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# On the Admissible Fundamental Groups of Curves over Algebraically Closed Fields of Characteristic $p > 0$

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# ON THE ADMISSIBLE FUNDAMENTAL GROUPS OF CURVES OVER ALGEBRAICALLY CLOSED FIELDS OF CHARACTERISTIC $p > 0$

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**Abstract**

In the present paper, we study the anabelian geometry of pointed stable curves over algebraically closed fields of positive characteristic. We prove that the semi-graph of anabelioids of PSC-type arising from a pointed stable curve over an algebraically closed field of positive characteristic can be reconstructed group-theoretically from its fundamental group. This result may be regarded as a mono-anabelian version of the combinatorial Grothendieck conjecture in positive characteristic. As an application, we prove that, if a pointed stable curve over an algebraic closure of a finite field satisfies certain conditions, then the isomorphism class of the admissible fundamental group of the pointed stable curve completely determines the isomorphism class of the pointed stable curve as a scheme.

Keywords: positive characteristic, pointed stable curve, admissible fundamental group, semi-graph of anabelioids, anabelian geometry.

Mathematics Subject Classification: Primary 14H30; Secondary 11G20.

## Introduction

The main question of interest in the anabelian geometry of curves is, roughly speaking, the following:

> how much geometric information about the isomorphism class of a curve is contained in various versions of its fundamental group?

In this paper, we study the anabelian geometry of curves over algebraically closed fields of positive characteristic, and prove that

> if a pointed stable curve over an algebraic closure of a finite field satisfies certain conditions, then the isomorphism class of the admissible fundamental group of the pointed stable curve completely determines the isomorphism class of the pointed stable curve as a scheme.

Let $X^\bullet := (X, D_X)$ be a pointed stable curve of type $(g_X, n_X)$ over an algebraically closed field $k$. Here, $X$ denotes the underlying scheme of $X^\bullet$, and $D_X$ denotes the set of marked points of $X^\bullet$. Write $\mathcal{G}_{X^\bullet}$ for the semi-graph of anabelioids of PSC-type arising from $X^\bullet$. We do not recall the theory of semi-graphs of anabelioids in the present paper. Roughly speaking, a semi-graph of anabelioids is a semi-graph (see [M3] for the definition of semi-graphs) which is equipped with a Galois category at each vertex and each edge, together with gluing isomorphisms that satisfy certain conditions; a semi-graph of anabelioids of PSC-type is a semi-graph of anabelioids that is isomorphic to the semi-graph of anabelioids that arises from a pointed stable curve defined over an algebraically closed field (cf. [HM], [M3], [M4]).

Suppose that the characteristic $\mathrm{char}(k)$ of $k$ is 0. Then the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet)$ (cf. Definition 1.2) of $X^\bullet$ depends only on $(g_X, n_X)$ and is known to admit a presentation as follows:

$$\pi_1^{\mathrm{adm}}(X^\bullet) \cong \langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid [a_1, b_1] \dots [a_{g_X}, b_{g_X}] c_1 \dots c_{n_X} = 1 \rangle^{\mathrm{pro}},$$

where $(-)^{\mathrm{pro}}$ denotes the profinite completion of $(-)$. Thus, we obtain that $(g_X, n_X)$ and $\mathcal{G}_{X^\bullet}$ are not completely determined by the isomorphism class of the profinite group $\pi_1^{\mathrm{adm}}(X^\bullet)$.

On the other hand, when $\mathrm{char}(k) = p > 0$, the situation is quite different from the characteristic 0 case. First, let us explain briefly some well-known results concerning the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$. From now on, $X^\bullet$ always denotes a pointed stable curve over an algebraically closed field $k$ of characteristic $p > 0$.

Suppose that $X^\bullet$ is smooth over $k$. By applying techniques based on subtle properties of wildly ramified coverings, A. Tamagawa proved that $(g_X, n_X)$ can be reconstructed group-theoretically from the étale fundamental group $\pi_1(X \setminus D_X)$ of $X \setminus D_X$, and moreover, that

> if $g_X = 0$ and $k = \overline{\mathbb{F}}_p$, then the isomorphism class of the profinite group $\pi_1(X \setminus D_X)$ completely determines the isomorphism class of the scheme $X \setminus D_X$

(cf. [T1]). Afterwards, by generalizing M. Raynaud's theory of theta divisors, Tamagawa proved that similar results hold if one replaces $\pi_1(X \setminus D_X)$ by the tame fundamental group $\pi_1^{\mathrm{tame}}(X \setminus D_X)$ of $X \setminus D_X$ (cf. [T2]). Since $\pi_1^{\mathrm{tame}}(X \setminus D_X)$ can be reconstructed group-theoretically from $\pi_1(X \setminus D_X)$ (cf. [T1, Corollary 1.10]), the tame fundamental group versions are stronger than the étale fundamental group versions. In the case of curves of higher genus, we have the following finiteness result:

> if $k = \overline{\mathbb{F}}_p$, then there are only finitely many isomorphism classes of smooth pointed stable curves over $k$ whose tame fundamental groups are isomorphic to $\pi_1^{\mathrm{tame}}(X \setminus D_X)$.

This finiteness result was proved by Raynaud, F. Pop, and M. Saïdi under certain conditions and by Tamagawa in full generality (cf. [R], [PS], [T3]). Note that, by the definition of the admissible fundamental group $\pi^{\mathrm{adm}}(-)$ (cf. Definition 1.2), we have a natural isomorphism $\pi_1^{\mathrm{tame}}(X \setminus D_X) \cong \pi_1^{\mathrm{adm}}(X^\bullet)$ if $X^\bullet$ is smooth over $k$.

In the present paper, we consider a generalization of the results of Tamagawa mentioned above to the case where $X^\bullet$ is an arbitrary pointed stable curve over an algebraically closed field $k$ of characteristic $p > 0$. We were motivated by the following Question.

**Question 0.1.** *Can $\mathcal{G}_{X^\bullet}$ be reconstructed group-theoretically from the profinite group $\pi_1^{\mathrm{adm}}(X^\bullet)$? If we assume further that $k = \overline{\mathbb{F}}_p$, then is the isomorphism class of the scheme $X \setminus D_X$ determined completely by the isomorphism class of the profinite group $\pi_1^{\mathrm{adm}}(X^\bullet)$?*

Next, we explain the main results of the present paper. In Section 5, we prove the following theorem (cf. Theorem 5.9).

**Theorem 0.2.** *Write $\mathcal{G}_{X^\bullet}$ for the semi-graph of anabelioids of PSC-type arising from $X^\bullet$. Then $p := \mathrm{char}(k)$ can be reconstructed group-theoretically from $\pi_1^{\mathrm{adm}}(X^\bullet)$. If, moreover, $p := \mathrm{char}(k) > 0$, then $\mathcal{G}_{X^\bullet}$ can be reconstructed group-theoretically from $\pi_1^{\mathrm{adm}}(X^\bullet)$.*

Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$, $v(\Gamma_{X^\bullet})$ for the set of vertices of $\Gamma_{X^\bullet}$. For each $v \in v(\Gamma_{X^\bullet})$, write $\widetilde{X}_v$ for the normalization of the irreducible component of $X$ corresponding to $v$ and

$$\widetilde{X}_v^\bullet := (\widetilde{X}_v, D_{\widetilde{X}_v})$$

for the smooth pointed stable curve over $k$ determined by $\widetilde{X}_v$ and the divisor of marked points $D_{\widetilde{X}_v}$ determined by the inverse images (via the natural morphism $\widetilde{X}_v \to X$) in $\widetilde{X}_v$ of the nodes and marked points of $X^\bullet$; $(g_v, n_v)$ for the type of $\widetilde{X}_v^\bullet$. Theorem 0.3 implies that the following data can be reconstructed group-theoretically from $\pi_1^{\mathrm{adm}}(X^\bullet)$:

- $g_X$, $n_X$, and $\Gamma_{X^\bullet}$;
- the conjugacy class of the inertia group of every marked point of $X^\bullet$ in $\pi_1^{\mathrm{adm}}(X^\bullet)$;
- the conjugacy class of the inertia group of every node of $X^\bullet$ in $\pi_1^{\mathrm{adm}}(X^\bullet)$;
- for each $v \in v(\Gamma_{X^\bullet})$, $g_v$, $n_v$, and the admissible fundamental group $\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet)$ of $\widetilde{X}_v^\bullet$.

Moreover, Theorem 0.2 can also be regarded as a **mono-anabelian** version of the combinatorial Grothendieck conjecture in positive characteristic (i.e., a group-theoretically algorithm for reconstructing semi-graphs of anabelioids of PSC-type from their fundamental groups — cf. Remark 5.9.1 for more details on the combinatorial Grothendieck conjecture, which plays a central role in combinatorial anabelian geometry).

We maintain the notations introduced above. By combining Tamagawa's results and Theorem 0.2, we obtain the following result, which is the main theorem of the present paper (see Theorem 6.3 for more details). Theorem 0.3 generalizes Tamagawa's results to the case of (possibly singular) pointed stable curves.

**Theorem 0.3.** *(a) Suppose that $k = \overline{\mathbb{F}}_p$, and $g_v = 0$ for each $v \in v(\Gamma_{X^\bullet})$. Then the isomorphism class of the profinite group $\pi_1^{\mathrm{adm}}(X^\bullet)$ completely determines the isomorphism class of the scheme $X \setminus D_X$.*

*(b) Suppose that $k = \overline{\mathbb{F}}_p$. Then there are only finitely many $k$-isomorphism classes of pointed stable curves over $k$ whose admissible fundamental groups are isomorphic to $\pi_1^{\mathrm{adm}}(X^\bullet)$.*

Finally, we mention that various versions of Theorem 0.3 (a) are also known in the case where $X^\bullet$ is a smooth pointed stable curve of type $(1, 1)$ (cf. Remark 6.2.1, [S], [T5]). These versions in the case of smooth pointed stable curves of $(1, 1)$ allow us to obtain a slightly more general form of Theorem 0.3 (a) (cf. Remark 6.3.1).

## 1 $p$-rank and $p$-average

In this section, we recall some definitions and results which will be used in the present paper.

**Definition 1.1.** Let $\mathbb{G} := (v(\mathbb{G}), e(\mathbb{G}), \{\zeta_e^{\mathbb{G}}\}_{e\in e(\mathbb{G})})$ be a semi-graph. Here, $v(\mathbb{G})$, $e(\mathbb{G})$, and $\{\zeta_e^{\mathbb{G}}\}_{e\in e(\mathbb{G})}$ denote the set of vertices of $\mathbb{G}$, the set of edges of $\mathbb{G}$, and the set of coincidence maps of $\mathbb{G}$, respectively.

(a) We define $e^{\mathrm{op}}(\mathbb{G})$ (resp. $e^{\mathrm{cl}}(\mathbb{G})$) to be the set of **open** (resp. **closed**) edges of $\mathbb{G}$.

(b) Let $v \in v(\mathbb{G})$. We shall call $\mathbb{G}$ **2-connected at** $v$ if $\mathbb{G} \setminus \{v\}$ is either empty or connected.

(c) We define an **one-point compactification** $\mathbb{G}^{\mathrm{cpt}}$ of $\mathbb{G}$ as follows: if $e^{\mathrm{op}}(\mathbb{G}) = \emptyset$, we set $\mathbb{G}^{\mathrm{cpt}} = \mathbb{G}$; otherwise, the set of vertices of $\mathbb{G}^{\mathrm{cpt}}$ is $v(\mathbb{G}^{\mathrm{cpt}}) := v(\mathbb{G}) \coprod \{v_\infty\}$, the set of edges of $\mathbb{G}^{\mathrm{cpt}}$ is $e(\mathbb{G}^{\mathrm{cpt}}) := e(\mathbb{G})$, and each edge $e \in e^{\mathrm{op}}(\mathbb{G}) \subseteq e(\mathbb{G}^{\mathrm{cpt}})$ connects $v_\infty$ with the vertex that is abutted by $e$.

(d) For each $v \in v(\mathbb{G})$, we set

$$b(v) := \sum_{e\in e(\mathbb{G})} b_e(v),$$

where $b_e(v) \in \{0, 1, 2\}$ denotes the number of times that $e$ meets $v$. Moreover, we set

$$v(\mathbb{G}^{\mathrm{cpt}})^{b\leq 1} := \{v \in v(\mathbb{G}) \subseteq v(\mathbb{G}^{\mathrm{cpt}}) \mid b(v) \leq 1\}.$$

We fix some notations. Let $k$ be an algebraically closed field and $X^\bullet = (X, D_X)$ a pointed stable curve of type $(g_X, n_X)$ over $k$. Here, $X$ denotes the underlying scheme of $X^\bullet$, and $D_X$ denotes the set of marked points of $X^\bullet$. Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$, and $\Gamma_X$ for the dual graph of $X$. Note that by the definitions of $\Gamma_{X^\bullet}$ and $\Gamma_X$, we have a natural embedding $\Gamma_X \hookrightarrow \Gamma_{X^\bullet}$; then we may identify $v(\Gamma_X)$ (resp. $e(\Gamma_X)$) with $v(\Gamma_{X^\bullet})$ (resp. $e^{\mathrm{cl}}(\Gamma_{X^\bullet})$) via the natural embedding $\Gamma_X \hookrightarrow \Gamma_{X^\bullet}$. Write $\Pi^{\mathrm{top}}_{X^\bullet}$ for the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$, and $r_X$ for $\dim_{\mathbb{C}}(\mathrm{H}^1(\Gamma_{X^\bullet}, \mathbb{C}))$.

**Definition 1.2.** Let $Y^\bullet := (Y, D_Y)$ be a pointed stable curve over $k$ and $f^\bullet : Y^\bullet \to X^\bullet$ a morphism of pointed stable curves over $\mathrm{Spec}\, k$.

We shall call $f^\bullet$ a **Galois admissible covering** over $\mathrm{Spec}\, k$ (or Galois admissible covering for short) if the following conditions hold: (i) there exists a finite group $G \subseteq \mathrm{Aut}_k(Y^\bullet)$ such that $Y^\bullet/G = X^\bullet$, and $f^\bullet$ is equal to the quotient morphism $Y^\bullet \to Y^\bullet/G$; (ii) for each $y \in Y^{\mathrm{sm}} \setminus D_Y$, $f^\bullet$ is étale at $y$, where $(-)^{\mathrm{sm}}$ denotes the smooth locus of $(-)$; (iii) for any $y \in Y^{\mathrm{sing}}$, the image $f^\bullet(y)$ is contained in $X^{\mathrm{sing}}$, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$; (iv) for each $y \in Y^{\mathrm{sing}}$, the local morphism between two nodes induced by $f^\bullet$ may be described as follows:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,f^\bullet(y)} \cong k[[u,v]]/uv & \to & \hat{\mathcal{O}}_{Y,y} \cong k[[s,t]]/st \\ u & \mapsto & s^n \\ v & \mapsto & t^n, \end{array}$$

where $(n, \mathrm{char}(k)) = 1$ if $\mathrm{char}(k) > 0$; moreover, write $D_y \subseteq G$ for the decomposition group of $y$ and $\#D_y$ for the cardinality of $D_y$; then $\tau(s) = \zeta_{\#D_y} s$ and $\tau(t) = \zeta_{\#D_y}^{-1} t$ for each $\tau \in D_y$, where $\zeta_{\#D_y}$ is a primitive $\#D_y$-th root of unit; (v) the local morphism between two marked points induced by $f^\bullet$ may be described as follows:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,f^\bullet(y)} \cong k[[a]] & \to & \hat{\mathcal{O}}_{Y,y} \cong k[[b]] \\ a & \mapsto & b^m, \end{array}$$

where $(m, \mathrm{char}(k)) = 1$ if $\mathrm{char}(k) > 0$ (i.e., a tamely ramified extension). Moreover, we shall call $f^\bullet$ an **admissible covering** if there exists a morphism of pointed stable curves $(f^\bullet)' : (Y^\bullet)' \to Y^\bullet$ over $\mathrm{Spec}\, k$ such that the composite morphism $f^\bullet \circ (f^\bullet)' : (Y^\bullet)' \to X^\bullet$ is a Galois admissible covering over $\mathrm{Spec}\, k$.

Let $Z^\bullet$ be the disjoint union of finitely many pointed stable curves over $\mathrm{Spec}\, k$. We shall call a morphism $Z^\bullet \to X^\bullet$ over $\mathrm{Spec}\, k$ **multi-admissible covering** if the restriction of $Z^\bullet \to X^\bullet$ to each connected component of $Z^\bullet$ is admissible. We use the notation $\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)$ to denote the category which consists of (empty object and) all the multi-admissible coverings of $X^\bullet$. It is well-known that $\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)$ is a Galois category. Thus, by choosing a base point $x \in X^{\mathrm{sm}} \setminus D_X$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ which is called the **admissible fundamental group** of $X^\bullet$. For simplicity of notation, we omit the base point and denote the admissible fundamental group by $\pi_1^{\mathrm{adm}}(X^\bullet)$. Note that we have a natural surjection $\pi_1^{\mathrm{adm}}(X^\bullet) \twoheadrightarrow \Pi_{X^\bullet}^{\mathrm{top}}$.

For more details on admissible coverings and the admissible fundamental groups for pointed stable curves, see [M1], [M2].

**Remark 1.2.1.** Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack of pointed stable curves of type $(g, n)$ over $\mathrm{Spec}\, \mathbb{Z}$ and $\mathcal{M}_{g,n}$ the open substack of $\overline{\mathcal{M}}_{g,n}$ parametrizing pointed smooth curves. Write $\overline{\mathcal{M}}_{g,n}^{\log}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g,n}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ relative to $\mathrm{Spec}\, \mathbb{Z}$.

The pointed stable curve $X^\bullet \to \mathrm{Spec}\, k$ induces a morphism $\mathrm{Spec}\, k \to \overline{\mathcal{M}}_{g_X,n_X}$. Write $s_X^{\log}$ for the log scheme whose underlying scheme is $\mathrm{Spec}\, k$, and whose log structure is the pulling-back log structure induced by the morphism $\mathrm{Spec}\, k \to \overline{\mathcal{M}}_{g_X,n_X}$. We obtain a natural morphism $s_X^{\log} \to \overline{\mathcal{M}}_{g_X,n_X}^{\log}$ induced by the morphism $\mathrm{Spec}\, k \to \overline{\mathcal{M}}_{g_X,n_X}$ and a stable log curve $X^{\log} := s_X^{\log} \times_{\overline{\mathcal{M}}_{g_X,n_X}^{\log}} \overline{\mathcal{M}}_{g_X,n_X+1}^{\log}$ over $s_X^{\log}$ whose underlying scheme is $X$. Then the admissible fundamental group $\Pi_{X^\bullet}$ of $X^\bullet$ is isomorphic to the geometric log étale fundamental group of $X^{\log}$ (i.e., $\mathrm{Ker}(\pi_1(X^{\log}) \to \pi_1(s_X^{\log}))$).

**Remark 1.2.2.** If $X^\bullet$ is smooth over $k$, by the definition of admissible fundamental groups, then we have a natural isomorphism from the admissible fundamental group of $X^\bullet$ to the tame fundamental group of $X \setminus D_X$.

In the remainder of this section, we suppose that the characteristic of $k$ is $p > 0$.

**Definition 1.3.** Write $\Pi_{X^\bullet}$ for $\pi_1^{\mathrm{adm}}(X^\bullet)$. We define the ***p*-rank** of $X^\bullet$ to be

$$\sigma(X^\bullet) := \dim_{\mathbb{F}_p}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_p) = \dim_{\mathbb{F}_p}((\Pi_{X^\bullet}^{\text{ét}})^{\mathrm{ab}} \otimes \mathbb{F}_p),$$

where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$, and $\Pi_{X^\bullet}^{\text{ét}}$ denotes the étale fundamental group of $X^\bullet$.

**Remark 1.3.1.** For each $v \in v(\Gamma_{X^\bullet})$, write $X_v$ for the irreducible components of $X$ corresponding to $v$. Then it is easy to prove that

$$\sigma(X^\bullet) = \sigma(X) = \sum_{v \in v(\Gamma_{X^\bullet})} \sigma(\widetilde{X_v}) + r_X,$$

where $\widetilde{(-)}$ denotes the normalization of $(-)$.

**Definition 1.4.** Let $\Pi$ be a profinite group, $n$ a natural number, and $\ell$ a prime number.

(a) We denote by $\Pi(n)$ the topological closure of the subgroup $[\Pi, \Pi]\Pi^n$ of $\Pi$. Note that $\Pi/\Pi(n) = \Pi^{\mathrm{ab}} \otimes (\mathbb{Z}/n\mathbb{Z})$.

(b) We set $\gamma_\ell := \dim_{\mathbb{F}_\ell}(\Pi/\Pi(n)) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

(c) Let $n$ be a natural number such that $[\Pi : \Pi(n)] < \infty$. We define **$\ell$-average** of $\Pi$ to be

$$\gamma_\ell^{\mathrm{av}}(n)(\Pi) := \gamma_\ell(\Pi(n))/[\Pi : \Pi(n)] \in \mathbb{Q}_{\geq 0} \cup \{\infty\}.$$

The following highly nontrivial result concerning $p$-average of $\Pi_{X^\bullet}$ was proved by Tamagawa (cf. [T4, Theorem 3.10]).

**Proposition 1.5.** *For any natural number $t \in \mathbb{N}$, we set*

$$\gamma_p^{\mathrm{av}}(p^t - 1)(X^\bullet) := \gamma_p^{\mathrm{av}}(p^t - 1)(\Pi_{X^\bullet}).$$

*Suppose that, for any $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma_{X^\bullet}^{\mathrm{cpt}})$, $\Gamma_{X^\bullet}^{\mathrm{cpt}}$ is 2-connected at $v$. Then we have*

$$\lim_{t \to \infty} \gamma_p^{\mathrm{av}}(p^t - 1)(X^\bullet) = g_X - r_X - \#(v(\Gamma_{X^\bullet}^{\mathrm{cpt}})^{b \leq 1}).$$

**Remark 1.5.1.** Tamagawa proved Proposition 1.5 as a main theorem of [T2] in the case where $X^\bullet$ is a smooth pointed stable curve over $k$ by developing a general theory of Raynaud's theta divisor; Tamagawa's result means that the genus of $X^\bullet$ can be reconstructed group-theoretically from the tame fundamental group of $X \setminus D_X$. Afterwards, in [T4], Tamagawa extends the result to the case where $X^\bullet$ is a certain pointed stable curve over $k$ by proving a result concerning the abelian injectivity of admissible fundamental groups.

# 2 The set of irreducible components

We maintain the notations introduced in Section 1. Let $X^\bullet$ be a pointed stable curve over an algebraically closed field $k$ of characteristic $p > 0$. In this section, we study the set of irreducible components of $X^\bullet$.

**Definition 2.1.** Let $Z^\bullet := (Z, D_Z)$ be any pointed stable curve over $\mathrm{Spec}\, k$. Write $\Gamma_{Z^\bullet}$ for the dual semi-graph of $Z^\bullet$. We shall call $Z^\bullet$ **untangled** (resp. **sturdy**) if each irreducible component of $Z^\bullet$ is smooth (resp. the genus of the normalization of each irreducible component of $Z^\bullet$ is $\geq 2$). We write $\mathrm{Irr}(Z^\bullet)$ (resp. $\mathrm{Nod}(Z^\bullet)$) for the set of irreducible components (resp. the set of nodes) of $Z$. We define a set of irreducible components of $Z$ to be

$$\mathrm{Irr}(Z^\bullet)^{\sigma > 0} := \{Z_v, v \in v(\Gamma_{Z^\bullet}) \mid \sigma(\widetilde{Z_v}) > 0\} \subseteq \mathrm{Irr}(Z^\bullet).$$

We have the following Proposition.

**Proposition 2.2.** *There exists a connected Galois admissible covering* $f^\bullet : Y^\bullet \to X^\bullet$ *over* $\mathrm{Spec}\, k$ *such that* $Y^\bullet$ *is untangled and sturdy, and* $\mathrm{Irr}(Y^\bullet)^{\sigma>0} = \mathrm{Irr}(Y^\bullet)$.

*Proof.* The proposition follows immediately from [M2, Lemma 2.9] and Proposition 1.5. □

Write $M_{X^\bullet}$ and $M_{X^\bullet}^{\mathrm{top}}$ for $\mathrm{H}^1_{\text{ét}}(X^\bullet, \mathbb{F}_p)$ and $\mathrm{H}^1(\Gamma_{X^\bullet}, \mathbb{F}_p)$, respectively. Note that there is a natural injection $M_{X^\bullet}^{\mathrm{top}} \hookrightarrow M_{X^\bullet}$ induced by the natural surjection $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\mathrm{top}}$. We set

$$M_{X^\bullet}^{\mathrm{ntop}} := \mathrm{coker}(M_{X^\bullet}^{\mathrm{top}} \hookrightarrow M_{X^\bullet}).$$

The elements of $M_{X^\bullet}$ correspond to étale, Galois abelian coverings of $X^\bullet$ of degree $p$. Let $V^* \subseteq M_{X^\bullet}$ be the subset of elements whose image in $M_{X^\bullet}^{\mathrm{ntop}}$ is not 0. Let $\alpha \in V^*$. Write $X^\bullet_\alpha \to X^\bullet$ for the étale covering correspond to $\alpha$. Then we obtain a morphism $\iota : V^* \to \mathbb{Z}$ that maps $\alpha \mapsto \#(\mathrm{Irr}(X^\bullet_\alpha))$. Let $V \subseteq V^*$ be the subset of elements $\alpha$ which $\iota$ attains its maximum (i.e., $\iota(\alpha) = p(\#\mathrm{Irr}(X^\bullet) - 1) + 1$). We define a pre-equivalence relation $\sim$ on $V$ as follows: let $\alpha, \beta \in V$; then $\alpha \sim \beta$ if , for each $\lambda, \mu \in \mathbb{F}_p^\times$ for which $\lambda\alpha + \mu\beta \in V^*$, we have $\lambda\alpha + \mu\beta \in V$. Then we have the following lemma.

**Lemma 2.3.** *The pre-equivalence relation* $\sim$ *on* $V$ *is an equivalence relation, and, moreover, the quotient set* $V/\sim$ *is naturally isomorphic to* $\mathrm{Irr}(X^\bullet)^{\sigma>0}$.

*Proof.* For any $\delta \in V$, $\iota(\delta)$ attains its maximum implies that there exists a unique irreducible component $I^\delta_{X^\bullet_\delta} \subseteq X^\bullet_\delta$ whose decomposition group is not trivial. We write $I^\delta_{X^\bullet} \subseteq X^\bullet$ for the image of $I^\delta_{X^\bullet_\delta}$ of the covering morphism $X^\bullet_\delta \to X^\bullet$. Note that $I^\delta_{X^\bullet} \in \mathrm{Irr}(X^\bullet)^{\sigma>0}$. Then $V = \emptyset$ if and only if $\mathrm{Irr}(X^\bullet)^{\sigma>0} = \emptyset$.

We suppose that $\mathrm{Irr}(X^\bullet)^{\sigma>0} \neq \emptyset$. Let $\alpha, \beta \in V$. If $I^\alpha_{X^\bullet} = I^\beta_{X^\bullet}$, then, for each $\lambda, \mu \in \mathbb{F}_p^\times$ for which $\lambda\alpha + \mu\beta \neq 0$, we have $I^{\lambda\alpha+\mu\beta}_{X^\bullet} = I^\alpha_{X^\bullet} = I^\beta_{X^\bullet}$. Thus, $\alpha \sim \beta$. On the other hand, if $\alpha \sim \beta$, we have $I^\alpha_{X^\bullet} = I^\beta_{X^\bullet}$; otherwise, there exist two irreducible components of $X^\bullet_{\alpha+\beta}$ whose decomposition groups are not trivial. Thus, $\alpha \sim \beta$ if and only if $I^\alpha_{X^\bullet} = I^\beta_{X^\bullet}$. This means that $\sim$ is an equivalence relation on $V$. Then we obtain a natural morphism $\kappa : V/\sim \to \mathrm{Irr}(X^\bullet)^{\sigma>0}$ that maps $\delta \mapsto I^\delta_{X^\bullet}$.

Let us prove that $\kappa$ is a bijection. It is easy to see that $\kappa$ is an injection. For any irreducible component $X_v \in \mathrm{Irr}(X^\bullet)^{\sigma>0}$, since the $p$-rank of the normalization of $X_v$ is not 0, we may construct an étale, Galois abelian covering $f^\bullet : Y^\bullet \to X^\bullet$ of degree $p$ such that $X_v$ is the unique irreducible component of $X^\bullet$ such that $(f^\bullet)^{-1}(X^\bullet_v)$ is connected. Then $\#(\mathrm{Irr}(Y^\bullet)) = p(\#(\mathrm{Irr}(X^\bullet)) - 1) + 1$. Thus, we obtain an element of $V$ corresponding to $Y^\bullet$. This means that $\kappa$ is a surjection. We complete the proof of the lemma. □

# 3 Geometry of admissible coverings

We maintain the notations introduced in the previous sections. Let $X^\bullet$ be a pointed stable curve over an algebraically closed field $k$ of characteristic $p > 0$. In this section, we study the admissible coverings of $X^\bullet$.

**Lemma 3.1.** *Let $\ell \neq 2$ be a prime number and*

$$\sum_{i=1}^{n} x_i = 0$$

*a linear indeterminate equation. Suppose that $n \geq 2$. Then there exists a solution $(a_1, \dots, a_n) \in (\mathbb{Z}/\ell\mathbb{Z})^{\oplus n}$ such that $a_i \neq 0$ for each $i = 1, \dots, n$.*

*Proof.* Trivial. □

**Condition 3.2.** *Let $Z^\bullet := (Z, D_Z)$ be any pointed stable curve over* $\mathrm{Spec}\, k$. *Write* $\mathrm{Cusp}(Z^\bullet)$ *for the set of marked points $D_Z$ of $Z^\bullet$. We shall say that $Z^\bullet$ satisfies Condition 3.2 if the following conditions hold: (a) $Z^\bullet$ is untangled and sturdy; (b) for each irreducible component $Z_v \subseteq Z$, if $Z_v \cap \mathrm{Nod}(Z^\bullet) \neq \emptyset$, we have $\#(Z_v \cap \mathrm{Nod}(Z^\bullet)) \geq 3$; (c) for each irreducible component $Z_v \subseteq Z$, if $Z_v \cap \mathrm{Cusp}(Z^\bullet) \neq \emptyset$, we have $\#(Z_v \cap \mathrm{Cusp}(Z^\bullet)) \geq 3$.*

We have the following propositions.

**Proposition 3.3.** *Suppose that* $\mathrm{Cusp}(X^\bullet) \neq \emptyset$, *and $X^\bullet$ satisfies Condition 3.2. Let $q \in \mathrm{Cusp}(X^\bullet)$. Then, for any prime number $\ell \neq 2$ distinct from $p$, there exists a Galois admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ of degree $\ell$ such that $f^\bullet$ is étale over $q$, and $f^\bullet$ is totally ramified over* $\mathrm{Cusp}(X^\bullet) \setminus \{q\}$.

*Proof.* Write $X_q$ for the irreducible component of $X$ which contains $q$. We set

$$\mathrm{Cusp}(X_q) := X_q \cap \mathrm{Cusp}(X^\bullet)$$

and

$$\mathrm{Sing}(X_q) := X_q \cap \mathrm{Nod}(X^\bullet).$$

If $X^\bullet$ is smooth over $\mathrm{Spec}\, k$, then $\#(\mathrm{Cusp}(X^\bullet) \setminus \{q\}) \geq 2$. Thus, the proposition follows from the structure of the maximal pro-$\ell$ quotient of the admissible fundamental group of $\Pi_{X^\bullet}$ and Lemma 3.1. Then, in order to prove the proposition, we may assume that $X^\bullet$ is a singular curve. Thus, the assumptions imply that $\#\mathrm{Irr}(X^\bullet) \geq 2$.

Since the maximal pro-$\ell$ quotient of admissible fundamental groups of pointed stable curves of type $(g, r)$ do not depend on the moduli, without the loss of generality, we may assume that $\#\mathrm{Irr}(X^\bullet) = 2$. Write $X_{\backslash q}$ for the irreducible component of $X$ distinct from $X_q$. We set

$$\mathrm{Cusp}(X_{\backslash q}) := X_{\backslash q} \cap \mathrm{Cusp}(X^\bullet)$$

and

$$\mathrm{Sing}(X_{\backslash q}) := X_{\backslash q} \cap \mathrm{Nod}(X^\bullet).$$

Moreover, we define two pointed stable curves over $\mathrm{Spec}\, k$ to be

$$X_q^\bullet := (X_q, \mathrm{Cusp}(X_q) \cup \mathrm{Sing}(X_q))$$

and

$$X_{\backslash q}^\bullet := (X_{\backslash q}, \mathrm{Cusp}(X_{\backslash q}) \cup \mathrm{Sing}(X_{\backslash q})).$$

Note that we have a natural bijection $\theta : \mathrm{Sing}(X_q) \xrightarrow{\sim} \mathrm{Sing}(X_{\backslash q})$ determined by $X^\bullet$.

Since $X^\bullet$ satisfies Condition 3.2, Lemma 3.1 implies that there exists a solution $(a_\nu)_{\nu\in\mathrm{Sing}(X_q)}$ (resp. $(b_\nu)_{\nu\in\mathrm{Cusp}(X_q)\setminus\{q\}}$, $(c_\nu)_{\nu\in\mathrm{Cusp}(X_{\backslash q})}$) of the linear indeterminate equation

$$\sum_{\nu\in\mathrm{Sing}(X_q)} x_\nu = 0 \text{ (resp. } \sum_{\nu\in\mathrm{Cusp}(X_q)\setminus\{q\}} x_\nu = 0, \sum_{\nu\in\mathrm{Cusp}(X_{\backslash q})} x_\nu = 0)$$

in $\mathbb{Z}/\ell\mathbb{Z}$ such that $a_\nu \neq 0$ (resp. $b_\nu \neq 0$, $c_\nu \neq 0$) for each $\nu \in \mathrm{Sing}(X_q)$ (resp. $\nu \in \mathrm{Cusp}(X_q) \setminus \{q\}$, $\nu \in \mathrm{Cusp}(X_{\backslash q})$). For any $\nu \in \mathrm{Sing}(X_q)$, we set $d_{\theta(\nu)} := -a_\nu$. Then $(d_{\theta(\nu)})_{\nu\in\mathrm{Sing}(X_q)}$ is a solution of the linear indeterminate equation

$$\sum_{\nu\in\mathrm{Sing}(X_{\backslash q})} x_\nu = 0$$

in $\mathbb{Z}/\ell\mathbb{Z}$.

Write $\Pi^{\ell,\mathrm{ab}}_{X_q^\bullet}$ (resp. $\Pi^{\ell,\mathrm{ab}}_{X_{\backslash q}^\bullet}$) for the abelianization of the maximal pro-$\ell$ quotient of the admissible fundamental group of $X_q^\bullet$ (resp. $X_{\backslash q}^\bullet$). Moreover, for each $\nu \in \mathrm{Sing}(X_q)$ (resp. $\nu \in \mathrm{Cusp}(X_q)$, $\nu \in \mathrm{Sing}(X_{\backslash q})$, $\nu \in \mathrm{Cusp}(X_{\backslash q})$), we write $\alpha_\nu$ (resp. $\beta_\nu$, $\delta_\nu$, $\gamma_\nu$) for a generator of the inertia group associated to $\nu$ in $\Pi^{\ell,\mathrm{ab}}_{X_q^\bullet}$ (resp. $\Pi^{\ell,\mathrm{ab}}_{X_q^\bullet}$, $\Pi^{\ell,\mathrm{ab}}_{X_{\backslash q}^\bullet}$, $\Pi^{\ell,\mathrm{ab}}_{X_{\backslash q}^\bullet}$). The structure of $\Pi^{\ell,\mathrm{ab}}_{X_q^\bullet}$ (resp. $\Pi^{\ell,\mathrm{ab}}_{X_{\backslash q}^\bullet}$) implies that we may construct a morphism from $\Pi^{\ell,\mathrm{ab}}_{X_q^\bullet}$ (resp. $\Pi^{\ell,\mathrm{ab}}_{X_{\backslash q}^\bullet}$) to $\mathbb{Z}/\ell\mathbb{Z}$ that maps $\alpha_\nu \mapsto a_\nu$ for $\nu \in \mathrm{Sing}(X_q^\bullet)$, $\beta_\nu \mapsto b_\nu$ for $\nu \in \mathrm{Cusp}(X_q^\bullet) \setminus \{q\}$, and $\beta_q \mapsto 0$ (resp. $\delta_\nu \mapsto d_{\theta(\nu)}$ for $d_{\theta(\nu)} \in \mathrm{Sing}(X_{\backslash q}^\bullet)$ and $\gamma_\nu \mapsto c_\nu$ for $\nu \in \mathrm{Cusp}(X_{\backslash q}^\bullet)$). Then we obtain two Galois admissible coverings

$$f_q^\bullet : Y_q^\bullet \to X_q^\bullet$$

and

$$f_{\backslash q}^\bullet : Y_{\backslash q}^\bullet \to X_{\backslash q}^\bullet$$

over $\mathrm{Spec}\, k$ of degree $\ell$; moreover, $f_q^\bullet$ is totally ramified over $(\mathrm{Cusp}(X_q) \cup \mathrm{Sing}(X_q)) \setminus \{q\}$ and étale over $q$, and $f_{\backslash q}^\bullet$ is totally ramified over $\mathrm{Cusp}(X_{\backslash q}) \cup \mathrm{Sing}(X_{\backslash q})$.

Thus, by gluing $f_q^\bullet$ and $f_{\backslash q}^\bullet$ together, we obtain a Galois admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ of degree $\ell$ such that $f^\bullet$ is étale over $q$, and $f^\bullet$ is totally ramified over $\mathrm{Cusp}(X^\bullet) \setminus \{q\}$. $\square$

Furthermore, similar arguments to the arguments given in the proof of Proposition 3.3 imply the following proposition holds.

**Proposition 3.4.** *Suppose that* $\mathrm{Nod}(X^\bullet) \neq \emptyset$, *and* $X^\bullet$ *satisfies Condition 3.2. Let* $q \in \mathrm{Nod}(Z)$. *Then, for any prime number* $\ell \neq 2$ *distinct from* $p$, *there exists a Galois admissible covering* $f^\bullet : Y^\bullet \to X^\bullet$ *of degree* $\ell$ *such that* $f^\bullet$ *is étale over* $q$, *and* $f^\bullet$ *is totally ramified over* $\mathrm{Nod}(X^\bullet) \setminus \{q\}$.

# 4 A result of pro-$\ell$ combinatorial anabelian geometry

Let $\ell$ be a prime number. In this section, we prove a result of pro-$\ell$ combinatorial anabelian geometry.

**Definition 4.1.** Let $\mathcal{G}$ be a semi-graph of anabelioids of PSC-type. Write $\Pi_{\mathcal{G}}$ for the fundamental group of $\mathcal{G}$ and $\Gamma_{\mathcal{G}}$ for the underlying semi-graph of $\mathcal{G}$.

(a) We shall call $\mathcal{G}$ untangled (resp. sturdy) if $\mathcal{G}$ is isomorphic to the semi-graph of anabelioids of PSC-type arising from a untangled (resp. sturdy) pointed stable curve over an algebraically closed field.

(b) For any open normal subgroup $H \subseteq \Pi_{\mathcal{G}}$, write $\mathcal{G}_H$ for the Galois covering of $\mathcal{G}$ determined by $H$, and write $\Gamma_{\mathcal{G}_H}$ for the underlying semi-graph of $\mathcal{G}_H$. We shall denote by $\Pi_{\mathcal{G}_H}^{\mathrm{ab/edge}}$ the quotient of $\Pi_{\mathcal{G}_H}^{\mathrm{ab}}$ by the closed subgroup generated by the images in $\Pi_{\mathcal{G}_H}^{\mathrm{ab}}$ of the edge-like subgroups (cf. [HM, Definition 1.3 (i)]).

In the remainder of this section, we suppose that $\mathcal{G}$ is the semi-graph of anabelioids of PSC-type arising from a pointed stable curve over an algebraically closed field of characteristic $p > 0$; moreover, we suppose that $\ell \neq p$, and we write $\mathcal{G}^{\ell}$ for the semi-graph of anabelioids of pro-$\ell$ PSC-type induced by $\mathcal{G}$ (cf. [M4, Definition 1.1 (i)]). Write $\Pi_{\mathcal{G}^{\ell}}$ for the fundamental group of $\mathcal{G}^{\ell}$. Then $\Pi_{\mathcal{G}^{\ell}}$ is naturally isomorphic to the maximal pro-$\ell$ quotient of $\Pi_{\mathcal{G}}$.

**Condition 4.2.** *For any open normal subgroup $H \subseteq \Pi_{\mathcal{G}^{\ell}}$, the set of vertices $v(\Gamma_{\mathcal{G}^{\ell}_H})$ of $\Gamma_{\mathcal{G}^{\ell}_H}$, the morphism $v(\Gamma_{\mathcal{G}^{\ell}_H}) \to v(\Gamma_{\mathcal{G}^{\ell}})$ induced by the Galois covering $\mathcal{G}^{\ell}_H \to \mathcal{G}^{\ell}$ determined by $H$, and $\Pi_{\mathcal{G}^{\ell}_H}^{\mathrm{ab/edge}}$ can be reconstructed group-theoretically from $\Pi_{\mathcal{G}^{\ell}}$.*

Then we have the following result.

**Proposition 4.3.** *Suppose that $\mathcal{G}^{\ell}$ satisfies Condition 4.2. Then $\mathcal{G}^{\ell}$ can be reconstructed group-theoretically from $\Pi_{\mathcal{G}^{\ell}}$.*

*Proof.* Since $\mathcal{G}^{\ell}$ satisfies Condition 4.2, the set of vertical-like groups of $\Pi_{\mathcal{G}^{\ell}}$ can be reconstructed group-theoretically from $\Pi_{\mathcal{G}^{\ell}}$; furthermore, [HM, Lemma 1.6] implies that the set of edges-like groups of $\Pi_{\mathcal{G}^{\ell}}$ can be reconstructed group-theoretically from $\Pi_{\mathcal{G}^{\ell}}$.

On the other hand, by applying [HM, Lemma 1.9 (ii)] (resp. [HM, Lemma 1.7] and [HM, Lemma 1.9 (i)] ), we have the set of vetices $v(\Gamma_{\mathcal{G}^{\ell}})$ (resp. the set of edges $e(\Gamma_{\mathcal{G}^{\ell}})$) of the underlying semi-graph $\Gamma_{\mathcal{G}^{\ell}}$ of $\mathcal{G}^{\ell}$ can be reconstructed group-theoretically from $\Pi_{\mathcal{G}^{\ell}}$. Moreover, [HM, Lemma 1.7] implies that the set of coincidence maps of $\Gamma_{\mathcal{G}^{\ell}}$ can be reconstructed group-theoretically from $\Pi_{\mathcal{G}^{\ell}}$. This completes the proof of the proposition. □

# 5 A mono-anabelian version of the Grothendieck conjecture for semi-graphs of anabelioids of PSC-type in positive characteristic

We maintain the notations introduced in the previous sections. Let $X^{\bullet}$ be a pointed stable curve over an algebraically closed field $k$. Write $\mathcal{G}_{X^{\bullet}}$ for the semi-graph of anabelioids of

PSC-type arising from $X^\bullet$. In this section, we will give a mono-anabelian reconstruction for $\mathcal{G}_{X^\bullet}$ from $\Pi_{X^\bullet}$.

For any open normal subgroup $H \subseteq \Pi_{X^\bullet}$, we write $X_H^\bullet \to X^\bullet$ for the Galois admissible covering of $X^\bullet$ determined by $H$, $\Gamma_{X_H^\bullet}$ for the dual semi-graph of $X_H^\bullet$, $r_{X_H}$ for $\dim_{\mathbb{C}} \mathrm{H}^1(\Gamma_{X_H^\bullet}, \mathbb{C})$, $g_{X_H}$ for the genus of $X_H^\bullet$, and $n_{X_H}$ for the cardinality of the set of marked points of $X_H^\bullet$. Then reconstructing $\mathcal{G}_{X^\bullet}$ group-theoretically from $\Pi_{X^\bullet}$ is equivalent to, for any open normal subgroup $H \subseteq \Pi_{X^\bullet}$, the morphism of dual semi-graphs $\Gamma_{X_H^\bullet} \to \Gamma_{X^\bullet}$ induced by the Galois admissible covering $X_H^\bullet \to X^\bullet$ determined by $H$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.

In this section, **we only assume that $\Pi_{X^\bullet}$ is the admissible fundamental group of a pointed stable curve $X^\bullet$ defined over an algebraically closed field $k$.** First, we have the following basic proposition.

**Proposition 5.1.** *The characteristic $p := \mathrm{char}(k)$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.*

*Proof.* For any prime number $\ell$, if $\dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_\ell)$ is a constant and $p > 0$, we have either

$$\mathrm{char}(k) = g_X = 2g_X + n_X - 1$$

or

$$\mathrm{char}(k) = g_X = 2g_X$$

holds. Thus, we obtain either $(g_X, n_X) = (0, 1)$ or $(g_X, n_X) = (0, 0)$ holds. Since $\Pi_{X^\bullet}$ is the admissible fundamental group of a pointed stable curve, this is a contradiction. Thus, if $\dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_\ell)$ is a constant, we have $p = 0$. Then we can detect whether $p > 0$ or not, group-theoretically from $\Pi_{X^\bullet}$. Moreover, if $p > 0$, then $p$ is the unique prime number such that $\dim_{\mathbb{F}_p}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_p) \neq \dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_\ell)$ for each prime number $\ell \neq p$. □

In the remainder of this section, we assume that $p := \mathrm{char}(k) > 0$. Next, let us introduce some conditions on semi-graphs.

**Condition 5.2.** *Let $\mathbb{G}$ be a semi-graph. We shall say that $\mathbb{G}$ satisfies Condition 5.2 if $\mathbb{G}^{\mathrm{cpt}}$ is 2-connected at each $v \in v(\mathbb{G}) \subseteq v(\mathbb{G}^{\mathrm{cpt}})$ and*

$$\#(v(\mathbb{G}^{\mathrm{cpt}})^{b \leq 1}) = 0.$$

**Remark 5.2.1.** If $\Gamma_{X^\bullet}$ satisfies Condition 5.2, Proposition 1.5 implies that

$$\lim_{t \to \infty} \gamma_p^{\mathrm{av}}(p^t - 1)(X^\bullet) = g_X - r_X.$$

**Lemma 5.3.** *There exists an open characteristic subgroup $N \subseteq \Pi_{X^\bullet}$ such that the following conditions hold: (a) the order of $N$ is prime to $p$; (b) $X_N^\bullet$ satisfies Condition 3.2; (c) $X_N^\bullet$ is untangled and sturdy, and $\Gamma_{X_N^\bullet}$ satisfies Condition 5.2; (d) $N$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.*

*Proof.* Let $\{\mathcal{G}_i\}_{i \in I}$ be a set of semi-graphs of anabelioids of PSC-type such that the following conditions hold: (i) $\Pi_{\mathcal{G}_i} \cong \Pi_{X^\bullet}$ for each $i \in I$; (ii) for any semi-graph of anabelioids of PSC-type $\mathcal{G}$, if $\Pi_{\mathcal{G}} \cong \Pi_{X^\bullet}$, then there exists $\mathcal{G}_i \in \{\mathcal{G}_i\}_{i \in I}$ such that $\mathcal{G} \cong \mathcal{G}_i$; (iii) for any $i, j \in I$, $\mathcal{G}_i \cong \mathcal{G}_j$ if and only if $i = j$. Since the set of isomorphism classes of the semi-graphs of anabelioids of PSC-type whose fundamental groups are isomorphic to $\Pi_{X^\bullet}$ is finite, we have $I$ is a finite set.

It is easy to see that, for each $i \in I$, we may construct a Galois covering $\mathcal{G}_{N_i} \to \mathcal{G}_i$ determined by an open normal subgroup $N_i \subseteq \Pi_{X^\bullet}$ such that $N_i$ is an open characteristic subgroup whose order is prime to $p$, $\mathcal{G}_{N_i}$ is isomorphic to the semi-graph of anabelioids of PSC-type arising from a pointed stable curve satisfying Condition 3.2, $\mathcal{G}_{N_i}$ is untangled and sturdy, and the underlying semi-graph $\Gamma_{\mathcal{G}_{N_i}}$ of $\mathcal{G}_{N_i}$ satisfies Condition 5.2. We set

$$N := \bigcap_{i \in I} N_i.$$

Then the lemma follows. □

If the dual semi-graph $\Gamma_{X^\bullet}$ satisfies Condition 5.2, we have the following result.

**Lemma 5.4.** *Write* $\Pi^{p\text{-top}}_{X^\bullet}$ *for the maximal pro-p quotient of* $\Pi^{\mathrm{top}}_{X^\bullet}$*. Suppose that* $\Gamma_{X^\bullet}$ *satisfies Condition 5.2. Then* $\Pi^{p\text{-top}}_{X^\bullet}$ *can be reconstructed group-theoretically from* $\Pi_{X^\bullet}$*; moreover,* $g_X$*,* $n_X$*, and* $r_X$ *can be reconstructed group-theoretically from* $\Pi_{X^\bullet}$*.*

*Proof.* Let $H$ be any open normal subgroup of $\Pi_{X^\bullet}$. We note that, if $\Pi_{X^\bullet}/H$ is a $p$-group, then the decomposition group of every irreducible component of $X^\bullet_H$ is trivial if and only if

$$g_{X_H} - r_{X_H} = \#(\Pi_{X^\bullet}/H)(g_X - r_X).$$

We set

$$\mathrm{Top}_p(\Pi_{X^\bullet}) := \{H \subseteq \Pi_{X^\bullet} \text{ open normal} \mid \Pi_{X^\bullet}/H \text{ is a } p\text{-group} \\ \text{and } g_{X_H} - r_{X_H} = \#(\Pi_{X^\bullet}/H)(g_X - r_X)\}.$$

Then $\Pi^{p\text{-top}}_{X^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$ as follows:

$$\Pi^{p\text{-top}}_{X^\bullet} = \Pi_{X^\bullet}/(\bigcap_{H \in \mathrm{Top}_p(\Pi_{X^\bullet})} H).$$

Since $\Gamma_{X^\bullet}$ satisfies Condition 5.2, we have $\Gamma_{X^\bullet_H}$ satisfies Condition 5.2 for each $H \in \mathrm{Top}_p(\Pi_{X^\bullet})$. Then $g_X - r_X$ and $g_{X_H} - r_{X_H}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$ and $H$, respectively. Thus, $\Pi^{p\text{-top}}_{X^\bullet}$ and $r_X = \dim_{\mathbb{C}}(\Pi^{p\text{-top},\mathrm{ab}}_{X^\bullet} \otimes \mathbb{C})$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Moreover, $g_X$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.

Next, we reconstruct $n_X$. Let $\ell \neq p$ be a prime number. Suppose that $\dim_{\mathbb{F}_\ell}(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{F}_\ell) \neq 2g_X$, then we have

$$n_X = \dim_{\mathbb{F}_\ell}(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{F}_\ell) - 2g_X + 1.$$

Suppose that $\dim_{\mathbb{F}_\ell}(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{F}_\ell) = 2g_X$. Then $n_X = 0$ if, for any open normal subgroup $H \subseteq \Pi_{X^\bullet}$, $\dim_{\mathbb{F}_\ell}(H^{\mathrm{ab}} \otimes \mathbb{F}_\ell) = 2g_{X_H}$. Otherwise, we have $n_X = 1$. This completes the proof of the lemma. □

Lemma 5.4 implies that the following corollary.

**Corollary 5.5.** *Suppose that $\Gamma_{X^\bullet}$ satisfies Condition 5.2. Then the natural exact sequence*

$$0 \to M_{X^\bullet}^{\mathrm{top}} \to M_{X^\bullet} \to M_{X^\bullet}^{\mathrm{ntop}} \to 0$$

*can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Moreover, $\mathrm{Irr}(X^\bullet)^{\sigma>0}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.*

*Proof.* Note that $M_{X^\bullet} = \mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{F}_p)$, $M_{X^\bullet}^{\mathrm{top}} = \mathrm{Hom}(\Pi_{X^\bullet}^{p\text{-top}}, \mathbb{F}_p)$, and $M_{X^\bullet}^{\mathrm{top}} \hookrightarrow M_{X^\bullet}$ is induced by the natural surjection $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{p\text{-top}}$. Then the corollary follows immediately from Lemma 2.3 and Lemma 5.4. □

Next, we reconstruct the set of vertices of $\Gamma_{X^\bullet}$ from $\Pi_{X^\bullet}$.

**Proposition 5.6.** *The set of vertices $v(\Gamma_{X^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Moreover, for any open normal subgroup $Q \subseteq \Pi_{X^\bullet}$, the morphism $v(\Gamma_{X_Q^\bullet}) \twoheadrightarrow v(\Gamma_{X^\bullet})$ on the sets of vertices induced by the admissible covering $X_Q^\bullet \to X^\bullet$ determined by $Q$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.*

*Proof.* Let $H \subseteq \Pi_{X^\bullet}$ be any open normal subgroup, and let $\{a_i\}_{i\in\Pi_{X^\bullet}/H} \subset \Pi_{X^\bullet}$ be a set of lifting of the elements of $\Pi_{X^\bullet}/H$ that the image of $a_i \in \Pi_{X^\bullet}$ under the quotient morphism $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}/H$ is $i \in \Pi_{X^\bullet}/H$. Write $M_{X_H^\bullet}$ for $\mathrm{H}^1_{\text{ét}}(X_H^\bullet, \mathbb{F}_p)$. Then, for any $i \in \Pi_{X^\bullet}/H$, the action of $i$ on $M_{X^\bullet}$ is conjugation action $a_i^{-1} M_{X_H^\bullet} a_i$. Thus, by applying Lemma 2.3, we have the action of $\Pi_{X^\bullet}/H$ on $M_{X_H^\bullet}$ induces an action of $\Pi_{X^\bullet}/H$ on the set $\mathrm{Irr}(X_H^\bullet)^{\sigma>0}$. Note that the action of $\Pi_{X^\bullet}/H$ on $\mathrm{Irr}(X_H^\bullet)^{\sigma>0}$ does not depend on the choices of $\{a_i\}_{i\in\Pi_{X^\bullet}/H}$. Thus, we obtain a morphism

$$\mathrm{Irr}(X_H^\bullet)^{\sigma>0} \twoheadrightarrow \mathrm{Irr}(X_H^\bullet)^{\sigma>0}/(\Pi_{X^\bullet}/H) \subseteq \mathrm{Irr}(X^\bullet).$$

By applying Lemma 5.3, we obtain a characteristic subgroup $N \subseteq \Pi_{X^\bullet}$ such that $\Gamma_{X_N^\bullet}$ satisfies Condition 5.2, and $N$ can be constructed group-theoretically from $\Pi_{X^\bullet}$. We set

$$\mathrm{Irr}_{X^\bullet} := \bigcup_{H\subseteq\Pi_{X^\bullet}\ \text{open normal}} \mathrm{Irr}(X_{H\cap N}^\bullet)^{\sigma>0}/(\Pi_{X^\bullet}/(H\cap N)).$$

Then we have $\mathrm{Irr}_{X^\bullet} \subseteq \mathrm{Irr}(X^\bullet)$. On the other hand, Proposition 2.2 implies that $\mathrm{Irr}_{X^\bullet} = \mathrm{Irr}(X^\bullet)$.

By applying Corollary 5.5, we have that $\mathrm{Irr}(X_{H\cap N}^\bullet)^{\sigma>0}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Moreover, since the action of $\Pi_{X^\bullet}/(H\cap N)$ on $\mathrm{Irr}(X_{H\cap N}^\bullet)^{\sigma>0}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$, we obtain $v(\Gamma_{X^\bullet}) = \mathrm{Irr}(X^\bullet)$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.

Let $Q \subseteq \Pi_{X^\bullet}$ be an open normal subgroup. We set $N_Q := Q \cap N$. Then, for any open normal subgroup $H \subseteq Q$, we have a natural morphism

$$\mathrm{Irr}(X_{H\cap N_Q}^\bullet)^{\sigma>0}/(Q/H\cap N_Q) \twoheadrightarrow \mathrm{Irr}(X_{H\cap N}^\bullet)^{\sigma>0}/(\Pi_{X^\bullet}/(H\cap N));$$

note that where $H \cap N_Q = H \cap N$. Moreover, we set

$$\mathrm{Irr}_{X_Q^\bullet} := \bigcup_{H\subseteq Q\ \text{open normal}} \mathrm{Irr}(X_{H\cap N_Q}^\bullet)^{\sigma>0}/(Q/(H\cap N_Q)).$$

Then we obtain a natural morphism

$$v(\Gamma_{X^\bullet_Q}) = \mathrm{Irr}(X^\bullet_Q) = \mathrm{Irr}_{X^\bullet_Q} \twoheadrightarrow \mathrm{Irr}_{X^\bullet} = \mathrm{Irr}(X^\bullet) = v(\Gamma_{X^\bullet}).$$

Since the morphism $\mathrm{Irr}(X^\bullet_{H\cap N_Q})^{\sigma>0}/(Q/H\cap N_Q) \twoheadrightarrow \mathrm{Irr}(X^\bullet_{H\cap N})^{\sigma>0}/(\Pi_{X^\bullet}/(H\cap N))$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Then the morphism $v(\Gamma_{X^\bullet_Q}) \twoheadrightarrow v(\Gamma_{X^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. This completes the proof of the proposition. □

Next, let us start to reconstruct $\mathcal{G}_{X^\bullet}$ from $\Pi_{X^\bullet}$.

**Lemma 5.7.** *Let $\ell$ be a prime number distinct from $p$. Write $\mathcal{G}^\ell_{X^\bullet}$ for the semi-graph of anabelioids of pro-$\ell$ PSC-type induced by $\mathcal{G}_{X^\bullet}$. Suppose that $\Gamma_{X^\bullet}$ satisfies Condition 5.2. Then $\mathcal{G}^\ell_{X^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.*

*Proof.* Let $H$ be any open normal subgroup of $\Pi_{X^\bullet_H}$. Since $\Gamma_{X^\bullet}$ satisfies Condition 5.2, $\Gamma_{X^\bullet_H}$ satisfies Condition 5.2 too. By applying Lemma 5.4, we obtain that $n_{X_H}$ and $r_{X_H}$ can be reconstructed group-theoretically from $H$; moreover, Proposition 5.6 implies that the set of vertices $v(\Gamma_{X^\bullet_H})$ of $\Gamma_{X^\bullet_H}$ and the morphism $v(\Gamma_{X^\bullet_H}) \twoheadrightarrow v(\Gamma_{X^\bullet})$ induced by the Galois covering $X^\bullet_H \to X^\bullet$ determined by $H$ can be reconstructed group-theoretically from $H$ and $\Pi_{X^\bullet}$. Then, by applying the Euler-Poincaré characteristic formula for $\Gamma_{X^\bullet}$, we obtain that

$$\#(e^{\mathrm{cl}}(\Gamma_{X^\bullet_H})) = r_{X_H} + \#(v(\Gamma_{X^\bullet_H})) - 1$$

can be reconstructed group-theoretically from $H$.

We set

$$\mathrm{Et}(\Pi_{X^\bullet}) := \{H \subseteq \Pi_{X^\bullet} \text{ open normal} \mid n_{X_H} + \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet_H})) = (\#(\Pi_{X^\bullet}/H))(n_X + \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet})))\}.$$

Then the étale fundamental group $\Pi^{\text{ét}}_{X^\bullet}$ of $X^\bullet$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$ as follows:

$$\Pi^{\text{ét}}_{X^\bullet} := \Pi_{X^\bullet}/\bigcap_{H\in\mathrm{Et}(\Pi_{X^\bullet})} H.$$

Note that $\Pi^{\text{ét},\mathrm{ab}}_{X^\bullet} = \Pi^{\mathrm{ab/edge}}_{\mathcal{G}_{X^\bullet}}$. Then $\Pi^{\mathrm{ab/edge}}_{\mathcal{G}^\ell_{X^\bullet}}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Thus, the lemma follows from Proposition 4.3 and Proposition 5.6. □

**Lemma 5.8.** *Suppose that $X^\bullet$ and $\mathcal{G}_{X^\bullet}$ satisfy Condition 3.2 and Condition 5.2, respectively. Then $\mathcal{G}_{X^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.*

*Proof.* Let $H \subseteq \Pi_{X^\bullet}$ be any open normal subgroup. In order to prove the lemma, we only need to prove that the morphism $\phi_H : \Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$ on dual semi-graphs induced by the Galois admissible covering $X^\bullet_H \to X^\bullet$ determined by $H$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$; moreover, Proposition 5.6 and Lemma 5.7 imply that it is sufficient to prove that the morphism $\phi_H|_{e(\Gamma_{X^\bullet_H})} : e(\Gamma_{X^\bullet_H}) \to e(\Gamma_{X^\bullet})$ on the sets of edges induced by $\phi_H$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.

Let $\ell \neq 2$ be a prime number distinct from $p$ such that $(\#(\Pi_{X^\bullet}/H), \ell) = 1$, and let $q$ be any marked point of $X^\bullet$. Write $e_q \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ for the open edge corresponding to

$q$. Since we assume that $X^\bullet$ satisfies Condition 3.2, Proposition 3.3 implies that there exists a Galois admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ whose Galois group is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ such that $f^\bullet$ is étale over $q$, and $f^\bullet$ is totally ramified over $\mathrm{Cusp}(X^\bullet) \setminus \{q\}$. Then we obtain a connected Galois admissible covering $g^\bullet : Y^\bullet_H := Y^\bullet \times_{X^\bullet} X^\bullet_H \to X^\bullet_H$. Here, $g^\bullet$ is the natural projection.

Write $\mathcal{G}_{X^\bullet_H}$ and $\mathcal{G}_{Y^\bullet_H}$ for the semi-graphs of anabelioids of PSC-type arising from $X^\bullet_H$ and $Y^\bullet_H$, respectively; moreover, write $\mathcal{G}^\ell_{X^\bullet_H}$ and $\mathcal{G}^\ell_{Y^\bullet_H}$ for the semi-graphs of anabelioids of pro-$\ell$ PSC-type induced by $\mathcal{G}_{X^\bullet_H}$ and $\mathcal{G}_{Y^\bullet_H}$, respectively. Then Lemma 5.7 implies that the morphism of dual semi-graphs $\psi_H : \Gamma_{Y^\bullet_H} \to \Gamma_{X^\bullet_H}$ induced by $g^\bullet$ can be reconstructed group-theoretically from $H$. Thus, we have

$$\phi_H^{-1}(e_q) = \{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet_H}) \mid \#(\psi_H^{-1}(e)) = \ell\}.$$

Then the morphism $\phi_H|_{e^{\mathrm{op}}(\Gamma_{X^\bullet_H})} : e^{\mathrm{op}}(\Gamma_{X^\bullet_H}) \to e^{\mathrm{op}}(\Gamma_{X^\bullet})$ induced by $\phi_H$ on the sets of open edges can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.

Together with Proposition 3.4, similar arguments to the arguments given in the proof above imply that the morphism $\phi_H|_{e^{\mathrm{cl}}(\Gamma_{X^\bullet_H})} : e^{\mathrm{cl}}(\Gamma_{X^\bullet_H}) \to e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ induced by $\phi_H$ on the sets of closed edges can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Then $\phi_H|_{e(\Gamma_{X^\bullet_H})} : e(\Gamma_{X^\bullet_H}) \to e(\Gamma_{X^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. This completes the proof of the lemma. □

Next, we prove the main theorem of the present section.

**Theorem 5.9.** *Let $X^\bullet$ be a pointed stable curve over an algebraically closed field $k$. Write $\Pi_{X^\bullet}$ for the admissible fundamental group of $X^\bullet$, and $\mathcal{G}_{X^\bullet}$ for the semi-graph of anabelioids of PSC-type $\mathcal{G}_{X^\bullet}$ arising from $X^\bullet$. Then $p := \mathrm{char}(k)$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Moreover, if $p := \mathrm{char}(k) > 0$, then $\mathcal{G}_{X^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.*

*Proof.* Proposition 5.1 implies that the characteristic of $k$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. We only prove the "moreover" part of the theorem.

Suppose that $p := \mathrm{char}(k) > 0$. Let $H \subseteq \Pi_{X^\bullet}$ be any open normal subgroup. Proposition 5.6 implies that, to verify the theorem, it is sufficient to prove that the morphism $\Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$ on the sets of edges induced by the Galois covering $X^\bullet_H \to X^\bullet$ determined by $H$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.

By applying Lemma 5.3, we obtain an open characteristic subgroup $N \subseteq \Pi_{X^\bullet}$ such that the conditions of Lemma 5.3. Write $H_N$ for $H \cap N$, $\mathcal{G}_{X^\bullet_{H_N}}$ for the semi-graph of anabelioids of PSC-type arising from $X^\bullet_{H_N}$. Since $X^\bullet_{H_N}$ and the dual semi-graph of $\Gamma_{X^\bullet_{H_N}}$ satisfy Condition 3.2 and Condition 5.2, respectively, then Lemma 5.8 implies that $\mathcal{G}_{X^\bullet_{H_N}}$ can be reconstructed group-theoretically from $H_N$.

Note that the natural action of $\Pi_{X^\bullet}/H_N$ on $\mathcal{G}_{X^\bullet_{H_N}}$ induces an action of $\Pi_{X^\bullet}/H_N$ on $\Gamma_{X^\bullet_{H_N}}$; moreover, we have $\Gamma_{X^\bullet} = \Gamma_{X^\bullet_{H_N}}/(\Pi_{X^\bullet}/H)$ and $\Gamma_{X^\bullet_H} = \Gamma_{X^\bullet_{H_N}}/(H/H_N)$. Thus, we obtain a natural morphism

$$\Gamma_{X^\bullet_H} = \Gamma_{X^\bullet_{H_N}}/(H/H_N) \to \Gamma_{X^\bullet} = \Gamma_{X^\bullet_{H_N}}/(\Pi_{X^\bullet}/H).$$

Thus, $\Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. This completes the proof of the theorem. □

**Remark 5.9.1.** The **bi-anabelian** combinatorial Grothendieck conjecture for semi-graphs of anabelioids of PSC-type can be formulated as follows.

*Let* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *be two semi-graphs of anabelioids of PSC-type associated to two pointed stable curves over algebraically closed fields* $k_1$ *and* $k_2$*, respectively,* $\Pi_{\mathcal{G}_1}$ *and* $\Pi_{\mathcal{G}_2}$ *the fundamental groups of* $\mathcal{G}_1$ *and* $\mathcal{G}_2$*, respectively,* $\alpha : \Pi_{\mathcal{G}_1} \overset{\sim}{\to} \Pi_{\mathcal{G}_2}$ *an isomorphism of profinite groups,* $I_1$ *and* $I_2$ *profinite groups,* $\rho_{I_1} : I_1 \to \mathrm{Out}(\Pi_{\mathcal{G}_1})$ *and* $\rho_{I_2} : I_1 \to \mathrm{Out}(\Pi_{\mathcal{G}_2})$ *outer Galois representations, and* $\beta : I_1 \overset{\sim}{\to} I_2$ *an isomorphism of profinite groups. Suppose that the diagram*

$$\begin{array}{ccc} I_1 & \xrightarrow{\rho_{I_1}} & \mathrm{Out}(\Pi_{\mathcal{G}_1}) \\ \beta \downarrow & & \downarrow \mathrm{Out}(\alpha) \\ I_2 & \xrightarrow{\rho_{I_2}} & \mathrm{Out}(\Pi_{\mathcal{G}_2}), \end{array}$$

*is commutative, where* $\mathrm{Out}(\alpha)$ *denotes the isomorphism induced by* $\alpha$*. Then we have* $\mathcal{G}_1 \cong \mathcal{G}_2$*.*

Let $\Sigma \subseteq \mathfrak{Primes}$ be a set of prime numbers which does not contain $\mathrm{char}(k_1)$ and $\mathrm{char}(k_2)$, where $\mathfrak{Primes}$ denotes the set of prime numbers. Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are two semi-graphs of anabelioids of pro-$\Sigma$ PSC-type. Then the bi-anabelian combinatorial Grothendieck conjecture was proved by S. Mochizuki in the case where $\rho_{I_1}$ and $\rho_{I_2}$ are outer Galois representations of IPSC-type (cf. [M4]), and by Y. Hoshi and Mochizuki in the case where $\rho_{I_1}$ and $\rho_{I_2}$ are certain outer Galois representations of NN-type (cf. [HM]). Furthermore, Theorem 5.9 may be regarded as a **mono-abelian** version of the combinatorial Grothendieck conjecture for the semi-graphs of anabelioids of PSC-type arising from pointed stable curves in positive characteristic (i.e., a group-theoretically algorithm for reconstructing semi-graphs of anabelioids of PSC-type from their fundamental groups).

**Remark 5.9.2.** Theorem 5.9 is a generalized version of a result of Tamagawa that the tame inertia groups associated to the cusps of smooth pointed stable curves can be reconstructed group-theoretically from their tame fundamental groups (cf. [T2, Theorem 5.2]).

# 6 The anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$

We maintain the notations introduced in Section 1. Let $X^\bullet$ be a pointed stable curve over an algebraically closed field $k$ of characteristic $p > 0$. In this section, we use Theorem 5.9 to prove some anabelian results for pointed stable curves in positive characteristic.

Let $\ell$ be any prime number and $\overline{\mathbb{F}}_\ell$ an algebraic closure of $\mathbb{F}_\ell$. We define two sets of rational points of moduli stacks as follows:

$$R_{g,n} := \bigcup_{\ell \in \mathfrak{Primes}} \mathcal{M}_{g,n}(\overline{\mathbb{F}}_\ell)$$

and

$$\overline{R}_{g,n} := \bigcup_{\ell \in \mathfrak{Primes}} \overline{\mathcal{M}}_{g,n}(\overline{\mathbb{F}}_\ell),$$

where $\overline{\mathcal{M}}_{g,n}$ denotes the moduli stack of pointed stable curve of type $(g,n)$ over $\operatorname{Spec}\mathbb{Z}$, and $\mathcal{M}_{g,n}$ denotes the open substack of $\overline{\mathcal{M}}_{g,n}$ parametrizing pointed smooth curves of type $(g,n)$. For any rational point $\mathfrak{q} \in \overline{R}_{g,n} : \operatorname{Spec}\overline{\mathbb{F}}_\ell \to \overline{\mathcal{M}}_{g,n}$, write $X^\bullet_{\mathfrak{q}} := (X_{\mathfrak{q}}, D_{X_{\mathfrak{q}}})$ for the pointed stable curve $\overline{\mathcal{M}}_{g,n+1} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathbb{F}}_\ell$ over $\overline{\mathbb{F}}_\ell$ determined by $\mathfrak{q}$. We define an equivalence relation $\sim^{\mathrm{sch}}$ on $\overline{R}_{g,n}$ as follows: if $\mathfrak{q}_1, \mathfrak{q}_2 \in \overline{R}_{g,n}$, then $\mathfrak{q}_1 \sim^{\mathrm{sch}} \mathfrak{q}_2$ if $X_{\mathfrak{q}_1} \setminus D_{X_{\mathfrak{q}_1}}$ and $X_{\mathfrak{q}_2} \setminus D_{X_{\mathfrak{q}_2}}$ are isomorphic as schemes (though not necessarily as $\overline{\mathbb{F}}_\ell$-schemes). Let FPG be the set of topologically finitely generated profinite groups. We define an equivalence relation $\sim^{\mathrm{pro}}$ on FPG as follows: if $G_1, G_2 \in \mathrm{FPG}$, then $G_1 \sim^{\mathrm{pro}} G_2$ if $G_1$ and $G_2$ are isomorphic as profinite groups. Then we obtain a natural morphism as follows:

$$\pi^{\mathrm{adm}}_{g,n} : \overline{R}_{g,n}/\sim^{\mathrm{sch}} \to \mathrm{FPG}/\sim^{\mathrm{pro}}$$

that maps the equivalence class of $\mathfrak{q}$ to the equivalence class of $\pi_1^{\mathrm{adm}}(X^\bullet_{\mathfrak{q}})$.

**Definition 6.1.** Let $S_1 \to S_2$ be a morphism of sets. We shall call the morphism $S_1 \to S_2$ quasi-finite if, for any $s_2 \in S_2$, $\#((S_1 \to S_2)^{-1}(s_2))$ is finite.

Then the following theorem was proved by Tamagawa (cf. [T2], [T3]).

**Theorem 6.2.** *(a) Suppose that $\overline{\mathbb{F}}_p \subseteq k$, and $X^\bullet$ is a smooth pointed stable curve over $k$. Let $X^\bullet_0$ be a smooth pointed stable curve over $\overline{\mathbb{F}}_p$ of genus $g_{X_0} = 0$. Then we can detect whether $X^\bullet$ is isomorphic to $X^\bullet_0 \times_{\overline{\mathbb{F}}_p} k$ or not, group-theoretically from $\Pi_{X^\bullet}$. In particular, the morphism*

$$\pi^{\mathrm{adm}}_{0,n}|_{R_{0,n}/\sim^{\mathrm{sch}}} : R_{0,n}/\sim^{\mathrm{sch}} \hookrightarrow \mathrm{FPG}/\sim^{\mathrm{pro}}$$

*induced by $\pi^{\mathrm{adm}}_{0,n}$ on the subset $R_{0,n}/\sim^{\mathrm{sch}}$ of $\overline{R}_{0,n}/\sim^{\mathrm{sch}}$ is an injection.*

*(b) Let $S$ be an $\mathbb{F}_p$-scheme, and $\eta$ and $s$ points of $S$ such that $s \in \overline{\{\eta\}}$ holds. We denote by $\overline{\eta}$ and $\overline{s}$ geometric points on $\eta$ and $s$, respectively. Let $\mathscr{X}^\bullet$ be a smooth pointed stable curve of type $(g,n)$ over $S$ and*

$$sp^{\mathrm{adm}}_{\eta,s} : \pi_1^{\mathrm{adm}}(\mathscr{X}^\bullet \times_\eta \overline{\eta}) \twoheadrightarrow \pi_1^{\mathrm{adm}}(\mathscr{X}^\bullet \times_s \overline{s})$$

*a specialization map. Suppose that $\mathscr{X}^\bullet \times_\eta \overline{\eta}$ cannot be defined over an algebraic closure of $\mathbb{F}_p$, and $\mathscr{X}^\bullet \times_s \overline{s}$ can be defined over an algebraic closure of $\mathbb{F}_p$. Then $sp^{\mathrm{adm}}_{\eta,s}$ is not an isomorphism. Moreover, the morphism*

$$\pi^{\mathrm{adm}}_{g,n}|_{R_{g,n}/\sim^{\mathrm{sch}}} : R_{g,n}/\sim^{\mathrm{sch}} \to \mathrm{FPG}/\sim^{\mathrm{pro}}$$

*induced by $\pi^{\mathrm{adm}}_{g,n}$ on the subset $R_{g,n}/\sim^{\mathrm{sch}}$ of $\overline{R}_{g,n}/\sim^{\mathrm{sch}}$ is quasi-finite.*

**Remark 6.2.1.** By replacing FPG (resp. $\pi^{\mathrm{adm}}(-)$) by the set of profinite groups (resp. $\pi_1(-)$ (i.e., the étale fundamental group of $(-)$)), we obtain the following natural morphism:

$$\pi_{g,n} : \overline{R}_{g,n}/\sim^{\mathrm{sch}} \to \mathrm{PG}/\sim^{\mathrm{pro}}$$

that maps the equivalence class of $\mathfrak{q}$ to the equivalence class of $\pi_1(X_{\mathfrak{q}} \setminus D_{X_{\mathfrak{q}}})$. Before Tamagawa proved Theorem 6.2 (a), he obtained an étale fundamental group version of Theorem 6.2 (a) (i.e., $\pi_{0,n}|_{R_{0,n}/\sim^{\mathrm{sch}}}$ is an injection) in a completely different way (by using wildly ramified coverings) (cf. [T1]). Note that, for any nonsingular pointed stable curve $Z^\bullet := (Z, D_Z)$ over an algebraically closed filed of positive characteristic, since $\pi_1^{\mathrm{adm}}(Z^\bullet)$ can be reconstructed group-theoretically from $\pi_1(Z \setminus D_Z)$ (cf. [T1, Corollary 1.10]), Theorem 6.2 (a) is stronger than the theorem of étale fundamental group version.

Recently, by following Tamagawa's idea, A. Sarashina proved that $\pi_{1,1}|_{R_{1,1}/\sim^{\mathrm{sch}}}$ is an injection (cf. [S], [T5, Theorem 6 (i)]). Moreover, by applying the theory of Tamagawa developed in [T2], Sarashina's result holds also for $\pi_{1,1}^{\mathrm{adm}}|_{R_{1,1}/\sim^{\mathrm{sch}}}$ (cf. [T5, Theorem 6 (ii)]).

**Remark 6.2.2.** Theorem 6.2 (b) was first proved by Raynaud (cf. [R]) and Pop-Saidi (cf. [PS]) under certain assumptions of Jacobian, and by Tamagawa in the fully general case (cf. [T3]).

Next, we prove our main theorem of the present paper. We generalize Theorem 6.2 as follows.

**Theorem 6.3.** *(a) Suppose that $\overline{\mathbb{F}}_p \subseteq k$, and $X^\bullet$ is a pointed stable curve of over $k$. Let $X_0^\bullet := (X_0, D_{X_0})$ be a pointed stable curve over $\overline{\mathbb{F}}_p$. Write $\Gamma_{X_0^\bullet}$ for the dual semi-graph of $X_0^\bullet$. For each $v \in v(\Gamma_{X_0^\bullet})$, write $\widetilde{(X_v)_0}$ for the normalization of the irreducible component of $X_0$ corresponding to $v$ and*

$$\widetilde{(X_0)}_v^\bullet := (\widetilde{(X_0)}_v, D_{\widetilde{(X_0)}_v})$$

*for the smooth pointed stable curve over $\overline{\mathbb{F}}_p$ determined by $\widetilde{(X_0)}_v$ and the divisor of marked points $D_{\widetilde{(X_0)}_v}$ determined by the inverse images (via the natural morphism $\widetilde{(X_0)}_v \to X_0$) in $\widetilde{(X_0)}_v$ of the nodes and marked points of $X_0^\bullet$; $(g_v, n_v)$ for the type of $\widetilde{X}_v^\bullet$. Suppose that $g_v = 0$ for each $v \in v(\Gamma_{X_0^\bullet})$. Then we can detect whether $X^\bullet$ is isomorphic to $X_0^\bullet \times_{\overline{\mathbb{F}}_p} k$ or not, group-theoretically from $\Pi_{X^\bullet}$. In particular, the morphism*

$$\pi_{0,n}^{\mathrm{adm}} : \overline{R}_{0,n}/\sim^{\mathrm{sch}} \hookrightarrow \mathrm{FPG}/\sim^{\mathrm{pro}}$$

*is an injection.*

*(b) Let $X_1^\bullet$ (resp. $X_2^\bullet$) be a pointed stable curves over an algebraically closed field $k_1$ (resp. $k_2$) of positive characteristic. Write $\mathcal{G}_{X_1^\bullet}$ (resp. $\mathcal{G}_{X_2^\bullet}$) for the semi-graph of anabelioids of PSC-type associated to $X_1^\bullet$ (resp. $X_2^\bullet$). Then $\pi_1^{\mathrm{adm}}(X_1^\bullet) \cong \pi_1^{\mathrm{adm}}(X_2^\bullet)$ if and only if $\mathrm{char}(k_1) = \mathrm{char}(k_2)$ and $\mathcal{G}_{X_1^\bullet} \cong \mathcal{G}_{X_2^\bullet}$. Moreover, we have the following result: Let $S$ be an $\mathbb{F}_p$-scheme, and $\eta$ and $s$ points of $S$ such that $s \in \overline{\{\eta\}}$ holds. We denote by $\overline{\eta}$ and $\overline{s}$ geometric points on $\eta$ and $s$, respectively. Let $\mathscr{X}^\bullet$ be a pointed stable curve of type $(g, n)$ over $S$,*

$$sp_{\eta,s}^{\mathrm{adm}} : \pi_1^{\mathrm{adm}}(\mathscr{X}^\bullet \times_\eta \overline{\eta}) \twoheadrightarrow \pi_1^{\mathrm{adm}}(\mathscr{X}^\bullet \times_s \overline{s})$$

*a specialization map. Suppose that $\mathscr{X}^\bullet \times_\eta \overline{\eta}$ cannot be defined over an algebraic closure of $\mathbb{F}_p$, and $\mathscr{X}^\bullet \times_s \overline{s}$ can be defined over an algebraic closure of $\mathbb{F}_p$. Then $sp_{\eta,s}^{\mathrm{adm}}$ is not an isomorphism. Furthermore, the morphism*

$$\pi_{g,n}^{\mathrm{adm}} : \overline{R}_{g,n}/\sim^{\mathrm{sch}} \to \mathrm{FPG}/\sim^{\mathrm{pro}}$$

*is quasi-finite.*

*Proof.* First, let us prove (a). We will prove that $X^\bullet \cong X_0^\bullet \times_{\overline{\mathbb{F}}_p} k$ if and only if $\Pi_{X^\bullet} \cong \pi_1^{\mathrm{adm}}(X_0^\bullet)$. Since the admissible fundamental groups of pointed stable curves do not depend on the base fields, we obtain the "only if" part of the theorem. Next, we prove the "if" part. Suppose that $\Pi_{X^\bullet} \cong \pi_1^{\mathrm{adm}}(X_0^\bullet)$. Write $\mathcal{G}_{X^\bullet}$ and $\mathcal{G}_{X_0^\bullet}$ for the semi-graphs of anabelioids of PSC-type arising from $X^\bullet$ and $X_0^\bullet$, respectively. By applying Theorem 5.9, we obtain $\mathcal{G}_{X^\bullet} \cong \mathcal{G}_{X_0^\bullet}$. We fix an isomorphism $\mathcal{G}_{X^\bullet} \overset{\sim}{\to} \mathcal{G}_{X_0^\bullet}$, and we may assume that $\mathcal{G} := \mathcal{G}_{X^\bullet} = \mathcal{G}_{X_0^\bullet}$ and $\Pi := \Pi_{X^\bullet} \cong \pi_1^{\mathrm{adm}}(X_0^\bullet)$. Write $\Gamma$ for the underlying semi-graph of $\mathcal{G}$, and $\Pi^{\mathrm{top}}$ for the profinite completion of the topological fundamental group of $\Gamma$. Note that there is a natural surjection $\Pi \twoheadrightarrow \Pi^{\mathrm{top}}$. Moreover, it is easy see that there exists a open normal group $H \subseteq \Pi$ such that $H \supseteq \ker(\Pi \twoheadrightarrow \Pi^{\mathrm{top}})$, and the semi-graph of anabelioids of PSC-type $\mathcal{G}_H$ determined by $H$ is untangled. To verify the theorem, by replacing $\mathcal{G}$ by $\mathcal{G}_H$, we may assume that $\mathcal{G}$ is untangled. Then every irreducible component of $X^\bullet$ and $X_0^\bullet$ is isomorphic to $\mathbb{P}^1$.

Let $v \in v(\Gamma)$. Write $X_v$ and $(X_0)_v$ of $X^\bullet$ and $X_0^\bullet$ for the irreducible component corresponding to $v$, respectively. We set

$$X_v^\bullet := (X_v, X_v \cap (\mathrm{Nod}(X^\bullet) \cap D_X))$$

and

$$(X_0)_v^\bullet := ((X_0)_v, (X_0)_v \cap (\mathrm{Nod}(X_0^\bullet) \cap D_{X_0})),$$

where $\mathrm{Nod}(-)$ denotes the set of nodes of $(-)$. Since we assume that $\mathcal{G}$ is untangled, we have $(X_0)_v^\bullet = \widetilde{(X_0)_v^\bullet}$. On the other hand, for any $v \in v(\Gamma)$, $(X_0)_v^\bullet$ can be defined over a finite field $\mathbb{F}_{p^{d_v}}$. Let $m \in \mathbb{N}$ such that $(\prod_{v \in v(\Gamma)} d_v)|m$. Thus, Theorem 5.9 and Theorem 6.2 (a) imply that $X_v^\bullet = \tau_v((X_0)_v^\bullet(m) \times_{\overline{\mathbb{F}}_p} k)$, where $\tau_v \in \mathrm{Aut}_k(X_v^\bullet)$, and $(-)(m)$ denotes the $m$-th Frobenius twist of $(-)$. Thus, by gluing $\{\tau_v\}_{v \in v(\Gamma)}$, we obtain $X^\bullet \cong X_0^\bullet(m) \times_{\overline{\mathbb{F}}_p} k \cong X_0^\bullet \times_{\overline{\mathbb{F}}_p} k$. This completes the proof of (a).

Next, we prove (b). The first part of (b) follows immediately from Theorem 5.9. The "moreover" part and the "furthermore" part follow immediately from Theorem 5.9 and Theorem 6.2 (b). $\square$

**Remark 6.3.1.** By Remark 6.2.1, we obtain the following generalized version of Theorem 6.3 (a).

*Suppose that $\overline{\mathbb{F}}_p \subseteq k$, and $X^\bullet$ is a pointed stable curve of over $k$. Let $X_0^\bullet := (X_0, D_{X_0})$ be a pointed stable curve over $\overline{\mathbb{F}}_p$. Write $\Gamma_{X_0^\bullet}$ for the dual semi-graph of $X_0^\bullet$. For each $v \in v(\Gamma_{X_0^\bullet})$, write $\widetilde{(X_v)}_0$ for the normalization of the irreducible component of $X_0$ corresponding to $v$ and*

$$\widetilde{(X_0)_v^\bullet} := (\widetilde{(X_0)_v}, D_{\widetilde{(X_0)_v}})$$

*for the smooth pointed stable curve over $\overline{\mathbb{F}}_p$ determined by $\widetilde{(X_0)_v}$ and the divisor of marked points $D_{\widetilde{(X_0)_v}}$ determined by the inverse images (via the natural morphism $\widetilde{(X_0)_v} \to X_0$) in $\widetilde{(X_0)_v}$ of the nodes and marked points of $X_0^\bullet$; $(g_v, n_v)$ for the type of $\widetilde{X_v^\bullet}$. Suppose that,* *for each* $v \in v(\Gamma_{X_0^\bullet})$, $\widetilde{(X_0)_v^\bullet}$ *is either a smooth pointed stable curve over* $\overline{\mathbb{F}}_p$ *of genus* $g_v = 0$ *or a smooth pointed stable curve over* $\overline{\mathbb{F}}_p$ *of type* $(1, 1)$*. Then we can detect whether* $X^\bullet$ *is isomorphic to* $X_0^\bullet \times_{\overline{\mathbb{F}}_p} k$ *or not, group-theoretically from* $\Pi_{X^\bullet}$*. In particular, the morphism*

$$\pi_{g,n}^{\mathrm{adm}} : \overline{R}_{g,n}/\sim^{\mathrm{sch}} \hookrightarrow \mathrm{FPG}/\sim^{\mathrm{pro}}$$

*is an injection if* $g = 0$ *or* $(g, n) = (1, 1)$*.*

**Remark 6.3.2.** The "moreover" part of Theorem 6.3 (b) can also be proved by applying Theorem 6.2 (b) and the geometry of stable reduction of admissible coverings. Then similar arguments to the arguments given in [T3, Theorem 8.6] imply that the "furthermore" part holds.


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