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On the Supersingular Divisors of Nilpotent Admissible Indigenous Bundles

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Abstract. — In the present paper, we give a characterization of the supersingular divisors [i.e., the zero loci of the Hasse invariants] of nilpotent admissible/ordinary indigenous bundles on hyperbolic curves. By applying the characterization, we also obtain lists of the nilpotent indigenous bundles on certain hyperbolic curves. Moreover, we prove the hyperbolic ordinariness of certain hyperbolic curves.

Contents

Introduction

Let p be an *odd* prime number, k an algebraically closed field of characteristic p, (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$, and

$$(X, D)$$

a *hyperbolic curve* of type (g, r) over k — i.e., a pair consisting of a projective smooth curve X of genus g over k and a reduced closed subscheme $D \subseteq X$ of X of degree r. The main objects of the present paper are *nilpotent* [cf. [5], Chapter II, Definition 2.4] *admissible* [cf. [5], Chapter II, Definition 2.4] *indigenous bundles* [cf. [5], Chapter I, Definition 2.2] and *nilpotent ordinary* [cf. [5], Chapter II, Definition 3.1] *indigenous bundles* on $(X, D)/k$ — i.e., suitable $\mathbb{P}^1$-bundles over X equipped with connections [relative to $(X, D)/k$]. A nilpotent admissible/ordinary indigenous bundle plays an important role in the theory

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of *hyperbolically ordinary curves* established due to *S. Mochizuki* [cf. [5]]; for instance, a nilpotent admissible indigenous bundle determines a "*canonical mod p^2 lifting*" of the Frobenius-twist of (X, D) [cf. [5], Chapter II]. In [2], *L. R. A. Finotti* studied nilpotent ordinary indigenous bundles on hyperbolic curves of type $(2, 0)$ [cf. also [3], Remark 6.1.2]. In [1], *I. I. Bouw* and *S. Wewers* studied nilpotent ordinary indigenous bundles on hyperbolic curves of type $(0, 4)$ [cf. also Remark 4.7.1].

By the theory of hyperbolically ordinary curves, one may define the *Hasse invariant* [cf. [5], Chapter II, Proposition 2.6, (3)] of a nilpotent admissible indigenous bundle [that is a global section of an invertible sheaf on X whose square is naturally isomorphic to $(\omega^{\log})^{\otimes p-1}$ — cf. §1, (1.c)]. We shall refer to the zero locus of the Hasse invariant of a nilpotent admissible indigenous bundle as the *supersingular divisor* [cf. [5], Chapter II, Proposition 2.6, (3)] of the nilpotent admissible indigenous bundle. The supersingular divisor is an important invariant of a nilpotent admissible indigenous bundle; for instance, the isomorphism class of a nilpotent admissible indigenous bundle on $(X, D)/k$ is *completely determined by the supersingular divisor* [cf. [5], Chapter II, Proposition 2.6, (4)].

In [3], the author of the present paper gave a characterization of the supersingular divisors of nilpotent admissible/ordinary indigenous bundles in the case where $(r, p) = (0, 3)$, i.e., on *projective* hyperbolic curves *of characteristic three*. The characterization of [3] asserts that if $(r, p) = (0, 3)$, then it holds that a given effective divisor on X coincides with the *supersingular divisor* of a nilpotent admissible indigenous bundle on X if and only if the divisor is *reduced* and may be obtained by forming the zero locus of a *Cartier eigenform* [cf. [3], Definition A.8, (ii)] associated to a square-trivialized invertible sheaf [cf. [3], Definition A.3] on X [cf. [3], Theorem B]; moreover, in this case, it holds that the nilpotent admissible indigenous bundle on X is *ordinary* if and only if either

• the underlying invertible sheaf of the square-trivialized invertible sheaf is *trivial*, and the Jacobian variety of X is *ordinary*, or

• the underlying invertible sheaf of the square-trivialized invertible sheaf is *nontrivial* [i.e., *of order two*], and the Prym variety associated to the underlying invertible sheaf is *ordinary*

[cf. [3], Theorem B].

In the present paper, we give another characterization of the supersingular divisors of nilpotent admissible/ordinary indigenous bundles on hyperbolic curves [in the case where (r, p) is not necessarily equal to $(0, 3)$]. The main result of the present paper is as follows [cf. Theorem 3.9]:

THEOREM A. — *Let us apply the notational conventions introduced in* §1*. By abuse of notation, write*

$$C\colon\ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

for the [*necessarily surjective*] *k-linear homomorphism obtained by applying "$\Gamma(X^F, - \otimes_{\mathcal{O}^F} (\omega^{\log})^F)$" to the Cartier operator associated to X/k and*

$$d\colon\ \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

for the k-linear homomorphism determined by the exterior differentiation operator. Let

$$E$$

be an effective divisor on X. Consider the following conditions:

(NA) *The divisor E is* **of NA-type** *relative to $(X, D)/k$ [cf. Definition 3.1], i.e., coincides with the* **supersingular divisor** *of a* **nilpotent admissible** *indigenous bundle on $(X, D)/k$.*

(NO) *The divisor E is* **of NO-type** *relative to $(X, D)/k$ [cf. Definition 3.1], i.e., coincides with the* **supersingular divisor** *of a* **nilpotent ordinary** *indigenous bundle on $(X, D)/k$.*

(R) *The divisor E is* **reduced** *and does* **not intersect** *the closed subscheme D.*

(1) *The divisor E is* **of degree** $p^{*} \deg \omega^{\log}$.

(2) *The composite*

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \hookrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \overset{C}{\twoheadrightarrow} \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is **surjective**.

(2′) *The composite*

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-2E)) \hookrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \overset{C}{\twoheadrightarrow} \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is **surjective**.

(3) *The subspace*

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

and the image of the k-linear homomorphism

$$d\colon \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

do **not generate** $\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$.

Then the following implications hold:

$$(\mathrm{NO}) \iff (1)+(2')+(3) \implies (\mathrm{NA}) \iff (1)+(2)+(3) \implies (\mathrm{R}).$$

By applying Theorem A, we obtain the following result concerning nilpotent indigenous bundles on certain hyperbolic curves [cf. Proposition 4.6; Proposition 5.2; Proposition 5.5; Proposition 5.7]:

THEOREM B. — *The following hold:*

(i) *Suppose that $(g, r, p) = (0, 4, 3)$. Then (X, D) has* **precisely three** *nilpotent indigenous bundles. Moreover, every nilpotent indigenous bundle on $(X, D)/k$ is* **ordinary**, *hence also* **admissible**. *The* **supersingular divisor** *of a nilpotent [necessarily admissible] indigenous bundle on $(X, D)/k$ coincides with the reduced effective divisor on X of degree two obtained by forming the* **fixed locus** *of one of the three nontrivial* **nonspecial** *[cf. Definition 4.5] automorphisms of (X, D) over k.*

(ii) *Suppose that* $(g, r, p) = (1, 1, 3)$. *Then* (X, D) *has* **precisely three** *nilpotent indigenous bundles. Moreover, every nilpotent indigenous bundle on* $(X, D)/k$ *is* **ordinary**, *hence also* **admissible**. *The* **supersingular divisor** *of a nilpotent* [*necessarily admissible*] *indigenous bundle on* $(X, D)/k$ *coincides with the reduced effective divisor on* X *of degree one determined by one of the three* **nontrivial 2-torsion points** *of the elliptic curve determined by* (X, D).

(iii) *Suppose that* $(g, r, p) = (1, 1, 5)$. *If the elliptic curve over* k *determined by* (X, D) *is* **ordinary** (*respectively,* **supersingular**), *then* (X, D) *has* **precisely five** (*respectively,* **four**) *nilpotent indigenous bundles. Moreover, every nilpotent indigenous bundle on* $(X, D)/k$ *is* **admissible**. *The* **supersingular divisor** *of a nilpotent* [*necessarily admissible*] *indigenous bundle on* $(X, D)/k$ *may be described explicitly* [*cf. Proposition* 5.5, (iii)]. *Finally, a nilpotent indigenous bundle on* $(X, D)/k$ *is* **ordinary** *if and only if one of the following two conditions is satisfied:*

(1) *The supersingular divisor of the nilpotent* [*necessarily admissible*] *indigenous bundle coincides with the reduced effective divisor on* X *of degree two determined by two of the three* **nontrivial 2-torsion points** *of the elliptic curve determined by* (X, D).

(2) *The elliptic curve determined by* (X, D) *is* **ordinary**.

(iv) *Suppose that* $(g, r, p) = (1, 1, 7)$. *Then* (X, D) *has* **at least one** *nilpotent* **ordinary** *indigenous bundle whose* **supersingular divisor** *coincides with the reduced effective divisor on* X *of degree three determined by the three* **nontrivial 2-torsion points** *of the elliptic curve determined by* (X, D).

Here, let us recall the following basic question in *p-adic Teichmüller theory* discussed in [6], Introduction, §2.1 [cf. [6], Introduction, §2.1, (1)]:

> Is every pointed stable curve *hyperbolically ordinary* [cf. [5], Chapter II, Definition 3.3]?

In the present paper, we prove [some portions of — cf. the discussion following Theorem C] the following result concerning the above basic question:

THEOREM C. — *If either*

$$(g, r) = (0, 3)$$

or

$$(g, r, p) \in \{(0, 4, 3), (1, 1, 3), (1, 1, 5), (1, 1, 7), (2, 0, 3)\},$$

then every hyperbolic curve of type (g, r) *over a connected noetherian scheme of characteristic* p *is* **hyperbolically ordinary**.

Note that:

• Theorem C in the case where $(g, r, p) = (0, 4, 3)$ (respectively, $(1, 1, 3)$; $(1, 1, 5)$; $(1, 1, 7)$) is proved in Corollary 4.7 (respectively, Corollary 5.3; Corollary 5.6; Corollary 5.8) [of the present paper].

• Theorem C in the case where $(g, r) = (0, 3)$ is a consequence of [5], Chapter II, Theorem 2.3 [cf. Proposition 4.2 of the present paper, as well as the discussion at the beginning of §4, (4.a), of the present paper]. In §4, (4.a), of the present paper, we give an alternative verification of Theorem C in the case where $(g, r) = (0, 3)$ by means of the main result of the present paper.

- Theorem C in the case where $(g, r, p) = (2, 0, 3)$ is the content of [3], Theorem D.

- Theorem C in the case where $(g, r, p) = (1, 1, 5)$ has already been verified in [6] [cf. Remark 5.6.1 of the present paper].

- Theorem C in the case where $(g, r, p) = (0, 4, 3)$ "*follows*" from [1], Proposition 6.4. However, unfortunately, the proof of [1], Lemma 6.3 — which implies [1], Proposition 6.4 — contains an *error* [cf. Remark 4.7.1 of the present paper].

Finally, in §A, we discuss the relationship between the zero loci of *square Hasse invariants* [cf. [5], Chapter II, Proposition 2.6, (1)] and the zero loci of *canonical sections* discussed in [1], §3.

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1. Notational Conventions

In the present §1, we introduce some notational conventions applied in the present paper:

(1.a). Let p be an *odd* prime number and k an algebraically closed field of characteristic p. We shall write

$$p^{*} \stackrel{\text{def}}{=} \frac{p-1}{2}.$$

If "$(-)$" is an object over k, then we shall write "$(-)^F$" for the object over k obtained by forming the base change of "$(-)$" via the absolute Frobenius morphism of k.

(1.b). Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$ and

$$(X, D)$$

a *hyperbolic curve* of type (g, r) over k, i.e., a pair consisting of a projective smooth curve X of genus g over k and a reduced closed subscheme $D \subseteq X$ of X of degree r. We shall write

$$\mathcal{O}$$

for the *structure sheaf* of X,

$$\omega$$

for the *cotangent sheaf* of X/k,

$$\tau$$

for the *tangent sheaf* of X/k, and

$$\Phi\colon\ X \longrightarrow X^F$$

for the *relative Frobenius morphism* of X/k.

(1.c). One verifies immediately that the pair (X, D) naturally determines a *log smooth* [cf. [4], (3.3)] *fine log scheme* [cf. [4], (2.3)] over k [cf. [4], Example (2.5)]. We shall write

$$\omega^{\log}$$

for the *cotangent sheaf* of the resulting log scheme over k [cf. [4], (1.7)] and

$$\tau^{\log} \stackrel{\text{def}}{=} \mathcal{H}om_{\mathcal{O}}(\omega^{\log}, \mathcal{O})$$

for the *tangent sheaf* of the resulting log scheme over k. Then one verifies easily that the natural morphism from the resulting log scheme to X determines *isomorphisms* of $\mathcal{O}$-modules

$$\omega(D) \xrightarrow{\sim} \omega^{\log}, \quad \tau(-D) \xrightarrow{\sim} \tau^{\log}.$$

We shall write

$$d\colon \mathcal{O} \longrightarrow \omega$$

for the exterior differentiation operator. By abuse of notation, we shall write

$$d\colon \mathcal{O} \longrightarrow \omega^{\log}$$

for the exterior differentiation operator obtained by forming the composite of d and the natural inclusion $\omega \hookrightarrow \omega^{\log}$. Note that let us observe that since (X, D) is *hyperbolic*, it holds that the invertible sheaf $\omega^{\log}$ on X is *ample*, i.e., that $\deg \omega^{\log}$ $(= 2g - 2 + r)$ is *positive*.

(1.d). Let $\mathcal{L}$ be an invertible sheaf on X. Then we have an *isomorphism* of $\mathcal{O}$-modules

$$\mathcal{L}^{\otimes p} \xrightarrow{\sim} \Phi^*\mathcal{L}^F; \quad l^{\otimes p} \mapsto \Phi^{-1}l^F$$

— where l is a local section of $\mathcal{L}$. By means of this isomorphism, we always identify $\mathcal{L}^{\otimes p}$ with $\Phi^*\mathcal{L}^F$.

(1.e). Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}^F$-module. Then one verifies easily that the k-linear homomorphism

$$\Phi^*\mathcal{E} = \mathcal{O} \otimes_{\mathcal{O}^F} \mathcal{E} \longrightarrow \omega^{\log} \otimes_{\mathcal{O}^F} \mathcal{E} = \omega^{\log} \otimes_{\mathcal{O}} \Phi^*\mathcal{E}; \quad f \otimes e \mapsto df \otimes e$$

is a *connection* on $\Phi^*\mathcal{E}$ [relative to $(X, D)/k$]. We shall write

$$d_{\mathcal{E}}$$

for this connection on $\Phi^*\mathcal{E}$.

(1.f). By applying [4], Theorem (4.12), to the log smooth fine log scheme over k determined by the pair (X, D) (respectively, the scheme X), we obtain an *exact* sequence of $\mathcal{O}^F$-modules

$$0 \longrightarrow \mathcal{O}^F \longrightarrow \Phi_*\mathcal{O} \xrightarrow{\Phi_* d} \Phi_*\omega^{\log} \xrightarrow{C^{\log}} (\omega^{\log})^F \longrightarrow 0$$

$$(\text{respectively, } 0 \longrightarrow \mathcal{O}^F \longrightarrow \Phi_*\mathcal{O} \xrightarrow{\Phi_* d} \Phi_*\omega \xrightarrow{C} \omega^F \longrightarrow 0).$$

We shall refer to the fourth arrow

$$C^{\log}\colon\ \Phi_*\omega^{\log} \longrightarrow (\omega^{\log})^F \quad (\text{respectively, } C\colon\ \Phi_*\omega \longrightarrow \omega^F)$$

as the *Cartier operator* associated to $(X, D)/k$ (respectively, X/k).

(1.g). We shall write

$$\mathcal{T} \ \stackrel{\text{def}}{=}\ \Phi^*(\tau^{\log})^F.$$

Thus, we have a *connection* on $\mathcal{T}$ [cf. (1.e)]

$$\nabla_{\mathcal{T}} \ \stackrel{\text{def}}{=}\ d_{(\tau^{\log})^F}\colon\ \mathcal{T} \longrightarrow \omega^{\log} \otimes_{\mathcal{O}} \mathcal{T}.$$

(1.h). We shall write

$$\mathcal{M}_{g,[r]}$$

for the moduli stack of hyperbolic curves of type (g, r) over k;

$$(\mathcal{X}_{g,[r]}, \mathcal{D}_{g,[r]})$$

for the universal hyperbolic curve over $\mathcal{M}_{g,[r]}$;

$$\mathcal{N}_{g,[r]}$$

for the moduli stack of *smooth nilcurves* [cf. the discussion preceding [6], Introduction, Theorem 0.1] of type (g, r) over k, i.e., the moduli stack of hyperbolic curves of type (g, r) over k equipped with nilpotent [cf. [5], Chapter II, Definition 2.4] indigenous bundles [cf. [5], Chapter I, Definition 2.2];

$$\mathcal{N}^{\text{adm}}_{g,[r]} \ \subseteq\ \mathcal{N}_{g,[r]}$$

for the *admissible locus* of $\mathcal{N}_{g,[r]}$, i.e., the [necessarily open] substack which parametrizes hyperbolic curves of type (g, r) over k equipped with nilpotent *admissible* [cf. [5], Chapter II, Definition 2.4] indigenous bundles;

$$\mathcal{N}^{\text{ord}}_{g,[r]} \ \subseteq\ \mathcal{N}^{\text{adm}}_{g,[r]}$$

for the *ordinary locus* of $\mathcal{N}_{g,[r]}$, i.e., the [necessarily open] substack which parametrizes hyperbolic curves of type (g, r) over k equipped with nilpotent *ordinary* [cf. [5], Chapter II, Definition 3.1] indigenous bundles;

$$\mathcal{M}_{g,r} \ \longrightarrow\ \mathcal{M}_{g,[r]}$$

for the connected finite étale Galois covering [whose Galois group is isomorphic to $\mathfrak{S}_r$] which trivializes the étale local system on $\mathcal{M}_{g,[r]}$ obtained by considering *"ordering on the r marked points"*;

$$(\mathcal{X}_{g,r}, \mathcal{D}_{g,r}) \ \stackrel{\text{def}}{=}\ (\mathcal{X}_{g,[r]}, \mathcal{D}_{g,[r]}) \times_{\mathcal{M}_{g,[r]}} \mathcal{M}_{g,r};$$

$\mathcal{N}^{\mathrm{ord}}_{g,r} \stackrel{\mathrm{def}}{=} \mathcal{N}^{\mathrm{ord}}_{g,[r]} \times_{\mathcal{M}_{g,[r]}} \mathcal{M}_{g,r} \subseteq \mathcal{N}^{\mathrm{adm}}_{g,r} \stackrel{\mathrm{def}}{=} \mathcal{N}^{\mathrm{adm}}_{g,[r]} \times_{\mathcal{M}_{g,[r]}} \mathcal{M}_{g,r} \subseteq \mathcal{N}_{g,r} \stackrel{\mathrm{def}}{=} \mathcal{N}_{g,[r]} \times_{\mathcal{M}_{g,[r]}} \mathcal{M}_{g,r}$.
Then the following facts are well-known:

(i) The forgetful morphism of stacks

$$\mathcal{N}_{g,[r]} \longrightarrow \mathcal{M}_{g,[r]}$$

is *finite flat of degree* p^{3g-3+r} [cf. [5], Chapter II, Theorem 2.3].

(ii) The open substack

$$\mathcal{N}^{\mathrm{adm}}_{g,[r]} \subseteq \mathcal{N}_{g,[r]}$$

coincides with the *smooth locus* of the structure morphism $\mathcal{N}_{g,[r]} \to \mathrm{Spec}(k)$ [cf. [5], Chapter II, Corollary 2.16].

(iii) The open substack

$$\mathcal{N}^{\mathrm{ord}}_{g,[r]} \subseteq \mathcal{N}_{g,[r]}$$

coincides with the *étale locus* of the forgetful morphism of stacks

$$\mathcal{N}_{g,[r]} \longrightarrow \mathcal{M}_{g,[r]}$$

[cf. [5], Chapter II, Proposition 2.12; [5], Chapter II, Theorem 2.13].

2. Review of FL-bundles

In [5], Chapter II, §1, *S. Mochizuki* studied the notion of an *FL-bundle* [cf. [5], Chapter II, Definition 1.3; Definition 2.2 of the present paper], which defines a section of the torsor of "mod p^2 liftings" of (X^F, D^F) and, moreover, also defines the torsor of "mod p^2 liftings" of Φ with respect to the resulting "mod p^2 liftings" of (X^F, D^F). In the present §2, let us review a portion of the theory of *FL-bundles* of [5], Chapter II, §1, from the point of view of the present paper.

Let us start our discussion with the exact sequence of $\mathcal{O}^F$-modules of §1, (1.f),

$$0 \longrightarrow \mathcal{O}^F \longrightarrow \Phi_*\mathcal{O} \stackrel{\Phi_* d}{\longrightarrow} \Phi_*\omega^{\log} \stackrel{C^{\log}}{\longrightarrow} (\omega^{\log})^F \longrightarrow 0.$$

Thus, by applying "$H^1(X^F, - \otimes_{\mathcal{O}^F} (\tau^{\log})^F)$, we obtain a sequence of k-vector spaces

$$H^1(X^F, (\tau^{\log})^F) \longrightarrow H^1(X, \mathcal{T}) \longrightarrow H^1(X, \omega^{\log} \otimes_{\mathcal{O}} \mathcal{T}).$$

LEMMA 2.1. — *In the above sequence*

$$H^1(X^F, (\tau^{\log})^F) \longrightarrow H^1(X, \mathcal{T}) \longrightarrow H^1(X, \omega^{\log} \otimes_{\mathcal{O}} \mathcal{T}),$$

the following hold:

(i) *The image of the composite of the two arrows is* **zero**.

(ii) *The first arrow is* **injective**.

(iii) *The kernel of the second arrow is naturally* **isomorphic** *to the relative first de Rham cohomology* $H^1_{\mathrm{DR}}(X, \mathcal{T}) \stackrel{\mathrm{def}}{=} H^1_{\mathrm{DR}}(X, (\mathcal{T}, \nabla_{\mathcal{T}}))$ *of* $(\mathcal{T}, \nabla_{\mathcal{T}})$:

$$H^1_{\mathrm{DR}}(X, \mathcal{T}) \stackrel{\sim}{\longrightarrow} \mathrm{Ker}\big(H^1(X, \mathcal{T}) \to H^1(X, \omega^{\log} \otimes_{\mathcal{O}} \mathcal{T})\big).$$

(iv) *The sequence under consideration determines a sequence of* **injections**

$$H^1(X^F, (\tau^{\log})^F) \hookrightarrow H^1_{\mathrm{DR}}(X, \mathcal{T}) \hookrightarrow H^1(X, \mathcal{T}).$$

(v) *The cokernel of the first arrow of* (iv) *is naturally* **isomorphic** *to* $k = \Gamma(X^F, \mathcal{O}^F)$, *hence also* **of dimension one**.

PROOF. — Assertion (i) is immediate. Assertions (ii), (iii) follow formally from the [easily verified] fact that

$$\Gamma(X, \mathcal{T}) = \Gamma(X, \omega^{\log} \otimes_{\mathcal{O}} \mathcal{T}) = \{0\}.$$

Assertion (iv) follows from assertions (i), (ii), (iii). Assertion (v) follows formally from the fact that $\Gamma(X, \omega^{\log} \otimes_{\mathcal{O}} \mathcal{T}) = \{0\}$, together with assertion (iii). This completes the proof of Lemma 2.1. □

DEFINITION 2.2. — Let $(\mathcal{E}, \nabla_{\mathcal{E}})$ be a pair consisting of a coherent $\mathcal{O}$-module $\mathcal{E}$ and a connection $\nabla_{\mathcal{E}}$ on $\mathcal{E}$ relative to $(X, D)/k$. Then we shall say that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is an *FL-bundle* on $(X, D)/k$ [cf. [5], Chapter II, Definition 1.3] if $(\mathcal{E}, \nabla_{\mathcal{E}})$ admits a structure of extension

$$0 \longrightarrow (\mathcal{T}, \nabla_{\mathcal{T}}) \longrightarrow (\mathcal{E}, \nabla_{\mathcal{E}}) \longrightarrow (\mathcal{O}, d) \longrightarrow 0$$

whose extension class $\in H^1_{\mathrm{DR}}(X, \mathcal{T})$ is not contained in the subspace $H^1(X^F, (\tau^{\log})^F) \subseteq H^1_{\mathrm{DR}}(X, \mathcal{T})$ [cf. Lemma 2.1, (iv)].

DEFINITION 2.3. — We shall say that an FL-bundle is *indigenous* if the projectivization of the FL-bundle is an indigenous bundle on $(X, D)/k$ [cf. [5], Chapter I, Definition 2.2].

The following proposition follows immediately from [5], Chapter II, Corollary 1.6:

PROPOSITION 2.4. — *Let* $(\mathcal{E}, \nabla_{\mathcal{E}})$ *be an* **FL-bundle** *on* $(X, D)/k$. *Then the horizontal invertible subsheaf "*$(\mathcal{T}, \nabla_{\mathcal{T}})$*" of* $(\mathcal{E}, \nabla_{\mathcal{E}})$ *in the extension of Definition* 2.2 *is the* **uniquely determined maximal horizontal invertible subsheaf** *of* $(\mathcal{E}, \nabla_{\mathcal{E}})$.

DEFINITION 2.5. — Let $(\mathcal{E}, \nabla_{\mathcal{E}})$ be an *FL-bundle* on $(X, D)/k$. Then we shall refer to the uniquely determined maximal horizontal invertible subsheaf of $(\mathcal{E}, \nabla_{\mathcal{E}})$ [cf. Proposition 2.4] as the *conjugate filtration* of $(\mathcal{E}, \nabla_{\mathcal{E}})$.

LEMMA 2.6. — *Let* $(\mathcal{E}, \nabla_{\mathcal{E}})$ *be an* **FL-bundle** *on* $(X, D)/k$. *Then the* **monodromy** *operator of* $\nabla_{\mathcal{E}}$ *at each point on* $D \subseteq X$ *is* **nilpotent**.

PROOF. — This follows from the existence of a structure of extension as in Definition 2.2, together with the [easily verified] fact that the monodromy operator of the connection $\nabla_{\mathcal{T}}$ (respectively, d) on $\mathcal{T}$ (respectively, $\mathcal{O}$) at each point on $D \subseteq X$ is *zero*. □

LEMMA 2.7. — *Let $(Y, D_Y) \to (X, D)$ be a finite flat tamely ramified covering between hyperbolic curves over k and $(\mathcal{E}, \nabla_{\mathcal{E}})$ an* **FL-bundle** *on $(X, D)/k$. Then it holds that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is* **indigenous** *if and only if the FL-bundle $(Y \to X)^*(\mathcal{E}, \nabla_{\mathcal{E}})$ on $(Y, D_Y)/k$ obtained by pulling back $(\mathcal{E}, \nabla_{\mathcal{E}})$ via $Y \to X$ is* **indigenous**.

PROOF. — Write (P, ∇_P), (Q, ∇_Q) for the projectivizations of $(\mathcal{E}, \nabla_{\mathcal{E}})$, $(Y \to X)^*(\mathcal{E}, \nabla_{\mathcal{E}})$, respectively. The *necessity* follows from [5], Chapter I, Proposition 2.3. To verify the *sufficiency*, suppose that (Q, ∇_Q) is *indigenous*. Then it follows immediately from the *uniqueness* discussed in [5], Chapter I, Proposition 2.4, that the *Hodge section* [cf. [5], Chapter I, Proposition 2.4] of the indigenous bundle (Q, ∇_Q) *descends* to a section of $P \to X$; moreover, one verifies easily from the various definitions involved that the resulting section of $P \to X$ is *of canonical height* $-\deg \omega^{\log}/2$ [cf. the discussion preceding [5], Chapter I, Definition 2.2]. Thus, in light of Lemma 2.6, we conclude that (P, ∇_P) is an *indigenous bundle* on $(X, D)/k$, as desired. □

LEMMA 2.8. — *Let $(Y, D_Y) \to (X, D)$ be a finite flat tamely ramified covering between hyperbolic curves over k and (P, ∇_P) an indigenous bundle on $(X, D)/k$. Then it holds that (P, ∇_P) is* **nilpotent** (*respectively,* **admissible**) [*cf.* [5]*, Chapter II, Definition* 2.4] *if and only if the indigenous bundle $(Y \to X)^*(P, \nabla_P)$ on $(Y, D_Y)/k$ obtained by pulling back (P, ∇_P) via $Y \to X$ is* **nilpotent** (*respectively,* **admissible**).

PROOF. — This follows immediately from the various definitions involved. □

One of the main results of the theory of *FL-bundles* is as follows [cf. [5], Chapter II, Proposition 2.5]:

THEOREM 2.9. — *The following hold:*

(i) *Let $(\mathcal{E}, \nabla_{\mathcal{E}})$ be an* **FL-bundle** *on $(X, D)/k$. Suppose that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is* **indigenous**. *Then the indigenous bundle on $(X, D)/k$ obtained by forming the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$ is* **nilpotent** *and* **admissible**.

(ii) *Let (P, ∇_P) be a* **nilpotent admissible indigenous bundle** *on $(X, D)/k$. Then the kernel of the dual of the p-curvature homomorphism $\mathcal{T} \to (P \to X)_* \tau_{P/X}$ — where we write $\tau_{P/X}$ for the tangent sheaf of P/X — of (P, ∇_P) [equipped with the connection determined by ∇_P] is an* **indigenous FL-bundle** *on $(X, D)/k$.*

(iii) *The constructions of* (i) *and* (ii) *determine a* **bijection** *between the set of isomorphism classes of* **indigenous FL-bundles** *on $(X, D)/k$ and the isomorphism classes of* **nilpotent admissible indigenous bundle** *on $(X, D)/k$.*

PROOF. — Let us first recall that if r is *even* [cf. the remark at the beginning of the discussion entitled "The Definition of the Verschiebung" in [5], Chapter II, §2], then these assertions follow immediately from [5], Chapter II, Proposition 2.5 [cf. also the proof of [5], Chapter II, Proposition 2.5]. Next, let us observe that one verifies easily that there exists a finite flat tamely ramified Galois covering $(Y, D_Y) \to (X, D)$ between hyperbolic curves over k such that "r" for (Y, D_Y) [i.e., the degree of the reduced closed subscheme $D_Y \subseteq Y$] is *even*, which thus implies that Theorem 2.9 for (Y, D_Y) holds.

Assertion (i) follows from assertion (i) for (Y, D_Y), together with Lemma 2.8. Next, we verify assertion (ii). Let us first observe that it follows immediately from a similar argument to the argument applied in the proof of [5], Chapter II, Proposition 2.5, that the kernel under consideration is an *FL-bundle*. Moreover, it follows from assertion (ii) for (Y, D_Y), together with Lemma 2.7, that the kernel under consideration is also *indigenous*. This completes the proof of assertion (ii). Assertion (iii) follows immediately from the various definitions involved. This completes the proof of Theorem 2.9. $\square$

3. A CHARACTERIZATION OF SUPERSINGULAR DIVISORS

In the present §3, we give a *characterization of the supersingular divisors* of nilpotent admissible/ordinary indigenous bundles [cf. Theorem 3.9; Corollary 3.11 below].

DEFINITION 3.1. — We shall say that an effective divisor on X is *of NA-type* (respectively, *of NO-type*) relative to $(X, D)/k$ if there exists a nilpotent admissible (respectively, nilpotent ordinary — cf. [5], Chapter II, Definition 3.1) indigenous bundle on $(X, D)/k$ whose supersingular divisor [cf. [5], Chapter II, Proposition 2.6, (3)] coincides with the effective divisor.

The following fact is well-known [cf. [5], Chapter II, Proposition 2.6, (2), (3); Proposition A.4 of the present paper]:

PROPOSITION 3.2. — *Let E be an effective divisor on X* **of NA-type** *relative to $(X, D)/k$. Then the following hold:*

(i) *The divisor E is* **of degree** $p^* \deg \omega^{\log}$.

(ii) *The divisor E is* **reduced**.

(iii) *It holds that $E \cap D = \emptyset$.*

Since a nilpotent *ordinary* indigenous bundle is *admissible* [cf. [5], Chapter II, Proposition 3.2], the following proposition holds:

PROPOSITION 3.3. — *If an effective divisor on X is* **of NO-type** *relative to $(X, D)/k$, then the divisor is* **of NA-type** *relative to $(X, D)/k$.*

Let

$$(\mathcal{E}, \nabla_{\mathcal{E}})$$

be an *FL-bundle* on $(X, D)/k$. Write

$$\mathcal{C} \subseteq \mathcal{E}$$

for the *conjugate filtration* of $(\mathcal{E}, \nabla_{\mathcal{E}})$ [cf. Definition 2.5] and fix horizontal isomorphisms

$$\mathcal{T} \xrightarrow{\sim} \mathcal{C}, \quad \mathcal{O} \xrightarrow{\sim} \mathcal{E}/\mathcal{C}.$$

By means of these horizontal isomorphisms, we identify $\mathcal{T}$, $\mathcal{O}$ with $\mathcal{C}$, $\mathcal{E}/\mathcal{C}$, respectively.

Let E be an effective divisor on X of degree $< -\deg \mathcal{T} = p \deg \omega^{\log}$. Then the natural inclusion $\mathcal{O}(-E) \hookrightarrow \mathcal{O}$ determines an exact sequence of $\mathcal{O}$-modules

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}(E) \longrightarrow \mathcal{T}(E)|_E \longrightarrow 0,$$

which thus determines an *exact* sequence of k-vector spaces

$$0 \longrightarrow \Gamma(E, \mathcal{T}(E)|_E) \longrightarrow H^1(X, \mathcal{T}) \longrightarrow H^1(X, \mathcal{T}(E)) \longrightarrow 0.$$

By means of the second arrow of this sequence, we regard $\Gamma(E, \mathcal{T}(E)|_E)$ as a subspace of $H^1(X, \mathcal{T})$:

$$\Gamma(E, \mathcal{T}(E)|_E) \subseteq H^1(X, \mathcal{T}).$$

DEFINITION 3.4. — We shall say that E is *liftable* with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$ if the natural inclusion $\mathcal{O}(-E) \hookrightarrow \mathcal{O}$ lifts to a [necessarily injective] homomorphism $\mathcal{O}(-E) \hookrightarrow \mathcal{E}$ of $\mathcal{O}$-modules [relative to the natural surjection $\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{C} = \mathcal{O}$].

Thus, one verifies immediately from the definition of the term "*liftable*" that the following lemma holds:

LEMMA 3.5. — *The following conditions are equivalent:*

(1) *The effective divisor E is* **liftable** *with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$.*

(2) *The FL-bundle $(\mathcal{E}, \nabla_{\mathcal{E}})$ has a structure of extension as in Definition* 2.2 *whose extension class $\in H^1_{\mathrm{DR}}(X, \mathcal{T})$ $(\subseteq H^1(X, \mathcal{T}))$ [cf. Lemma* 2.1, (iv)*] is* **contained** *in the subspace $\Gamma(E, \mathcal{T}(E)|_E) \subseteq H^1(X, \mathcal{T})$.*

LEMMA 3.6. — *If E is* **liftable** *with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$, then it holds that $p^* \deg \omega^{\log} \leq \deg E$.*

PROOF. — Since E is *liftable* with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$, the natural inclusion $\mathcal{O}(-E) \hookrightarrow \mathcal{O}$ lifts to a homomorphism $\mathcal{O}(-E) \hookrightarrow \mathcal{E}$. Now we may assume without loss of generality, by replacing E by a suitable effective subdivisor of E, that the lifting $\mathcal{O}(-E) \hookrightarrow \mathcal{E}$ is *locally split*. Then since $\det \mathcal{E} \cong \mathcal{T}$, it holds that $\mathcal{E}/\mathcal{O}(-E) \cong \mathcal{T}(E)$.

Let us consider the homomorphism of $\mathcal{O}$-modules obtained by forming the composite

$$\mathcal{O}(-E) \hookrightarrow \mathcal{E} \stackrel{\nabla_{\mathcal{E}}}{\to} \omega^{\log} \otimes_{\mathcal{O}} \mathcal{E} \twoheadrightarrow \omega^{\log} \otimes_{\mathcal{O}} (\mathcal{E}/\mathcal{O}(-E)) \cong \omega^{\log} \otimes_{\mathcal{O}} \mathcal{T}(E).$$

Then it follows immediately from Proposition 2.4 that this composite is *injective*. Thus, we obtain that

$$-\deg E = \deg \mathcal{O}(-E) \leq \deg(\omega^{\log} \otimes_{\mathcal{O}} \mathcal{T}(E)) = (1-p) \deg \omega^{\log} + \deg E,$$

which thus implies the desired inequality. This completes the proof of Lemma 3.6. $\square$

PROPOSITION 3.7. — *The following conditions are equivalent:*

(1) *The FL-bundle* $(\mathcal{E}, \nabla_{\mathcal{E}})$ *is* **indigenous**.

(2) *There exists an effective divisor on* X *of degree* $p^{*} \deg \omega^{\log}$ *which is* **liftable** *with respect to* $(\mathcal{E}, \nabla_{\mathcal{E}})$.

Moreover, in this case, the effective divisor of (2) **coincides** *with the* **supersingular divisor** *of the* **nilpotent admissible indigenous bundle** *on* $(X, D)/k$ *obtained by forming the projectivization of* $(\mathcal{E}, \nabla_{\mathcal{E}})$ [*cf. Theorem* 2.9, (i)].

PROOF. — First, we verify the implication (1) $\Rightarrow$ (2). Suppose that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is *indigenous.* Write $\mathcal{L} \subseteq \mathcal{E}$ for the *Hodge filtration* of $(\mathcal{E}, \nabla_{\mathcal{E}})$ [i.e., the invertible subsheaf which defines the *Hodge section* of the indigenous bundle obtained by forming the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$]. Then it follows immediately from the definition of an indigenous bundle that the homomorphism of $\mathcal{O}$-modules obtained by forming the composite

$$\mathcal{L} \hookrightarrow \mathcal{E} \overset{\nabla_{\mathcal{E}}}{\to} \omega^{\log} \otimes_{\mathcal{O}} \mathcal{E} \twoheadrightarrow \omega^{\log} \otimes_{\mathcal{O}} (\mathcal{E}/\mathcal{L})$$

is an *isomorphism.* In particular, since $(\mathcal{E}/\mathcal{L}) \otimes_{\mathcal{O}} \mathcal{L} \cong \det \mathcal{E} \cong \mathcal{T}$, it holds that $\deg \mathcal{L} = -p^{*} \deg \omega^{\log}$; moreover, the homomorphism of $\mathcal{O}$-modules obtained by forming the composite

$$\mathcal{L} \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{C} = \mathcal{O}$$

is thus *injective* [cf. also Proposition 2.4]. Thus, there exists an effective divisor F on X of degree $-\deg \mathcal{L} = p^{*} \deg \omega^{\log}$ such that the injection $\mathcal{L} \hookrightarrow \mathcal{O}$ determines an isomorphism $\mathcal{L} \overset{\sim}{\to} \mathcal{O}(-F)$. In particular, condition (2) is satisfied. This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that E is of degree $p^{*} \deg \omega^{\log}$ and *liftable* with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$. Since E is *liftable* with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$, the natural inclusion $\mathcal{O}(-E) \hookrightarrow \mathcal{O}$ lifts to a homomorphism $\mathcal{O}(-E) \hookrightarrow \mathcal{E}$. Let us observe that it follows immediately from Lemma 3.6 that this lifting $\mathcal{O}(-E) \hookrightarrow \mathcal{E}$ is *locally split*; moreover, since $\det \mathcal{E} \cong \mathcal{T}$, it holds that $\mathcal{E}/\mathcal{O}(-E) \cong \mathcal{T}(E)$.

Consider the homomorphism of $\mathcal{O}$-modules obtained by forming the composite

$$\mathcal{O}(-E) \hookrightarrow \mathcal{E} \overset{\nabla_{\mathcal{E}}}{\to} \omega^{\log} \otimes_{\mathcal{O}} \mathcal{E} \twoheadrightarrow \omega^{\log} \otimes_{\mathcal{O}} (\mathcal{E}/\mathcal{O}(-E)) \cong \omega^{\log} \otimes_{\mathcal{O}} \mathcal{T}(E).$$

Since E is *of degree* $p^{*} \deg \omega^{\log}$, and this composite is *injective* [cf. Proposition 2.4], this composite is in fact an *isomorphism*, which thus implies that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is *indigenous* [cf. also Lemma 2.6]. This completes the proof of the implication (2) $\Rightarrow$ (1).

The final assertion follows immediately from the proof of the implication (1) $\Rightarrow$ (2), together with a similar argument to the argument applied in the verification of [3], Proposition B.4 [cf. also Proposition A.3, (iv), and Lemma A.10, (i), of the present paper]. This completes the proof of Proposition 3.7. □

PROPOSITION 3.8. — *It holds that* E *is* **of NA-type** *relative to* $(X, D)/k$ *if and only if the following three conditions are satisfied:*

(1) *It holds that* $\deg E = p^{*} \deg \omega^{\log}$.

(2) *It holds that* $H^1(X^F, (\tau^{\log})^F) \cap \Gamma(E, \mathcal{T}(E)|_E) = \{0\}$.

(3) *It holds that* $H^1_{\mathrm{DR}}(X, \mathcal{T}) \cap \Gamma(E, \mathcal{T}(E)|_E) \neq \{0\}$.

PROOF. — First, we verify the *sufficiency*. Take a *nonzero* element $c \in H^1_{\mathrm{DR}}(X, \mathcal{T}) \cap \Gamma(E, \mathcal{T}(E)|_E)$ [cf. condition (3)]. Then it follows from condition (2) that $c \notin H^1(X^F, (\tau^{\log})^F)$. In particular, the class c determines an *FL-bundle* on $(X, D)/k$. Thus, it follows, in light of Lemma 3.5, from the implication (2) ⇒ (1) of Proposition 3.7, together with condition (1), that the projectivization of the FL-bundle is a(n) [necessarily *nilpotent admissible* — cf. Theorem 2.9, (i)] *indigenous bundle* on $(X, D)/k$. Moreover, it follows from the final assertion of Proposition 3.7 that the supersingular divisor of the nilpotent admissible indigenous bundle *coincides* with E. Thus, the divisor E is *of NA-type* relative to $(X, D)/k$. This completes the proof of the *sufficiency*.

Finally, we verify the *necessity*. Suppose that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is *indigenous*, and that E *coincides* with the supersingular divisor of the nilpotent admissible indigenous bundle on $(X, D)/k$ determined by $(\mathcal{E}, \nabla_{\mathcal{E}})$ [cf. Theorem 2.9, (i), (iii)]. Then it follows from Proposition 3.2, (i), that condition (1) is satisfied. Next, let us observe that it follows from the definition of an FL-bundle that the conjugate filtration $\mathcal{C} \subseteq \mathcal{E}$ of $(\mathcal{E}, \nabla_{\mathcal{E}})$, together with the identifications $\mathcal{C} = \mathcal{T}$, $\mathcal{E}/\mathcal{C} = \mathcal{O}$, determines an extension class $c_{\mathcal{E}} \in H^1(X, \mathcal{T})$ such that $c_{\mathcal{E}} \notin H^1(X^F, (\tau^{\log})^F)$, $c_{\mathcal{E}} \in H^1_{\mathrm{DR}}(X, \mathcal{T})$. Moreover, let us observe that it follows, in light of Lemma 3.5, from the implication (1) ⇒ (2) of Proposition 3.7 and the final assertion of Proposition 3.7 that $c_{\mathcal{E}} \in \Gamma(E, \mathcal{T}(E)|_E)$ [which thus implies that condition (3) is satisfied]. Thus, to complete the verification of the *necessity*, it suffices to verify condition (2), i.e., $H^1(X^F, (\tau^{\log})^F) \cap \Gamma(E, \mathcal{T}(E)|_E) = \{0\}$.

Assume that there exists a *nonzero* element $a \in H^1(X^F, (\tau^{\log})^F) \cap \Gamma(E, \mathcal{T}(E)|_E)$. Then it is immediate that $c_{\mathcal{E}} + a \notin H^1(X^F, (\tau^{\log})^F)$, $c_{\mathcal{E}} + a \in H^1_{\mathrm{DR}}(X, \mathcal{T})$, and $c_{\mathcal{E}} + a \in \Gamma(E, \mathcal{T}(E)|_E)$. Thus, it follows immediately, in light of Lemma 3.5, from the implication (2) ⇒ (1) of Proposition 3.7 and the final assertion of Proposition 3.7 that the class $c_{\mathcal{E}} + a \in H^1(X, \mathcal{T})$ determines an *FL-bundle* $(\mathcal{E}', \nabla_{\mathcal{E}'})$ on $(X, D)/k$ such that the projectivization of $(\mathcal{E}', \nabla_{\mathcal{E}'})$ is a(n) [necessarily *nilpotent admissible* — cf. Theorem 2.9, (i)] *indigenous bundle* whose supersingular divisor *coincides* with E. In particular, it follow from [5], Chapter II, Proposition 2.6, (4), together with Theorem 2.9, (iii), that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is *isomorphic* to $(\mathcal{E}', \nabla'_{\mathcal{E}})$. On the other hand, it follows immediately from Proposition 2.4 that this isomorphism restricts to an isomorphism between the respective *conjugate filtrations* of $\mathcal{E}$ and $\mathcal{E}'$, which thus implies that $c_{\mathcal{E}} + a \in H^1(X, \mathcal{T})$ is a *k-multiple* of $c_{\mathcal{E}}$ — in contradiction to the fact that $a \in H^1(X^F, (\tau^{\log})^F) \setminus \{0\}$ and $c_{\mathcal{E}} \notin H^1(X^F, (\tau^{\log})^F)$. This completes the proof of the *necessity*, hence also of Proposition 3.8. □

It follows from the definitions of the two subspaces

$$H^1_{\mathrm{DR}}(X, \mathcal{T}),\ \Gamma(E, \mathcal{T}(E)|_E) \ \subseteq\ H^1(X, \mathcal{T})$$

[cf. also Lemma 2.1, (iii)] that condition (2) (respectively, (3)) of the statement of Proposition 3.8 is *equivalent* to the condition that

$$\mathrm{Ker}\big(H^1(X^F, (\tau^{\log})^F) \ \hookrightarrow\ H^1(X, \mathcal{T}) \twoheadrightarrow\ H^1(X, \mathcal{T}(E))\big) =\ \{0\}$$

$$(\text{respectively,}\ \ \mathrm{Ker}\big(H^1(X, \mathcal{T}) \to H^1(X, \omega^{\log} \otimes_{\mathcal{O}} \mathcal{T}) \oplus H^1(X, \mathcal{T}(E))\big) \ \neq\ \{0\}).$$

Thus, in light of Proposition 3.2 and Proposition 3.3, by applying the *Serre duality*, together with [5], Chapter II, Lemma 2.11, we obtain the following theorem, which is the main result of the present paper:

THEOREM 3.9. — *In the notational conventions introduced in* §1*, by abuse of notation, write*

$$C\colon\ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \ \twoheadrightarrow\ \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

for the [*necessarily surjective*] *k-linear homomorphism obtained by applying "$\Gamma(X^F, - \otimes_{\mathcal{O}^F} (\omega^{\log})^F)$" to the Cartier operator associated to X/k and*

$$d\colon\ \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \ \longrightarrow\ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

for the k-linear homomorphism determined by the exterior differentiation operator. Let

$$E$$

be an effective divisor on X. Consider the following conditions:

(NA) *The divisor E is* **of NA-type** *relative to $(X, D)/k$.*

(NO) *The divisor E is* **of NO-type** *relative to $(X, D)/k$.*

(R) *The divisor E is* **reduced** *and does* **not intersect** *the closed subscheme D.*

(1) *The divisor E is* **of degree** $p^{*} \deg \omega^{\log}$.

(2) *The composite*

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \ \hookrightarrow\ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \ \overset{C}{\twoheadrightarrow}\ \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is **surjective**.

(2′) *The composite*

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-2E)) \ \hookrightarrow\ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \ \overset{C}{\twoheadrightarrow}\ \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is **surjective** [*or, alternatively, an* **isomorphism** — *cf. Remark* 3.9.1*,* (i)*,* (iii)*, below*].

(3) *The subspace*

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \ \subseteq\ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

and the image of the k-linear homomorphism

$$d\colon\ \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \ \longrightarrow\ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

do **not generate** $\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$.

Then the following implications hold:

$$(\mathrm{NO}) \iff (1)+(2')+(3) \implies (\mathrm{NA}) \iff (1)+(2)+(3) \implies (\mathrm{R}).$$

PROOF. — Let us recall that we have already verified [cf. the discussion preceding Theorem 3.9] that the implications

$$(\mathrm{NO}) \implies (\mathrm{NA}) \iff (1)+(2)+(3) \implies (\mathrm{R}).$$

hold. Thus, to complete the verification of Theorem 3.9, it suffices to verify the implications

$$(1)+(2')+(3) \implies (\mathrm{NO}) \implies (2').$$

To verify the implication $(1)+(2')+(3) \Rightarrow (\mathrm{NO})$, suppose that conditions (1), (2′), and (3) are satisfied. Then since [it is immediate that] the implication $(2') \Rightarrow (2)$ holds, it follows from the implication $(1)+(2)+(3) \Rightarrow (\mathrm{NA})$ that E is *of NA-type*. In particular, the divisor $2E$ coincides with the zero locus of the *square Hasse invariant* [cf. [5], Chapter

II, Proposition 2.6, (1)] of a nilpotent admissible indigenous bundle on $(X, D)/k$. Thus, it follows from condition (2′), together with [5], Chapter II, Proposition 2.12, that the nilpotent admissible indigenous bundle is *ordinary*, which thus implies that condition (NO) is satisfied. This completes the proof of the implication (1) + (2′) + (3) ⇒ (NO).

To verify the implication (NO) ⇒ (2′), suppose that the condition (NO) is satisfied. In particular, the divisor $2E$ coincides with the zero locus of the *square Hasse invariant* of a nilpotent *ordinary* indigenous bundle on $(X, D)/k$. Thus, it follows from [5], Chapter II, Proposition 2.12, that condition (2′) is satisfied. This completes the proof of the implication (NO) ⇒ (2′), hence also of Theorem 3.9. □

REMARK 3.9.1. — In Theorem 3.9, we consider the two k-linear homomorphisms

$$\begin{aligned} C\colon\ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) &\twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F)), \\ d\colon\ \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) &\rightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \end{aligned}$$

and the two subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-2E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)).$$

Let us first observe that it follows from the *Riemann-Roch formula* that:

(i) The domain, codomain of the k-linear homomorphism

$$C\colon\ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

are of dimension

$$\begin{aligned} 1 - g + (p+1)\deg\omega^{\log} - r &= (2p+1)\cdot g - (2p+1) + pr, \\ \dim \mathcal{M}_{g,[r]} &= 3g - 3 + r, \end{aligned}$$

respectively.

(ii) The domain, codomain of the k-linear homomorphism

$$d\colon\ \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

are of dimension

$$\begin{aligned} 1 - g + p\deg\omega^{\log} - r &= (2p-1)\cdot g - (2p-1) + (p-1)\cdot r, \\ 1 - g + (p+1)\deg\omega^{\log} - r &= (2p+1)\cdot g - (2p+1) + pr, \end{aligned}$$

respectively.

(iii) If condition (1) of the statement of Theorem 3.9 is satisfied, then the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-2E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

of $\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$ are of dimension

$$\begin{aligned} \dim \mathcal{M}_{g,[r]} &= 3g - 3 + r, \\ 1 - g + (p^{*}+2)\deg\omega^{\log} - r &= (2p^{*}+3)\cdot g - (2p^{*}+3) + (p^{*}+1)\cdot r, \end{aligned}$$

respectively.

Next, let us recall that it follows immediately from the various definitions involved [cf. also the discussion preceding Lemma 2.1] that:

(iv) The image of the composite

$$\Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \xrightarrow{d} \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \overset{C}{\twoheadrightarrow} \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is *zero*.

(v) The kernel of the k-linear homomorphism

$$d\colon\ \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

is of dimension

$$\dim_k H^1(X^F, \mathcal{O}^F) = g.$$

Finally, let us observe that it follows from Lemma 2.1, (v), that:

(vi) The cokernel of the k-linear homomorphism

$$d\colon\ \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

is of dimension

$$1 + \dim_k \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F)) = 3g - 2 + r.$$

DEFINITION 3.10. In the situation of Theorem 3.9:

(i) We shall write

$$V_{(X,D)} \overset{\text{def}}{=} \operatorname{Coker}\bigl(d\colon \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \to \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))\bigr).$$

(ii) We shall write

$$V_{(X,D)}[2E] \subseteq V_{(X,D)}[E] \subseteq V_{(X,D)}$$

for the subspaces of $V_{(X,D)}$ determined by the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-2E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)),$$

respectively.

(iii) We shall write

$$C\colon\ V_{(X,D)} \twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

for the surjective k-linear homomorphism determined by the homomorphism C in the statement of Theorem 3.9 [cf. Remark 3.9.1, (iv)].

It follows from Remark 3.9.1, (vi), that the kernel of the surjective k-linear homomorphism of Definition 3.10, (iii),

$$C\colon\ V_{(X,D)} \twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is *of dimension one*. Thus, the following corollary follows immediately from Theorem 3.9, together with Remark 3.9.1, (i), (iii):

COROLLARY 3.11. — *In the situation of Theorem* 3.9, *let E be an effective divisor on X of degree $p^* \deg \omega^{\log}$. Then the following hold:*

(i) *It holds that E is* **of NA-type** *relative to $(X, D)/k$ if and only if the composite*

$$V_{(X,D)}[E] \hookrightarrow V_{(X,D)} \overset{C}{\twoheadrightarrow} \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is an **isomorphism**, *i.e., the subspace $V_{(X,D)}[E] \subseteq V_{(X,D)}$ determines a* **splitting** *of $C\colon V_{(X,D)} \twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$.*

(ii) *It holds that E is* **of NO-type** *relative to $(X, D)/k$ if and only if the two composites*

$$V_{(X,D)}[E] \hookrightarrow V_{(X,D)} \overset{C}{\twoheadrightarrow} \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F)),$$

$$V_{(X,D)}[2E] \hookrightarrow V_{(X,D)} \overset{C}{\twoheadrightarrow} \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

are **isomorphisms**, *i.e., the subspaces $V_{(X,D)}[E]$, $V_{(X,D)}[2E] \subseteq V_{(X,D)}$ determine* **splittings** *of $C\colon V_{(X,D)} \twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$, respectively.*

4. EXPLICIT COMPUTATIONS IN CASES OF GENUS ZERO

In the present §4, we apply the characterization of Corollary 3.11 to some hyperbolic curves *of genus zero*.

In the present §4, suppose that

$$g = 0,$$

which thus implies that

$$\deg \omega^{\log} = r - 2.$$

Thus, there exists a function $t \in \Gamma(X \setminus D, \mathcal{O}^\times)$ which determines an isomorphism over k

$$\mathrm{Spec}\Big(k\Big[t, \frac{1}{t}, \frac{1}{t-1}, \frac{1}{t-a_1}, \ldots, \frac{1}{t-a_{r-3}}\Big]\Big) \xrightarrow{\sim} X \setminus D$$

for some distinct $r-3$ elements $a_1, \ldots, a_{r-3} \in k \setminus \{0, 1\}$ of $k \setminus \{0, 1\}$. By means of this isomorphism, let us identify the left-hand side with the right-hand side. We shall write

$$f_0(t) \overset{\text{def}}{=} t \cdot (t-1) \cdot (t-a_1) \cdots (t-a_{r-3}) \in \Gamma(X \setminus D, \mathcal{O}^\times)$$

and

$$\omega_0 \in \Gamma(X, \omega^{\log})$$

for the uniquely determined global section of $\omega^{\log}$ whose restriction to $X \setminus D$ is given by

$$\frac{dt}{f_0(t)} = \frac{dt}{t \cdot (t-1) \cdot (t-a_1) \cdots (t-a_{r-3})} \in \Gamma(X \setminus D, \omega^{\log}).$$

Write, moreover, for each integer d,

$$k[t]^{\leq d} \overset{\text{def}}{=} \{ f(t) \in k[t] \mid \deg f(t) \leq d \}.$$

[Thus, one verifies easily that the equality

$$\dim_k k[t]^{\leq d} = \max\{0,\, d+1\}$$

holds.] Then one verifies immediately that there exist isomorphisms of k-vector spaces

$$\begin{array}{ccc} k[t]^{\leq p(r-2)-2} & \xrightarrow{\sim} & \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \\ f(t) & \mapsto & f(t)dt \otimes \omega_0^{\otimes p}, \end{array}$$

$$\begin{array}{ccc} k[t]^{\leq p(r-2)-r} & \xrightarrow{\sim} & \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \\ g(t) & \mapsto & g(t)dt \otimes \omega_0^{\otimes p-1}, \end{array}$$

$$\begin{array}{ccc} k[t^F]^{\leq r-4} & \xrightarrow{\sim} & \Gamma(X, ((\omega^{\log})^F)^{\otimes 2}(-D^F)) \\ h(t^F) & \mapsto & h(t^F)dt^F \otimes \omega_0^F. \end{array}$$

Moreover, one also verifies immediately that the sequence of k-vector spaces

$$\Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \xrightarrow{d} \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \xrightarrow{C} \Gamma(X, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

corresponds, relative to above isomorphisms, to the following sequence of k-vector spaces:

$$\begin{array}{ccccc} k[t]^{\leq p(r-2)-r} & \longrightarrow & k[t]^{\leq p(r-2)-2} & \longrightarrow & k[t^F]^{\leq r-4} \\ g(t) & \mapsto & \dfrac{d}{dt}(g(t)\cdot f_0(t)) & & \\ & & f(t) & \mapsto & -\dfrac{d^{p-1}}{dt^{p-1}} f(t)\Big|_{t^p=t^F}. \end{array}$$

Next, let

$$e_1, \ldots, e_{p^*(r-2)} \ \in \ k \setminus \{0, 1, a_1, \ldots, a_{r-3}\}$$

be distinct $p^*(r-2)$ $(= p^* \deg \omega^{\log})$ elements of $k \setminus \{0, 1, a_1, \ldots, a_{r-3}\}$. Write

$$E \ = \ \sum_{i=1}^{p^*(r-2)} [e_i]$$

for the [necessarily reduced effective] divisor on X *of degree* $p^*(r-2)$ $(= p^* \deg \omega^{\log})$ — where we write "$[-]$" for the principal divisor defined by the closed point of X corresponding to "$(-)$" — and

$$f_E(t) \ \stackrel{\text{def}}{=} \ (t-e_1)\cdots(t-e_{p^*(r-2)}) \ \in \ \Gamma(X \setminus (D \cup E), \mathcal{O}^\times).$$

Then one verifies immediately that the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-2E)) \ \subseteq \ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \ \subseteq \ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

correspond, relative to the above isomorphism

$$k[t]^{\leq p(r-2)-2} \ \xrightarrow{\sim} \ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)),$$

to the subspaces

$$f_E(t)^2 \cdot k[t]^{\leq r-4} \ \stackrel{\text{def}}{=} \ \{\, f(t)\cdot f_E(t)^2 \in k[t]^{\leq p(r-2)-2} \mid f(t) \in k[t]^{\leq r-4} \,\}$$

$$\subseteq \ f_E(t)\cdot k[t]^{\leq (p^*+1)(r-2)-2} \ \stackrel{\text{def}}{=} \ \{\, f(t)\cdot f_E(t) \in k[t]^{\leq p(r-2)-2} \mid f(t) \in k[t]^{\leq (p^*+1)(r-2)-2} \,\}$$

$$\subseteq \ k[t]^{\leq p(r-2)-2},$$

respectively. Thus, by Corollary 3.11, we obtain the following:

PROPOSITION 4.1. — *It holds that E is* **of NA-type** (*respectively,* **of NO-type**) *relative to $(X, D)/k$ if and only if the following two conditions* (1), (2) (*respectively,* (1), (2′)) *are satisfied:*

(1) *The k-linear homomorphism*

$$\begin{array}{ccc} f_E(t) \cdot k[t]^{\leq (p^* + 1)(r-2)-2} & \longrightarrow & k[t^F]^{\leq r-4} \\ f_E(t) \cdot f(t) & \mapsto & -\dfrac{d^{p-1}}{dt^{p-1}}(f_E(t) \cdot f(t))\Big|_{t^p = t^F} \end{array}$$

is **surjective**.

(2) *The subspace*

$$f_E(t) \cdot k[t]^{\leq (p^* + 1)(r-2)-2} \subseteq k[t]^{\leq p(r-2)-2}$$

and the image of the k-linear homomorphism

$$\begin{array}{ccc} k[t]^{\leq p(r-2)-r} & \longrightarrow & k[t]^{\leq p(r-2)-2} \\ g(t) & \mapsto & \dfrac{d}{dt}(g(t) \cdot f_0(t)) \end{array}$$

do **not generate** $k[t]^{\leq p(r-2)-2}$.

(2′) *The subspace*

$$f_E(t) \cdot k[t]^{\leq (p^* + 1)(r-2)-2} \subseteq k[t]^{\leq p(r-2)-2}$$

is **contained** *in the subspace of $k[t]^{\leq p(r-2)-2}$ generated by the subspace*

$$f_E(t)^2 \cdot k[t]^{\leq r-4} \subseteq k[t]^{\leq p(r-2)-2}$$

and the image of the k-linear homomorphism

$$\begin{array}{ccc} k[t]^{\leq p(r-2)-r} & \longrightarrow & k[t]^{\leq p(r-2)-2} \\ g(t) & \mapsto & \dfrac{d}{dt}(g(t) \cdot f_0(t)). \end{array}$$

(4.a). In the present (4.a), suppose that

$$(g, r) = (0, 3),$$

which thus implies that

$$\deg \omega^{\log} = 1.$$

In this situation, it follows from §1, (1.h), (i), (iii), that

- the hyperbolic curve (X, D) over k has a *unique* nilpotent indigenous bundle, and
- the unique nilpotent indigenous bundle is *ordinary.*

Moreover, it is well-known that the projectivization of the relative first de Rham cohomology [equipped with the Gauss-Manin connection] of the Legendre family of elliptic curves over $X \setminus D$ determines a *nilpotent ordinary indigenous bundle* on $(X, D)/k$. In particular, the supersingular divisor of the unique nilpotent ordinary indigenous bundle on $(X, D)/k$ coincides with the divisor determined by the *Hasse polynomial*

$$\chi_{\mathrm{Hss}}(t) \stackrel{\mathrm{def}}{=} \sum_{i=0}^{p^*} \binom{p^*}{i}^2 \cdot t^i.$$

In summary, in this situation, we already obtained the following assertion:

PROPOSITION 4.2. — *There exists a* **precisely one divisor of NA-type** — *relative to* $(X, D)/k$ — *on* X. *The divisor of NA-type is* **of NO-type** *relative to* $(X, D)/k$ *and obtained by forming the* **zero locus** *of the* **Hasse polynomial** $\chi_{\mathrm{Hss}}(t)$.

In the remainder of (4.a), let us verify the assertion that

the zero locus of $\chi_{\mathrm{Hss}}(t)$ satisfies conditions (1), (2′) of Proposition 4.1,

which thus gives an alternative verification of the assertion that

the zero locus of $\chi_{\mathrm{Hss}}(t)$ is *of NO-type* [hence also *of NA-type*] relative to $(X, D)/k$

by means of the characterization of Corollary 3.11.

To verify the assertion that the zero locus of $\chi_{\mathrm{Hss}}(t)$ satisfies conditions (1), (2′) of Proposition 4.1, let us first observe that since $r - 4 < 0$, it holds that

$$k[t^F]^{\leq r-4} = \{0\}, \quad f_E(t)^2 \cdot k[t]^{\leq r-4} = \{0\}.$$

In particular, condition (1) of Proposition 4.1 is always satisfied; moreover, condition (2′) of Proposition 4.1 is *equivalent* to the following assertion:

($\dagger_1$): The subspace

$$\chi_{\mathrm{Hss}}(t) \cdot k[t]^{\leq p^{*}-1} \subseteq k[t]^{\leq p-2}$$

is *contained* in the image of the k-linear homomorphism

$$\begin{array}{ccc} k[t]^{\leq p-3} & \longrightarrow & k[t]^{\leq p-2} \\ g(t) & \mapsto & \dfrac{d}{dt}(g(t) \cdot t \cdot (t-1)). \end{array}$$

Next, to verify the assertion ($\dagger_1$), for each $f(t) \in k[t]^{\leq p-2}$, let us write

$$\int f(t)dt \in k[t]^{\leq p-1}$$

for the uniquely determined element of $k[t]^{\leq p-1}$ such that

$$\frac{d}{dt}\int f(t)dt = f(t) \quad \text{and} \quad \int f(t)dt\Big|_{t=0} = 0,$$

i.e., the uniquely determined "*indefinite integral*" of degree $\leq p-1$ whose *constant of integration is zero*. Then one verifies easily that, to verify the assertion ($\dagger_1$), it suffices to verify that:

($\dagger_2$): For each $0 \leq n \leq p^{*} - 1$, it holds that $\displaystyle\int t^n \cdot \chi_{\mathrm{Hss}}(t)dt\Big|_{t=1} = 0$.

Next, to verify the assertion ($\dagger_2$), for each $0 \leq n_1, n_2 \leq p^{*}-1$ such that $n_1+n_2 \leq p^{*}-1$, let us write

$$I(n_1, n_2) \stackrel{\mathrm{def}}{=} \int t^{n_1} \cdot \Big(\overbrace{\int \cdots \int}^{n_2} \chi_{\mathrm{Hss}}(t)\, \overbrace{dt \cdots dt}^{n_2}\Big)dt\Big|_{t=1}.$$

Thus, the assertion ($\dagger_2$) is *equivalent* to the assertion that $I(n, 0) = 0$ for each $0 \leq n \leq p^{*} - 1$. In particular, to verify the assertion ($\dagger_2$), it suffices to verify that:

($\dagger_3$): For each $0 \leq n_1,\ n_2 \leq p^* - 1$ such that $n_1 + n_2 \leq p^* - 1$, it holds that $I(n_1, n_2) = 0$.

Let us observe that, for each $0 \leq n \leq p^* - 1$, since

$$\overbrace{\int \cdots \int}^{n+1} \chi_{\mathrm{Hss}}(t) \overbrace{dt \cdots dt}^{n+1} = \sum_{i=0}^{p^*} \binom{p^*}{i}^2 \cdot \frac{1}{(i+1)\cdots(i+n+1)} \cdot t^{i+n+1}$$

$$= \frac{1}{(p^*+1)\cdots(p^*+n+1)} \cdot \sum_{i=0}^{p^*} \binom{p^*}{i} \cdot \binom{p^*+n+1}{i+n+1} \cdot t^{i+n+1},$$

it follows from "*Vandermonde's convolution*" that

$$I(0,n) = \frac{1}{(p^*+1)\cdots(p^*+n+1)} \cdot \sum_{i=0}^{p^*} \binom{p^*}{i} \cdot \binom{p^*+n+1}{i+n+1}$$

$$= \frac{1}{(p^*+1)\cdots(p^*+n+1)} \cdot \binom{p^*+p^*+n+1}{p^*}$$

$$= \frac{1}{(p^*+1)\cdots(p^*+n+1)} \cdot \binom{p+n}{p^*} = 0.$$

This completes the proof of the fact that $I(n_1, n_2) = 0$ if $n_1 = 0$. Thus, the assertion ($\dagger_3$) follows from *induction on* n_1, together with the equality

$$I(n_1, n_2) = t^{n_1}|_{t=1} \cdot I(0, n_2) - n_1 \cdot I(n_1 - 1, n_2 + 1)$$

obtained by "*partial integration*". This completes the proof of the assertion that the zero locus of $\chi_{\mathrm{Hss}}(t)$ satisfies conditions (1), (2′) of Proposition 4.1.

(4.b). In the present (4.b), suppose that

$$(g, r, p) = (0, 4, 3),$$

which thus implies that

$$p^* = 1, \quad \deg \omega^{\log} = 2.$$

Write

$$a \overset{\mathrm{def}}{=} a_1 \in k \setminus \{0, 1\}.$$

[So $f_0(t) = t \cdot (t-1) \cdot (t-a)$.] Then, by Proposition 4.1, we obtain the following:

LEMMA 4.3. — *It holds that E is* **of NO-type** *relative to $(X, D)/k$ if and only if the following two conditions are satisfied:*

(1) *The k-linear homomorphism*

$$\begin{array}{ccc} f_E(t) \cdot k[t]^{\leq 2} & \longrightarrow & k[t^F]^{\leq 0} \\ f_E(t) \cdot f(t) & \mapsto & -\dfrac{d^2}{dt^2}(f_E(t) \cdot f(t)) \end{array}$$

is **surjective**.

(2) *The subspace*

$$f_E(t) \cdot k[t]^{\leq 2} \subseteq k[t]^{\leq 4}$$

is **contained** *in the subspace of* $k[t]^{\leq 4}$ *generated by the subspace*

$$f_E(t)^2 \cdot k[t]^{\leq 0} \subseteq k[t]^{\leq 4}$$

and the image of the k*-linear homomorphism*

$$\begin{array}{ccc} k[t]^{\leq 2} & \longrightarrow & k[t]^{\leq 4} \\ g(t) & \mapsto & \frac{d}{dt}(g(t) \cdot f_0(t)). \end{array}$$

Here, let us recall the following well-known fact concerning automorphisms of (X, D) over k:

PROPOSITION 4.4. — *The following hold:*

(i) *The homomorphism of* [*finite*] *groups*

$$\mathrm{Aut}_{\mathcal{M}_{0,4}}(\mathcal{X}_{0,4}, \mathcal{D}_{0,4}) \longrightarrow \mathrm{Aut}_k(X, D)$$

obtained by considering restrictions, relative to some choice of an ordering on the 4 *marked points of* (X, D)*, is* **injective**.

(ii) *The finite group* $\mathrm{Aut}_{\mathcal{M}_{0,4}}(\mathcal{X}_{0,4}, \mathcal{D}_{0,4})$ *is* **isomorphic** *to* $\mathbb{Z}/2 \times \mathbb{Z}/2$.

(iii) *The three* [*cf.* (ii)] *nontrivial automorphisms of* (X, D) *contained in the image of the injective* [*cf.* (i)] *homomorphism of* (i) *are the three automorphisms determined by the following three automorphisms of* $X \setminus D$ *over* k*:*

$$\sigma_0\colon\ t \mapsto \frac{t-a}{t-1}, \quad \sigma_1\colon\ t \mapsto \frac{a}{t}, \quad \sigma_\infty\colon\ t \mapsto a \cdot \frac{t-1}{t-a}.$$

In particular, the image of the injective homomorphism of (i) *does* **not depend** *on the choice of an ordering on the* 4 *marked points of* (X, D).

DEFINITION 4.5. — We shall refer to an automorphism of the hyperbolic curve (X, D) over k which is contained in the image of the homomorphism of Proposition 4.4, (i) [cf. also the final assertion of Proposition 4.4, (iii)], as a *nonspecial* automorphism of (X, D).

Let σ be a nontrivial *nonspecial* automorphism of (X, D). Now I *claim* that

> the reduced effective divisor on X of degree 2 $(= p^* \deg \omega^{\log})$ obtained by forming the *fixed locus* of σ is *of NO-type* relative to $(X, D)/k$.

To verify this *claim*, let us take "E" of the discussion preceding Proposition 4.1 to be the reduced effective divisor on X obtained by forming the *fixed locus* of σ.

First, let us observe that it follows from Proposition 4.4, (iii), that we may assume without loss of generality, by applying a suitable change of coordinate, that the automorphism σ is the automorphism determined by σ_1 of Proposition 4.4, (iii). Thus, we obtain that

$$f_E(t) = t^2 - a.$$

Since

$$-\frac{d^2}{dt^2} f_E(t) = -\frac{d^2}{dt^2}(t^2 - a) = 1,$$

it holds that E satisfies condition (1) of Lemma 4.3. Next, to verify the assertion that E satisfies condition (2) of Lemma 4.3, let us observe that the following equalities hold:

$$f_E(t) \;=\; \frac{1}{a}\Big(f_E(t)^2 + \frac{d}{dt}\big((t+1)\cdot(t+a)\cdot f_0(t)\big)\Big),$$

$$t\cdot f_E(t) \;=\; \frac{d}{dt}(t\cdot f_0(t)),$$

$$t^2\cdot f_E(t) \;=\; 2\cdot f_E(t)^2 + \frac{d}{dt}\big((t+1)\cdot(t+a)\cdot f_0(t)\big).$$

Thus, we conclude that E satisfies condition (2) of Lemma 4.3. In particular, it follows from Lemma 4.3 that E is *of NO-type* relative to $(X,D)/k$, as desired. This completes the proof of the above *claim*.

Next, let us recall that it follows immediately from §1, (1.h), (i), that the hyperbolic curve (X,D) over k has at most 3 $(=p^{3g-3+r})$ nilpotent indigenous bundles. Thus, the above *claim*, together with §1, (1.h), (i), (iii), leads us to the following list of the nilpotent indigenous bundles on $(X,D)/k$:

PROPOSITION 4.6. — *The following hold:*

(i) *The hyperbolic curve (X,D) over k has* **precisely three** *nilpotent indigenous bundles.*

(ii) *Every nilpotent indigenous bundles on $(X,D)/k$ is* **ordinary***, hence also* **admissible***.*

(iii) *The* **supersingular divisor** *of a nilpotent* [*necessarily admissible — cf.* (ii)] *indigenous bundle on $(X,D)/k$ coincides with the reduced effective divisor obtained by forming the* **fixed locus** *of one of the three nontrivial* **nonspecial** *automorphisms of (X,D) over k.*

REMARK 4.6.1. — By Proposition 4.6, (i), (ii), the following assertion holds:

> Every sufficiently general hyperbolic curve of type $(0,4)$ over k has *precisely three* nilpotent ordinary indigenous bundles.

On the other hand, this assertion has already been verified [cf. [6], Chapter V, Corollary 1.3, (3)].

The following corollary follows from Proposition 4.6, (i), (ii), together with [5], Chapter II, Proposition 3.4:

COROLLARY 4.7. — *Every hyperbolic curve of type $(0,4)$ over a connected noetherian scheme of characteristic 3 is* **hyperbolically ordinary** [*cf.* [5]*, Chapter II, Definition* 3.3].

REMARK 4.7.1. — In the present Remark 4.7.1, let us discuss §6.2 of [1]. In the remainder of the present Remark 4.7.1, suppose that we are in the situation of §1, (1.h). [In particular, the field "k" is *not necessarily of characteristic three.*]

(i) [1], Lemma 6.3, asserts that the forgetful morphism $\mathcal{N}_{0,4} \to \mathcal{M}_{0,4}$ of stacks admits a *splitting*. Thus, since [it has already been verified that] $\mathcal{N}_{0,4}$ is *smooth* over k, it follows from §1, (1.h), (i), (iii), that the restriction of the morphism $\mathcal{N}_{0,4} \to \mathcal{M}_{0,4}$ of stacks to the ordinary locus $\mathcal{N}_{0,4}^{\mathrm{ord}} \subseteq \mathcal{N}_{0,4}$ is *surjective* [cf. [1], Proposition 6.4], and, moreover, the stack $\mathcal{N}_{0,4}$ is *not connected* [cf. [1], Corollary 6.5]. In particular, one may conclude that Corollary 4.7 holds [even if $p > 3$].

(ii) In the first and second paragraphs of the proof of [1], Lemma 6.3, the authors of [1] claimed that

> there exists a *nonzero* vector $(u_0, \ldots, u_{p-1})$ in the field $k_\lambda \stackrel{\mathrm{def}}{=} k(\lambda)$ of rational functions in λ over k such that the recursion (6.5) of [1], i.e.,
>
> $$\lambda \cdot (i+1)^2 \cdot u_{i+1} \;=\; (1+\lambda)\cdot(i^2+i+1)\cdot u_i - i^2 \cdot u_{i-1} \quad (i \in \{0, \ldots, p-1\})$$
>
> — where we write $u_{-1} \stackrel{\mathrm{def}}{=} u_p \stackrel{\mathrm{def}}{=} 0$ — holds.

However, this assertion is *false* in general. Indeed, if we are in the situation in which $p = 3$, then the above recursion is *equivalent* to the equality

$$\begin{pmatrix} 1+\lambda & -\lambda & 0 \\ 1 & 0 & \lambda \\ 0 & -1 & 1+\lambda \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

On the other hand, the determinant of the left-hand matrix is equal to $-\lambda \cdot (1+\lambda) \neq 0$. Thus, there is *no nonzero* vector (u_0, u_1, u_2) in k_λ which satisfies the recursion (6.5) of [1]. [Note that in the fourth paragraph of the proof of [1], Lemma 6.3, it is asserted that the v_i's also satisfy the recursion (6.5) of [1]. However, the author of the present paper *cannot find* any reason which implies that the v_i's satisfy the recursion (6.5) of [1].]

(iii) As a consequence of the discussion of (ii), the proof given in [1] of [1], Lemma 6.3 — hence also of [1], Proposition 6.4; [1], Corollary 6.5 — must be considered *incomplete.*

(iv) On the other hand, by a straightforward computation of a similar recursion to the recursion (6.5) of [1] which arises from the differential operator $L_{\lambda,\beta}$ of (6.3) of [1], one can verify the *validity* of [1], Lemma 6.3, at least in the case where $p = 3$, which thus implies [cf. the discussion of (i)] [1], Proposition 6.4, in the case where $p = 3$, as well as [1], Corollary 6.5, in the case where $p = 3$. In particular, one may conclude that Corollary 4.7 of the present paper may also be deduced from the consideration of §6.2 of [1].

(v) However, after pointing out the error discussed in (ii) to the authors of [1], the author of the present paper was informed by *I. I. Bouw* [who is one of the authors of [1]] that she could verify that [1], Lemma 6.3, in the case where $p \in \{11, 13\}$ is in fact *false* by a straightforward computation of a similar recursion to the recursion (6.5) of [1] which arises from the differential operator $L_{\lambda,\beta}$ of (6.3) of [1].

5. Explicit Computations in Cases of Once-punctured Elliptic Curves

In the present §5, we apply the characterization of Theorem 3.9 to some *once-punctured elliptic curves.*

In the present §5, suppose that

$$(g, r) \;=\; (1, 1),$$

which thus implies that

$$\deg \omega^{\log} \;=\; 1.$$

Thus, there exist functions s, $t \in \Gamma(X \setminus D, \mathcal{O})$ which determine an isomorphism over k

$$\mathrm{Spec}(k[s,t]/(s^2 - t \cdot (t-1) \cdot (t-a)) \;\xrightarrow{\sim}\; X \setminus D$$

for some element $a \in k \setminus \{0,1\}$ of $k \setminus \{0,1\}$. By means of this isomorphism, let us identify the left-hand side with the right-hand side. We shall write

$$f_0(t) \;\stackrel{\mathrm{def}}{=}\; t \cdot (t-1) \cdot (t-a) \;\in\; \Gamma(X \setminus D, \mathcal{O}),$$

$$f_0'(t) \;\stackrel{\mathrm{def}}{=}\; \frac{d}{dt} f_0(t) \;=\; 3t^2 - 2(1+a)t + a,$$

$$U \;\stackrel{\mathrm{def}}{=}\; \mathrm{Spec}\Big(k\Big[s, \frac{1}{s}, t\Big]/(s^2 - f_0(t))\Big) \;\subseteq\; X \setminus D$$

for the largest open subscheme of $X \setminus D$ on which the function $s \in \Gamma(X \setminus D, \mathcal{O})$ is invertible, and

$$\omega_0 \;\in\; \Gamma(X, \omega) \;=\; \Gamma(X, \omega^{\log})$$

for the uniquely determined global section of ω ($\subseteq \omega^{\log}$) whose restriction to $U \subseteq X$ is given by

$$\frac{dt}{s} \;\in\; \Gamma(U, \omega^{\log}).$$

Write, moreover, for each integer d,

$$k[s,t]^{\leq d} \;\stackrel{\mathrm{def}}{=}\; \Big\{\, f(s,t) = \sum_{i,j} c_{i,j} \cdot s^i \cdot t^j \in k[s,t] \;\Big|\; c_{i,j} = 0 \text{ if } 3i + 2j > d \,\Big\}$$

and

$$V^{\leq d} \;\subseteq\; k\Big[s, \frac{1}{s}, t\Big]/(s^2 - f_0(t))$$

for the subspace obtained by forming the image of $k[s,t]^{\leq d} \subseteq k[s,t]$. [Thus, one verifies easily that the equality

$$\dim_k V^{\leq d} \;=\; \begin{cases} d & \text{if } d \geq 1 \\ 1 & \text{if } d = 0 \\ 0 & \text{if } d \leq -1 \end{cases}$$

holds.] Then one verifies immediately that we have isomorphisms of k-vector spaces

$$\begin{array}{ccc} V^{\leq p} & \xrightarrow{\sim} & \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \\ f(s,t) & \mapsto & f(s,t) \cdot \omega_0^{\otimes p+1}, \end{array}$$

$$\begin{array}{ccc} V^{\leq p-1} & \xrightarrow{\sim} & \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \\ g(s,t) & \mapsto & g(s,t) \cdot \omega_0^{\otimes p}, \end{array}$$

$$\begin{array}{ccc} V^{\leq 0} & \xrightarrow{\sim} & \Gamma(X, ((\omega^{\log})^F)^{\otimes 2}(-D^F)) \\ c & \mapsto & c \cdot (\omega_0^F)^{\otimes 2}. \end{array}$$

Moreover, one also verifies immediately that the sequence of k-vector spaces

$$\Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \xrightarrow{d} \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \xrightarrow{C} \Gamma(X, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

corresponds, relative to the above isomorphisms, to the following sequence of k-vector spaces:

$$\begin{array}{cccccc} V^{\leq p-1} & \longrightarrow & V^{\leq p} & & \longrightarrow & V^{\leq 0} \\ g(s,t) & \mapsto & s \cdot \dfrac{d}{dt} g(s,t) \ \Big(= \dfrac{1}{s^p} \cdot f_0(t)^{p^*+1} \cdot \dfrac{d}{dt} g(s,t) \Big) & & & \\ & & f(s,t) & & \mapsto & -\dfrac{d^{p-1}}{dt^{p-1}} (f(s,t) \cdot f_0(t)^{p^*}). \end{array}$$

Note that one verifies easily that the first arrow of this sequence is given by

$$\begin{array}{ccc} V^{\leq p-1} & \longrightarrow & V^{\leq p} \\ t^n & \mapsto & n \cdot t^{n-1} \cdot s \\ t^n \cdot s & \mapsto & G_n(t) \stackrel{\text{def}}{=} n \cdot t^{n-1} \cdot f_0(t) + t^n \cdot \dfrac{f_0'(t)}{2}. \end{array}$$

Thus, we obtain the following:

Lemma 5.1. — *Let E be a reduced effective divisor on X of degree p^* $(= p^* \deg \omega^{\log})$. Then it holds that E satisfies condition* (3) *of Theorem* 3.9 *if and only if the subspace of $V^{\leq p}$ corresponding, relative to the above isomorphism*

$$V^{\leq p} \xrightarrow{\sim} \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)),$$

to the subspace

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

and

$$t^n \cdot s \quad (0 \leq n \leq p^* - 1), \quad G_m(t) \quad (0 \leq m \leq p^* - 2)$$

do **not generate** $V^{\leq p}$.

(5.a). In the present (5.a), suppose that

$$(g, r, p) = (1, 1, 3),$$

which thus implies that

$$p^* = 1.$$

Let us first consider the principal divisor [i.e., the reduced effective divisor of degree $1 = p^* \deg \omega^{\log}$] on X defined by the closed point of $X \setminus D$ which is *not a* 2*-torsion* point of the elliptic curve over k determined by (X, D). One verifies easily that such a closed point of $X \setminus D$ is defined by the maximal ideal

$$(s - c_2,\, t - c_1) \subseteq k[s,t]/(s^2 - f_0(t))$$

for some pair (c_1, c_2) of elements of k such that $f_0(c_1) \neq 0$ and $c_2^2 = f_0(c_1)$. Write

$$E_{(c_1,c_2)} \subseteq X$$

for the principal divisor defined by this closed point. Then one verifies immediately that the subspace

$$\Gamma(X, (\omega^{\log})^{\otimes 4}(-D - E_{(c_1,c_2)})) \subseteq \Gamma(X, (\omega^{\log})^{\otimes 4}(-D))$$

corresponds, relative to the isomorphism

$$V^{\leq 3} \xrightarrow{\sim} \Gamma(X, (\omega^{\log})^{\otimes 4}(-D))$$

discussed above, to the subspace

$$\langle t - c_1,\ s - c_2 \rangle \subseteq V^{\leq 3}.$$

Thus, since [it is immediate from the fact that $c_2 \neq 0$ that] the subspace $\langle t - c_1,\ s - c_2 \rangle$ and

$$s$$

generate $V^{\leq 3}$, it follows from Lemma 5.1 that $E_{(c_1,c_2)}$ does *not satisfy* condition (3) of Theorem 3.9. Thus, it follows from Theorem 3.9 that $E_{(c_1,c_2)}$ is *not of NA-type* relative to $(X, D)/k$.

Next, let us consider the principal divisor [i.e., the reduced effective divisor of degree $1 = p^{*} \deg \omega^{\log}$] on X defined by the closed point of $X \setminus D$ which is a [necessarily nontrivial] 2-*torsion* point of the elliptic curve over k determined by (X, D). Let $c \in k$ be a solution of the equation "$f_0(t) = 0$", i.e., an element of $\{0, 1, a\}$. In the remainder of (5.a), write

$$E \subseteq X$$

for the principal divisor defined by the maximal ideal

$$(s,\ t - c) \subseteq k[s, t]/(s^2 - f_0(t)).$$

Now I *claim* that

> the reduced effective divisor E on X of degree 1 ($= p^{*} \deg \omega^{\log}$) is *of NO-type* relative to $(X, D)/k$.

To verify this *claim*, let us first observe one verifies easily that we may assume without loss of generality, by applying a suitable change of coordinate, that $c = 0$. Then one verifies immediately that the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes 4}(-D - 2E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes 4}(-D - E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes 4}(-D))$$

correspond, relative to the isomorphism

$$V^{\leq 3} \xrightarrow{\sim} \Gamma(X, (\omega^{\log})^{\otimes 4}(-D))$$

discussed above, to the subspaces

$$\langle t \rangle \subseteq \langle t,\ s \rangle \subseteq V^{\leq 3}.$$

Since

$$-\frac{d^2}{dt^2}(t \cdot f_0(t)) = -\frac{d^2}{dt^2}(t^2 \cdot (t - 1) \cdot (t - a)) = a \neq 0,$$

it holds that E satisfies condition ($2'$) of Theorem 3.9. Moreover, since [it is immediate that] the subspace of $V^{\leq 3}$ generated by $\langle t,\ s \rangle$ and

$$s$$

is *of dimension* ≤ 2 (< 3), it follows from Lemma 5.1 that E satisfies condition (3) of Theorem 3.9. Thus, it follows from Theorem 3.9 that E is *of NO-type* relative to $(X, D)/k$, as desired. This completes the proof of the above *claim*.

Next, let us recall that it follows immediately from §1, (1.h), (i), that the hyperbolic curve (X, D) over k has at most 3 ($= p^{3g-3+r}$) nilpotent indigenous bundles. Thus, the above *claim*, together with §1, (1.h), (i), (iii), leads us to the following list of the nilpotent indigenous bundles on $(X, D)/k$:

PROPOSITION 5.2. — *The following hold:*

(i) *The hyperbolic curve* (X, D) *over* k *has* **precisely three** *nilpotent indigenous bundles.*

(ii) *Every nilpotent indigenous bundles on* $(X, D)/k$ *is* **ordinary**, *hence also* **admissible**.

(iii) *The* **supersingular divisor** *of a nilpotent [necessarily admissible — cf.* (ii)*] indigenous bundle on* $(X, D)/k$ *coincides with the reduced effective divisor on* X *of degree one determined by one of the three* **nontrivial 2-torsion points** *of the elliptic curve determined by* (X, D).

REMARK 5.2.1. — By Proposition 5.2, (i), (ii), the following assertion holds:

> Every sufficiently general hyperbolic curve of type $(1, 1)$ over k has *precisely three* nilpotent ordinary indigenous bundles.

On the other hand, this assertion has already been verified [cf. [6], Chapter V, Corollary 1.3, (3)].

The following corollary follows from Proposition 5.2, (i), (ii), together with [5], Chapter II, Proposition 3.4:

COROLLARY 5.3. — *Every hyperbolic curve of type* $(1, 1)$ *over a connected noetherian scheme of characteristic* 3 *is* **hyperbolically ordinary**.

Let us observe that it follows from Proposition 5.2, (ii), that

$$\mathcal{N}^{\mathrm{ord}}_{1,[1]} \;=\; \mathcal{N}^{\mathrm{adm}}_{1,[1]} \;=\; \mathcal{N}_{1,[1]}.$$

Next, let us recall that the morphism of stacks

$$\mathcal{X}_{1,[1]} \;\longrightarrow\; \mathcal{M}_{1,[1]}$$

forms a *family of elliptic curves* over $\mathcal{M}_{1,[1]}$ whose identity section is given by $\mathcal{D}_{1,[1]} \subseteq \mathcal{X}_{1,[1]}$. For each positive integer n, we shall write

$$\mathcal{X}_{1,[1]}[n] \;\longrightarrow\; \mathcal{M}_{1,[1]}$$

for the kernel of the endomorphism of $\mathcal{X}_{1,[1]}$ over $\mathcal{M}_{1,[1]}$ obtained by multiplication by n. [So $\mathcal{X}_{1,[1]}[1] = \mathcal{D}_{1,[1]}$.] Then it follows from Proposition 5.2, (iii), that, by considering *supersingular divisors*, we obtain a *dominant* morphism of stacks

$$\mathcal{N}^{\mathrm{ord}}_{1,[1]} \;=\; \mathcal{N}^{\mathrm{adm}}_{1,[1]} \;=\; \mathcal{N}_{1,[1]} \;\longrightarrow\; \mathcal{X}_{1,[1]}[2] \setminus \mathcal{D}_{1,[1]}$$

over $\mathcal{M}_{1,[1]}$ [i.e., the "$(1, [1])$-version" of the *Hasse defect morphism* — cf. [3], Definition C.1]. Thus, both $\mathcal{N}^{\mathrm{ord}}_{1,[1]} = \mathcal{N}^{\mathrm{adm}}_{1,[1]} = \mathcal{N}_{1,[1]}$ and $\mathcal{X}_{1,[1]}[2] \setminus \mathcal{D}_{1,[1]}$ are *finite étale* and *of degree three* over $\mathcal{M}_{1,[1]}$ [cf. §1, (1.h), (i), (iii)], we obtain the following:

COROLLARY 5.4. — *There exists a natural* **isomorphism** *of stacks*

$$\mathcal{N}^{\mathrm{ord}}_{1,[1]} \;=\; \mathcal{N}^{\mathrm{adm}}_{1,[1]} \;=\; \mathcal{N}_{1,[1]} \;\xrightarrow{\sim}\; \mathcal{X}_{1,[1]}[2] \setminus \mathcal{D}_{1,[1]}$$

over $\mathcal{M}_{1,[1]}$.

(5.b). In the present (5.b), suppose that

$$(g, r, p) \;=\; (1, 1, 5),$$

which thus implies that

$$p^{\circledast} \;=\; 2.$$

Let $c_1,\, c_2 \in k$ be two distinct solutions of the equation "$f_0(t) = 0$", i.e., two distinct elements of $\{0, 1, a\}$. Write

$$E_1 \;\subseteq\; X$$

for the reduced effective divisor of degree 2 ($= p^{\circledast} \deg \omega^{\log}$) defined by the ideal

$$(s,\, (t - c_1) \cdot (t - c_2)) \;\subseteq\; k[s,t]/(s^2 - f_0(t)).$$

Now I claim that the following assertion holds:

($\dagger_1$): The reduced effective divisor E_1 on X of degree 2 ($= p^{\circledast} \deg \omega^{\log}$) is *of NO-type* relative to $(X, D)/k$.

To verify the assertion ($\dagger_1$), let us first observe that one verifies easily that we may assume without loss of generality, by applying a suitable change of coordinate, that $(c_1, c_2) = (0, 1)$. Then one verifies immediately that the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes 6}(-D - 2E_1)) \;\subseteq\; \Gamma(X, (\omega^{\log})^{\otimes 6}(-D - E_1)) \;\subseteq\; \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

correspond, relative to the isomorphism

$$V^{\leq 5} \;\xrightarrow{\sim}\; \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

discussed above, to the subspaces

$$\langle t \cdot (t-1) \rangle \;\subseteq\; \langle s,\, t \cdot (t-1),\, t \cdot s \rangle \;\subseteq\; V^{\leq 5}.$$

Since

$$-\frac{d^4}{dt^4}(t \cdot (t-1) \cdot f_0(t)^2) \;=\; -\frac{d^4}{dt^4}(t^3 \cdot (t-1)^3 \cdot (t-a)^2) \;=\; 3 \cdot a \cdot (a-1) \;\neq\; 0,$$

it holds that E_1 satisfies condition (2$'$) of Theorem 3.9. Moreover, since [it is immediate that] the subspace of $V^{\leq 5}$ generated by $\langle s,\, t \cdot (t-1),\, t \cdot s \rangle$ and

$$s, \quad G_0(t) \;=\; 3 \cdot f_0'(t), \quad t \cdot s$$

is *of dimension* ≤ 4 (< 5), it follows from Lemma 5.1 that E_1 satisfies condition (3) of Theorem 3.9. Thus, it follows from Theorem 3.9 that E_1 is *of NO-type* relative to $(X, D)/k$, as desired. This completes the proof of the above assertion ($\dagger_1$).

Next, let $c \in k$ be a solution of the equation "$f_0'(t) = 0$". [So the equality

$$c^2 + (1 + a) \cdot c + 2a \;=\; 0$$

holds.] Write

$$E_2 \;\subseteq\; X$$

for the reduced effective divisor of degree 2 ($= p^{*} \deg \omega^{\log}$) defined by the ideal

$$(t-c) \ \subseteq \ k[s,t]/(s^2 - f_0(t)).$$

Now I claim that the following assertion holds:

> ($\dagger_2$): The reduced effective divisor E_2 on X of degree 2 ($= p^{*} \deg \omega^{\log}$) is *of NA-type* relative to $(X,D)/k$.

To verify the assertion ($\dagger_2$), let us first observe that one verifies immediately that the subspace

$$\Gamma(X, (\omega^{\log})^{\otimes 6}(-D - E_2)) \ \subseteq \ \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

corresponds, relative to the isomorphism

$$V^{\leq 5} \ \xrightarrow{\sim} \ \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

discussed above, to the subspace

$$\langle t-c,\ (t-c)^2,\ (t-c)\cdot s \rangle \ \subseteq \ V^{\leq 5}.$$

Here, let us observe that it holds that

$$-\frac{d^4}{dt^4}((t-c)\cdot f_0(t)^2) \ = \ -\frac{d^4}{dt^4}((t-c)\cdot t^2 \cdot (t-1)^2 \cdot (t-a)^2)$$

$$= \ 3\cdot a \cdot (1+a) - c\cdot (a^2 - a + 1).$$

If $a^2 - a + 1 = 0$ [which thus implies that a is a *primitive sixth root of unity*], then it is immediate that $3\cdot a\cdot(1+a) - c\cdot(a^2-a+1) \neq 0$; moreover, if $a^2 - a + 1 \neq 0$, and $3\cdot a\cdot(1+a) - c\cdot(a^2-a+1) = 0$, then the equality $c^2 + (1+a)\cdot c + 2a = 0$ implies that

$$a^2 \cdot (a-1)^2 \ = \ 0$$

— in contradiction to the fact that $a \notin \{0,1\}$. Thus, we conclude that

$$-\frac{d^4}{dt^4}((t-c)\cdot f_0(t)^2) \ \neq \ 0$$

— which thus implies that E_2 satisfies condition (2) of Theorem 3.9.

Next, let us observe that one verifies easily that if $c' \in k$ is *not a solution* of the equation "$f_0'(t) = 0$", then $t - c' \in V^{\leq 5}$ is *not contained* in the subspace of $V^{\leq 5}$ generated by $\langle t-c,\ (t-c)^2,\ (t-c)\cdot s\rangle$ and

$$s, \quad G_0(t) \ = \ 3\cdot f_0'(t), \quad t\cdot s.$$

In particular, it follows from Lemma 5.1 that E_2 satisfies condition (3) of Theorem 3.9. Thus, it follows from Theorem 3.9 that E_2 is *of NA-type* relative to $(X,D)/k$, as desired. This completes the proof of the above assertion ($\dagger_2$).

Next, I claim that the following assertion holds:

> ($\dagger_3$): If, moreover, the elliptic curve over k determined by (X,D) is *supersingular* [i.e., the equality $a^2 - a + 1 = 0$ holds — cf. the Hasse polynomial "$\chi_{\mathrm{Hss}}(t)$" discussed in §4, (4.a), in the case where $p = 5$], then the divisor E_2 is *not of NO-type* relative to $(X,D)/k$.

To verify the assertion ($\dagger_3$), suppose that the equality $a^2 - a + 1 = 0$ holds. Then one verifies easily that the equation "$f_0'(t) = 0$" has a *multiple root*, which thus implies that

$$f_0'(t) \ = \ 3 \cdot (t-c)^2.$$

Next, let us observe that one verifies immediately that the subspace

$$\Gamma(X, (\omega^{\log})^{\otimes 6}(-D - 2E_2)) \ \subseteq \ \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

corresponds, relative to the isomorphism

$$V^{\leq 5} \ \xrightarrow{\sim} \ \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

discussed above, to the subspace

$$\langle (t-c)^2 \rangle \ \subseteq \ V^{\leq 5}.$$

Thus, since $G_0(t)$ $(= 3 \cdot f_0'(t) = -(t-c)^2)$ is *contained* in the image of d, hence also the kernel of C [cf. Remark 3.9.1, (iv)], it holds that E_2 does *not satisfy* condition (2′) of Theorem 3.9. Thus, it follows from Theorem 3.9 that E_2 is *not of NO-type* relative to $(X, D)/k$, as desired. This completes the proof of the above assertion ($\dagger_3$).

By the assertions ($\dagger_1$), ($\dagger_2$), and ($\dagger_3$), we obtain the following list of the nilpotent indigenous bundles on $(X, D)/k$:

PROPOSITION 5.5. — *Write A for the elliptic curve over k determined by (X, D). Then the following hold:*

(i) *If A is* **ordinary** (*respectively,* **supersingular**), *then the hyperbolic curve (X, D) over k has* **precisely five** (*respectively,* **four**) *nilpotent indigenous bundles.*

(ii) *Every nilpotent indigenous bundles on $(X, D)/k$ is* **admissible**.

(iii) *The* **supersingular divisor** *of a nilpotent [necessarily admissible — cf.* (ii)] *indigenous bundle on $(X, D)/k$ coincides with the reduced effective divisor on X of degree two determined by either*

(a) *two of the three* **nontrivial 2-torsion points** *of A or*

(b) *one of the solutions of the equation "$f_0'(t) = 0$". [Note that if A is* **ordinary** (*respectively,* **supersingular**), *then the equation "$f_0'(t) = 0$" has exactly two* (*respectively, one*) *solution(s).]*

(iv) *It holds that a nilpotent [necessarily admissible — cf.* (ii)] *indigenous bundle on $(X, D)/k$ is* **ordinary** *if and only if one of the following two conditions is satisfied:*

(1) *The supersingular divisor of the nilpotent indigenous bundle is given by* (a) *of* (iii).

(2) *The elliptic curve A is* **ordinary**.

In particular, if A is **ordinary** (*respectively,* **supersingular**), *then (X, D) has* **precisely five** (*respectively,* **three**) *nilpotent* **ordinary** *indigenous bundles.*

PROOF. — First, we verify Proposition 5.5 in the case where A is *ordinary.* Suppose that A is *ordinary.* Then it follows from the assertions ($\dagger_1$) and ($\dagger_2$) that the hyperbolic curve (X, D) has *at least five* nilpotent admissible indigenous bundles. Thus, it follows immediately from §1, (1.h), (i), that assertion (i) — hence also assertions (ii), (iii) [cf. the assertions ($\dagger_1$) and ($\dagger_2$)] — holds; moreover, it follows immediately from §1, (1.h), (iii), that every nilpotent indigenous bundle is *ordinary*, which thus implies that assertion (iv) holds. This completes the proof of Proposition 5.5 in the case where A is *ordinary*.

Next, we verify Proposition 5.5 in the case where A is *supersingular*. Suppose that A is *supersingular*. Then it follows from the assertions ($\dagger_1$) and ($\dagger_2$) that the hyperbolic curve (X, D) has *at least four* nilpotent admissible indigenous bundles; moreover, it follows from the assertion ($\dagger_3$) that one of the four nilpotent admissible indigenous bundles is *not ordinary*. Thus, it follows immediately from §1, (1.h), (i), (iii), that assertion (i) — hence also assertions (ii), (iii) [cf. the assertions ($\dagger_1$) and ($\dagger_2$)] — holds; moreover, it follows immediately from the assertions ($\dagger_1$) and ($\dagger_3$) that assertion (iv) holds. This completes the proof of Proposition 5.5 in the case where A is *supersingular*, hence also of Proposition 5.5. □

REMARK 5.5.1. — By Proposition 5.5, (iv), the following assertion holds:

> Every sufficiently general hyperbolic curve of type $(1, 1)$ over k has *precisely five* nilpotent ordinary indigenous bundles.

On the other hand, this assertion has already been verified [cf. [6], Chapter V, Corollary 1.3, (3)].

The following corollary follows from Proposition 5.5, (iv), together with [5], Chapter II, Proposition 3.4 [cf. Remark 5.6.1 below]:

COROLLARY 5.6. — *Every hyperbolic curve of type* $(1, 1)$ *over a connected noetherian scheme of characteristic* 5 *is* **hyperbolically ordinary**.

REMARK 5.6.1. — Note that Corollary 5.6 has already been verified in the second remark of [6], Chapter IV, §1.3.

(5.c). In the present (5.c), suppose that

$$(g, r, p) \;=\; (1, 1, 7),$$

which thus implies that

$$p^{*} \;=\; 3.$$

Write

$$E \;\subseteq\; X$$

for the reduced effective divisor of degree 3 ($= p^{*} \deg \omega^{\log}$) defined by the ideal

$$(s) \;\subseteq\; k[s, t]/(s^2 - f_0(t)).$$

Now I *claim* that

> the reduced effective divisor E on X of degree 3 ($= p^{*} \deg \omega^{\log}$) is *of NO-type* relative to $(X, D)/k$.

To verify this *claim*, let us first observe one verifies immediately that the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes 8}(-D-2E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes 8}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes 8}(-D))$$

correspond, relative to the isomorphism

$$V^{\leq 7} \xrightarrow{\sim} \Gamma(X, (\omega^{\log})^{\otimes 8}(-D))$$

discussed above, to the subspaces

$$\langle f_0(t) \rangle \subseteq \langle s, t \cdot s, f_0(t), t^2 \cdot s \rangle \subseteq V^{\leq 7}.$$

Since

$$-\frac{d^6}{dt^6}(f_0(t) \cdot f_0(t)^3) = -\frac{d^6}{dt^6}(t^4 \cdot (t-1)^4 \cdot (t-a)^4) = -a^2 \cdot (a-1)^2 \neq 0,$$

it holds that E satisfies condition (2′) of Theorem 3.9. Moreover, since [it is immediate that] the subspace of $V^{\leq 7}$ generated by $\langle s, t \cdot s, f_0(t), t^2 \cdot s \rangle$ and

$$s, \quad G_0(t) = 4 \cdot f_0'(t), \quad t \cdot s, \quad G_1(t) = f_0(t) + 4 \cdot t \cdot f_0'(t), \quad t^2 \cdot s$$

is *of dimension* ≤ 6 (< 7), it follows from Lemma 5.1 that E satisfies condition (3) of Theorem 3.9. Thus, it follows from Theorem 3.9 that E is *of NO-type* relative to $(X, D)/k$, as desired. This completes the proof of the above *claim*.

By the above *claim*, we obtain the following:

PROPOSITION 5.7. — *The hyperbolic curve* (X, D) *over* k *has a* **nilpotent ordinary indigenous bundle** *whose* **supersingular divisor** *coincides with the reduced effective divisor on* X *of degree three determined by the three* **nontrivial 2-torsion points** *of the elliptic curve determined by* (X, D).

The following corollary follows from Proposition 5.7, together with [5], Chapter II, Proposition 3.4:

COROLLARY 5.8. — *Every hyperbolic curve of type* $(1, 1)$ *over a connected noetherian scheme of characteristic* 7 *is* **hyperbolically ordinary**.

APPENDIX A. CANONICAL SECTIONS AND SQUARE HASSE INVARIANTS

In the present §A, we apply the notational conventions introduced in §1. In the present §A, we discuss the relationship between the zero loci of *square Hasse invariants* [cf. [5], Chapter II, Proposition 2.6, (1)] and the zero loci of *canonical sections* discussed in [1], §3. Moreover, we also verify that the application of the discussions of [1], §3, leads us to the relationship [cf. Proposition A.3 below] between the zero loci of *square Hasse invariants*, *generalized supersingular divisors* [cf. Definition A.2, (iii), below], and *spiked loci* [cf. Definition A.2, (iv), below].

DEFINITION A.1. — We shall say that an indigenous bundle on $(X, D)/k$ is *active* [cf. [6], Chapter II, Definition 1.1] if the p-curvature homomorphism of the indigenous bundle is a nonzero homomorphism.

DEFINITION A.2. — Let (P, ∇_P) be a *nilpotent active* indigenous bundle on $(X, D)/k$.

(i) We shall write

$$E_{\mathrm{sH}}$$

for the divisor on X obtained by forming the zero locus of the *square Hasse invariant* of (P, ∇_P).

(ii) Since (P, ∇_P) is nilpotent and active, one verifies immediately that there exists a uniquely determined horizontal section of $P \to X$. We shall refer to the uniquely determined horizontal section of $P \to X$ as the *conjugate section* of (P, ∇_P).

(iii) We shall write

$$E_{\mathrm{gss}}$$

for the divisor on X obtained by pulling back the conjugate section via the Hodge section of (P, ∇_P) and refer to E_{gss} as the *generalized supersingular divisor* of (P, ∇_P).

(iv) We shall write

$$E_{\mathrm{spk}}$$

for the divisor on X obtained by forming the zero locus of the p-curvature homomorphism of (P, ∇_P) and refer to E_{spk} as the *spiked locus* of (P, ∇_P) [cf. [6], Chapter II, Definition 3.1]. Thus, it follows from the various definitions involved that (P, ∇_P) is admissible if and only if $E_{\mathrm{spk}} = \emptyset$.

The purpose of the present §A is to verify the following proposition:

PROPOSITION A.3. — *Let (P, ∇_P) be a* **nilpotent active** *indigenous bundle on $(X, D)/k$. Then the following hold:*

(i) *The divisor E_{gss} [cf. Definition* A.2, *(iii)] is* **reduced**. *Moreover, it holds that $E_{\mathrm{gss}} \cap D = E_{\mathrm{gss}} \cap E_{\mathrm{spk}} = \emptyset$ [cf. Definition* A.2, *(iv)].*

(ii) *It holds that $E_{\mathrm{spk}} \cap D = \emptyset$. Moreover, there exists a divisor $\underline{E}_{\mathrm{spk}}$ on X such that $E_{\mathrm{spk}} = p\underline{E}_{\mathrm{spk}}$.*

(iii) *It holds that $E_{\mathrm{sH}} = 2E_{\mathrm{gss}} + E_{\mathrm{spk}}$ [cf. Definition* A.2, *(i)].*

(iv) *If (P, ∇_P) is* **admissible**, *then the divisor E_{gss} coincides with the* **supersingular divisor** *of (P, ∇_P).*

The following proposition, which seem to be well-known to experts, follows immediately from Proposition A.3, (i), (iv):

PROPOSITION A.4. — *The* **supersingular divisor** *of a nilpotent admissible indigenous bundle on $(X, D)/k$ is* **reduced** *and does* **not intersect** *the closed subscheme D.*

Since [we have assumed that] $p \geq 3$, the following proposition follows immediately from Proposition A.3:

PROPOSITION A.5. — *Let* (P, ∇_P) *be a* **nilpotent active** *indigenous bundle on* $(X, D)/k$. *Then the following conditions are equivalent:*

(1) *The indigenous bundle* (P, ∇_P) *is* **admissible**.

(2) *There exists a* **reduced** *divisor* $\underline{E}_{\mathrm{sH}}$ *on* X *such that* $E_{\mathrm{sH}} = 2\underline{E}_{\mathrm{sH}}$.

(3) *For each closed point* $x \in X$ *of* X, *the order of* E_{sH} *at* x *is* $\in \{0, 2\}$.

(4) *For each closed point* $x \in X$ *of* X, *the order of* E_{sH} *at* x *is* ≤ 2.

In the remainder of the present §A, let us prove Proposition A.3 by means of the discussions of [1], §3. In the present §A, for a closed point $x \in X$ of X and a positive integer d, we shall write

$$x_d$$

for the natural closed immersion from the [uniquely determined] closed subscheme of X of length d whose underlying set consists of x into X. [So x_1 is the natural closed immersion from the reduced closed subscheme of X determined by $x \in X$ into X.]

Let (P, ∇_P) be a *nilpotent active* indigenous bundle on $(X, D)/k$. Let us first observe that, to verify Proposition A.3, we may assume without loss of generality, by replacing (X, D) by a suitable connected finite flat tamely ramified covering of (X, D), that r is *even*. Then it follows from [5], Chapter I, Proposition 2.6, that there exists an *indigenous vector bundle* [cf. [5], Chapter I, Definition 2.2]

$$(\mathcal{E}, \nabla_{\mathcal{E}})$$

on $(X, D)/k$ whose projectivization is isomorphic to (P, ∇_P). Write

$$\mathcal{H} \subseteq \mathcal{E}$$

for the *Hodge filtration* of $(\mathcal{E}, \nabla_{\mathcal{E}})$ [i.e., the invertible subsheaf which defines the Hodge section of (P, ∇_P)]. Thus, it follows from the definition of an indigenous bundle that the homomorphism of $\mathcal{O}$-modules obtained by forming the composite

$$\mathcal{H} \hookrightarrow \mathcal{E} \stackrel{\nabla_{\mathcal{E}}}{\to} \omega^{\log} \otimes_{\mathcal{O}} \mathcal{E} \twoheadrightarrow \omega^{\log} \otimes_{\mathcal{O}} (\mathcal{E}/\mathcal{H})$$

is an *isomorphism*. In particular, since $\det \mathcal{E} \cong \mathcal{O}$ [cf. the discussion preceding [5], Chapter I, Definition 2.2], we have the following:

LEMMA A.6. — *It holds that*

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{H}, \mathcal{E}/\mathcal{H}) \cong \tau^{\log}, \quad \mathcal{H}^{\otimes 2} \cong \omega^{\log}.$$

Next, write

$$\mathcal{P}\colon \mathcal{T} \longrightarrow \mathcal{E}nd_{\mathcal{O}}(\mathcal{E})$$

for the *p-curvature* homomorphism of $(\mathcal{E}, \nabla_{\mathcal{E}})$. Let us recall that, in this situation, the *square Hasse invariant* of (P, ∇_P) is defined as the composite

$$\mathcal{T} \xrightarrow{\mathcal{P}} \mathcal{E}nd_{\mathcal{O}}(\mathcal{E}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{H}, \mathcal{E}/\mathcal{H}) \cong \tau^{\log}$$

[cf. Lemma A.6]. Thus, by the definition, we have the following:

LEMMA A.7. — *The following hold:*

(i) *The divisor* E_{sH} [*cf. Definition* A.2, (i)] *is* **of degree** $(p-1)\deg\omega^{\log}$.

(ii) *For a closed point* $x \in X$ *of* X *and a positive integer* d, *the following conditions are equivalent:*

(1) *It holds that* $\mathrm{ord}_x E_{\mathrm{sH}} \geq d$.

(2) *For each local section* ∂ *of* $\mathcal{T}$ *at* x, *the endomorphism* $x_d^*\mathcal{P}(\partial)$ *of* $x_d^*\mathcal{E}$ **preserves** *the submodule* $x_d^*\mathcal{H} \subseteq x_d^*\mathcal{E}$.

Next, write

$$\mathcal{C} \ \subseteq \ \mathcal{E}$$

for the *conjugate filtration* of $(\mathcal{E}, \nabla_{\mathcal{E}})$ [i.e., the invertible subsheaf which defines the conjugate section of (P, ∇_P) — cf. Definition A.2, (ii) — or, equivalently, the uniquely determined maximal horizontal invertible subsheaf of $(\mathcal{E}, \nabla_{\mathcal{E}})$] and

$$\nabla_{\mathcal{C}}, \quad \nabla_{\mathcal{Q}}$$

for the connections on

$$\mathcal{C}, \quad \mathcal{Q} \ \stackrel{\mathrm{def}}{=} \ \mathcal{E}/\mathcal{C}$$

induced by $\nabla_{\mathcal{E}}$, respectively.

LEMMA A.8. — *The following hold:*

(i) *The* ***p*-curvature** *homomorphisms, as well as the* **monodromy** *operators at each point on* $D \subseteq X$, *of* $(\mathcal{C}, \nabla_{\mathcal{C}})$, $(\mathcal{Q}, \nabla_{\mathcal{Q}})$ *are* **zero**, *respectively.*

(ii) *It holds that*

$$(\mathcal{C}, \nabla_{\mathcal{C}}) \otimes (\mathcal{Q}, \nabla_{\mathcal{Q}}) \ \cong \ (\mathcal{O}, d).$$

PROOF. — Assertion (i) follows from our assumption that the *p-curvature* homomorphism, as well as the *monodromy* operator at each point on $D \subseteq X$, of (P, ∇_P) is *nilpotent*. Assertion (ii) follows from the fact that $\det(\mathcal{E}, \nabla_{\mathcal{E}}) \cong (\mathcal{O}, d)$ [cf. the discussion preceding [5], Chapter I, Definition 2.2]. $\square$

In this situation, the *canonical section* of $(\mathcal{E}, \nabla_{\mathcal{E}})$ introduced in the discussion preceding [1], Lemma 3.4, may be defined as follows:

DEFINITION A.9. — We shall refer to the composite

$$\mathcal{H} \ \hookrightarrow \ \mathcal{E} \ \twoheadrightarrow \ \mathcal{Q}$$

as the *canonical section* of $(\mathcal{E}, \nabla_{\mathcal{E}})$ [cf. the discussion preceding [1], Lemma 3.4].

Thus, by definition, we have the following:

LEMMA A.10. — *The following hold:*

(i) *The divisor on X obtained by forming the zero locus of the* **canonical section** *coincides with the* **generalized supersingular divisor** E_{gss} *of* (P, ∇_P) *[cf. Definition* A.2, (iii)].

(ii) *For a closed point $x \in X$ of X, the following conditions are equivalent:*

(1) *It holds that* $\mathrm{ord}_x E_{\text{gss}} \geq 1$.

(2) *The subspace $x_1^*\mathcal{H} \subseteq x_1^*\mathcal{E}$ is* **contained** *in the subspace $x_1^*\mathcal{C} \subseteq x_1^*\mathcal{E}$.*

It follows from Lemma A.8, (i), that the p-curvature homomorphism $\mathcal{P}$ of $(\mathcal{E}, \nabla_{\mathcal{E}})$ factors through the subsheaf $\mathcal{H}om_{\mathcal{O}}(\mathcal{Q}, \mathcal{C}) \subseteq \mathcal{E}nd_{\mathcal{O}}(\mathcal{E})$:

$$\mathcal{P}\colon \mathcal{T} \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{Q}, \mathcal{C}) \cong \mathcal{C}^{\otimes 2}$$

[cf. Lemma A.8, (ii)]. In this situation, the *spiked locus* E_{spk} of (P, ∇_P) [cf. Definition A.2, (iv)] is defined as the zero locus of the resulting homomorphism $\mathcal{P}\colon \mathcal{T} \to \mathcal{C}^{\otimes 2}$. Thus, by definition, we have the following:

LEMMA A.11. — *The following hold:*

(i) *The* **p-curvature** *homomorphism determines an* **isomorphism** *of $\mathcal{O}$-modules*

$$\mathcal{T}(E_{\text{spk}}) \xrightarrow{\sim} \mathcal{C}^{\otimes 2}.$$

(ii) *For a closed point $x \in X$ of X and a positive integer d, the following conditions are equivalent:*

(1) *It holds that* $\mathrm{ord}_x E_{\text{spk}} \geq d$.

(2) *For each local section ∂ of $\mathcal{T}$ at x, the image of the endomorphism $x_d^*\mathcal{P}(\partial)$ of $x_d^*\mathcal{E}$ is* **zero**.

By [1], §3, together with Lemma A.10, (i), we obtain the following:

LEMMA A.12. — *The following hold:*

(i) *The divisor E_{gss} is* **of degree** $((p-1)\deg \omega^{\log} - \deg E_{\text{spk}})/2$.

(ii) *The divisor E_{gss} is* **reduced**.

(iii) *It holds that $E_{\text{gss}} \cap D = E_{\text{gss}} \cap E_{\text{spk}} = \emptyset$.*

(iv) *It holds that $E_{\text{spk}} \cap D = \emptyset$.*

(v) *There exists a divisor $\underline{E}_{\text{spk}}$ on X such that $E_{\text{spk}} = p\underline{E}_{\text{spk}}$.*

PROOF. — Assertion (i) follows from Lemma A.6; Lemma A.8, (ii); Lemma A.11, (i). Assertion (ii) is the content of the first assertion of [1], Lemma 3.4. The equality $E_{\text{gss}} \cap D = \emptyset$ (respectively, $E_{\text{gss}} \cap E_{\text{spk}} = \emptyset$) in assertion (iii) is the content of the second assertion of [1], Lemma 3.4 (respectively, [1], Proposition 3.6, (iv)). Assertion (iv) follows from [1], Proposition 3.6, (i). Assertion (v) is the content of [1], Proposition 3.6, (iii). □

In the remainder of the present §A, let us give a proof of Proposition A.3. Assertion (i) of Proposition A.3 follows from Lemma A.12, (ii), (iii). Assertion (ii) of Proposition A.3 follows from Lemma A.12, (iv), (v).

Next, we verify assertion (iii) of Proposition A.3.

LEMMA A.13. — *The following hold:*

(i) *It holds that $E_{\text{spk}} \leq E_{\text{sH}}$.*

(ii) *It holds that $E_{\text{gss}} \leq E_{\text{sH}}$.*

PROOF. — Assertion (i) follows from Lemma A.7, (ii), and Lemma A.11, (ii). Next, we verify assertion (ii). Let $x \in X$ be a closed point of X which is *contained* in the support of E_{gss}. Then it follows from Lemma A.12, (iii), that x is *not contained* in the support of the divisor $E_{\text{spk}}+D$. Thus, it follows from the definition of E_{spk}, together with Lemma A.8, (i), that, for each local generator ∂ of $\mathcal{T}$ at x, the *kernel* of the endomorphism $x_1^*\mathcal{P}(\partial)$ of $x_1^*\mathcal{E}$ *coincides* with the subspace $x_1^*\mathcal{C} \subseteq x_1^*\mathcal{E}$. In particular, it follows from Lemma A.10, (ii), that, for each local section ∂ of $\mathcal{T}$ at x, the image of the restriction of the endomorphism $x_1^*\mathcal{P}(\partial)$ of $x_1^*\mathcal{E}$ to the subspace $x_1^*\mathcal{H} \subseteq x_1^*\mathcal{E}$ is *zero*. Thus, it follows from Lemma A.7, (ii), that x is *contained* in the support of E_{sH}, which thus implies [cf. Lemma A.12, (ii)] that $E_{\text{gss}} \leq E_{\text{sH}}$, as desired. This completes the proof of assertion (ii), hence also of Lemma A.13. □

It follows immediately, in light of Lemma A.7, (i), and Lemma A.12, (i), (iii), from Lemma A.13 that, to complete the verification of assertion (iii) of Proposition A.3, it suffices to verify that $2E_{\text{gss}} \leq E_{\text{sH}}$. To this end, let us take a closed point $x \in X$ of X which is *contained* in the support of E_{gss} [which thus implies that x is *not contained* in the support of the divisor $E_{\text{spk}} + D$ — cf. Lemma A.12, (iii)]. Thus, since $\text{ord}_x E_{\text{gss}} = 1$ [cf. Lemma A.12, (ii)], to verify that $2E_{\text{gss}} \leq E_{\text{sH}}$, it suffices to verify that $\text{ord}_x E_{\text{sH}} \geq 2$.

To verify that $\text{ord}_x E_{\text{sH}} \geq 2$, let us fix respective local generators $e_{\mathcal{H}}$ $e_{\mathcal{C}}$, ∂ of $\mathcal{H}$, $\mathcal{C}$, $\mathcal{T}$ at x. Now let us observe that since the p-curvature homomorphism of $\nabla_{\mathcal{Q}}$ is *zero* [cf. Lemma A.8, (i)], there exists a local generator $e_{\mathcal{Q}}$ of $\mathcal{Q}$ at x such that $\nabla_{\mathcal{Q}}(e_{\mathcal{Q}}) = 0$. Write, moreover, $e_{\mathcal{E}} \stackrel{\text{def}}{=} \nabla_{\mathcal{E}}(\partial)(e_{\mathcal{H}})$ for the local section of $\mathcal{E}$ at x obtained by forming the image of $e_{\mathcal{H}}$ via $\nabla_{\mathcal{E}}(\partial)$. Thus, it follows from the definition of an indigenous bundle that the pair $(e_{\mathcal{H}}, e_{\mathcal{E}})$ forms a *basis* of $\mathcal{E}$ at x.

LEMMA A.14. — *Write*

- *$f_{\mathcal{Q}}$ for the local section of $\mathcal{O}$ at x such that the image of $e_{\mathcal{H}}$ via the natural surjection $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ coincides with $f_{\mathcal{Q}} \cdot e_{\mathcal{Q}}$,*
- *$f_{\mathcal{C}}$ for the local section of $\mathcal{O}$ at x such that the image $\mathcal{P}(\partial)(e_{\mathcal{H}})$ of $e_{\mathcal{H}}$ via $\mathcal{P}(\partial)$ coincides with $f_{\mathcal{C}} \cdot e_{\mathcal{C}}$ [cf. the discussion preceding Lemma* A.11], *and*
- *$f_{\mathcal{H}}$, $f_{\mathcal{E}}$ for the local sections of $\mathcal{O}$ at x such that*

$$\mathcal{P}(\partial)(e_{\mathcal{H}}) \;=\; f_{\mathcal{C}} \cdot e_{\mathcal{C}} \;=\; f_{\mathcal{H}} \cdot e_{\mathcal{H}} + f_{\mathcal{E}} \cdot e_{\mathcal{E}}.$$

[*Thus, it is immediate that the equalities*

$$\text{ord}_x E_{\text{gss}} \;=\; \text{ord}_x f_{\mathcal{Q}}, \quad \text{ord}_x E_{\text{sH}} \;=\; \text{ord}_x f_{\mathcal{E}}$$

hold.] Then the following hold:

(i) *It holds that* $\mathrm{ord}_x f_{\mathcal{Q}} = 1$.

(ii) *It holds that* $\mathrm{ord}_x f_{\mathcal{C}} \geq \mathrm{ord}_x f_{\mathcal{Q}}$.

(iii) *It holds that* $\mathrm{ord}_x f_{\mathcal{E}} \geq 1$.

(iv) *It holds that* $\mathrm{ord}_x f_{\mathcal{H}} \geq 1$.

(v) *It holds that* $\mathrm{ord}_x f_{\mathcal{E}} \geq 2$.

PROOF. — Assertion (i) follows from Lemma A.12, (ii). Assertion (ii) follows from the discussion preceding Lemma A.11. Assertion (iii) follows from Lemma A.13, (ii). Assertion (iv) follows from assertions (i), (ii), (iii), together with the equality $f_{\mathcal{C}} \cdot e_{\mathcal{C}} = f_{\mathcal{H}} \cdot e_{\mathcal{H}} + f_{\mathcal{E}} \cdot e_{\mathcal{E}}$ in the statement of Lemma A.14.

Finally, we verify assertion (v). Let us observe that since [we have assumed that] $\nabla_{\mathcal{Q}}(e_{\mathcal{Q}}) = 0$, the image of $e_{\mathcal{H}}$ in $\mathcal{Q}$ is given by $f_{\mathcal{Q}} \cdot e_{\mathcal{Q}}$, and the natural surjection $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ is *horizontal*, it holds that the image of $e_{\mathcal{E}} = \nabla_{\mathcal{E}}(\partial)(e_{\mathcal{H}})$ in $\mathcal{Q}$ *coincides* with $\partial f_{\mathcal{Q}} \cdot e_{\mathcal{Q}}$. Thus, by considering the image of "$f_{\mathcal{C}} \cdot e_{\mathcal{C}} = f_{\mathcal{H}} \cdot e_{\mathcal{H}} + f_{\mathcal{E}} \cdot e_{\mathcal{E}}$" in $\mathcal{Q}$, we obtain that

$$0 \;=\; f_{\mathcal{H}} \cdot f_{\mathcal{Q}} \cdot e_{\mathcal{Q}} + f_{\mathcal{E}} \cdot \partial f_{\mathcal{Q}} \cdot e_{\mathcal{Q}}.$$

Next, let us observe that since $\mathrm{ord}_x f_{\mathcal{Q}} = 1$ [cf. assertion (i)], it holds that $\mathrm{ord}_x \partial f_{\mathcal{Q}} = 0$. Thus, it follows from assertions (i), (iv), that

$$\mathrm{ord}_x f_{\mathcal{E}} \;=\; \mathrm{ord}_x(f_{\mathcal{H}} \cdot f_{\mathcal{Q}}) \;\geq\; 2,$$

as desired. This completes the proof of assertion (v), hence also of Lemma A.14. □

It follows from Lemma A.14, (v), that $\mathrm{ord}_x E_{\mathrm{sH}} \geq 2$, as desired. This completes the proof of assertion (iii) of Proposition A.3. Assertion (iv) of Proposition A.3 follows immediately from assertion (iii) of Proposition A.3 [cf. also [5], Chapter II, Proposition 2.6, (3)]. This completes the proof of Proposition A.3.

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