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Group-theoreticity of Numerical Invariants and Distinguished Subgroups of Configuration Space Groups

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ABSTRACT. Let Σ be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers. In this paper, we prove that various objects that arise from the *geometry* of the configuration space of a hyperbolic curve over an algebraically closed field of characteristic zero may be *reconstructed group-theoretically* from the pro-Σ *fundamental group* of the configuration space. Let X be a hyperbolic curve of type (g, r) over a field k of characteristic zero. Thus, X is obtained by removing from a proper smooth curve of genus g over k a closed subscheme [i.e., the "divisor of cusps"] of X whose structure morphism to $\mathrm{Spec}(k)$ is finite étale of degree r; $2g-2+r > 0$. Write X_n for the n-th configuration space associated to X, i.e., the complement of the various diagonal divisors in the fiber product over k of n copies of X. Then, when k is *algebraically closed*, we show that the *triple* (n, g, r) and the *generalized fiber subgroups* — i.e., the subgroups that arise from the various *natural morphisms* $X_n \to X_m$ $[m < n]$, which we refer to as *generalized projection morphisms* — of the pro-Σ *fundamental group* Π_n of X_n may be *reconstructed group-theoretically* from Π_n. This result *generalizes* results obtained previously by the first and third authors and A. Tamagawa to the case of *arbitrary hyperbolic curves* [i.e., without restrictions on (g, r)]. As an application, in the case where $(g, r) = (0, 3)$ and $n \geq 2$, we conclude that there exists a *direct product decomposition*

$$\mathrm{Out}(\Pi_n) \ = \ \mathrm{GT}^{\Sigma} \times \mathfrak{S}_{n+3}$$

— where we write "$\mathrm{Out}(-)$" for the group of outer automorphisms [i.e., *without any auxiliary restrictions*!] of the profinite group in parentheses and GT^{Σ} (respectively, $\mathfrak{S}_{n+3}$) for the pro-Σ *Grothendieck-Teichmüller group* (respectively, symmetric group on $n+3$ letters). This direct product decomposition may be applied to obtain a *simplified purely group-theoretic equivalent definition* — i.e., as the *centralizer* in $\mathrm{Out}(\Pi_n)$ of the *union of the centers of the open subgroups* of $\mathrm{Out}(\Pi_n)$ — of GT^{Σ}. One of the key notions underlying the theory of the present paper is the notion of a pro-Σ *log-full subgroup* — which may be regarded as a sort of *higher-dimensional analogue* of the notion of a pro-Σ *cuspidal inertia subgroup of a surface group* — of Π_n. In the final section of the present paper, we show that, when X and k satisfy certain conditions concerning *"weights"*, the pro-l log-full subgroups may be *reconstructed group-theoretically* from the natural outer action of the absolute Galois group of k on the geometric pro-l fundamental group of X_n.

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Contents

Introduction

Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$; k an algebraically closed field; X a hyperbolic curve of type (g, r) over k, i.e., the open subscheme of a proper smooth curve of genus g over k obtained by removing r closed points. In the following discussion, we shall write "$\Pi_{(-)}$" for the [log] étale fundamental group of a connected locally noetherian [fs log] scheme [for some choice of basepoint]. If the characteristic of k is *zero*, then Π_X is a *surface group* [cf. the discussion entitled "Topological Groups" in §0]; in particular, if $r > 0$, then

Π_X is a *free profinite group of rank* $2g + r - 1$.

Thus, at least in the case of k of *characteristic zero*,

the *isomorphism class* of the profinite group Π_X is *insufficient* to determine (g, r).

[Note that, if the characteristic of k is *positive*, then

the *isomorphism class* of Π_X *completely determines* (g, r)

— cf. [Tama], Theorem 0.1.] On the other hand, if, instead of just considering the étale fundamental group of the given hyperbolic curve, one considers the étale fundamental groups of the various *configuration spaces* associated to the hyperbolic curve, then the following Fact is known [cf. [CbTpI], Theorem 1.8; [MT], Corollary 6.3]:

Fact. *Let Σ be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers. For $\square \in \{\circ, \bullet\}$, let $X^\square$ be a hyperbolic curve of type $(g^\square, r^\square)$ over an algebraically closed field of characteristic zero; $n^\square$ a positive integer; $X^\square_{n^\square}$ the $n^\square$-th configuration space of $X^\square$; $\Pi^\square_{n^\square}$ the maximal pro-Σ quotient of $\Pi_{X^\square_{n^\square}}$;*

$$\alpha : \Pi^\circ_{n^\circ} \xrightarrow{\sim} \Pi^\bullet_{n^\bullet}$$

an isomorphism of profinite groups. Suppose that

$$\{(g^\circ, r^\circ), (g^\bullet, r^\bullet)\} \cap \{(0, 3), (1, 1)\} = \emptyset.$$

Then the following hold:

(i) *The equality $n^\circ = n^\bullet$ holds. If, moreover, $n \overset{\mathrm{def}}{=} n^\circ = n^\bullet \geq 2$, then $(g^\circ, r^\circ) = (g^\bullet, r^\bullet)$.*

(ii) *The isomorphism α induces a* **bijection** *between the set of* **fiber subgroups** *of Π_n° and the set of* **fiber subgroups** *of $\Pi_n^\bullet$.*

Here, we recall that a fiber subgroup of $\Pi_n^\square$ is defined to be the kernel of the natural [outer] surjection

$$\Pi_n^\square \twoheadrightarrow \Pi_m^\square$$

— where $1 \leq m \leq n$ — induced by a projection $X_n^\square \to X_m^\square$ obtained by forgetting some of the factors. In this paper, we first generalize the above Fact to include the case of *arbitrary hyperbolic curves* [cf. Theorem 2.5 for more details]:

Theorem A. (Group-theoreticity of the dimension, genus, number of cusps, and generalized fiber subgroups). *Let Σ be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers. For $\square \in \{\circ, \bullet\}$, let $X^\square$ be a hyperbolic curve of type $(g^\square, r^\square)$ over an algebraically closed field of characteristic zero; $n^\square$ a positive integer; $X_{n^\square}^\square$ the $n^\square$-th configuration space of $X^\square$; $\Pi_{n^\square}^\square$ the maximal pro-Σ quotient of $\Pi_{X_{n^\square}^\square}$;*

$$\alpha : \Pi_{n^\circ}^\circ \xrightarrow{\sim} \Pi_{n^\bullet}^\bullet$$

an isomorphism of profinite groups. Then the following hold:

(i) *The equality $n^\circ = n^\bullet$ holds. Moreover, if $n \overset{\mathrm{def}}{=} n^\circ = n^\bullet \geq 2$, then $(g^\circ, r^\circ) = (g^\bullet, r^\bullet)$.*

(ii) *If $n \geq 2$ [cf.* (i)*], then α induces a* **bijection** *between the set of* **generalized fiber subgroups** *of Π_n° [cf. Definition 2.1,* (ii)*] and the set of* **generalized fiber subgroups** *of $\Pi_n^\bullet$.*

Here, note that Theorem A, (ii), *fails to hold* if one uses ["classical"] *fiber subgroups* instead of *generalized fiber subgroups.* Indeed, one verifies immediately [cf. Definition 2.1, (i); Remark 2.1.1] that the following holds:

> Let $n \geq 2$ be a positive integer; X_n the n-th configuration space of X; $\Pi_n \overset{\mathrm{def}}{=} \Pi_{X_n}$. Suppose that $(g, r) \in \{(0, 3), (1, 1)\}$. Then for any ["classical"] fiber subgroup $F \subseteq \Pi_n$, there exists an *automorphism* $\alpha \in \mathrm{Aut}(\Pi_n)$ — which arises from an k-*automorphism* $\in \mathrm{Aut}_k(X_n)$ — such that $\alpha(F) \subseteq \Pi_n$ is *not* a ["classical"] fiber subgroup of Π_n.

We also remark that, in Theorem 2.5, below, we give *explicit group-theoretic algorithms* for reconstructing the triple (n, g, r), as well as the generalized fiber subgroups of Π_n, from Π_n.

Next, we apply Theorem A, (ii), to prove the following result [cf. Corollary 2.6 for more details]. This result may be regarded as a *generalization* of [CbTpII], Theorem B, (i), to the case of *arbitrary hyperbolic curves.*

Corollary B. (Structure of the group of outer automorphisms of a configuration space group). *Let Σ be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers; X a* *hyperbolic curve of type* (g,r) *over an algebraically closed field of characteristic zero;* $n \geq 2$ *a positive integer;* X_n *the* n*-th configuration space of* X*;* Π_n *the maximal pro-*Σ *quotient of* Π_{X_n}*;*

$$n^* \stackrel{\text{def}}{=} \begin{cases} n+r & \text{if } (g,r) \in \{(0,3),(1,1)\}; \\ n & \text{if } (g,r) \notin \{(0,3),(1,1)\}; \end{cases}$$

$\mathfrak{S}_{n^*}$ *the symmetric group on* n^* *letters. Write* $\mathrm{Out}(\Pi_n)$ *for the group of outer automorphisms [i.e., without any auxiliary restrictions!] of the profinite group* Π_n*. Let us regard* $\mathfrak{S}_{n^*}$ *as a subgroup of* $\mathrm{Out}(\Pi_n)$ *via the natural inclusion* $\mathfrak{S}_{n^*} \hookrightarrow \mathrm{Out}(\Pi_n)$ *induced by the natural action of* $\mathfrak{S}_{n^*}$ *on* X_n *[cf. Remark 2.1.1]. Suppose that* $(r,n) \neq (0,2)$*. Then we have an equality*

$$\mathrm{Out}(\Pi_n) \;=\; \mathrm{Out}^{\mathrm{gF}}(\Pi_n) \times \mathfrak{S}_{n^*}$$

— where we write $\mathrm{Out}^{\mathrm{gF}}(\Pi_n)$ *for the group of outer automorphisms of* Π_n *that stabilize arbitrary generalized fiber subgroups of* Π_n*.*

We remark that, in Corollaries 2.6, 2.10, below, we give *explicit group-theoretic algorithms* for reconstructing the subgroup $\mathfrak{S}_{n^*} \subseteq \mathrm{Out}(\Pi_n)$ from Π_n.

In particular, by restricting Corollary B to the case where $(g,r) = (0,3)$, we obtain the following result [cf. Corollary 2.8 for more details]:

Corollary C. (Simplified group-theoretic approach to the pro-Σ Grothendieck-Teichmüller group). *In the notation of Corollary B, suppose that* $(g,r) = (0,3)$*. Then* $\mathrm{Out}^{\mathrm{gF}}(\Pi_n)$ *may be naturally identified with the* **pro-Σ Grothendieck-Teichmüller group** GT^{Σ} *[cf. Definition 2.7]. In particular, we have an equality*

$$\mathrm{Out}(\Pi_n) \;=\; \mathrm{GT}^{\Sigma} \times \mathfrak{S}_{n+3}.$$

Moreover, we have

$$\mathfrak{S}_{n+3} \;=\; Z_{\mathrm{Out}(\Pi_n)}(\mathrm{GT}^{\Sigma}) \;=\; Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n)),$$
$$\mathrm{GT}^{\Sigma} \;=\; Z_{\mathrm{Out}(\Pi_n)}(Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n)))$$

— where we write $Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n))$ *for the* **local center** *of* $\mathrm{Out}(\Pi_n)$ *[cf. the discussion entitled "Topological Groups" in §0].*

In our proof of the equality "$n^\circ = n^\bullet$" stated in Theorem A, (i), we focus on a certain special kind of point — called a *log-full* point [cf. Definition 1.1] — of [the underlying scheme of] the log configuration space of a stable log curve that gives rise to the given hyperbolic curve [cf. the discussion entitled "Curves" in §0]. Roughly speaking, a log-full point is defined to be a closed point of the log configuration space at which the log structure is the "*most concentrated*". For instance, if $X^{\log}$ is a stable log curve over an algebraically closed field equipped with an fs log structure, then the set of log-full points of $X^{\log}$ *coincides* with the set of *cusps* and *nodes* of $X^{\log}$. In particular, the notion of a log-full point of the log configuration space of a stable log curve may be considered as a sort of *higher-dimensional analogue* of the notion of a cusp/node.

In the following discussion, for simplicity, we consider the case of *smooth curves*. Let l be a prime number; Σ a set of prime numbers which is equal to either $\{l\}$ or the set of all prime numbers; k an algebraically closed field of characteristic $\notin \Sigma$;

$$X^{\log} \to \mathrm{Spec}(k)$$

a stable log curve [i.e., where we regard "Spec(k)" as being equipped with the trivial log structure] such that the interior [i.e., the open subscheme of points at which the log structure of $X^{\log}$ is trivial] of $X^{\log}$ is *affine*. Then let us recall that a cusp of $X^{\log}$ determines [up to conjugation]

$$\text{an } \textit{inertia subgroup} \ \subseteq \ \Pi^{\Sigma}_{X^{\log}} \text{ [noncanonically] isomorphic to } \widehat{\mathbb{Z}}^{\Sigma}$$

— where we write $(-)^{\Sigma}$ for the maximal pro-Σ quotient of a profinite group $(-)$.

In a similar vein, if n is a positive integer, then a log-full point of the n-th log configuration space $X^{\log}_n$ of $X^{\log}$ determines [up to conjugation] what we shall refer to [cf. Definition 3.4] as

$$\text{a } \textit{log-full} \text{ subgroup } \ \subseteq \ \Pi_n \overset{\mathrm{def}}{=} \Pi^{\Sigma}_{X^{\log}_n} \text{ [noncanonically] isomorphic to } (\widehat{\mathbb{Z}}^{\Sigma})^{\oplus n}.$$

Here, we observe that the *dimension* "n" appears as the *rank* of a log-full subgroup. Thus, a log-full subgroup may be regarded as a sort of *group-theoretic manifestation* of the dimension. In fact, this point of view plays an important role in the proof of Theorem 1.6.

The notion of a log-full subgroup also plays an important role in our approach to the following Problem:

Problem. Can one give a purely group-theoretic algorithm for reconstructing from Π_n the *inertia subgroups* of Π_n associated to the various *log divisors* of $X^{\log}_n$ [cf. the discussion entitled "Curves" in §0]?

In fact, since [as is easily verified] each inertia subgroup [$\cong \widehat{\mathbb{Z}}^{\Sigma}$] of Π_n associated to a log divisor of $X^{\log}_n$ appears as a *direct summand* of some log-full subgroup [$\cong (\widehat{\mathbb{Z}}^{\Sigma})^{\oplus n}$] of Π_n, it is natural to divide this Problem into steps (P1), (P2), as follows:

(P1): Can one give a purely group-theoretic algorithm for reconstructing from Π_n the *log-full subgroups* of Π_n?

(P2): Can one give a purely group-theoretic algorithm for reconstructing from Π_n, together with the auxiliary data constituted by the set of log-full subgroups of Π_n, the *direct summands* of a given log-full subgroup that arise as inertia subgroups associated to *log divisors*?

In the present paper, we prove a result that yields a *partial affirmative answer* to (P1), in the form of a *sufficient condition* for the reconstructibility of *log-full subgroups* [cf. Theorem 3.8; Corollary 3.9; Proposition 3.11; Corollary 3.12, (ii); Remark 3.12.1, for more details]:

Theorem D. (Group-theoretic preservation of log-full subgroups). *Let l be a prime number; n a positive integer. For $\Box \in \{\circ, \bullet\}$, let $k_\Box$ be a field of characteristic $\neq l$; $G_{k_\Box}$ the absolute Galois group of $k_\Box$ [for a suitable* *choice of algebraic closure of* $k_\square$*];* $\chi_{k_\square} : G_{k_\square} \to \mathbb{Z}_l^\times$ *the* l*-adic cyclotomic character associated to* $k_\square$*;* $X_\square^{\log} \to \mathrm{Spec}(k_\square)$ *a smooth log curve;* $(X_\square^{\log})_n$ *the* n*-th log configuration space of* $X_\square^{\log}$*;* $\Pi_{(X_\square^{\log})_n/k_\square}$ *the kernel of the natural [outer] surjection* $\Pi_{(X_\square^{\log})_n} \twoheadrightarrow G_{k_\square}$*;* $\Delta_n^\square \stackrel{\mathrm{def}}{=} \Pi^{(l)}_{(X_\square^{\log})_n/k_\square}$*;* $\Pi_n^\square$ *the quotient of* $\Pi_{(X_\square^{\log})_n}$ *by the kernel of the natural surjection* $\Pi_{(X_\square^{\log})_n/k_\square} \twoheadrightarrow \Delta_n^\square$*. Thus, the natural conjugation action of* $\Pi_n^\square$ *on* $\Delta_n^\square$ *determines a* **natural outer Galois action** $G_{k_\square} \to \mathrm{Out}(\Delta_n^\square)$*. Set* $n_{\min} = 3$ *if* $(g, r) \neq (0, 3)$*;* $n_{\min} = 2$ *if* $(g, r) = (0, 3)$*. Suppose* **either** *that* $n \geq n_{\min}$ **or** *that the following conditions are satisfied:*

(a) $k_\square$ *is* **strongly** $\boldsymbol{l}$**-cyclotomically full** *[cf. Definition 3.1,* (iii)*].*

(b) *Let* $J \subseteq \Delta_1^\square$ *be a characteristic open subgroup. Observe that* $\Pi_1^\square$ *naturally acts on* J *by conjugation, hence on* $J^{\mathrm{ab/edge}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ *[cf. the discussion entitled "Topological groups" in §0]. Write* $\rho_J : \Pi_1^\square \to \mathrm{Aut}_{\mathbb{Q}_l}(J^{\mathrm{ab/edge}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$ *for this action of* $\Pi_1^\square$ *on* $J^{\mathrm{ab/edge}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$*. Then there exists an element* $g \in \Pi_1^\square$ *such that* $\rho_J(g)$ *is* $\boldsymbol{\chi_{k_\square}(g)}$**-transverse** *[cf. Definition 3.1,* (ii)*], where, by abuse of notation, we write* $\chi_{k_\square}(-)$ *for the restriction of* $\chi_{k_\square}(-)$*, as defined above, via the natural [outer] surjection* $\Pi_1^\square \twoheadrightarrow G_{k_\square}$*.*

Let

$$\alpha : \Delta_n^\circ \xrightarrow{\sim} \Delta_n^\bullet$$

be an **isomorphism** *of profinite groups that is* **compatible** *with the respective natural outer Galois actions* $G^\circ \to \mathrm{Out}(\Delta_n^\circ)$*,* $G^\bullet \to \mathrm{Out}(\Delta_n^\bullet)$ *relative to some isomorphism of profinite groups* $G^\circ \xrightarrow{\sim} G^\bullet$*. Then for any* **log-full subgroup** $A \subseteq \Delta_n^\circ$ *of* Δ_n°*,* $\alpha(A) \subseteq \Delta_n^\bullet$ *is a* **log-full subgroup** *of* $\Delta_n^\bullet$*.*

Theorem D may be considered as a sort of *higher-dimensional analogue* of the reconstruction of *inertia subgroups of surface groups* given in [CmbGC], Corollary 2.7, (i). Finally, we remark that a *complete affirmative answer* to (P2) may be found in [Higashi].

0. Notations and conventions

Numbers:

The notation $\mathbb{Q}$ will be used to denote the field of *rational numbers*. The notation $\mathbb{Z}$ will be used to denote the set, group, or ring of *rational integers*. The notation $\mathbb{N}$ will be used to denote the set or additive monoid of *non-negative rational integers*. The notation $\mathfrak{Primes}$ will be used to denote the set of all *prime numbers*. Let Σ be a nonempty subset of $\mathfrak{Primes}$. Then we shall write

$$\widehat{\mathbb{Z}}^\Sigma$$

for the *pro-*Σ *completion* of $\mathbb{Z}$. If $l \in \mathfrak{Primes}$, then we shall write $\mathbb{Z}_l \stackrel{\mathrm{def}}{=} \widehat{\mathbb{Z}}^{\{l\}}$. The notation $\mathbb{Q}_l$ will be used to denote the quotient field of the ring $\mathbb{Z}_l$. For a field k, we shall denote by $\mathrm{ch}(k)$ the characteristic of k.

Schemes:

If S is a *scheme*, and s is a point of the *underlying set* of the scheme S, then, for simplicity, we shall write

$$s \in S.$$

Log Schemes:

We refer to [KK1], [KK2] for basic facts concerning *log schemes* and *log structures*. In this paper, log structures are always considered on the étale topoi of schemes [or algebraic stacks]. If $X^{\log}$ is a log scheme, then we shall write $\mathcal{M}_X$ for the sheaf of monoids that defines the log structure of $X^{\log}$, X for the underlying scheme of $X^{\log}$. If $f^{\log} : X^{\log} \to Y^{\log}$ is a morphism of log schemes, then we shall write $f : X \to Y$ for the associated underlying morphism of schemes; we shall refer to the image of $\mathcal{M}_X$ in the cokernel of the morphism induced on *groupifications* $f^*\mathcal{M}_Y^{\mathrm{gp}} \to \mathcal{M}_X^{\mathrm{gp}}$ by the morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ determined by $f^{\log}$ as the *relative characteristic* of $f^{\log}$. If $X^{\log}$ is a log scheme, then we shall refer to the relative characteristic of $X^{\log} \to X$, where X is regarded as a log scheme equipped with the trivial log structure, as the *characteristic* of $X^{\log}$.

Let $X^{\log}$ be a log scheme, and $\overline{x}$ a geometric point of X. Then we shall denote by $I(\overline{x}, \mathcal{M}_X)$ the ideal of $\mathcal{O}_{X,\overline{x}}$ generated by the image of $\mathcal{M}_{X,\overline{x}} \setminus \mathcal{O}_{X,\overline{x}}^{\times}$ via the morphism of monoids $\mathcal{M}_{X,\overline{x}} \to \mathcal{O}_{X,\overline{x}}$ induced by the morphism $\mathcal{M}_X \to \mathcal{O}_X$ which defines the log structure of $X^{\log}$.

If $X^{\log}$, $Y^{\log}$ are fs [i.e., fine saturated] log schemes over an fs log scheme $Z^{\log}$, then we shall denote by $X^{\log} \times_{Z^{\log}} Y^{\log}$ the *fiber product* of $X^{\log}$ and $Y^{\log}$ over $Z^{\log}$ in the category of fs log schemes; we shall refer to as the *interior* of $X^{\log}$ the open subscheme of points at which the log structure of $X^{\log}$ is trivial.

Curves:

Let n be a positive integer; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; $X \to S$ a *hyperbolic curve of type* (g, r) [cf. [MT], §0]; P_n the fiber product of n copies of X over S. Then we shall refer to as the *n-th configuration space* of $X \to S$ the S-scheme

$$X_n \to S$$

representing the open subfunctor

$$T \mapsto \{\ (f_1, \ldots, f_n) \in P_n(T) \mid\ f_i \neq f_j \ \text{ if } \ i \neq j\ \}$$

of the functor represented by P_n.

We shall write $\overline{\mathcal{M}}_{g,r}$ for the *moduli stack of r-pointed stable curves of genus g* over $\mathbb{Z}$ [where we assume the marked points to be *ordered*], $\mathcal{M}_{g,r}$ for the open substack of $\overline{\mathcal{M}}_{g,r}$ which parametrizes *smooth curves*, $\overline{\mathcal{M}}_{g,r}^{\log}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the log structure determined by the divisor at infinity $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$, and $\mathcal{C}_{g,[r]} \to \mathcal{M}_{g,[r]}$ for the stack-theoretic quotient of the morphism $\mathcal{M}_{g,r+1} \to \mathcal{M}_{g,r}$ [i.e., determined by forgetting the $(r+1)$-th marked point] by the action of the symmetric group on r letters on the labels of the [first r] marked points. Let k be a field. Then we shall write $(\mathcal{M}_{g,r})_k \stackrel{\text{def}}{=} \mathcal{M}_{g,r} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(k)$.

Let $X^{\log} \to S^{\log}$ be a *stable log curve of type* (g, r) [cf. [CmbGC], §0]. Then we shall refer to as the *n-th log configuration space* of $X^{\log} \to S^{\log}$ the $S^{\log}$-log scheme

$$X_n^{\log} \to S^{\log}$$

obtained by pulling back the morphism $\overline{\mathcal{M}}_{g,r+n}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ determined by forgetting the last n marked points via the classifying morphism $T^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ of $X^{\log} \times_S T \to T^{\log} \stackrel{\text{def}}{=} S^{\log} \times_S T$ for a suitable finite étale covering T of S [i.e., over which the *divisor of cusps* of $X^{\log} \to S^{\log}$ becomes *split*] and then descending from T to S. Note that, when the log structure on $S^{\log}$ is *trivial* [i.e., "$S^{\log} = S$"], in which case the *interior* U of $X^{\log}$ may be regarded as a hyperbolic curve over S, the *interior* of the n-th log configuration space $X_n^{\log}$ may be identified with the n-th configuration space U_n associated to U. When $S^{\log}$ is the *spectrum of a field* equipped with the *trivial log structure*, we shall refer to the divisors of the underlying scheme X_n of $X_n^{\log}$ that lie in the complement of the interior of $X_n^{\log}$ as *log divisors* of $X_n^{\log}$ [or, depending on the context of the discussion, of U_n].

Let Σ be a nonempty subset of $\mathfrak{Primes}$; $S^{\log}$ an fs log scheme whose underlying scheme is the spectrum of an algebraically closed field of characteristic $\notin \Sigma$; $X^{\log} \to S^{\log}$ a stable log curve. Then the pointed stable curve determined by the log structure of $X^{\log}$ defines a *semi-graph of anabelioids of pro-*Σ *PSC-type* $\mathcal{G}_{X^{\log}}$ [cf. [CmbGC], Definition 1.1, (i)]. We shall write

$$\mathcal{E}(X^{\log})$$

for the set of closed points of X which correspond to the edges [i.e., "nodes" and "cusps"] of the underlying semi-graph of $\mathcal{G}_{X^{\log}}$.

Topological Groups:

Let G be a *topological group*. Then we shall use the notation

$$\text{Aut}(G), \quad \text{Out}(G)$$

introduced in the discussion entitled "Topological groups" in [CbTpI], §0. Thus, if G is a *topologically finitely generated profinite group*, then $\text{Aut}(G)$ and $\text{Out}(G)$ admit a *natural profinite topology*. Here, we recall from [NS], Theorem 1.1, that, in fact, if G is a topologically finitely generated profinite group, then $\text{Aut}(G)$ and $\text{Out}(G)$ *remain unaffected* if one *replaces* G by the *discrete* topological group determined by G. This result [NS], Theorem 1.1, also implies that, if G is a topologically finitely generated profinite group, then the natural profinite topologies on $\text{Aut}(G)$ and $\text{Out}(G)$ may be described as the topologies determined by the *subgroups of finite index* of $\text{Aut}(G)$ and $\text{Out}(G)$.

Let Σ be a nonempty subset of $\mathfrak{Primes}$, G a profinite group. Then we shall denote by G^{Σ} the *maximal pro-*Σ *quotient* of G. For $l \in \mathfrak{Primes}$, we shall write $G^{(l)} \stackrel{\text{def}}{=} G^{\{l\}}$. We shall say that G is *almost pro-l* if G admits an open pro-l subgroup.

We shall denote by G^{ab} the *abelianization* of a profinite group G, i.e., the quotient of G by the closure of the commutator subgroup of G. If H is a closed subgroup of a profinite group G, then we shall write $Z_G(H)$ (respectively, $C_G(H)$) for the *centralizer* (respectively, *commensurator*) of H in G [cf. the discussion entitled "Topological groups" in [CbTpI], §0]. We shall say that the closed subgroup H is *commensurably terminal* in G if $H = C_G(H)$. We shall write

$$Z_G^{\mathrm{loc}}(H) \overset{\mathrm{def}}{=} \varinjlim_{H' \subseteq H} Z_G(H') \subseteq G$$

— where $H' \subseteq H$ ranges over the open subgroups of H. We shall refer to $Z_G^{\mathrm{loc}}(H)$ as the *local centralizer* of H in G; $Z^{\mathrm{loc}}(G) \overset{\mathrm{def}}{=} Z_G^{\mathrm{loc}}(G)$ as the *local center* of G.

Let Σ be a nonempty subset of $\mathfrak{Primes}$. For a connected locally noetherian scheme X (respectively, fs log scheme $X^{\log}$; semi-graph of anabelioids $\mathcal{G}$ of pro-Σ PSC-type), we shall denote by

$$\pi_1(X) \quad (\text{respectively, } \pi_1(X^{\log});\ \Pi_{\mathcal{G}})$$

the *étale fundamental group* of X (respectively, *log fundamental group* of $X^{\log}$; *PSC-fundamental group* of $\mathcal{G}$ [cf. [CmbGC], Definition 1.1, (ii)]) [for some choice of basepoint]. We shall denote by

$$\Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$$

the quotient of $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ by the closed subgroup generated by the images in $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ of the edge-like subgroups [cf. [CmbGC], Definition 1.1, (ii)] of $\Pi_{\mathcal{G}}$ [cf. [NodNon], Definition 1.3, (i)].

If K is a field, then we shall write G_K for the *absolute Galois group* of K, i.e., $\pi_1(\mathrm{Spec}(K))$.

Suppose that either $\Sigma = \{l\}$ or $\Sigma = \mathfrak{Primes}$. We shall say that a profinite group G is a *[pro-Σ] surface group* (respectively, a *[pro-Σ] configuration space group*) if G is isomorphic to the maximal pro-Σ quotient of the étale fundamental group of a hyperbolic curve (respectively, the configuration space of a hyperbolic curve) over an algebraically closed field of characteristic zero.

1. Group-theoretic reconstruction of the dimension

In this section, we introduce the notion of a *log-full point* of an fs log scheme [cf. Definition 1.1]. We then discuss some elementary properties of the log-full points of the *log configuration space of a stable log curve* [cf. Proposition 1.3]. As an application, we give a *group-theoretic characterization* of the *dimension* of a [log] configuration space [cf. Theorem 1.6].

Definition 1.1. Let $X^{\log}$ be an fs log scheme. Then we shall say that a point $x \in X$ is *log-full* if, for any geometric point $\overline{x}$ lying over x, the equality

$$\dim(\mathcal{O}_{X,\overline{x}}/I(\overline{x}, \mathcal{M}_X)) = 0$$

[cf. the discussion entitled "Log Schemes" in §0] holds.

Proposition 1.2. (Properties of closed points of log configuration spaces). *Let n be a positive integer; Σ a subset of $\mathfrak{Primes}$ which is either of cardinality one or equal to $\mathfrak{Primes}$; k an algebraically closed field of characteristic $\notin \Sigma$; $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k)$; $S^{\log}$ an fs log scheme whose underlying scheme is S; $X^{\log}$ a stable log curve over $S^{\log}$; $X_n^{\log}$ the n-th log configuration space of $X^{\log}$; $X_0^{\log} \stackrel{\mathrm{def}}{=} S^{\log}$; $X_n^{\log} \to X_{n-1}^{\log}$ the projection morphism obtained by forgetting the factor labeled n, when $n \geq 2$; $X_1^{\log} \to X_0^{\log}$ the stable log curve $X^{\log} \to S^{\log}$; $\mathcal{P}_n$ the groupification of the relative characteristic of $X_n^{\log} \to S^{\log}$; $\mathcal{P}_0 \stackrel{\mathrm{def}}{=} \{0\}$; $x_n \in X_n$ a closed point; x_{n-1} the image of x_n in $X_{n-1}^{\log}$; $x_n^{\log}$ (respectively, $x_{n-1}^{\log}$) the log scheme obtained by restricting the log structure of $X_n^{\log}$ (respectively, $X_{n-1}^{\log}$) to the [reduced, artinian] closed subscheme of X_n (respectively, X_{n-1}) determined by x_n (respectively, x_{n-1}). Moreover, we shall write*

$$(X_n^{\log})_{x_{n-1}} \stackrel{\mathrm{def}}{=} X_n^{\log} \times_{X_{n-1}^{\log}} x_{n-1}^{\log} \to x_{n-1}^{\log}$$

for the stable log curve obtained by base-changing $X_n^{\log} \to X_{n-1}^{\log}$ via the natural inclusion $x_{n-1}^{\log} \hookrightarrow X_{n-1}^{\log}$; $\pi_1(X_n^{\log}/S^{\log})$ (respectively, $\pi_1(x_n^{\log}/S^{\log})$) for the kernel of the natural [outer] surjection $\pi_1(X_n^{\log}) \twoheadrightarrow \pi_1(S^{\log})$ (respectively, $\pi_1(x_n^{\log}) \twoheadrightarrow \pi_1(S^{\log})$); $\Delta_n \stackrel{\mathrm{def}}{=} \pi_1(X_n^{\log}/S^{\log})^{\Sigma}$; $\Delta_{x_n} \stackrel{\mathrm{def}}{=} \pi_1(x_n^{\log}/S^{\log})^{\Sigma}$; Π_n (respectively, Π_{x_n}) for the quotient of $\pi_1(X_n^{\log})$ (respectively, $\pi_1(x_n^{\log})$) by the kernel of the natural surjection $\pi_1(X_n^{\log}/S^{\log}) \twoheadrightarrow \Delta_n$ (respectively, $\pi_1(x_n^{\log}/S^{\log}) \twoheadrightarrow \Delta_{x_n}$). In particular, we have a commutative diagram of [outer] homomorphisms of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{x_n} & \longrightarrow & \Pi_{x_n} & \longrightarrow & \pi_1(S^{\log}) & \longrightarrow & 1 \\ & & \downarrow \iota^{\Delta}_{x_n} & & \downarrow \iota^{\Pi}_{x_n} & & \| & & \\ 1 & \longrightarrow & \Delta_n & \longrightarrow & \Pi_n & \longrightarrow & \pi_1(S^{\log}) & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact, and $\iota^{\Delta}_{x_n}$, $\iota^{\Pi}_{x_n}$ are the [outer] homomorphisms induced by the natural inclusion $x_n^{\log} \hookrightarrow X_n^{\log}$. Then the following hold:

(i) *Let us regard, by abuse of notation, x_n (respectively, x_{n-1}) as a geometric point of X_n (respectively, X_{n-1}). Then the following* **inequalities** *hold:*

$$\mathrm{rk}(\mathcal{P}_{n-1,x_{n-1}}) \ \leq\ \mathrm{rk}(\mathcal{P}_{n,x_n}) \ \leq\ \mathrm{rk}(\mathcal{P}_{n-1,x_{n-1}}) + 1.$$

Here, in the first (respectively, second) inequality, equality holds if and only if $x_n \notin \mathcal{E}((X_n^{\log})_{x_{n-1}})$ (respectively, $x_n \in \mathcal{E}((X_n^{\log})_{x_{n-1}})$).

(ii) $\mathrm{rk}(\mathcal{P}_{n,x_n}) \leq n$.

(iii) $\Delta_{x_n} \cong \widehat{\mathbb{Z}}^{\Sigma}(1)^{\oplus \mathrm{rk}(\mathcal{P}_{n,x_n})}$. *[Here, the "(1)" denotes a "Tate twist".]*

(iv) *The [outer] homomorphism $\iota^{\Delta}_{x_n}$ is* **injective**. *In particular, $\iota^{\Pi}_{x_n}$ is also* **injective**.

Proof. First, we consider assertion (i). To verify the *inequalities*, it suffices to show that

$$0 \leq \mathrm{rk}(\mathcal{P}_{n/n-1,x_n}) \leq 1$$

— where we write $\mathcal{P}_{n/n-1}$ for the *groupification of the relative characteristic* of $X_n^{\log} \to X_{n-1}^{\log}$. But this follows immediately from [KF], Lemma 1.4. Here, note that $\mathrm{rk}(\mathcal{P}_{n/n-1,x_n}) = 0$ (respectively, $\mathrm{rk}(\mathcal{P}_{n/n-1,x_n}) = 1$) holds if and only if $x_n \notin \mathcal{E}((X_n^{\log})_{x_{n-1}})$ (respectively, $x_n \in \mathcal{E}((X_n^{\log})_{x_{n-1}})$) [cf. [KF], Lemma 1.6; [KF], Proof of Theorem 1.3]. This completes the proof assertion (i). Assertion (ii) is an immediate consequence of assertion (i). Assertion (iii) follows from [Hsh], Proposition B.5. Finally, we consider assertion (iv). First, we verify the case $n = 1$. Suppose that $x_1 \notin \mathcal{E}(X^{\log})$. Then since $\Delta_{x_1} \cong \{1\}$ [cf. assertions (i), (iii)], it follows that $\iota_{x_1}^{\Delta}$ is *injective*. Thus, we may assume that $x_1 \in \mathcal{E}(X^{\log})$. Then the injectivity of $\iota_{x_1}^{\Delta}$ follows from [the evident pro-Σ generalization of] [SemiAn], Proposition 2.5, (i) [cf. also [CmbGC], Remark 1.1.3]. This completes the proof of assertion (iv) in the case $n = 1$. Thus, it remains to verify assertion (iv) in the case $n \geq 2$. To this end, let us first observe that the projection morphism $X_n^{\log} \to X_{n-1}^{\log}$ induces [outer] surjections

$$\Delta_n \twoheadrightarrow \Delta_{n-1}; \ \Delta_{x_n} \twoheadrightarrow \Delta_{x_{n-1}}$$

— where "Δ_{n-1}", "$\Delta_{x_{n-1}}$" are defined in the same manner as "Δ_n", "Δ_{x_n}", respectively. Write

$$\Delta_{n/n-1} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Delta_n \twoheadrightarrow \Delta_{n-1}); \ \Delta_{x_n/x_{n-1}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Delta_{x_n} \twoheadrightarrow \Delta_{x_{n-1}}).$$

Thus, we have a commutative diagram of [outer] homomorphisms of profinite groups as follows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{x_n/x_{n-1}} & \longrightarrow & \Delta_{x_n} & \longrightarrow & \Delta_{x_{n-1}} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \iota_{n/n-1}} & & \downarrow{\scriptstyle \iota_{x_n}^{\Delta}} & & \downarrow{\scriptstyle \iota_{x_{n-1}}^{\Delta}} & & \\ 1 & \longrightarrow & \Delta_{n/n-1} & \longrightarrow & \Delta_n & \longrightarrow & \Delta_{n-1} & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and $\iota_{x_{n-1}}^{\Delta}$ (respectively, $\iota_{n/n-1}$) is the [outer] homomorphism induced by the natural inclusion $x_{n-1}^{\log} \hookrightarrow X_{n-1}^{\log}$ (respectively, induced by the [outer] homomorphism $\iota_{x_n}^{\Delta}$). [Here, we recall that the image of $\iota_{n/n-1}$ is *commensurably terminal* in $\Delta_{n/n-1}$ [cf. [CmbGC], Proposition 1.2, (ii)]. This implies that $\iota_{n/n-1}$ is *well-defined* as an *outer* homomorphism.] On the other hand, since [cf. [MT], Proposition 2.2, (i); the discussion of *"specialization isomorphisms"* in the subsection entitled "The Etale Fundamental Group of a Log Scheme" in [CmbCusp], §0] one may identify $\iota_{n/n-1}$ with the "$\iota_{x_1}^{\Delta}$" induced by the natural inclusion

$$x_n^{\log} \hookrightarrow (X_n^{\log})_{x_{n-1}},$$

it follows [from the case $n = 1$] that $\iota_{n/n-1}$ is *injective*. Therefore, by *induction* on n, we conclude that $\iota_{x_n}^{\Delta}$ is *injective*. This completes the proof of assertion (iv). □

Proposition 1.3. (Properties of the log-full points of a log configuration space). *In the notation of Proposition 1.2, the following hold:*

(i) $x_n \in X_n$ *is* **log-full** *if and only if* $\mathrm{rk}(\mathcal{P}_{n,\overline{x}_n}) = n$.

(ii) *If* $x_n \in X_n$ *is* **log-full***, then* x_{n-1} *is also a* **log-full** *point of* X_{n-1}.

(iii) *If* $x_n \in X_n$ *is* **log-full***, then* $\Delta_{x_n} \cong \widehat{\mathbb{Z}}^{\Sigma}(1)^{\oplus n}$.

Proof. First, we verify assertion (i). Fix a *clean chart* $P \to k$ of $S^{\log}$ [cf., e.g., [Hsh], Definition B.1, (ii)]. Write $T^{\log}$ for the log scheme whose underlying scheme is $\mathrm{Spec}(k[[P]])$, and whose log structure is defined by the natural inclusion $P \hookrightarrow k[[P]]$; $S^{\log} \hookrightarrow T^{\log}$ for the [strict] closed immersion determined by the maximal ideal of $k[[P]]$. Thus, it follows immediately from the well-known deformation theory of stable log curves that we may assume without loss of generality that there exists a *stable log curve*

$$Y^{\log} \to T^{\log}$$

whose base-change via $S^{\log} \hookrightarrow T^{\log}$ is isomorphic to [hence may be identified with] $X^{\log} \to S^{\log}$. Write $Y_n^{\log}$ for the n-th log configuration space of $Y^{\log}$. Since $T^{\log}$ is *log regular* [cf. [KK2], Definition 2.1], and $Y_n^{\log} \to T^{\log}$ is *log smooth*, it follows that $Y_n^{\log}$ is also *log regular* [cf. [KK2], Theorem 8.2]. Thus, we obtain an *equality*

$$(*_n) \qquad \dim(\mathcal{O}_{Y_n,x_n}) \ = \ \dim(\mathcal{O}_{Y_n,x_n}/I(x_n,\mathcal{M}_{Y_n})) \ + \ \mathrm{rk}(\mathcal{Q}_{n,x_n})$$

— where we regard x_n as a geometric point of Y_n; $\mathcal{Q}_n$ denotes the *groupification of the characteristic* of $Y_n^{\log}$ [cf. [KK2], Definition 2.1]. On the other hand, it follows from the various definitions involved that we have *equalities*

$$\dim(\mathcal{O}_{Y_n,x_n}) = \dim(\mathcal{O}_{T,x_0}) + n, \quad \mathrm{rk}(\mathcal{Q}_{n,x_n}) = \dim(\mathcal{O}_{T,x_0}) + \mathrm{rk}(\mathcal{P}_{n,x_n}).$$

— where we regard x_0 as a geometric point of T. Combining these equalities with $(*_n)$, we obtain an *equality*

$$\dim(\mathcal{O}_{Y_n,x_n}/I(x_n,\mathcal{M}_{Y_n})) = n - \mathrm{rk}(\mathcal{P}_{n,x_n}).$$

Thus, since $\dim(\mathcal{O}_{X_n,x_n}/I(x_n,\mathcal{M}_{X_n})) = \dim(\mathcal{O}_{Y_n,x_n}/I(x_n,\mathcal{M}_{Y_n}))$, we conclude that $x_n \in X_n$ is log-full if and only if $\mathrm{rk}(\mathcal{P}_{n,x_n}) = n$. This completes the proof of assertion (i). Assertion (ii) follows from assertion (i) and the fact that $\mathrm{rk}(\mathcal{P}_{n,x_n}) = n$ implies $\mathrm{rk}(\mathcal{P}_{n-1,x_{n-1}}) = n-1$ [cf. Proposition 1.2, (i), (ii)]. Assertion (iii) follows from assertion (i) and Proposition 1.2, (iii). □

Proposition 1.4. (Abelian closed subgroups of a pro-l surface group). *Let l be a prime number; G a pro-l surface group. Then every* **nontrivial abelian closed subgroup** *of G is isomorphic to $\mathbb{Z}_l$.*

Proof. Let $H \subseteq G$ be a *nontrivial abelian closed subgroup* of G. Here, note that H is *of infinite index* in G. [Indeed, this follows from the fact that every *open* subgroup of G is also a *pro-l surface group*, hence *nonabelian*.] Thus, it follows from the well-known fact that the *l-cohomological dimension* of a closed infinite index subgroup of a pro-l surface group is ≤ 1 [cf., e.g., the proof of [MT], Theorem 1.5] that H is a *free pro-l group* [cf. [RZ], Theorem 7.7.4]. Since H ($\neq \{1\}$) is *abelian*, we thus conclude that $H \cong \mathbb{Z}_l$. $\square$

Lemma 1.5. (Upper bound for the rank of free abelian pro-l closed subgroups of a pro-l configuration space group). *Let l be a prime number; n a positive integer; s a positive integer; X a hyperbolic curve over an algebraically closed field of characteristic $\neq l$; X_n the n-th configuration space of X; $\Pi_n \stackrel{\text{def}}{=} \pi_1(X_n)^{(l)}$; $H \subseteq \Pi_n$ a closed subgroup such that $H \cong \mathbb{Z}_l^{\oplus s}$. Then the* **inequality**

$$s \leq n$$

holds.

Proof. Let us prove the inequality by induction on n. When $n = 1$, the inequality follows from Proposition 1.4. Now suppose that $n \geq 2$, and that the *induction hypothesis* is in force. Write

$$p : \Pi_n \twoheadrightarrow \Pi_1$$

for the natural [outer] surjection induced by the projection morphism $X_n \to X$ obtained by forgetting the factors with labels $\neq n$. Then, by Proposition 1.4, it follows that either

$$p(H) = \{1\} \quad \text{or} \quad p(H) \cong \mathbb{Z}_l.$$

In particular, $H' \stackrel{\text{def}}{=} H \cap \text{Ker}(p)$ ($\subseteq \text{Ker}(p)$) satisfies either

$$H' \cong \mathbb{Z}_l^{\oplus s} \quad \text{or} \quad H' \cong \mathbb{Z}_l^{\oplus s-1}.$$

Here, we note that $\text{Ker}(p)$ may be identified with the maximal pro-l quotient of the étale fundamental group of the $(n-1)$-*st configuration space* of some hyperbolic curve over an algebraically closed field of characteristic $\neq l$ [cf. [MT], Proposition 2.4, (i)]. Thus, by applying the induction hypothesis, we obtain that either

$$s \leq n-1 \quad \text{or} \quad s-1 \leq n-1,$$

which implies that $s \leq n$, as desired. $\square$

Theorem 1.6. (Group-theoretic characterization of the dimension of a configuration space). *Let l be a prime number; n a positive integer; X a hyperbolic curve over an algebraically closed field k of characteristic $\neq l$; X_n the n-th configuration space of X; $\Pi_n \stackrel{\text{def}}{=} \pi_1(X_n)^{(l)}$. Then the following* **equality** *holds:*

$$n = \max\{s \in \mathbb{N} \mid \exists a \textit{ closed subgroup of } \Pi_n \text{ which is isomorphic to } \mathbb{Z}_l^{\oplus s}\}.$$

Proof. In light of Lemma 1.5, to verify the assertion, it suffices to show the following *claim*:

There exists a closed subgroup H of Π_n such that $H \cong \mathbb{Z}_l^{\oplus n}$.

Thus, we proceed to verify this *claim*. First, we observe that, by applying a suitable *specialization isomorphism* [cf. the discussion preceding [CmbCusp], Definition 2.1, as well as [CbTpI], Remark 5.6.1], one may conclude that there exists a *stable log curve* $Z^{\log} \to S^{\log}$, where S is the spectrum of an algebraically closed field of characteristic $\neq l$, and we write $Z_n^{\log}$ for the n-th log configuration space of $Z^{\log}$, satisfying the following properties:

$$\mathcal{E}(Z^{\log}) \neq \emptyset, \quad \mathrm{Ker}(\pi_1(Z_n^{\log}) \to \pi_1(S^{\log}))^{(l)} \cong \Pi_n.$$

[Here, we note that, although the algebraically closed field "$k_\circ = k$" of the discussion preceding [CmbCusp], Definition 2.1, is assumed to be of characteristic zero, one verifies immediately that, if one restricts one's attention to pro-l fundamental groups, then this discussion generalizes immediately to the case where "$k_\circ = k$" is of characteristic $\neq l$.] Thus, by applying Propositions 1.2, (i); 1.3, (i), it follows that there exists a *log-full* point $z_n \in Z_n$. In particular, by applying Proposition 1.3, (iii) to z_n, we conclude that there exists a closed subgroup H of Π_n such that $H \cong \mathbb{Z}_l^{\oplus n}$. This completes the proof of the *claim*, hence also of Theorem 1.6. □

2. Group-theoretic reconstruction of the genus, number of cusps, and generalized fiber subgroups

In this section, we prove that, if n is a positive integer ≥ 2, then the triple (n, g, r) may be recovered *group-theoretically* from any configuration space group that arises from the n-th configuration space of a hyperbolic curve of type (g, r) over an algebraically closed field of characteristic zero [cf. Theorem 2.5, (i)]. Moreover, by combining this result with a result of Nakamura and Takao [cf. [NT], Theorem D], we generalize [MT], Corollary 6.3 — by introducing the notion of a *generalized fiber subgroup* [cf. Definition 2.1, (ii)] — to the case of *arbitrary hyperbolic curves* [cf. Theorem 2.5, (iii)].

Definition 2.1. Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$; $0 < n' < n$ positive integers; Σ a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers; k an algebraically closed field of characteristic $\notin \Sigma$; X a hyperbolic curve of type (g, r) over k. Write $n'' \overset{\mathrm{def}}{=} n - n'$; X_n for the n-th configuration space of X; $\Pi_n \overset{\mathrm{def}}{=} \pi_1(X_n)^{\Sigma}$.

(i) We define a *generalized projection morphism of length n'* [or, alternatively, a *generalized projection morphism of co-length n''*]

$$X_n \to X_{n''}$$

to be a morphism that may be described as follows:

(a) Suppose that $(g, r) \in \{(0, 3), (1, 1)\}$. Then it follows immediately from the definition of X_n [cf. the discussion entitled "Curves" in §0] that the projection morphism $X_{n+1} \to X_n$ to the first n factors of X_{n+1} determines, in a natural way, a(n) [smooth] $(n + r)$-pointed stable curve of genus g over X_n. One verifies immediately that our assumption that $(g, r) \in$

$\{(0,3),(1,1)\}$ implies that the [smooth] r-pointed stable curve of genus g determined by forgetting *any subset of cardinality* n of the set of $n+r$ *marked points* of the $(n+r)$-pointed stable curve of genus g over X_n determined by the projection morphism $X_{n+1} \to X_n$ is *naturally isomorphic* to the [smooth] r-pointed stable curve of genus g determined by X. We shall refer to the morphism

$$X_n \to X_{n''}$$

obtained by forgetting *some subset of cardinality* n' of the set of $n+r$ *marked points* of the $(n+r)$-pointed stable curve of genus g over X_n determined by the projection morphism $X_{n+1} \to X_n$ discussed above as a *generalized projection morphism of length* n'.

(b) Suppose that $(g,r) \notin \{(0,3),(1,1)\}$. Then we shall refer to the projection morphism $X_n \to X_{n''}$ obtained by forgetting some collection of n' factors as a *generalized projection morphism of length* n'.

Note that, in the case where $(g,r) \notin \{(0,3),(1,1)\}$, the notion of a generalized projection morphism coincides with the notion of a *projection morphism* as defined in [MT], Definition 2.1, (ii). By contrast, in the case where $(g,r) \in \{(0,3),(1,1)\}$, one verifies immediately that there exist generalized projection morphisms that are *not projection morphisms* in the sense of [MT], Definition 2.1, (ii).

(ii) We shall refer to

$$\mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_{n''})$$

— where $\Pi_n \twoheadrightarrow \Pi_{n''}$ is the natural [outer] surjection induced by a generalized projection morphism of length n' — as a *generalized fiber subgroup of* Π_n *of length* n' [or, alternatively, as a *generalized fiber subgroup of* Π_n *of co-length* n'']. Note that, in the case where $(g,r) \notin \{(0,3),(1,1)\}$, the notion of a generalized fiber subgroup of Π_n coincides with the notion of a *fiber subgroup* of Π_n as defined in [MT], Definition 2.3, (iii). By contrast, in the case where $(g,r) \in \{(0,3),(1,1)\}$, one verifies immediately that there exist generalized fiber subgroups of Π_n that are *not fiber subgroups* of Π_n in the sense of [MT], Definition 2.3, (iii).

(iii) We shall refer to a fiber subgroup of Π_n of co-length one in the sense of [MT], Definition 2.3, (iii), as a *co-surface subgroup* of Π_n. We shall refer to a closed normal subgroup H of Π_n such that

Π_n/H is a [pro-Σ] surface group which is not isomorphic to a free pro-Σ group of rank two

as a *quasi-co-surface subgroup* of Π_n.

(iv) We shall say that an automorphism α of Π_n is *gF-admissible* if, for every generalized fiber subgroup F of Π_n, it holds that $\alpha(F) = F$. We shall say that an outer automorphism of Π_n is *gF-admissible* if it is arises from a gF-admissible automorphism. Write

$$\mathrm{Out}^{\mathrm{gF}}(\Pi_n)$$

for the group of gF-admissible outer automorphisms of Π_n.

Remark 2.1.1. In the notation of Definition 2.1, let

$$n^* \stackrel{\mathrm{def}}{=} \begin{cases} n+r & \text{if } (g,r) \in \{(0,3),(1,1)\}; \\ n & \text{if } (g,r) \notin \{(0,3),(1,1)\}. \end{cases}$$

Then one verifies immediately that the natural action of the **symmetric group** on n^* letters

$$\mathfrak{S}_{n^*}$$

on the *set of marked points with labels*

$$\in \{n+r-n^*+1, n+r-n^*+2, \dots, n+r\}$$

[a set of cardinality n^*] of the $(n+r)$-pointed stable curve $X_{n+1} \to X_n$ [determined by the projection morphism to the first n factors of X_{n+1} — cf. the discussion of Definition 2.1, (i); [MT], Remark 2.1.2] induces a natural action of $\mathfrak{S}_{n^*}$ on X_n [i.e., the base space of this $(n+r)$-pointed stable curve]. Moreover, this action induces an *action* of $\mathfrak{S}_{n^*}$ on the set of *generalized fiber subgroups* of Π_n of *co-length* n'', which is easily verified to be *transitive*.

The content of the following Proposition is, in essence, well-known.

Proposition 2.2. (Abelianization of a configuration space group). *Let l be a prime number; $n \geq 2$ an integer; k an algebraically closed field of characteristic $\neq l$; X a hyperbolic curve of type (g,r) over k; X_n the n-th configuration space of X; $\Pi_n \stackrel{\mathrm{def}}{=} \pi_1(X_n)^{(l)}$;*

$$\varphi : \Pi_n^{\mathrm{ab}} \twoheadrightarrow \overbrace{\Pi_1^{\mathrm{ab}} \times \cdots \times \Pi_1^{\mathrm{ab}}}^{n}$$

the surjection induced by the natural open immersion $X_n \hookrightarrow \overbrace{X \times_k \cdots \times_k X}^{n}$; $\Pi_{n/1}$ the kernel of the natural [outer] surjection $\Pi_n \twoheadrightarrow \Pi_1$ induced by the first projection $X_n \to X$. Then the following hold:

(i) *If $g = 0$, then the first projection $X_n \to X$ induces an* **exact** *sequence*

$$1 \longrightarrow \Pi_{n/1}^{\mathrm{ab}} \longrightarrow \Pi_n^{\mathrm{ab}} \longrightarrow \Pi_1^{\mathrm{ab}} \longrightarrow 1.$$

In particular, it follows that Π_n^{ab} is a **finitely-generated free $\mathbb{Z}_l$-module** *of rank*

$$\mathrm{rk}_{\mathbb{Z}_l}(\Pi_n^{\mathrm{ab}}) = \sum_{j=r-1}^{n+r-2} j = n(r-1) + \frac{n(n-1)}{2} > n(r-1).$$

(ii) *If $g \neq 0$, then φ is an* **isomorphism**. *In particular, it follows that* Π_n^{ab} *is a* **finitely-generated free $\mathbb{Z}_l$-module** *of rank*

$$\mathrm{rk}_{\mathbb{Z}_l}(\Pi_n^{\mathrm{ab}}) \;=\; \begin{cases} n(2g+r-1), & \text{if } r > 0 \\ 2gn, & \text{if } r = 0. \end{cases}$$

(iii) $g \neq 0$ *if and only if* φ *is an* **isomorphism**.

(iv) $(g, r) = (1, 1)$ *if and only if* $\mathrm{rk}_{\mathbb{Z}_l}(\Pi_n^{\mathrm{ab}}) = 2n$ *[cf.* (i), (ii)*]*.

Proof. First, we consider assertion (i). We begin by verifying the *exactness* of the sequence of the first display in the statement of assertion (i). Write $\Pi_n^{\mathrm{ab}/1}$ for the quotient of Π_n by the kernel of the natural surjection $\Pi_{n/1} \twoheadrightarrow \Pi_{n/1}^{\mathrm{ab}}$. Thus, we have a *tautological* exact sequence

$$1 \longrightarrow \Pi_{n/1}^{\mathrm{ab}} \longrightarrow \Pi_n^{\mathrm{ab}/1} \longrightarrow \Pi_1 \longrightarrow 1.$$

Next, recall from the well-known structure theory of surface groups that Π_1 is a *free* pro-l group. Thus, the exact sequence of the preceding display admits a *splitting* $s : \Pi_1 \to \Pi_n^{\mathrm{ab}/1}$. Now we *claim* that

> *Claim 2.2A*: The natural conjugation action of Π_1 on $\Pi_{n/1}^{\mathrm{ab}}$ is *trivial*.

Note that the *exactness* of the sequence of the first display in the statement of assertion (i) follows *formally* from *Claim 2.2A*, together with the existence of the *splitting* $s : \Pi_1 \to \Pi_n^{\mathrm{ab}/1}$.

To verify *Claim 2.2A*, we reason as follows. Let $\overline{K}$ be an *algebraic closure* of the *function field* K of X. Thus, the *geometric fiber* $Z \stackrel{\mathrm{def}}{=} X_2 \times_X \mathrm{Spec}(\overline{K})$ of the first projection $X_2 \to X$ is a *hyperbolic curve of genus* 0 over $\overline{K}$. For simplicity, we assume that $\overline{K}$ has been chosen so that Π_1 may be regarded as a *quotient* of the Galois group $\mathrm{Gal}(\overline{K}/K)$. Now observe that the $(n-1)$-st configuration space Z_{n-1} of Z may be naturally identified with the *geometric fiber* $X_n \times_X \mathrm{Spec}(\overline{K})$ of the first projection $X_n \to X$, and hence that $\Pi_{n/1}$ may be regarded as the *maximal pro-l quotient of the étale fundamental group of* Z_{n-1}. Since Z is *of genus* 0, it follows that Z_{n-1} embeds as an open subvariety of the *fiber product* over $\overline{K}$ of $n-1$ copies of the *projective line* $\mathbb{P}^1_{\overline{K}}$ over $\overline{K}$. Since the étale fundamental group of such a fiber product is well-known to be *trivial*, it follows immediately from the *Zariski-Nagata purity theorem* that $\Pi_{n/1}$ is *topologically normally generated* by the *inertia subgroups* $\subseteq \Pi_{n/1}$ associated to the various *log divisors* [cf. the discussion entitled "Curves" in §0] of Z_{n-1}. Moreover, one verifies immediately that each *such log divisor* D_Z arises, by base-changing via the natural morphism $\mathrm{Spec}(\overline{K}) \to X$, from some *log divisor* D_X of X_n [where we regard X_n as an object over X by means of the first projection $X_n \to X$] that corresponds to a *discrete valuation* v of the function field of X_n that induces the *trivial valuation* of K.

Next, let us observe that it follows immediately from the well-known *geometry of moduli stacks of pointed stable curves of genus* 0 that K embeds as a *subfield* of the *residue field* L of the *discrete valuation ring* $\mathcal{O}_v$ determined by v which is *algebraically closed* in L. [Indeed, the log divisor D_X may be identified with a *suitable fiber product over* k *of log configuration spaces* associated to smooth log curves of genus 0 over k. Moreover, the morphism from D_X to the unique [up to isomorphism] *compactification* of X induced by the first projection $X_n \to X$ is easily seen to *coincide* with the [*manifestly geometrically connected!*] projection morphism that arises from one of the projection morphisms associated to one of the log configuration spaces that appears in such a fiber product over k.]

Thus, the *image* $I_Z \subseteq \Pi^{\mathrm{ab}}_{n/1}$ of any inertia subgroup $\subseteq \Pi_{n/1}$ associated to D_Z may be thought of as the *image* $\subseteq \Pi^{\mathrm{ab}}_{n/1}$ of the *maximal pro-l quotient* G_v ($\cong \mathbb{Z}_l$) *of the absolute Galois group* of the quotient field of some *strict henselization* of $\mathcal{O}_v$, whose residue field we denote by $\overline{L}$. Here, we observe that we may assume without loss of generality that L, K, and $\overline{K}$ are subfields of $\overline{L}$ such that $L \cap \overline{K} = K$ [so $\mathrm{Gal}(\overline{K}/K)$ may be regarded as a quotient of $\mathrm{Gal}(\overline{L}/L)$]. Since the natural conjugation action of $\mathrm{Gal}(\overline{L}/L)$ on G_v is the *action via the cyclotomic character* [hence factors through the quotients $\mathrm{Gal}(\overline{L}/L) \twoheadrightarrow \mathrm{Gal}(\overline{K}/K) \twoheadrightarrow \Pi_1$], it follows from the fact that k is *algebraically closed* that this natural conjugation action is *trivial*, hence that the *natural conjugation action* of Π_1 [which we regard as a *quotient* of $\mathrm{Gal}(\overline{L}/L)$!] on $I_Z \subseteq \Pi^{\mathrm{ab}}_{n/1}$ is *trivial*. Since [cf. the above discussion] $\Pi^{\mathrm{ab}}_{n/1}$ is *topologically generated* by such subgroups $I_Z \subseteq \Pi^{\mathrm{ab}}_{n/1}$, we thus conclude that the *natural conjugation action* of Π_1 on $\Pi^{\mathrm{ab}}_{n/1}$ [cf. Claim 2.2A] is *trivial*. This completes the proof of *Claim 2.2A*.

The final portion of assertion (i) now follows formally, by *induction* on n, from the *exactness* of the sequence of the first display in the statement of assertion (i), together with the well-known structure theory of surface groups. This completes the proof of assertion (i).

Next, we consider assertion (ii). First, let us prove that φ is an *isomorphism* in the case where $n = 2$. In this case, we shall denote by $\Pi_{2/1}$ (respectively, M) the *kernel* of the natural [outer] surjection $\Pi_2 \to \Pi_1$ induced by the first projection $X_2 \to X$ (respectively, the *kernel* of φ). In particular, we have a commutative diagram as follows:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & J & \xrightarrow{\alpha} & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Pi^{\mathrm{ab}}_{2/1} & \longrightarrow & \Pi^{\mathrm{ab}}_{2} & \longrightarrow & \Pi^{\mathrm{ab}}_{1} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \| & & \\
1 & \longrightarrow & \Pi^{\mathrm{ab}}_{1} & \xrightarrow{\beta} & \Pi^{\mathrm{ab}}_{1} \times \Pi^{\mathrm{ab}}_{1} & \longrightarrow & \Pi^{\mathrm{ab}}_{1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

— where the middle and lower horizontal sequences are the *exact* sequences induced by the first projection $X \times_k X \to X$; ψ is the *surjection*, whose *kernel* we denote by J, induced by φ; α is the *surjection* induced by the morphism $\Pi_{2/1}^{\mathrm{ab}} \to \Pi_2^{\mathrm{ab}}$; β is the *injection* induced by the morphism $\Pi_{2/1}^{\mathrm{ab}} \to \Pi_2^{\mathrm{ab}}$. To verify that φ is an isomorphism, it suffices to show that $M = \{0\}$. First, let us observe that the structure of J may be described explicitly by applying the well-known structure theory of surface groups. If $r = 0$, then [since $J = \{0\}$] it follows immediately that $M = \{0\}$. Thus, it suffices to show that $M = \{0\}$ under the assumption that $r > 0$. Suppose that $M \neq \{0\}$. Then since $J \cong \mathbb{Z}_l$, it holds that M is a *free* A*-module of rank one*, where either

$$A = \mathbb{Z}_l \quad \text{or} \quad A = \mathbb{Z}_l/l^m\mathbb{Z}_l \ \text{ for some } m > 0.$$

Thus, we obtain a commutative diagram as follows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{2/1} & \longrightarrow & \Pi_2 & \longrightarrow & \Pi_1 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Pi_{2/1}^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} A & \longrightarrow & \Pi_2^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} A & \longrightarrow & \Pi_1^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} A & \longrightarrow & 1
\end{array}$$

— where the vertical arrows are the natural surjections induced by $1 \in A$; the horizontal sequences are induced by the first projection $X \times_k X \to X$; the upper horizontal sequence is *exact by definition*. Next, let us *observe* that it follows immediately from the definition of A that $\alpha \otimes A$ is an *isomorphism*. This *observation*, together with the *split injectivity* of $\beta : \Pi_1^{\mathrm{ab}} \to \Pi_1^{\mathrm{ab}} \times \Pi_1^{\mathrm{ab}}$, implies that the lower horizontal sequence of the diagram of the preceding display is *exact*. Thus, it follows immediately that the *composite*

$$\rho : \Pi_1 \to \mathrm{Aut}(\Pi_{2/1}^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} A)$$

of the outer representation $\Pi_1 \to \mathrm{Out}(\Pi_{2/1})$ with the natural morphism $\mathrm{Out}(\Pi_{2/1}) \to \mathrm{Aut}(\Pi_{2/1}^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} A)$ is *trivial*. On the other hand, we conclude from the evident "$\otimes_{\mathbb{Z}_l} A$ *version*" of [CbTpI], Lemma 1.3, (iv) — which follows immediately [cf. the proof of [CbTpI], Lemma 1.3, (iv)] from the evident "$\otimes_{\mathbb{Z}_l} A$ *version*" of [CbTpI], Lemma 1.3, (iii), together with the well-known structure theory of surface groups — that

> $\mathrm{Ker}(\rho)$ coincides with the *kernel of the natural surjection* [induced by $1 \in A$]
>
> $$\Pi_1 \twoheadrightarrow (\Pi_1^{\mathrm{cpt}})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} A$$
>
> — where we write Π_1^{cpt} for the maximal pro-l quotient of the étale fundamental group of the smooth compactification of X.

In particular, we obtain that

$$(A^{\oplus 2g} \cong) \quad (\Pi_1^{\mathrm{cpt}})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} A = \{0\}.$$

Thus, since $g \neq 0$, we conclude that $M = \{0\}$, a contradiction. This completes the proof that φ is an *isomorphism* in the case where $n = 2$.

Next, we consider the case where $n \geq 3$. First, we make the following *observation*, which follows immediately from the *Zariski-Nagata purity theorem*:

> $\mathrm{Ker}(\varphi) = \{0\}$ if and only if, for every pair of integers (i, j) such that $1 \leq i < j \leq n$, the image, via the natural surjection $\Pi_n \twoheadrightarrow \Pi_n^{\mathrm{ab}}$, of the *inertia subgroup* $I[i, j] \subseteq \Pi_n$ determined, up to Π_n-conjugacy, by the *diagonal divisor* X_n corresponding to the i-th and j-th factors is *trivial.*

Denote by $\Pi[i, j]$ the *kernel* of the natural [outer] surjection $\Pi_n \twoheadrightarrow \Pi_{n-2}$ determined by the projection $f_{i,j} : X_n \to X_{n-2}$ obtained by forgetting the factors labeled i, j. In particular, we may *identify* $\Pi[i, j]$ with the pro-l fundamental group of the *second configuration space* Z_2 of a hyperbolic curve Z of type $(g, r + n - 2)$ over k. Here, we note that, relative to this *identification*, we may identify the *inertia subgroup* of $\Pi[i, j]$ determined, up to Π_n-conjugacy, by the *diagonal divisor* of $Z \times_k Z$ with $I[i, j]$. Thus, by applying the fact that φ is an *isomorphism* in the case where $n = 2$, together with the above *observation*, we conclude that the image of $I[i, j] \subseteq \Pi[i, j]$ via the natural surjection $\Pi[i, j] \twoheadrightarrow (\Pi[i, j])^{\mathrm{ab}}$ is *trivial*, hence that the image of $I[i, j] \subseteq \Pi_n$ via the natural surjection $\Pi_n \twoheadrightarrow \Pi_n^{\mathrm{ab}}$ is *trivial.* Therefore, again by the above *observation*, we obtain that $\mathrm{Ker}(\varphi) = \{0\}$. This completes the proof that φ is an *isomorphism* in the case of *arbitrary* n. The final portion of assertion (ii) now follows formally from the fact that φ is an *isomorphism*, together with the well-known structure theory of surface groups. This completes the proof of assertion (ii). Assertion (iii) follows from assertions (i), (ii) by considering *ranks.*

Finally, we consider assertion (iv). Suppose that $(g, r) = (1, 1)$. Then since $\mathrm{rk}_{\mathbb{Z}_l}(\Pi_1^{\mathrm{ab}}) = 2$, and, moreover, φ is an *isomorphism* [cf. assertion (iii)], it follows that $\mathrm{rk}_{\mathbb{Z}_l}(\Pi_n^{\mathrm{ab}}) = 2n$. Conversely, suppose that $\mathrm{rk}_{\mathbb{Z}_l}(\Pi_n^{\mathrm{ab}}) = 2n$. Then since φ is *surjective*, it follows that

$$\mathrm{rk}_{\mathbb{Z}_l}(\Pi_1^{\mathrm{ab}} \times \cdots \times \Pi_1^{\mathrm{ab}}) \leq 2n.$$

Thus, since $\mathrm{rk}_{\mathbb{Z}_l}(\Pi_1^{\mathrm{ab}}) \geq 2$, we have:

$$\mathrm{rk}_{\mathbb{Z}_l}(\Pi_1^{\mathrm{ab}} \times \cdots \times \Pi_1^{\mathrm{ab}}) = 2n, \qquad \mathrm{rk}_{\mathbb{Z}_l}(\Pi_1^{\mathrm{ab}}) = 2.$$

In particular, the first (respectively, second) equality implies that the surjection φ is, in fact, an *isomorphism*, hence that $g \neq 0$ [cf. assertion (iii)] (respectively, that either $(g, r) = (0, 3)$ or $(g, r) = (1, 1)$ holds). Therefore, we conclude that $(g, r) = (1, 1)$. This completes the proof of assertion (iv). □

Proposition 2.3. (Properties of quasi-co-surface subgroups). *Let $n \geq 2$ be an integer; Σ a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers; X a hyperbolic curve of type (g, r) over an algebraically closed field of characteristic $\notin \Sigma$; X_n the n-th configuration space of X; $\Pi_n \stackrel{\mathrm{def}}{=} \pi_1(X_n)^{\Sigma}$. Then the following hold:*

(i) *Every quasi-co-surface subgroup of* Π_n *contains a* **co-surface subgroup** *of* Π_n.

(ii) *It holds that* $(g, r) \notin \{(0, 3), (1, 1)\}$ *if and only if there exists a* **quasi-co-surface subgroup** *of* Π_n.

(iii) *Suppose that* $(g, r) \notin \{(0, 3), (1, 1)\}$. *Then a subgroup of* Π_n *is a* **co-surface subgroup** *of* Π_n *if and only if it is a* **minimal quasi-co-surface subgroup** *of* Π_n.

Proof. First, we consider assertion (i). Let H be a *quasi-co-surface subgroup* of Π_n;

$$\phi : \Pi_n \twoheadrightarrow G \stackrel{\text{def}}{=} \Pi_n/H$$

the natural surjection. Then we consider the following *claim*:

> *Claim 2.3A*: Any *two fiber subgroups* $J_1, J_2 \subseteq \Pi_n$ *of length one* [cf. [MT], Definition 2.3, (iii)] such that $\phi(J_1) \neq \{1\}$, $\phi(J_2) \neq \{1\}$ necessarily *coincide*.

Indeed, suppose that there exist *two distinct* fiber subgroups $J_1, J_2 \subseteq \Pi_n$ of length one such that $\phi(J_1) \neq \{1\}$, $\phi(J_2) \neq \{1\}$. Then since $\phi(J_1)$ and $\phi(J_2)$ are *topologically finitely generated closed normal subgroups* of the *surface group* G, it follows from [MT], Theorem 1.5, that $\phi(J_1)$ and $\phi(J_2)$ are *open* in G. Next, let us observe that it follows immediately from [MT], Proposition 2.4, (v), that there exists a closed normal subgroup $N \subseteq G$ that is topologically normally generated by a *single element* such that the *images* of $\phi(J_1)$ and $\phi(J_2)$ in G/N *commute*. Thus, we conclude that G/N has an *abelian open subgroup*, hence that G is *nearly abelian* [cf. [MT], Definition 6.1]. On the other hand, since G is a *surface group*, it follows from [MT], Proposition 6.2, that G is a *free pro-*Σ *group of rank two* — in contradiction to the definition of a *quasi-co-surface subgroup* [cf. Definition 2.1, (iii)]. This completes the proof of *Claim 2.3A*. It follows formally from Claim 2.3A that there exist $n-1$ [distinct] fiber subgroups of Π_n of length one whose *images* via ϕ are *trivial*. Thus, the closed normal subgroup of Π_n topologically generated by these $n-1$ fiber subgroups is a *co-surface subgroup* of Π_n that is *contained* in H. This completes the proof of assertion (i).

Next, we consider the following *claim*:

> *Claim 2.3B*: Suppose that $(g, r) \notin \{(0, 3), (1, 1)\}$. Then every *co-surface subgroup* of Π_n is a *quasi-co-surface subgroup* of Π_n.

Claim 2.3B follows immediately from the various definitions involved.

Next, we consider assertion (ii). The *necessity* portion of assertion (ii) follows immediately from Claim 2.3B. Thus, it suffices to verify the *sufficiency* portion of assertion (ii). Suppose that there exists a *quasi-co-surface subgroup* H of Π_n. Then it follows from assertion (i) that there exists a *co-surface subgroup* $N \subseteq \Pi_n$ such that $N \subseteq H$. Thus, we have a *surjection*

$$(\Pi_n/N)^{\text{ab}} \twoheadrightarrow (\Pi_n/H)^{\text{ab}}$$

of *free* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules.* Since $\mathrm{rk}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_n/H)^{\mathrm{ab}} > 2$, we thus conclude that

$$\mathrm{rk}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_n/N)^{\mathrm{ab}} > 2,$$

hence that $(g, r) \notin \{(0,3),(1,1)\}$. This completes the proof of the *sufficiency* portion of assertion (ii).

Finally, we consider assertion (iii). Let H be a *co-surface subgroup* of Π_n. First, we observe that H is a *quasi-co-surface subgroup* of Π_n [cf. Claim 2.3B]. Next, let us verify the *minimality* of H. Let $N \subseteq \Pi_n$ be a *quasi-co-surface subgroup* such that $N \subseteq H$. Then it follows from assertion (i) that there exists a *co-surface subgroup* K of Π_n such that $K \subseteq N$. Since both K and H are *co-surface subgroups* of Π_n, we conclude that $K = H$ [cf. [MT], Proposition 2.4, (ii)], hence that $N = H$. This completes the proof of the *minimality* of H. Conversely, let H be a *minimal quasi-co-surface subgroup* of Π_n. Then, by assertion (i), there exists a *co-surface subgroup* $N \subseteq \Pi_n$ such that $N \subseteq H$. Since N is *quasi-co-surface subgroup* of Π_n [cf. Claim 2.3B], we conclude from the *minimality* of H that $N = H$. In particular, H is a *co-surface subgroup* of Π_n. This completes the proof of assertion (iii). □

Definition 2.4. Let $n \geq 2$ be an integer.

(i) Write

$$\mathcal{L}_{(0,3,n)} \quad (\text{respectively, } \mathcal{L}_{(1,1,n)})$$

for the graded Lie algebra over $\mathbb{Q}_l$ defined by the generators

$$X_{ij} \quad (1 \leq i < j \leq n+3)$$

$$(\text{respectively, } \quad X_1^{(k)}, Y_1^{(k)} \quad (1 \leq k \leq n))$$

[all of which are of *weight* 1] and the relations described in [Nk], Proposition 4.1.1, in the case where the "n" of *loc. cit.* is taken to be $n+3$ (respectively, described in [NTU], 2.8.2, in the case where the "g" of *loc. cit.* is taken to be 1, the "n" of *loc. cit.* is taken to be 1, and the "r" of *loc. cit.* is taken to be n).

(ii) For integers $1 \leq a < b < c < d \leq n+3$ (respectively, $1 \leq e < f \leq n+1$), we denote by

$$W_{(0,3,n)}^{(a,b,c,d)} \quad (\text{respectively, } W_{(1,1,n)}^{(e,f)})$$

the $\mathbb{Q}_l$-vector subspace of $\mathcal{L}_{(0,3,n)}$ generated by the set

$$\{X_{ij} \mid 1 \leq i < j \leq n+3 \text{ and } \{i,j\} \not\subseteq \{a,b,c,d\}\ \}$$

(respectively, the $\mathbb{Q}_l$-vector subspace of $\mathcal{L}_{(1,1,n)}$ generated by the set

$$\{X_1^{(k)}, Y_1^{(k)} \mid 1 \leq k \leq n+1 \text{ and } k \notin \{e,f\}\ \}$$

— where $X_1^{(n+1)} \stackrel{\mathrm{def}}{=} -\sum_{k=1}^{n} X_1^{(k)}$ and $Y_1^{(n+1)} \stackrel{\mathrm{def}}{=} -\sum_{k=1}^{n} Y_1^{(k)}$). Write

$$S_{(0,3,n)} \stackrel{\mathrm{def}}{=} \{W_{(0,3,n)}^{(a,b,c,d)} \mid 1 \leq a < b < c < d \leq n+3\}$$

$$(\text{respectively, } S_{(1,1,n)} \stackrel{\mathrm{def}}{=} \{W_{(1,1,n)}^{(e,f)} \mid 1 \leq e < f \leq n+1\}).$$

Theorem 2.5. (Group-theoreticity of the dimension, genus, number of cusps, and generalized fiber subgroups). *Let l be a prime number; Σ a set of prime numbers which is either equal to $\{l\}$ or $\mathfrak{Primes}$; (g,r) a pair of nonnegative integers such that $2g-2+r>0$; X a hyperbolic curve of type (g,r) over an algebraically closed field k of characteristic $\notin \Sigma$; n a positive integer; X_n the n-th configuration space of X;*

$$\Pi_n \stackrel{\mathrm{def}}{=} \pi_1(X_n)^{\Sigma}.$$

Then:

(i) *One may reconstruct n group-theoretically from Π_n as the* **maximum** *of the set*

$$\{s \in \mathbb{N} \mid \exists a \text{ closed subgroup of } \Pi_n \text{ which is isomorphic to } \mathbb{Z}_l^{\oplus s}\}$$

[cf. Theorem 1.6].

Moreover, if $n \geq 2$ [cf. (i)*], then the following hold:*

(ii) *One may determine whether or not it holds that*

$$(g,r) \notin \{(0,3),(1,1)\}$$

group-theoretically from Π_n by considering whether or not there exists a **quasi-co-surface subgroup** *of Π_n [cf. Proposition 2.3,* (ii)*]. Moreover, one may determine whether or not it holds that*

$$(g,r) = (1,1)$$

group-theoretically from Π_n by considering whether or not it holds that

$$\mathrm{rk}_{\mathbb{Z}_l}(\Pi_n^{(l)})^{\mathrm{ab}} = 2n$$

[cf. (i)*; Proposition 2.2,* (iv)*].*

(iii) *Suppose that $(g,r) \notin \{(0,3),(1,1)\}$ [cf.* (ii)*]. Then the* **generalized fiber subgroups** *of Π_n of* **co-length one** *may be characterized group-theoretically as the* **minimal quasi-co-surface subgroups** *of Π_n [cf. Proposition 2.3,* (iii)*].*

(iv) *Suppose that $(g,r) \in \{(0,3),(1,1)\}$ [cf.* (ii)*]. Write*

$$T_{(g,r,n)}$$

for the set of kernels of the natural surjections

$$(\Pi_n^{(l)})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \twoheadrightarrow (\Pi_1^{(l)})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

arising from generalized projection morphisms $X_n \to X$ of co-length one. Then one may reconstruct $T_{(g,r,n)}$ as the set of **images** *of the $\mathbb{Q}_l$-vector subspaces $W \subseteq \mathcal{L}_{(g,r,n)}$ [cf. Definition 2.4,* (i)*], for $W \in S_{(g,r,n)}$ [cf. Definition 2.4,* (ii)*], via the various* **surjections**

$$\mathcal{L}_{(g,r,n)} \stackrel{\sim}{\to} \mathrm{Gr}(\Pi_n^{(l)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \twoheadrightarrow (\Pi_n^{(l)})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

— where $\mathrm{Gr}(\Pi_n^{(l)})$ is the graded Lie algebra over $\mathbb{Z}_l$ associated to the **lower central series** *of $\Pi_n^{(l)}$; the "$\stackrel{\sim}{\to}$" is an isomorphism of graded Lie algebras over $\mathbb{Q}_l$ [where we recall that such isomorphisms* **exist**

— *cf.* [Nk], *Proposition 4.1.1;* [NTU], *2.8.2]; the "$\twoheadrightarrow$" is the natural surjection. In particular, one may reconstruct the* **generalized fiber subgroups** *of* Π_n *of* **co-length one** *group-theoretically as the* **topologically finitely generated closed normal subgroups** H *of* Π_n *satisfying the following conditions:*

(iv-a) Π_n/H *is* **elastic** *[cf.* [AbsTopI], *Definition 1.1,* (ii)*].*

(iv-b) *The* $\mathbb{Q}_l$*-vector subspace of* $(\Pi_n^{(l)})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ *generated by the image of* H *via the natural morphism* $\Pi_n \to (\Pi_n^{(l)})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ **coincides** *with some element of* $T_{(g,r,n)}$.

(v) *Let* $m < n$ *be a positive integer,* F *a generalized fiber subgroup of* Π_n *of co-length* m*. Then* F *may be naturally identified with*

$$\pi_1(Z_{n-m})^\Sigma$$

— *where* Z_{n-m} *is the* $(n-m)$*-th* **configuration space** *of a hyperbolic curve* Z *of type* $(g, r+m)$ *over an algebraically closed field of characteristic* $\notin \Sigma$*. Moreover, relative to this identification, the generalized fiber subgroups of* Π_n *of co-length* $m+1$ *which are contained in* F *may be identified with the* **generalized fiber subgroups** *of* $F \cong \pi_1(Z_{n-m})^\Sigma$ *of* **co-length one** *[cf. Remark 2.1.1;* [MT], *Remark 2.1.2;* [MT], *Proposition 2.2, (i);* [MT], *Proposition 2.4, (i), (ii)]. In particular, by applying* (iii), (iv) *inductively, one may reconstruct the* **generalized fiber subgroups** *of* Π_n *[of arbitrary length] group-theoretically from* Π_n.

(vi) *One may reconstruct* (g,r) *group-theoretically from* Π_n *as follows:*

(vi-a) *First, let us observe that, by applying* (v)*, one may reconstruct the* **quotients**

$$(\Pi_n \twoheadrightarrow) \;\; \Pi_n^{(l)} \twoheadrightarrow \Pi_2^{(l)} \twoheadrightarrow \Pi_1^{(l)}$$

determined, respectively, by some generalized fiber subgroup of co-length 2 *and some generalized fiber subgroup of co-length* 1 *that contains the generalized fiber subgroup of co-length* 2*. Write* $\Pi_{2/1}^{(l)}$ *for the kernel of the surjection* $\Pi_2^{(l)} \twoheadrightarrow \Pi_1^{(l)}$.

(vi-b) *In particular, one may determine whether or not*

$$r > 0$$

by considering whether or not $\Pi_1^{(l)}$ *[cf.* (vi-a)*] is* **free**.

(vi-c) *When* $r = 0$ *[cf.* (vi-b)*], one may reconstruct* (g,r) *as*

$$g \;=\; \tfrac{1}{2} \cdot \mathrm{rk}_{\mathbb{Z}_l}(\Pi_1^{(l)})^{\mathrm{ab}}, \quad r \;=\; 0.$$

(vi-d) *When* $r > 0$ *[cf.* (vi-b)*], note that, by considering the kernel of the natural action*

$$\Pi_1^{(l)} \to \mathrm{Aut}((\Pi_{2/1}^{(l)})^{\mathrm{ab}})$$

induced by the surjection $\Pi_2^{(l)} \twoheadrightarrow \Pi_1^{(l)}$ *appearing in* (vi-a)*, one may reconstruct*

$$I \stackrel{\text{def}}{=} \text{Ker}((\Pi_1^{(l)})^{\text{ab}} \twoheadrightarrow (\overline{\Pi}_1^{(l)})^{\text{ab}})$$

— where we write $\overline{\Pi}_1$ *for the étale fundamental group of the smooth compactification* $\overline{X}$ *of* X*; the surjection* $(\Pi_1^{(l)})^{\text{ab}} \twoheadrightarrow (\overline{\Pi}_1^{(l)})^{\text{ab}}$ *is the morphism induced by the natural immersion* $X \hookrightarrow \overline{X}$ *[cf.* [CbTpI]*, Lemma 1.3,* (iv)*]. Thus, one may reconstruct* (g, r) *as*

$$g \;=\; \tfrac{1}{2} \cdot \left\{ \text{rk}_{\mathbb{Z}_l}(\Pi_1^{(l)})^{\text{ab}} - \; \text{rk}_{\mathbb{Z}_l} I \right\}, \quad r \;=\; \text{rk}_{\mathbb{Z}_l} I \;+\; 1.$$

Proof. Assertions (i), (ii), (iii), (v), and (vi) follow immediately from the various results cited in the statements of these assertions, together with the various definitions involved. Thus, it remains to verify assertion (iv). First, we consider the reconstruction of generalized fiber subgroups of Π_n of co-length one. It follows from Remark 2.1.1; [MT], Theorem 1.5; [MT], Proposition 2.2, (ii); [MT], Proposition 2.4, (i), that every generalized fiber subgroup of Π_n of co-length one is a *topologically finitely generated* closed normal subgroup of Π_n satisfying conditions (iv-a), (iv-b). Conversely, let H be a topologically finitely generated closed normal subgroup of Π_n satisfying conditions (iv-a), (iv-b). Then it follows from condition (iv-b) that there exists a generalized projection morphism $p_X : X_n \to X$ such that the image of H via the composite

$$\Pi_n \stackrel{p_\Pi}{\twoheadrightarrow} \Pi_1 \stackrel{p^{\text{ab}}}{\to} (\Pi_1^{(l)})^{\text{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

— where the first arrow is the natural [outer] surjection induced by p_X; the second arrow is the natural morphism — is *trivial.* Now we *claim* that $p_\Pi(H) = \{1\}$. Indeed, suppose that $p_\Pi(H) \neq \{1\}$. Then since $p_\Pi(H)$ is a nontrivial topologically finitely generated closed normal subgroup of Π_1, it follows from [MT], Theorem 1.5, that $p_\Pi(H)$ is *open* in Π_1. But this implies that p^{ab} factors through the *finite* quotient $\Pi_1 \twoheadrightarrow \Pi_1/p_\Pi(H)$. Thus, we conclude that $\text{Im}(p^{\text{ab}})$ is *finite* — a contradiction. This completes the proof of the *claim.* In light of the *claim*, it holds that $H \subseteq \text{Ker}(p_\Pi)$. Now suppose that $\text{Ker}(p_\Pi)/H \neq \{1\}$. Then since $\text{Ker}(p_\Pi)/H$ is a nontrivial topologically finitely generated closed normal subgroup of Π_n/H, it follows from condition (iv-a) that $\Pi_n/\text{Ker}(p_\Pi) \xrightarrow{\sim} \Pi_1$ is *finite* — a contradiction. Thus, we conclude that $H = \text{Ker}(p_\Pi)$, hence that H is a generalized fiber subgroup of Π_n of co-length one, as desired.

Thus, to complete the verification of assertion (iv), it remains to consider the reconstruction of the set $T_{(g,r,n)}$. We begin by recalling the *explicit constructions* in the case where

- $\underline{(g, r) = (0, 3)}$: cf. [Nk], (3.1.3); [Nk], Proposition 4.1.1;
- $\underline{(g, r) = (1, 1)}$: cf. [NTU], 2.8.2; [NT], 2.4.1, 2.4.2, 3.2; [NT], Proof of Theorem (3.1) by using (3.3-6), (4).

In particular, let us observe that it follows immediately in the case where

- $\underline{(g, r) = (0, 3)}$, from the *explicit description* of the "A_{ij}" in [Nk], (3.1.3) [which correspond to the "X_{ij}" in [Nk], Proposition 4.1.1],

- $\underline{(g,r)=(1,1)}$, from the *explicit description* of the "$X_i^{(k)}$", "$X_{g+i}^{(k)}$" at the beginning of [NT], 3.2 [which correspond to the "$X_i^{(k)}$", "$Y_i^{(k)}$" of [NTU], 2.8.2] in terms of the "ϕ_k" of [NT], 2.4.1, 2.4.2, together with the discussion [NT], Proof of Theorem (3.1) by using (3.3-6), (4),

that — relative to the isomorphism of graded Lie algebras over $\mathbb{Q}_l$

$$\alpha_{(g,r,n)} : \mathcal{L}_{(g,r,n)} \xrightarrow{\sim} \mathrm{Gr}(\Pi_n^{(l)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

implicit in these explicit descriptions — each subspace $W \in S_{(g,r,n)}$ maps into *some* subspace $T \in T_{(g,r,n)}$. On the other hand, an elementary computation of the *dimensions* of W and T shows that this implies that W maps *isomorphically* onto T. Moreover, one computes immediately that $S_{(g,r,n)}$ and $T_{(g,r,n)}$ are of the *same cardinality.* We thus conclude that $\alpha_{(g,r,n)}$ induces a *bijection* $S_{(g,r,n)} \xrightarrow{\sim} T_{(g,r,n)}$.

Thus, to complete the proof of assertion (iv), it remains to verify the following claim:

> *Claim 2.5A*: Let
>
> $$\beta : \mathcal{L}_{(g,r,n)} \xrightarrow{\sim} \mathrm{Gr}(\Pi_n^{(l)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$
>
> be an isomorphism of graded Lie algebras over $\mathbb{Q}_l$. Then the set of images of the $\mathbb{Q}_l$-vector subspaces $W \subseteq \mathcal{L}_{(g,r,n)}$, for $W \in S_{(g,r,n)}$, via the composite
>
> $$\overline{\beta} : \mathcal{L}_{(g,r,n)} \xrightarrow{\sim} \mathrm{Gr}(\Pi_n^{(l)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \twoheadrightarrow (\Pi_n^{(l)})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$
>
> — where the first arrow is β; the second arrow is the natural surjection — *coincides* with $T_{(g,r,n)}$.

Let us verify Claim 2.5A. In the following, we write $\mathfrak{S}_\square$ for the symmetric group on $\square$ letters.

First, suppose that $(g,r) = (0,3)$. In this case, recall that we have an isomorphism

$$\mathrm{Aut}(\mathrm{Gr}(\Pi_n^{(l)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \cong \mathbb{Q}_l^\times \times \mathfrak{S}_{n+3}$$

[cf. [NT], Theorem D]. Moreover, the action of $\mathbb{Q}_l^\times \times \mathfrak{S}_{n+3}$ on $\mathrm{Gr}(\Pi_n^{(l)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is *described explicitly* in the discussion preceding the statement of [Nk], Lemma 4.1.2 [cf. also [Nk], Proof of Lemma 4.1.2]. It follows immediately from this explicit description [cf. also the above discussion of $\alpha_{(g,r,n)}$] that $\beta \circ \alpha_{(g,r,n)}^{-1} \in \mathrm{Aut}(\mathrm{Gr}(\Pi_n^{(l)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$ induces a *permutation* of the subspaces $\in T_{(g,r,n)}$. Thus, we conclude that the set of images of the $\mathbb{Q}_l$-vector subspaces $W \subseteq \mathcal{L}_{(g,r,n)}$, for $W \in S_{(g,r,n)}$, via $\overline{\beta}$ *coincides* with $T_{(g,r,n)}$, as desired.

Next, suppose that $(g,r) = (1,1)$. In this case, recall that we have an isomorphism

$$\mathrm{Aut}(\mathrm{Gr}(\Pi_n^{(l)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \cong \mathrm{GSp}(2, \mathbb{Q}_l) \times \mathfrak{S}_{n+1}$$

— where we write $\mathrm{GSp}(2, \mathbb{Q}_l)$ for the group of symplectic similitudes of $\mathbb{Q}_l \times \mathbb{Q}_l$, equipped with the standard symplectic form [cf. [NT], Theorem D]. Moreover, it follows immediately from the *explicit description of the permutation* "$\tau \in S_{r+1}$" in the discussion entitled "(3), (4):" in [NT], Proof of Theorem (3.1) by using (3.3-6), that $\beta \circ \alpha_{(g,r,n)}^{-1} \in \mathrm{Aut}(\mathrm{Gr}(\Pi_n^{(l)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$ induces a *permutation* of the subspaces $\in T_{(g,r,n)}$. Thus, we conclude that the set of images of the $\mathbb{Q}_l$-vector subspaces $W \subseteq \mathcal{L}_{(g,r,n)}$, for $W \in S_{(g,r,n)}$, via $\overline{\beta}$ *coincides* with $T_{(g,r,n)}$, as desired. □

Corollary 2.6. (Structure of the group of outer automorphisms of a configuration space group I). *In the notation of Remark 2.1.1, Theorem 2.5, suppose that* $n \geq 2$ *and* $(r,n) \neq (0,2)$. *Then the following hold:*

(i) *Every element of* $\mathrm{Out}(\Pi_n)$ *induces a* **permutation** *on the set — whose cardinality is* n^* *— of* **generalized fiber subgroups** *of* Π_n *of* **length one**. *Here, we recall from Theorem 2.5,* (v), *that this set of generalized fiber subgroups may be reconstructed* **group-theoretically** *from* Π_n. *Thus, by considering such permutations, we obtain the following exact sequence of profinite groups:*

$$1 \longrightarrow \mathrm{Out}^{\mathrm{gF}}(\Pi_n) \longrightarrow \mathrm{Out}(\Pi_n) \longrightarrow \mathfrak{S}_{n^*} \longrightarrow 1.$$

Moreover, the injection

$$\iota_n : \mathfrak{S}_{n^*} \hookrightarrow \mathrm{Out}(\Pi_n)$$

— induced by the natural action of $\mathfrak{S}_{n^*}$ *on* X_n *[cf. Remark 2.1.1] — determines a* **section** *of the above surjection* $\mathrm{Out}(\Pi_n) \twoheadrightarrow \mathfrak{S}_{n^*}$ *and hence a natural isomorphism*

$$\mathrm{Out}^{\mathrm{gF}}(\Pi_n) \times \mathfrak{S}_{n^*} \xrightarrow{\sim} \mathrm{Out}(\Pi_n).$$

In the following, we shall use ι_n *to regard* $\mathfrak{S}_{n^*}$ *as a subgroup of* $\mathrm{Out}(\Pi_n)$. *In particular, we have inclusions*

$$\mathfrak{S}_{n^*} \subseteq Z_{\mathrm{Out}(\Pi_n)}(\mathrm{Out}^{\mathrm{gF}}(\Pi_n)) \subseteq Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n)).$$

(ii) *Suppose that* $(g,r) \notin \{(0,4),(1,1),(1,2),(2,0)\}$. *Then*

$$\mathfrak{S}_{n^*} = Z_{\mathrm{Out}(\Pi_n)}(\mathrm{Out}^{\mathrm{gF}}(\Pi_n)) = Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n))$$

— i.e., one may reconstruct the subgroup $\mathfrak{S}_{n^*} \subseteq \mathrm{Out}(\Pi_n)$ *[cf.* (i)*] group-theoretically from* Π_n *as the* **local center** $Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n))$ *of* $\mathrm{Out}(\Pi_n)$.

(iii) *If* $n^* \geq 3$, *then we have an equality*

$$\mathrm{Out}^{\mathrm{gF}}(\Pi_n) = Z_{\mathrm{Out}(\Pi_n)}(\mathfrak{S}_{n^*}).$$

Proof. First, we consider assertion (i). The fact that ι_n determines a section of the surjection $\mathrm{Out}(\Pi_n) \twoheadrightarrow \mathfrak{S}_{n^*}$ follows immediately from the definitions. Next, let us recall [cf. [CbTpII], Theorem 2.3, (iv)] that [since it follows immediately from the definitions that $\mathrm{Out}^{\mathrm{gF}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$]

$$\iota_n(\mathfrak{S}_n) \subseteq Z_{\mathrm{Out}(\Pi_n)}(\mathrm{Out}^{\mathrm{F}}(\Pi_n)) \subseteq Z_{\mathrm{Out}(\Pi_n)}(\mathrm{Out}^{\mathrm{gF}}(\Pi_n)).$$

Thus, since $n \geq 2$, by conjugating the *composite inclusion* of the preceding display by *arbitrary elements* of $\mathfrak{S}_{n^*}$, we conclude that

$$\iota_n(\mathfrak{S}_{n^*}) \ \subseteq \ Z_{\mathrm{Out}(\Pi_n)}(\mathrm{Out}^{\mathrm{gF}}(\Pi_n)).$$

Now the fact that ι_n determines a natural isomorphism $\mathrm{Out}^{\mathrm{gF}}(\Pi_n) \times \mathfrak{S}_{n^*} \overset{\sim}{\to} \mathrm{Out}(\Pi_n)$ follows formally.

Next, we consider assertion (ii). In light of the inclusions of the final display of the statement of assertion (i), to verify assertion (ii), it suffices to verify the following *claim*:

> *Claim 2.6A*: Suppose that $(g, r) \notin \{(0, 4), (1, 1), (1, 2), (2, 0)\}$. Then we have
>
> $$Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n)) \ \subseteq \ \mathfrak{S}_{n^*}.$$

First, we assume that $(g, r) = (0, 3)$. Let L be a *number field*;

$$\rho_L^{\mathrm{tpd}} : G_L \ \to \ \mathrm{Out}(\Pi_n)$$

the natural pro-Σ outer Galois representation associated to the n-th configuration space T_n of $T \overset{\mathrm{def}}{=} \mathbb{P}^1_L \setminus \{0, 1, \infty\}$; $\sigma \in Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n))$. Thus, there exists an *open* subgroup H of $\mathrm{Out}(\Pi_n)$ such that $\sigma \in Z_{\mathrm{Out}(\Pi_n)}(H)$. In particular, by replacing L by a suitable finite field extension of L, we may assume that $\mathrm{Im}(\rho_L^{\mathrm{tpd}}) \subseteq H$. Then it follows from [Nk], Theorem A; [IN], Theorem C [or, alternatively, from assertion (i); [CbTpII], Theorem A, (i); [LocAn], Theorem A], that

$$\sigma \in Z_{\mathrm{Out}(\Pi_n)}(\mathrm{Im}(\rho_L^{\mathrm{tpd}})) \ = \ \mathfrak{S}_{n^*} \times Z_{\mathrm{Out}^{\mathrm{gF}}(\Pi_n)}(\mathrm{Im}(\rho_L^{\mathrm{tpd}})) \ = \ \mathfrak{S}_{n^*}.$$

This completes the proof of Claim 2.6A in the case where $(g, r) = (0, 3)$.

Thus, in the remainder of the proof of Claim 2.6A, we may assume that

$$(g, r) \notin \{(0, 3), (0, 4), (1, 1), (1, 2), (2, 0)\}.$$

Thus, $n = n^*$, $\mathrm{Out}^{\mathrm{gF}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)$. Let $\sigma \in Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n))$. Thus, there exists an *open* subgroup H of $\mathrm{Out}^{\mathrm{F}}(\Pi_n)$ such that $\sigma \in Z_{\mathrm{Out}(\Pi_n)}(H)$. Moreover, by assertion (i), we may assume without loss of generality that $\sigma \in \mathrm{Out}^{\mathrm{F}}(\Pi_n)$. Now we *claim* that

> *Claim 2.6B*: It holds that
>
> $$\sigma \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$
>
> — where we write $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ for the group of FC-admissible outer automorphisms of Π_n [cf. [CmbCusp], Definition 1.1, (ii)].

Indeed, since, by assertion (i), H is *open* in $\mathrm{Out}(\Pi_n)$, our assumption that $\sigma \in Z_{\mathrm{Out}(\Pi_n)}(H)$ implies that Claim 2.6B follows from [CmbGC], Proposition 2.4, (v); [CmbGC], Corollary 2.7, (i).

Next, let us write

$$\rho_{g,r,n} : \Pi_{(\mathcal{M}_{g,r})_k} \ \to \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

for the outer representation induced by the [1-]morphism

$$(\mathcal{M}_{g,r+n})_k \ \to \ (\mathcal{M}_{g,r})_k$$

obtained by forgetting the last n marked points. Then since, by assertion (i), H is *open* in $\mathrm{Out}(\Pi_n)$, there exists an *open* subgroup N of $\Pi_{(\mathcal{M}_{g,r})_k}$ such that $\sigma \in Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_n)}(\rho_{g,r,n}(N))$. Write

$$\phi_n : \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \rightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$$

for the homomorphism induced by the first projection $X_n \rightarrow X$. Then it follows from [CbTpI], Theorem D, (i), together with our *assumption on* (g, r), that

$$\phi_n(\sigma) \in Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_1)}(\phi_n(\rho_{g,r,n}(N))) = \{1\}.$$

Thus, since ϕ_n is *injective* [cf. [NodNon], Theorem B], we conclude that $\sigma = 1$. This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (i) and the well-known fact that, since $n^* \geq 3$, $\mathfrak{S}_{n^*}$ is *center-free*. □

Definition 2.7. In the notation of Corollary 2.6, suppose that $(g, r) = (0, 3)$. Write

$$\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$$

for the group of FC-admissible outer automorphisms of Π_n [cf. [CmbCusp], Definition 1.1, (ii)] that commute with the image of $\iota_n : \mathfrak{S}_{n+3} \hookrightarrow \mathrm{Out}(\Pi_n)$ [cf. [CmbCusp], Definition 1.1, (vi)]. Then [by a slight abuse of the notational conventions established in the discussion entitled "Topological Groups" in §0 concerning the superscript "Σ"] we shall write

$$\mathrm{GT}^{\Sigma} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FCS}}(\Pi_2)$$

and refer to GT^{Σ} as the *pro-*Σ *Grothendieck-Teichmüller group* [even though it is, in fact, not necessarily a pro-Σ group!]. Note that if $\Sigma = \mathfrak{Primes}$, then GT^{Σ} may be identified with the Grothendieck-Teichmüller group as defined in more classical works [cf. [CmbCusp], Definition 1.11, (i); [CmbCusp], Remark 1.11.1; [CmbCusp], Corollary 1.12, (ii); [CmbCusp], Corollary 4.2, (i)].

Corollary 2.8. (Simplified group-theoretic approach to the pro-Σ Grothendieck-Teichmüller group). *In the notation of Corollary 2.6, suppose that* $(g, r) = (0, 3)$. *Then the following hold:*

(i) *The natural homomorphism* $\mathrm{Out}^{\mathrm{F}}(\Pi_n) \rightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_1)$ *— where we write* $\mathrm{Out}^{\mathrm{F}}((-))$ *for the group of F-admissible outer automorphisms of* $(-)$ *[cf.* [CmbCusp], *Definition 1.1,* (ii)*] — induced by the first projection* $X_n \rightarrow X$ *induces a natural isomorphism*

$$\mathrm{Out}^{\mathrm{gF}}(\Pi_n) \stackrel{\sim}{\rightarrow} \mathrm{GT}^{\Sigma}$$

In particular, we have a natural isomorphism

$$\mathrm{GT}^{\Sigma} \times \mathfrak{S}_{n+3} \stackrel{\sim}{\rightarrow} \mathrm{Out}(\Pi_n)$$

— where we recall that

$$\mathfrak{S}_{n+3} = Z_{\mathrm{Out}(\Pi_n)}(\mathrm{Out}^{\mathrm{gF}}(\Pi_n)) = Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n))$$

[cf. Remark 2.1.1; Corollary 2.6, (i), (ii)*].*

(ii) *We have a natural isomorphism*

$$Z_{\mathrm{Out}(\Pi_n)}(Z^{\mathrm{loc}}(\Pi_n)) \ \overset{\sim}{\to} \ \mathrm{GT}^{\Sigma}.$$

Proof. First, we consider assertion (i). Let us observe that we have

$$\begin{aligned}\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) &= \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \cap Z_{\mathrm{Out}(\Pi_n)}(\mathfrak{S}_{n+3}) \\ &= \mathrm{Out}^{\mathrm{F}}(\Pi_n) \cap \mathrm{Out}^{\mathrm{gF}}(\Pi_n) \\ &= \mathrm{Out}^{\mathrm{gF}}(\Pi_n)\end{aligned}$$

— where we write $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ for the group of *FC-admissible* outer automorphisms of Π_n [cf. [CmbCusp], Definition 1.1, (ii)]; the second equality follows by applying Corollary 2.6, (iii), and [CbTpII], Theorem A, (ii). Thus, assertion (i) follows immediately from [CmbCusp], Corollary 4.2, (ii). Assertion (ii) follows from assertion (i) and Corollary 2.6, (ii), (iii). ☐

Corollary 2.9. (Structure of the group of outer automorphisms of a configuration space group II). *Write* $(\mathcal{C}_{g,[r]})_{\overline{\eta}} \to \overline{\eta}$ *for the result of base-changing the* **tautological curve** $\mathcal{C}_{g,[r]} \to \mathcal{M}_{g,[r]}$ *[cf. the discussion entitled "Curves" in §0] via some* **dominant morphism** $\overline{\eta} \to \mathcal{M}_{g,[r]}$*, where* $\overline{\eta}$ *is the spectrum of an algebraically closed field* Ω*. Then, in the notation of Corollary 2.6, we have inclusions*

$$\mathfrak{S}_{n^*} \ \subseteq \ Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n)) \ \subseteq \ \mathfrak{S}_{n^*} \times G_{g,r}$$

— where we write $G_{g,r} \subseteq \mathrm{Out}(\Pi_n)$ *for the subgroup determined by the natural action of the group* $\mathrm{Aut}_{\overline{\eta}}((\mathcal{C}_{g,[r]})_{\overline{\eta}})$ *of automorphisms of* $(\mathcal{C}_{g,[r]})_{\overline{\eta}}$ *over* $\overline{\eta}$ *on the n-th configuration space of* $(\mathcal{C}_{g,[r]})_{\overline{\eta}}$ *over* $\overline{\eta}$*. Here, we recall that*

$$G_{g,r} = \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \textit{if} \ \ (g,r) = (0,4); \\ \mathbb{Z}/2\mathbb{Z} & \textit{if} \ \ (g,r) \in \{(1,1),(1,2),(2,0)\}. \end{cases}$$

Proof. The first inclusion follows from Corollary 2.6, (i). Thus, it remains to verify the second inclusion. Let $\sigma \in Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n))$. Note that, by replacing σ by the product of σ with an element of $\mathfrak{S}_{n^*}$, we may assume, without loss of generality, that $\sigma \in \mathrm{Out}^{\mathrm{gF}}(\Pi_n)$ [cf. Corollary 2.6, (i)]. Thus, it follows from [CmbGC], Proposition 2.4, (v); [CmbGC], Corollary 2.7, (i), that

$$\sigma \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

— where we write $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ for the group of *FC-admissible* outer automorphisms of Π_n [cf. [CmbCusp], Definition 1.1, (ii)]. Next, let us recall from [NodNon], Theorem B, that the natural morphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ is *injective*. Thus, it follows from [LocAn], Theorem A, applied to a suitable sub-p-adic subfield of Ω, that $\sigma \in G_{g,r}$. This completes the proof of Corollary 2.9. ☐

Corollary 2.10. (Structure of the group of outer automorphisms of a configuration space group III). *In the notation of Corollary 2.6, the following hold:*

(i) *Suppose that* $(g, r) \in \{(0, 4), (1, 2)\}$ *and* $\Sigma = \{l\}$. *Then one may reconstruct the subgroup* $\mathfrak{S}_2 \subseteq \mathrm{Out}(\Pi_2)$ *[cf. Corollary 2.6,* (i)*] group-theoretically from* Π_2 *as follows:*

(i-a) *First, we note that, by applying Theorem 2.5,* (v), *one may reconstruct the* **two quotients**

$$p_1, p_2 : \Pi_2 \twoheadrightarrow \Pi_1$$

determined by the two fiber subgroups of co-length one. Here, we note that in fact, Theorem 2.5, (v), *only yields a reconstruction of the* **kernels** *of these quotients, i.e., not an "identification isomorphism" [cf. the use of the* **same** *notation for the codomains of* p_1, p_2*] between the codomains of* p_1, p_2*; on the other hand, one verifies easily that this abuse of notation [which was made for the sake of keeping the notation simple] does not have any substantive effect on the argument to follow.*

(i-b) *In light of the reconstruction of the quotients* $p_1, p_2 : \Pi_2 \twoheadrightarrow \Pi_1$ *in* (i-a), *one may apply [the group-theoretic reconstruction algorithms implicit in the proof of]* [CbTpI], *Lemma 1.6, to reconstruct the set of* **conjugacy classes of cuspidal inertia subgroups** *of* Π_1.

(i-c) *Write* Z *for the smooth compactification of* X*;* $Z^{\log}$ *for the log scheme whose underlying scheme is* Z*, and whose log structure is determined by the cusps of* X*;* $Z_2^{\log}$ *for the second log configuration space of* $Z^{\log}$*;* $Z_2^{\log} \to Z^{\log}$ *for the natural projection corresponding to* p_1. *Recall that, for every* **cusp** $c \in Z(k)$, *the log structure on* $Z_2^{\log}$ *determines on the* **fiber** $(Z_2)_c$ *of the morphism* $Z_2^{\log} \to Z^{\log}$ *over* c *a structure of* **pointed stable curve**, *which consists of*

- *precisely one irreducible component of genus* 0 *with precisely* 2 *cusps [called the "minor cuspidal component"]* E_c *and*
- *precisely one irreducible component of genus* g *with precisely* $r-1$ *cusps [called the "major cuspidal component"]* F_c

[cf. [CmbCusp], *Definition 1.4,* (i)*]. Write* $\Pi_{2/1} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(p_1)$. *Thus,* $\Pi_{2/1}$ *may be identified with the PSC-fundamental group of the semi-graph of anabelioids of pro-*Σ *PSC-type determined by the pointed stable curve* $(Z_2)_c$ *[cf.* [CmbGC], *Definition 1.1,* (i), (ii)*].*

(i-d) *In the notation of* (i-c), *the* **verticial subgroups**

$$\Pi_{E_c} \subseteq \Pi_{2/1} \quad (\textit{respectively}, \ \Pi_{F_c} \subseteq \Pi_{2/1})$$

associated to E_c *(respectively,* F_c*) [cf.* [CmbCusp], *Definition 1.4,* (ii)*] may be characterized as follows: Let* $I_c \subseteq \Pi_1$ *be a cuspidal inertia subgroup associated to* c *[cf.* (i-b)*]. Write*

$$\rho_c : I_c \to \mathrm{Out}(\Pi_{2/1})$$

for the composite of the inclusion $I_c \hookrightarrow \Pi_1$ *with the outer representation* $\Pi_1 \to \mathrm{Out}(\Pi_{2/1})$ *induced by* p_1*;* $\Pi_{I_c} \stackrel{\mathrm{def}}{=} \Pi_{2/1} \stackrel{\mathrm{out}}{\rtimes} I_c$ *[cf. the discussion entitled "Topological groups" in* [CbTpI], *§0]. Thus, we have a natural exact sequence of profinite groups*

$$1 \longrightarrow \Pi_{2/1} \longrightarrow \Pi_{I_c} \longrightarrow I_c \longrightarrow 1.$$

We shall say that a closed subgroup $H \subseteq \Pi_{2/1}$ *is a* **section-centralizer** *if it may be written in the form* $Z_{\Pi_{I_c}}(s(I))$ *for some section* $s : I_c \to \Pi_{I_c}$ *of the natural surjection* $\Pi_{I_c} \twoheadrightarrow I_c$*. Then the verticial subgroups* $\Pi_{E_c} \subseteq \Pi_{2/1}$ *(respectively,* $\Pi_{F_c} \subseteq \Pi_{2/1}$*) may be characterized as the* **maximal section-centralizers** *whose rank — as a free pro-*Σ *group — is* **two** *(respectively,* **three***) [cf.* [CbTpII], *Corollary 1.7,* (ii)*].*

(i-e) *One may reconstruct the image* J_c *of* $I_c \subseteq \Pi_1$ *in* Π_1^{ab} *as the image of* $\Pi_{E_c} \subseteq \Pi_{2/1}$ *[cf.* (i-c), (i-d)*] via the surjection*

$$\Pi_{2/1} \stackrel{p_2|_{\Pi_{2/1}}}{\twoheadrightarrow} \Pi_1 \twoheadrightarrow \Pi_1^{\mathrm{ab}}$$

— where the second arrow is the natural surjection. In particular, one may reconstruct the $\widehat{\mathbb{Z}}^{\Sigma}$*-submodule*

$$J_c \times J_c \subseteq \Pi_1^{\mathrm{ab}} \times \Pi_1^{\mathrm{ab}}.$$

(i-f) *One may reconstruct the unique nontrivial element of the subgroup* $\mathfrak{S}_2 \subseteq \mathrm{Out}(\Pi_2)$ *as the unique nontrivial element* $\alpha \in Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_2))$ *— whose image in* $\mathrm{Aut}(\Pi_1^{\mathrm{ab}} \times \Pi_1^{\mathrm{ab}})$ *[cf. Theorem 2.5,* (v)*] we denote by* $\overline{\alpha}$ *— such that, for each cusp* $c \in Z(k)$*, i.e., for each cuspidal inertia subgroup* $I_c \subseteq \Pi_1$ *[cf.* (i-b)*],*

$$\overline{\alpha}(J_c \times J_c) = J_c \times J_c.$$

(ii) *Suppose that* $(g, r) = (1, 1)$ *and* $\Sigma = \{l\}$*. Then one may reconstruct the subgroup* $\mathfrak{S}_2 \subseteq \mathfrak{S}_3 \subseteq \mathrm{Out}(\Pi_2)$ *[cf. Corollary 2.6,* (i)*] — i.e., which is well-defined up to* **conjugation** *in* $\mathfrak{S}_3$ *— group-theoretically from* Π_2 *as follows:*

(ii-a) *First, we note that, by applying Theorem 2.5,* (v), *one may reconstruct the* **three quotients**

$$p_1, p_2, p_3 : \Pi_2 \twoheadrightarrow \Pi_1$$

determined by the three generalized fiber subgroups of co-length one. Here, we note that in fact, Theorem 2.5, (v), *only yields a reconstruction of the* **kernels** *of these quotients, i.e., not an "identification isomorphism" [cf. the use of the* **same** *notation for the codomains of* p_1*,* p_2*,* p_3*] between the codomains of* p_1*,* p_2*,* p_3*; on the other hand, one verifies easily that this abuse of notation [which was made for the sake of keeping the notation simple] does not have any substantive effect on the argument to follow. Write*

$$\overline{p}_1, \overline{p}_2, \overline{p}_3 : \Pi_2^{\mathrm{ab}} \twoheadrightarrow \Pi_1^{\mathrm{ab}}$$

for the natural surjections induced by p_1, p_2, p_3, *respectively.*

(ii-b) *Observe that* $\overline{p}_3$ *factors as the composite of the isomorphism*

$$(\overline{p}_1, \overline{p}_2) : \Pi_2^{\mathrm{ab}} \xrightarrow{\sim} \Pi_1^{\mathrm{ab}} \times \Pi_1^{\mathrm{ab}}$$

[cf. Proposition 2.2, (ii)*] with* **some** *surjection*

$$p_- : \Pi_1^{\mathrm{ab}} \times \Pi_1^{\mathrm{ab}} \twoheadrightarrow \Pi_1^{\mathrm{ab}}.$$

whose kernel coincides with the **diagonal**

$$\delta \subseteq \Pi_1^{\mathrm{ab}} \times \Pi_1^{\mathrm{ab}}.$$

In particular, one may reconstruct δ *as* $\mathrm{Ker}(p_-)$ *and hence the* **natural identification** *between the* **copy** *of* Π_1^{ab} *that appears as the codomain of* $\overline{p}_1$ *and the* **copy** *of* Π_1^{ab} *that appears as the codomain of* $\overline{p}_2$ *as the composite isomorphism*

$$\Pi_1^{\mathrm{ab}} \xleftarrow{\sim} \delta \xrightarrow{\sim} \Pi_1^{\mathrm{ab}}$$

— where the first (respectively, second) arrow denotes the arrow induced by $\overline{p}_1$ *(respectively,* $\overline{p}_2$*).*

(ii-c) *One may reconstruct the unique nontrivial element of the subgroup* $\mathfrak{S}_2 \subseteq \mathrm{Out}(\Pi_2)$ *[which is well-defined up to* **conjugation** *in* $\mathfrak{S}_3$*, i.e., which corresponds to* **permutation** *of the projections* p_1, p_2, p_3*] as the unique element* $\alpha \in Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_2))$ *— whose image in* $\mathrm{Aut}(\Pi_2^{\mathrm{ab}}) \xrightarrow{\sim} \mathrm{Aut}(\Pi_1^{\mathrm{ab}} \times \Pi_1^{\mathrm{ab}})$ *we denote by* $\overline{\alpha}$ *— such that, for any* $(x, y) \in \Pi_1^{\mathrm{ab}} \times \Pi_1^{\mathrm{ab}}$,

$$\overline{\alpha}((x, y)) = (y, x).$$

(iii) *Suppose that* $(g, r) = (2, 0)$ *[which implies that* $n \geq 3$*] and* $\Sigma = \{l\}$*. Note that one may reconstruct the* **quotients**

$$\Pi_n \twoheadrightarrow \Pi_3 \twoheadrightarrow \Pi_2$$

determined by fiber subgroups of Π_n *of co-length three and two, respectively [cf. Theorem 2.5,* (v)*]. In the remainder of the present item* (iii)*, we* **fix** *such quotients and write* $F_1 \subseteq \Pi_3$ *for the kernel of the surjection* $\Pi_3 \twoheadrightarrow \Pi_2$*. Then one may reconstruct the subgroup* $\mathfrak{S}_2 \subseteq \mathrm{Out}(\Pi_2)$ *[cf. Corollary 2.6,* (i)*] group-theoretically from* Π_3 *as follows:*

(iii-a) *Let* $F_2 \neq F_1$ *be a fiber subgroup of* Π_3 *of co-length two;* F_{12} *the unique fiber subgroup of* Π_3 *of co-length one that contains* F_1 *and* F_2 *[cf. Theorem 2.5,* (v)*]. Here, let us recall that* Π_3 *may be identified with the maximal pro-*l *quotient of the étale fundamental group of the third configuration space* X_3 *of* X*. Moreover, relative to this identification,* F_{12}/F_1, F_{12}/F_2 *may be naturally identified with the maximal pro-*l *quotient of the étale fundamental group of a hyperbolic curve* X^* *of type* $(2, 1)$ *over* k *[i.e., a geometric fiber of a projection morphism* $X_2 \to X$*];* F_{12} *may be naturally identified with the maximal pro-*l *quotient of the étale fundamental group of the second configuration space*

X_2^* of X^*. In particular, relative to these identifications, one may reconstruct the **quotients**

$$q_1, q_2 : \Pi_2^* \stackrel{\text{def}}{=} \pi_1(X_2^*)^{(l)} \twoheadrightarrow \Pi_1^* \stackrel{\text{def}}{=} \pi_1(X^*)^{(l)}$$

corresponding to the quotients $F_{12} \twoheadrightarrow F_{12}/F_2$, $F_{12} \twoheadrightarrow F_{12}/F_1$, respectively [i.e., corresponding to the two projection morphisms $X_2^* \to X^*$]. Here, we note that in fact, Theorem 2.5, (v)*, only yields a reconstruction of the* **kernels** *of these quotients, i.e., not an "identification isomorphism" [cf. the use of the* **same** *notation for the codomains of q_1, q_2] between the codomains of q_1, q_2; on the other hand, one verifies easily that this abuse of notation [which was made for the sake of keeping the notation simple] does not have any substantive effect on the argument to follow.*

(iii-b) *In light of the reconstruction of the quotients $q_1, q_2 : \Pi_2^* \twoheadrightarrow \Pi_1^*$ in* (iii-a)*, one may apply [the group-theoretic reconstruction algorithms implicit in the proof of]* [CbTpI]*, Lemma 1.6, to reconstruct the* **conjugacy class of cuspidal inertia subgroups** *of Π_1^* associated to the unique cusp of X^*.*

(iii-c) *One may reconstruct the* **conjugacy class of decomposition groups** *of Π_2 associated to the* **diagonal divisor** *$\subseteq X \times_k X$ as the set of* **normalizers** *in Π_2 of the cuspidal inertia subgroups of Π_1^* [$= F_{12}/F_1 \subseteq \Pi_3/F_1 = \Pi_2$] associated to the unique cusp of X^* [cf.* (iii-b)*].*

(iii-d) *Write F_{13} for the unique fiber subgroup of Π_3 of co-length one that contains F_1 but is $\neq F_{12}$ [cf. Theorem 2.5,* (v)*];*

$$p_2, p_3 : \Pi_2 = \Pi_3/F_1 \twoheadrightarrow \Pi_1$$

for the quotients determined by the quotients $\Pi_3/F_1 \twoheadrightarrow \Pi_3/F_{13}$, $\Pi_3/F_1 \twoheadrightarrow \Pi_3/F_{12}$, respectively;

$$\overline{p}_2, \overline{p}_3 : \Pi_2^{\text{ab}} \twoheadrightarrow \Pi_1^{\text{ab}}$$

for the natural surjections induced by p_2, p_3, respectively. Here, we note that in fact, Theorem 2.5, (v)*, only yields a reconstruction of the* **kernels** *of these quotients, i.e., not an "identification isomorphism" [cf. the use of the* **same** *notation for the codomains of p_2, p_3] between the codomains of p_2, p_3; on the other hand, one verifies easily that this abuse of notation [which was made for the sake of keeping the notation simple] does not have any substantive effect on the argument to follow. Then one may reconstruct the* **diagonal**

$$\delta \subseteq \Pi_1^{\text{ab}} \times \Pi_1^{\text{ab}}$$

as the image of any decomposition group associated to the diagonal divisor of $X \times_k X$ [cf. (iii-c)*] via the surjection*

$$\Pi_2 \twoheadrightarrow \Pi_2^{\text{ab}} \stackrel{(\overline{p}_2, \overline{p}_3)}{\twoheadrightarrow} \Pi_1^{\text{ab}} \times \Pi_1^{\text{ab}}$$

— *where the first arrow is the natural surjection. In particular, one may reconstruct the* **natural identification** *between the* **copy** *of* Π_1^{ab} *that appears as the codomain of* $\overline{p}_2$ *and the* **copy** *of* Π_1^{ab} *that appears as the codomain of* $\overline{p}_3$ *as the composite isomorphism*

$$\Pi_1^{\mathrm{ab}} \overset{\sim}{\leftarrow} \delta \overset{\sim}{\rightarrow} \Pi_1^{\mathrm{ab}}$$

— *where the first (respectively, second) arrow denotes the arrow induced by* $\overline{p}_2$ *(respectively,* $\overline{p}_3$*).*

(iii-e) *One may reconstruct the unique nontrivial element of the subgroup* $\mathfrak{S}_2 \subseteq \mathrm{Out}(\Pi_2)$ *as the unique element* $\alpha \in Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_2))$ — *whose image in* $\mathrm{Aut}(\Pi_1^{\mathrm{ab}} \times \Pi_1^{\mathrm{ab}})$ *[cf. Theorem 2.5,* (v)*] we denote by* $\overline{\alpha}$ — *such that, for any* $(x, y) \in \Pi_1^{\mathrm{ab}} \times \Pi_1^{\mathrm{ab}}$,

$$\overline{\alpha}((x, y)) \;=\; (y, x).$$

(iv) *One may reconstruct the subgroup* $\mathfrak{S}_{n^*} \subseteq \mathrm{Out}(\Pi_n)$ *as the subgroup of* $Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n))$ *generated by the elements* α *satisfying the following conditions:*

- *The outer automorphism of* $\Pi_n^{(l)}$ *induced by* α **preserves** *some generalized fiber subgroup* F *of* $\Pi_n^{(l)}$ *of co-length two.*
- *The outer automorphism of* $\Pi_2^{(l)}$ *induced by* α *via the surjection* $\Pi_n^{(l)} \twoheadrightarrow \Pi_2^{(l)}$ *determined by* F *is contained in* $\mathfrak{S}_2 \subseteq \mathrm{Out}(\Pi_2^{(l)})$ *[cf. the descriptions of* (i-f), (ii-c), (iii-e)*].*

Proof. Assertion (i) (respectively, (ii); (iii)) follows immediately — with the *exception* of (i-f) (respectively, (ii-c), (iii-e)) — from the various results cited in the statement of assertion (i) (respectively, (ii); (iii)), together with the various definitions involved. On the other hand, assertion (iv), as well as (i-f), (ii-c), (iii-e), follow immediately, by considering the various *transpositions* in $\mathfrak{S}_{n^*}$, from Corollary 2.9. □

3. Group-theoretic reconstruction of log-full subgroups

Let l be a prime number; n a positive integer; Σ a set of prime numbers which is either equal to $\{l\}$ or $\mathfrak{Primes}$; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; k a field of characteristic $\notin \Sigma$; $\overline{k}$ an algebraic closure of k; $S \overset{\mathrm{def}}{=} \mathrm{Spec}(k)$; $S^{\log}$ an fs log scheme whose underlying scheme is S; $X^{\log}$ a stable log curve of type (g, r) over $S^{\log}$; $X_n^{\log}$ the n-th log configuration space of $X^{\log}$; Δ_n the maximal pro-Σ quotient of the kernel of the natural [outer] surjection $\pi_1(X_n^{\log}) \twoheadrightarrow \pi_1(S^{\log})$. In this section, we introduce the notion of a *pro-*Σ *log-full subgroup* of Δ_n [cf. Definition 3.4] and show that, when $\Sigma = \{l\}$, and, moreover, $X^{\log}$ and k satisfy certain conditions concerning "*weights*", the pro-Σ log-full subgroups may be *reconstructed group-theoretically* from the natural outer action of $\pi_1(S^{\log})$ on Δ_n [cf. Theorem 3.8]. We also introduce [cf. Definition 3.10] and study [cf. Corollary 3.12] the notion of a *log-full subgroup of smooth type* and prove that, in the case where $(g,r) = (1,1)$, log-full subgroups of *smooth type* [cf. Definition 3.10] may be *distinguished* from log-full subgroups of *singular type* [cf. Definition 3.10] by means of a *very simple group-theoretic criterion* [cf. Proposition 3.14], which makes *essential use* of the assumption $(g,r) = (1,1)$.

Definition 3.1. In the notation introduced above:

(i) Write $\mathcal{G}$ for the *semi-graph of anabelioids of pro-*Σ *PSC-type* [cf. [CmbGC], Definition 1.1, (i)] determined by the stable log curve $X^{\log} \to S^{\log}$. [Thus, Δ_1 may be identified with the *PSC-fundamental group* [cf. [CmbGC], Definition 1.1, (ii)] $\Pi_{\mathcal{G}}$ of $\mathcal{G}$.] Write $\mathrm{Out}^{\mathrm{FC}}(-)$ for the group of *FC-admissible* outer automorphisms of $(-)$ [cf. [CmbCusp], Definition 1.1, (ii)],

$$\chi_1 : \mathrm{Out}^{\mathrm{FC}}(\Delta_1) \to \mathbb{Z}_l^\times$$

for the *pro-l cyclotomic character* associated to "$\mathcal{G}_{\rightsquigarrow \mathrm{Node}(\mathcal{G})}$" [cf. [CbTpI], Definitions 2.8 and 3.8], and

$$\chi_n : \mathrm{Out}^{\mathrm{FC}}(\Delta_n) \to \mathbb{Z}_l^\times$$

for the composite with χ_1 of the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Delta_n) \to \mathrm{Out}^{\mathrm{FC}}(\Delta_1)$ [cf. [CmbCusp], Proposition 1.2, (iii); [CbTpI], Corollary 1.9, (i)] induced by any projection morphism $X_n^{\log} \to X_1^{\log}$ of co-length one.

(ii) Let V be a finite dimensional $\mathbb{Q}_l$-vector space, α a $\mathbb{Q}_l$-linear automorphism of V, $\beta \in \mathbb{Q}_l^\times$. Then we shall say that α is *β-transverse* if, for every positive integer N and every eigenvalue $\lambda \in \overline{\mathbb{Q}}_l^\times$ of α^N, it holds that $\lambda \neq \beta^N$. [Thus, if α is *β-transverse*, and M is a *positive integer*, then α^M is *β^M-transverse*.]

(iii) We shall say that a field K is *l-cyclotomically full* if the image of the l-adic cyclotomic character $\chi_K : G_K \to \mathbb{Z}_l^\times$ associated to K is infinite. We shall say that a field K is *strongly l-cyclotomically full* if there exists a normal noetherian domain A such that K is a subfield of the field of fractions L of A, and, moreover, the subset

$$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{ch}(k(\mathfrak{p})) > 0, \text{ and } k(\mathfrak{p}) \text{ is } l\text{-cyclotomically full}\}$$

— where we write $k(\mathfrak{p})$ for the residue field of $\mathfrak{p}$ — of $\mathrm{Spec}(A)$ is Zariski dense in $\mathrm{Spec}(A)$.

Proposition 3.2. (Computation of characters). *Set*

$$m \stackrel{\mathrm{def}}{=} \begin{cases} 2n(g+r-1), & \text{if } g > 0 \text{ and } r > 0, \\ 2ng, & \text{if } g > 0 \text{ and } r = 0, \\ 2n(r-1) + n(n-1), & \text{if } g = 0. \end{cases}$$

Write

$$\delta_n : \mathrm{Out}^{\mathrm{FC}}(\Delta_n) \to \mathbb{Z}_l^\times$$

for the square of the character obtained by forming the composite of the natural morphism $\mathrm{Out}^{\mathrm{FC}}(\Delta_n) \to \mathrm{Aut}(\Delta_n^{\mathrm{ab}})$ *with the determinant homomorphism* $\mathrm{Aut}(\Delta_n^{\mathrm{ab}}) \to \mathbb{Z}_l^{\times}$. *Then* δ_n **coincides** *with the* $\boldsymbol{m}$**-th power** *of* χ_n.

Proof. This follows immediately from Proposition 2.2, (i), (ii); [CmbGC], Proposition 1.3. □

Proposition 3.3. (Basic properties of [strongly] $\boldsymbol{l}$**-cyclotomically full fields).** *Let* K *be a field,* K' *a* **finitely generated** *field extension of* K. *Then the following hold:*

(i) *If* K *is* $\boldsymbol{l}$**-cyclotomically full**, *then* $\mathrm{ch}(K) \neq l$.

(ii) *If* K *is* **strongly** $\boldsymbol{l}$**-cyclotomically full**, *then* K *is* $\boldsymbol{l}$**-cyclotomically full**.

(iii) *Suppose that* $\mathrm{ch}(K) > 0$. *Then* K *is* $\boldsymbol{l}$**-cyclotomically full** *if and only if* K *is* **strongly** $\boldsymbol{l}$**-cyclotomically full**.

(iv) K *is* $\boldsymbol{l}$**-cyclotomically full** *if and only if* K' *is.*

(v) K *is* **strongly** $\boldsymbol{l}$**-cyclotomically full** *if and only if* K' *is.*

(vi) *Suppose that* K *is* **strongly** $\boldsymbol{l}$**-cyclotomically full**. *Write* $T \stackrel{\mathrm{def}}{=} \mathbb{P}^1_K \setminus \{0, 1, \infty\}$. *Let* Y *be a hyperbolic curve over* K *that admits a finite étale Galois covering* $Y \to T$ *of degree* d, *where* d *is* **invertible** *in* K. *Write* Z *for the [*K*-]smooth compactification of* Y,
$$\rho : G_K \to \mathrm{Aut}_{\mathbb{Q}_l}(H^1_{\text{ét}}(Z \times_K \overline{K}, \mathbb{Q}_l))$$
for the Galois representation on the first l*-adic étale cohomology module of* Z. *Then there exists an element* $g \in G_K$ *such that* $\rho(g)$ *is* $\boldsymbol{\chi_K(g)}$**-transverse**.

(vii) *Suppose that* K *is a mixed characteristic, nonarchimedean local field, i.e., a finite extension of the field* $\mathbb{Q}_p$ *of* p*-adic numbers. Then* K *is* $\boldsymbol{l}$**-cyclotomically full**, *but* **not strongly** $\boldsymbol{l}$**-cyclotomically full**.

(viii) *Suppose that* K *is either the field* $\mathbb{Q}$ *of rational numbers or a finite field of characteristic* $\neq l$. *Then every finitely generated field extension of* K *is* **strongly** $\boldsymbol{l}$**-cyclotomically full**.

Proof. Assertion (i) follows immediately from the well-known fact that if $\mathrm{ch}(K) = l$, then the cyclotomic character $\chi_K : G_K \to \mathbb{Z}_l^{\times}$ is *trivial.* Next, we consider assertion (ii). Suppose that K is *strongly* l*-cyclotomically full.* We apply the notation of Definition 3.1, (iii). Then let us observe that to verify that K is l-cyclotomically full, it suffices to show that L is l-cyclotomically full. Also, we observe that we may assume without loss of generality that l is *invertible* in A. Since A is *normal*, this assumption implies that the l-adic cyclotomic character $\chi_L : G_L \to \mathbb{Z}_l^{\times}$ associated to

L may be thought of as the restriction via the natural [outer] surjection $G_L \twoheadrightarrow \pi_1(\mathrm{Spec}(A))$ induced by the injection $A \hookrightarrow L$ of the l-adic cyclotomic character $\chi_A : \pi_1(\mathrm{Spec}(A)) \to \mathbb{Z}_l^\times$ associated to A. On the other hand, since [cf. Definition 3.1, (iii)] there exists a prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $k(\mathfrak{p})$ is *l-cyclotomically full*, we thus conclude that the image of χ_A, hence also of χ_L, is *infinite*. This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (ii) and Definition 3.1, (iii). Since the natural outer homomorphism $G_{K'} \to G_K$ is [well-known to be] *open*, assertion (iv) follows immediately from Definition 3.1, (iii).

Assertion (v) in the case where $\mathrm{ch}(K) > 0$ follows immediately from assertions (iii) and (iv). Next, we consider assertion (v) in the case where $\mathrm{ch}(K) = 0$, and K' is a *finite* extension of K [i.e., $[K' : K] < \infty$]. We apply the notation of Definition 3.1, (iii). Let L' be one of the residue fields of the artinian ring $L \otimes_K K'$. Then observe that since $\mathrm{ch}(K) = \mathrm{ch}(L) = 0$, L' is a *finite separable* field extension of L. But, since A is *noetherian*, this implies that the *normalization* A' of A in L' is a *finite A-module*. Thus, assertion (v) in the case where $\mathrm{ch}(K) = 0$ and $[K' : K] < \infty$ follows immediately from Definition 3.1, (iii), together with assertion (iv). Finally, to complete the proof of assertion (v), it suffices to consider the case where $\mathrm{ch}(K) = 0$ and $[K' : K] = \infty$. Note that K' may be identified with a finite field extension of some *rational function field* $K(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are indeterminates. Thus, by the portion of assertion (v) that has already been verified, to complete the verification assertion (v), we may assume without loss of generality that $K' = K(t_1, \ldots, t_n)$. Moreover, by *induction* on n, we may assume without loss of generality that $n = 1$. Next, let us observe that, in the notation of Definition 3.1, (iii), $L(t_1)$ — which is a field extension of $K' = K(t_1)$ — is the field of fractions of a normal noetherian domain $A[t_1]$. Thus, since the *sufficiency* portion of assertion (v) is immediate, it suffices to verify that if K is strongly l-cyclotomically full, then so is K'. To verify that K' is strongly l-cyclotomically full, it suffices to show that the subset

$$\{\mathfrak{q} \in \mathrm{Spec}(A[t_1]) \mid \mathrm{ch}(k(\mathfrak{q})) > 0, \text{ and } k(\mathfrak{q}) \text{ is } l\text{-cyclotomically full}\}$$

— where we write $k(\mathfrak{q})$ for the residue field of $\mathfrak{q}$ — of $\mathrm{Spec}(A[t_1])$ is *Zariski dense* in $\mathrm{Spec}(A[t_1])$. But this follows immediately from assertion (iv), together with the fact that the morphism $\mathrm{Spec}(A[t_1]) \to \mathrm{Spec}(A)$ induced by the natural injection $A \hookrightarrow A[t_1]$ is *flat*, hence *open* [[EGA], Théorème 2.4.6]. This completes the proof of assertion (v).

Next, we consider assertion (vi). We apply the notation of Definition 3.1, (iii). Then one may *base-change* the diagram $Z \supseteq Y \to T$ of *smooth curves over* K to a diagram of *smooth curves over* L. Next, we observe that, by possibly replacing A by $A[\frac{1}{f}]$ for a suitable nonzero $f \in A$, we may assume without loss of generality — i.e., by applying the natural [outer] surjection $G_L \twoheadrightarrow \pi_1(\mathrm{Spec}(A))$ induced by the injection $A \hookrightarrow L$ — that this diagram of smooth curves over L arises from a diagram of *smooth curves over* A, and that d is *invertible* in A. Thus, by restricting to a point $\mathfrak{p} \in \mathrm{Spec}(A)$, we obtain a diagram of *smooth curves over* $k(\mathfrak{p})$. In particular, we conclude that we may assume without loss of generality [cf. [SGA1], EXPOSÉ X, Corollaires 1.8; 3.9] that $\mathrm{ch}(K) > 0$, and K is *strongly l-cyclotomically full* [cf. assertion (iii)]. Also, we observe that after possibly replacing K by a *finite* field extension of K [cf. assertion (v)], we may assume without loss of generality that ρ factors through the *maximal pro-l quotient* of G_K. Thus, since T is defined over a *finite field*, we may assume without loss of generality [cf. [SGA1], EXPOSÉ X, Corollaire 1.8; [SGA1], EXPOSÉ XIII, Corollaire 5.3] that K is a *finite field*, and that ρ factors through the *maximal pro-l quotient* of G_K. But then assertion (vi) follows immediately from the *"Riemann hypothesis" for abelian varieties over finite fields* [cf., e.g., [Mumf], p. 206]. This completes the proof of assertion (vi).

Next, we consider assertion (vii). When $l \neq p$, the fact that K is *l-cyclotomically full* follows immediately from the fact that the *residue field* of K is *finite* [together with well-known facts concerning the structure of finite fields]. When $l = p$, the fact that K is *l-cyclotomically full* follows immediately from well-known facts concerning *ramification* in *cyclotomic extensions* of $\mathbb{Q}_p$ [cf. also assertion (iv)]. Next, *assume* that K is *strongly l-cyclotomically full*. Observe that, by assertion (v), from the point of view of *deriving a contradiction* to this assumption, we may always replace K by a finite extension of K. Let E be a *Tate curve* over K [i.e., a smooth, proper curve of genus 1 over K that has *stable, multiplicative reduction* over the ring of integers of K] that is *defined over a number field*. Thus, by applying the theory of *Belyi maps* [cf. [Belyi]], we conclude that there exists a *dense open subscheme* $U \subseteq E$ that admits a finite étale morphism $U \to T$, where T is as in assertion (vi). Note, moreover, that, by possibly replacing K by a finite extension of K, we may assume without loss of generality that there exists a finite étale covering $Y \to U$ of hyperbolic curves over K such that the composite finite étale morphism $Y \to U \to T$ is *Galois*. On the other hand, the well-known appearance of the *cyclotomic character* in the Galois action on the Tate module of a Tate curve [cf. [Serre], IV, §A.1.2] implies that this finite étale Galois covering $Y \to T$ yields a *contradiction* to the conclusion of assertion (vi). This completes the proof of assertion (vii). Finally, since finite fields of characteristic $\neq l$ are clearly *l-cyclotomically full*, assertion (viii) follows immediately from assertions (iii), (v). $\square$

Definition 3.4. Let $x_n \in X_n \times_k \overline{k}$ be a *log-full* closed point [cf. Definition 1.1]. Then we shall refer to any Δ_n-conjugate of the image of the natural inclusion $\Delta_{x_n} \hookrightarrow \Delta_n$ of Proposition 1.2, (iv), as a [pro-Σ] *log-full subgroup* of Δ_n associated to x_n. We shall refer to an open subgroup of a log-full subgroup of Δ_n associated to x_n as a [pro-Σ] *quasi-log-full subgroup* of Δ_n associated to x_n.

Proposition 3.5. **(Projections of log-full subgroups).** *Let n'' be a positive integer such that $n'' < n$, A a* **log-full** *subgroup of Δ_n. Write $p : \Delta_n \twoheadrightarrow \Delta_{n''}$ for the natural [outer] surjection induced by the* **projection morphism** *$X_n^{\log} \to X_{n''}^{\log}$ obtained by forgetting $n - n''$ of the n factors of $X_n^{\log}$. Then $p(A)$ is a* **log-full** *subgroup of $\Delta_{n''}$.*

Proof. Proposition 3.5 follows immediately from Proposition 1.3, (ii) [cf. also the *commutative diagram* in the proof of Proposition 1.2]. □

Remark 3.5.1. Observe that, in Proposition 3.5, the natural [outer] surjection $p : \Delta_n \twoheadrightarrow \Delta_{n''}$ may in fact be taken to be the natural [outer] surjection induced by a *"generalized projection morphism"*. That is to say, although the notion of a "generalized projection morphism" is only defined in Definition 2.1, (i), in the case where $X^{\log}$ is *smooth*, one verifies immediately that a similar definition may be made in the case of *arbitrary* $X^{\log}$ [i.e., as in the present §3]. Alternatively, in the notation of Remark 2.1.1, one may observe that, by a similar argument to the argument applied in Remark 2.1.1, the *symmetric group* on n^* letters $\mathfrak{S}_{n^*}$ *acts naturally* on $X_n^{\log}$, hence induces a natural outer action of $\mathfrak{S}_{n^*}$ on Δ_n; thus, in Proposition 3.5, one may take the natural [outer] surjection $p : \Delta_n \twoheadrightarrow \Delta_{n''}$ to be the *composite* of a surjection p as in the statement of Proposition 3.5 with the outer automorphism of Δ_n determined by an element of $\mathfrak{S}_{n^*}$.

Lemma 3.6. (Commensurable terminality). *Every* **log-full** *subgroup of* Δ_n *is* **commensurably terminal** *in* Δ_n*.*

Proof. Let us first observe that, to verify Lemma 3.6, we may assume without loss of generality that k is algebraically closed. Let A be a *log-full* subgroup of Δ_n. Note that the inclusion $A \subseteq C_{\Delta_n}(A)$ is clear. We prove the inclusion $C_{\Delta_n}(A) \subseteq A$ by induction on n. The case where $n = 1$ follows from [CmbGC], Proposition 1.2, (ii). Next, suppose that $n > 1$, and that the *induction hypothesis* is in force. Write $p : \Delta_n \twoheadrightarrow \Delta_{n-1}$ for the natural [outer] surjection induced by the projection morphism $X_n^{\log} \to X_{n-1}^{\log}$ obtained by forgetting the last factor. Then since $p(A)$ is a *log-full* subgroup of Δ_{n-1} [cf. Proposition 1.3, (ii)], by applying the induction hypothesis, we conclude that $C_{\Delta_{n-1}}(p(A)) \subseteq p(A)$. Thus, we obtain that

$$p(A) \;\subseteq\; p(C_{\Delta_n}(A)) \;\subseteq\; C_{\Delta_{n-1}}(p(A)) \;\subseteq\; p(A),$$

hence that $p(A) = p(C_{\Delta_n}(A))$. Therefore, to verify the inclusion $C_{\Delta_n}(A) \subseteq A$, it suffices to show that

$$\Delta_{n/n-1} \cap A \;=\; \Delta_{n/n-1} \cap C_{\Delta_n}(A)$$

— where we write $\Delta_{n/n-1} \stackrel{\text{def}}{=} \mathrm{Ker}(p)$. Let $x_n \in X_n$ be a *log-full* point of X_n such that $A = \Delta_{x_n}$ [cf. Definition 3.4]. Write $x_{n-1} \in X_{n-1}$ for the image of x_n via $X_n^{\log} \to X_{n-1}^{\log}$, $x_{n-1}^{\log}$ for the log scheme obtained by restricting the log structure of $X_{n-1}^{\log}$ to the [reduced, artinian] closed subscheme of X_{n-1} determined by x_{n-1}, and $(X_n^{\log})_{x_{n-1}} \stackrel{\text{def}}{=} X_n^{\log} \times_{X_{n-1}^{\log}} x_{n-1}^{\log}$. Then we observe that — by applying [Hsh], Theorem 2, to the *completion* of the natural projection morphism $X_n^{\log} \to X_{n-1}^{\log}$ along the natural projection morphism $(X_n^{\log})_{x_{n-1}} \to x_{n-1}^{\log}$ — we may identify $\Delta_{n/n-1}$ with the maximal pro-Σ quotient of the kernel of the [outer] surjection $\pi_1((X_n^{\log})_{x_{n-1}}) \twoheadrightarrow \pi_1(x_{n-1}^{\log})$. Next, let us observe that $\Delta_{n/n-1} \cap A$ is a *log-full* subgroup of $\Delta_{n/n-1}$ that arises from the log-full point of the underlying scheme of $(X_n^{\log})_{x_{n-1}}$ determined by x_n [cf. the *commutative diagram* in the proof of Proposition 1.2]. Thus, it follows from the case where $n = 1$ that $C_{\Delta_{n/n-1}}(\Delta_{n/n-1} \cap A) \subseteq \Delta_{n/n-1} \cap A$, hence that

$$\begin{aligned} \Delta_{n/n-1} \cap A \ &\subseteq \ \Delta_{n/n-1} \cap C_{\Delta_n}(A) \\ &\subseteq \ C_{\Delta_{n/n-1}}(\Delta_{n/n-1} \cap A) \ \subseteq \ \Delta_{n/n-1} \cap A. \end{aligned}$$

This completes the proof of the equality $\Delta_{n/n-1} \cap A \ = \ \Delta_{n/n-1} \cap C_{\Delta_n}(A)$ and hence of Lemma 3.6. ☐

Lemma 3.7. (Open subgroups of closed subgroups of profinite groups via commensurable terminality). *For $i \in \{1, 2, 3\}$, let G_i be a profinite group; $H_i \subseteq G_i$ an* **abelian** *closed subgroup of G_i; $I_2 \subseteq G_2$ an* **abelian** *closed subgroup of G_2. Suppose that H_1 is* **commensurably terminal** *in G_1, and that there exists a commutative diagram of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \xrightarrow{p} & G_3 & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are exact, and the vertical arrows are the natural injections. Suppose, moreover, that $I_1 \stackrel{\text{def}}{=} I_2 \cap G_1$ (respectively, $I_3 \stackrel{\text{def}}{=} p(I_2)$) is an **open subgroup** *of H_1 (respectively, of H_3). Then it holds that $Z^{\text{loc}}_{p^{-1}(H_3)}(H_1) \ = \ H_2$. In particular, I_2 is an* **open subgroup** *of H_2.*

Proof. Write $K_2 \stackrel{\text{def}}{=} p^{-1}(H_3)$. In particular, it holds that $H_1 \subseteq H_2 \subseteq K_2$. In light of our assumption that I_1 (respectively, I_3) is an *open subgroup* of H_1 (respectively, of H_3), to complete the verification of Lemma 3.7, it suffices to show that

$$I_2 \subseteq Z^{\text{loc}}_{K_2}(H_1) = H_2.$$

First, let us observe that the inclusion $H_2 \subseteq Z^{\text{loc}}_{K_2}(H_1)$ follows from our assumption that H_2 is *abelian*. On the other hand, since H_1 is *abelian* and *commensurably terminal* in G_1, the following inclusions hold:

$$H_1 \ \subseteq \ G_1 \cap Z^{\text{loc}}_{K_2}(H_1) \ = \ Z^{\text{loc}}_{G_1}(H_1) \ \subseteq \ C_{G_1}(H_1) \ = \ H_1.$$

Thus, we obtain that $H_1 = G_1 \cap Z^{\mathrm{loc}}_{K_2}(H_1)$. In particular, we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_1 & \longrightarrow & K_2 & \longrightarrow & H_3 & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & H_1 & \longrightarrow & Z^{\mathrm{loc}}_{K_2}(H_1) & \longrightarrow & p(Z^{\mathrm{loc}}_{K_2}(H_1)) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are the *natural injections*. It follows immediately from this commutative diagram that the inclusion $H_3 \hookrightarrow p(Z^{\mathrm{loc}}_{K_2}(H_1))$ is, in fact, *bijective*, hence that $Z^{\mathrm{loc}}_{K_2}(H_1) = H_2$. Finally, since I_1 is *open* in H_1, and I_2 is *abelian*, we conclude that

$$I_2 \subseteq Z^{\mathrm{loc}}_{K_2}(H_1) = H_2.$$

This completes the proof of Lemma 3.7. $\square$

Theorem 3.8. (Group-theoretic characterization of log-full subgroups). *Let l be a prime number; n a positive integer; k a field of characteristic $\neq l$; $\chi_k : G_k \to \mathbb{Z}_l^\times$ the l-adic cyclotomic character associated to k; $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k)$; $S^{\log}$ an fs log scheme whose underlying scheme is S; $X^{\log}$ a stable log curve over $S^{\log}$; $\mathcal{G}$ the semi-graph of anabelioids of pro-l PSC-type determined by $X^{\log} \to S^{\log}$ [cf.* [CmbGC]*, Definition 1.1,* (i)*]; $X_n^{\log}$ the n-th log configuration space of $X^{\log}$; $G \stackrel{\mathrm{def}}{=} \pi_1(S^{\log})$; $\pi_1(X_n^{\log}/S^{\log})$ the kernel of the natural [outer] surjection $\pi_1(X_n^{\log}) \twoheadrightarrow G$; $\Delta_n \stackrel{\mathrm{def}}{=} \pi_1(X_n^{\log}/S^{\log})^{(l)}$; Π_n the quotient of $\pi_1(X_n^{\log})$ by the kernel of the natural surjection $\pi_1(X_n^{\log}/S^{\log}) \twoheadrightarrow \Delta_n$. Moreover, suppose that the following two conditions are satisfied:*

(a) *k is* **strongly *l*-cyclotomically full**.

(b) *Let $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}} \cong \Delta_1$ be a characteristic open subgroup. Observe that Π_1 naturally acts on $\Pi_{\mathcal{G}'}$ by conjugation, hence on $\Pi_{\mathcal{G}'}^{\mathrm{ab/edge}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ [cf. the discussion entitled "Topological groups" in §0]. Write $\rho_{\mathcal{G}'} : \Pi_1 \to \mathrm{Aut}_{\mathbb{Q}_l}(\Pi_{\mathcal{G}'}^{\mathrm{ab/edge}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$ for this action of Π_1 on $\Pi_{\mathcal{G}'}^{\mathrm{ab/edge}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Then there exists an element $g \in \Pi_1$ such that $\rho_{\mathcal{G}'}(g)$ is* **$\boldsymbol{\chi_k(g)}$-transverse** *[cf. Definition 3.1,* (ii)*], where, by abuse of notation, we write $\chi_k(-)$ for the restriction of $\chi_k(-)$, as defined above, via the natural [outer] surjection $\Pi_1 \twoheadrightarrow G \twoheadrightarrow G_k$.*

Let A be a closed subgroup of Δ_n. Then A is **quasi-log-full** *(respectively,* **log-full***) if and only if A satisfies the following two conditions:*

(i) *The pro-l group A is* **abelian** *and [necessarily — cf.* [MT]*, Remark 1.2.2;* [MT]*, Proposition 2.2,* (i)*]* **free** *of* **rank n** *(respectively, is a* **maximal abelian** *closed subgroup of Δ_n [hence necessarily* **free** *of* **rank n** *— cf. Theorem 1.6;* [MT]*, Remark 1.2.2;* [MT]*, Proposition 2.2,* (i)*]).*

(ii) *There exists a closed subgroup H of Π_n such that*

(ii-1) *the image of H via the surjection $\Pi_n \twoheadrightarrow G$ is an* **open** *subgroup of G;*

(ii-2) *the conjugation action of H on Δ_n* **preserves** *$A \subseteq \Delta_n$, and the resulting action of H on A is given by a* **character** *$\chi_H : H \to \mathbb{Z}_l^\times$;*

(ii-3) *the* **m-th power** *of χ_H [cf.* (ii-2)*], where we write m for the positive integer "m" appearing in the statement of Proposition 3.2,* **coincides** *with the restriction $\delta_n|_H$ of δ_n [cf. Proposition 3.2] to H [via the homomorphism $H \to \mathrm{Out}^{\mathrm{FC}}(\Delta_n)$ determined by the conjugation action of H on Δ_n].*

Here, we note that, in the situation of (ii-3)*, the definition of $\delta_n|_H$ is* **manifestly group-theoretic***, and that a* **group-theoretic** *characterization of the integer m may be obtained by applying Theorem 2.5,* (i), (vi)*, together with [when $n = 1$] assumption (a); Proposition 3.3,* (ii)*;* [CmbGC]*, Corollary 2.7,* (i).

Proof. First, we note that it follows from Lemma 3.6 that the *equivalence*

A is *log-full* $\Leftrightarrow$ A satisfies the *resp'd* portion of conditions (i) and (ii)

follows immediately from the *equivalence*

A is *quasi-log-full* $\Leftrightarrow$ A satisfies the *non-resp'd* portion of conditions (i) and (ii).

Thus, in the remainder of the proof of Theorem 3.8, we prove the *non-resp'd equivalence.* Suppose that A satisfies conditions (i), (ii). Then we prove that A is *quasi-log-full* by *induction on n.* First, we consider the case where $n = 1$. In this case, since A is *nontrivial procyclic* [cf. condition (i)], it follows from assumption (b); condition (ii); Proposition 3.2; [NodNon], Lemma 1.6, that A is contained in a *log-full* subgroup $\subseteq \Delta_1$, hence that A is *quasi-log-full.*

Next, we suppose that $n \geq 2$, and that the *induction hypothesis* is in force. Let $\overline{k}$ be an algebraic closure of k, $\overline{S} \overset{\mathrm{def}}{=} \mathrm{Spec}(\overline{k})$. In this case, we observe that we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta_{n/1} \longrightarrow \Delta_n \overset{p}{\longrightarrow} \Delta_1 \longrightarrow 1$$

— where we write p for the [outer] surjection induced by the first projection $X_n^{\log} \to X^{\log}$ and $\Delta_{n/1} \overset{\mathrm{def}}{=} \mathrm{Ker}(p)$. In particular, it follows from Lemma 1.5 and condition (i) [cf. also [MT], Remark 1.2.2; [MT], Proposition 2.4, (i)] that $p(A)$ is a *free*, *abelian* closed subgroup of Δ_1 of rank 1. Moreover, since $\mathbb{Z}_l^\times$ admits a *torsion-free open subgroup*, one verifies immediately, by applying Proposition 3.2, that, after possibly replacing H [cf. condition (ii)] by a suitable open subgroup of H, it holds that the image of H via the natural surjection $\Pi_n \twoheadrightarrow \Pi_1$ induced by the first projection $X_n^{\log} \to X^{\log}$ satisfies "condition (ii) for the closed subgroup $p(A) \subseteq \Delta_1$". In particular, it follows from the case where $n = 1$ [which has already been verified] that $p(A)$ is *quasi-log-full*, hence, that there exists a *log-full* point $\overline{x} \in X \times_S \overline{S}$ such that $p(A)$ is an open subgroup of a log-full subgroup $\Delta_{\overline{x}} \subseteq \Delta_1$ associated to $\overline{x}$.

Write $x \in X$ for the image of $\overline{x}$ via the natural morphism $X \times_S \overline{S} \to X$, $S_x^{\log}$ for the log scheme obtained by restricting the log structure of $X^{\log}$ to the closed subscheme of X determined by x. Then, by considering the base-change of the first projection $X_2^{\log} \to X^{\log}$ via the natural inclusion $S_x^{\log} \hookrightarrow X^{\log}$, we obtain a stable log curve

$$X_x^{\log} \stackrel{\text{def}}{=} X_2^{\log} \times_{X^{\log}} S_x^{\log} \to S_x^{\log}.$$

Next, observe that the stable log curve $X_x^{\log}$ may be obtained by "attaching" the *tripod* "$\mathbb{P}^1_{S_x} \setminus \{0, 1, \infty\}$", in a suitable fashion, to the stable log curve $X^{\log} \times_{S^{\log}} S_x^{\log}$. Thus, by applying Proposition 3.3, (ii), (v), (vi), to suitable connected finite étale coverings of the *tripod*, we conclude from assumptions (a), (b) [cf. also [NodNon], Lemma 1.4; [CmbGC], Remark 1.1.3] that *"assumptions (a), (b) for the stable log curve $X_x^{\log}$" hold.*

Next, let us observe that the fs log scheme over $S_x^{\log}$

$$X_n^{\log} \times_{X^{\log}} S_x^{\log} \to S_x^{\log}$$

— where the fiber product denotes the base-change of the first projection $X_n^{\log} \to X^{\log}$ via the natural inclusion $S_x^{\log} \hookrightarrow X^{\log}$ — may be naturally identified with the $(n-1)$*-st log configuration space* $(X_x)_{n-1}^{\log}$ of $X_x^{\log}$. On the other hand, since the *kernel* of the natural [outer] surjection $\pi_1((X_x)_{n-1}^{\log}) \to \pi_1(S_x^{\log})$ may be identified with the kernel of the natural [outer] surjection $\pi_1(X_n^{\log}) \twoheadrightarrow \pi_1(X^{\log})$ [cf. [Hsh], Theorem 2, which we think of as being applied to the *completion* of the natural projection morphism $X_n^{\log} \to X^{\log}$ along the natural projection morphism $(X_x)_{n-1}^{\log} \to S_x^{\log}$], we obtain a *commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{n/1} & \longrightarrow & \Pi_n & \longrightarrow & \Pi_1 & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Delta_{n/1} & \longrightarrow & \Pi_{n-1}^x & \longrightarrow & G^x & \longrightarrow & 1
\end{array}$$

— where we write $G^x \stackrel{\text{def}}{=} \pi_1(S_x^{\log})$ and Π_{n-1}^x for the quotient of $\pi_1((X_x)_{n-1}^{\log})$ by the kernel of the natural surjection $\text{Ker}(\pi_1(X_n^{\log}) \twoheadrightarrow \pi_1(X^{\log})) \twoheadrightarrow \Delta_{n/1}$; the horizontal sequences are *exact*; the middle (respectively, right-hand) vertical arrow is the [outer] homomorphism induced by $(X_x)_{n-1}^{\log} \to X_n^{\log}$ (respectively, $S_x^{\log} \to X^{\log}$).

Thus, $A \cap \Delta_{n/1}$ is a *free abelian* closed subgroup of $\Delta_{n/1}$ of rank $n-1$; moreover, one verifies immediately, by applying Proposition 3.2, that [since $\mathbb{Z}_l^\times$ admits a *torsion-free open subgroup*] there exists an open subgroup of the closed subgroup of Π_{n-1}^x obtained by pulling back $H \subseteq \Pi_n$ via the arrow $\Pi_{n-1}^x \to \Pi_n$ of the above commutative diagram with respect to which "condition (ii) for the closed subgroup $A \cap \Delta_{n/1} \subseteq \Delta_{n/1}$" is satisfied. In particular, we conclude from the *induction hypothesis* that $A \cap \Delta_{n/1}$ is *quasi-log-full*, hence that there exists a *log-full* point $\overline{y} \in (X_x)_{n-1} \times_{S_x} \overline{S}$, such that $A \cap \Delta_{n/1}$ is an *open* subgroup of a log-full subgroup $\Delta_{\overline{y}} \subseteq \Delta_{n/1}$ associated to $\overline{y}$.

Next, let us write $\overline{z} \in X_n \times_S \overline{S}$ for the image of $\overline{y} \in (X_x)_{n-1} \times_{S_x} \overline{S}$ via the morphism

$$(X_x)_{n-1} \times_{S_x} \overline{S} \to (X_n \times_S S_x) \times_{S_x} \overline{S} = X_n \times_S \overline{S}$$

obtained by considering the base-change of the natural morphism $(X_x)_{n-1} \to X_n \times_S S_x$ via the natural morphism $\overline{S} \to S_x$. Here, note that since $\overline{x}$ [$\in X \times_S \overline{S}$] and $\overline{y}$ [$\in (X_x)_{n-1} \times_{S_x} \overline{S}$] are *log-full*, it follows from Proposition 1.3, (i), that $\overline{z} \in X_n \times_S \overline{S}$ is *log-full*. Moreover, one verifies easily that there exists a log-full subgroup $\Delta_{\overline{z}} \subseteq \Delta_n$ — among its various Δ_n-conjugates — associated to $\overline{z}$ that fits into a *commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{n/1} & \longrightarrow & \Delta_n & \stackrel{p}{\longrightarrow} & \Delta_1 & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Delta_{\overline{y}} & \longrightarrow & \Delta_{\overline{z}} & \longrightarrow & \Delta_{\overline{x}} & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact* [cf. the *exactness* discussed above of the sequence $1 \to \Delta_{n/1} \to \Pi_n \to \Pi_1 \to 1$; the *commutative diagram* in the proof of Proposition 1.2], and the vertical arrows are the natural injections. Therefore, since $p(A)$ (respectively, $A \cap \Delta_{n/1}$) is an *open subgroup* of $\Delta_{\overline{x}}$ (respectively, $\Delta_{\overline{y}}$), it follows from Lemmas 3.6, 3.7 [where we take the "commutative diagram" to be the commutative diagram of the above display and "I_2" to be A] that A is an *open subgroup* of $\Delta_{\overline{z}}$, hence that A is *quasi-log-full*, as desired.

Conversely, suppose that A is *quasi-log-full*, i.e., that A is an open subgroup of a *log-full* subgroup $\Delta_{\overline{z}} \subseteq \Delta_n$ associated to a log-full point $\overline{z} \in X_n \times_S \overline{S}$. Then since $\Delta_{\overline{z}} \cong \mathbb{Z}_l(1)^{\oplus n}$ [cf. Proposition 1.3, (iii)], we conclude that A satisfies condition (i). Write $S_z^{\log}$ for the log scheme obtained by restricting the log structure of $X_n^{\log}$ to the image of $\overline{z}$ via the natural projection $X_n \times_S \overline{S} \to X_n$, H for the image of the composite

$$\pi_1(S_z^{\log}) \to \pi_1(X_n^{\log}) \twoheadrightarrow \Pi_n$$

— where the first arrow is the [outer] homomorphism induced by the natural morphism $S_z^{\log} \to X_n^{\log}$; the second arrow is the natural surjection. Then one verifies easily [cf. Proposition 3.2] that H satisfies condition (ii). This completes the proof of Theorem 3.8. □

Corollary 3.9. (Group-theoretic preservation of log-full subgroups). *Let l be a prime number; n a positive integer. For $\square \in \{\circ, \bullet\}$, let $S_\square^{\log}$ be an fs log scheme whose underlying scheme is the spectrum of a field $k_\square$ of characteristic $\neq l$; $X_\square^{\log}$ a stable log curve over $S_\square^{\log}$; $(X_\square^{\log})_n$ the n-th log configuration space of $X_\square^{\log}$; $\pi_1((X_\square^{\log})_n/S_\square^{\log})$ the kernel of the natural [outer] surjection $\pi_1((X_\square^{\log})_n) \twoheadrightarrow G^\square \stackrel{\mathrm{def}}{=} \pi_1(S_\square^{\log})$; $\Delta_n^\square \stackrel{\mathrm{def}}{=} \pi_1((X_\square^{\log})_n/S_\square^{\log})^{(l)}$; $\Pi_n^\square$ the quotient of $\pi_1((X_\square^{\log})_n)$ by the kernel of the natural surjection $\pi_1((X_\square^{\log})_n/S_\square^{\log}) \twoheadrightarrow \Delta_n^\square$. Thus, the natural conjugation action of $\Pi_n^\square$ on $\Delta_n^\square$ determines a* **natural outer Galois action** *$G^\square \to \mathrm{Out}(\Delta_n^\square)$. Suppose that, for each $\square \in \{\circ, \bullet\}$, the above collection of data [i.e., with "$\square$'s" omitted] satisfies assumptions (a), (b) in the statement of Theorem 3.8. Let*

$$\alpha : \Delta_n^\circ \stackrel{\sim}{\to} \Delta_n^\bullet$$

be an **isomorphism** *of profinite groups that is* **compatible** *with the respective natural outer Galois actions $G^\circ \to \mathrm{Out}(\Delta_n^\circ)$, $G^\bullet \to \mathrm{Out}(\Delta_n^\bullet)$ relative to some isomorphism of profinite groups $G^\circ \stackrel{\sim}{\to} G^\bullet$. Then for any* **quasi-log-full** *(respectively,* **log-full***) subgroup $A \subseteq \Delta_n^\circ$ of Δ_n°, $\alpha(A) \subseteq \Delta_n^\bullet$ is a* **quasi-log-full** *(respectively,* **log-full***) subgroup of $\Delta_n^\bullet$.*

Proof. Corollary 3.9 follows immediately from Theorem 3.8. $\square$

Finally, in the remainder of the present §3, we introduce [cf. Definition 3.10, below] and study [cf. Corollary 3.12, below] the notion of a *log-full subgroup of smooth type* and show that, at least in the case of *once-punctured elliptic curves*, the condition that a log-full subgroup be of *smooth type* admits a *simple group-theoretic characterization* [cf. Proposition 3.14, Corollary 3.15, below].

Definition 3.10. Let $A \subseteq \Delta_n$ be a *log-full subgroup* [cf. Definition 3.4] of Δ_n. For $i \in \{1, \ldots, n^*\}$ [cf. Remark 2.1.1], write $p_i : \Delta_n \twoheadrightarrow \Delta_1$ for the natural [outer] surjection induced by the generalized projection morphism [cf. Definition 2.1, (i); Remark 3.5.1] $X_n^{\log} \to X^{\log}$ determined by forgetting the marked points with labels $\in \{1, \ldots, n^*\} \setminus \{i\}$. We shall say that A is of *smooth type* if, for every $i \in \{1, \ldots, n^*\}$, the log-full subgroup [cf. Proposition 3.5; Remark 3.5.1] $p_i(A) \subseteq \Delta_1$ is a *cuspidal edge-like subgroup* [cf. Definition 3.1, (i); [CmbGC], Definition 1.1, (ii)] of Δ_1. We shall refer to a log-full subgroup of Δ_n which is not of smooth type as a log-full subgroup of *singular type*.

Remark 3.10.1. Observe that it follows immediately from Definition 3.10 that if there *exists* a log-full subgroup of Δ_n which is of *smooth type*, then $r > 0$.

Proposition 3.11. (Alternative characterization of log-full subgroups of smooth type). *Fix a clean chart* $P \to k$ *of* $S^{\log}$ *[cf., e.g.,* [Hsh]*, Definition B.1,* (ii)*]. Write* $T^{\log}$ *for the log scheme whose underlying scheme is* $\mathrm{Spec}(k[[P]])$*, and whose log structure is defined by the natural inclusion* $P \hookrightarrow k[[P]]$*;* $S^{\log} \hookrightarrow T^{\log}$ *for the [strict] closed immersion determined by the maximal ideal of* $k[[P]]$*. Let* $Y^{\log} \to T^{\log}$ *be a generically smooth stable log curve whose base-change via* $S^{\log} \hookrightarrow T^{\log}$ *is isomorphic to [hence may be identified with]* $X^{\log} \to S^{\log}$*;* $Y_{\overline{\eta}}$ *the hyperbolic curve determined by the interior of some geometric generic fiber of* $Y^{\log} \to T^{\log}$*;* $(Y_{\overline{\eta}})_n$ *the* n*-th configuration space of* $Y_{\overline{\eta}}$*;* A *a log-full subgroup of* Δ_n*. Here, recall that we have a natural* **specialization isomorphism**

$$\pi_1((Y_{\overline{\eta}})_n)^{(l)} \xrightarrow{\sim} \Delta_n^{(l)}$$

[cf. the discussion preceding [CmbCusp]*, Definition 2.1, as well as* [CbTpI]*, Remark 5.6.1]. Then the following conditions are equivalent:*

(i) A *is of* **smooth type***.*

(ii) *For each* $l \in \Sigma$*, the image of* A *in* $\Delta_n^{(l)}$ *coincides with the image, via the above specialization isomorphism, of* **some log-full subgroup** *of* $\pi_1((Y_{\overline{\eta}})_n)^{(l)}$*.*

Proof. Proposition 3.11 follows immediately from Proposition 3.5; [CmbGC], Proposition 1.2, (i), together with the fact that a stable log curve is *smooth* in a suitable neighborhood of a *cusp*. □

Remark 3.11.1. Note that it follows immediately from Proposition 3.11 that, in fact, in Definition 3.10, an *equivalent definition* is obtained if one replaces "n^*" by "n".

Corollary 3.12. (Preservation of log-full subgroups of smooth type by FC-admissible outer automorphisms). *Let* $A \subseteq \Delta_n$ *be a* **log-full subgroup of smooth type***,* σ *an automorphism of* Δ_n*. Set* $n_{\min} = 3$ *if* $(g, r) \neq (0, 3)$*;* $n_{\min} = 2$ *if* $(g, r) = (0, 3)$*. Then:*

(i) *If* σ *is* **FC-admissible** *[cf.* [CmbCusp]*, Definition 1.1,* (ii)*], then* $\sigma(A) \subseteq \Delta_n$ *is a* **log-full subgroup of smooth type***.*

(ii) *If* $n \geq n_{\min}$*, then* $\sigma(A) \subseteq \Delta_n$ *is a* **log-full subgroup of smooth type** *[i.e., even if* σ *is not assumed to be FC-admissible!].*

Proof. We begin by observing that, by applying the *alternative characterization* of Proposition 3.11, we may assume without loss of generality that $X^{\log}$ is a *smooth log curve*. Then assertion (i) follows immediately from the well-known structure of *fibers of* $X_n^{\log} \to X_{n-1}^{\log}$ *over log-full points* [cf. [CbTpI], Lemma 5.4, (i), (ii)] — which allows one to interpret *log-full subgroups* of Δ_{n-1} as groups "Dehn(−)" of *profinite Dehn-multi-twists* [cf. [CbTpI], Proposition 5.6, (ii)] — by applying *induction on* n, together with [CbTpI], Theorem 5.14, (iii) [concerning the *normalizer* of "Dehn(−)"];

[CmbGC], Proposition 2.4, (v) [concerning the *graphic fullness* of such *normalizers*]; [CmbGC], Corollary 2.7, (ii) [concerning the *graphicity* implied by this *graphic fullness*]. Finally, assertion (ii) follows formally from assertion (i), together with Remark 3.10.1; Corollary 2.6, (i); [CbTpII], Theorem A, (ii). □

Remark 3.12.1. At first glance, it may appear to be unclear whether or not Corollary 3.12, (ii), is applicable to situations such as the situation in the statement of Theorem D of the Introduction, i.e., situations involving *two distinct collections of data*, labeled by $\circ$ and $\bullet$. In fact, however, Corollary 3.12, (ii), may be applied to such situations, by considering the following argument: The *existence* of the isomorphism α of Theorem D implies [cf. Theorem 2.5, (vi)] that the type "(g, r)" associated to the data labeled by $\square \in \{\circ, \bullet\}$ is *independent* of the choice of $\square$. In particular, it follows from the [well-known!] *connectedness of the moduli stack of hyperbolic curves of type* (g, r) that there *exists at least one* isomorphism of profinite groups

$$\alpha_{\mathrm{mod}} : \Delta_n^\circ \xrightarrow{\sim} \Delta_n^\bullet$$

[i.e., arising from the classical theory of the étale fundamental group!] that *maps log-full subgroups to log-full subgroups*. On the other hand, the given isomorphism α differs from α_{mod} by composition with an *automorphism* $\alpha_\circ$ of Δ_n°. Thus, the fact that $\alpha_\circ$, hence also α, maps log-full subgroups to log-full subgroups follows from a *direct application* of Corollary 3.12, (ii).

Lemma 3.13. (Group-theoretic characterization of edge-like subgroups of smooth type for once-punctured elliptic curves). *Suppose that* $(g, r) = (1, 1)$*, and that* X *is* **singular***. Let* A *be an* **edge-like subgroup** *[cf. Definition 3.1,* (i)*;* [CmbGC]*, Definition 1.1,* (ii)*] — i.e., a* **log-full subgroup** *— of* $\Delta_1 \xrightarrow{\sim} \Pi_{\mathcal{G}}$*. Then the following conditions are equivalent:*

(i) *A is a* **cuspidal** *edge-like subgroup [cf. Definition 3.1,* (i)*;* [CmbGC]*, Definition 1.1,* (ii)*] of* $\Pi_{\mathcal{G}}$*.*
(ii) *The image of A via the natural surjection* $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab}}$ *is* **trivial***.*

Proof. Lemma 3.13 follows immediately from the following well-known *fact*:

> If A is a *cuspidal* (respectively, *nodal*) edge-like subgroup of $\Pi_{\mathcal{G}}$, then the natural surjection $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab}}$ maps A to $\{0\}$ (respectively, A isomorphically onto its [nonzero] image).

In the following, we give a brief sketch of a proof of this well-known *fact*, for the convenience of the reader. First, observe that we may assume without loss of generality that $\Sigma = \{l\}$. Write $p_{\mathcal{G}} : \Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab}}$ for the natural surjection. One verifies easily that $p_{\mathcal{G}}$ maps *cuspidal* edge-like subgroups of $\Pi_{\mathcal{G}}$ to $\{0\}$ [cf. the *specialization isomorphism* reviewed in the statement of Proposition 3.11; the well-known structure theory of surface groups]. Thus, to complete the verification of the *fact* of the above display, it remains to consider the case where A is *nodal*. In this case, let us verify that $A \xrightarrow{\sim} p_{\mathcal{G}}(A)$.

Suppose that $p_{\mathcal{G}}(A)$ is *finite*, hence *trivial* [cf. the *specialization isomorphism* reviewed in the statement of Proposition 3.11; Proposition 2.2, (ii)]. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}$. Let $\Pi_v \subseteq \Pi_{\mathcal{G}}$ be a *verticial* subgroup [cf. [CmbGC], Definition 1.1, (ii)] associated to the unique vertex of $\mathbb{G}$. Then it follows immediately from the well-known structure theory of surface groups [i.e., in this case, the maximal pro-l quotient of the étale fundamental group of the affine line!] that $p_{\mathcal{G}}(\Pi_v) = \{0\}$. Thus, we conclude that the *rank two free* $\mathbb{Z}_l$*-module* $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ [cf. the *specialization isomorphism* reviewed in the statement of Proposition 3.11; Proposition 2.2, (ii)] is *isomorphic* to the abelianization of the pro-l completion of the *topological fundamental group* of $\mathbb{G}$, i.e., $\mathbb{Z}_l$ [cf. [CmbGC], Remark 1.1.4], a contradiction. $\square$

Proposition 3.14. (Group-theoretic characterization of log-full subgroups of smooth type). *Suppose that $(g, r) = (1, 1)$, and that X is* **singular**. *Let $A \subseteq \Delta_n$ be a log-full subgroup of Δ_n. Then the following conditions are equivalent:*

(i) *A is of* **smooth type** *[cf. Definition 3.10].*

(ii) *The image of A via the natural surjection $\Delta_n \twoheadrightarrow \Delta_n^{\mathrm{ab}}$ is* **trivial**.

Proof. Proposition 3.14 follows immediately, in light of Proposition 2.2, (ii), from Lemma 3.13. $\square$

Corollary 3.15. (Group-theoretic preservation of log-full subgroups of smooth type). *Let l be a prime number; n a positive integer; Σ a set of prime numbers which is either equal to $\{l\}$ or $\mathfrak{Primes}$. For $\square \in \{\circ, \bullet\}$, let $S_{\square}^{\log}$ be an fs log scheme whose underlying scheme is the spectrum of an algebraically closed field of characteristic $\notin \Sigma$; $X_{\square}^{\log}$ a stable log curve of type $(1, 1)$ over $S_{\square}^{\log}$ such that $X_{\square}$ is* **singular**; *$(X_{\square}^{\log})_n$ the n-th log configuration space of $X_{\square}^{\log}$; $\Delta_n^{\square}$ the maximal pro-Σ quotient of the kernel of the natural [outer] surjection $\pi_1((X_{\square}^{\log})_n) \twoheadrightarrow \pi_1(S_{\square}^{\log})$; $\alpha : \Delta_n^{\circ} \xrightarrow{\sim} \Delta_n^{\bullet}$ an isomorphism of profinite groups satisfying the following condition:*

> *a closed subgroup $A \subseteq \Delta_n^{\circ}$ of Δ_n° is* **log-full** *if and only if $\alpha(A) \subseteq \Delta_n^{\bullet}$ is* **log-full** *[cf. Definition 3.4].*

Then a log-full subgroup $A \subseteq \Delta_n^{\circ}$ of Δ_n° is of **smooth type** *(respectively,* **singular type***) if and only if $\alpha(A) \subseteq \Delta_n^{\bullet}$ is of* **smooth type** *(respectively,* **singular type***) [cf. Definition 3.10].*

Proof. Corollary 3.15 follows immediately from Proposition 3.14. $\square$

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