beta)$, we have that

$$-\frac{1}{\pi}\,H \;=\; \begin{pmatrix} 2a-\frac{ab-1}{a+b} & \frac{ab-1}{a+b} & 0 \\ \frac{ab-1}{a+b} & 2b-\frac{ab-1}{a+b}-\frac{bc-1}{b+c} & \frac{bc-1}{b+c} \\ 0 & \frac{bc-1}{b+c} & 2c-\frac{bc-1}{b+c} \end{pmatrix}.$$

It is sufficient to show that the above matrix is positive definite. By Lemma C.6 below, it is sufficient to show that

$$\begin{pmatrix} 2a-\frac{ab-1}{a+b} & \frac{ab-1}{a+b} \\ \frac{ab-1}{a+b} & b-\frac{ab-1}{a+b} \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} b-\frac{bc-1}{b+c} & \frac{bc-1}{b+c} \\ \frac{bc-1}{b+c} & 2c-\frac{bc-1}{b+c} \end{pmatrix}$$

are positive definite. Further, by Lemma C.7 below, they are positive definite. Therefore, H is negative definite, as required. □

The following two lemmas are used in the above proof of Lemma C.1.

Lemma C.6. *Let α_i and β_i be real numbers. If $\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_2 & \alpha_3\end{pmatrix}$ and $\begin{pmatrix}\beta_3 & \beta_2\\ \beta_2 & \beta_1\end{pmatrix}$ are positive definite, then*

$$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_2 & \alpha_3{+}\beta_3 & \beta_2 \\ 0 & \beta_2 & \beta_1 \end{pmatrix}$$

is positive definite.

Proof. Since $\big(\alpha_1\big)$ is obtained from $\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_2 & \alpha_3\end{pmatrix}$ by restricting $\mathbb{R}^2$ to $\mathbb{R}{\times}\{0\}$, $\big(\alpha_1\big)$ is positive definite, *i.e.*, $\alpha_1>0$. Hence, $\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_2 & \alpha_3\end{pmatrix}$ is related to $\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_3-\alpha_2^2/\alpha_1\end{pmatrix}$ by an elementary transformation as a quadratic form. Therefore, $\alpha_3 - \alpha_2^2/\alpha_1 > 0$. In the same way, we obtain that $\beta_1 > 0$ and $\beta_3 - \beta_2^2/\beta_1 > 0$. Further, the required matrix is related to the following matrix by elementary transformations,

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_3 - \frac{\alpha_2^2}{\alpha_1} + \beta_3 - \frac{\beta_2^2}{\beta_1} & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}.$$

Since the diagonal entries are positive, the required matrix is positive definite. □

Lemma C.7. *Let a, b and c be positive real numbers. Then, the following matrices are positive definite,*

$$\begin{pmatrix} 2a - \frac{ab-1}{a+b} & \frac{ab-1}{a+b} \\ \frac{ab-1}{a+b} & b - \frac{ab-1}{a+b} \end{pmatrix}, \qquad \begin{pmatrix} b - \frac{bc-1}{b+c} & \frac{bc-1}{b+c} \\ \frac{bc-1}{b+c} & 2c - \frac{bc-1}{b+c} \end{pmatrix}.$$

Proof. We show the lemma for the first matrix, as follows. Its trace is given by

$$2a + b - 2 \cdot \frac{ab-1}{a+b} \;=\; \frac{(2a+b)(a+b) - 2ab + 2}{a+b} \;=\; \frac{2a^2 + b^2 + ab + 2}{a+b} \;>\; 0.$$

Further, its determinant is given by

$$2ab - (2a+b) \cdot \frac{ab-1}{a+b} \;=\; \frac{2ab(a+b) - (2a+b)(ab-1)}{a+b} \;=\; \frac{ab^2 + 2a + b}{a+b} \;>\; 0.$$

Hence, the two eigenvalues of this matrix are positive, and the first matrix of the lemma is positive definite.

It is shown that the second matrix is also positive definite from the above argument replacing a with c. □

D Estimate of $|(q)_n|$ and restriction of Δ to Δ'

In this section, we give an estimate of $\frac{1}{N} \log|(q)_n|$ in Lemma D.2. By using this estimate, we explain why we can restrict Δ to Δ' in Sections 3.1, 4.1 and 5.1.

We explain the motivation to show Lemma D.2. By (6) and (11), we have that

$$\log\big|(q)_n\big| \;=\; \mathrm{Re}\,\varphi\big(\frac{1}{2N}\big) - \mathrm{Re}\,\varphi\big(\frac{2n+1}{2N}\big) \;=\; \frac{1}{2}\log N - \mathrm{Re}\,\varphi\big(\frac{2n+1}{2N}\big).$$

We arbitrarily fix a sufficiently small $\delta > 0$. Then, we can show by Proposition 2.1 that

$$\mathrm{Re}\,\frac{1}{N}\varphi\big(\frac{2n+1}{2N}\big) \;=\; \mathrm{Re}\,\frac{1}{2\pi\sqrt{-1}}\,\mathrm{Li}_2\big(e^{(2n+1)/(2N)}\big) \;+O\big(\frac{1}{N^2}\big) \;=\; \Lambda\big(\frac{2n+1}{2N}\big) \;+O\big(\frac{1}{N^2}\big),$$

for sufficiently large N and any integer n such that $\delta \le \frac{2n+1}{2N} \le 1-\delta$, where $O(\frac{1}{N^2})$ means that the absolute value of the error term is bounded by C_1/N^2 with some constant C_1, which is independent of N and n (but might be dependent on δ). Hence, there exists $C > 0$ such that

$$\Big| \frac{1}{N} \log \big|(q)_n\big| + \Lambda\big(\frac{2n+1}{2N}\big) \Big| \;\leq\; C \cdot \frac{\log N}{N}$$

for sufficiently large N and any integer n such that $\delta \leq \frac{2n+1}{2N} \leq 1-\delta$. The aim of Lemma D.2 is to extend this estimate to the case where $\delta = 0$.

Before we show Lemma D.2, we show the following lemma.

Lemma D.1. *There exist $C_1, C_2 > 0$ such that*

$$\Big| \Lambda\big(\frac{1}{N}\big) \Big| \;\leq\; C_1 \cdot \frac{\log N}{N}, \qquad \Big| \Lambda\big(\frac{1}{2N}\big) \Big| \;\leq\; C_2 \cdot \frac{\log N}{N},$$

for any sufficiently large N. That is,

$$\Lambda\big(\frac{1}{N}\big) \;=\; O\big(\frac{\log N}{N}\big), \qquad \Lambda\big(\frac{1}{2N}\big) \;=\; O\big(\frac{\log N}{N}\big).$$

Proof. Since $\Lambda'(t) = -\log 2 \sin \pi t$, we have that

$$\Lambda(t) \;=\; \int_0^t \big(-\log 2 \sin \pi s \big) ds.$$

Let t be sufficiently small. Since $\sin x \geq x - \frac{1}{6}x^3 = x(1 - \frac{1}{6}x^2)$ for sufficiently small x,

$$\begin{aligned} \Lambda(t) \;&\leq\; \int_0^t \big(-\log 2\pi s \,(1 - \frac{1}{6}\pi^2 s^2)\big) ds \;=\; \int_0^t \big(-\log 2\pi s - \log(1 - \frac{1}{6}\pi^2 s^2)\big) ds \\ &\leq\; \int_0^t \big(-\log 2\pi s - 2s^2\big) ds \;=\; -t \,\log 2\pi t + t + \frac{2}{3}\,t^3, \end{aligned}$$

where we obtain the second inequality, since $-\log(1 - \frac{1}{6}\pi^2 s^2) = \frac{1}{6}\pi^2 s^2 + O(s^4) \leq 2s^2$ for sufficiently small s. Hence,

$$\Lambda\big(\frac{1}{N}\big) \;\leq\; -\frac{1}{N} \log \frac{2\pi}{N} + \frac{1}{N} + \frac{2}{3N^3} \;=\; O\big(\frac{\log N}{N}\big).$$

Since $\Lambda(t)$ is non-negative for sufficiently small t, we obtain the first formula of the lemma.

We obtain the second formula of the lemma from the first formula by replacing N with $2N$. □

Lemma D.2. *There exists $C > 0$ such that*

$$\Big| \frac{1}{N} \log \big|(q)_n\big| + \Lambda\big(\frac{2n+1}{2N}\big) \Big| \;\leq\; C \cdot \frac{\log N}{N}$$

for any sufficiently large N and any integer n such that $0 \leq n < N$. That is,

$$\frac{1}{N} \log \big|(q)_n\big| \;=\; -\Lambda\big(\frac{2n+1}{2N}\big) \;+ O\big(\frac{\log N}{N}\big)$$

as $N \to \infty$ fixing a value of n/N.

Proof. When $n = 0$, we can verify the lemma by Lemma D.1. So, we assume that $1 \le n \le N-1$, in the following of this proof. From the definition of $(q)_n$, we have that

$$\log\big|(q)_n\big| \;=\; \log \prod_{1\le j\le n} \big|1-q^j\big| \;=\; \sum_{1\le j\le n} \log 2\sin\pi\frac{j}{N} \;=\; -\sum_{1\le j\le n} \Lambda'\big(\frac{j}{N}\big). \tag{100}$$

Further, we note that $\Lambda'(t)$ is a convex function for $0<t<1$, since $\Lambda'''(t) = \pi^2/\sin^2\pi t \ge \pi^2$ for $0<t<1$.

We give a lower bound of "$\frac{1}{N}\log\big|(q)_n\big| + \Lambda\big(\frac{2n+1}{2N}\big)$", as follows. Since $\Lambda'(t)$ is a convex function, the tangent line of the graph of $\Lambda'(t)$ at $t = N/j$ is located under the graph of $\Lambda'(t)$. Hence, by integrating them for $\frac{2j-1}{2N} \le t \le \frac{2j+1}{2N}$, we obtain that

$$\frac{1}{N}\Lambda'\big(\frac{j}{N}\big) \;\le\; \int_{(2j-1)/(2N)}^{(2j+1)/(2N)} \Lambda'(t)\,dt.$$

Further, by making the sum of the above formula over $1\le j\le n$, we have that

$$\frac{1}{N}\sum_{1\le j\le n}\Lambda'\big(\frac{j}{N}\big) \;\le\; \int_{1/(2N)}^{(2n+1)/(2N)} \Lambda'(t)\,dt \;=\; \Lambda\big(\frac{2n+1}{2N}\big) - \Lambda\big(\frac{1}{2N}\big).$$

Therefore, by (100),

$$\frac{1}{N}\log\big|(q)_n\big| + \Lambda\big(\frac{2n+1}{2N}\big) \;\ge\; \Lambda\big(\frac{1}{2N}\big) \;=\; O\big(\frac{\log N}{N}\big), \tag{101}$$

where we obtain the last equality by Lemma D.1.

We give an upper bound of "$\frac{1}{N}\log\big|(q)_n\big| + \Lambda\big(\frac{2n+1}{2N}\big)$", as follows. Since $\Lambda'(t)$ is a convex function, the line connecting $\big(\frac{j}{N}, \Lambda'(\frac{j}{N})\big)$ and $\big(\frac{j+1}{N}, \Lambda'(\frac{j+1}{N})\big)$ is located over the graph of $\Lambda'(t)$. Hence, by integrating them for $\frac{j}{N}\le t\le\frac{j+1}{N}$, we obtain that

$$\int_{j/N}^{(j+1)/N} \Lambda'(t)\,dt \;\le\; \frac{1}{2N}\Big(\Lambda'\big(\frac{j}{N}\big) + \Lambda'\big(\frac{j+1}{N}\big)\Big).$$

Further, by making the sum of the above formula over $1\le j\le n$, we have that

$$\int_{1/N}^{n/N} \Lambda'(t)\,dt \;\le\; \frac{1}{N}\sum_{1\le j\le n}\Lambda'\big(\frac{j}{N}\big) \;-\frac{1}{2N}\Big(\Lambda'\big(\frac{1}{N}\big) + \Lambda'\big(\frac{n}{N}\big)\Big).$$

Furthermore, in a similar way as above, we have that

$$\int_{n/N}^{(2n+1)/(2N)} \Lambda'(t)\,dt \;\le\; \frac{1}{4N}\Big(\Lambda'\big(\frac{n}{N}\big) + \Lambda'\big(\frac{2n+1}{2N}\big)\Big).$$

Hence, as the sum of the above two formulas, we obtain the following inequality,

$$\begin{aligned}
\Lambda\big(\frac{2n+1}{2N}\big) - \Lambda\big(\frac{1}{2N}\big) \;&=\; \int_{1/(2N)}^{(2n+1)/(2N)} \Lambda'(t)\,dt \\
\le\; \frac{1}{N}\sum_{1\le j\le n}\Lambda'\big(\frac{j}{N}\big) \;&-\frac{1}{2N}\Lambda'\big(\frac{1}{N}\big) - \frac{1}{4N}\Lambda'\big(\frac{n}{N}\big) + \frac{1}{4N}\Lambda'\big(\frac{2n+1}{N}\big).
\end{aligned}$$

Therefore, by (100),

$$\begin{aligned} &\frac{1}{N}\,\log\big|(q)_n\big| + \Lambda\big(\frac{2n+1}{2N}\big) \\ &\le\ \Lambda\big(\frac{1}{2N}\big)\ -\frac{1}{2N}\,\Lambda'\big(\frac{1}{N}\big) - \frac{1}{4N}\,\Lambda'\big(\frac{n}{N}\big) + \frac{1}{4N}\,\Lambda'\big(\frac{2n+1}{N}\big)\ =\ O\big(\frac{\log N}{N}\big), \end{aligned} \tag{102}$$

where we obtain the last equality by Lemma D.1 and from the fact that $\Lambda'(t) = -\log 2\sin\pi t$, noting that $1 \le n \le N-1$.

Therefore, by (101) and (102), we obtain the required formula of the lemma. □

Restriction of Δ to Δ' for the $\overline{6_1}$ knot

We explain why we can restrict Δ to Δ' in Section 3.1, that is, we show the equality of (32).

Proof of the equality of (32). By Lemma D.2, we can show that

$$\frac{1}{N}\,\log\big|\text{the summand of (29)}\big|\ =\ \mathrm{Re}\,\hat{V}\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big)\ + O\big(\frac{\log N}{N}\big). \tag{103}$$

We note that this convergence is uniform, *i.e.*, the absolute value of the error term is bounded by $C\cdot\frac{\log N}{N}$, where C is a constant which is independent of i,j,k. We recall that Δ' is a domain in Δ satisfying (30) for some $\varepsilon>0$. Then, in order to show the equality of (32), it is sufficient to show that the following sum is of the order $O(e^{-N\varepsilon'})$ for some $\varepsilon'>0$,

$$\begin{aligned} &\Bigg|\sum_{\substack{i,j,k\in\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta-\Delta'}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big) - N\varsigma\Big)\Bigg| \\ &\le \sum_{\substack{i,j,k\in\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta-\Delta'}} \Big|\exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big) - N\varsigma\Big)\Big| \\ &= \sum_{\substack{i,j,k\in\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta-\Delta'}} \exp\Big(N\cdot\mathrm{Re}\,V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big) - N\varsigma_R\Big). \end{aligned} \tag{104}$$

By (103), the convergence of $\mathrm{Re}\,V(\cdot)$ to $\mathrm{Re}\,\hat{V}(\cdot)$ is uniform. Hence, for any $\varepsilon_1>0$,

$$\mathrm{Re}\,V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big)\ \le\ \mathrm{Re}\,\hat{V}\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big) + \varepsilon_1$$

for sufficiently large N. Therefore, by choosing sufficiently small ε_1, we have that

$$(104)\ \le \sum_{\substack{i,j,k\in\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N})\in\Delta-\Delta'}} \exp\Big(N\cdot\mathrm{Re}\,\hat{V}\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N}\big) - N\varsigma_R + N\varepsilon_1\Big)\ =\ O\big(e^{-N\varepsilon_2}\big),$$

for some $\varepsilon_2>0$. Hence, we obtain the equality of (32). □

Restriction of Δ to Δ' for the 6_2 knot and the 6_3 knot

In similar ways as above, we can restrict Δ to Δ' for the sums of $\langle 6_2 \rangle_N$ and $\langle 6_3 \rangle_N$. That is, we can show (54) and (80) by using Lemma D.2 in similar ways as in the above proof for the $\overline{6_1}$ knot.

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