RIMS-1866

An Extended Probabilistic Serial Mechanism to the Random Assignment Problem with Multi-unit Demands and Polymatroidal Supplies

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November 2016

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November 21, 2016

Abstract

The authors have recently extended the probabilistic serial (PS) mechanism, due to Bogomolnaia and Moulin, of the random assignment problem to that with unit demands on a matroidal family of goods. In the present paper we consider a further extension of the PS mechanism for multi-unit demands on polymatroids and examine the properties of the extended PS mechanism. We show the ordinal efficiency and envy-freeness of the extended PS mechanism and also give a useful sufficient condition for the obtained solution to be a weakly sd Nash equilibrium. Moreover, we show that the extended PS mechanism is weakly strategy-proof when the underlying polymatroid is a matroid and agents have unit demands.

Keywords: Random assignment, probabilistic serial mechanism, ordinal preference, matchings, polymatroids, independent flows, submodular optimization, Nash equilibrium

1. Introduction

Problems of allocating indivisible goods to agents having preferences over the goods, in a fair and efficient manner without money, have long been investigated in the literature (see, e.g., [23, 26, 1, 5, 17, 18, 4, 14, 15, 3, 24]). A seminal paper of Bogomolnaia and Moulin

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[5] shows a probabilistic serial mechanism to give a solution to the problem, called the random assignment problem, by using lotteries. Most of the investigated problems about the random assignment problem treat a fixed feasible (available) set of goods while they are extended to the case of multi-unit demands [2, 6, 15, 18, 3].

Another extension of the random assignment problem is made by the present authors [12] to the case where we are given a family $\mathcal{B}$ of feasible sets of indivisible goods which forms a family of bases of a matroid (see [25, 22]). Paper [6] also investigated extensions of the ordinary random assignment problem with additional constraints, which are different from our (poly)matroidal ones. They considered a bihierarchy structure by means of laminar families to guarantee the existence of a lottery that realizes the required randomized solution. Actually, the laminar family they considered generates a polymatroid of multi-terminal network flows (see [20] and [11, Sec. 2.2]) and they imposed two polymatroids of multi-terminal network flows on $N \times E$, where N is the set of agents and E is the set of goods. Hence the integrality property of the bihierarchy model [6] follows from the integrality property of network flows.

In this paper we consider the random assignment problem where each agent has a multi-unit demand and the set of feasible (available) vectors of multiple goods forms the set of integral bases of a polymatroid. This is a common generalization of the problem with multi-unit demands considered by [2, 15, 18] and that with a matroidal feasible set of goods by the authors [12].

We show that the results obtained in [12] can naturally be extended to the present problem with polymatroidal supplies. That is, our extended probabilistic serial (extended PS) mechanism gives a solution that is efficient and envy-free with respect to the partial order defined by the stochastic dominance relation employed by Bogomolnaia and Moulin [5]. Moreover, we show that the extended PS mechanism gives us a solution that is a weakly sd Nash equilibrium ([8, 16]) under a certain practically useful condition. This is new even for the ordinary random assignment problem with multi-unit demands. Furthermore, we show that the extended PS mechanism is weakly strategy-proof when the underlying polymatroid is a matroid and agents have unit demands.

The well-known Birkhoff-von Neumann theorem on bi-stochastic matrices shows that every bi-stochastic matrix is expressed as a convex combination of permutation matrices, which plays a crucial rôle in designing the probabilistic serial mechanism developed by Bogomolnaia and Moulin [5]. On the other hand, our extended PS mechanism heavily depends on the results of submodular optimization such as the integrality of the independent flow polyhedra ([9, 11]), which generalizes the Birkhoff-von Neumann theorem.

The present paper is organized as follows. Definitions and some preliminaries to be used in the paper are given in Section 2. In Section 3 we describe the random assignment problem with multi-unit demands and polymatroidal supplies. We first treat the random assignment problem as an allocation of divisible goods in Section 4 and give the procedure Extended_Random_Assignment to obtain a solution called the extended PS solution.

We also show the ordinal efficiency and the envy-freeness of the extended PS solution. In Section 5 we discuss the strategy-proofness. Though the extended PS mechanism is not weakly strategy-proof in general, we give a useful sufficient condition that guarantees that the obtained solution is a weakly sd Nash equilibrium and also show the weak strategy-proofness in case of unit demands and matroidal supplies. Section 6 gives a randomized mechanism that generates the extended PS solution by means of lotteries on deterministic assignments of indivisible goods. Section 7 concludes the paper.

2. Definitions and Preliminaries

Let N be a finite set of *agents* and E be that of *goods*. Suppose that $|N| = n$ and $|E| = m$, where $|\cdot|$ denotes the cardinality. For any subset $X \subseteq E$ denote by χ_X the characteristic vector of X in $\mathbb{R}^E$, i.e., $\chi_X(e) = 1$ for $e \in X$ and $\chi_X(e) = 0$ for $e \in E \setminus X$. We also write χ_e instead of $\chi_{\{e\}}$ for $e \in E$.

A pair (E, ρ) of a finite nonempty set E and a function $\rho : 2^E \to \mathbb{R}_{\geq 0}$ is called a *polymatroid* if the following three conditions hold (see, e.g., [7, 11, 25]).

1. $\rho(\emptyset) = 0$.
2. For any $X, Y \in 2^E$ with $X \subseteq Y$ we have $\rho(X) \leq \rho(Y)$.
3. For any $X, Y \in 2^E$ we have $\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$.[1]

The set E is called the *ground set* and the function ρ is called the *rank function* of the polymatroid (E, ρ). We assume $\rho(E) > 0$ in the sequel.

For a given polymatroid (E, ρ), let $\mathrm{B}(\rho)(\subseteq \mathbb{R}^E)$ be the *base polytope* of the polymatroid (see, e.g., [11]), which is given by

$$\mathrm{B}(\rho) = \{x \in \mathbb{R}^E \mid \forall X \subset E : x(X) \leq \rho(X),\ x(E) = \rho(E)\}, \tag{2.1}$$

where for any $X \subseteq E$ we define $x(X) = \sum_{e \in X} x(e)$. It should be noted that $\mathrm{B}(\rho) \subseteq \mathbb{R}^E_{\geq 0}$. Also consider the lower hereditary closure of the base polytope $\mathrm{B}(\rho)$ given by

$$\mathrm{P}(\rho) = \{x \in \mathbb{R}^E \mid \forall X \subseteq E : x(X) \leq \rho(X)\}, \tag{2.2}$$

which is called the *submodular polyhedron* associated with ρ. The polytope $\mathrm{P}_{(+)}(\rho) \equiv \mathrm{P}(\rho) \cap \mathbb{R}^E_{\geq 0}$ is called the *independence polytope* of polymatroid (E, ρ) and each vector in $\mathrm{P}_{(+)}(\rho)$ is called an *independent vector*. Given a vector $x \in \mathrm{P}(\rho)$, a subset X of E is called *tight* for x (or x*-tight* for short) if we have $x(X) = \rho(X)$, and there exists a unique

[1] A set function satisfying these inequalities is called a *submodular function* and the negative of a submodular function is called a *supermodular function*. A function that is submodular and at the same time supermodular is called a *modular function*.

maximal x-tight set, denoted by $\mathrm{sat}(x)$, which is equal to the union of all tight sets for x. We also have

$$\mathrm{sat}(x) = \{e \in E \mid \forall \alpha > 0 : x + \alpha\chi_e \notin \mathrm{P}(\rho)\}, \tag{2.3}$$

which is the set of elements $e \in E$ for which we cannot increase $x(e)$ without leaving $\mathrm{P}(\rho)$. Moreover, for $x \in \mathrm{P}(\rho)$ and $e \in \mathrm{sat}(x)$ define

$$\mathrm{dep}(x, e) = \{e' \in E \mid \exists \alpha > 0 : x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{P}(\rho)\}. \tag{2.4}$$

The following fact is fundamental in the theory of polymatroids and submodular functions.

- Given any $x \in \mathrm{P}(\rho)$ and $X, Y \subseteq E$, if we have $x(X) = \rho(X)$ and $x(Y) = \rho(Y)$, then $x(X \cup Y) = \rho(X \cup Y)$ and $x(X \cap Y) = \rho(X \cap Y)$. That is, the set of x-tight sets is closed with respect to the set union and intersection.

Because of this fact $\mathrm{sat}(x)$ for $x \in \mathrm{P}(\rho)$ is the unique maximal x-tight set and $\mathrm{dep}(x, e)$ for $e \in \mathrm{sat}(x)$ is the unique minimal x-tight set that includes e. (See [11] for more details about these concepts and related facts.)

For any polymatroid (E, ρ) with an integer-valued rank function ρ define

$$\mathrm{B}_{\mathbb{Z}}(\rho) = \mathrm{B}(\rho) \cap \mathbb{Z}^E, \qquad \mathrm{P}_{\mathbb{Z}}(\rho) = \mathrm{P}(\rho) \cap \mathbb{Z}^E. \tag{2.5}$$

It is known (see, e.g., [11]) that when (E, ρ) is a polymatroid with an integer-valued rank function ρ, $\mathrm{B}(\rho)$ (resp. $\mathrm{P}(\rho)$) is the convex hull of $\mathrm{B}_{\mathbb{Z}}(\rho)$ (resp. $\mathrm{P}_{\mathbb{Z}}(\rho)$).

Simple examples of polymatroids are given as follows. They will be used to show the behavior of our solution in the next section.

Matroids: A matroid (E, ρ) is a polymatroid with an integer-valued rank function having the property that $\rho(\{e\}) \leq 1$ for all $e \in E$. This is also the one treated in our previous paper ([12]). Matroids arise from various combinatorial objects such as graphs, networks, and matrices (see, e.g., [22, 25]). □

Symmetric polymatroids: Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing concave function with $g(0) = 0$. Define $\rho : 2^E \to \mathbb{R}$ by $\rho(X) = g(|X|)$. Then (E, ρ) is a polymatroid. Note that the concavity of g corresponds to the law of diminishing marginal utility in economics. □

Linear polymatroids: Let V be a vector space. Let E be a finite set and for each $e \in E$ let F_e be a finite set of vectors in V. Define $\rho : 2^E \to \mathbb{R}$ by $\rho(X) = \mathrm{rank}(\bigcup_{e \in X} F_e)$ for all $X \subseteq E$. Then (E, ρ) is a polymatroid with the integer-valued rank function ρ. □

Polymatroids of multi-terminal network flows ([20],[11, Sec 2.2]; also see [10, 13]): Let $\mathcal{N} = (G = (V, A), s, T, c)$ be a network, where $G = (V, A)$ is a graph with a vertex set V and an arc set A, $s \in V$ is a source, $T \subset V \setminus \{s\}$ is a set of sink terminals, and

$c : A \to \mathbb{R}_{>0}$ is a capacity function of the network. We suppose that there exists no arc leaving T. A function $\varphi : A \to \mathbb{R}_{\geq 0}$ is called a feasible flow in $\mathcal{N}$ if it satisfies the capacity constraints

$$0 \leq \varphi(a) \leq c(a) \qquad (\forall a \in A) \tag{2.6}$$

and the flow conservation constraints

$$\partial\varphi(v) = 0 \qquad (\forall v \in V \setminus (\{s\} \cup T)), \tag{2.7}$$

where the boundary $\partial\varphi : V \to \mathbb{R}$ of flow φ is defined by

$$\partial\varphi(v) = \sum_{(v,w)\in A} \varphi(v,w) - \sum_{(w,v)\in A} \varphi(w,v) \qquad (\forall v \in V). \tag{2.8}$$

Also define the out-flow $\partial^-\varphi : T \to \mathbb{R}_{\geq 0}$ of φ by

$$\partial^-\varphi(v) = \sum_{(w,v)\in A} \varphi(w,v) \; (= -\partial\varphi(v)) \qquad (\forall v \in T). \tag{2.9}$$

Then the set of out-flows $\partial^-\varphi$ of all feasible flows in $\mathcal{N}$ is the independence polytope, in $\mathbb{R}^T_{\geq 0}$, of a polymatroid on T.

When the underlying graph $G = (V, A)$ is a star such that $V = \{s\} \cup T$ and $A = \{(s,t) \mid t \in T\}$, we have a polymatroid on T with a modular rank function ρ such that $\rho(X) = \sum_{v\in X} c(s,v)$ for all $X \subseteq T$. Any polymatroid of this kind has a unique base and vice versa. $\square$

3. Model Description

Let $N = \{1, 2, \cdots, n\}$ be a set of agents and E be a set of goods. Each good $e \in E$ should be considered as a type of good and the number of available good e can be more than one. Each agent $i \in N$ wants to obtain a certain amount of goods, denoted by $d(i) \in \mathbb{Z}_{>0}$, in total. We refer to $d(i)$ as the *demand* of agent i. The vector $d = (d(i) \mid i \in N) \in \mathbb{Z}^N_{>0}$ is called the *demand vector*. For each $i \in N$ and $e \in E$ let $x^i(e)$ be the number of copies of good e that agent i obtains. Then we must have

$$x^i(E) \equiv \sum_{e\in E} x^i(e) = d(i) \tag{3.1}$$

and the sum of vectors $\sum_{i\in N} x^i$ must be available in the market. Let the set of all available vectors of goods in the market be given by $\mathbf{B} \subseteq \mathbb{Z}^E_{\geq 0}$. Hence we must have

$$\sum_{i\in N} x^i \in \mathbf{B}. \tag{3.2}$$

We suppose that each agent $i \in N$ has an ordinal *preference* $\succ_i$ over set E of goods, which is a linear ordering of E. Let agent i's preference be given by

$$L^i : \; e_1^i \succ_i e_2^i \succ_i \cdots \succ_i e_m^i, \tag{3.3}$$

where $\{e_1^i, e_2^i, \cdots, e_m^i\} = E$ and e_1^i is the most favorite good for agent i. Let $\mathcal{L}$ be the profile of preferences L^i $(i \in N)$.

We consider a problem of allocating multiple goods to agents in an efficient and fair manner (to be defined later) under the constraints (3.1) and (3.2) and the preference profile $\mathcal{L} = (L^i \mid i \in N)$. In order to give a solution to the problem without money we will introduce lotteries, so that we call the problem a *random assignment problem* as in the literature [5, 17, 4]. The *random assignment problem* is denoted by $\mathbf{RA} = (N, E, \mathcal{L}, d, \mathbf{B})$. The problem to be considered in the present paper includes the following as special cases.

(a) The ordinary random assignment problem considered in the literature is mostly the case where $d = \mathbf{1} \in \mathbb{Z}_{>0}^N$ and $\mathbf{B} = \{\mathbf{1}\} \subseteq \mathbb{Z}_{>0}^E$ (e.g., [5, 17, 4]). Here $\mathbf{1}$ denotes a vector of all ones of appropriate dimension (determined by the context).

(b) Kojima [18], Aziz [2], and Heo [15] considered a multi-unit demand case where $d \in \mathbb{Z}_{>0}^N$ and $\mathbf{B} = \{b\} \subseteq \mathbb{Z}_{>0}^E$ for some $b \in \mathbb{Z}_{>0}^E$.

(c) In the unit-demand problem on the full preference domain considered by Katta and Sethuraman [17], if all agents have the same (common) set of indifference classes of goods and each agent has a strict preference on the set of the indifference classes, the problem reduces to a unit demand problem with strict preference domains on the set E^* of the common indifference classes of goods, by considering each indifference class as a type of goods, where $d = \mathbf{1} \in \mathbb{Z}_{>0}^N$ and $\mathbf{B} = \{b\} \subseteq \mathbb{Z}_{>0}^{E^*}$ for some $b \in \mathbb{Z}_{>0}^{E^*}$.

(d) The authors [12] considered a matroidal extension of (a) where $d = \mathbf{1} \in \mathbb{Z}_{>0}^N$ and $\mathbf{B}$ is the set of all characteristic vectors of bases of a matroid on E. In this model the constraint (3.1) is relaxed as $\sum_{e \in E} x^i(e) \in \{0, 1\}$ for all $i \in N$.

In the present paper, we consider a common generalization of the above models (a)–(d) by considering multi-unit demands and polymatroidal available vectors instead of matroidal ones. That is, we consider the case where $d \in \mathbb{Z}_{>0}^N$ and $\mathbf{B} = \mathrm{B}_{\mathbb{Z}}(\rho)$, the set of integral vectors in the base polytope of a polymatroid (E, ρ) with an integer-valued rank function ρ. Note that when $\mathbf{B}$ is a singleton set as in (a), (b), and (c) above, the underlying polymatroid (E, ρ) has the unique base and the rank function ρ is modular.

4. Random Assignment with Multi-unit Demands and Polymatroidal Supplies

In the seminal paper [5] of Bogomolnaia and Moulin they proposed a new solution through what is called the *probabilistic serial* (PS) mechanism from the point of view of stochastic dominance when there is only one feasible set of goods. In [12] we extended the PS mechanism so as to deal with the case where we are given a set of feasible sets of goods that forms a family of bases of a matroid. In the present paper we further extend the results of [12] to the case where agents have multi-unit demands and the set of feasible (available) vectors is the set $\mathrm{B}_{\mathbb{Z}}(\rho)$ of integral vectors of a base polytope.

Let us consider a random assignment problem given by $\mathbf{RA} = (N, E, \mathcal{L} = (L^i \mid i \in N), d = (d(i) \mid i \in N) \in \mathbb{Z}^N_{>0}, \mathrm{B}_{\mathbb{Z}}(\rho) \subseteq \mathbb{Z}^E_{\geq 0})$. The constraints (3.1) and (3.2) imply $\sum_{i \in N} d(i) = \rho(E)$. However, in order to treat the case where $\sum_{i \in N} d(i) \geq \rho(E)$ as considered in [12] we relax (3.1) as

$$x^i(E) \equiv \sum_{e \in E} x^i(e) \leq d(i) \qquad (\forall i \in N). \tag{4.1}$$

4.1. The random assignment as an allocation of divisible goods

First, we consider the base polytope $\mathrm{B}(\rho)$ (the convex hull of $\mathrm{B}_{\mathbb{Z}}(\rho)$) as a set of *divisible* goods and find an allocation of the divisible goods in an efficient and fair manner, where the precise definitions of efficiency and fairness will be given in Sections 4.2 and 4.3.

Let P be an $N \times E$ real matrix satisfying the following three conditions:

1. $P(i, e) \geq 0$ for all $i \in N$ and $e \in E$.

2. For each agent $i \in N$ we have

$$\sum_{e \in E} P(i, e) \leq d(i). \tag{4.2}$$

   (Except for the integrality condition this is equivalent to (4.1) with $x^i(e) = P(i, e)$ for all $i \in N$ and $e \in E$.)

3. Regarding each ith row $P_i \equiv (P(i, e) \mid e \in E)$ of P as a vector in $\mathbb{R}^E_{\geq 0}$, we have

$$\sum_{i \in N} P_i \in \mathrm{B}(\rho). \tag{4.3}$$

   (Except for the integrality condition this is equivalent to (3.2) with $x^i(e) = P(i, e)$ for all $i \in N$ and $e \in E$.)

Then we call P a *random assignment matrix* (or a *random assignment* for short).

Let (E, ρ) be a polymatroid with $\rho(E) \le \sum_{i \in N} d(i)$, and we consider the case where $\mathbf{B} = \mathrm{B}(\rho)$ in the following. Let us define the base x^*_P associated with a random assignment matrix P by

$$x^*_P \equiv \sum_{i \in N} P_i. \tag{4.4}$$

Recall that for each $i \in N$ agent i's preference is given by (3.3), where $\{e^i_1, e^i_2, \cdots, e^i_m\} = E$ and e^i_1 is the most favorite good for agent i, and $\mathcal{L}$ is the profile of preferences L^i $(i \in N)$. Based on the collection (a multiset) of the first (most favorite) elements e^i_1 of all agents $i \in N$, define a nonnegative integral vector $b(\mathcal{L}) \in \mathbb{Z}^E_{\ge 0}$ by

$$b(\mathcal{L}) = \sum_{i \in N} d(i)\chi_{e^i_1}, \tag{4.5}$$

where note that we may have $e^i_1 = e^j_1$ for distinct $i, j \in N$ and $d(i)$ is the integral demand of agent $i \in N$.

We also denote the random assignment problem by $\mathbf{RA} = (N, E, \mathcal{L} = (L^i \mid i \in N), d = (d(i) \mid i \in N), (E, \rho))$. Our random assignment algorithm by the extended PS mechanism is described as follows. During the execution of the following algorithm the current preference lists L^i may get shorter because of removal of exhausted (or saturated) goods.

Extended_Random_Assignment

Input: A random assignment problem $\mathbf{RA} = (N, E, \mathcal{L}, d, (E, \rho))$.

Output: A random assignment matrix $P \in \mathbb{R}^{N \times E}_{\ge 0}$.

Step 0: For each $i \in N$ put $x^i \leftarrow \mathbf{0} \in \mathbb{R}^E$ (the zero vector), and $x^* \leftarrow \mathbf{0} \in \mathbb{R}^E$.
Put $S_0 \leftarrow \emptyset$, $p \leftarrow 1$, and $\lambda_0 \leftarrow 0$.

Step 1: For current (updated) $\mathcal{L} = (L^i \mid i \in N)$, using $b(\mathcal{L})$ in (4.5), compute

$$\lambda_p = \max\{t \ge \lambda_{p-1} \mid x^* + (t - \lambda_{p-1})b(\mathcal{L}) \in \mathrm{P}(\rho)\}. \tag{4.6}$$

For each $i \in N$ put $x^i \leftarrow x^i + (\lambda_p - \lambda_{p-1})d(i)\chi_{e^i_1}$.
Put $x^* \leftarrow x^* + (\lambda_p - \lambda_{p-1})b(\mathcal{L})$ and $S_p \leftarrow \mathrm{sat}(x^*)$.

Step 2: Put $T_p \leftarrow S_p \setminus S_{p-1}$.
Update L^i $(i \in N)$ by removing all elements of T_p from current L^i $(i \in N)$.

Step 3: If $\rho(S_p) < \rho(E)$, then put $p \leftarrow p + 1$ and go to Step 1.
Otherwise $(\rho(S_p) = \rho(E))$ put $P(i, e) \leftarrow x^i(e)$ for all $i \in N$ and $e \in E$.
Return P.

As in [5], the parameter t can be considered as time and each agent $i \in N$ eats the current top good e_1^i at the rate $d(i)$ per unit time.

To see the behavior of the procedure Extended_Random_Assignment let us consider illustrative examples as follows.

Example 1: Consider $N = \{1, 2, 3, 4\}$ and $E = \{a, b, c, d\}$. Let (E, ρ) be a polymatroid with a rank function given by

$$\rho(X) = \begin{cases} 4|X| & \text{if } |X| \le 2 \\ 8 & \text{if } |X| > 2 \end{cases} \qquad (\forall X \subseteq E). \tag{4.7}$$

Note that (E, ρ) here is a symmetric polymatroid. Suppose that preferences of all agents are given as follows.

$i \in N$	preference L^i
1	$a \succ_1 b \succ_1 c \succ_1 d$
2	$a \succ_2 c \succ_2 b \succ_2 d$
3	$a \succ_3 c \succ_3 d \succ_3 b$
4	$b \succ_4 a \succ_4 d \succ_4 c$

Let $d = (4, 2, 1, 1)$ be a demand vector. Then by Extended_Random_Assignment we have

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} \frac{16}{7} & \frac{12}{7} & 0 & 0 \\ \frac{8}{7} & 0 & \frac{6}{7} & 0 \\ \frac{4}{7} & 0 & \frac{3}{7} & 0 \\ 0 & \frac{4}{7} + \frac{3}{7} & 0 & 0 \end{pmatrix} \end{array},$$

where

$$b(\mathcal{L}) = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ (4+2+1, \ 1, \ 0, \ 0) \end{array}, \quad S_1 = \{a\}, \quad \lambda_1 = \frac{4}{7} \ \text{ for } \ p = 1$$

and

$$b(\mathcal{L}) = (0, 4+1, 2+1, 0), \quad S_2 = \{a, b, c, d\}, \quad \lambda_2 = \lambda_1 + \frac{3}{7} \ \text{ for } \ p = 2$$

to get the random assignment matrix P given above. Also, vectors $x^*_{\lambda_p}$, which are the restriction of x^*_P on $T_p = S_p \setminus S_{p-1}$ for $p = 1, 2$, are given by

$$\begin{aligned} &T_1 = \{a\}, \quad T_2 = \{b, c, d\}, \\ &x^*_{\lambda_1}(a) = 4, \quad x^*_{\lambda_2}(b) = \tfrac{19}{7}, \quad x^*_{\lambda_2}(c) = \tfrac{9}{7}, \quad x^*_{\lambda_2}(d) = 0. \end{aligned}$$

Hence $x^*_P = (4, \frac{19}{7}, \frac{9}{7}, 0)$. Note that $\emptyset$, $\{a\}$, $\{a, b, c\}$, and $\{a, b, c, d\}(= \mathrm{sat}(x^*_P))$ are tight sets for x^*_P. $\square$

Example 2: Let N, E, L^i $(i \in N)$, and d be the same as Example 1, while let $\rho : 2^E \to \mathbb{Z}_{\geq 0}$ be a function defined by

$$\rho(X) = \begin{cases} 4|X| & \text{if } |X| \leq 2 \text{ and } X \neq \{a,b\} \\ 4 & \text{if } X = \{a,b\} \\ 8 & \text{if } |X| > 2 \end{cases} \qquad (\forall X \subseteq E). \tag{4.8}$$

Then we get

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \end{array},$$

where

$$\begin{aligned} b(\mathcal{L}) &= (4+2+1, 1, 0, 0), \quad S_1 = \{a,b\}, \quad \lambda_1 = \tfrac{1}{2} \ \text{ for } \ p = 1, \\ b(\mathcal{L}) &= (0, 0, 4+2+1, 1), \quad S_2 = \{a,b,c,d\}, \quad \lambda_2 = \lambda_1 + \tfrac{1}{2} \ \text{ for } \ p = 2. \end{aligned}$$

Also, vectors $x^*_{\lambda_p}$ on $T_p = S_p \setminus S_{p-1}$ for $p = 1, 2$ are given by

$$\begin{aligned} &T_1 = \{a,b\}, \quad T_2 = \{c,d\}, \\ &x^*_{\lambda_1}(a) = \tfrac{7}{2}, \quad x^*_{\lambda_1}(b) = \tfrac{1}{2}, \quad x^*_{\lambda_2}(c) = \tfrac{7}{2}, \quad x^*_{\lambda_2}(d) = \tfrac{1}{2}. \end{aligned}$$

Hence $x^*_P = (\frac{7}{2}, \frac{1}{2}, \frac{7}{2}, \frac{1}{2})$. We have tight sets $\emptyset$, $\{a,b\}$ and $\{a,b,c,d\}$ for x^*_P. □

We will show that the random assignment matrix P obtained by Extended_Random_Assignment is an efficient and envy-free allocation of divisible goods in $\mathrm{B}(\rho)$, where precise definitions of efficiency and envy-freeness will be given below.

4.2. Ordinal efficiency

Let P and Q be random assignment matrices for Problem $\mathbf{RA} = (N, E, \mathcal{L} = (L^i \mid i \in N), d, (E, \rho))$. For each agent $i \in N$ with preference relation $\succ_i$ given by $e^i_1 \succ_i \cdots \succ_i e^i_m$, define a relation (*sd-dominance* relation[2]) $\succeq^{\mathrm{d}}_i$ between the ith rows P_i and Q_i of P and Q, respectively, as follows.

$$P_i \succeq^{\mathrm{d}}_i Q_i \quad \Longleftrightarrow \quad \forall \ell \in \{1, \cdots, m\} : \ \sum_{k=1}^{\ell} P(i, e^i_k) \geq \sum_{k=1}^{\ell} Q(i, e^i_k). \tag{4.9}$$

The random assignment matrix P is *sd-dominated* by Q if we have $Q_i \succeq^{\mathrm{d}}_i P_i$ for all $i \in N$ and $P \neq Q$. We say that P is *ordinally efficient* if P is not sd-dominated by any other random assignment ([5]). The following theorem can be shown in a very similar way as the corresponding one in [5] and [12].

[2] sd stands for (first-order) stochastic dominance employed in [5].

Theorem 4.1: *The procedure* Extended_Random_Assignment *computes a random assignment matrix P that is ordinally efficient.*

(Proof) By Procedure Extended_Random_Assignment we get a random assignment P together with a chain $S_0 = \emptyset \subset S_1 \subset \cdots \subset S_p = E$. Let Q be an arbitrary random assignment and suppose that $Q = P$ or Q sd-dominates P. It suffices to prove $Q = P$.

At the qth execution of Step 1 of Extended_Random_Assignment define

$$F_q = \{i \in N \mid e_1^i \in T_q\}. \tag{4.10}$$

Let us denote e_1^i (the top element in current L^i) at the qth execution of Step 1 by $e_1^i(q)$ and suppose that for some integer $q^* \geq 1$ we have

$$Q(i, e_1^i(q)) = P(i, e_1^i(q)) \qquad (\forall q = 1, \cdots, q^* - 1, \ \forall i \in F_q) \tag{4.11}$$

and we execute the q^*th Step 1. Then, because of Step 1 of Extended_Random_Assignment we have

$$\sum_{i \in F_q} P(i, e_1^i(q)) = \rho(S_q) - \rho(S_{q-1}) \qquad (q = 1, \cdots, q^*). \tag{4.12}$$

Since $Q = P$ or Q sd-dominates P, it follows from (4.11) that $Q(i, e_1^i(q^*)) \geq P(i, e_1^i(q^*))$ for all $i \in F_{q^*}$. Hence from (4.11) and (4.12) we must have

$$Q(i, e_1^i(q^*)) = P(i, e_1^i(q^*)) \qquad (\forall i \in F_{q^*}), \tag{4.13}$$

since we have $\sum_{i \in F_{q^*}} Q(i, e_1^i(q^*)) \leq \rho(S_{q^*}) - \rho(S_{q^*-1})$. (Here, $\sum_{q=1}^{q^*} \sum_{i \in F_q} Q(i, e_1^i(q)) \leq \rho(S_{q^*})$.)

Now, note that when $q^* = 1$, (4.11) is void (and thus holds). Hence, by induction on $q = 1, \cdots, p$, we have shown $Q = P$. □

4.3. Envy-freeness

We say a random assignment P is *normalized envy-free* ([15]) with respect to a profile of ordinal preferences $\succ_i$ for all $i \in N$ if for all $i, j \in N$ we have $\frac{1}{d(i)} P_i \succeq_i^{\mathrm{d}} \frac{1}{d(j)} P_j$.

We have the following theorem on normalized envy-freeness of the extended PS mechanism. The proof is actually a direct adaptation of the one given by Bogomolnaia and Moulin [5] and Schulman and Vazirani [24] for a non-matroidal problem setting (also see [15, 12]). It should be noted that by Extended_Random_Assignment every agent $i \in N$ eats $d(i)$ units of goods per unit time.

Theorem 4.2: *The procedure* Extended_Random_Assignment *computes a random assignment matrix P that is normalized envy-free.*

(Proof) It suffices to show that for any $i \in N$ and $k \in \{1, \cdots, m\}$ we have

$$\frac{1}{d(i)} \sum_{\ell=1}^{k} P(i, e_\ell^i) \geq \frac{1}{d(j)} \sum_{\ell=1}^{k} P(j, e_\ell^i) \qquad (\forall j \in N). \tag{4.14}$$

Define

$$t_k^i = \frac{1}{d(i)} \sum_{\ell=1}^{k} P(i, e_\ell^i). \tag{4.15}$$

When good e_k^i is removed after an execution of Step 1, all goods e_ℓ^i $(\ell = 1, \cdots, k)$ have been removed from E. It follows that for all $j \in N$ the time spent by agent j to eat e_ℓ^i $(\ell = 1, \cdots, k)$ given by the sum of possible values $\frac{1}{d(j)} P(j, e_\ell^i)$ for goods e_ℓ^i $(\ell = 1, \cdots, k)$ is within t_k^i. Hence we must have

$$t_k^i \geq \frac{1}{d(j)} \sum_{\ell=1}^{k} P(j, e_\ell^i) \qquad (\forall j \in N). \tag{4.16}$$

□

5. Strategy-proofness

It is known that the extension of the PS mechanism of Bogomolnaia and Moulin to the case of multi-unit demands cannot be weakly strategy-proof in general ([6, 15, 18, 3]). Therefore, our polymatroidal extension is not weakly strategy-proof in general either.

Note that a solution mechanism M is *weakly strategy-proof* if for *every* input preference profile $\mathcal{L}$ the mechanism M gives a solution P such that every misreport of every agent i's preference results in a solution Q satisfying that Q_i does not sd-dominate P_i for i. Here the strategy-proofness is concerned with the mechanism.

5.1. Weakly sd Nash equilibrium

Let us consider the concept of a *weakly sd Nash equilibrium* ([8, 16]), which is a property of the obtained solution. For a *given input profile* we say that the solution P obtained by the mechanism M is called a *weakly sd Nash equilibrium* if every misreport of every agent i's preference results in a solution Q satisfying that Q_i does not sd-dominate P_i for i. The solution of the extended PS mechanism for multi-unit demands was investigated from the point of view of the weakly sd Nash equilibrium in [8, 16].

We examine our polymatroidal extension and give a certain (useful) sufficient condition for our solution to be a weakly sd Nash equilibrium. The result, Theorem 5.4 given below, seems to be new even for the ordinary multi-demand case where the base polytope consists of a single base, i.e., $\mathrm{B}(\rho) = \{b\}$ for some $b \in \mathbb{Z}^E_{>0}$. At the end of this section (Sec. 5.2) we also show the weak strategy-proofness in the special case of unit demands and matroidal supplies considered in [12].

5.1.1. Lemmas

We first prepare a few lemmas to prove Theorem 5.4 concerned with a condition for our extended PS solution to be weakly sd Nash equilibrium.

For any vector $x \in \mathbb{R}^E$ and any set $A \subseteq E$ define $x^A \in \mathbb{R}^A$ by $x^A(e) = x(e)$ for all $e \in A$.

Lemma 5.1: *Let* $\mathbf{P} = (E, \rho)$ *be a polymatroid. For any vectors* $x, y \in \mathrm{P}(\rho)$ *satisfying* $x(e) \geq y(e)$ $(\forall e \in E \setminus \mathrm{sat}(x))$, *we have* $\mathrm{sat}(x) \supseteq \mathrm{sat}(y)$.

(Proof) Suppose that we are given vectors $x, y \in \mathrm{P}(\rho)$ satisfying $x(e) \geq y(e)$ $(\forall e \in E \setminus \mathrm{sat}(x))$. Increase the values of $y(e)$ for all $e \in \mathrm{sat}(x)$ as much as possible while keeping the vector within $\mathrm{P}(\rho)$, and let us denote by y' the resulting vector in $\mathrm{P}(\rho)$. Then, we have $\mathrm{sat}(y') \supseteq \mathrm{sat}(x)$. Letting $A = \mathrm{sat}(x)$ and $B = \mathrm{sat}(y')$, if $\rho(A) > y'(A)$, then

$$\rho(B) - \rho(A) < y'(B) - y'(A) \leq x(B) - x(A).$$

Since $\rho(A) = x(A)$, this implies $\rho(B) < x(B)$, a contradiction. Hence $\rho(A) = y'(A)$ and $(y')^A$ and x^A are bases of $\mathbf{P} \cdot A = (A, \rho^A)$, the restriction of $\mathbf{P}$ to A. Hence $(y')^{E \setminus A} \in \mathrm{P}(\rho_A)$, where ρ_A is the rank function of the contraction $(E \setminus A, \rho_A)$ of $\mathbf{P}$ by A. Since $x(e) \geq y'(e)$ $(\forall e \in E \setminus A)$ and $\mathrm{sat}(x) = A$, we have $\mathrm{sat}(y) \subseteq \mathrm{sat}(y') \subseteq \mathrm{sat}(x) = A$. $\square$

Let us consider the 'eating process' (due to Bogomolnaia and Moulin [5]). By the procedure Extended_Random_Assignment we have *critical times*

$$\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_q = \rho(E)/d(N)$$

computed by (4.6), where $d(N) = \sum_{i \in N} d(i)$. At each critical time $\lambda_k > 0$, $\mathcal{L}$ is updated by removing all the saturated goods from $\mathcal{L}$.

For each time t with $\lambda_k \leq t \leq \lambda_{k+1}$ for $k \in \{0, \cdots, q-1\}$ we put

$$x^*_t = x^*_{\lambda_k} + (t - \lambda_k) b(\mathcal{L}_t),$$

where $\mathcal{L}_t = (L^i_t \mid i \in N)$ denotes the current $\mathcal{L} = (L^i \mid i \in N)$ at time t and

$$b(\mathcal{L}_t) = \sum_{i \in N} d(i) \chi_{e^i_1}$$

with e^i_1 being the top element (good) of current L^i_t. We put $x^*_0 = \mathbf{0}$. Note that we have

$$\mathrm{sat}(x^*_t) = \mathrm{sat}(x^*_{\lambda_k}) \qquad (\forall t \in [\lambda_k, \lambda_{k+1}),\ \forall k \in \{0, \cdots, q-1\}).$$

Now, suppose that agent $1 \in N$ has a preference list L^1 and misreports her preference as $\bar{L}^1$. Put $\bar{\mathcal{L}} = (\bar{L}^i \mid i \in N)$ with $\bar{L}^i = L^i$ for $i \in N \setminus \{1\}$. For any original object p (a parameter, a vector, etc.) defined under preference profile $\mathcal{L}$, let us denote by $\bar{p}$ the object p defined under misreported preference profile $\bar{\mathcal{L}}$.

For each $e \in E$ define $N_P(e) = \{i \in N \mid P(i, e) > 0\}$. For each $e \in E$ let $t(e)$ be the time when good e is exhausted (or saturated). Also for each $e \in E$ and $i \in N_P(e)$ let $t_0^i(e)$ be the time when agent i starts eating good e (or the time when e becomes the top element of current L^i).

Lemma 5.2: *Let L^1 and $\bar{L}^1$ be given by*

$$L^1 : w_1 \succ \cdots \succ w_s \succ a \succ \cdots , \tag{5.1}$$
$$\bar{L}^1 : w_1 \succ \cdots \succ w_s \succ z_1 \succ \cdots \succ z_{s'} \succ a \succ \cdots \tag{5.2}$$

with $P(1, a) > 0$ for some integers $s \geq 0$ and $s' \geq 1$. Suppose that $\bar{t}(a) > t(a)$. Then, the following three statements hold during the execution of Extended_Random_Assignment *with current time $t < t(a)$.*

(a) *For each $i \in N \setminus \{1\}$ we have $e_1^i \succeq_i \bar{e}_1^i$, where e_1^i and $\bar{e}_1^i$ are the top elements of current L_t^i for $\mathcal{L}_t$ and current $\bar{L}_t^i$ for $\bar{\mathcal{L}}_t$, respectively, and $\succeq_i$ is the order of original L^i.*

(b) *For each $e \in E \setminus (\mathrm{sat}(\bar{x}_t^*) \cup \{a\})$ we have $x_t^*(e) \leq \bar{x}_t^*(e)$.*

(c) $\mathrm{sat}(x_t^*) \subseteq \mathrm{sat}(\bar{x}_t^*)$.

Moreover, we have

(d) $\bar{t}_0^i(a) \leq t_0^i(a) \quad (\forall i \in N_P(a) \setminus \{1\})$.

(Proof) We can easily see from the procedure Extended_Random_Assignment and the definition of $\mathcal{L}$ and $\bar{\mathcal{L}}$ that (a) implies

$$b(\bar{\mathcal{L}}_t)(e) \geq b(\mathcal{L}_t)(e) \qquad (\forall t \in [0, t(a)), \ \forall e \in E \setminus (\mathrm{sat}(\bar{x}_t^*) \cup \{a\})),$$

which implies (b). Moreover, (c) follows from (b) and Lemma 5.1, where we restrict the ground set of polymatroid (E, ρ) to $E \setminus \{a\}$ since a is not saturated for both x_t^* and $\bar{x}_t^*$ for $t < t(a)$ by the assumption. Also (d) easily follows from (a) (for all $t \in [0, t(a))$).

Hence it suffices to show that (a) holds for all $t \in [0, t(a))$, by induction on the indices k of critical times λ_k and $\bar{\lambda}_k$. First, note that (a) holds for $t \in [0, \min\{\lambda_1, \bar{\lambda}_1, t(a)\})$ since $e_1^i = \bar{e}_1^i$ for all $i \in N \setminus \{1\}$.

We consider the following three cases (A), (B), and (C).

(A) Suppose that (a) holds for all $t \in [0, \lambda_k)$ for some $k \geq 1$ with $\lambda_k < t(a)$ and that λ_k is not equal to any critical time for $\bar{\mathcal{L}}$, i.e., $\bar{\lambda}_p < \lambda_k < \bar{\lambda}_{p+1}$ for some p. Then it follows from Lemma 5.1 that at time $t = \lambda_k$, if e_1^i for $i \in N \setminus \{1\}$ becomes saturated, then e_1^i belongs to $\mathrm{sat}(\bar{x}_t^*)$ and the new (non-saturated) e_1^i satisfies $e_1^i \succeq^i \bar{e}_1^i$. (Here we employ Lemma 5.1 by restricting the polymatroid to $E \backslash \{a\}$ since a is not saturated for x_t^* with $t < t(a)$.) Hence (a) holds for $t = \lambda_k$ and then so does for all $t \in [0, \min\{\lambda_{k+1}, \bar{\lambda}_{p+1}, t(a)\})$.

(B) Suppose that (a) holds for all $t \in [0, \bar{\lambda}_k)$ for some $k \geq 1$ with $\bar{\lambda}_k < t(a)$ and that $\lambda_p < \bar{\lambda}_k < \lambda_{p+1}$ for some p. Then at time $t = \bar{\lambda}_k$ we have $\mathrm{sat}(\bar{x}_t^*)$ enlarged and newly saturated $\bar{e}_1^i$ is replaced by the next non-saturated one in $\bar{L}_t^i$, current $\bar{L}^i$. Hence (a) holds for $t = \bar{\lambda}_k$ and then so does for all $t \in [0, \min\{\lambda_{p+1}, \bar{\lambda}_{k+1}, t(a)\})$.

(C) Suppose that (a) holds for all $t \in [0, \lambda_k)$ for some $k \geq 1$ with $\lambda_k < t(a)$ and that $\lambda_k = \bar{\lambda}_p$ for some $p \geq 1$. Then at time $t = \bar{\lambda}_p (= \lambda_k)$ we have $\mathrm{sat}(\bar{x}_t^*)$ enlarged and newly saturated $\bar{e}_1^i$ is replaced by the next non-saturated one in $\bar{L}_t^i$. Also, at time $t = \lambda_k (= \bar{\lambda}_p)$, if e_1^i becomes saturated, then e_1^i belongs to updated $\mathrm{sat}(\bar{x}_t^*)$ and the new non-saturated e_1^i satisfies $e_1^i \succeq^i \bar{e}_1^i$ for possibly new non-saturated $\bar{e}_1^i$ (due to Lemma 5.1). Hence (a) holds for $t = \lambda_k (= \bar{\lambda}_p)$ and then so does for all $t \in [0, \min\{\lambda_{k+1}, \bar{\lambda}_{p+1}, t(a)\})$.

This completes the proof of the present lemma by induction. □

Lemma 5.3: *Under the same assumption as in Lemma* 5.2*, we have*

$$x_{t(a)}^*(a) \geq \bar{x}_{\bar{t}(a)}^*(a). \tag{5.3}$$

(Proof) Suppose that $\bar{t}(a) > t(a)$ and let p and q be integers such that $\bar{\lambda}_p < t(a) \leq \bar{\lambda}_{p+1}$ and $\lambda_q = t(a)$. It follows from Lemma 5.2 that for all $t \in [\bar{\lambda}_p, t(a))$

$$\mathrm{sat}(x_t^*) \subseteq \mathrm{sat}(\bar{x}_t^*), \qquad x_t^*(e) \leq \bar{x}_t^*(e) \quad (\forall e \in E \setminus (\mathrm{sat}(\bar{x}_t^*) \cup \{a\})). \tag{5.4}$$

Also we see from Lemma 5.2 and the continuity of x_t^* in t that at $t = t(a)$ we have

$$x_{t(a)}^*(e) \leq \bar{x}_{t(a)}^*(e) \quad (\forall e \in E \setminus (\mathrm{sat}(\bar{x}_{t(a)}^*) \cup \{a\})). \tag{5.5}$$

Define $\bar{y}_\epsilon = \bar{x}_{\bar{t}(a)-\epsilon}^*$ for any ϵ with $0 < \epsilon \leq \bar{t}(a)$. Then, for a sufficiently small $\epsilon > 0$ we have $a \notin \mathrm{sat}(\bar{y}_\epsilon)$ and

$$\bar{x}_{t(a)}^*(e) = \bar{y}_\epsilon(e) \quad (\forall e \in \mathrm{sat}(\bar{x}_{t(a)}^*)), \tag{5.6}$$

$$\bar{x}_{t(a)}^*(e) \leq \bar{y}_\epsilon(e) \quad (\forall e \in E \setminus \mathrm{sat}(\bar{x}_{t(a)}^*)). \tag{5.7}$$

Increase the values of $x_{t(a)}^*(e)$ for all $e \in \mathrm{sat}(\bar{y}_\epsilon)$ as much as possible while keeping the vector within $\mathrm{P}(\rho)$. Let y^* be the resulting independent vector. Then we have

$$\mathrm{sat}(y^*) \supseteq \mathrm{sat}(\bar{y}_\epsilon) \cup \{a\}. \tag{5.8}$$

Put $A = \mathrm{sat}(y^*)$ and $B = \mathrm{sat}(\bar{y}_\epsilon)$. Then,

$$x_{t(a)}^*(A) - x_{t(a)}^*(B) = y^*(A) - y^*(B) \geq \rho(A) - \rho(B) > \bar{y}_\epsilon(A) - \bar{y}_\epsilon(B), \tag{5.9}$$

where the last inequality follows from the fact that $a \in A$ and $a \notin \mathrm{sat}(\bar{y}_\epsilon)$. Hence from (5.5)–(5.9) we have

$$x^*_{t(a)}(a) > \bar{y}_\epsilon(a) = \bar{x}^*_{\bar{t}(a)-\epsilon}(a). \tag{5.10}$$

Since (5.10) holds for any sufficiently small $\epsilon > 0$ and $\bar{x}^*_t$ is continuous in t, we have $x^*_{t(a)}(a) \geq \bar{x}^*_{\bar{t}(a)}(a)$. □

5.1.2. Proof of Theorem 5.4

Suppose that we are given the extended PS solution P. Recall that $N_P(e) = \{i \in N \mid P(i,e) > 0\}$ for all $e \in E$.

Theorem 5.4: *Given the extended PS solution P, if we have $|N_P(e)| \neq 1$ for all $e \in E$, then the solution P is a weakly sd Nash equilibrium.*

(Proof) Suppose that $|N_P(e)| \neq 1$ for all $e \in E$. Recall that for each $e \in E$, $t(e)$ is the time when good e is exhausted (or saturated). Also for each $e \in E$ and $i \in N_P(e)$, $t^i_0(e)$ is the time when agent i starts eating good e (or the time when e becomes the top element of current L^i).

For the extended PS solution P, if $P(1,a) = 0$ for some $a \in E$, shifting good a in L^1 toward the end of L^1 does not change the solution P. Hence we can assume

(†) goods e with $P(1,e) > 0$ appear consecutively in L^1 from the top of L^1.

Now suppose that for agent $1 \in N$ her preference is given by

$$L^1 : a \succ \cdots \tag{5.11}$$

with $P(1,a) > 0$ and she misreports her preference as

$$\bar{L}^1 : z_1 \succ \cdots \succ z_s \succ a \succ \cdots \tag{5.12}$$

with some integer $s \geq 1$. Let $\bar{P}$ be the PS solution obtained under the misreport. When L^1 is replaced by $\bar{L}^1$, we denote $t(e)$ and $t^i_0(e)$ by $\bar{t}(e)$ and $\bar{t}^i_0(e)$, respectively, for all $e \in E$ and $i \in N$, and also denote x^*_t by $\bar{x}^*_t$.

Suppose that $\bar{P}_1$ sd-dominates P_1 or is equal to P_1 (i.e., $\bar{P}_1 \succeq^{\mathrm{d}}_1 P_1$), where recall $\bar{P}_1 = (\bar{P}(1,e) \mid e \in E)$ and $P_1 = (P(1,e) \mid e \in E)$. Then it suffices to prove $\bar{P}_1 = P_1$.

(I) Suppose that $\bar{t}(a) > t(a)$. Then from Lemma 5.2 (d) and Lemma 5.3 we have

$$\bar{t}^i_0(a) \leq t^i_0(a) \qquad (\forall i \in N_P(a) \setminus \{1\}). \tag{5.13}$$

$$x^*_{t(a)}(a) \geq \bar{x}^*_{\bar{t}(a)}(a), \tag{5.14}$$

Now, since $\bar{t}(a) > t(a)$ and $N_P(a) \setminus \{1\} \neq \emptyset$ by the assumption, it follows from (5.13) and (5.14) that $\bar{P}(1,a) < P(1,a)$, a contradiction. Hence we have $\bar{t}(a) \leq t(a)$ and

$$\bar{P}(1,a) = \bar{t}(a) - (\bar{P}(1,z_1) + \cdots + \bar{P}(1,z_s)) \leq t(a) = P(1,a). \tag{5.15}$$

Since from the assumption that $\bar{P}_1 \succeq_1^{\mathrm{d}} P_1$ we must have $\bar{P}(1,a) \geq P(1,a)$, it follows from (5.15) that

$$\bar{P}(1,a) = P(1,a), \qquad \bar{P}(1,z_1) = \cdots = \bar{P}(1,z_s) = 0. \tag{5.16}$$

The latter relation in (5.16) implies

- elements $z_1, \cdots, z_s$ are saturated at time $t = 0$ for $\bar{\mathcal{L}}$.

Since shifting elements $z_1, \cdots, z_s$ toward the end of $\bar{L}^1$ does not change $\bar{P}$, it suffices to consider that L^1 and $\bar{L}^1$ are given as

$$L^1 : a \succ b \succ \cdots, \tag{5.17}$$
$$\bar{L}^1 : a \succ z'_1 \succ \cdots \succ \cdots \succ z'_{s'} \succ b \succ \cdots \tag{5.18}$$

for some $\{z'_1, \cdots, z'_{s'}\} \subseteq E \setminus \{a,b\}$ with an integer $s' \geq 0$.
(II) If $P(1,b) = 0$, then it easily follows from the assumption (†) that $P_1 = \bar{P}_1$. Hence suppose $P(1,b) > 0$. Then by the same arguments as in (I), using Lemma 5.3 again, we can show

1. $\bar{t}(b) \leq t(b)$,
2. $\bar{P}(1,b) = P(1,b), \qquad \bar{P}(1,z'_1) = \cdots = \bar{P}(1,z'_{s'}) = 0,$
3. elements $z'_1, \cdots, z'_{s'}$ are saturated at time $t = \bar{t}(a)(= t(a))$ for $\bar{\mathcal{L}}$

and it suffices to consider the case where there is no element between a and b in $\bar{L}^1$.
(III) Further repeating this argument, we can show that $\bar{P}_1 = P_1$. □

Theorem 5.4 is rephrased as follows. (Note that matrix $P \in \mathbb{R}^{N \times E}$ has the row set N and the column set E.)

- If no column of P contains exactly one non-zero entry, the extended PS solution P is a weakly sd Nash equilibrium.

Theorem 5.4 has very useful practical implications from the point of view of strategy-proofness. The condition that $|N_P(e)| \neq 1$ ($\forall e \in E$) is very likely to be satisfied when the number $|N|$ of 'agents' is significantly large, compared with the number $|E|$ of 'types of goods' such as the assignment of students to courses.

Related non-manipulability result was also obtained by Kojima and Manea [19], assuming the availability of utility functions. They gave a sufficient condition for their extended PS solution to be a weakly sd Nash equilibrium, which can be checked by the given data including utility functions. On the other hand, our condition can easily be checked by the extended PS solution computed without using any additional information about utility functions.

5.2. Weak strategy-proofness in case of unit demands and matroidal supplies

We show that when the polymatroid (E, ρ) is a matroid and agents have unit demands, the extended PS mechanism is weakly strategy-proof, where the matroidal $\{0, 1\}$ property plays a crucial rôle.

Consider the random assignment problem $\mathbf{RA} = (N, E, \mathcal{L} = (L^i \mid i \in N), d, (E, \rho))$ and suppose that the underlying polymatroid (E, ρ) is a matroid and agents have unit demands, i.e., $d = \mathbf{1}$, which is treated by the authors [12]. We assume that $\rho(E) = |N|$.

Lemma 5.5: *Suppose that* L^1 *and* $\bar{L}^1$ *are given by* (5.1) *and* (5.2) *and that* $P(1, a) > 0$. *Suppose that* $\bar{t}(a) > t(a)$. *Then we have* $\bar{P}(1, a) \leq P(1, a)$. *Moreover, we have* $\bar{P}(1, a) = P(1, a)(> 0)$ *only when* $\bar{P}(1, z_1) = \cdots = \bar{P}(1, z_{s'}) = 0$.

(Proof) Because of Theorem 5.4 it suffices to consider the case where $|N_P(a)| = 1$. Suppose that L^1 and $\bar{L}^1$ are given by (5.1) and (5.2) and that $P(1, a) > 0$.

Suppose $|N_P(a)| = 1$, i.e., $N_P(a) = \{a\}$. From Lemma 5.3 we have

$$P(1, a) = x^*_{t(a)}(a) \geq \bar{x}^*_{\bar{t}(a)}(a) \geq \bar{P}(1, a). \tag{5.19}$$

If $|\bar{N}_{\bar{P}}(a)| \geq 2$, then the last inequality in (5.19) should hold with strict inequality. Hence it suffices to consider the case where $|\bar{N}_{\bar{P}}(a)| = 1 = |N_P(a)|$. Moreover, since $\bar{t}(a) > t(a)$ by the assumption, it follows from (5.19) that

$$\bar{P}(1, z_1) + \cdots + \bar{P}(1, z_{s'}) > 0. \tag{5.20}$$

We show that this leads us to $\bar{P}(1, a) < P(1, a)$.

Increase the values of $x^*_{t(a)}(e)$ for all $e \in \mathrm{sat}(\bar{x}^*_{t(a)})$ as much as possible while keeping the vector within $\mathrm{P}(\rho)$. (Here note that $a \notin \mathrm{sat}(\bar{x}^*_{t(a)})$ since $\bar{t}(a) > t(a)$.) Let z^* be the resulting independent vector. Then, since $X \equiv \mathrm{sat}(\bar{x}^*_{t(a)})$ and $Z \equiv \mathrm{dep}(x^*_{t(a)}, a)$ are tight for z^*, we have

$$z^*(X \cup Z) = \rho(X \cup Z). \tag{5.21}$$

Consider the following two cases (i) and (ii).

Case (i): $(Z \setminus \{a\}) \setminus X \neq \emptyset$. In this case, it follows from (5.5) that

$$\bar{x}^*_{\bar{t}(a)}(e) > \bar{x}^*_{t(a)}(e) \geq x^*_{t(a)}(e) \quad (\forall e \in (Z \setminus \{a\}) \setminus X). \tag{5.22}$$

(Here note that for all $e \in (Z \setminus \{a\}) \setminus X$ we have $\bar{x}^*_{t(a)}(e) \geq x^*_{t(a)}(e) > 0$, where we have $x^*_{t(a)}(e) > 0$ because of the definition of $Z = \mathrm{dep}(x^*_{t(a)}, a)$, and hence e is the top element of current L^i_t of at least one agent $i \in N \setminus \{1\}$ for $\bar{\mathcal{L}}_t$ (as well as for $\mathcal{L}_t$) at time $t = t(a)$.) Hence, if $\bar{x}^*_{\bar{t}(a)}(a) = x^*_{t(a)}(a)$, then from (5.21) and (5.22) we have

$$\bar{x}^*_{\bar{t}(a)}(X \cup Z) > z^*(X \cup Z) = \rho(X \cup Z), \tag{5.23}$$

a contradiction. We thus have $\bar{P}(1, a) = \bar{x}^*_{\bar{t}(a)}(a) < x^*_{t(a)}(a) = P(1, a)$.

Case (ii): $(Z \setminus \{a\}) \setminus X = \emptyset$. In this case, it follows from (5.21) that

$$P(1, a) = x^*_{t(a)}(a) = z^*(X \cup \{a\}) - z^*(X) = \rho(X \cup \{a\}) - \rho(X) = 1, \tag{5.24}$$

where note that $X \cup Z = X \cup \{a\}$ and $P(1, a) > 0$. It follows from (5.20) that $\bar{P}(1, a) = \bar{x}^*_{\bar{t}(a)}(a) < 1 = P(1, a)$. □

It should be noted that the above proof in Case (ii) depends on the matroidal $\{0, 1\}$ property.

Theorem 5.6: *When the underlying polymatroid (E, ρ) is a matroid and agents have unit demands, the extended PS mechanism is weakly strategy-proof.*

(Proof) The present theorem can be shown similarly as Theorem 5.4, based on Lemma 5.5. □

6. Randomized Mechanism

Now we consider the assignment problem with *indivisible* goods.

Suppose that given a random assignment matrix P obtained as the output of the procedure Extended_Random_Assignment, P can be represented by a convex combination of assignment matrices $Q^{(k)} \in \mathrm{B}_{\mathbb{Z}}(\rho)$ $(k \in K)$ with a finite index set K as

$$P = \sum_{k \in K} \nu_k Q^{(k)}, \tag{6.1}$$

where each row sum of $Q^{(k)}$ satisfies the demand constraint (4.2) (because of Theorem 6.2 shown below). Here, each $Q^{(k)}$ $(k \in K)$ is a feasible *deterministic* assignment, and we see that the convex combination coefficients $\nu_k > 0$ $(k \in K)$ with $\sum_{k \in K} \nu_k = 1$ serve as the probability distribution over the deterministic assignments $Q^{(k)}$ ($k \in K$), which realizes a required lottery to get the solution P as an expected assignment matrix.

Let us consider Example 2 given in Section 4.1. The random assignment matrix P obtained in Example 2 is expressed as

$$
\begin{aligned}
P &= \begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{2}Q^{(1)} + \frac{1}{2}Q^{(2)}.
\end{aligned} \tag{6.2}
$$

Note that both $Q^{(1)}$ and $Q^{(2)}$ satisfy the demand constraints with demand vector (4,2,1,1). The solution P is realized by choosing $Q^{(1)}$ or $Q^{(2)}$ at random (with equal probability $\frac{1}{2}$).

Now we will show that we can always compute a required convex combination representation (6.1) in an efficient way. With the aid of polymatroidal results achieved in [9, 10, 11] we can construct a randomized mechanism to attain P. The procedure is essentially the same as the one given in [12] for a special case of a matroidal family of feasible sets of goods.

6.1. Random assignments and independent flows

Consider a complete bipartite graph $G = (S^+, S^-; A)$ with a vertex set $V = S^+ \cup S^-$ given by

$$S^+ = N, \qquad S^- = E \tag{6.3}$$

and an arc set A given by

$$A = N \times E. \tag{6.4}$$

For every arc $a \in A$ we consider its capacity $c(a) = +\infty$. The vertex set $S^+{=}N$ is the set of entrances and $S^-{=}E$ is the set of exits. Denote by $\mathcal{N} = (G = (S^+, S^-, A), d, c, (E, \rho))$ the network with the polymatroidal constraints on the exit set $S^- = E$ defined as follows. (See Figure 1.) Consider a nonnegative flow $\varphi : A \to \mathbb{R}_{\geq 0}$ in $\mathcal{N}$ and define $\partial^{\pm}\varphi : S^{\pm} \to \mathbb{R}_{\geq 0}$ by

$$
\begin{aligned}
\partial^+\varphi(i) &= \sum\{\varphi(i,e) \mid e \in E\} \qquad (\forall i \in S^+ = N), && (6.5) \\
\partial^-\varphi(e) &= \sum\{\varphi(i,e) \mid i \in N\} \qquad (\forall e \in S^- = E). && (6.6)
\end{aligned}
$$

If $\varphi : A \to \mathbb{R}_{\geq 0}$ satisfies

$$\partial^+\varphi(i) \leq d(i) \quad (\forall i \in N), \qquad \partial^-\varphi \in \mathrm{B}(\rho), \tag{6.7}$$

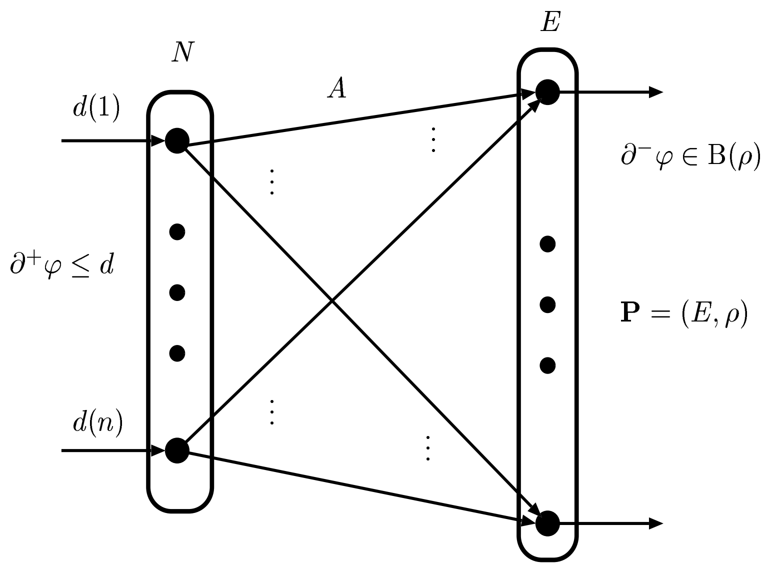

Figure 1: An independent-flow network $\mathcal{N}$.

then we call φ an *independent flow* in $\mathcal{N}$.

Given a random assignment matrix P computed by Extended_Random_Assignment, put

$$\varphi_P(a) = P(i, e) \qquad (\forall a = (i, e) \in A). \tag{6.8}$$

Then the flow $\varphi_P : A \to \mathbb{R}_{\geq 0}$ is an independent flow in $\mathcal{N}$ and satisfies

$$\partial^+\varphi_P(i) \leq d(i) \quad (\forall i \in N), \tag{6.9}$$

$$\partial^-\varphi_P \in \mathrm{B}(\rho), \tag{6.10}$$

where note that $\partial^-\varphi_P = x^*_P$. It should also be noted that if $d(N) = \rho(E)$, then (6.10) together with (6.9) implies

$$\partial^+\varphi_P(i) = d(i) \qquad (\forall i \in N). \tag{6.11}$$

Define a polytope P^* by

$$P^* = \{\varphi \mid \varphi \text{ is an independent flow in } \mathcal{N}\}. \tag{6.12}$$

It should be noted that P^* is nonempty since φ_P satisfies (6.9) and (6.10) and hence $\varphi_P \in P^*$.

The following integrality property holds true for independent flows in $\mathcal{N}$ ([9, 11]), which plays a crucial rôle in our problem setting. Note that P^* is a polytope, i.e., a bounded polyhedron, by definition.

Proposition 6.1: *The polytope P^* defined by* (6.12) *is integral, i.e., every extreme point of P^* is an integral vector.*

When a random assignment matrix Q is integral, we call Q an *assignment matrix*. By Proposition 6.1 we have

Theorem 6.2: *For the random assignment matrix P computed by* Extended_Random_Assignment *there exist assignment matrices $Q^{(k)}$ $(k \in K)$ and convex combination coefficients ν_k $(k \in K)$ such that*

$$P = \sum_{k \in K} \nu_k Q^{(k)}. \tag{6.13}$$

Since the present paper is a generalization of our previous paper [12], Theorem 6.2 is also a generalization of the Birkhoff-von Neumann theorem on bi-stochastic matrices.

6.2. Computing the probability distribution

In the following we show how to efficiently compute an expression (6.13). It is basically a standard procedure to obtain an expression of a given point in polytope P^* by a convex combination of its extreme points, but it is crucial how efficiently we can compute an end point of the intersection of a line and a base polytope ([21]) and can identify the unique minimal face of P^* containing any given point in P^* ([11]).

For the random assignment matrix P (or independent flow φ_P) and base $x^*_P \in \mathrm{B}(\rho)$ computed by Extended_Random_Assignment we first consider the unique minimal face of P^* containing φ_P.

Denote by $\mathcal{D}(x^*_P)$ the set of all tight sets for x^*_P in $\mathrm{B}(\rho)$, where $\mathcal{D}(x^*_P)$ is closed with respect to the binary operations of set union and intersection and is a distributive lattice (see [11]). Let a maximal chain of $\mathcal{D}(x^*_P)$ be given by

$$\hat{\mathcal{C}} : \hat{S}_0 = \emptyset \subset \cdots \subset \hat{S}_p = E. \tag{6.14}$$

The chain of tight sets obtained during the execution of Extended_Random_Assignment is a subchain of (6.14). A maximal chain $\hat{\mathcal{C}}$ is determined by the dependence structure associated with $\mathrm{dep}(x^*_P, e)$ for all $e \in E$ and can be computed in strongly polynomial time ([11, 12]).

For each $q = 1, \cdots, p$ consider the minor, denoted by $\mathbf{P}_q$, of polymatroid (E, ρ) obtained by its restriction to $\hat{S}_q$ followed by the contraction of $\hat{S}_{q-1}$. The minor $\mathbf{P}_q$ is the polymatroid on $\hat{T}_q \equiv \hat{S}_q \setminus \hat{S}_{q-1}$ with the rank function ρ_q given by

$$\rho_q(X) = \rho(X \cup \hat{S}_{q-1}) - \rho(\hat{S}_{q-1}) \qquad (\forall X \subseteq \hat{T}_q). \tag{6.15}$$

Also denote by x_q^* the restriction of x_P^* to $\hat{T}_q (= \hat{S}_q \setminus \hat{S}_{q-1})$. Then x_q^* is a base of the polymatroid $(\hat{T}_q, \rho_q)$, i.e., $x_q^* \in \mathrm{B}(\rho_q)$. Note that x_P^* is a base of the direct sum $\oplus_{q=1}^p \mathbf{P}_q$ of minors $\mathbf{P}_q$ $(q = 1, \cdots, p)$. Let $\hat{\rho}$ be the rank function of polymatroid $\oplus_{q=1}^p \mathbf{P}_q$. It should be noted that because of the maximality of chain $\hat{\mathcal{C}}$, for each $q = 1, \cdots, p$ the base polytope $\mathrm{B}(\rho_q) \subseteq \mathbb{R}^{\hat{T}_q}$ is of dimension $|\hat{T}_q| - 1$ and base x_q^* is within the relative interior of $\mathrm{B}(\rho_q)$ and that x_P^* is within the relative interior of the base polytope $\mathrm{B}(\hat{\rho})$ of $\oplus_{q=1}^p \mathbf{P}_q$, which is the unique minimal face of $\mathrm{B}(\rho)$ containing x_P^*. (See [11, Chapter II].)

Put

$$\hat{A}^0 = \{a \in A \mid \varphi_P(a) = 0\}, \tag{6.16}$$
$$\hat{A}^+ = A \setminus \hat{A}^0, \tag{6.17}$$
$$\hat{I} = \{i \in N \mid \partial^+ \varphi_P(i) = d(i)\}. \tag{6.18}$$

Then, define a face of P^* containing φ_P by

$$P^*(\varphi_P) = \{\varphi \in P^* \mid \forall i \in \hat{I} : \partial^+ \varphi(i) = d(i),\ \forall a \in \hat{A}^0 : \varphi(a) = 0,\ \partial^- \varphi \in \mathrm{B}(\hat{\rho})\}. \tag{6.19}$$

We can show the following lemma.

Lemma 6.3: *The polytope $P^*(\varphi_P)$ is the unique minimal face of P^* containing φ_P. Moreover, $P^*(\varphi_P)$ restricted in $\mathbb{R}^{\hat{A}^+}$ is the set of independent flows satisfying* (6.10) *in the network $\hat{\mathcal{N}} = (\hat{G} = (S^+, S^-; \hat{A}^+), d, c, (E, \hat{\rho}))$.*

(Proof) In the system of inequalities (and equations) that defines P^* of (6.12), the given φ_P satisfies $\partial^+ \varphi_P(i) = d(i)$ for all $i \in \hat{I}$, $\varphi_P(a) = 0$ for all $a \in \hat{A}^0$, and

$$\partial^- \varphi_P(X) = \rho(X) \qquad (\forall X \in \mathcal{D}(x_P^*)), \tag{6.20}$$

which includes all the inequalities for P^* satisfied with equality by φ_P. Note that (6.20) is implied by

$$\partial^- \varphi_P(X) = \rho(X) \qquad (\forall X \in \mathcal{C}(x_P^*)), \tag{6.21}$$

since $\mathcal{D}(x_P^*)$ is a distributive lattice and ρ is modular on $\mathcal{D}(x_P^*)$. Also note that the system of equations (6.21) together with $\partial^- \varphi_P \in \mathrm{B}(\rho)$ is equivalent to $\partial^- \varphi_P \in \mathrm{B}(\hat{\rho})$. Hence (6.19) defines the unique minimal face of P^* containing φ_P.

Moreover, the latter statement holds true because of the definition of the network $\hat{\mathcal{N}}$. □

Based on Lemma 6.3, we can compute the expression of x_P^* as a convex combination of integral vectors in $\mathrm{B}_{\mathbb{Z}}(\rho)$ corresponding to assignments $Q^{(k)}$ $(k \in K)$ and then accordingly P can be expressed by a convex combination of $Q^{(k)}$ $(k \in K)$ as (6.13).

6.3. Randomized mechanism

Now we have the following procedure.

Randomized_Mechanism

1. Compute P by Extended_Random_Assignment.
2. Compute an expression (6.13).
3. Choose an assignment from among $Q^{(k)}$ $(k \in K)$ by the lottery with the probability distribution ν_k $(k \in K)$.

Consequently, we can show the following theorem.

Theorem 6.4: *For the random assignment matrix P computed by* Extended_Random_Assignment*, the above described* Randomized_Mechanism *generates an assignment in strongly polynomial time whose expectation is equal to the PS solution P.*

(Proof) We begin with base $x_1^* \equiv x_P^* \in \mathrm{B}(\hat{\rho})$ and independent flow $\hat{\varphi}_1 \equiv$ (the restriction of φ_P on $\hat{A}^+$) in network $\hat{\mathcal{N}} = (\hat{G} = (S^+, S^-; \hat{A}^+), d, c, (E, \hat{\rho}))$ such that $x_1^* = \partial^- \hat{\varphi}_1$. If φ_P is already integer-valued, we are done. Hence we assume that φ_P is not integer-valued. Perform the following procedure to compute an expression (6.13).

---

1. Put $t \leftarrow 1$.
2. Find an integer-valued independent flow φ_t in $\hat{\mathcal{N}}$.
3. Compute
$$\beta_t^* = \max\{\beta > 0 \mid \hat{\varphi}_t + \beta(\hat{\varphi}_t - \varphi_t) \in P^*(\hat{\varphi}_t)\}. \tag{6.22}$$
4. Put $\hat{\varphi}_{t+1} \leftarrow \hat{\varphi}_t + \beta_t^*(\hat{\varphi}_t - \varphi_t)$ and $x_{t+1}^* \leftarrow x_t^* + \beta_t^*(x_t^* - \partial^- \varphi_t)$.
5. If flow $\hat{\varphi}_{t+1}$ is not integer-valued, then put $t \leftarrow t+1$, update $\hat{\mathcal{N}}$ for the current base x_t^* and flow $\hat{\varphi}_t$, and go to Step 2.
   Otherwise put $\varphi_{t+1} \leftarrow \hat{\varphi}_{t+1}$.
   Return φ_s for all $s = 1, \cdots, t+1$ and β_s^* for all $s = 1, \cdots, t$.

During the execution of the above procedure, $P^*(\hat{\varphi}_t)$ appearing in (6.22) is the unique minimal face of P^* containing $\hat{\varphi}_t$, due to Lemma 6.3. At the tth execution of Step 3 with current rank function $\hat{\rho}$ we have the unique minimal face $\mathrm{B}(\hat{\rho})$ of $\mathrm{B}(\rho)$ containing x_t^*. Then β_t^* in (6.22) is the maximum value of β that satisfies

$$\partial^+\hat{\varphi}_t(i) \le d(i) \qquad (\forall i \in N), \tag{6.23}$$

$$\hat{\varphi}_t(a) + \beta(\hat{\varphi}_t(a) - \varphi_t(a)) \ge 0 \qquad (\forall a \in \hat{A}_t^+), \tag{6.24}$$

$$x_t^* + \beta(x_t^* - \partial^-\varphi_t) \in \mathrm{B}(\hat{\rho}), \tag{6.25}$$

where $\hat{A}_t^+ = \{a \in A \mid \hat{\varphi}_t(a) > 0\}$. Note that since $\hat{\varphi}_t$ is within the relative interior of $P^*(\hat{\varphi}_t)$, we get $\beta_t^* > 0$. We can compute β_t^* in strongly polynomial time (due to Nagano [21]) by using any strongly polynomial submodular function minimization algorithm. Also note that the final value of t is $\mathrm{O}(|N||E|)$ since every execution of Step 3 and Step 4 makes at least one strict inequality in (6.23) or (6.24) hold with equality or makes the length of a maximal chain $\mathcal{C}(x_{t+1}^*)$ greater than that of $\mathcal{C}(x_t^*)$.

We regard each φ_s $(s = 1, \cdots, t+1)$ as a flow in the original network $\mathcal{N}$ by putting $\varphi_s(a) = 0$ for all $a \in A \setminus \hat{A}_s^+$, and similarly for $\hat{\varphi}_s$ $(s = 1, \cdots, t+1)$. From the output φ_s for all $s = 1, \cdots, t+1$ and β_s^* for all $s = 1, \cdots, t$ we have

$$\hat{\varphi}_{s+1} = (1 + \beta_s^*)\hat{\varphi}_s - \beta_s^*\varphi_s \qquad (\forall s = 1, \cdots t), \tag{6.26}$$

or

$$\hat{\varphi}_s = (1 + \beta_s^*)^{-1}(\hat{\varphi}_{s+1} + \beta_s^*\varphi_s) \qquad (\forall s = 1, \cdots t). \tag{6.27}$$

Eliminating $\hat{\varphi}_s$ for $s = 1, \cdots, t$ and using $\hat{\varphi}_{t+1} = \varphi_{t+1}$, we can obtain the following expression.

$$\varphi_P(= \hat{\varphi}_1) = \sum_{s=1}^{t+1} \nu_s \varphi_s \tag{6.28}$$

for some convex combination coefficients ν_s $(s = 1, \cdots, t+1)$. Each integer-valued flow φ_s gives a desired assignment matrix $Q^{(s)}$, and ν_s $(s = 1, \cdots, t+1)$ the desired probability distribution on the set of assignment matrices $Q^{(s)}$ $(s = 1, \cdots, t+1)$. Note that (6.28) is equivalent to

$$P = \sum_{s=1}^{t+1} \nu_s Q^{(s)}, \tag{6.29}$$

which thus can be computed in strongly polynomial time. □

It should be noted that defining $\underline{u} = \lfloor \partial^-\varphi_P \rfloor$ and $\overline{u} = \lceil \partial^-\varphi_P \rceil$, we may replace the original base polytope $\mathrm{B}(\rho)$ by its vector minor (the restriction by $\overline{u}$ and the contraction by $\underline{u}$)

$$\mathrm{B}(\rho)_{\underline{u}}^{\overline{u}} = \{x \in \mathrm{B}(\rho) \mid \underline{u} \le x \le \overline{u}\}, \tag{6.30}$$

where for any real z $\lfloor z \rfloor$ and $\lceil z \rceil$ are, respectively, the integer z^* nearest to z satisfying $z^* \le z$ and $z \le z^*$, and for any $x \in \mathbb{R}^E$ $\lfloor x \rfloor = (\lfloor x(e) \rfloor \mid e \in E)$ and $\lceil x \rceil = (\lceil x(e) \rceil \mid e \in$

E). Also we define lower and upper capacities $\underline{c}$ and $\overline{c}$ on arcs in A as $\underline{c}(a) = \lfloor \varphi_P(a) \rfloor$ and $\overline{c}(a) = \lceil \varphi_P(a) \rceil$ for all $a \in A$, and consider the independent flows with these capacities and base polytope $\mathrm{B}(\rho)^{\overline{u}}_{\underline{u}}$. Then, we can adapt the procedure Randomized_Mechanism to the independent flow network modified above and the obtained $Q^{(k)}$ $(k \in K)$ become closer to P than those obtained for the original network $\mathcal{N}$. This may give a favorable lottery in practice, especially for a polymatroid (E, ρ) with large $\rho(E)$ or $d(N)$.

7. Conclusion

We have considered the random assignment problem with multi-unit demands and polymatroidal supplies and have shown that the probabilistic serial (PS) mechanism can naturally be extended to give an ordinally efficient and normalized envy-free solution. The obtained results are natural extensions of those given in [12], while we have shown a sufficient condition (Theorem 5.4) that guarantees that the computed PS solution is a weakly sd Nash equilibrium, which is practically useful for problems with a large number of agents. We have also shown the weak strategy-proofness (Theorem 5.6) of the extended PS mechanism in case of unit demands and matroidal supplies.

Acknowledgments

S. Fujishige's work is supported by JSPS KAKENHI Grant Number JP25280004 and Y. Sano's work by JSPS KAKENHI Grant Numbers JP15K20885, JP16H03118.

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