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Introduction to Mono-anabelian Geometry

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Abstract. — The present article is based on the four hours mini-courses "Introduction to Mono-anabelian Geometry" which the author gave at the conference "Fundamental Groups in Arithmetic Geometry" (Paris, 2016). The purpose of the present article is to introduce *mono-anabelian geometry* by focusing on mono-anabelian geometry for *mixed-characteristic local fields*, which provides elementary but nontrivial examples of typical discussions in the study of mono-anabelian geometry.

Contents

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INTRODUCTION

The present article is based on the four hours mini-courses "Introduction to Mono-anabelian Geometry" which the author gave at the conference "Fundamental Groups in Arithmetic Geometry" (Paris, 2016). In the present article, we discuss *mono-anabelian geometry.*

Anabelian geometry is, in a word, an area of arithmetic geometry in which one studies the geometry of geometric objects of interest from the point of view of arithmetic fundamental groups. Put another way, roughly speaking, anabelian geometry discusses the issue of how much information concerning the geometry of geometric objects of interest is contained in the knowledge of the arithmetic fundamental groups.

The classical point of view of anabelian geometry [i.e., more precisely, of *Grothendieck's anabelian conjecture*] centers around a *comparison* between *two* geometric objects of interest via the arithmetic fundamental groups. In fact, in a discussion of anabelian geometry, typically, one fixes *two* geometric objects and discusses the relationship between a certain set of morphisms [e.g., the set of isomorphisms] between the fixed two objects and a certain set of homomorphisms [e.g., the set of isomorphisms] between the étale fundamental groups. In particular, roughly speaking, the classical point of view of anabelian geometry may be summarized as the study of some properties such as *faithfulness/fullness* of the [restriction, to a certain suitable category of geometric objects, of the] functor of taking arithmetic fundamental groups. Moreover, the term "*group-theoretic*" [that often appears in discussions of anabelian geometry] is, in the classical point of view, defined simply to mean "*preserved by an arbitrary isomorphism between the arithmetic fundamental groups under consideration*". In [9], this classical point of view is referred to as "*bi-anabelian geometry*".

$$\underline{\text{bi-anabelian geometry}}$$
$$\pi_1^{\text{ét}}(X_\circ) \xrightarrow{\sim} \pi_1^{\text{ét}}(X_\bullet) \overset{?}{\Longrightarrow} \text{[objects related to] } X_\circ \xrightarrow{\sim} \text{[objects related to] } X_\bullet$$

By contrast, *mono-anabelian geometry* centers around the task of establishing a "*group-theoretic software*" [i.e., "*group-theoretic algorithm*"] whose input data consists of a *single abstract topological group* isomorphic to the arithmetic fundamental group of a *single* geometric object of interest. In particular, a *mono-anabelian reconstruction algorithm* [i.e., a "group-theoretic algorithm" discussed in mono-anabelian geometry] has the virtue of being *free* of any mention of some "fixed reference model" copy of geometric objects [as the above "$X_\circ$" for "$\pi_1^{\text{ét}}(X_\circ)$" in the case of bi-anabelian geometry]. In the point of view of mono-anabelian geometry, the term "*group-theoretic algorithm*" is used to mean that "*the algorithm in a discussion is phrased in language that only depends on the topological group structure of the arithmetic fundamental group under consideration*" [cf., e.g., [9], Introduction; [9], Remark 1.9.8; Remarks following [9], Corollary 3.7, for more details concerning bi-anabelian/mono-anabelian geometry].

$$\begin{array}{c}\underline{\text{mono-anabelian geometry}}\\ \text{a topological group isomorphic to } \pi_1^{\text{ét}}(X)\\ ?\Big\Downarrow\\ \text{object(s) isomorphic to [objects related to] } X\end{array}$$

The purpose of the present article is to introduce *mono-anabelian geometry* by focusing on mono-anabelian geometry for *mixed-characteristic local fields*, i.e., *MLF* [cf. Definition 1.1], which provides elementary but nontrivial examples of typical discussions in the study of mono-anabelian geometry.

In §1, we introduce some notational conventions related to MLF and recall some basic facts concerning objects that arise from MLF. These basic facts will be applied in other sections of the present article. In §2, we recall some results in *bi-anabelian geometry* for MLF. Some of fundamental results in bi-anabelian geometry for MLF are as follows [cf. Theorem 2.2]:

- The isomorphism class of an MLF is *not determined* by the isomorphism class of the absolute Galois group of the MLF.

- There exists an outer automorphism of the absolute Galois group of an MLF which does *not arise* from any automorphism of the MLF.

These results lead us to an interest in a study of *conditions* for an outer isomorphism between the absolute Galois groups of MLF to arise from an isomorphism between the original MLF. In §2, we also recall such *conditions* [cf. Theorem 2.3; Remark 2.3.2].

In §3 and §4, we establish some *mono-anabelian reconstruction algorithms* for MLF, i.e., some "group-theoretic algorithms" whose input data consist of a group *of MLF-type* [i.e., an abstract group *isomorphic to the absolute Galois group of an MLF* — cf. Definition 3.1]. For instance, by applying the mono-anabelian reconstruction algorithms discussed in §4, one may construct, from a group G of MLF-type, G-monoids

$$\overline{\mathcal{O}}^{\times}(G) \ \subseteq\ \overline{\mathcal{O}}^{\triangleright}(G) \ \subseteq\ \overline{k}^{\times}(G)$$

which "correspond" to the $\mathrm{Gal}(\overline{k}/k)$-monoids

$$\mathcal{O}^{\times}_{\overline{k}} \ \subseteq\ \mathcal{O}^{\triangleright}_{\overline{k}} \ \subseteq\ \overline{k}^{\times}$$

— where k is an MLF, and $\overline{k}$ is an algebraic closure of k — [i.e.,

- the multiplicative $\mathrm{Gal}(\overline{k}/k)$-module $\mathcal{O}^{\times}_{\overline{k}}$ of units of the ring of integers of $\overline{k}$,
- the multiplicative $\mathrm{Gal}(\overline{k}/k)$-monoid $\mathcal{O}^{\triangleright}_{\overline{k}}$ of nonzero integers of $\overline{k}$, and
- the multiplicative $\mathrm{Gal}(\overline{k}/k)$-module $\overline{k}^{\times}$ of nonzero elements of k]

respectively [cf. Summary 3.15, Summary 4.3].

One important aspect of mono-anabelian geometry is the technique of *mono-anabelian transport*. In order to explain mono-anabelian transport, in §5, we introduce the notion of an *MLF-pair* [cf. Definition 5.3]. Some types of MLF-pairs are discussed in the present article. For instance, an *MLF*$^{\times}$*-pair* is defined to be a collection of data $G \curvearrowright M$ consisting of a monoid M, a group G, and an action of G on M such that there exists an isomorphism [in the evident sense] of $G \curvearrowright M$ with the collection of data

$$\mathrm{Gal}(\overline{k}/k) \curvearrowright \mathcal{O}^{\times}_{\overline{k}}$$

for some MLF k and some algebraic closure $\overline{k}$ of k. If $G \curvearrowright M$ is an MLF-pair, then we shall refer to the group G, the monoid M as the *étale-like portion*, the *Frobenius-like portion* of $G \curvearrowright M$, respectively [cf. Definition 5.4].

In §6, we discuss a phenomenon of *cyclotomic synchronization* for MLF-pairs. Let us recall that a *cyclotome* refers to an "object isomorphic to the object $\widehat{\mathbb{Z}}(1)$". Various [a priori independent] cyclotomes often appear in studies of arithmetic geometry. For instance, if Ω is an algebraically closed field of characteristic zero, then each of

- the cyclotome

$$\Lambda(\Omega) \stackrel{\text{def}}{=} \varprojlim_{n} \operatorname{Ker}(\Omega^{\times} \stackrel{n}{\to} \Omega^{\times})$$

— where the projective limit is taken over the positive integers n — associated to Ω and

- the dual

$$\Lambda(C) \stackrel{\text{def}}{=} \operatorname{Hom}_{\widehat{\mathbb{Z}}}\bigl(H^2_{\text{ét}}(C, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}}\bigr)$$

over $\widehat{\mathbb{Z}}$ of the second étale cohomology $H^2_{\text{ét}}(C, \widehat{\mathbb{Z}})$ of a projective smooth curve C over Ω

gives an example of a cyclotome.

If one works with certain scheme/ring structures of objects related to cyclotomes under consideration, then one may obtain a phenomenon of *cyclotomic synchronization*, i.e., synchronization of cyclotomes. For instance, in the case of the above examples $\Lambda(\Omega)$ and $\Lambda(C)$, the homomorphism $\operatorname{Pic}(C) \to H^2(C, \Lambda(\Omega))$ obtained by considering the *first Chern classes* yields an isomorphism

$$\bigl(\operatorname{Pic}(C)/\operatorname{Pic}^0(C)\bigr) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \stackrel{\sim}{\longrightarrow} \operatorname{Hom}_{\widehat{\mathbb{Z}}}\bigl(\Lambda(C), \Lambda(\Omega)\bigr);$$

thus, an invertible sheaf on C *of degree one* determines a *cyclotomic synchronization* [i.e., an isomorphism between cyclotomes] $\Lambda(C) \stackrel{\sim}{\to} \Lambda(\Omega)$ by means of which one usually [i.e., in the usual point of view of arithmetic geometry] *identifies* $\Lambda(C)$ with $\Lambda(\Omega)$. On the other hand, if one works in a situation in which objects related to cyclotomes under consideration *lose* a certain portion of the rigidity that arises from scheme/ring structures, then the task of establishing a phenomenon of cyclotomic synchronization is *nontrivial*.

In our study of MLF-pairs, one may construct, from a *single* MLF-pair, [a priori independent] *two cyclotomes*, i.e., a cyclotome constructed from the *étale-like* portion and a cyclotome constructed from the *Frobenius-like* portion [cf. Definition 5.9; Proposition 5.10]. In §6, we establish a cyclotomic synchronization which relates the Frobenius-like cyclotome to the étale-like cyclotome [cf. Definition 6.6, Proposition 6.7].

In §7, we discuss *Kummer poly-isomorphisms* and *mono-anabelian transport*. A *Kummer poly-isomorphism* often refers to a collection of isomorphisms between [moonids constructed, via some functorial algorithms, from] *Frobenius-like portions* and *mono-anabelian étale-like monoids* [i.e., monoids constructed, via some mono-anabelian reconstruction algorithms, from the *étale-like portions*]. In §7, we establish Kummer poly-isomorphisms for MLF-pairs [cf. Definition 7.4]. Finally, we discuss the technique of mono-anabelian transport [cf. Remark 7.6.1].

Some details of discussions given in the portion from §3 to §7 of the present article may be found in, for instance, [7], §1, §2; [9], §3, §5; [10], §2; [2], §1.

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0. NOTATIONAL CONVENTIONS

SETS. — Let G be a group and S a set equipped with an action of G. Then we shall write $S^G \subseteq S$ for the subset of G-invariants of S.

Let S be a finite set. Then we shall write $\sharp S$ for the *cardinality* of S.

MONOIDS. — In the present article, a "monoid" always means a "commutative monoid". Let M be a monoid. [The monoid operation of M will be written *multiplicatively*]. We shall write $M^\times \subseteq M$ for the abelian group of invertible elements of M. We shall write M^{gp} for the *groupification* of M [i.e., the abelian group obtained by forming the monoid of equivalence classes with respect to the relation $\sim$ on $M \times M$ defined by, for (a_1, b_1), $(a_2, b_2) \in M \times M$, $(a_1, b_1) \sim (a_2, b_2)$ if there exists an element $c \in M$ of M such that $ca_1b_2 = ca_2b_1$]. We shall write M^{pf} for the *perfection* of M [i.e., the monoid obtained by forming the injective limit of the injective system of monoids

$$\cdots \longrightarrow M \longrightarrow M \longrightarrow \cdots$$

given by assigning to each positive integer n a copy of M, which we denote by I_n, and to each two positive integers n, m such that n divides m the homomorphism $I_n = M \to I_m = M$ given by multiplication by m/n]. We shall write $M^\circledast \stackrel{\mathrm{def}}{=} M \cup \{*_M\}$; we regard $M^\circledast$ as a *monoid* by $m \cdot *_M \stackrel{\mathrm{def}}{=} *_M$, $*_M \cdot m \stackrel{\mathrm{def}}{=} *_M$, $*_M \cdot *_M \stackrel{\mathrm{def}}{=} *_M$ for every $m \in M$.

MODULES. — Let M be a module. If n is a positive integer, then we shall write $M[n] \subseteq M$ for the submodule obtained by forming the kernel of the endomorphism of M given by multiplication by n. We shall write $M_{\mathrm{tor}} \stackrel{\mathrm{def}}{=} \bigcup_{n \geq 1} M[n] \subseteq M$ for the submodule of torsion elements of M and

$$M^\wedge \stackrel{\mathrm{def}}{=} \varprojlim_n M/(n \cdot M)$$

— where the projective limit is taken over the positive integers n. [So if M is finitely generated, then $M^\wedge$ is nothing but the profinite completion of M.]

TOPOLOGICAL GROUPS. — Let G be a topological group. Then we shall write G^{ab} for the *abelianization* of G [i.e., the quotient of G by the closure of the commutator subgroup of G], $G^{\text{ab-tor}} \stackrel{\mathrm{def}}{=} (G^{\mathrm{ab}})_{\mathrm{tor}} \subseteq G^{\mathrm{ab}}$, and $G^{\mathrm{ab/tor}}$ for the quotient of G^{ab} by the closure of $G^{\text{ab-tor}} \subseteq G^{\mathrm{ab}}$.

Let G be a profinite group. Then we shall say that G is *slim* if, for every open subgroup $H \subseteq G$ of G, the centralizer of H in G is trivial.

Let n be a nonnegative integer, G a profinite group, and M a topological G-module. Then we shall write $H^n(G, M)$ for the *n-th continuous group cohomology* of G with coefficients in M and

$$ {}_{\infty}H^n(G, M) \stackrel{\text{def}}{=} \varinjlim_{H \subseteq G} H^n(H, M) $$

— where the injective limit is taken over the open subgroups $H \subseteq G$ of G.

RINGS. — In the present article, a "ring" always means a "commutative ring". Let R be a ring. Then we shall write R_+ for the underlying additive module of R and $R^\times \subseteq R$ for the multiplicative module of units of R. If R is an integral domain, then we shall write $R^\rhd \subseteq R$ for the multiplicative monoid of nonzero elements of R. [So it holds that $R^\times = R^\rhd$ if and only if R is a field.]

FIELDS. — Let K be a field. Then we shall write $\boldsymbol{\mu}(K) \stackrel{\text{def}}{=} (K^\times)_{\text{tor}}$ for the group of roots of unity in K and $K_\times$ for the underlying multiplicative monoid of K. [So we have a natural isomorphism $(K^\times)^{\circledast} \xrightarrow{\sim} K_\times$ of monoids]. If, moreover, K is algebraically closed and of characteristic zero, then we shall write

$$ \Lambda(K) \stackrel{\text{def}}{=} \varprojlim_{n} \boldsymbol{\mu}(K)[n] = \varprojlim_{n} K^\times[n] $$

— where the projective limits are taken over the positive integers n — and refer to $\Lambda(K)$ as the *cyclotome* associated to K. Thus, the cyclotome has a natural structure of profinite, hence also topological, module and is, as an abstract topological module, isomorphic to $\widehat{\mathbb{Z}}$.

CATEGORIES. — Let A be an object of a category. Then we shall refer to an object of the category isomorphic to A as an *isomorph* of A.

Let A, B, and C be objects of a category. Then we shall refer to a nonempty set of isomorphisms from A to B in the category as a *poly-isomorphism* [from A to B]. [So one may regard a single isomorphism as a poly-isomorphism, i.e., of cardinality one.] Let $f \colon A \xrightarrow{\sim} B$ be a poly-isomorphism [i.e., from A to B] and $g \colon B \xrightarrow{\sim} C$ a poly-isomorphism [i.e., from B to C]. We shall write $g \circ f \colon A \xrightarrow{\sim} C$ for the poly-isomorphism [i.e., from A to C] obtained by forming the set $\{ \mathfrak{g} \circ \mathfrak{f} \mid \mathfrak{f} \in f,\ \mathfrak{g} \in g \}$ and refer to $g \circ f$ as the *composite* of f and g. We shall write $f^{-1} \colon B \xrightarrow{\sim} A$ for the poly-isomorphism [i.e., from B to A] obtained by forming the set $\{ \mathfrak{f}^{-1} \mid \mathfrak{f} \in f \}$ and refer to f^{-1} as the *inverse* of f.

1. GENERALITIES ON MLF

In the present §1, let us introduce some notational conventions related to *mixed-characteristic local fields*, i.e., *MLF* [cf. Definition 1.1 below], and recall some basic facts concerning objects that arise from MLF. These basic facts will be applied in other sections of the present article.

DEFINITION 1.1. — We shall refer to a finite extension of $\mathbb{Q}_p$ for some prime number p as an *MLF*. Here, "MLF" is to be understood as an abbreviation for "mixed-characteristic local field".

In the remainder of the present §1, let

$$k$$

be an MLF. Then we shall write

- $\mathcal{O}_k \subseteq k$ for the ring of integers of k,
- $\mathfrak{m}_k \subseteq \mathcal{O}_k$ for the maximal ideal of $\mathcal{O}_k$,
- $\underline{k} \stackrel{\text{def}}{=} \mathcal{O}_k/\mathfrak{m}_k$ for the residue field of $\mathcal{O}_k$,
- $U_k^{(n)} \stackrel{\text{def}}{=} 1 + \mathfrak{m}_k^n \subseteq \mathcal{O}_k^\times$ [where n is a positive integer] for the n-th higher unit group of k,
- μ_k for the [uniquely determined] Haar measure on [the locally compact topological module] k_+ such that $\mu_k((\mathcal{O}_k)_+) = 1$, and
- $p_k \stackrel{\text{def}}{=} \text{char}(\underline{k})$ for the residue characteristic of k.

Thus, one verifies easily that $[k : \mathbb{Q}_{p_k}]$ and $[\underline{k} : \mathbb{F}_{p_k}]$ are finite. We shall write

- $d_k \stackrel{\text{def}}{=} [k : \mathbb{Q}_{p_k}]$ and
- $f_k \stackrel{\text{def}}{=} [\underline{k} : \mathbb{F}_{p_k}]$.

We shall write, moreover,

- $e_k \stackrel{\text{def}}{=} \sharp(k^\times/(\mathcal{O}_k^\times \cdot \mathbb{Q}_{p_k}^\times))$ for the absolute ramification index of k,
- $\log_k \colon \mathcal{O}_k^\times \to k_+$ for the p_k-adic logarithm, and
- $\mathcal{I}_k \stackrel{\text{def}}{=} (2p_k)^{-1} \cdot \log_k(\mathcal{O}_k^\times) \subseteq k_+$ for the log-shell of k.

In the following lemma, let us recall some basic facts concerning the above objects.

LEMMA 1.2. — *The following hold:*

(i) *The topological module $k^\times$ is* **isomorphic** *to the topological module*

$$\big(\mathbb{Z}/(p_k^{f_k} - 1)\mathbb{Z}\big) \oplus (\mathbb{Z}/p_k^a\mathbb{Z}) \oplus \mathbb{Z}_{p_k}^{\oplus d_k} \oplus \mathbb{Z}$$

for some nonnegative integer a. Moreover, the topological submodule $\mathcal{O}_k^\times \subseteq k^\times$ of $k^\times$ **corresponds**, *relative to each of such an isomorphism, to the kernel of the fourth projection*

$$\big(\mathbb{Z}/(p_k^{f_k} - 1)\mathbb{Z}\big) \oplus (\mathbb{Z}/p_k^a\mathbb{Z}) \oplus \mathbb{Z}_{p_k}^{\oplus d_k} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}.$$

(ii) *The topological submodule $U_k^{(1)} \subseteq \mathcal{O}_k^\times$ of $\mathcal{O}_k^\times$ is the* **maximal pro-p_k** *submodule of $\mathcal{O}_k^\times$.*

(iii) *It holds that $d_k = f_k \cdot e_k$.*

(iv) *The p_k-adic logarithm* $\log_k \colon \mathcal{O}_k^{\times} \to k_+$ *determines an* **isomorphism** *of topological modules*

$$(\mathcal{O}_k^{\times})^{\mathrm{pf}} \xrightarrow{\sim} k_+.$$

(v) *It holds that* $\mathrm{Ker}(\log_k) = \boldsymbol{\mu}(k)$.

(vi) *It holds that* $(\mathcal{O}_k)_+ \subseteq \mathcal{I}_k$.

PROOF. — Assertions (i), (ii) follow immediately from [4], Chapter II, Proposition 5.3; [4], Chapter II, Proposition 5.7, (i). Assertion (iii) follows from [4], Chapter II, Proposition 6.8. Assertions (iv), (v) follow immediately from [4], Chapter II, Proposition 5.5, together with assertion (i). Assertion (vi) follows immediately from [4], Chapter II, Proposition 5.5, together with the [easily verified] fact that $e_k > e_k/(p_k - 1)$ (respectively, $2e_k > e_k/(p_k - 1)$) if $p_k \neq 2$ (respectively, $p_k = 2$). □

Next, let us recall some basic facts concerning the measure μ_k.

LEMMA 1.3. — *The following hold:*

(i) *Let* S, $T \subseteq k_+$ *be* **compact open** *subsets of* k_+. *Then the measure* μ_k *satisfies the following conditions:*

(1) *If* $S \cap T = \emptyset$, *then it holds that* $\mu_k(S \cup T) = \mu_k(S) + \mu_k(T)$.

(2) *For each* $a \in k_+$, *it holds that* $\mu_k(S + a) = \mu_k(S)$.

(3) *If* S *is* **contained** *in* $\mathcal{O}_k^{\times}$ $(\subseteq k_+)$, *and the natural surjection* $S \twoheadrightarrow \log_k(S)$ *determined by* $\log_k$ *is* **bijective**, *then it holds that* $\mu_k(\log_k(S)) = \mu_k(S)$.

(ii) *It holds that*

$$\mu_k(\mathcal{O}_k^{\times}) \;=\; 1 - p_k^{-f_k} \;=\; p_k^{-f_k} \cdot (p_k^{f_k} - 1).$$

(iii) *It holds that*

$$\mu_k(\mathcal{I}_k) \;=\; p_k^{\epsilon_k \cdot d_k - f_k} / \sharp \boldsymbol{\mu}(k)^{(p_k)}$$

— *where we write*

$$\epsilon_k \stackrel{\mathrm{def}}{=} \begin{cases} 1 & \text{if } p_k \neq 2 \\ 2 & \text{if } p_k = 2 \end{cases}$$

and $\boldsymbol{\mu}(k)^{(p_k)}$ *for the* p_k*-Sylow subgroup of* [*the finite — cf. Lemma* 1.2, (i) — *abelian group*] $\boldsymbol{\mu}(k)$.

PROOF. — First, we verify assertion (i). The assertion that μ_k satisfies conditions (1), (2) follows from the definition of a Haar measure. Next, we verify the assertion that μ_k satisfies condition (3). Let us recall that the system $\{U_k^{(n)}\}_{n \geq 1}$ forms a basis of neighborhoods of the identity element $1 \in \mathcal{O}_k^{\times}$ of $\mathcal{O}_k^{\times}$ [cf. the discussion preceding [4], Chapter II, Proposition 3.10]. Thus, it follows immediately — in light of the [easily verified] fact that the endomorphism of k_+ given by multiplication by an element of $\mathcal{O}_k^{\times}$ is a *topological automorphism* which restricts to an *automorphism* of $(\mathcal{O}_k)_+ \subseteq k_+$ — from conditions (1), (2) that, to verify the assertion that μ_k satisfies condition (3), it suffices to verify that there exists a positive integer n_0 such that if $n > n_0$, then

$$\mu_k\big(\log_k(U_k^{(n)})\big) \;=\; \mu_k(U_k^{(n)}).$$

On the other hand, it follows from [4], Chapter II, Proposition 5.5, together with condition (2), that if $n > e_k/(p_k - 1)$, then

$$\mu_k\bigl(\log_k(U_k^{(n)})\bigr) \;=\; \mu_k(\mathfrak{m}_k^n) \;=\; \mu_k(1+\mathfrak{m}_k^n) \;=\; \mu_k(U_k^{(n)}).$$

This completes the proof of the assertion that μ_k satisfies condition (3), hence also of assertion (i).

Next, we verify assertion (ii). Since [one verifies easily that] $\mathcal{O}_k^\times = (\mathcal{O}_k)_+ \setminus \mathfrak{m}_k$, it follows from condition (1) of assertion (i) that

$$\mu_k(\mathcal{O}_k^\times) \;=\; \mu_k((\mathcal{O}_k)_+) - \mu_k(\mathfrak{m}_k) \;=\; \mu_k((\mathcal{O}_k)_+)\cdot(1 - [(\mathcal{O}_k)_+ : \mathfrak{m}_k]^{-1}) \;=\; 1 - p_k^{-f_k},$$

as desired. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Let us first recall from Lemma 1.2, (v), that $\mathrm{Ker}(\log_k) = \boldsymbol{\mu}(k)$. Thus, it follows from conditions (1), (3) of assertion (i) that

$$\mu_k(\mathcal{I}_k) \;=\; [\mathcal{I}_k : \log_k(\mathcal{O}_k^\times)]\cdot\mu_k\bigl(\log(\mathcal{O}_k^\times)\bigr) \;=\; [\mathcal{I}_k : \log_k(\mathcal{O}_k^\times)]\cdot\sharp\bigl(\boldsymbol{\mu}(k)\bigr)^{-1}\cdot\mu_k(\mathcal{O}_k^\times).$$

In particular, since

$$\sharp\boldsymbol{\mu}(k) \;=\; \sharp\boldsymbol{\mu}(k)^{(p_k)}\cdot(p_k^{f_k} - 1)$$

[cf. Lemma 1.2, (i)], it follows from Lemma 1.2, (iv); assertion (ii) that

$$\mu_k(\mathcal{I}_k) \;=\; p_k^{\epsilon_k\cdot d_k}\cdot\sharp\bigl(\boldsymbol{\mu}(k)^{(p_k)}\bigr)^{-1}\cdot(p_k^{f_k}-1)^{-1}\cdot p_k^{-f_k}\cdot(p_k^{f_k}-1) \;=\; p_k^{\epsilon_k\cdot d_k - f_k}/\sharp\boldsymbol{\mu}(k)^{(p_k)},$$

as desired. This completes the proof of assertion (iii). □

Next, let

$$\overline{k}$$

be an algebraic closure of k. Then we shall write

- $\mathcal{O}_{\overline{k}} \subseteq \overline{k}$ for the ring of integers of $\overline{k}$,
- $\underline{\overline{k}}$ for the residue field of $\mathcal{O}_{\overline{k}}$,
- $\log_{\overline{k}}\colon \mathcal{O}_{\overline{k}}^\times \to \overline{k}_+$ for the p_k-adic logarithm,
- $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$,
- $I_k \subseteq G_k$ for the inertia subgroup of G_k, and
- $P_k \subseteq I_k$ for the wild inertia subgroup of G_k.

Thus, $\underline{\overline{k}}$ is an algebraic closure of $\underline{k}$ such that the absolute Galois group $\mathrm{Gal}(\underline{\overline{k}}/\underline{k})$ of $\underline{k}$ with respect to $\underline{\overline{k}}$ is naturally identified with the quotient G_k/I_k. We shall write

- $\mathrm{Frob}_k \in \mathrm{Gal}(\underline{\overline{k}}/\underline{k}) \xleftarrow{\sim} G_k/I_k$ for the [$\sharp\underline{k}$-th power] Frobenius element.

We shall write, moreover,

- $\mathrm{Br}(k) \overset{\mathrm{def}}{=} H^2(G_k, \overline{k}^\times)$ [where we regard $\overline{k}^\times$ as a discrete G_k-module] for the Brauer group of k.

The following lemma is one of fundamental results concerning the structure of the topological group G_k.

LEMMA 1.4. — *The following hold:*

(i) *The topological group G_k is* **topologically finitely generated***.*

(ii) *For a subgroup $H \subseteq G_k$ of G_k, it holds that H is* **open** *in G_k if and only if H is* **of finite index** *in G_k.*

PROOF. — Assertion (i) follows from [5], Theorem 7.4.1. Assertion (ii) follows from [6], Theorem 1.1, together with assertion (i). □

The following lemma is one of fundamental results concerning the structure of the tame quotient G_k/P_k of G_k.

LEMMA 1.5. — *The following hold:*

(i) *The quotient G_k/I_k is* **topologically generated** *by $\mathrm{Frob}_k \in G_k/I_k$ and, as an abstract topological group,* **isomorphic** *to $\widehat{\mathbb{Z}}$. In particular, the quotient G_k/I_k is* **abelian***.*

(ii) *There exists an* **isomorphism** *of topological groups*

$$I_k/P_k \xrightarrow{\sim} \Lambda(\overline{k})^{(p'_k)}$$

— where we write $\Lambda(\overline{k})^{(p'_k)}$ for the quotient of $\Lambda(\overline{k})$ by the pro-p_k-Sylow subgroup of $\Lambda(\overline{k})$. Moreover, each of such an isomorphism is **G_k-equivariant** *[i.e., with respect to the action of G_k on I_k/P_k by conjugation and the natural action of G_k on $\Lambda(\overline{k})^{(p'_k)}$].*

(iii) *The action of G_k on I_k/P_k by conjugation determines an* **injection**

$$G_k/I_k \hookrightarrow \prod_{l\colon \textit{prime}, l\neq p_k} \mathbb{Z}_l^\times = \mathrm{Aut}(I_k/P_k)$$

[cf. (ii)] which maps $\mathrm{Frob}_k \in G_k/I_k$ to $p_k^{f_k} \in \prod \mathbb{Z}_l^\times$.

PROOF. — Assertion (i) is discussed in the discussion following [5], Proposition 7.5.1. Assertion (ii) follows immediately from [5], Proposition 7.5.2. Assertion (iii) follows immediately — in light of assertion (ii) — from [5], Lemma 7.5.4, (ii), together with the definition of $\mathrm{Frob}_k \in G_k/I_k$. □

Next, let us recall an *explicit description of the Brauer group* of an MLF as follows.

LEMMA 1.6. — *Write $k_{\mathrm{ur}} \subseteq \overline{k}$ for the maximal unramified extension of k in $\overline{k}$ [i.e., $k_{\mathrm{ur}} = \overline{k}^{I_k}$], $\mathcal{O}_{k_{\mathrm{ur}}} \subseteq k_{\mathrm{ur}}$ for the ring of integers of k_{ur}, and $V \stackrel{\mathrm{def}}{=} k_{\mathrm{ur}}^\times/\mathcal{O}_{k_{\mathrm{ur}}}^\times$. Then the following hold:*

(i) *The monoids $\mathcal{O}_{k_{\mathrm{ur}}}^\triangleright/\mathcal{O}_{k_{\mathrm{ur}}}^\times \subseteq V$ are, as abstract monoids,* **isomorphic** *to $\mathbb{N} \subseteq \mathbb{Z}$. Moreover, the action of G_k on V determined by the action of G_k on $k_{\mathrm{ur}}^\times$ is* **trivial***.*

(ii) *For each positive integer n, the natural homomorphism*

$$H^2\big(G_k, \boldsymbol{\mu}(\overline{k})[n]\big) \longrightarrow \mathrm{Br}(k)$$

determines an **isomorphism**

$$H^2\big(G_k, \boldsymbol{\mu}(\overline{k})[n]\big) \xrightarrow{\sim} \mathrm{Br}(k)[n].$$

(iii) *The natural homomorphism*

$$H^2(G_k/I_k, k_{\mathrm{ur}}^\times) \longrightarrow \mathrm{Br}(k)$$

is an **isomorphism**.

(iv) *The natural homomorphism*

$$H^2(G_k/I_k, k_{\mathrm{ur}}^\times) \longrightarrow H^2(G_k/I_k, V)$$

is an **isomorphism**.

(v) *The homomorphism*

$$H^1(G_k/I_k, V^{\mathrm{pf}}/V) \longrightarrow H^2(G_k/I_k, V)$$

determined by the exact sequence of G_k/I_k-modules

$$0 \longrightarrow V \longrightarrow V^{\mathrm{pf}} \longrightarrow V^{\mathrm{pf}}/V \longrightarrow 0$$

is an **isomorphism**.

(vi) *The homomorphism*

$$H^1(G_k/I_k, V^{\mathrm{pf}}/V) \longrightarrow V^{\mathrm{pf}}/V$$

obtained by mapping

$$\chi \in \mathrm{Hom}(G_k/I_k, V^{\mathrm{pf}}/V) = H^1(G_k/I_k, V^{\mathrm{pf}}/V)$$

[*cf.* (i)] *to*

$$\chi(\mathrm{Frob}_k) \in V^{\mathrm{pf}}/V$$

is an **isomorphism**.

(vii) *The various isomorphisms of* (iii), (iv), (v), (vi) *determine an* **isomorphism**

$$\mathrm{Br}(k) \xrightarrow{\sim} V^{\mathrm{pf}}/V$$

[*which thus determines an* **isomorphism**

$$H^2(G_k, \Lambda(\overline{k})) \xrightarrow{\sim} V^\wedge$$

— *cf.* (i), (ii)].

Proof. — Assertion (i) follows immediately from the [easily verified] fact that the natural inclusion $k \hookrightarrow k_{\mathrm{ur}}$ determines *isomorphisms* $\mathcal{O}_k^\triangleright/\mathcal{O}_k^\times \xrightarrow{\sim} \mathcal{O}_{k_{\mathrm{ur}}}^\triangleright/\mathcal{O}_{k_{\mathrm{ur}}}^\times$, $k^\times/\mathcal{O}_k^\times \xrightarrow{\sim} V$. Assertion (ii) follows immediately from "*Hilbert Theorem 90*" [cf. [5], Theorem 6.2.1]. Assertion (iii) follows from [11], §1.1, Theorem 1. Assertion (iv) follows from [11], §1.1, Theorem 2. Assertion (v) follows from the discussion following [11], §1.1, Theorem 2. Assertion (vi) follows from the discussion preceding [11], §1.1, Corollary. Assertion (vii) follows from [11], §1.1, Corollary. □

Next, let us recall *local class field theory* as follows.

LEMMA 1.7. — *There exists an* **injective** *homomorphism*

$$\mathrm{rec}_k\colon\ k^\times \hookrightarrow G_k^{\mathrm{ab}}$$

which satisfies the following conditions:

(1) *The homomorphism* rec_k *determines a* **commutative diagram**

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}_k^\times & \longrightarrow & k^\times & \longrightarrow & k^\times/\mathcal{O}_k^\times & \longrightarrow & 1 \\
 & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\
1 & \longrightarrow & \mathrm{Im}(I_k \hookrightarrow G_k \twoheadrightarrow G_k^{\mathrm{ab}}) & \longrightarrow & G_k^{\mathrm{ab}} \times_{G_k/I_k} \mathrm{Frob}_k^{\mathbb{Z}} & \longrightarrow & \mathrm{Frob}_k^{\mathbb{Z}} & \longrightarrow & 1 \\
 & & \| & & \cap\downarrow & & \cap\downarrow & & \\
1 & \longrightarrow & \mathrm{Im}(I_k \hookrightarrow G_k \twoheadrightarrow G_k^{\mathrm{ab}}) & \longrightarrow & G_k^{\mathrm{ab}} & \longrightarrow & G_k/I_k & \longrightarrow & 1
\end{array}$$

[*cf. Lemma* 1.5, (i)] — *where the horizontal sequences are* **exact**, *the upper vertical arrows are* **isomorphisms**, *the lower vertical arrows are* **injective**, *and the right-hand upper vertical arrow* $k^\times/\mathcal{O}_k^\times \xrightarrow{\sim} \mathrm{Frob}_k^{\mathbb{Z}}$ *determines an* **isomorphism**

$$\mathcal{O}_k^{\triangleright}/\mathcal{O}_k^\times \xrightarrow{\sim} \mathrm{Frob}_k^{\mathbb{N}}.$$

In particular, the homomorphism rec_k *determines an* **isomorphism** *of topological modules*

$$(k^\times)^\wedge \xrightarrow{\sim} G_k^{\mathrm{ab}}.$$

(2) *Let* $K \subseteq \overline{k}$ *be a finite extension of* k. [*So* K *is an* **MLF**, *and* $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/K) \subseteq G_k$ *is an open subgroup of* G_k.] *Then the Norm map* $\mathrm{Nm}_{K/k}\colon K^\times \to k^\times$ *and the natural homomorphism* $G_K^{\mathrm{ab}} \to G_k^{\mathrm{ab}}$ *fit into the following* **commutative diagram***:*

$$\begin{array}{ccc}
K^\times & \xrightarrow{\mathrm{Nm}_{K/k}} & k^\times \\
\mathrm{rec}_K\downarrow & & \mathrm{rec}_k\downarrow \\
G_K^{\mathrm{ab}} & \longrightarrow & G_k^{\mathrm{ab}}.
\end{array}$$

(3) *Let* $K \subseteq \overline{k}$ *be a finite extension of* k. [*So* K *is an* **MLF**, *and* $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/K) \subseteq G_k$ *is an open subgroup of* G_k.] *Then the natural inclusion* $k^\times \hookrightarrow K^\times$ *and the transfer map* $\mathrm{Tf}_{G_K \subseteq G_k}\colon G_k^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ *fit into the following* **commutative diagram***:*

$$\begin{array}{ccc}
k^\times & \xrightarrow{\subseteq} & K^\times \\
\mathrm{rec}_k\downarrow & & \mathrm{rec}_K\downarrow \\
G_k^{\mathrm{ab}} & \xrightarrow{\mathrm{Tf}_{G_K \subseteq G_k}} & G_K^{\mathrm{ab}}.
\end{array}$$

(4) *Let* L *be an MLF,* $\overline{L}$ *an algebraic closure of* L*, and* $\iota\colon k \xrightarrow{\sim} L$ *an isomorphism of fields. Then the diagram*

$$\begin{array}{ccc}
k^\times & \xrightarrow{\iota} & L^\times \\
\mathrm{rec}_k\downarrow & & \mathrm{rec}_L\downarrow \\
G_k^{\mathrm{ab}} & \longrightarrow & \mathrm{Gal}(\overline{L}/L)^{\mathrm{ab}}
\end{array}$$

— *where the lower horizontal arrow is an isomorphism induced by* ι — **commutes**.

PROOF. — This assertion follows immediately from the various assertions in [11], §2. □

Finally, as an application of *local class field theory*, let us verify the *slimness* of the absolute Galois group of an MLF.

LEMMA 1.8. — *The following hold:*

(i) *The absolute Galois group* G_k *is* **center-free**.

(ii) *The absolute Galois group* G_k *is* **slim**.

PROOF. — First, we verify assertion (i). Let $\gamma \in G_k$ be an element of the *center* of G_k. Let us observe that, for each finite Galois extension K of k contained in $\overline{k}$, it follows immediately from the *injectivity* of the homomorphism rec_K of Lemma 1.7, together with Lemma 1.7, (4), that the action of the quotient of G_k by $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/K)$ [i.e., $\mathrm{Gal}(K/k)$] on G_K^{ab} by conjugation is *faithful*, which thus implies that $\gamma \in G_K$. Thus, by allowing "K" to vary, we conclude that $\gamma = 1$, as desired. This completes the proof of assertion (i).

Finally, we verify assertion (ii). Let K be a finite extension of k contained in $\overline{k}$ and $\gamma \in G_k$ an element of the *centralizer* of $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/K)$ in G_k. Let us observe that, to verify that $\gamma = 1$, we may assume without loss of generality, by replacing K by a suitable finite extension of K contained in $\overline{k}$, that K is *Galois* over k. Thus, it follows immediately from the *injectivity* of the homomorphism rec_K of Lemma 1.7, together with Lemma 1.7, (4), that the action of $G_k/G_K = \mathrm{Gal}(K/k)$ on G_K^{ab} by conjugation is *faithful*, which thus implies that $\gamma \in G_K$. In particular, it follows from assertion (i) that $\gamma = 1$, as desired. This completes the proof of assertion (ii). □

2. BI-ANABELIAN RESULTS FOR MLF

In the present §2, let us recall some results in *bi-anabelian geometry* for MLF. In the present §2, for $\square \in \{\circ, \bullet\}$, let $k_\square$ be an MLF and $\overline{k}_\square$ an algebraic closure of $k_\square$; write $G_\square \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square)$. Thus, we have a natural map

$$\phi = \phi_{\overline{k}_\circ/k_\circ, \overline{k}_\bullet/k_\bullet} \colon \mathrm{Isom}(k_\bullet, k_\circ) \longrightarrow \mathrm{Isom}(G_\circ, G_\bullet)/\mathrm{Inn}(G_\bullet).$$

Typically, *bi-anabelian geometry* [i.e., the classical point of view of anabelian geometry] discusses some properties such as *faithfulness/fullness* of the [restriction, to a certain suitable category of geometric objects, of the] functor of taking arithmetic fundamental groups. Put another way, in a discussion of bi-anabelian geometry, one usually fixes two schemes/rings of interest [e.g., the two fields $k_\circ$ and $k_\bullet$ in our case] and discusses the relationship between a certain set of morphisms [e.g., the set of isomorphisms] between the fixed two schemes/rings and a certain set of homomorphisms [e.g., the set of isomorphisms] between the arithmetic fundamental groups. In particular, roughly speaking, bi-anabelian geometry — i.e., for isomorphisms between MLF — may be summarized as the study of the above map ϕ.

The following proposition asserts the *injectivity* of the map ϕ.

PROPOSITION 2.1. — *The map ϕ is* **injective**.

PROOF. — Since [one verifies easily that] every automorphism of the field $\overline{k}_\square$ is an *automorphism over* $\mathbb{Q}_{p_{k_\square}}$ $(\subseteq \overline{k}_\square)$, this assertion follows immediately, by considering the difference of two elements of $\mathrm{Isom}(k_\bullet, k_\circ)$ whose images via ϕ coincide, from Lemma 1.8, (ii). $\square$

The following theorem is fundamental in bi-anabelian geometry for MLF.

THEOREM 2.2. — *The following hold:*

(i) *There exists a pair "$(\overline{k}_\circ/k_\circ, \overline{k}_\bullet/k_\bullet)$" which satisfies the following condition: The domain of $\phi_{\overline{k}_\circ/k_\circ,\overline{k}_\bullet/k_\bullet}$ is* **empty**, *but the codomain of $\phi_{\overline{k}_\circ/k_\circ,\overline{k}_\bullet/k_\bullet}$ is* **nonempty**.

(ii) *There exists a pair "$(\overline{k}_\circ/k_\circ, \overline{k}_\bullet/k_\bullet)$" which satisfies the following condition: The domain of $\phi_{\overline{k}_\circ/k_\circ,\overline{k}_\bullet/k_\bullet}$ is* **nonempty** [*which thus implies that the codomain of $\phi_{\overline{k}_\circ/k_\circ,\overline{k}_\bullet/k_\bullet}$ is* **nonempty**], *but the map $\phi_{\overline{k}_\circ/k_\circ,\overline{k}_\bullet/k_\bullet}$ is* **not surjective**.

PROOF. — Assertion (i) follows from the examples discussed in [3], §2. Assertion (ii) follows from the discussion given at the final portion of [5], Chapter VII, §5. $\square$

REMARK 2.2.1. — In [3], a *necessary and sufficient condition* for the pair "$(\overline{k}_\circ/k_\circ, \overline{k}_\bullet/k_\bullet)$" to satisfy that the codomain of $\phi_{\overline{k}_\circ/k_\circ,\overline{k}_\bullet/k_\bullet}$ is *nonempty* was discussed.

The map ϕ is always *injective* [cf. Proposition 2.1] but *not surjective* in general [cf. Theorem 2.2]. Thus, in the study of bi-anabelian geometry for MLF, one often discusses conditions for an outer isomorphism $G_\circ \xrightarrow{\sim} G_\bullet$ [i.e., an element of the codomain of ϕ] to be contained in the image of ϕ. The following theorem gives such conditions.

THEOREM 2.3. — *Let $\alpha\colon G_\circ \xrightarrow{\sim} G_\bullet$ be an isomorphism of topological groups* [*which thus implies that $p_{k_\circ} = p_{k_\bullet}$ — cf. Remark* 2.3.1 *below*]. *Then the following five conditions are equivalent:*

(1) *The outer isomorphism determined by α* [*i.e., the element of the codomain of ϕ determined by α*] *is* **contained** *in the image of ϕ.*

(2) *The isomorphism α is* **compatible** *with the respective* **ramification filtrations** [*cf.* [11], §4.1] *of $G_\circ$, $G_\bullet$.*

(3) *For $\square \in \{\circ, \bullet\}$, write $(\widehat{\overline{k}}_\square)_+$ for the topological $G_\square$-module obtained by forming the underlying additive module of the $p_{k_\square}$-adic completion $\widehat{\overline{k}}_\square$ of $\overline{k}_\square$. Write, moreover, $\alpha^*(\widehat{\overline{k}}_\bullet)_+$ for the topological $G_\circ$-module obtained by considering the action of $G_\circ$ on $(\widehat{\overline{k}}_\bullet)_+$ via α. Then there exists a* **$G_\circ$-equivariant topological isomorphism** $(\widehat{\overline{k}}_\circ)_+ \xrightarrow{\sim} \alpha^*(\widehat{\overline{k}}_\bullet)_+$.

(4) *Write* $\alpha^*(\mathcal{O}_{\overline{k}_\bullet})_+$ *for the* $G_\circ$*-module obtained by considering the action of* $G_\circ$ *on* $(\mathcal{O}_{\overline{k}_\bullet})_+$ *via* α*. Then there exists a* $\boldsymbol{G_\circ}$**-equivariant isomorphism** $(\mathcal{O}_{\overline{k}_\circ})_+ \xrightarrow{\sim} \alpha^*(\mathcal{O}_{\overline{k}_\bullet})_+$.

(5) *For every finite-dimensional* **Hodge-Tate** *representation* $\rho_\bullet$ *of* $G_\bullet$ *over* $\mathbb{Q}_{p_{k_\circ}} = \mathbb{Q}_{p_{k_\bullet}}$*, the finite-dimensional representation of* $G_\circ$ *over* $\mathbb{Q}_{p_{k_\circ}} = \mathbb{Q}_{p_{k_\bullet}}$ *obtained by forming the composite* $\rho_\bullet \circ \alpha$ *is* **Hodge-Tate**.

Proof. — The equivalence (1) ⇔ (2) follows from [7], Theorem. The equivalence (1) ⇔ (3) follows from [8], Theorem 3.5, (ii). The implications (1) ⇒ (4) ⇒ (3) are immediate; thus, by the equivalence (1) ⇔ (3) already discussed, we obtain the equivalence (1) ⇔ (4). The equivalence (1) ⇔ (5) follows from [1], Theorem. □

Remark 2.3.1. — Suppose that the codomain of ϕ is *nonempty*, i.e., that there exists an isomorphism $\alpha\colon G_\circ \xrightarrow{\sim} G_\bullet$ of topological groups. Then it is well-known that it holds that $p_{k_\circ} = p_{k_\bullet}$. This fact also follows from Proposition 3.6 of the present article.

Remark 2.3.2.

(i) One may find other conditions for "α" as in Theorem 2.3 *equivalent* to condition (1) of Theorem 2.3 in, for instance, [8], §3; [1], §3.

(ii) We have considered, in Theorem 2.3, some conditions for an *outer isomorphism* between the absolute Galois groups of MLF to arise from an isomorphism between the original MLF. On the other hand, one may consider a condition for an *outer open homomorphism* between the absolute Galois groups of MLF to arise from an homomorphism between the original MLF. One may also find such conditions in, for instance, [8], §3; [1], §3. For instance, a similar equivalence to the equivalence (1) ⇔ (5) of Theorem 2.3 still holds even if one considers an *open homomorphism* [i.e., as opposed to an *isomorphism*] from $G_\circ$ to $G_\bullet$ [cf. [1], Theorem].

3. Mono-anabelian Reconstruction for MLF: I

Let us recall that, as discussed at the beginning of §2, *bi-anabelian geometry* centers around a *comparison* between *two fixed schemes/rings* of interest via the arithmetic fundamental groups. By contrast, *mono-anabelian geometry* centers around the task of establishing a "*group-theoretic software*" [i.e., "*group-theoretic algorithm*"] whose input data consists of a *single abstract* [*topological*] *group* isomorphic to the arithmetic fundamental group of a scheme/ring of interest [cf., e.g., [9], Introduction; [9], Remark 1.9.8; Remarks following [9], Corollary 3.7, for more details concerning bi-anabelian/mono-anabelian geometry].

$$\begin{array}{ccc} \underline{\text{bi-anabelian geometry}} & & \\ \pi_1^{\text{ét}}(X_\circ) \xrightarrow{\sim} \pi_1^{\text{ét}}(X_\bullet) & \stackrel{?}{\Longrightarrow} & \text{[objects related to] } X_\circ \xrightarrow{\sim} \text{[objects related to] } X_\bullet \end{array}$$

$$\begin{array}{ccc} \underline{\text{mono-anabelian geometry}} & & \\ \text{an isomorph of } \pi_1^{\text{ét}}(X) & \stackrel{?}{\Longrightarrow} & \text{isomorph(s) of [objects related to] } X \end{array}$$

In the present §3, let us establish some *mono-anabelian reconstruction algorithms* for MLF. In particular, we discuss some "*group-theoretic algorithms*" [cf. Remark 3.15.1 below] whose input data consist of an abstract group *isomorphic to the absolute Galois group of an MLF.*

DEFINITION 3.1. — We shall refer to an isomorph, as a group, of the absolute Galois group of an MLF as a group *of MLF-type.*

In the remainder of the present §3, let

$$G$$

be a group *of MLF-type*,

$$k$$

an MLF, and

$$\overline{k}$$

an algebraic closure of k. We shall also apply the notational conventions related to k and $\overline{k}$ introduced in §1.

DEFINITION 3.2. — We shall say that a subgroup of G is *open* if the subgroup is of finite index in G.

PROPOSITION 3.3. — *The following hold:*

(i) *The* **open** *subgroups of* G [*i.e., in the sense of Definition* 3.2] *determine a structure of* **profinite group** *of* G*.*

(ii) *Every isomorphism* $G \xrightarrow{\sim} G_k$ *of groups is an* **isomorphism of topological groups** *with respect to the structure of* **profinite group** *of* G *of* (i)*.*

PROOF. — Assertion (i) follows from Lemma 1.4, (ii). Assertion (ii) follows from assertion (i). □

In the remainder of the present article, we always regard a group *of MLF-type* as a *profinite*, hence also *topological*, group by Proposition 3.3, (i). Note that it follows from the various definitions involved that every open subgroup of a group *of MLF-type* is *of MLF-type.*

LEMMA 3.4. — *The following hold:*

(i) *There exists a* **uniquely determined** *prime number l such that*

$$\log_l\big(\sharp(G^{\mathrm{ab/tor}}/l \cdot G^{\mathrm{ab/tor}})\big) \;\geq\; 2.$$

(ii) *The subquotient $G^{\text{ab-tor}}$ of G is* **finite**.

PROOF. — Let us observe that it follows from Lemma 1.7, (1), together with Lemma 1.2, (i), that the topological module G_k^{ab} is *isomorphic* to the topological module

$$\big(\mathbb{Z}/(p_k^{f_k}-1)\mathbb{Z}\big) \oplus (\mathbb{Z}/p_k^a\mathbb{Z}) \oplus \mathbb{Z}_{p_k}^{\oplus d_k} \oplus \widehat{\mathbb{Z}}$$

for some nonnegative integer a. In particular, the topological module $G_k^{\mathrm{ab/tor}}$ (respectively, $G_k^{\text{ab-tor}}$) is *isomorphic* to the topological module

$$\mathbb{Z}_{p_k}^{\oplus d_k} \oplus \widehat{\mathbb{Z}} \quad \text{(respectively, } \big(\mathbb{Z}/(p_k^{f_k}-1)\mathbb{Z}\big) \oplus (\mathbb{Z}/p_k^a\mathbb{Z})\text{)}.$$

Thus, assertion (i) (respectively, (ii)) holds. □

DEFINITION 3.5.

(i) We shall write

$$p(G)$$

for the uniquely determined [cf. Lemma 3.4, (i)] prime number such that

$$\log_{p(G)}\Big(\sharp\big(G^{\mathrm{ab/tor}}/p(G) \cdot G^{\mathrm{ab/tor}}\big)\Big) \;\geq\; 2.$$

(ii) We shall write

$$d(G) \stackrel{\mathrm{def}}{=} \log_{p(G)}\Big(\sharp\big(G^{\mathrm{ab/tor}}/p(G) \cdot G^{\mathrm{ab/tor}}\big)\Big) - 1,$$

$$f(G) \stackrel{\mathrm{def}}{=} \log_{p(G)}\Big(1 + \sharp\big((G^{\text{ab-tor}})^{(p(G)')}\big)\Big) \quad (\neq 0)$$

— where we write $(G^{\text{ab-tor}})^{(p(G)')}$ for the quotient of $G^{\text{ab-tor}}$ by the $p(G)$-Sylow subgroup of [the finite — cf. Lemma 3.4, (ii) — abelian group] $G^{\text{ab-tor}}$ — and

$$e(G) \stackrel{\mathrm{def}}{=} d(G)/f(G).$$

(iii) We shall write

$$I(G) \stackrel{\mathrm{def}}{=} \bigcap_{N \subseteq G} N$$

— where N ranges over the normal open subgroups of G [so N is of MLF-type — cf. the discussion following Proposition 3.3] such that $e(N) = e(G)$;

$$P(G) \stackrel{\mathrm{def}}{=} \bigcap_{N \subseteq G} N$$

— where N ranges over the normal open subgroups of G [so N is of MLF-type — cf. the discussion following Proposition 3.3] such that $e(N)/e(G)$ is a positive integer prime to $p(G)$.

PROPOSITION 3.6. — *It holds that*

$$p_k \ = \ p(G_k), \quad d_k \ = \ d(G_k), \quad f_k \ = \ f(G_k), \quad e_k \ = \ e(G_k),$$
$$I_k \ = \ I(G_k), \quad P_k \ = \ P(G_k).$$

PROOF. — The assertions for p_k, d_k, and f_k follow from Lemma 1.7, (1), together with Lemma 1.2, (i) [cf. also the explicit descriptions of $G_k^{\mathrm{ab/tor}}$ and $G_k^{\mathrm{ab\text{-}tor}}$ given in the proof of Lemma 3.4]. The assertion for e_k follows — in light of the assertions for d_k and f_k — from Lemma 1.2, (iii). The assertions for I_k and P_k follow — in light of the assertions for p_k and e_k — from the definitions of I_k and P_k. □

LEMMA 3.7. — *The following hold:*

(i) *The quotients $G/I(G)$ and $I(G)/P(G)$ are* **abelian**.

(ii) *There exists a* **uniquely determined** *element $\gamma \in G/I(G)$ of $G/I(G)$ such that the action of γ on [the* **abelian** *— cf.* (i) *— group] $I(G)/P(G)$ by conjugation is given by multiplication by $p(G)^{f(G)}$.*

PROOF. — Assertion (i) follows — in light of Proposition 3.6 — from Lemma 1.5, (i), (ii). Assertion (ii) follows — in light of Proposition 3.6 — from Lemma 1.5, (iii). □

DEFINITION 3.8. — We shall write

$$\mathrm{Frob}(G) \ \in \ G/I(G)$$

for the uniquely determined [cf. Lemma 3.7, (ii)] element of $G/I(G)$ such that the action of $\mathrm{Frob}(G)$ on $I(G)/P(G)$ by conjugation is given by multiplication by $p(G)^{f(G)}$ [cf. Lemma 3.7, (i)].

PROPOSITION 3.9. — *It holds that*

$$\mathrm{Frob}_k \ = \ \mathrm{Frob}(G_k)$$

in $G_k/I_k = G_k/I(G_k)$ [cf. Proposition 3.6].

PROOF. — This assertion follows — in light of Proposition 3.6 — from Lemma 1.5, (iii). □

DEFINITION 3.10.

(i) We shall write

$$\mathcal{O}^{\times}(G) \ \stackrel{\mathrm{def}}{=} \ \mathrm{Im}\big(I(G) \hookrightarrow G \twoheadrightarrow G^{\mathrm{ab}}\big) \ \subseteq \ G^{\mathrm{ab}}.$$

By considering the topology induced by the topology of $I(G)$, we regard $\mathcal{O}^{\times}(G)$ as a profinite, hence also topological, module.

(ii) We shall write

$$U^{(1)}(G) \ \subseteq \ \mathcal{O}^{\times}(G)$$

for the maximal pro-$p(G)$ submodule of the profinite module $\mathcal{O}^\times(G)$.

(iii) We shall write

$$\underline{k}^\times(G) \stackrel{\text{def}}{=} \mathcal{O}^\times(G)/U^{(1)}(G).$$

(iv) We shall write

$$k^\times(G) \stackrel{\text{def}}{=} G^{\text{ab}} \times_{G/I(G)} \text{Frob}(G)^{\mathbb{Z}} \subseteq G^{\text{ab}}$$

[cf. Lemma 3.7, (i)],

$$\text{rec}(G)\colon\ k^\times(G) \hookrightarrow G^{\text{ab}}$$

for the natural inclusion, and

$$\mathcal{O}^\rhd(G) \stackrel{\text{def}}{=} G^{\text{ab}} \times_{G/I(G)} \text{Frob}(G)^{\mathbb{N}} \subseteq k^\times(G)$$

[cf. Lemma 3.7, (i)]. By considering the topologies induced by the topology of $\mathcal{O}^\times(G)$ [cf. (i)], we regard $\mathcal{O}^\rhd(G)$, $k^\times(G)$ as topological monoid, module, respectively.

(v) We shall write

$$k_\times(G) \stackrel{\text{def}}{=} k^\times(G)^\circledast$$

and

$$\underline{k}_\times(G) \stackrel{\text{def}}{=} \underline{k}^\times(G)^\circledast.$$

(vi) We shall write

$$k_+(G) \stackrel{\text{def}}{=} \mathcal{O}^\times(G)^{\text{pf}}$$

and

$$\mathcal{I}(G) \stackrel{\text{def}}{=} \big(2\cdot p(G)\big)^{-1}\cdot \text{Im}\big(\mathcal{O}^\times(G) \to k_+(G)\big) \subseteq k_+(G).$$

By considering the topologies induced by the topology of $\mathcal{O}^\times(G)$ [cf. (i)], we regard $k_+(G)$, $\mathcal{I}(G)$ as topological modules, respectively.

PROPOSITION 3.11. — *The following hold:*

(i) *The injective homomorphism* $\text{rec}_k\colon k^\times \hookrightarrow G_k^{\text{ab}}$ *of Lemma* 1.7 *determines a* **commutative diagram** *of topological monoids*

$$\begin{array}{ccccccccc}
U_k^{(1)} & \xrightarrow{\subseteq} & \mathcal{O}_k^\times & \xrightarrow{\subseteq} & \mathcal{O}_k^\rhd & \xrightarrow{\subseteq} & k^\times & \xrightarrow{\text{rec}_k} & G_k^{\text{ab}} \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \| \\
U^{(1)}(G_k) & \xrightarrow{\subseteq} & \mathcal{O}^\times(G_k) & \xrightarrow{\subseteq} & \mathcal{O}^\rhd(G_k) & \xrightarrow{\subseteq} & k^\times(G_k) & \xrightarrow{\text{rec}(G_k)} & G_k^{\text{ab}}
\end{array}$$

— *where the horizontal arrows are* **injective**, *and the vertical arrows are* **isomorphisms**.

(ii) *The isomorphism* $k^\times \xrightarrow{\sim} k^\times(G_k)$ *in the diagram of* (i) *determines an* **isomorphism** *of monoids*

$$k_\times \xrightarrow{\sim} k_\times(G_k).$$

(iii) *The left-hand square of the diagram of* (i) *determines* **isomorphisms** *of monoids*

$$\underline{k}^\times \xrightarrow{\sim} \underline{k}^\times(G_k), \quad \underline{k}_\times \xrightarrow{\sim} \underline{k}_\times(G_k).$$

(iv) *The isomorphism* $\mathcal{O}_k^\times \xrightarrow{\sim} \mathcal{O}^\times(G_k)$ *in the diagram of* (i) *and the natural homomorphisms* $\mathcal{O}^\times(G_k) \to \mathcal{I}(G_k) \hookrightarrow k_+(G_k)$ *fit into a* **commutative diagram** *of topological modules*

$$\begin{array}{ccccc} \mathcal{O}_k^\times & \xrightarrow{\log_k} & \mathcal{I}_k & \xrightarrow{\subseteq} & k_+ \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ \mathcal{O}^\times(G_k) & \longrightarrow & \mathcal{I}(G_k) & \xrightarrow{\subseteq} & k_+(G_k). \end{array}$$

— *where the vertical arrows are* **isomorphisms**.

PROOF. — Assertion (i) follows immediately — in light of Proposition 3.6; Proposition 3.9 — from Lemma 1.7, (1), together with Lemma 1.2, (ii). Assertions (ii), (iii) are immediate. Assertion (iv) follows — in light of Proposition 3.6 — from Lemma 1.2, (iv). ☐

LEMMA 3.12. — *The following hold:*

(i) *The topological module* $k_+(G)$ *is* **locally compact**.

(ii) *The topological submodule* $\mathcal{I}(G) \subseteq k_+(G)$ *of* $k_+(G)$ *is* **compact** *and* **open**.

(iii) *For each positive real number* a, *there exists a* **uniquely determined** *Haar measure on* $k_+(G)$ [*cf.* (i)] *that assigns the compact open* [*cf.* (ii)] *subset* $\mathcal{I}(G) \subseteq k_+(G)$ *to* a.

PROOF. — Assertions (i), (ii) follow from Proposition 3.11, (iv). Assertion (iii) follows from assertions (i), (ii). ☐

DEFINITION 3.13. — We shall write

$$\mu(G)$$

for the uniquely determined [cf. Lemma 3.12, (iii)] Haar measure on $k_+(G)$ such that

$$\mu(G)\big(\mathcal{I}(G)\big) \ = \ p(G)^{\epsilon(G)\cdot d(G)-f(G)}/\sharp(k^\times(G)_{\mathrm{tor}})^{(p(G))}$$

— where we write

$$\epsilon(G) \ \stackrel{\mathrm{def}}{=} \ \begin{cases} 1 & \text{if } p(G) \neq 2 \\ 2 & \text{if } p(G) = 2 \end{cases}$$

and $(k^\times(G)_{\mathrm{tor}})^{(p(G))}$ for the $p(G)$-Sylow subgroup of [the finite — cf. Lemma 1.2, (i); Proposition 3.11, (i) — abelian group] $k^\times(G)_{\mathrm{tor}}$.

PROPOSITION 3.14. — *It holds that*

$$\mu_k \ = \ \mu(G_k)$$

relative to the right-hand vertical arrow $k_+ \xrightarrow{\sim} k_+(G_k)$ *of the diagram of Proposition* 3.11, (iv).

PROOF. — This assertion follows — in light of Proposition 3.6; Proposition 3.11, (i), (iv) — from Lemma 1.3, (iii). ☐

The various assertions discussed in the present §3 may be summarized as follows.

SUMMARY 3.15. — *There exist* **functorial group-theoretic algorithms** [*cf. Remark* 3.15.1 *below*] *for constructing, from a group* G **of MLF-type**,

- *a topology on* G,
- *a prime number* $p(G)$,
- *positive integers* $d(G)$, $f(G)$, *and* $e(G)$,
- *subgroups* $P(G) \subseteq I(G) \subseteq G$ *of* G,
- *an element* $\mathrm{Frob}(G) \in G/I(G)$ *of* $G/I(G)$,
- *topological monoids* $U^{(1)}(G) \subseteq \mathcal{O}^{\times}(G) \subseteq \mathcal{O}^{\triangleright}(G) \subseteq k^{\times}(G) \overset{\mathrm{rec}(G)}{\hookrightarrow} G^{\mathrm{ab}}$,
- *monoids* $\underline{k}^{\times}(G) \subseteq \underline{k}_{\times}(G)$ *and* $k_{\times}(G)$,
- *topological modules* $\mathcal{I}(G) \subseteq k_{+}(G)$, *and*
- *a measure* $\mu(G)$ *on* $k_{+}(G)$

which **"correspond"** *to*

- *the profinite topology on* G_k,
- *the prime number* p_k,
- *the positive integers* d_k, f_k, *and* e_k,
- *the subgroups* $P_k \subseteq I_k \subseteq G_k$ *of* G_k,
- *the element* $\mathrm{Frob}_k \in G_k/I_k$ *of* G_k/I_k,
- *the topological monoids* $U_k^{(1)} \subseteq \mathcal{O}_k^{\times} \subseteq \mathcal{O}_k^{\triangleright} \subseteq k^{\times} \overset{\mathrm{rec}_k}{\hookrightarrow} G_k^{\mathrm{ab}}$,
- *the monoids* $\underline{k}^{\times} \subseteq \underline{k}_{\times}$ *and* $k_{\times}$,
- *the topological modules* $\mathcal{I}_k \subseteq k_{+}$, *and*
- *the measure* μ_k *on* k_{+},

respectively.

REMARK 3.15.1. — As discussed in [9], Remark 1.9.8, in *bi-anabelian geometry*, the term "*group-theoretic*" is usually used simply to mean "*preserved by an arbitrary isomorphism between the arithmetic fundamental groups under consideration*"; on the other hand, in *mono-anabelian geometry*, the term "*group-theoretic algorithm*" is used to mean that "*the algorithm in a discussion is phrased in language that only depends on the topological group structure of the arithmetic fundamental group under consideration*".

REMARK 3.15.2.

(i) It follows from Theorem 2.2, (ii); the equivalence (1) ⇔ (2) of Theorem 2.3 that there exist an MLF L, an algebraic closure $\overline{L}$ of L, and an automorphism α of $G_L \overset{\mathrm{def}}{=}$

$\mathrm{Gal}(\overline{L}/L)$ *not compatible* with the *ramification filtration* of G_L. By this fact, one may conclude that

> the ramification filtration of the absolute Galois group of an MLF should be considered to be *"not group-theoretic"*.

(ii) It follows from Theorem 2.2, (ii); the equivalence (1) $\Leftrightarrow$ (5) of Theorem 2.3 that there exist an MLF L, an algebraic closure $\overline{L}$ of L, an automorphism α of $G_L \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{L}/L)$, and a finite-dimensional *Hodge-Tate* representation ρ of G_L over $\mathbb{Q}_{p_L}$ such that the composite $\rho \circ \alpha$ is *not Hodge-Tate* [cf. also [1], Remark 3.3.1]. By this fact, one may conclude that

> *Hodge-Tate-ness* of the p-adic representations of the absolute Galois group of an MLF should be considered to be *"not group-theoretic"*.

4. Mono-anabelian Reconstruction for MLF: II

In the present §4, we continue to establish *mono-anabelian reconstruction algorithms* for MLF. Let

$$G$$

be a group *of MLF-type*,

$$k$$

an MLF, and

$$\overline{k}$$

an algebraic closure of k. We shall also apply the notational conventions related to k and $\overline{k}$ introduced in §1.

Definition 4.1.

(i) We shall write

$$\overline{\mathcal{O}}^{\times}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H\subseteq G} \mathcal{O}^{\times}(H), \quad \underline{\overline{k}}^{\times}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H\subseteq G} \underline{k}^{\times}(H), \quad \overline{k}^{\times}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H\subseteq G} k^{\times}(H),$$

$$\overline{\mathcal{O}}^{\triangleright}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H\subseteq G} \mathcal{O}^{\triangleright}(H), \quad \overline{k}_{\times}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H\subseteq G} k_{\times}(H), \quad \underline{\overline{k}}_{\times}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H\subseteq G} \underline{k}_{\times}(H),$$

$$\overline{k}_{+}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H\subseteq G} k_{+}(H)$$

— where the injective limits are taken over the open subgroups $H \subseteq G$ of G [so H is of MLF-type — cf. the discussion following Proposition 3.3], and the transition morphisms in the limits are given by the homomorphisms determined by the transfer maps. We regard these seven monoids as G-monoids by the actions of G obtained by conjugation.

(ii) We shall write

$$\boldsymbol{\mu}(G) \stackrel{\mathrm{def}}{=} \overline{k}^{\times}(G)_{\mathrm{tor}}.$$

We regard $\boldsymbol{\mu}(G)$ as a G-module by the action of G induced by the action of G on $\overline{k}^{\times}(G)$ [cf. (i)].

(iii) We shall write

$$\Lambda(G) \stackrel{\text{def}}{=} \varprojlim_{n} \boldsymbol{\mu}(G)[n]$$

— where the projective limit is taken over the positive integers n — and refer to $\Lambda(G)$ as the *cyclotome* associated to G. We regard the cyclotome $\Lambda(G)$ as a profinite, hence also topological, module [by the easily verified finiteness of $\boldsymbol{\mu}(G)[n]$ for each n]. Moreover, we also regard the cyclotome $\Lambda(G)$ as a [topological] G-module by the action of G induced by the action of G on $\boldsymbol{\mu}(G)$ [cf. (ii)].

PROPOSITION 4.2. — *The following hold:*

(i) *The various injective homomorphisms* $\mathrm{rec}_K \colon K^\times \hookrightarrow G_K^{\mathrm{ab}}$ *— where K ranges over the finite extensions of k contained in $\overline{k}$ — of Lemma* 1.7 *determine a* **commutative diagram** *of G_k-monoids*

$$\begin{array}{ccccccc}
\mathcal{O}_{\overline{k}}^{\times} & \stackrel{\subseteq}{\longrightarrow} & \mathcal{O}_{\overline{k}}^{\rhd} & \stackrel{\subseteq}{\longrightarrow} & \overline{k}^{\times} & \stackrel{\subseteq}{\longrightarrow} & \overline{k}_{\times} \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
\overline{\mathcal{O}}^{\times}(G_k) & \stackrel{\subseteq}{\longrightarrow} & \overline{\mathcal{O}}^{\rhd}(G_k) & \stackrel{\subseteq}{\longrightarrow} & \overline{k}^{\times}(G_k) & \stackrel{\subseteq}{\longrightarrow} & \overline{k}_{\times}(G_k)
\end{array}$$

— where the horizontal arrows are **injective***, and the vertical arrows are* **isomorphisms***.*

(ii) *The left-hand square of the diagram of* (i) *determines* **isomorphisms** *of G_k-monoids*

$$\underline{\overline{k}}^{\times} \xrightarrow{\sim} \underline{\overline{k}}^{\times}(G_k), \quad \underline{\overline{k}}_{\times} \xrightarrow{\sim} \underline{\overline{k}}_{\times}(G_k).$$

(iii) *The isomorphism $\mathcal{O}_{\overline{k}}^{\times} \xrightarrow{\sim} \overline{\mathcal{O}}^{\times}(G_k)$ in the diagram of* (i) *and the natural homomorphism $\overline{\mathcal{O}}^{\times}(G_k) \to \overline{k}_{+}(G_k)$ fit into a* **commutative diagram** *of G_k-modules*

$$\begin{array}{ccc}
\mathcal{O}_{\overline{k}}^{\times} & \stackrel{\log_{\overline{k}}}{\longrightarrow} & \overline{k}_{+} \\
\wr\downarrow & & \wr\downarrow \\
\overline{\mathcal{O}}^{\times}(G_k) & \longrightarrow & \overline{k}_{+}(G_k)
\end{array}$$

— where the vertical arrows are **isomorphisms***.*

(iv) *The isomorphism $\overline{k}^{\times} \xrightarrow{\sim} \overline{k}^{\times}(G_k)$ in the diagram of* (i) *determines an* **isomorphism** *of G_k-modules*

$$\boldsymbol{\mu}(\overline{k}) \xrightarrow{\sim} \boldsymbol{\mu}(G_k),$$

hence also an **isomorphism** *of topological G_k-modules*

$$\Lambda(\overline{k}) \xrightarrow{\sim} \Lambda(G_k).$$

PROOF. — These assertions follow — in light of Proposition 3.11, (i), (ii), (iii), (iv) — from Lemma 1.7, (3), (4). ☐

The various assertions discussed in the present §4 may be summarized as follows.

SUMMARY 4.3. — *There exist* **functorial group-theoretic algorithms** [*cf. Remark* 3.15.1] *for constructing, from a group* G **of MLF-type**,

- *G-monoids* $\overline{\mathcal{O}}^{\times}(G) \subseteq \overline{\mathcal{O}}^{\triangleright}(G) \subseteq \overline{k}^{\times}(G) \subseteq \overline{k}_{\times}(G)$,
- *G-monoids* $\underline{\overline{k}}^{\times}(G) \subseteq \underline{\overline{k}}_{\times}(G)$,
- *a G-module* $\overline{k}_{+}(G)$,
- *a G-module* $\boldsymbol{\mu}(G)$, *and*
- *a topological G-module* $\Lambda(G)$

which **"correspond"** *to*

- *the G_k-monoids* $\mathcal{O}_{\overline{k}}^{\times} \subseteq \mathcal{O}_{\overline{k}}^{\triangleright} \subseteq \overline{k}^{\times} \subseteq \overline{k}_{\times}$,
- *the G_k-monoids* $\underline{\overline{k}}^{\times} \subseteq \underline{\overline{k}}_{\times}$,
- *the G_k-module* $\overline{k}_{+}$,
- *the G_k-module* $\boldsymbol{\mu}(\overline{k})$, *and*
- *the topological G_k-module* $\Lambda(\overline{k})$,

respectively.

REMARK 4.3.1.

(i) It follows from Theorem 2.2, (ii); the equivalence (1) $\Leftrightarrow$ (3) of Theorem 2.3 that there exist an MLF L, an algebraic closure $\overline{L}$ of L, and an automorphism α of $G_L \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{L}/L)$ such that the automorphism of $\overline{L}_{+}$ induced by α [cf. Proposition 4.2, (iii)] is *not compatible* with the *p_L-adic topology* of $\overline{L}_{+}$. By this fact, one may conclude that

> the p-adic topology on the underlying additive module of an algebraic closure of an MLF should be considered to be *"not group-theoretic"*

[cf. also Remark 4.3.2, (ii), below].

(ii) It follows from Theorem 2.2, (ii); the equivalence (1) $\Leftrightarrow$ (4) of Theorem 2.3 that there exist an MLF L, an algebraic closure $\overline{L}$ of L, and an automorphism α of $G_L \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{L}/L)$ such that the automorphism of $\overline{L}_{+}$ induced by α [cf. Proposition 4.2, (iii)] does *not restrict* to an automorphism of the submodule $(\mathcal{O}_{\overline{L}})_{+} \subseteq \overline{L}_{+}$ of $\overline{L}_{+}$. By this fact, one may conclude that

> the underlying additive module of the ring of integers of an algebraic closure of an MLF should be considered to be *"not group-theoretic"*.

(iii) Assume that there exists a *functorial group-theoretic algorithm* for constructing, from a group G *of MLF-type*, the submodule of $k_{+}(G)$ which "corresponds" to the submodule $(\mathcal{O}_k)_{+} \subseteq k_{+}$ of k_{+}. Then, by applying a similar construction to the constructions of Definition 4.1, (i), one may obtain a *functorial group-theoretic algorithm* for constructing, from a group G *of MLF-type*, the submodule of $\overline{k}_{+}(G)$ which "corresponds" to the submodule $(\mathcal{O}_{\overline{k}})_{+} \subseteq \overline{k}_{+}$ of $\overline{k}_{+}$ — in *contradiction* to the discussion of (ii). By this fact, one may conclude that the underlying additive module of the ring of integers of an MLF should be considered to be *"not group-theoretic"*.

[Note that if one works with a *special* case, then a *functorial group-theoretic algorithm* for constructing the submodule of $k_+(G)$ which "corresponds" to the submodule $(\mathcal{O}_k)_+ \subseteq k_+$ of k_+ may be established. For instance, if $p(G) \neq 2$, and $e(G) = 1$, then it follows immediately from Lemma 1.2, (vi); Lemma 1.3, (iii), that the submodule $\mathcal{I}(G) \subseteq k_+(G)$ of $k_+(G)$ in fact "corresponds" to the submodule $(\mathcal{O}_k)_+ \subseteq k_+$ of k_+.]

Remark 4.3.2.

(i) Let us observe that the field $\overline{k}$, as well as the underlying additive module $\overline{k}_+$ of $\overline{k}$, admits the following two natural topologies:

(a) the *p_k-adic topology*

(b) the "*injective limit-topology*" determined by the p_k-adic topologies of the various subfields of $\overline{k}$ finite over k

Let us also observe that these two topologies do *not coincide*. One verification of this fact is given as follows: Let us consider the map

$$\mathrm{Tr}\colon\ \overline{k}_+ \longrightarrow k_+$$

defined by mapping $a \in \overline{k}_+$ to $d_K^{-1} \cdot \mathrm{Tr}_{K/k}(a) \in k_+$ if $a \in K_+$ for a finite extension K of k contained in $\overline{k}$ — where we write $\mathrm{Tr}_{K/k}$ for the trace map with respect to the finite extension K/k. Then one verifies immediately that this map Tr is [well-defined and] a *k-linear homomorphism*. Moreover, since [one verifies easily that] the restriction of the map Tr to each finite extension of k contained in $\overline{k}$ is *continuous* with respect to the respective p_k-adic topologies on the finite extension and on k_+, the map Tr is *continuous* with respect to the topology (b) on $\overline{k}_+$ and the p_k-adic topology on k_+.

Next, let us observe that it follows immediately from the well-known *surjectivity* of the trace map with respect to a finite extension of finite fields that,

for each positive integer n, there exists an element $a_n \in (\mathcal{O}_{K_n})_+$ of $(\mathcal{O}_{K_n})_+$ — where we write $K_n \subseteq \overline{k}$ for the [uniquely determined] *unramified* finite extension of k such that $d_{K_n} = p^n \cdot d_k$ — such that $\mathrm{Tr}_{K_n/k}(a_n) \in \mathcal{O}_k^\times$ $(\subseteq (\mathcal{O}_k)_+)$.

Then it is immediate that the sequence $(p^n \cdot a_n)_{n\geq 1}$ *converges* to $0 \in \overline{k}_+$ with respect to the topology (a) on $\overline{k}_+$. On the other hand, since $\mathrm{Tr}(p^n \cdot a_n) = p^{-n} \cdot d_k^{-1} \cdot \mathrm{Tr}_{K_n/k}(p^n \cdot a_n) = d_k^{-1} \cdot \mathrm{Tr}_{K_n/k}(a_n) \in d_k^{-1} \cdot \mathcal{O}_k^\times$, the sequence $(\mathrm{Tr}(p^n \cdot a_n))_{n\geq 1}$ does *not converge* to $\mathrm{Tr}(0) = 0 \in k_+$ with respect to the p_k-adic topology on k_+. In particular, the map Tr is *not continuous* with respect to the topology (a) on $\overline{k}_+$ and the p_k-adic topology on k_+. Thus, we conclude that the topology (a) on $\overline{k}_+$ does *not coincide* with the topology (b) on $\overline{k}_+$.

(ii) It follows from Summary 3.15 that one has a *functorial group-theoretic algorithm* for constructing, from a group G *of MLF-type*, the *topological* module $k_+(G)$, i.e., the module $k_+(G)$ equipped with the "*$p(G)$-adic topology*". Thus, it follows from the construction of $\overline{k}_+(G)$ that one has a *functorial group-theoretic algorithm* for constructing, from a group G *of MLF-type*, the module $\overline{k}_+(G)$ equipped with the *"injective limit-topology"* *determined by the "$p(G)$-adic topologies" of the various submodules* $k_+(H) = \overline{k}_+(G)^H$ of $\overline{k}_+(G)$ — where H ranges over the open subgroups $H \subseteq G$ of G. On the other hand, as already discussed in (i), the p_k-adic topology on $\overline{k}_+$ does *not coincide* with the *"injective limit-topology" determined by the p_k-adic topologies of the various submodules* $K_+ = (\overline{k}_+)^{G_K}$ of $\overline{k}_+$ — where K ranges over the finite extensions of k contained in $\overline{k}$.

$$G \not\rightsquigarrow \text{ the topology on } \overline{k}_+(G) \text{ which "corresponds" to the topology (a) of (i)}$$
— cf. Remark 4.3.1, (i)

$$G \rightsquigarrow \text{ the topology on } \overline{k}_+(G) \text{ which "corresponds" to the topology (b) of (i)}$$
— cf. Summary 3.15; Definition 4.1, (i)

(iii) The consideration of the *continuity* of the map Tr of (i) is in fact important in the study of *p-adic Hodge theory*. The discussion of (i) asserts that the restriction of the map Tr to the [uniquely determined] *unramified* $\mathbb{Z}_{p_k}$-extension of k is *not continuous* with respect to the respective p_k-adic topologies. On the other hand, in [12], it was proved that the restriction of the map Tr to an arbitrary *totally ramified* $\mathbb{Z}_{p_k}$-extension of k is *continuous* with respect to the respective p_k-adic topologies [cf. the discussion preceding [12], Proposition 7]. Moreover, this *continuity* is one of the crucial ingredients of the computation of an important continuous group cohomology "$H^i(G_k, \widehat{\overline{k}}_+(j))$" in the *p-adic Hodge theory* established in [12].

REMARK 4.3.3. — By Theorem 2.2; Summary 3.15; Summary 4.3, one may conclude that a group of MLF-type may not be directly related to a sort of *ring structure* but may be related to *one of underlying additive and underlying multiplicative structures* of a ring structure. Put another way, from the point of view of the terminology of [10], a group of MLF-type may/should be regarded as a *mono-analytic* object [cf. [10], §2.7, (vii)] [i.e., as opposed to an *arithmetic holomorphic* object — cf. [10], §2.7, (vii)]. Moreover, the passage

- from the Galois group $G_k \subseteq \mathrm{Aut}(\overline{k})$ which is, by definition, related to the ring structure of the field $\overline{k}$
- to an *abstract* group isomorphic to G_k [i.e., as opposed to a situation in which one regards the group as a group related to the ring structure under consideration as in the case of G_k]

may be regarded as an analogue of the passage

- from the field $\overline{k}$ equipped with the natural action of G_k
- to an *abstract* monoid isomorphic to either the underlying additive module $\overline{k}_+$ or the underlying multiplicative monoid $\overline{k}_\times$ [i.e., as opposed to a situation in which one regards the monoid as a monoid arising from the field $\overline{k}$] equipped with an action of the abstract group isomorphic to G_k.

5. MLF-PAIRS

One important aspect of mono-anabelian geometry is the technique of *mono-anabelian transport* [cf. [10], §2]. In order to explain mono-anabelian transport, in the present §5, we introduce the notion of an *MLF-pair* [cf. Definition 5.3 below].

DEFINITION 5.1.

(i) We shall refer to a collection of data

$$G \curvearrowright M$$

consisting of a monoid M, a group G, and an action of G on M as a *group-monoid-pair.*

(ii) Let $G_\circ \curvearrowright M_\circ$, $G_\bullet \curvearrowright M_\bullet$ be group-monoid-pairs. Then we shall refer to a pair $\alpha = (\alpha_G, \alpha_M)$ consisting of isomorphisms $\alpha_G \colon G_\circ \xrightarrow{\sim} G_\bullet$, $\alpha_M \colon M_\circ \xrightarrow{\sim} M_\bullet$ compatible with the respective actions of $G_\circ$, $G_\bullet$ on $M_\circ$, $M_\bullet$ as an *isomorphism* from $G_\circ \curvearrowright M_\circ$ to $G_\bullet \curvearrowright M_\bullet$.

DEFINITION 5.2. — Let k be an MLF and $\overline{k}$ an algebraic closure of k. Then we shall refer to the group-monoid-pair

$$G_k \curvearrowright \overline{k}^\times \quad (\text{respectively, } G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\rhd}; \quad G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\times})$$

consisting of the monoid $\overline{k}^\times$ (respectively, $\mathcal{O}_{\overline{k}}^{\rhd}$; $\mathcal{O}_{\overline{k}}^{\times}$), the group $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$, and the natural action of G_k on $\overline{k}^\times$ (respectively, $\mathcal{O}_{\overline{k}}^{\rhd}$; $\mathcal{O}_{\overline{k}}^{\times}$) as the *model MLF$^{\diamond}$-pair* (respectively, *model MLF$^{\rhd}$-pair*; *model MLF$^{\times}$-pair*) [associated to $\overline{k}/k$].

DEFINITION 5.3. — We shall refer to an isomorph, as a group-monoid-pair [i.e., in the sense of Definition 5.1, (ii)], of a model MLF$^{\diamond}$-pair (respectively, model MLF$^{\rhd}$-pair; model MLF$^{\times}$-pair) as an *MLF$^{\diamond}$-pair* (respectively, *MLF$^{\rhd}$-pair*; *MLF$^{\times}$-pair*).

REMARK 5.3.1. — The notion of an MLF$^{\diamond}$-pair (respectively, MLF$^{\rhd}$-pair; MLF$^{\times}$-pair) in the sense of Definition 5.3 is the same as the notion of an MLF-Galois $\mathbb{TLG}$-pair of mono-analytic type (respectively, MLF-Galois $\mathbb{TM}$-pair of mono-analytic type; MLF-Galois $\mathbb{TCG}$-pair of mono-analytic type) in the sense of [9], Definition 3.1, (ii) [cf. also Remark 5.7.2 below].

In the remainder of the present §5, let

$$\square \in \{\diamond, \rhd, \times\}$$

and

$$G \curvearrowright M$$

an MLF$^{\square}$-pair.

DEFINITION 5.4. — We shall refer to the group G (respectively, monoid M) as the *étale-like portion* [cf. [10], §2.7, (iii)] (respectively, *Frobenius-like portion* [cf. [10], §2.7, (ii)]) of the MLF$^{\square}$-pair $G \curvearrowright M$.

REMARK 5.4.1. — It follows from the various definitions involved that the étale-like portion of an MLF$^{\square}$-pair is a group *of MLF-type*.

DEFINITION 5.5. — We shall write

$$\mathrm{Ind}^{\square} \stackrel{\mathrm{def}}{=} \begin{cases} \{\pm 1\} & \text{if } \square = \diamond \\ \{1\} & \text{if } \square = \triangleright \\ \widehat{\mathbb{Z}}^{\times} & \text{if } \square = \times. \end{cases}$$

Let us regard $\mathrm{Ind}^{\square}$ as a subgroup of $\widehat{\mathbb{Z}}^{\times}$ [in an evident way].

REMARK 5.5.1. — Let us observe that, for each element of $\mathrm{Ind}^{\square}$, the identity automorphism of G and the automorphism of $M^{\square}$ given by multiplication by the element of $\mathrm{Ind}^{\square}$ [i.e., the automorphism given by "$x \mapsto x^{\lambda}$" for the fixed "λ" $\in \mathrm{Ind}^{\square} \subseteq \widehat{\mathbb{Z}}^{\times}$] form an *automorphism* of the MLF$^{\square}$-pair $G \curvearrowright M$. In particular, we have an *injective* homomorphism

$$\mathrm{Ind}^{\square} \hookrightarrow \mathrm{Aut}_G(M) \quad \big(\subseteq \mathrm{Aut}(G \curvearrowright M)\big).$$

LEMMA 5.6. — *Suppose that $\square = \diamond$. Then the following hold:*

(i) *For each open subgroup $H \subseteq G$ of G, there exist infinitely many prime numbers l such that the pro-l completion of the submodule $M^H \subseteq M$ of H-invariants of M is, as an abstract pro-l group,* **isomorphic** *to $\mathbb{Z}_l$.*

(ii) *Write*

$$M(\times) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G} J(H, l_H) \subseteq M$$

— where the injective limit is taken over the open subgroups $H \subseteq G$ of G, l_H is a prime number as in (i) *[for the open subgroup H], and $J(H, l_H) \subseteq M^H$ for the submodule of M^H obtained by forming the kernel of the natural homomorphism from M^H to the pro-l_H completion of M^H [which is, as an abstract pro-l_H group, isomorphic to $\mathbb{Z}_{l_H}$]. Then the submodule $M(\times) \subseteq M$ does* **not depend** *on the choices of the various "l_H" and is* ***G*-stable**.

PROOF. — Let us observe that it follows from the various definitions involved that, for each open subgroup $H \subseteq G$ of G, the module M^H is, as an abstract module, *isomorphic* to "$k^{\times}$" for some MLF. Thus, these assertions follow immediately from Lemma 1.2, (i). $\square$

PROPOSITION 5.7. — *The following hold:*

(i) *Suppose that $\square = \triangleright$. Then the group-monoid-pair $G \curvearrowright M^{\mathrm{gp}}$ obtained by replacing M in $G \curvearrowright M$ by M^{gp} is an* **MLF$^{\diamond}$-pair**.

(ii) *Suppose that* $\Box = \diamond$. *Then the group-monoid-pair* $G \curvearrowright M(\times)$ *obtained by replacing* M *in* $G \curvearrowright M$ *by* $M(\times)$ *of Lemma* 5.6, (ii), *is an* **MLF$^{\times}$-pair**.

PROOF. — Assertion (i) follows from the [easily verified] fact that the natural inclusion $\mathcal{O}^{\triangleright}_{\overline{k}} \hookrightarrow \overline{k}^{\times}$ determines an *isomorphism* $(\mathcal{O}^{\triangleright}_{\overline{k}})^{\mathrm{gp}} \xrightarrow{\sim} \overline{k}^{\times}$. Assertion (ii) follows immediately — in light of Lemma 1.2, (i) — from the definition of $M(\times)$. $\Box$

REMARK 5.7.1. — The content of Proposition 5.7 may be summarized as follows:

$$\begin{array}{ccccc} \text{an MLF}^{\triangleright}\text{-pair} & \Longrightarrow & \text{an MLF}^{\diamond}\text{-pair} & \Longrightarrow & \text{an MLF}^{\times}\text{-pair} \\ G \curvearrowright M & \rightsquigarrow & G \curvearrowright M^{\mathrm{gp}} & & \\ & & G \curvearrowright M & \rightsquigarrow & G \curvearrowright M(\times). \end{array}$$

REMARK 5.7.2.

(i) Suppose that $\Box = \times$. Then it follows from the various definitions involved that, for each open subgroup $H \subseteq G$ of G, the module M^H "corresponds" to the module "$\mathcal{O}^{\times}$" for some MLF. Thus, by considering the submodules of M^H *of finite index*, one obtains a topology on M^H which "*corresponds*" to the p-adic topology on "$\mathcal{O}^{\times}$". In particular, one may regard M as an object of the category "$\mathbb{TCG}$" of [9], Definition 3.1, (i).

(ii) Suppose that $\Box = \diamond$ (respectively, $\Box = \triangleright$). Then it follows from the various definitions involved that, for each open subgroup $H \subseteq G$ of G, the submonoid $M^H \subseteq M$ of H-invariants of M *contains* the submodule $M^{\mathrm{gp}}(\times)^H$ of H-invariants of $M^{\mathrm{gp}}(\times)$ [cf. Proposition 5.7, (i), (ii)]; moreover, the pair of monoids $M^{\mathrm{gp}}(\times)^H \subseteq M^H$ "*corresponds*" to the pair of monoids "$\mathcal{O}^{\times} \subseteq k^{\times}$" (respectively, "$\mathcal{O}^{\times} \subseteq \mathcal{O}^{\triangleright}$") for some MLF. Thus, by considering the topology on M^H induced by the topology of $M^{\mathrm{gp}}(\times)^H$ discussed in (i) [cf. Proposition 5.7, (i), (ii)], one obtains a topology on M^H which "*corresponds*" to the p-adic topology on "$k^{\times}$" (respectively, "$\mathcal{O}^{\triangleright}$"). In particular, one may regard M as an object of the category "$\mathbb{TLG}$" (respectively, "$\mathbb{TM}$") of [9], Definition 3.1, (i).

DEFINITION 5.8. — We shall write

$$M^{\Box}(G) \stackrel{\text{def}}{=} \begin{cases} \overline{k}^{\times}(G) & \text{if } \Box = \diamond \\ \overline{\mathcal{O}}^{\triangleright}(G) & \text{if } \Box = \triangleright \\ \overline{\mathcal{O}}^{\times}(G) & \text{if } \Box = \times \end{cases}$$

[cf. Definition 4.1, (i); Remark 5.4.1]. Thus, the group-monoid-pair

$$G \curvearrowright M^{\Box}(G)$$

is an MLF$^{\Box}$-pair.

DEFINITION 5.9.

(i) We shall refer to the group-monoid-pair

$$G \curvearrowright \Lambda(G)$$

[cf. Definition 4.1, (iii); Remark 5.4.1], i.e., obtained by considering the cyclotome associated to G, as the *étale-like cyclotome* associated to $G \curvearrowright M$.

(ii) We shall write

$$\Lambda(M) \stackrel{\mathrm{def}}{=} \varprojlim_{n} M^{\times}[n] = \varprojlim_{n} M^{\mathrm{gp}}[n]$$

— where the projective limis are taken over the positive integers n — and refer to the group-monoid-pair

$$G \curvearrowright \Lambda(M)$$

obtained by considering the action of G on $\Lambda(M)$ induced by the action of G on M as the *Frobenius-like cyclotome* associated to $G \curvearrowright M$. We regard the cyclotome $\Lambda(M)$ as a profinite, hence also topological, G-module [by the easily verified finiteness of $M^{\times}[n]$ for each n].

REMARK 5.9.1. — One verifies easily from the various definitions involved that we have a natural identification of the étale-like cyclotome $G \curvearrowright \Lambda(G)$ with the Frobenius-like cyclotome $G \curvearrowright \Lambda(M^{\square}(G))$ associated to the MLF$^{\square}$-pair $G \curvearrowright M^{\square}(G)$ constructed from G in Definition 5.8.

PROPOSITION 5.10. — *Let k be an MLF and $\overline{k}$ an algebraic closure of k. Suppose that the MLF$^{\square}$-pair $G \curvearrowright M$ is the* **model** *MLF$^{\square}$-pair associated to $\overline{k}/k$ [which thus implies that $G = G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ and $M \subseteq \overline{k}^{\times}$]. Then the following hold:*

(i) *The isomorphism $\Lambda(\overline{k}) \xrightarrow{\sim} \Lambda(G)$ of topological G-modules of Proposition* 4.2, (iv), *induces an* **isomorphism** *of the* **étale-like cyclotome**

$$G = G_k \curvearrowright \Lambda(G)$$

with the group-monoid-pair

$$G_k \curvearrowright \Lambda(\overline{k})$$

obtained by considering the **cyclotome** *associated to $\overline{k}$.*

(ii) *The natural inclusion $\boldsymbol{\mu}(\overline{k}) \hookrightarrow M$ induces an* **isomorphism** *of the* **Frobenius-like cyclotome**

$$G = G_k \curvearrowright \Lambda(M)$$

with the group-monoid-pair

$$G_k \curvearrowright \Lambda(\overline{k})$$

obtained by considering the **cyclotome** *associated to $\overline{k}$.*

PROOF. — These assertions are immediate. □

6. CYCLOTOMIC SYNCHRONIZATION FOR MLF-PAIRS

Let us recall that a *cyclotome* refers to an "isomorph of the object $\widehat{\mathbb{Z}}(1)$". Various [a priori independent] cyclotomes often appear in studies of arithmetic geometry. For instance, if Ω is an algebraically closed field of characteristic zero, then each of

- the cyclotome $\Lambda(\Omega)$ associated to Ω,
- the étale fundamental group

$$\pi_1^{\text{ét}}\Big(\text{Spec}\big(\Omega((t))\big)\Big)$$

of the spectrum of the field $\Omega((t))$ of fractions of the ring of formal power series with coefficients in Ω, and

- the dual

$$\Lambda(C) \stackrel{\text{def}}{=} \text{Hom}_{\widehat{\mathbb{Z}}}\big(H^2_{\text{ét}}(C, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}}\big)$$

over $\widehat{\mathbb{Z}}$ of the second étale cohomology $H^2_{\text{ét}}(C, \widehat{\mathbb{Z}})$ of a projective smooth curve C over Ω gives an example of a cyclotome. In our case, we constructed, from a *single* MLF$^{\square}$-pair $G \curvearrowright M$ [where $\square \in \{\diamond, \triangleright, \times\}$], [a priori independent] *two* cyclotomes, i.e.,

- the *étale-like cyclotome* $\Lambda(G)$ and
- the *Frobenius-like cyclotome* $\Lambda(M)$

[cf. Definition 5.9, Proposition 5.10].

If one works with certain *arithmetic holomorphic structures* [cf. [10], §2.7, (vii)] related to cyclotomes under consideration, then one may obtain a phenomenon of *cyclotomic synchronization*, i.e., synchronization of cyclotomes. For instance, in the case of the above examples $\Lambda(\Omega)$ and $\Lambda(C)$, the homomorphism $\text{Pic}(C) \to H^2(C, \Lambda(\Omega))$ obtained by considering the *first Chern classes* yields an isomorphism

$$\big(\text{Pic}(C)/\text{Pic}^0(C)\big) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \stackrel{\sim}{\longrightarrow} \text{Hom}_{\widehat{\mathbb{Z}}}\big(\Lambda(C), \Lambda(\Omega)\big);$$

thus, an invertible sheaf on C *of degree one* determines a *cyclotomic synchronization* [i.e., an isomorphism between cyclotomes] $\Lambda(C) \stackrel{\sim}{\to} \Lambda(\Omega)$ by means of which one usually [i.e., in the usual point of view of arithmetic geometry] *identifies* $\Lambda(C)$ with $\Lambda(\Omega)$. In our case, if our MLF$^{\square}$-pair $G \curvearrowright M$ is a *model MLF$^{\square}$-pair* [i.e., an MLF$^{\square}$-pair directly related to the ring structure of an MLF], then, by means of the *arithmetic holomorphic structures* related to the objects involved [more concretely, essentially, by means of *local class field theory*], one may construct a natural isomorphism between the Frobenius-like cyclotome $\Lambda(M)$ and the étale-like cyclotome $\Lambda(G)$ [cf. Definition 6.2 below].

On the other hand, if one works with only *mono-analytic structures* [cf. [10], §2.7, (vii)], then the task of establishing a phenomenon of cyclotomic synchronization is *nontrivial.* In the present §6, we establish a cyclotomic synchronization that one may apply to an arbitrary [i.e., a not necessarily model] MLF$^{\square}$-pair [cf. Definition 6.6, Proposition 6.7 below].

In the present §6, let

$$\square \in \{\diamond, \triangleright, \times\}$$

and

$$G \curvearrowright M$$

an MLF$^{\Box}$-pair.

PROPOSITION 6.1. — *The set*

$$\mathrm{Isom}_G\big(\Lambda(M), \Lambda(G)\big)$$

*of G-equivariant isomorphisms $\Lambda(M) \xrightarrow{\sim} \Lambda(G)$ [is nonempty and] forms a $\widehat{\mathbb{Z}}^{\times}$-**torsor**, i.e., relative to the natural action of $\widehat{\mathbb{Z}}^{\times} = \mathrm{Aut}(\Lambda(G))$ on $\Lambda(G)$.*

PROOF. — This follows immediately from Proposition 5.10, (i), (ii). $\Box$

DEFINITION 6.2. — Suppose that the MLF$^{\Box}$-pair $G \curvearrowright M$ is the model MLF$^{\Box}$-pair. Then, by considering the composite of the isomorphisms of Proposition 5.10, (i), (ii) [i.e, the isomorphisms "$\Lambda(G) \xrightarrow{\sim} \Lambda(\overline{k})$", "$\Lambda(M) \xrightarrow{\sim} \Lambda(\overline{k})$"], we obtain a G-equivariant isomorphism

$$\Lambda(M) \xrightarrow{\sim} \Lambda(G).$$

We shall refer to this composite $\Lambda(M) \xrightarrow{\sim} \Lambda(G)$ [that is an element of the $\widehat{\mathbb{Z}}^{\times}$-torsor discussed in Proposition 6.1] as the *holomorphic cyclotomic synchronization isomorphism* associated to the model MLF$^{\Box}$-pair $G \curvearrowright M$.

REMARK 6.2.1. — It follows from the various definitions involved that the *holomorphic cyclotomic synchronization isomorphism* of Definition 6.2 *depends* on the *ring structure* [i.e., the *arithmetic holomorphic structure* — cf. [10], §2.7, (vii)] of a given model MLF$^{\Box}$-pair. In particular, one *cannot apply* the construction of the holomorphic cyclotomic synchronization isomorphism of Definition 6.2 to an abstract [i.e., a not necessarily model] MLF$^{\Box}$-pair.

LEMMA 6.3. — *Suppose that $\Box \in \{\diamond, \rhd\}$. Write $M(\times) \stackrel{\mathrm{def}}{=} M^{\mathrm{gp}}(\times)$ [cf. Proposition 5.7, (i), (ii)]. [Thus, one verifies easily that $M(\times) \subseteq M \subseteq M^{\mathrm{gp}}$.] Write, moreover, $V(G \curvearrowright M) \stackrel{\mathrm{def}}{=} M^{I(G)}/M(\times)^{I(G)}$ [cf. Definition 3.5, (iii); Remark 5.4.1]. [Thus, one also verifies easily that there exists a natural identification $V(G \curvearrowright M)^{\mathrm{gp}} = (M^{\mathrm{gp}})^{I(G)}/M(\times)^{I(G)}$.] Then the following hold:*

(i) *The monoid $V(G \curvearrowright M)$ is, as an abstract monoid,* **isomorphic** *to*

$$\begin{cases} \mathbb{Z} & \text{if } \Box = \diamond \\ \mathbb{N} & \text{if } \Box = \rhd. \end{cases}$$

Moreover, the action of G on $V(G \curvearrowright M)$ induced by the action of G on M is **trivial**.

(ii) *For each positive integer n, the natural homomorphism*

$$H^2(G, M^{\times}[n]) \longrightarrow H^2(G, M^{\mathrm{gp}})$$

determines an **isomorphism**

$$H^2(G, M^\times[n]) \stackrel{\sim}{\longrightarrow} H^2(G, M^{\mathrm{gp}})[n].$$

(iii) *The natural homomorphism*

$$H^2\big(G/I(G), (M^{\mathrm{gp}})^{I(G)}\big) \longrightarrow H^2(G, M^{\mathrm{gp}})$$

is an **isomorphism***.*

(iv) *The natural homomorphism*

$$H^2\big(G/I(G), (M^{\mathrm{gp}})^{I(G)}\big) \longrightarrow H^2\big(G/I(G), V(G \curvearrowright M)^{\mathrm{gp}}\big)$$

is an **isomorphism***.*

(v) *The homomorphism*

$$H^1\Big(G/I(G), \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\mathrm{pf}}/V(G \curvearrowright M)^{\mathrm{gp}}\Big) \longrightarrow H^2\big(G/I(G), V(G \curvearrowright M)^{\mathrm{gp}}\big)$$

determined by the exact sequence of $G/I(G)$-modules

$$0 \longrightarrow V(G \curvearrowright M)^{\mathrm{gp}} \longrightarrow \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\mathrm{pf}} \longrightarrow \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\mathrm{pf}}/V(G \curvearrowright M)^{\mathrm{gp}} \longrightarrow 0$$

is an **isomorphism***.*

(vi) *The homomorphism*

$$H^1\Big(G/I(G), \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\mathrm{pf}}/V(G \curvearrowright M)^{\mathrm{gp}}\Big) \longrightarrow \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\mathrm{pf}}/V(G \curvearrowright M)^{\mathrm{gp}}$$

obtained by mapping

$$\begin{aligned}\chi \in\ & \mathrm{Hom}\Big(G/I(G), \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\mathrm{pf}}/V(G \curvearrowright M)^{\mathrm{gp}}\Big)\\ =\ & H^1\Big(G/I(G), \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\mathrm{pf}}/V(G \curvearrowright M)^{\mathrm{gp}}\Big)\end{aligned}$$

[*cf.* (i)] *to*

$$\chi\big(\mathrm{Frob}(G)\big) \in \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\mathrm{pf}}/V(G \curvearrowright M)^{\mathrm{gp}}$$

[*cf. Definition* 3.8*; Remark* 5.4.1] *is an* **isomorphism***.*

(vii) *The various isomorphisms of* (iii)*,* (iv)*,* (v)*,* (vi) *determine an* **isomorphism**

$$H^2(G, M^{\mathrm{gp}}) \stackrel{\sim}{\longrightarrow} \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\mathrm{pf}}/V(G \curvearrowright M)^{\mathrm{gp}}$$

which thus determines an **isomorphism**

$$H^2\big(G, \Lambda(M)\big) \stackrel{\sim}{\longrightarrow} \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^\wedge$$

[*cf.* (i)*,* (ii)]*.*

PROOF. — These assertions follow — in light of Proposition 3.6; Proposition 3.9 — from Lemma 1.6, (i), (ii), (iii), (iv), (v), (vi), (vii). □

DEFINITION 6.4. — Suppose that $\square \in \{\diamond, \triangleright\}$. Then we shall write

$$\mathrm{inv}^\wedge(G \curvearrowright M)\colon\ H^2\big(G, \Lambda(M)\big) \stackrel{\sim}{\longrightarrow} \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^\wedge$$

for the second displayed isomorphism of Lemma 6.3, (vii).

LEMMA 6.5. — *Suppose that* $\Box \in \{\diamond, \rhd\}$. *Recall the* $\widehat{\mathbb{Z}}^{\times}$**-torsor**

$$\mathrm{Isom}_G\big(\Lambda(M), \Lambda(G)\big)$$

discussed in Proposition 6.1. *Then the subset of* $\mathrm{Isom}_G(\Lambda(M), \Lambda(G))$ *consisting of isomorphisms* $\alpha \in \mathrm{Isom}_G(\Lambda(M), \Lambda(G))$ *which satisfy the following condition* [*is nonempty and*] *forms an* $\mathbf{Ind}^{\Box}$**-torsor**: *The composite*

$$\big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\wedge} \overset{\mathrm{inv}^{\wedge}(G \curvearrowright M)}{\overset{\sim}{\longleftarrow}} H^2\big(G, \Lambda(M)\big)$$

$$\overset{H^2(\mathrm{id}_G, \alpha)}{\overset{\sim}{\longrightarrow}} H^2\big(G, \Lambda(G)\big) \overset{\mathrm{inv}^{\wedge}(G \curvearrowright M^{\Box}(G))}{\overset{\sim}{\longrightarrow}} \Big(V\big(G \curvearrowright M^{\Box}(G)\big)^{\mathrm{gp}}\Big)^{\wedge}$$

[*cf. Remark* 5.9.1] — *where we write* $H^2(\mathrm{id}_G, \alpha)$ *for the isomorphism induced by* α — **maps** *the submonoid*

$$V(G \curvearrowright M) \subseteq \big(V(G \curvearrowright M)^{\mathrm{gp}}\big)^{\wedge}$$

[*cf. Lemma* 6.3, (i)] *to the submonoid*

$$V\big(G \curvearrowright M^{\Box}(G)\big) \subseteq \Big(V\big(G \curvearrowright M^{\Box}(G)\big)^{\mathrm{gp}}\Big)^{\wedge}.$$

PROOF. — Let us first observe that one verifies immediately from Proposition 5.10, (i), (ii), that the subset discussed in Lemma 6.5 is *nonempty*. Next, let us observe that, by Lemma 6.3, (i), there exists a natural isomorphism of $\mathrm{Aut}(V(G \curvearrowright M))$ with $\mathrm{Ind}^{\Box}$. Thus, it follows immediately from the definition of the displayed composite in the statement of Lemma 6.5 that the subset discussed in Lemma 6.5 forms an $\mathrm{Ind}^{\Box}$-*torsor*, as desired. $\Box$

DEFINITION 6.6. — If $\Box = \times$, then we shall write

$$\mathfrak{syn}_{\Lambda}(G \curvearrowright M) \overset{\mathrm{def}}{=} \mathrm{Isom}_G\big(\Lambda(M), \Lambda(G)\big).$$

If $\Box \in \{\diamond, \rhd\}$, then we shall write

$$\mathfrak{syn}_{\Lambda}(G \curvearrowright M) \subseteq \mathrm{Isom}_G\big(\Lambda(M), \Lambda(G)\big)$$

for the $\mathrm{Ind}^{\Box}$-torsor discussed in Lemma 6.5. [So, for each $\Box \in \{\diamond, \rhd, \times\}$, the set $\mathfrak{syn}_{\Lambda}(G \curvearrowright M)$ forms an $\mathrm{Ind}^{\Box}$-torsor — cf. also Proposition 6.1]. We shall refer to the $\mathrm{Ind}^{\Box}$-torsor $\mathfrak{syn}_{\Lambda}(G \curvearrowright M)$ as the *cyclotomic synchronization poly-isomorphism* associated to $G \curvearrowright M$.

PROPOSITION 6.7. — *Suppose that the* $MLF^{\Box}$*-pair* $G \curvearrowright M$ *is the* **model** $MLF^{\Box}$*-pair. Then the* $\mathrm{Ind}^{\Box}$*-torsor* $\mathfrak{syn}_{\Lambda}(G \curvearrowright M)$ **coincides** *with the* $\mathrm{Ind}^{\Box}$*-orbit of the* **holomorphic cyclotomic synchronization isomorphism** *associated to* $G \curvearrowright M$.

PROOF. — This assertion follows immediately from the definition of $\mathfrak{syn}_{\Lambda}(G \curvearrowright M)$. $\Box$

REMARK 6.7.1.

(i) Let us observe that one verifies easily from the definition of the injective homomorphism of Remark 5.5.1 that the composite

$$\mathrm{Ind}^{\square} \hookrightarrow \mathrm{Aut}(G \curvearrowright M) \to \mathrm{Aut}\big(G \curvearrowright \Lambda(M)\big) \to \mathrm{Aut}\big(\Lambda(M)\big) = \widehat{\mathbb{Z}}^{\times}$$

coincides with the *natural inclusion* $\mathrm{Ind}^{\square} \hookrightarrow \widehat{\mathbb{Z}}^{\times}$, but the image of the composite

$$\mathrm{Ind}^{\square} \hookrightarrow \mathrm{Aut}(G \curvearrowright M) \to \mathrm{Aut}(G) \to \mathrm{Aut}\big(\Lambda(G)\big) = \widehat{\mathbb{Z}}^{\times}$$

consists of $1 \in \widehat{\mathbb{Z}}^{\times}$.

(ii) Let

$$\mathfrak{s}^{\square}\colon\ G^{\dagger} \curvearrowright M^{\dagger} \ \rightsquigarrow\ \mathfrak{s}^{\square}(G^{\dagger} \curvearrowright M^{\dagger}) \subseteq \mathrm{Isom}_{G^{\dagger}}\big(\Lambda(M^{\dagger}), \Lambda(G^{\dagger})\big)$$

be a *functorial assignment* which assigns each MLF$^{\square}$-pair $G^{\dagger} \curvearrowright M^{\dagger}$ [for a fixed $\square \in \{\diamond, \rhd, \times\}$] to a subset of the $\widehat{\mathbb{Z}}^{\times}$-torsor $\mathrm{Isom}_{G^{\dagger}}(\Lambda(M^{\dagger}), \Lambda(G^{\dagger}))$. Then, by the observation of (i), one may verify easily that

> the subset $\mathfrak{s}^{\square}(G^{\dagger} \curvearrowright M^{\dagger}) \subseteq \mathrm{Isom}_{G^{\dagger}}(\Lambda(M^{\dagger}), \Lambda(G^{\dagger}))$ is *stable* by the action of $\mathrm{Ind}^{\square} \subseteq \widehat{\mathbb{Z}}^{\times}$.

(iii) Let us recall that the cyclotomic synchronization poly-isomorphism "$\mathfrak{syn}_{\Lambda}(G \curvearrowright M)$" [which forms — by allowing "$G \curvearrowright M$" to vary — a *functorial assignment* as discussed in (ii)]

> ($*$) *contains* the holomorphic cyclotomic synchronization isomorphism [i.e., if one considers the cyclotomic synchronization poly-isomorphism of a *model MLF$^{\square}$-pair*] [cf. Proposition 6.7].

By the observation of (ii), together with Proposition 6.7, one may conclude that the cyclotomic synchronization poly-isomorphism "$\mathfrak{syn}_{\Lambda}(G \curvearrowright M)$" [which forms — by allowing "$G \curvearrowright M$" to vary — a *functorial assignment* as discussed in (ii)] is "*minimal*" among functorial assignments as discussed in (ii) which satisfy the condition ($*$). Put another way, roughly speaking, the cyclotomic synchronization poly-isomorphism of Definition 6.6 may be considered to be "*best*" among functorial assignments as discussed in (ii).

7. MONO-ANABELIAN TRANSPORT FOR MLF-PAIRS

One important aspect of the technique of *mono-anabelian transport* is the notion of a *Kummer poly-isomorphism*. A *Kummer poly-isomorphism* often refers to a poly-isomorphism between [monoids constructed, via some functorial algorithms, from] *Frobenius-like portions* and *mono-anabelian étale-like monoids* [i.e., monoids constructed, via some mono-anabelian reconstruction algorithms, from the *étale-like portions*]. In the present §7, we establish Kummer poly-isomorphisms [cf. Definition 7.4 below] and discuss mono-anabelian transport for MLF-pairs [cf. Remark 7.6.1 below].

In the present §7, let

$$\square \in \{\diamond, \rhd, \times\}$$

and

$$G \curvearrowright M$$

an MLF$^{\square}$-pair.

LEMMA 7.1. — *The following hold:*

(i) *For each open subgroup $H \subseteq G$ of G, the homomorphism*

$$(M^H \subseteq) \quad (M^{\text{gp}})^H \longrightarrow H^1\big(H, \Lambda(M)\big)$$

determined by the exact sequences of H-modules

$$1 \longrightarrow M^{\text{gp}}[n] \longrightarrow M^{\text{gp}} \xrightarrow{n} M^{\text{gp}} \longrightarrow 1$$

— where n ranges over the positive integers — is **injective**.

(ii) *The homomorphism*

$$(M \subseteq) \quad M^{\text{gp}} \longrightarrow {}_{\infty}H^1\big(G, \Lambda(M)\big)$$

obtained by forming the injective limit of the homomorphisms of (i) *for the open subgroups $H \subseteq G$ of G is* **injective**.

PROOF. — First, we verify assertion (i). Let us first observe that, by definition, the kernel of the homomorphism discussed in assertion (i) coincides with the submodule of $(M^{\text{gp}})^H$ consisting of *divisible* elements of $(M^{\text{gp}})^H$. Next, let us observe that the module $(M^{\text{gp}})^H$ is, as an abstract module, *isomorphic* to "$k^\times$" (respectively, "$k^\times$"; "$\mathcal{O}^\times$") for some MLF if $\square = \diamond$ (respectively, $\square = \triangleright$; $\square = \times$). Thus, assertion (i) follows from Lemma 1.2, (i). This completes the proof of assertion (i). Assertion (ii) follows from assertion (i). $\square$

DEFINITION 7.2. — We shall write

$${}_{\infty}\text{Kmm}(G \curvearrowright M)\colon \ M \hookrightarrow {}_{\infty}H^1\big(G, \Lambda(M)\big)$$

for the injection [obtained by the injection] discussed in Lemma 7.1, (ii).

LEMMA 7.3. — *The* Ind$^{\square}$*-orbit of G-equivariant isomorphisms*

$${}_{\infty}H^1\big(G, \Lambda(M)\big) \xrightarrow{\sim} {}_{\infty}H^1\big(G, \Lambda(G)\big)$$

induced by the Ind$^{\square}$*-orbit of G-equivariant isomorphisms*

$$\mathfrak{syn}_{\Lambda}(G \curvearrowright M)\colon \ \Lambda(M) \xrightarrow{\sim} \Lambda(G)$$

restricts *— relative to the injections ${}_{\infty}\text{Kmm}(G \curvearrowright M)$, ${}_{\infty}\text{Kmm}(G \curvearrowright M^{\square}(G))$ of Definition* 7.2 [*cf. also Remark* 5.9.1] *— to an* **Ind$^{\square}$-orbit of G-equivariant isomorphisms**

$$M \xrightarrow{\sim} M^{\square}(G).$$

PROOF. — This assertion follows immediately from Proposition 6.7. $\square$

DEFINITION 7.4. — We shall write

$$\kappa(G \curvearrowright M)\colon\ (G \curvearrowright M) \ \xrightarrow{\sim}\ \big(G \curvearrowright M^{\square}(G)\big)$$

for the $\mathrm{Ind}^{\square}$-orbit of G-equivariant isomorphisms discussed in Lemma 7.3 and refer to $\kappa(G \curvearrowright M)$ as the *Kummer poly-isomorphism* associated to $G \curvearrowright M$.

Some of the various assertions discussed in §5, §6, and §7 may be summarized as follows.

SUMMARY 7.5. — *One may construct, from an MLF*$^{\square}$*-pair* $G \curvearrowright M$,

- *the "ëtale-like" MLF*$^{\square}$*-pair* $G \curvearrowright M^{\square}(G)$,
- *the étale-like cyclotome* $G \curvearrowright \Lambda(G)$,
- *the Frobenius-like cyclotome* $G \curvearrowright \Lambda(M)$,
- *the cyclotomic synchronization poly-isomorphism* $\mathfrak{syn}_{\Lambda}(G \curvearrowright M)\colon \Lambda(M) \xrightarrow{\sim} \Lambda(G)$ [*i.e., a certain poly-isomorphism which forms an* $\mathrm{Ind}^{\square}$*-orbit of* G*-equivariant isomorphisms* $\Lambda(M) \xrightarrow{\sim} \Lambda(G)$], *and*
- *the Kummer poly-isomorphism* $\kappa(G \curvearrowright M)\colon M \xrightarrow{\sim} M^{\square}(G)$ [*i.e., a certain poly-isomorphism which forms an* $\mathrm{Ind}^{\square}$*-orbit of* G*-equivariant isomorphisms* $M \xrightarrow{\sim} M^{\square}(G)$].

THEOREM 7.6. — *Let* $\square \in \{\diamond, \triangleright, \times\}$*;* $G_{\circ} \curvearrowright M_{\circ}$, $G_{\bullet} \curvearrowright M_{\bullet}$ *MLF*$^{\square}$*-pairs. Thus, we have a natural map*

$$F\colon\ \mathrm{Isom}(G_{\circ} \curvearrowright M_{\circ}, G_{\bullet} \curvearrowright M_{\bullet}) \ \longrightarrow\ \mathrm{Isom}(G_{\circ}, G_{\bullet}).$$

Moreover, by considering the action of $\mathrm{Ind}^{\square}$ *on* $M_{\bullet}$ [*cf. Remark* 5.5.1], *we have a natural action of* $\mathrm{Ind}^{\square}$ *on the set* $\mathrm{Isom}(G_{\circ} \curvearrowright M_{\circ}, G_{\bullet} \curvearrowright M_{\bullet})$ *over* $\mathrm{Isom}(G_{\circ}, G_{\bullet})$ *relative to the map* F. *Then the following hold:*

(i) *The map* F *is* **surjective**.

(ii) *Every fiber of the map* F *forms an* $\mathbf{Ind}^{\square}$**-torsor**.

(iii) *If* $\square = \diamond$, *then every fiber of the map* F *is* **of cardinality two**.

(iv) *If* $\square = \triangleright$, *then the map* F *is* **bijective**.

PROOF. — First, we verify assertion (i). Let $\alpha\colon G_{\circ} \xrightarrow{\sim} G_{\bullet}$ be an isomorphism [i.e., an element of the codomain of F]. Then one verifies easily from the *functoriality* of the mono-anabelian reconstruction algorithm of Definition 4.1, (i), that α induces an isomorphism between the "étale-like" MLF$^{\square}$-pairs

$$\big(G_{\circ} \curvearrowright M^{\square}(G_{\circ})\big) \ \overset{\left(\alpha, M^{\square}(\alpha)\right)}{\xrightarrow{\sim}}\ \big(G_{\bullet} \curvearrowright M^{\square}(G_{\bullet})\big).$$

Thus, by considering respective elements $\iota_{\circ}$, $\iota_{\bullet}$ of the Kummer poly-isomorphisms $\kappa(G_{\circ} \curvearrowright M_{\circ})$, $\kappa(G_{\bullet} \curvearrowright M_{\bullet})$, we obtain isomorphisms

$$(G_{\circ} \curvearrowright M_{\circ}) \ \overset{\iota_{\circ}}{\xrightarrow{\sim}}\ \big(G_{\circ} \curvearrowright M^{\square}(G_{\circ})\big)$$

$$\xrightarrow[\sim]{\left(\alpha, M^{\square}(\alpha)\right)} \quad \big(G_{\bullet} \curvearrowright M^{\square}(G_{\bullet})\big) \ \overset{\iota_{\bullet}}{\underset{\sim}{\longleftarrow}} \ (G_{\bullet} \curvearrowright M_{\bullet})$$

such that the image of the composite $(G_{\circ} \curvearrowright M_{\circ}) \xrightarrow{\sim} (G_{\bullet} \curvearrowright M_{\bullet})$ of these isomorphisms via F *coincides* with the original isomorphism α. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that, to verify assertion (ii), by considering the difference of two elements of the domain of F whose images via F coincide, it suffices to verify the following assertion:

> Let (α_G, α_M) be an automorphism of the MLF$^{\square}$-pair $G \curvearrowright M$. Suppose that $\alpha_G = \mathrm{id}_G$. Then it holds that $\alpha_M \in \mathrm{Ind}^{\square}$ $(\subseteq \mathrm{Aut}_G(M))$.

To this end, let us observe that one verifies easily from the *functoriality* of the Kummer poly-isomorphism of Definition 7.4 that the Kummer poly-isomorphism associated to $G \curvearrowright M$

$$\kappa(G \curvearrowright M)\colon\ (G \curvearrowright M) \ \xrightarrow{\sim}\ \big(G \curvearrowright M^{\square}(G)\big)$$

is *compatible* with the automorphism (α_G, α_M), i.e., the diagram of *poly-isomorphisms*

$$\begin{array}{ccc}
(G \curvearrowright M) & \xrightarrow{\kappa(G \curvearrowright M)} & \big(G \curvearrowright M^{\square}(G)\big) \\
{\scriptstyle (\alpha_G, \alpha_M)}\big\downarrow & & \big(\alpha_G, M^{\square}(\alpha_G)\big)\big\downarrow \\
(G \curvearrowright M) & \xrightarrow{\kappa(G \curvearrowright M)} & \big(G \curvearrowright M^{\square}(G)\big)
\end{array}$$

commutes. Next, let us observe that since [we have assumed that] $\alpha_G = \mathrm{id}_G$, the right-hand vertical arrow of this diagram is the [poly-isomorphism consisting of the] *identity automorphism*. Thus, it holds that $(\alpha_G, \alpha_M) \in \kappa(G \curvearrowright M)^{-1} \circ \kappa(G \curvearrowright M)$, which thus implies that $\alpha_M \in \mathrm{Ind}^{\square}$ $(\subseteq \mathrm{Aut}_G(M))$, as desired. This completes the proof of assertion (ii).

Assertions (iii), (iv) follow from assertions (i), (ii), together with the [easily verified] fact that $\sharp\mathrm{Ind}^{\diamond} = 2$ and $\sharp\mathrm{Ind}^{\triangleright} = 1$. $\square$

REMARK 7.6.1. — The content of Theorem 7.6, as well as the proof of Theorem 7.6, gives some examples of the technique of *mono-anabelian transport*.

(i) In order to explain the technique of *mono-anabelian transport* from the point of view of Theorem 7.6, (i), let us recall the proof of Theorem 7.6, (i), i.e., the *surjectivity* of the map F of Theorem 7.6, as follows:

Theorem 7.6, (i), asserts that, roughly speaking, for two MLF$^{\square}$-pairs, an "étale-like link" between the MLF$^{\square}$-pairs [i.e., an isomorphism between the *étale-like portions* of the MLF$^{\square}$-pairs] induces a "Frobenius-like link" between the MLF$^{\square}$-pairs [i.e., an isomorphism between the *Frobenius-like portions* of the MLF$^{\square}$-pairs] compatible with the given "étale-like link".

$$\begin{array}{ccc}
\text{étale-like portion} & \xrightarrow{\sim} & \text{étale-like portion} \\
\curvearrowright & ?\Big\Downarrow & \curvearrowright \\
\text{Frobenius-like portion} & & \text{Frobenius-like portion}
\end{array}$$

Suppose that we are in a situation described by the following diagram to recall the proof of Theorem 7.6, (i):

$$\begin{array}{ccc} \text{étale-like portion} & \xrightarrow{\sim} & \text{étale-like portion} \\ \curvearrowright & & \curvearrowright \\ \text{Frobenius-like portion} & & \text{Frobenius-like portion.} \end{array}$$

In order to obtain a "Frobenius-like link" from our "étale-like link", let us first apply the *mono-anabelian reconstruction algorithm* discussed in Summary 4.3 to each of the étale-like portions to construct a mono-anabelian étale-like monoid [i.e., the "étale-like copy $M^{\square}(G)$" — cf. Definition 5.8 — of the Frobenius-like portion "M"].

$$\text{étale-like portion} \underset{\text{reconstruction algorithm}}{\overset{\text{mono-anabelian}}{\rightsquigarrow}} \text{mono-anabelian étale-like monoid}$$

Here, let us recall that we do *not require* the "étale-link" to be *compatible* with any sort of ring structures. On the other hand, observe that the mono-anabelian reconstruction algorithms discussed in §3 and §4 may be applied *without* existence of some "fixed reference model" [as $\overline{k}_\circ/k_\circ$ and $\overline{k}_\bullet/k_\bullet$ for $G_\circ = \mathrm{Gal}(\overline{k}_\circ/k_\circ)$ and $G_\bullet = \mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$ in the case of bi-anabelian geometry discussed in §2]. Put another way, the mono-anabelian reconstruction algorithms discussed in §3 and §4 have the virtue of being *free* of any mention of some "fixed reference model" copy of ring-theoretic objects. In particular, by the mono-anabelian property of our algorithm, we obtain, from the given "étale-like link", an isomorphism between the mono-anabelian étale-like monoids *compatible* with the given "étale-like link".

$$\begin{array}{ccc} \text{étale-like portion} & \xrightarrow{\sim} & \text{étale-like portion} \\ \curvearrowright & \text{mono-anab.} \Downarrow \text{property} & \curvearrowright \\ \text{mono-anabelian étale-like monoid} & \xrightarrow{\sim} & \text{mono-anabelian étale-like monoid} \end{array}$$

Next, in order to relate the Frobenius-like portions to the mono-anabelian étale-like monoids, let us discuss *cyclotomic synchronization poly-isomorphisms*, which induce *Kummer poly-isomorphisms.* Let us recall that, as in the discussion of §6, we can construct a *cyclotomic synchronization poly-isomorphism* "$\mathfrak{syn}_\Lambda$" [cf. Definition 6.6] [i.e., a poly-isomorphism between the Frobenius-like cyclotome and the étale-like cyclotome].

$$\text{Frobenius-like cyclotome} \underset{\text{synchronization}}{\overset{\text{cyclotomic}}{\xrightarrow{\sim}}} \text{étale-like cyclotome}$$

Thus, by applying a sort of the *Kummer theory* [i.e., by considering the injection "${}_\infty\mathrm{Kmm}$" — cf. Definition 7.2], we obtain a *Kummer poly-isomorphism* "κ" [cf. Definition 7.4] [i.e., a poly-isomorphism between the Frobenius-like portion and the mono-anabelian étale-like monoid] from the above cyclotomic synchronization poly-isomorphism.

$$\text{Frobenius-like cyclotome} \underset{\text{synchronization}}{\overset{\text{cyclotomic}}{\xrightarrow{\sim}}} \text{étale-like cyclotome}$$

$$\overset{\text{Kummer theory}}{\rightsquigarrow} \quad \text{Frobenius-like portion} \quad \underset{\text{poly-isomorphism}}{\overset{\text{Kummer}}{\xrightarrow{\sim}}} \quad \text{mono-anabelian étale-like monoid}$$

Thus, we are in a situation described by the following diagram:

$$\begin{array}{ccc}
\text{étale-like portion} & \xrightarrow{\sim} & \text{étale-like portion} \\
 & \text{mono-anab.} \Downarrow \text{property} & \\
\text{mono-anabelian étale-like monoid} & \xrightarrow{\sim} & \text{mono-anabelian étale-like monoid} \\
\text{Kummer poly-} \uparrow \text{isomorphism} & & \text{Kummer poly-} \uparrow \text{isomorphism} \\
\text{Frobenius-like portion} & & \text{Frobenius-like portion.}
\end{array}$$

In particular, by considering the composite of the two lower vertical arrows and the middle horizontal arrow of this diagram, we obtain a "Frobenius-like link" *compatible* with the given "étale-like link".

Thus, in summary, by means of

- *Kummer-detachment* [cf. [10], §2.7, (vi)], i.e., the passage, via Kummer poly-isomorphisms, from Frobenius-like structures to corresponding étale-like structures, and

- *étale-transport*, i.e., the passage, via the mono-anabelian property of mono-anabelian reconstruction algorithms, from the "left-hand side" to the "right-hand side",

one may *transport*, via "étale-like links", Frobenius-like portions from the "left-hand side" to the "right-hand side".

$$\begin{array}{ccc}
\text{étale-like portion} & \xrightarrow{\sim} & \text{étale-like portion} \\
\curvearrowright & \Downarrow & \curvearrowright \\
\text{Frobenius-like portion} & \xrightarrow{\sim} & \text{Frobenius-like portion}
\end{array}$$

(ii) In order to explain the technique of *mono-anabelian transport* from the point of view of Theorem 7.6, (ii), let us recall the proof of Theorem 7.6, (ii), as follows:

Theorem 7.6, (ii), asserts that, roughly speaking, for an isomorphism between $\mathrm{MLF}^{\square}$-pairs, one may compute the effect on the Frobenius-like portions of the "Frobenius-like link" [cf. (i)] from the point of view of the effect on the étale-like portions of the "étale-like link" [cf. (i)]. Suppose that we are in a situation described by the following diagram to recall the proof of Theorem 7.6, (ii):

$$\begin{array}{ccc}
\text{étale-like portion} & \xrightarrow{\sim} & \text{étale-like portion} \\
\curvearrowright & & \curvearrowright \\
\text{Frobenius-like portion} & \xrightarrow{\sim} & \text{Frobenius-like portion.}
\end{array}$$

In order to compute the effect of the "Frobenius-like link" from the point of view of the effect of the "étale-like link", let us first apply the *mono-anabelian reconstruction* *algorithm* discussed in Summary 4.3 to each of the étale-like portions to construct a mono-anabelian étale-like monoid [cf. (i)].

$$\text{étale-like portion} \underset{\text{reconstruction algorithm}}{\overset{\text{mono-anabelian}}{\rightsquigarrow}} \text{mono-anabelian étale-like monoid}$$

Here, let us recall that we do *not require* the "links" to be *compatible* with any sort of ring structures. On the other hand, observe that the mono-anabelian reconstruction algorithms discussed in §3 and §4 may be applied *without* existence of some "fixed reference model" [as $\overline{k}_\circ/k_\circ$ and $\overline{k}_\bullet/k_\bullet$ for $G_\circ = \mathrm{Gal}(\overline{k}_\circ/k_\circ)$ and $G_\bullet = \mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$ in the case of bi-anabelian geometry discussed in §2]. Put another way, the mono-anabelian reconstruction algorithms discussed in §3 and §4 have the virtue of being *free* of any mention of some "fixed reference model" copy of ring-theoretic objects. In particular, by the mono-anabelian property of our algorithm, we obtain, from the "étale-like link", an isomorphism between the mono-anabelian étale-like monoids *compatible* with the "étale-like link".

$$\begin{array}{ccc}
\text{étale-like portion} & \xrightarrow{\sim} & \text{étale-like portion} \\
\curvearrowright & \text{mono-anab.} \Big\Downarrow \text{property} & \curvearrowright \\
\text{mono-anabelian étale-like monoid} & \xrightarrow{\sim} & \text{mono-anabelian étale-like monoid}
\end{array}$$

Next, in order to relate the Frobenius-like portions to the mono-anabelian étale-like monoids, let us discuss *cyclotomic synchronization poly-isomorphisms*, which induce *Kummer poly-isomorphisms*. As in the discussion of (i), by applying a sort of the *Kummer theory*, we obtain a *Kummer poly-isomorphism* "κ" from the cyclotomic synchronization poly-isomorphism "$\mathfrak{syn}_\Lambda$".

$$\text{Frobenius-like cyclotome} \underset{\text{synchronization}}{\overset{\text{cyclotomic}}{\xrightarrow{\sim}}} \text{étale-like cyclotome}$$

$$\overset{\text{Kummer theory}}{\rightsquigarrow} \quad \text{Frobenius-like portion} \underset{\text{poly-isomorphism}}{\overset{\text{Kummer}}{\xrightarrow{\sim}}} \text{mono-anabelian étale-like monoid}$$

Thus, we are in a situation described by the following diagram:

$$\begin{array}{ccc}
\text{étale-like portion} & \xrightarrow{\sim} & \text{étale-like portion} \\
 & \text{mono-anab.} \Big\Downarrow \text{property} & \\
\text{mono-anabelian étale-like monoid} & \xrightarrow{\sim} & \text{mono-anabelian étale-like monoid} \\
\text{Kummer poly-} \Big\uparrow \text{isomorphism} & & \text{Kummer poly-} \Big\uparrow \text{isomorphism} \\
\text{Frobenius-like portion} & \xrightarrow{\sim} & \text{Frobenius-like portion.}
\end{array}$$

Here, let us observe that, *in our case*, by the constructions of the various objects under consideration, the lower square [of *poly-isomorphisms*] of this diagram *commutes*. In particular, by this *commutativity*, we can compute the effect on the Frobenius-like portions of the "Frobenius-like link" by considering the composite of the Kummer poly-isomorphism, a(n) [poly-]isomorphism that arises from the mono-anabelian property of the mono-anabelian reconstruction algorithm, and the inverse of the Kummer poly-isomorphism.

Thus, in summary, by means of

- *Kummer-detachment*, i.e., the passage, via Kummer poly-isomorphisms, from Frobenius-like structures to corresponding étale-like structures, and
- *étale-transport*, i.e., the passage, via the mono-anabelian property of mono-anabelian reconstruction algorithms, from the "left-hand side" to the "right-hand side",

one may *compute* the effect of the "Frobenius-like link" from the point of view of the effect of the "étale-like link" by comparing the "Frobenius-like link" with the *transport* as discussed in (i).

$$\begin{array}{ccc} \text{étale-like portion} & \xrightarrow{\sim} & \text{étale-like portion} \\ \curvearrowright & & \curvearrowright \\ \text{Frobenius-like portion} & \xrightarrow{\sim} & \text{Frobenius-like portion} \end{array}$$

$$\rightsquigarrow \quad \text{computation of the effect of the "Frobenius-like link"}$$

COROLLARY 7.7. — *Let $\square \in \{\Diamond, \rhd, \times\}$ and $G \curvearrowright M$ an MLF$^{\square}$-pair. Then the natural homomorphism*

$$\mathrm{Aut}(G \curvearrowright M) \longrightarrow \mathrm{Aut}(G)$$

fits into the following **exact sequence***:*

$$1 \longrightarrow \mathrm{Ind}^{\square} \longrightarrow \mathrm{Aut}(G \curvearrowright M) \longrightarrow \mathrm{Aut}(G) \longrightarrow 1.$$

PROOF. — This assertion is the content of Theorem 7.6 in the case where $(G_{\circ} \curvearrowright M_{\circ}) = (G_{\bullet} \curvearrowright M_{\bullet})$. $\square$

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