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On the quantum SU(2) invariant at $q = \exp(4\pi\sqrt{-1}/N)$ and the twisted Reidemeister torsion for some closed 3-manifolds

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On the quantum SU(2) invariant at $q = \exp(4\pi\sqrt{-1}/N)$ and the twisted Reidemeister torsion for some closed 3-manifolds

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Abstract

The perturbative expansion of the Chern–Simons path integral predicts a formula of the asymptotic expansion of the quantum invariant of a 3-manifold. When $q = \exp(2\pi\sqrt{-1}/N)$, there have been some researches, where the asymptotic expansion of the quantum SU(2) invariant is presented by a sum of contributions from SU(2) flat connections whose coefficients are square roots of the Reidemeister torsions. When $q = \exp(4\pi\sqrt{-1}/N)$, it is conjectured recently that the quantum SU(2) invariant of a closed hyperbolic 3-manifold M is of exponential order of N whose growth is given by the complex volume of M. The first author showed in the previous work that this conjecture holds for the hyperbolic 3-manifold M_p obtained from S^3 by p surgery along the figure-eight knot.

In this paper, we show that a square root of the Reidemeister torsion appears as a coefficient in the semi-classical approximation of the asymptotic expansion of the quantum SU(2) invariant of M_p at $q = \exp(4\pi\sqrt{-1}/N)$. Further, when $q = \exp(4\pi\sqrt{-1}/N)$, we show that the semi-classical approximation of the asymptotic expansion of the quantum SU(2) invariant of some Seifert 3-manifolds M is presented by a sum of contributions from some of $\mathrm{SL}_2\mathbb{C}$ flat connections on M, and square roots of the Reidemeister torsions appear as coefficients of such contributions.

1 Introduction

In the late 1980s, Witten [45] proposed the quantum G invariant of a closed 3-manifold M for a simple compact Lie group G, which is formally presented by the Chern–Simons path integral. By the operator formalism of the Chern–Simons path integral, we obtain a rigorous construction of the quantum invariant; in particular, the quantum SU(2) invariant of M can be defined to be a linear sum of the colored Jones polynomials of a link L at an Nth primitive root of unity q, where L is a link such that M is obtained from S^3 by integral surgery along L; for details, see *e.g.* [32]. Further, by the perturbative expansion of the Chern–Simons path integral, we obtain a formula which predicts the asymptotic expansion of the quantum invariant at $N \to \infty$.

When $q = e^{2\pi\sqrt{-1}/N}$, there have been researches [1, 2, 3, 4, 10, 13, 14, 15, 42, 43], where the asymptotic expansion of the quantum invariant is presented by a sum of contributions from SU(2) flat connections on M and such contributions are obtained from stationary phase method for SU(2) connections. In the semi-classical approximation of the expansion, a square root of the Reidemeister torsion appears as a coefficient of such a contribution. It is known that this expansion is of polynomial order of N in the case of $q = e^{2\pi\sqrt{-1}/N}$.

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When $q = e^{4\pi\sqrt{-1}/N}$, we can also define the quantum SU(2) invariant for odd N; we note that the quantum SU(2) and SO(3) invariants are equal when $q = e^{4\pi\sqrt{-1}/N}$ (see Appendix A), and we simply call them the quantum invariant and denote it by $\hat{\tau}_N(M)$ in this paper. Recently, Chen–Yang [8] observed that $\hat{\tau}_N(M)$ is of exponential order as $N \to \infty$ for some hyperbolic 3-manifolds obtained by surgery along the figure-eight knot and the 5_2 knot, and conjectured that $\hat{\tau}_N(M)$ is of order $e^{N\varsigma(M)}$ for a closed hyperbolic 3-manifold M, where $\varsigma(M)$ is a (normalized) complex volume whose real part is given by the hyperbolic volume of M and imaginary part is given by the Chern–Simons invariant of M. This conjecture is a "volume conjecture" for closed hyperbolic 3-manifolds. From the viewpoint of mathematical physics, this expansion can be regarded as a perturbative expansion at a $\mathrm{SL}_2\mathbb{C}$ flat connection corresponding to the holonomy representation of the hyperbolic structure of M.

Let p be an integer, and let M_p be the 3-manifold obtained from S^3 by p surgery along the figure-eight knot. It is known that M_p is hyperbolic if and only if $|p| > 4$. The first author [35] showed that the quantum invariant $\hat{\tau}_N(M_p)$ of M_p for odd N is expanded as $N \to \infty$ in the following form,

$$\begin{aligned}\hat{\tau}_N(M_p) &= (-1)^p\, e^{-\frac{\pi\sqrt{-1}}{4}pN} \sqrt{-1}^{\,\mathrm{sign}(p)\frac{N-3}{2}}\, e^{N\varsigma(M_p)} N^{3/2}\, \omega(M_p) \\ &\quad \times \Big(1 + \sum_{i=1}^{d} \kappa_i(M_p) \cdot \big(\frac{4\pi\sqrt{-1}}{N}\big)^i + O\big(\frac{1}{N^{d+1}}\big)\Big),\end{aligned} \tag{1}$$

for any integer $d \geq 1$, where $\omega(M_p)$ and $\kappa_i(M_p)$ are constants determined by M_p.

Theorem 1.1. *Let p be an integer with $|p| > 4$, and let M_p be the 3-manifold given above. Then,*

$$\omega(M_p)^2 \;=\; \pm\frac{1}{16\pi^2}\,\mathrm{Tor}(M_p),$$

where $\mathrm{Tor}(M)$ denotes the twisted Reidemeister torsion of M with $\mathfrak{sl}_2$ coefficient twisted by the adjoint action of the holonomy representation of the hyperbolic structure of M.

We give a proof of the theorem in Section 2.4.

Let p_1 and p_2 be coprime odd integers ≥ 3. Let M_{p_1,p_2} be the Seifert 3-manifold[1] obtained from S^3 by surgery along the framed link (23). We classify irreducible representations of $\pi_1(M_{p_1,p_2})$ to $\mathrm{SL}_2\mathbb{C}$ up to conjugation of $\mathrm{SL}_2\mathbb{C}$ in Proposition 3.16, where we show that any irreducible representation, denoted by ρ_{k_1,k_2}, is conjugate to a SU(2) representation or a $\mathrm{SL}_2\mathbb{R}$ representation.

Theorem 1.2. *Let p_1 and p_2 be coprime odd integers ≥ 3. Then, the quantum invariant*

[1]We note that $M_{3,7}$ is homeomorphic to M_{-1}, which is homeomorphic to M_1 with the opposite orientation; see *e.g.* [18, Remark (5) of Problem 1.77] and Remark C.2.

$\hat{\tau}_N(M_{p_1,p_2})$ of M_{p_1,p_2} for odd N is expanded as $N\to\infty$ in the following form,

$$\begin{aligned}\hat{\tau}_N(M_{p_1,p_2}) &= e^{\frac{\pi\sqrt{-1}}{4}}\,N^{3/2}\,(-1)^{\frac{N-1}{2}}\,e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N}\\ &\times\Bigg(\frac{1}{2}\sum_{\substack{\mathrm{SU}(2)\text{ rep}\\ \rho_{k_1,k_2}}}+\sum_{\substack{\mathrm{SL}_2\mathbb{R}\text{ rep}\\ \rho_{k_1,k_2}\\ \frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}}}\Bigg)\Big(e^{\pi\sqrt{-1}\,\mathrm{CS}(M_{p_1,p_2};\,\mathrm{ad}\circ\rho_{k_1,k_2})\,N}\,\omega(M_{p_1,p_2};\,\rho_{k_1,k_2})\Big)\;+O(N),\end{aligned}$$

where CS denotes the Chern–Simons invariant (see also Remark 3.27), and we put

$$\omega(M_{p_1,p_2};\,\rho_{k_1,k_2}) \;=\; (-1)^{\frac{k_1+k_2}{2}}\,\frac{\sqrt{2}}{\pi\sqrt{p_1p_2}}\,\sin\frac{k_1\pi}{p_1}\,\sin\frac{k_2\pi}{p_2}\,.$$

Further, we have that

$$\omega(M_{p_1,p_2};\,\rho_{k_1,k_2})^2 \;=\; \pm\frac{1}{16\pi^2}\,\mathrm{Tor}(M_{p_1,p_2};\,\mathrm{ad}\circ\rho_{k_1,k_2}).$$

We give a proof of the theorem in Section 3.6.

Remark 1.3. By refining the proof of Theorem 1.2, we can show that the full asymptotic expansion of $\hat{\tau}_N(M_{p_1,p_2})$ is given by the following form,

$$\begin{aligned}\hat{\tau}_N(M_{p_1,p_2}) &= e^{\frac{\pi\sqrt{-1}}{4}}\,N^{3/2}\,(-1)^{\frac{N-1}{2}}\,e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N}\\ &\quad\times\Bigg(\frac{1}{2}\sum_{\substack{\mathrm{SU}(2)\text{ rep}\\ \rho_{k_1,k_2}}}+\sum_{\substack{\mathrm{SL}_2\mathbb{R}\text{ rep}\\ \rho_{k_1,k_2}\\ \frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}}}\Bigg)\Big(e^{\pi\sqrt{-1}\,\mathrm{CS}(M_{p_1,p_2};\,\mathrm{ad}\circ\rho_{k_1,k_2})\,N}\,\omega(M_{p_1,p_2};\,\rho_{k_1,k_2})\Big)\\ &\qquad\qquad\times\Big(1+\frac{\lambda_{1,\rho_{k_1,k_2}}}{N}+\frac{\lambda_{2,\rho_{k_1,k_2}}}{N^2}+\cdots\Big)\\ &\quad+N\sum_k e^{\pi\sqrt{-1}\,c_kN}\,\lambda'_k\Big(1+\frac{\lambda'_{1,k}}{N}+\frac{\lambda'_{2,k}}{N^2}+\cdots\Big),\end{aligned}$$

with some rational constants c_k and some complex constants $\lambda_{i,\rho_{k_1,k_2}}$, λ'_i, $\lambda'_{i,k}$. It is expected that the last line is the contribution from abelian representations of $\pi_1(M_{p_1,p_2})$; see [7, 23, 43], and see also Remark 3.31.

Theorem 1.1 means that $\omega(M_p)$ of (1) is a constant multiple of a square root of the Reidemeister torsion, as we can expect from the perturbative expansion of the Chern–Simons path integral at a $\mathrm{SL}_2\mathbb{C}$ flat connection. Theorem 1.2 means that the semi-classical approximation of the asymptotic expansion of the quantum invariant of a Seifert 3-manifold M_{p_1,p_2} is presented by a sum of contributions from $\mathrm{SL}_2\mathbb{C}$ flat connections whose coefficients are given by square roots of Reidemeister torsions at these flat connections, as we can expect from the perturbative expansion of the Chern–Simons path integral. We note that this sum is a sum over some of $\mathrm{SL}_2\mathbb{C}$ flat connections, unlike the case of $q=e^{2\pi\sqrt{-1}/N}$, noting that, when $q=e^{2\pi\sqrt{-1}/N}$, the corresponding sum is a sum over SU(2) flat connections. It would be a problem how we choose such $\mathrm{SL}_2\mathbb{C}$ flat connections for a general Seifert 3-manifold, noting that the irreducible $\mathrm{SL}_2\mathbb{C}$ representations appearing in Theorem 1.2 are not all irreducible $SL_2\mathbb{C}$ representations; see Remark 1.6 below. See also Appendix B, for the semi-classical approximation of the quantum invariant of the lens space $L(p,1)$ for odd p; in this case, there are only abelian representations, and the asymptotic expansion is of order N.

Theorem 1.1 is shown, as follows. The figure-eight knot complement is presented by gluing two ideal tetrahedra, which are parameterized by complex parameters z and w. The hyperbolic structure of M_p can be described by using these parameters. The Reidemeister torsion and $\omega(M_p)$ can be presented by using these parameters, and we can show Theorem 1.1 by calculating them concretely.

Theorem 1.2 is shown, as follows. The quantum invariant of a Seifert 3-manifold can be calculated explicitly by using Gauss sums, and we can obtain a concrete formula presenting the semi-classical approximation of the quantum invariant. Further, by calculating the Reidemeister torsion and the Chern–Simons invariant concretely, we can show Theorem 1.2.

Remark 1.4. As for invariants of knots, it is conjectured, as the volume conjecture [16, 30], that the Nth colored Jones polynomial of a hyperbolic knot K at $q = e^{2\pi\sqrt{-1}/N}$ (which is equal to the Kashaev invariant) is of order $e^{N\varsigma(S^3-K)}$, where $\varsigma(S^3-K)$ is a (normalized) complex volume of S^3-K. As a refinement of the volume conjecture, the asymptotic expansion of the Kashaev invariant is studied in [33, 34, 36]. Further, the authors showed in [37] that a square root of the Reidemeister torsion appears as a coefficient of the semi-classical approximation of this expansion for some two-bridge knots.

Remark 1.5. As we mentioned above, it is expected that the asymptotic expansion of the quantum invariant of a closed 3-manifold M is presented by a sum of contributions from $SL_2\mathbb{C}$ flat connections on M, which are alternatively given by $SL_2\mathbb{C}$ representations of $\pi_1(M)$. In this remark, we consider when $\pi_1(M)$ tends to have a finite number of $SL_2\mathbb{C}$ representations.

When M has an incompressible torus, we can "bend" a representation ρ of $\pi_1(M)$ along the torus. That is, when M is presented by the form $M_1 \underset{T^2}{\cup} M_2$, $\pi_1(M)$ is isomorphic to $\pi_1(M_1) \underset{\pi_1(T^2)}{*} \pi_1(M_2)$, and ρ is presented by the form $\rho_1 * \rho_2$, which can be deformed as $\rho_1 * g\rho_2 g^{-1}$ with $g \in SL_2\mathbb{C}$ which is commutative with $\rho\big(\pi_1(T^2)\big)$. Since there are 1-dimensional possibilities of such g, in this case, the moduli space of $SL_2\mathbb{C}$ representations of $\pi_1(M)$ tends to have a positive dimension.[2]

When M does not have an incompressible torus, it is suggested by the JSJ decomposition and the geometrization theorem (see *e.g.* [11, 39]) that typical cases of M are a closed hyperbolic 3-manifold, a Seifert 3-manifold with three singular fibers with a base orbifold of genus 0, and a lens space. In these cases, $\pi_1(M)$ tends to have a finite number of $SL_2\mathbb{C}$ representations. In this sense, these cases are basic cases. In this paper, we study examples of these cases in Theorem 1.1, Theorem 1.2 and Appendix B respectively. We conjecture that, in these cases, the asymptotic expansion of the quantum invariant is presented by the forms of formulas in (1), Remark 1.3, Proposition B.1 respectively.

[2] In this case, it might be expected that the semi-classical approximation of the asymptotic expansion of the quantum invariant of M can be described by using the Chern–Simons invariant and the Reidemeister torsion of M_1 and M_2. It is shown in [29, 31] that such description holds for the behavior of the colored Jones polynomial of iterated torus knots.

Remark 1.6. We note that the representations appearing in the formula of Theorem 1.2 are not necessarily all representations of $\pi_1(M_{p_1,p_2})$. For example, by Proposition 3.16, $\pi_1(M_{3,p})$ has representations $\rho_{1,1}$, $\rho_{1,3}$, $\rho_{1,5}$, $\cdots$, $\rho_{1,p-2}$, where $\rho_{1,k}$ is a SU(2) representation if $\frac{1}{6} < \frac{k}{p} < \frac{5}{6}$, and a $SL_2\mathbb{R}$ representation if $\frac{k}{p} < \frac{1}{6}$ or $\frac{5}{6} < \frac{k}{p}$. Hence, when $\frac{5}{6} < \frac{k}{p}$, $\rho_{1,k}$ does not appear in the formula of Theorem 1.2. See Example 3.32 for concrete numerical calculation to verify this phenomenon. As mentioned before, it would be a problem how we choose representations which contribute to the semi-classical approximation of the quantum invariant for a general Seifert 3-manifold.

The paper is organized, as follows. In Section 2, we give a proof of Theorem 1.1. We review that the hyperbolic structure of M_p is given as a union of two ideal tetrahedra, which are parameterized by complex parameters z and w (Section 2.1). In terms of z and w, we calculate the Reidemeister torsion (Section 2.2) and review a formula of $\omega(M_p)$ (Section 2.3). By using them, we give a proof of Theorem 1.1 (Section 2.4). In Section 3, we give a proof of Theorem 1.2. We calculate the semi-classical approximation of the quantum invariant of M_{p_1,p_2} (Section 3.3), and give formulas of the Reidemeister torsion and the Chern–Simons invariant (Section 3.5). By using them, we give a proof of Theorem 1.2 (Section 3.6). In Appendix A, we review that the quantum SU(2) and SO(3) invariants are equal, when $q = e^{4\pi\sqrt{-1}/N}$. In Appendix B, we calculate the semi-classical approximation of the quantum invariant of the lens space $L(p, 1)$ for odd p. In Appendix C, we review equivalences between some Seifert 3-manifolds.

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2 Proof of Theorem 1.1

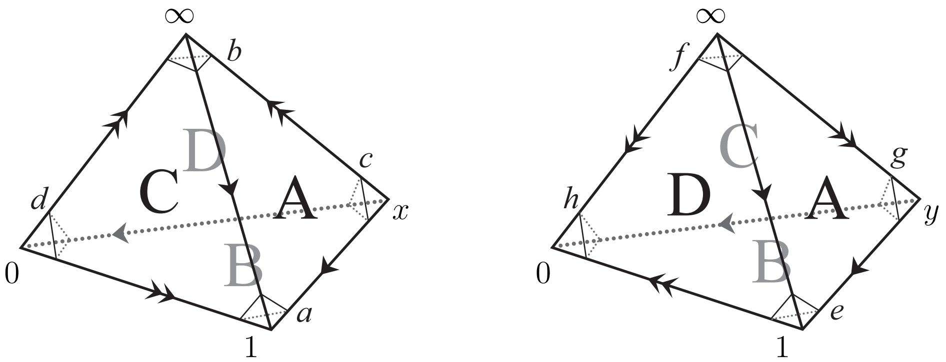
We recall that M_p denotes the 3-manifold obtained from S^3 by p surgery along the figure-eight knot for an integer p with $|p| > 4$. In this section, we give a proof of Theorem 1.1, which relates the Reidemeister torsion and the semi-classical limit of the quantum invariant of M_p.

In Section 2.1, we calculate the holonomy representation of the hyperbolic structure of M_p. In Section 2.2, we show a formula of the Reidemeister torsion of M_p. In Section 2.3, we review a formula of the semi-classical limit of the quantum invariant of M_p. In Section 2.4, we give a proof of Theorem 1.1.

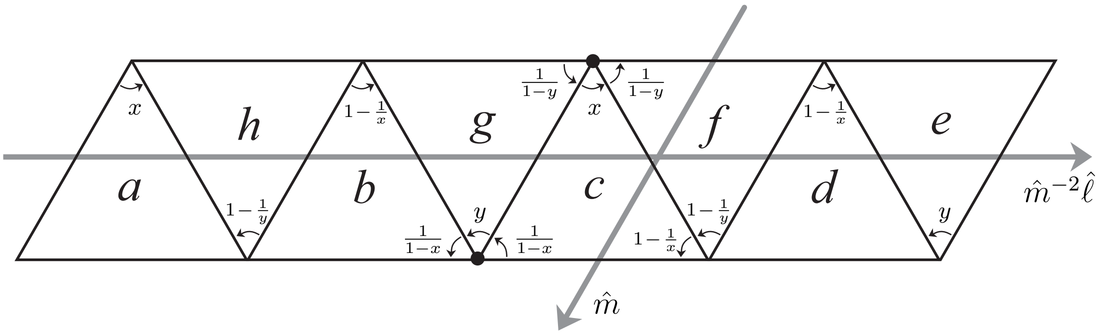
2.1 The holonomy representation of M_p

In this section, we review the hyperbolic structures of the figure-eight knot complement and M_p, following [44] (see also [35]). We also review their holonomy representations, following [28].

We review the hyperbolic structure of M_p given in [44]. The figure-eight knot complement can be expressed as the union of the following two ideal tetrahedra.



Here, the 4 faces "A", "B", "C", "D" are glued respectively, and the shapes of the tetrahedra are given by complex parameters x and y; we fix their concrete values later. The boundary torus of a tubular neighborhood of the figure-eight knot is expressed as the union of 8 triangles "a", "b", $\cdots$, "h", which appear in neighborhoods of the vertices of the above ideal tetrahedra.

 (2)

As shown in [44], the holonomy m of the meridian and the holonomy ℓ of the longitude are given by

$$\hat{m} \;=\; \frac{1-\frac{1}{x}}{\frac{1}{1-y}} \;=\; -\frac{(1-x)(1-y)}{x}\,, \qquad \hat{m}^{-2}\hat{\ell} \;=\; \frac{x^2(1-\frac{1}{x})^2}{y^2(1-\frac{1}{y})^2} \;=\; \frac{(1-x)^2}{(1-y)^2}\,. \tag{3}$$

To obtain the hyperbolic structure of the figure-eight knot complement, we require that

$$\begin{aligned} &\operatorname{Im} x > 0, \qquad \operatorname{Im} y > 0, \\ &x\, y\, \Big(\frac{1}{1-x}\Big)^2 \Big(\frac{1}{1-y}\Big)^2 \;=\; 1, \\ &\operatorname{Arg}(x) + \operatorname{Arg}(y) + 2 \operatorname{Arg}\Big(\frac{1}{1-x}\Big) + 2 \operatorname{Arg}\Big(\frac{1}{1-y}\Big) \;=\; 2\pi. \end{aligned} \tag{4}$$

Further, to obtain the hyperbolic structure of M_p, we require that

$$\hat{m}^p\, \hat{\ell} \;=\; 1, \qquad (p+2) \operatorname{Arg}(\hat{m}) \;+\; \operatorname{Arg}(\hat{m}^{-2}\hat{\ell}) \;=\; 2\pi. \tag{5}$$

By the rigidity of the hyperbolic structure, there exists a unique hyperbolic structure of M_p, which can be given by a unique solution of the above formulas.

We review how we fix concrete such values of x and y, following [35]. Putting $x = e^{4\pi\sqrt{-1}\,(s-t)}$ and $y = e^{4\pi\sqrt{-1}\,(t+s)}$, the above formulas are rewritten,

$$\begin{aligned} \log\big(1 - e^{4\pi\sqrt{-1}(s-t)}\big) - \log\big(1 - e^{-4\pi\sqrt{-1}(t+s)}\big) + 4\pi\sqrt{-1}\big(\frac{p}{2}\,t - s\big) &= 0, \\ -\log\big(1 - e^{4\pi\sqrt{-1}(s-t)}\big) - \log\big(1 - e^{-4\pi\sqrt{-1}(t+s)}\big) - 4\pi\sqrt{-1}\,t &= 0, \\ 0 < \mathrm{Re}\,(t+s) < \frac{1}{4}\,, \quad 0 < \mathrm{Re}\,(s-t) < \frac{1}{4}\,, \quad \mathrm{Re}\,t \ge 0, \end{aligned}$$

where we choose the branch of the log in the way that $-\pi < \mathrm{Im}\,\log(\cdot) < \pi$. As shown in [35], there exists a unique solution (t_0, s_0) of the above formulas; we show some numerical solutions below.

p	(t_0, s_0)
6	$(0.0743075... - \sqrt{-1}\cdot 0.0382219...\,,\ 0.1128050... - \sqrt{-1}\cdot 0.0314723...)$
7	$(0.0640105... - \sqrt{-1}\cdot 0.0283809...\,,\ 0.1065380... - \sqrt{-1}\cdot 0.0212048...)$
8	$(0.0566257... - \sqrt{-1}\cdot 0.0221934...\,,\ 0.1022661... - \sqrt{-1}\cdot 0.0152090...)$
9	$(0.0509104... - \sqrt{-1}\cdot 0.0179265...\,,\ 0.0991274... - \sqrt{-1}\cdot 0.0113510...)$
10	$(0.0462978... - \sqrt{-1}\cdot 0.0148180...\,,\ 0.0967225... - \sqrt{-1}\cdot 0.0087183...)$

Further, putting $z = e^{-4\pi\sqrt{-1}\,t}$ and $w = e^{-4\pi\sqrt{-1}\,s}$, we rewrite the above equations in terms of z and w. Since $x = \frac{z}{w}$ and $y = \frac{1}{zw}$, (4) is rewritten,

$$(1 - zw)(w - z) \;=\; zw, \tag{6}$$

which is further rewritten,

$$w + w^{-1} \;=\; z + z^{-1} - 1. \tag{7}$$

Further, (3) is rewritten,

$$\hat{m} \;=\; \frac{1}{z}\,, \qquad \hat{\ell} \;=\; \frac{(1-x)^4}{x^2} \;=\; \frac{(w-z)^4}{z^2 w^2}\,. \tag{8}$$

Furthermore, by using (6), the first equation of (5) is rewritten

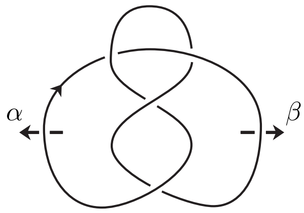
$$z^{p/2}(1 - zw) \;=\; w - z. \tag{9}$$

Moreover, as shown in [35], (6) and (9) are rewritten,

$$z^{p/2} + z^{-p/2} - z^2 - z^{-2} + z + z^{-1} + 2 \;=\; 0, \qquad w \;=\; \frac{z + z^{p/2}}{1 + z^{1+p/2}}\,. \tag{10}$$

We can obtain numerical solutions from the above equations.

We review representations of $\pi_1(S^3 - K)$ to $\mathrm{SL}_2\mathbb{C}$, following [28]. Let K be the figure-eight knot in S^3.



The fundamental group of the knot complement is presented by

$$\pi_1(S^3 - K) \;=\; \langle \alpha, \beta \mid \alpha\beta^{-1}\alpha^{-1}\beta\alpha = \beta\alpha\beta^{-1}\alpha^{-1}\beta \rangle.$$

For a complex parameter u, putting $m = e^u$, we set the representation $\rho_u : \pi_1(S^3 - K) \to \mathrm{SL}_2\mathbb{C}$ by

$$\rho_u(\alpha) \;=\; \begin{pmatrix} m^{1/2} & 1 \\ 0 & m^{-1/2} \end{pmatrix}, \quad \rho_u(\beta) \;=\; \begin{pmatrix} m^{1/2} & 0 \\ -d & m^{-1/2} \end{pmatrix},$$

where d satisfies that

$$d^2 \;-\; (m + m^{-1} - 3)d \;-\; (m + m^{-1} - 3) \;=\; 0. \tag{11}$$

It is known [41] that any nonabelian representation $\pi_1(S^3 - K) \to \mathrm{SL}_2\mathbb{C}$ can be given by this form, up to conjugation. The longitude λ corresponding to the meridian α is given by $\lambda = \alpha\beta^{-1}\alpha\beta\alpha^{-2}\beta\alpha\beta^{-1}\alpha^{-1}$. It follows from concrete calculation that

$$\rho_u(\lambda) \;=\; \begin{pmatrix} m - 1 - 2m^{-1} + m^{-2} + d(m - m^{-1}) & -(m^{1/2} + m^{-1/2})(m + m^{-1} - 3 - 2d) \\ 0 & m^2 - 2m - 1 + m^{-1} - d(m - m^{-1}) \end{pmatrix}.$$

We set a parameter ℓ by putting

$$\ell^{1/2} \;=\; m - 1 - 2m^{-1} + m^{-2} + d(m - m^{-1}).$$

Then, we note that, by using (11), we can verify that

$$\ell^{-1/2} \;=\; m^2 - 2m - 1 + m^{-1} - d(m - m^{-1}).$$

Hence, when $\mathrm{SL}_2\mathbb{C}$ acts on $\mathbb{C} \cup \{\infty\}$ by Möbius transformation, the holonomy of the actions of $\rho_u(\alpha)$ and $\rho_u(\lambda)$ are m and ℓ. We choose the meridian and the longitude for $\pi_1(S^3 - K)$ by putting them to be the inverses of the meridian and the longitude of (2). Then, we have that $m = \hat{m}^{-1} = z$ by (8), and $\ell = \hat{\ell}^{-1}$. Hence, by (11),

$$d^2 \;-\; (z + z^{-1} - 3)d \;-\; (z + z^{-1} - 3) \;=\; 0.$$

By using (6), we can verify that the solutions of the above equation for d are given by $d = w^{\pm 1} - 1$. Further, by using (6), we can verify that

$$\ell^{1/2} \;=\; \begin{cases} \dfrac{zw}{(w - z)^2} & \text{when } d = w - 1, \\[2ex] \dfrac{(w - z)^2}{zw} & \text{when } d = w^{-1} - 1. \end{cases}$$

Therefore, by (8), we have that $d = w - 1$. Hence, ρ_u is presented by

$$\rho_u(\alpha) \;=\; \begin{pmatrix} z^{1/2} & 1 \\ 0 & z^{-1/2} \end{pmatrix}, \quad \rho_u(\beta) \;=\; \begin{pmatrix} z^{1/2} & 0 \\ 1-w & z^{-1/2} \end{pmatrix}, \quad \rho_u(\lambda) \;=\; \begin{pmatrix} \frac{zw}{(w-z)^2} & * \\ 0 & \frac{(w-z)^2}{zw} \end{pmatrix}.$$

We note that, since

$$d \;=\; \frac{1}{2}\big(z + z^{-1} - 3 \pm \sqrt{(z+z^{-1}+1)(z+z^{-1}-3)}\,\big) \;=\; w - 1,$$

we have that

$$\pm\sqrt{(z+z^{-1}+1)(z+z^{-1}-3)} \;=\; 2w - z - z^{-1} + 1 \;=\; w - w^{-1}, \tag{12}$$

where we obtain the last equality by (7).

2.2 The Reidemeister torsion of M_p

In this section, we calculate the twisted cohomological Reidemeister torsion of M_p with $\mathfrak{sl}_2$ coefficient twisted by the adjoint action of the holonomy representation of the hyperbolic structure of M_p. We remark that a formula for the Reidemeister torsion of 3-manifold obtained by $1/n$-surgery along the figure-eight knot is known [21, 22].

We review formulas of the cohomological Reidemeister torsion the figure-eight knot complement, following [28]. We denote by $\mathrm{Tor}\,(S^3 - K;\, \mathrm{ad}\circ\rho_u)$ the cohomological Reidemeister torsion associated with the meridian α with $\mathfrak{sl}_2$ coefficient twisted by the adjoint action of ρ_u. It is known [38] (see also [28]) that

$$\mathrm{Tor}\,(S^3 - K;\, \mathrm{ad}\circ\rho_u) \;=\; \pm\frac{2}{\sqrt{(m+m^{-1}+1)(m+m^{-1}-3)}} \;=\; \pm\frac{2}{w - w^{-1}}, \tag{13}$$

where we obtain the last equality by (12), noting that $m = z$.

We calculate the Reidemeister torsion of M_p; we recall that M_p denotes the 3-manifold obtained from S^3 by p surgery along the figure-eight knot for an integer p with $|p| > 4$. The fundamental group $\pi_1(M_p)$ is presented by

$$\pi_1(M_p) \;=\; \big\langle \alpha, \beta \;\big|\; \alpha\beta^{-1}\alpha^{-1}\beta\alpha = \beta\alpha\beta^{-1}\alpha^{-1}\beta,\;\; \alpha^p\lambda = 1 \big\rangle.$$

We consider ρ_u which induces a representation $\rho_u : \pi_1(M_p) \to \mathrm{SL}_2\mathbb{C}$. We denote by $\mathrm{Tor}\,(M_p)$ the cohomological Reidemeister torsion with $\mathfrak{sl}_2$ coefficient twisted by the adjoint action of the holonomy representation ρ_u corresponding to the hyperbolic structure of M_p. We introduce a parameter v so that the $(1,1)$ entry of the matrix $\rho(\alpha^p\lambda)$ equals $e^{v/2}$, *i.e.*,

$$\log v \;=\; (p+2)\log z + 2\log w - 4\log(w - z). \tag{14}$$

By [38, Page 108], we have

$$\begin{aligned} \mathrm{Tor}\,(M_p) \;&=\; \pm\,\mathrm{Tor}\,(S^3 - K;\, \mathrm{ad}\circ\rho_u)\left(\frac{dv}{du}\right)^{-1}\big((\mathrm{trace}\,\rho_u(\alpha))^2 - 4\big) \\ &=\; \pm\,\mathrm{Tor}\,(S^3 - K;\, \mathrm{ad}\circ\rho_u)\left(\frac{dv}{du}\right)^{-1}\big(z + z^{-1} - 2\big), \end{aligned} \tag{15}$$

noting that since Porti [38] uses the homological Reidemeister torsion, we need to take its inverse. We compute $\frac{dv}{du}$, as follows. By (7), we have that

$$\frac{dw}{dz} \cdot \frac{d}{dw}\big(w + w^{-1}\big) \;=\; \frac{d}{dz}\big(z + z^{-1}\big).$$

Hence,

$$\frac{dw}{dz} \;=\; \frac{w^2(z^2-1)}{z^2(w^2-1)}\,.$$

Further, by (14),

$$\frac{dv}{dz} \;=\; \frac{p+2}{z} + \frac{4}{w-z} + \frac{dw}{dz}\Big(\frac{2}{w} - \frac{4}{w-z}\Big).$$

Further, since $z = e^u$, we have that $\frac{d}{du} = z\frac{d}{dz}$. Therefore, we obtain that

$$\frac{dv}{du} \;=\; p + \frac{2(z+w)(zw+1)}{z\,(w^2-1)}\,. \tag{16}$$

Hence, by (13), (15) and (16), we obtain that

$$\mathrm{Tor}\,(M_p) \;=\; \pm\frac{2(z+z^{-1}-2)}{w-w^{-1}}\Big(p + \frac{2(z+w)(zw+1)}{z\,(w^2-1)}\Big)^{-1}. \tag{17}$$

2.3 The semi-classical limit of the quantum invariant of M_p

In this section, we recall the formula of $\omega(M_p)^2$ given in [35].

We put

$$\begin{aligned}
H_{11} &= 4\pi\sqrt{-1}\,\Big(\frac{z}{w-z} - \frac{zw}{1-zw} + \frac{p}{2}\Big) \;=\; 4\pi\sqrt{-1}\,\Big(w^{-1} - w + \frac{p}{2}\Big),\\
H_{12} &= 4\pi\sqrt{-1}\,\Big(-\frac{z}{w-z} - \frac{zw}{1-zw} - 1\Big) \;=\; 4\pi\sqrt{-1}\,\big(z - z^{-1}\big),\\
H_{22} &= 4\pi\sqrt{-1}\,\Big(\frac{z}{w-z} - \frac{zw}{1-zw}\Big) \;=\; 4\pi\sqrt{-1}\,\big(w^{-1} - w\big),
\end{aligned}$$

where we obtain the right equalities by (6) and (7). Then, as shown in [35],

$$\begin{aligned}
\omega(M_p)^2 &= (1-\frac{1}{z})^2 z\,\big(H_{11}H_{22} - H_{12}^2\big)^{-1}\\
&= -\frac{z+z^{-1}-2}{16\pi^2}\Big(\frac{p}{2}\big(w^{-1}-w\big) + \big(w^{-1}-w\big)^2 - \big(z-z^{-1}\big)^2\Big)^{-1}.
\end{aligned} \tag{18}$$

2.4 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. We show that

$$\frac{1}{\mathrm{Tor}\,(M_p)} \;=\; \pm\frac{1}{16\pi^2\,\omega(M_p)^2}\,.$$

By (17) and (18), it is sufficient to show that

$$-\frac{w-w^{-1}}{2}\Big(p+\frac{2(z+w)(zw+1)}{z\,(w^2-1)}\Big) \;=\; \frac{p}{2}\big(w^{-1}-w\big)+\big(w^{-1}-w\big)^2-\big(z-z^{-1}\big)^2.$$

Hence, it is sufficient to show that

$$-\frac{(z+w)(zw+1)}{zw} \;=\; \big(w^{-1}-w\big)^2-\big(z-z^{-1}\big)^2. \qquad (19)$$

The difference of the two sides of the above equation is calculated as

$$\begin{aligned} &-\frac{(z+w)(zw+1)}{zw} \;-\; \big(w^{-1}-w\big)^2+\big(z-z^{-1}\big)^2 \\ &= \frac{(z+w)(zw+1)}{zw}\big(z+z^{-1}-w-w^{-1}-1\big) \;=\; 0, \end{aligned}$$

where we obtain the last equality by (7). Therefore, we obtain (19), as required. □

Example 2.1. We show numerical experiment of Theorem 1.1 for M_8. We obtain the values of z and w as solutions of (10),

$$\begin{aligned} z &= 0.573013413202... - \sqrt{-1}\;0.494098312716...\;, \\ w &= 0.232785615938... - \sqrt{-1}\;0.792551992515...\;. \end{aligned}$$

By (17), the Reidemeister torsion of M_8 is numerically given by

$$\mathrm{Tor}\,(M_8) \;=\; \pm\big(-0.063010779425... - \sqrt{-1}\;0.044929069636...\big).$$

In the following table, we show some numerical values of the quantum invariant of M_8 normalized in such a way that the resulting sequence should converges to $\omega(M_8)$, where we can numerically calculate $\varsigma(M_8)$ as shown in [35].

N	$\hat{\tau}_N(M_8)\sqrt{-1}^{\,-(N-3)/2}e^{-N\varsigma(M_8)}N^{-3/2}$
101	$0.003316724659... - \sqrt{-1}\;0.021953933824...$
201	$0.005041422357... - \sqrt{-1}\;0.021586460174...$
501	$0.006067785821... - \sqrt{-1}\;0.021301349281...$
1001	$0.006408009983... - \sqrt{-1}\;0.021195494445...$

Hence, by Theorem 1.1, we can guess that the value of $\omega(M_8)$ is obtained as a constant multiple of a square root of the Reidemeister torsion as

$$\omega(M_8) \;=\; 0.006747146303... - \sqrt{-1}\;0.021084221715...\;,$$

which satisfies that

$$16\pi^2\omega(M_8)^2 \;=\; -0.063010779425... - \sqrt{-1}\;0.044929069636...\;,$$

noting that, for a given value of the Reidemeister torsion, there is an ambiguity of a choice of the sign of $\omega(M_8)$ as a constant multiple of a square root of the Reidemeister torsion. We note that, for a given hyperbolic 3-manifold M, such numerical observation of values of the quantum invariant of M might be able to suggest how we should choose an appropriate sign of a square root of the Reidemeister torsion (see Remark 3.28) and how we should choose a sign of the Reidemeister torsion.

Remark 2.2. If we could choose an appropriate sign of $\omega(M)$ for a hyperbolic 3-manifold M, the formula (1) suggests that there might be an invariant $I_N(M) \in \mathbb{Z}/8\mathbb{Z}$ of M such that

$$\hat{\tau}_N(M) \ \sim \ e^{\frac{\pi\sqrt{-1}}{4} I_N(M)} N^{3/2} \omega(M).$$

As we mention in Remark 3.28, it might be related to the "spectral flow" of the holonomy representation of M.

3 Proof of Theorem 1.2

We recall that M_{p_1,p_2} denotes the 3-manifold obtained from S^3 by surgery along the framed link (23). In this section, we give a proof of Theorem 1.2, which relates the Reidemeister torsion and the semi-classical limit of the quantum invariant of M_{p_1,p_2}.

In Section 3.1, we show some formulas which we use later. In Section 3.2, we calculate the quantum invariant of M_{p_1,p_2}. In Section 3.3, we calculate the semi-classical limit of the quantum invariants of M_{p_1,p_2}. In Section 3.4, we give a classification of $\mathrm{SL}_2\mathbb{C}$ representations of $\pi_1(M_{p_1,p_2})$. In Section 3.5, we give formulas for the Chern–Simons invariant and the Reidemeister torsion of M_{p_1,p_2}. In Section 3.6, we give a proof of Theorem 1.2.

In this section, we put $A = e^{\frac{\pi\sqrt{-1}}{N}}$, $a = e^{\frac{2\pi\sqrt{-1}}{N}}$ and $[n] = \frac{a^n - a^{-n}}{a - a^{-1}}$.

3.1 Some formulas of Gauss sums

In this section, we show some formulas of Gauss sums, which we use to calculate quantum invariants in Section 3.2.

Lemma 3.1. *For any integer f,*

$$\sum_{\substack{1 \le i < N \\ i \text{ is odd}}} A^{f(i^2-1)} [mi][i] \ = \ \frac{(-A)^{-f}}{(a-a^{-1})^2} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} a^{2fj^2} \big(a^{2(m+1)j} - a^{2(m-1)j} \big).$$

Proof. The left-hand side of the required formula is equal to

$$\begin{aligned} &\frac{1}{2} \sum_{\substack{-N < i < N \\ i \text{ is odd}}} A^{f(i^2-1)} \frac{(a^{mi} - a^{-mi})(a^i - a^{-i})}{(a-a^{-1})^2} \\ &= \frac{(-A)^{-f}}{2(a-a^{-1})^2} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} a^{2fj^2} \big(a^{2mj} - a^{-2mj} \big) \big(a^{2j} - a^{-2j} \big) \end{aligned}$$

$$= \frac{(-A)^{-f}}{(a-a^{-1})^2} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} a^{2fj^2}\big(a^{2(m+1)j} - a^{2(m-1)j}\big),$$

which is equal to the right-hand side of the required formula, where we obtain the first equality by putting $i = 2j - N$. This completes the proof of the lemma. $\square$

By Lemma 3.1 and (36), we have that

$$\begin{aligned}
\frac{1}{c_+} \sum_{\substack{1 \le i < N \\ i \text{ is odd}}} A^{2(i^2-1)}[mi][i] &= \frac{A^{-2}}{c_+(a-a^{-1})^2} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} a^{4j^2}\big(a^{2(m+1)j} - a^{2(m-1)j}\big) \\
&= \frac{A^{-2}}{c_+(a-a^{-1})^2}\, a^{-\bar{4}\,(m^2+1)}\big(a^{-\bar{2}\,m} - a^{\bar{2}\,m}\big) \sum_{j \in \mathbb{Z}/N\mathbb{Z}} a^{j^2} \\
&= (-1)^{\frac{N^2-1}{8}} \frac{A}{a-a^{-1}}\, a^{-\bar{4}\,(m^2+1)}\big(a^{-\bar{2}\,m} - a^{\bar{2}\,m}\big). \qquad (20)
\end{aligned}$$

Let N and M be positive integers such that NM is even, and let ℓ be an integer. Then, it is known as Gauss sum reciprocity formula (see [6, 15]) that

$$\frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} e^{\frac{\pi\sqrt{-1}}{N} Mn^2 + \frac{2\pi\sqrt{-1}}{N}\ell n} = \frac{e^{\frac{\pi\sqrt{-1}}{4}}}{\sqrt{M}} \sum_{m \in \mathbb{Z}/M\mathbb{Z}} e^{-\frac{\pi\sqrt{-1}}{MN}(Nm+\ell)^2}. \qquad (21)$$

Lemma 3.2. *Let f and p be coprime positive odd integers. Let g be a map $\mathbb{Z}/pN\mathbb{Z} \to \mathbb{C}$. Then,*

$$\begin{aligned}
&\sum_{m \in \mathbb{Z}/pN\mathbb{Z}} g(m) \sum_{\substack{1 \le k < N \\ k \text{ is odd}}} A^{f(k^2-1)}[mk][k] \\
&= \frac{-A^{-f}\, e^{\frac{\pi\sqrt{-1}}{4}} e^{-\frac{\pi\sqrt{-1}}{4} fN} e^{-\frac{\pi\sqrt{-1}}{fN}} \sqrt{N}}{(a-a^{-1})^2\sqrt{f}} \sum_{m \in \mathbb{Z}/fpN\mathbb{Z}} g(m)\, e^{-\frac{2\pi\sqrt{-1}}{fN}\,\bar{2}\,m^2}\big(e^{\frac{2\pi\sqrt{-1}}{fN} m} - e^{-\frac{2\pi\sqrt{-1}}{fN} m}\big)
\end{aligned}$$

Proof. By Lemma 3.1, we have that

$$\begin{aligned}
&\sum_{m \in \mathbb{Z}/pN\mathbb{Z}} g(m) \sum_{\substack{1 \le k < N \\ k \text{ is odd}}} A^{f(k^2-1)}[mk][k] \\
&= \sum_{m \in \mathbb{Z}/pN\mathbb{Z}} g(m) \frac{(-A)^{-f}}{(a-a^{-1})^2} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} a^{2fj^2}\big(a^{2(m+1)j} - a^{2(m-1)j}\big) \\
&= \frac{(-1)^f A^{-f}}{(a-a^{-1})^2} \sum_{n \in \mathbb{Z}/pN\mathbb{Z}} g(\bar{2}\,n) \sum_{j \in \mathbb{Z}/N\mathbb{Z}} a^{2fj^2}\big(a^{(n+2)j} - a^{(n-2)j}\big) \\
&= \frac{(-1)^f A^{-f}\, e^{\frac{\pi\sqrt{-1}}{4}}\sqrt{N}}{2\,(a-a^{-1})^2\sqrt{f}} \sum_{n \in \mathbb{Z}/pN\mathbb{Z}} g(\bar{2}\,n) \sum_{k \in \mathbb{Z}/4f\mathbb{Z}} \big(e^{-\frac{\pi\sqrt{-1}}{4fN}(Nk+n+2)^2} - e^{-\frac{\pi\sqrt{-1}}{4fN}(Nk+n-2)^2}\big)
\end{aligned}$$

$$= \frac{(-1)^f A^{-f} e^{\frac{\pi\sqrt{-1}}{4}} \sqrt{N}}{2(a-a^{-1})^2\sqrt{f}} \sum_{n \in \mathbb{Z}/pN\mathbb{Z}} g(\overline{2}\, n) \sum_{k \in \mathbb{Z}/4f\mathbb{Z}} \big(e^{-\frac{\pi\sqrt{-1}}{4fN}(Npk+n+2)^2} - e^{-\frac{\pi\sqrt{-1}}{4fN}(Npk+n-2)^2} \big)$$
$$= \frac{(-1)^f A^{-f} e^{\frac{\pi\sqrt{-1}}{4}} \sqrt{N}}{2(a-a^{-1})^2\sqrt{f}} \sum_{\ell \in \mathbb{Z}/4fpN\mathbb{Z}} g(\overline{2}\, \ell) \big(e^{-\frac{\pi\sqrt{-1}}{4fN}(\ell+2)^2} - e^{-\frac{\pi\sqrt{-1}}{4fN}(\ell-2)^2} \big)$$
$$= \frac{-(-1)^f A^{-f} e^{\frac{\pi\sqrt{-1}}{4}} e^{-\frac{\pi\sqrt{-1}}{fN}} \sqrt{N}}{2(a-a^{-1})^2\sqrt{f}} \sum_{\ell \in \mathbb{Z}/4fpN\mathbb{Z}} g(\overline{2}\, \ell)\, e^{-\frac{\pi\sqrt{-1}}{4fN}\ell^2} \big(e^{\frac{\pi\sqrt{-1}}{fN}\ell} - e^{-\frac{\pi\sqrt{-1}}{fN}\ell} \big), \qquad (22)$$

where we obtain the second equality by putting $m = \overline{2}\, n$, and obtain the third equality by (21), and obtain the fourth equality by replacing k with pk, and obtain the fifth equality by putting $\ell = Npk + n$.

We calculate the sum of (22) by replacing ℓ with $\ell + 2fpN$

$$\sum_{\ell \in \mathbb{Z}/4fpN\mathbb{Z}} g(\overline{2}\, \ell)\, e^{-\frac{\pi\sqrt{-1}}{4fN}\ell^2} \big(e^{\frac{\pi\sqrt{-1}}{fN}\ell} - e^{-\frac{\pi\sqrt{-1}}{fN}\ell} \big)$$
$$= \sum_{\ell \in \mathbb{Z}/4fpN\mathbb{Z}} g(\overline{2}\, \ell)\, e^{-\frac{\pi\sqrt{-1}}{4fN}(\ell+2fpN)^2} \big(e^{\frac{\pi\sqrt{-1}}{fN}\ell} - e^{-\frac{\pi\sqrt{-1}}{fN}\ell} \big)$$
$$= \sum_{\ell \in \mathbb{Z}/4fpN\mathbb{Z}} g(\overline{2}\, \ell)\, (-1)^{\ell+f} e^{-\frac{\pi\sqrt{-1}}{4fN}\ell^2} \big(e^{\frac{\pi\sqrt{-1}}{fN}\ell} - e^{-\frac{\pi\sqrt{-1}}{fN}\ell} \big)$$
$$= \sum_{\substack{\ell \in \mathbb{Z}/4fpN\mathbb{Z} \\ \ell+f \text{ is even}}} g(\overline{2}\, \ell)\, e^{-\frac{\pi\sqrt{-1}}{4fN}\ell^2} \big(e^{\frac{\pi\sqrt{-1}}{fN}\ell} - e^{-\frac{\pi\sqrt{-1}}{fN}\ell} \big),$$

where we obtain the fourth formula by making the average of the first and third formulas.

Hence, we can calculate the formula (22) by putting $\ell = 2k + fpN$,

$$\frac{A^{-f} e^{\frac{\pi\sqrt{-1}}{4}} e^{-\frac{\pi\sqrt{-1}}{fN}} \sqrt{N}}{2(a-a^{-1})^2\sqrt{f}} \sum_{\substack{\ell \in \mathbb{Z}/4fpN\mathbb{Z} \\ \ell \text{ is odd}}} g(\overline{2}\, \ell)\, e^{-\frac{\pi\sqrt{-1}}{4fN}\ell^2} \big(e^{\frac{\pi\sqrt{-1}}{fN}\ell} - e^{-\frac{\pi\sqrt{-1}}{fN}\ell} \big)$$
$$= \frac{-A^{-f} e^{\frac{\pi\sqrt{-1}}{4}} e^{-\frac{\pi\sqrt{-1}}{fN}} \sqrt{N}}{2(a-a^{-1})^2\sqrt{f}} \sum_{k \in \mathbb{Z}/2fpN\mathbb{Z}} g(k)\, e^{-\frac{\pi\sqrt{-1}}{4fN}(2k+fpN)^2} \big(e^{\frac{2\pi\sqrt{-1}}{fN}k} - e^{-\frac{2\pi\sqrt{-1}}{fN}k} \big)$$
$$= \frac{-A^{-f} e^{\frac{\pi\sqrt{-1}}{4}} e^{-\frac{\pi\sqrt{-1}}{4}fN} e^{-\frac{\pi\sqrt{-1}}{fN}} \sqrt{N}}{2(a-a^{-1})^2\sqrt{f}} \sum_{k \in \mathbb{Z}/2fpN\mathbb{Z}} g(k)\, e^{-\frac{\pi\sqrt{-1}}{fN}k^2} (-1)^k \big(e^{\frac{2\pi\sqrt{-1}}{fN}k} - e^{-\frac{2\pi\sqrt{-1}}{fN}k} \big).$$

Further, since the summand is invariant when we replace k with $k + fpN$, the above formula is calculated as

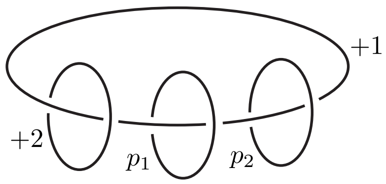
$$\begin{aligned} &\frac{-A^{-f}\, e^{\frac{\pi\sqrt{-1}}{4}} e^{-\frac{\pi\sqrt{-1}}{4}fN} e^{-\frac{\pi\sqrt{-1}}{fN}}\sqrt{N}}{(a-a^{-1})^2\sqrt{f}} \sum_{k\in\mathbb{Z}/fpN\mathbb{Z}} g(k)\, e^{-\frac{\pi\sqrt{-1}}{fN}k^2}(-1)^k\big(e^{\frac{2\pi\sqrt{-1}}{fN}k} - e^{-\frac{2\pi\sqrt{-1}}{fN}k}\big) \\ &= \frac{-A^{-f}\, e^{\frac{\pi\sqrt{-1}}{4}} e^{-\frac{\pi\sqrt{-1}}{4}fN} e^{-\frac{\pi\sqrt{-1}}{fN}}\sqrt{N}}{(a-a^{-1})^2\sqrt{f}} \sum_{k\in\mathbb{Z}/fpN\mathbb{Z}} g(k)\, e^{-\frac{2\pi\sqrt{-1}}{fN}\overline{2}\,k^2}\big(e^{\frac{2\pi\sqrt{-1}}{fN}k} - e^{-\frac{2\pi\sqrt{-1}}{fN}k}\big). \end{aligned}$$

Therefore, we obtain the required formula of the lemma. □

3.2 The quantum invariant of M_{p_1,p_2}

In this section, we calculate the quantum invariant of the Seifert 3-manifold M_{p_1,p_2}. We note that the quantum SU(2) and SO(3) invariants are equal when $q = e^{4\pi\sqrt{-1}/N}$ (see Appendix A), and we simply call them the quantum invariant.

Let p_1 and p_2 be coprime odd integers ≥ 3. We denote by M_{p_1,p_2} the Seifert 3-manifold obtained from S^3 by surgery along the following framed link,



(23)

where $+2, p_1, p_2, +1$ denote the framings of these components of the link. We assume that the linking matrix of this link is positive definite.

As shown in (38), the quantum invariant of M_{p_1,p_2} is given by

$$\hat{\tau}_N(M_{p_1,p_2}) \;=\; \frac{1}{c_+^4} \sum_{\substack{1\leq i,j,k,m<N \\ i,j,k,m \text{ are odd}}} A^{2(i^2-1)+p_1(j^2-1)+p_2(k^2-1)+(m^2-1)}\, \frac{[mi][mj][mk]}{[m]}\,[i][j][k].$$

By putting $m = 2n-N$, we have that

$$\begin{aligned} &\hat{\tau}_N(M_{p_1,p_2}) \\ &= \frac{-A^{-1}}{2\,c_+^4} \sum_{\substack{1\leq i,j,k<N \\ i,j,k \text{ are odd}}} \sum_{\substack{n\in\mathbb{Z}/N\mathbb{Z} \\ n\neq 0}} A^{2(i^2-1)+p_1(j^2-1)+p_2(k^2-1)}\, a^{2n^2}\, \frac{[2ni][2nj][2nk]}{[2n]}\,[i][j][k] \\ &= \frac{-A^{-1}}{2\,c_+^4} \sum_{\substack{1\leq i,j,k<N \\ i,j,k \text{ are odd}}} \sum_{\substack{n\in\mathbb{Z}/N\mathbb{Z} \\ n\neq 0}} A^{2(i^2-1)+p_1(j^2-1)+p_2(k^2-1)}\, a^{\overline{2}\,n^2}\, \frac{[ni][nj][nk]}{[n]}\,[i][j][k], \end{aligned}$$

where we obtain the second equality by replacing n with $\overline{2}\,n$. By (20), we have that

$$\hat{\tau}_N(M_{p_1,p_2}) \;=\; \frac{(-1)^{\frac{N^2-1}{8}}}{2\,c_+^3(a-a^{-1})} \sum_{\substack{n\in\mathbb{Z}/N\mathbb{Z} \\ n\neq 0}} a^{\overline{4}\,(n^2-1)}\big(a^{\overline{2}\,n}-a^{-\overline{2}\,n}\big) \sum_{\substack{1\leq j,k<N \\ j,k \text{ are odd}}} A^{p_1(j^2-1)+p_2(k^2-1)}\, \frac{[nj][nk]}{[n]}\,[j][k].$$

Further, by applying Lemma 3.2 twice, we have that

$$\hat{\tau}_N(M_{p_1,p_2}) = \frac{(-1)^{\frac{N^2-1}{8}}}{2\,c_+^3(a-a^{-1})} \sum_{\substack{n\in\mathbb{Z}/p_1p_2N\mathbb{Z} \\ n\neq 0}} a^{\overline{4}\,(n^2-1)} \frac{a^{\overline{2}\,n}-a^{-\overline{2}\,n}}{[n]}$$

$$\times \frac{A^{-p_1} e^{\frac{\pi\sqrt{-1}}{4}} e^{-\frac{\pi\sqrt{-1}}{4}p_1N} e^{-\frac{\pi\sqrt{-1}}{p_1N}} \sqrt{N}}{(a-a^{-1})^2\sqrt{p_1}} e^{-\frac{2\pi\sqrt{-1}}{p_1N}\overline{2}\,n^2} (e^{\frac{2\pi\sqrt{-1}}{p_1N}n} - e^{-\frac{2\pi\sqrt{-1}}{p_1N}n})$$

$$\times \frac{A^{-p_2} e^{\frac{\pi\sqrt{-1}}{4}} e^{-\frac{\pi\sqrt{-1}}{4}p_2N} e^{-\frac{\pi\sqrt{-1}}{p_2N}} \sqrt{N}}{(a-a^{-1})^2\sqrt{p_2}} e^{-\frac{2\pi\sqrt{-1}}{p_2N}\overline{2}\,n^2} (e^{\frac{2\pi\sqrt{-1}}{p_2N}n} - e^{-\frac{2\pi\sqrt{-1}}{p_2N}n})$$

$$= \frac{(-1)^{\frac{N^2-1}{8}} N A^{-p_1-p_2} a^{-\overline{4}}}{2\,c_+^3(a-a^{-1})^4\sqrt{p_1p_2}} e^{\frac{\pi\sqrt{-1}}{2}} e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N} e^{-\frac{\pi\sqrt{-1}}{p_1N}} e^{-\frac{\pi\sqrt{-1}}{p_2N}}$$

$$\times \sum_{\substack{n\in\mathbb{Z}/p_1p_2N\mathbb{Z} \\ n\neq 0}} a^{\overline{4}\,n^2} \frac{a^{\overline{2}\,n}-a^{-\overline{2}\,n}}{a^n-a^{-n}} e^{-\frac{2\pi\sqrt{-1}}{p_1N}\overline{2}\,n^2} e^{-\frac{2\pi\sqrt{-1}}{p_2N}\overline{2}\,n^2} (e^{\frac{2\pi\sqrt{-1}}{p_1N}n} - e^{-\frac{2\pi\sqrt{-1}}{p_1N}n})(e^{\frac{2\pi\sqrt{-1}}{p_2N}n} - e^{-\frac{2\pi\sqrt{-1}}{p_2N}n}).$$

By replacing n with $2n$, we have that

$$\hat{\tau}_N(M_{p_1,p_2}) = \frac{(-1)^{\frac{N^2-1}{8}} N A^{-p_1-p_2} a^{-\overline{4}}}{2\,c_+^3(a-a^{-1})^4\sqrt{p_1p_2}} e^{\frac{\pi\sqrt{-1}}{2}} e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N} e^{-\frac{\pi\sqrt{-1}}{p_1N}} e^{-\frac{\pi\sqrt{-1}}{p_2N}}$$

$$\times \sum_{n\in\mathbb{Z}/p_1p_2N\mathbb{Z}} \frac{a^{n^2} e^{-\frac{4\pi\sqrt{-1}}{p_1N}n^2} e^{-\frac{4\pi\sqrt{-1}}{p_2N}n^2}}{a^n+a^{-n}} (e^{\frac{4\pi\sqrt{-1}}{p_1N}n} - e^{-\frac{4\pi\sqrt{-1}}{p_1N}n})(e^{\frac{4\pi\sqrt{-1}}{p_2N}n} - e^{-\frac{4\pi\sqrt{-1}}{p_2N}n})$$

Further, by (36), we have that

$$\hat{\tau}_N(M_{p_1,p_2}) = \frac{(-1)^{\frac{N^2-1}{8}} A^{9-p_1-p_2} a^{-\overline{4}}}{2\,(a-a^{-1})\sqrt{p_1p_2}\sqrt{N}} e^{-\frac{\pi\sqrt{-1}}{4}} e^{\frac{\pi\sqrt{-1}}{4}(3-p_1+p_2)N} e^{-\frac{\pi\sqrt{-1}}{p_1N}} e^{-\frac{\pi\sqrt{-1}}{p_2N}}$$

$$\times \sum_{n\in\mathbb{Z}/p_1p_2N\mathbb{Z}} \frac{a^{n^2} e^{-\frac{4\pi\sqrt{-1}}{p_1N}n^2} e^{-\frac{4\pi\sqrt{-1}}{p_2N}n^2}}{a^n+a^{-n}} (e^{\frac{4\pi\sqrt{-1}}{p_1N}n} - e^{-\frac{4\pi\sqrt{-1}}{p_1N}n})(e^{\frac{4\pi\sqrt{-1}}{p_2N}n} - e^{-\frac{4\pi\sqrt{-1}}{p_2N}n})$$

$$= \frac{(-1)^{\frac{N^2-1}{8}} A^{9-p_1-p_2}}{2\,(a-a^{-1})\sqrt{p_1p_2}\sqrt{N}} e^{\frac{3\pi\sqrt{-1}}{4}} e^{\frac{\pi\sqrt{-1}}{4}(1-p_1+p_2)N} e^{-\frac{\pi\sqrt{-1}}{p_1N}} e^{-\frac{\pi\sqrt{-1}}{p_2N}} e^{-\frac{\pi\sqrt{-1}}{2N}}$$

$$\times \sum_{n\in\mathbb{Z}/p_1p_2N\mathbb{Z}} \frac{a^{n^2} e^{-\frac{4\pi\sqrt{-1}}{p_1N}n^2} e^{-\frac{4\pi\sqrt{-1}}{p_2N}n^2}}{a^n+a^{-n}} (e^{\frac{4\pi\sqrt{-1}}{p_1N}n} - e^{-\frac{4\pi\sqrt{-1}}{p_1N}n})(e^{\frac{4\pi\sqrt{-1}}{p_2N}n} - e^{-\frac{4\pi\sqrt{-1}}{p_2N}n}),$$

where we obtain the second equality since $\overline{4} = \left(\frac{N+1}{2}\right)^2$ and $a^{-\overline{4}} = -e^{-\frac{\pi\sqrt{-1}}{2}N} e^{-\frac{\pi\sqrt{-1}}{2N}}$. Hence, we obtain the following proposition.

Proposition 3.3. *Let p_1 and p_2 be coprime positive odd integers such that $\frac{1}{p_1}+\frac{1}{p_2}<\frac{1}{2}$.*

Then, the quantum invariant of M_{p_1,p_2} is presented by

$$\begin{aligned}\hat{\tau}_N(M_{p_1,p_2}) \;&=\; \frac{(-1)^{\frac{N^2-1}{8}} A^{9-p_1-p_2}}{2\,(a-a^{-1})\sqrt{p_1p_2}\sqrt{N}}\, e^{\frac{3\pi\sqrt{-1}}{4}}\, e^{\frac{\pi\sqrt{-1}}{4}(1-p_1-p_2)N}\, e^{-\frac{\pi\sqrt{-1}}{p_1N}}\, e^{-\frac{\pi\sqrt{-1}}{p_2N}}\, e^{-\frac{\pi\sqrt{-1}}{2N}} \\ &\quad\times \sum_{n\,\in\,\mathbb{Z}/p_1p_2N\mathbb{Z}} b^{(p_1p_2-2p_1-2p_2)n^2}\, \frac{(b^{2p_1n}-b^{-2p_1n})(b^{2p_2n}-b^{-2p_2n})}{b^{p_1p_2n}+b^{-p_1p_2n}}\,,\end{aligned}$$

where we put $b = e^{\frac{2\pi\sqrt{-1}}{p_1p_2N}}$.

We note that we assume that $\frac{1}{p_1}+\frac{1}{p_2}<\frac{1}{2}$ in order that the linking matrix of the link (23) is positive definite.

As a particular case of the above proposition, we obtain the following proposition.

Proposition 3.4. *Let p be an odd integer > 6. Then, the quantum invariant of $M_{3,p}$ is presented by*

$$\begin{aligned}\hat{\tau}_N(M_{3,p}) \;&=\; \frac{(-1)^{\frac{N^2-1}{8}} A^{6-p}}{2\,(a-a^{-1})\sqrt{3p}\sqrt{N}}\, e^{\frac{3\pi\sqrt{-1}}{4}}\, e^{-\frac{\pi\sqrt{-1}}{4}(p+2)N}\, e^{-\frac{\pi\sqrt{-1}}{pN}}\, e^{-\frac{5\pi\sqrt{-1}}{6N}} \\ &\quad\times \sum_{n\,\in\,\mathbb{Z}/3pN\mathbb{Z}} b^{(p-6)n^2}\, \frac{(b^{6n}-b^{-6n})(b^{2pn}-b^{-2pn})}{b^{3pn}+b^{-3pn}}\,,\end{aligned}$$

where we put $b = e^{\frac{2\pi\sqrt{-1}}{3pN}}$.

We show some examples of the proposition below.

Example 3.5. The quantum invariant of $M_{3,7}$ is presented by

$$\hat{\tau}_N(M_{3,7}) \;=\; \frac{(-1)^{\frac{N^2-1}{8}} A^{-1}\, e^{\frac{3\pi\sqrt{-1}}{4}}}{(a-a^{-1})\sqrt{21}\sqrt{N}}\, e^{-\frac{\pi\sqrt{-1}}{4}N}\, e^{-\frac{41\pi\sqrt{-1}}{42N}} \sum_{n\,\in\,\mathbb{Z}/21N\mathbb{Z}} b^{n^2}\, \frac{b^{20n}-b^{8n}}{b^{21n}+b^{-21n}}\,,$$

where we put $b = e^{\frac{2\pi\sqrt{-1}}{21N}}$.

Example 3.6. The quantum invariant of $M_{3,11}$ is presented by

$$\hat{\tau}_N(M_{3,11}) \;=\; \frac{(-1)^{\frac{N^2-1}{8}} A^{-5}\, e^{\frac{3\pi\sqrt{-1}}{4}}}{(a-a^{-1})\sqrt{33}\sqrt{N}}\, e^{-\frac{5\pi\sqrt{-1}}{4}N}\, e^{-\frac{61\pi\sqrt{-1}}{66N}} \sum_{n\,\in\,\mathbb{Z}/33N\mathbb{Z}} b^{5n^2}\, \frac{b^{28n}-b^{16n}}{b^{33n}+b^{-33n}}\,,$$

where we put $b = e^{\frac{2\pi\sqrt{-1}}{33N}}$.

Example 3.7. The quantum invariant of $M_{3,5}$ is presented by

$$\hat{\tau}_N(M_{3,5}) \;=\; \frac{(-1)^{\frac{N+1}{2}}\,(-1)^{\frac{N^2-1}{8}} A^{-5}\, e^{\frac{3\pi\sqrt{-1}}{4}}}{(a-a^{-1})\sqrt{15}\sqrt{N}}\, e^{\frac{\pi\sqrt{-1}}{4}N}\, e^{-\frac{31\pi\sqrt{-1}}{30N}} \sum_{n\,\in\,\mathbb{Z}/15N\mathbb{Z}} b^{-n^2}\, \frac{b^{16n}-b^{4n}}{b^{15n}+b^{-15n}}\,,$$

where we put $b = e^{\frac{2\pi\sqrt{-1}}{15N}}$. We note that, in this case, the signature of the linking matrix of the link (23) is 2, and we obtain the above formula as $\frac{c_+}{c_-}$ times the formula of Proposition 3.4, where the value of $\frac{c_+}{c_-}$ is given in (37).

3.3 The semi-classical limit of the quantum invariants of M_{p_1,p_2}

In this section, we calculate the semi-classical limit of the quantum invariants of M_{p_1,p_2} in Proposition 3.11.

In order to show Proposition 3.11, we show Lemma 3.8 below. The former part of the proof of Lemma 3.8 is calculated by using residues of integrals, following a method of [23].

Lemma 3.8. *Let p and δ be coprime positive odd integers, and let k be an integer. We put $b = e^{\frac{2\pi\sqrt{-1}}{pN}}$. Then, the asymptotic behavior of the left-hand side of the following formula as $N \to \infty$ (where N is odd) is given by*

$$\sum_{n \in \mathbb{Z}/pN\mathbb{Z}} b^{\delta n^2} \frac{b^{4kn}}{b^{pn} + b^{-pn}} = -N\sqrt{-1}\,(-1)^{\frac{\delta-1}{2}}\, e^{\frac{\pi\sqrt{-1}}{8}\,\overline{\delta}\,pd}\, \frac{(-1)^{\frac{p-1}{2}} + \sqrt{-1}^{\,\delta d}}{4}$$
$$\times \sum_{-\frac{p}{4}<c<\frac{p}{4}} (-1)^c\, e^{\frac{2\pi\sqrt{-1}}{p}\,\overline{\delta}\,c^2 d} \big(e^{\frac{8\pi\sqrt{-1}}{p}\,\overline{\delta}\,ck} + e^{-\frac{8\pi\sqrt{-1}}{p}\,\overline{\delta}\,ck} \big) + O\big(N^{1/2}\big),$$

where $\overline{\delta}$ denotes the inverse of δ in $\mathbb{Z}/16p\mathbb{Z}$.

Proof. We put the left-hand side of the required formula to be S,

$$S = \sum_{n \in \mathbb{Z}/pN\mathbb{Z}} b^{\delta n^2} \frac{b^{4kn}}{b^{pn} + b^{-pn}},$$

and we put

$$f_N(\beta, x) = \frac{e^{\frac{2\pi\sqrt{-1}}{pN}\delta\beta^2}\, e^{\frac{2\pi\sqrt{-1}}{N}\beta x}}{e^{\frac{2\pi\sqrt{-1}}{N}\beta} + e^{-\frac{2\pi\sqrt{-1}}{N}\beta}},$$

noting that the summand of the sum of S is given by $f_N(n, \frac{4k}{p})$. Further, in order to calculate this sum, we put

$$h_N(\beta, x) = \frac{f_N(\beta, x)}{e^{2\pi\sqrt{-1}\beta} - 1} = \frac{e^{\frac{2\pi\sqrt{-1}}{pN}\delta\beta^2}\, e^{\frac{2\pi\sqrt{-1}}{N}\beta x}}{(e^{\frac{2\pi\sqrt{-1}}{N}\beta} + e^{-\frac{2\pi\sqrt{-1}}{N}\beta})(e^{2\pi\sqrt{-1}\beta} - 1)}.$$

We note that this function satisfies that

$$h_N(\beta + pN, x) = h_N(\beta, x + 2\delta N)\, e^{2\pi\sqrt{-1}\,px}.$$

We consider the following integral,

$$\Theta_N(x) = \int_C h_N(\beta, x)\, d\beta,$$

where the contour C is the oriented line from $(-1-\sqrt{-1})\infty$ to $(1+\sqrt{-1})\infty$ through the origin. Then,

$$\Theta_N(x + 2\delta N) - \Theta_N(x) = \int_{C+2\delta N} h_N(\beta, x)\, d\beta - \int_C h_N(\beta, x)\, d\beta$$
$$= 2\pi\sqrt{-1} \sum_{n=0}^{pN-1} \operatorname*{Res}_{\beta=n} h_N(\beta, x) + 2\pi\sqrt{-1} \sum_{\substack{0<c<4p \\ c \text{ is odd}}} \operatorname*{Res}_{\beta=\frac{cN}{4}} h_N(\beta, x)$$

$$= \sum_{n=0}^{pN-1} f_N(n,x) \ + 2\pi\sqrt{-1} \sum_{\substack{0<c<4p \\ c \text{ is odd}}} \operatorname*{Res}_{\beta=\frac{cN}{4}} h_N(\beta,x),$$

where we obtain the second equality since $h_N(\beta,x)$ has poles at $\beta = n$ and $\beta = \frac{cN}{4}$. Further, since

$$h_N(\beta, x+2\delta N) \ = \ h_N(\beta,x)\, e^{4\pi\sqrt{-1}\,\beta\delta},$$

the above formula is also calculated in another way,

$$\begin{aligned}\Theta(x+2\delta N) - \Theta(x) \ &= \ \int_C f_N(\beta,x)\, \frac{e^{4\pi\sqrt{-1}\,\beta\delta}-1}{e^{2\pi\sqrt{-1}\,\beta}-1}\, d\beta \\ &= \ \int_C f_N(\beta,x)\, \big(1+e^{2\pi\sqrt{-1}\,\beta}+e^{4\pi\sqrt{-1}\,\beta}+\cdots+e^{2\pi\sqrt{-1}\,\beta(2\delta-1)}\big)\, d\beta \\ &= \ \sum_{\ell=0}^{2\delta-1} \int_C f_N(\beta, x+\ell N)\, d\beta.\end{aligned}$$

Therefore, by the above two formulas of "$\Theta(x+2\delta N)-\Theta(x)$", we obtain that

$$\sum_{n=0}^{pN-1} f_N(n,x) \ = \ \sum_{\ell=0}^{2\delta-1} \int_C f_N(\beta, x+\ell N)\, d\beta \ - 2\pi\sqrt{-1} \sum_{\substack{0<c<4p \\ c \text{ is odd}}} \operatorname*{Res}_{\beta=\frac{cN}{4}} h_N(\beta,x).$$

Hence, the sum of the problem is calculated as

$$\begin{aligned} S \ &= \ \sum_{n\,\in\,\mathbb{Z}/pN\mathbb{Z}} f_N\big(n,\ \frac{4k}{p}\big) \\ &= \ \sum_{\ell=0}^{2\delta-1} \int_C f_N\big(\beta,\ \frac{4k}{p}+\ell N\big)\, d\beta \ - 2\pi\sqrt{-1} \sum_{\substack{0<c<4p \\ c \text{ is odd}}} \operatorname*{Res}_{\beta=\frac{cN}{4}} h_N\big(\beta,\ \frac{4k}{p}\big).\end{aligned}$$

Further, by changing the variable putting $y = \frac{2\pi\sqrt{-1}}{N}\beta$, we have that

$$\begin{aligned} S \ = \ & \frac{N}{2\pi\sqrt{-1}} \sum_{\ell=0}^{2\delta-1} \int_{C'} \frac{e^{\frac{4k}{p}y}}{e^y+e^{-y}}\, e^{N(\frac{\delta}{2\pi\sqrt{-1}\,p}y^2+\ell y)}\, dy \\ & - N \sum_{\substack{0<c<4p \\ c \text{ is odd}}} \operatorname*{Res}_{y=\frac{\pi\sqrt{-1}\,c}{2}} \frac{e^{\frac{N}{2\pi\sqrt{-1}}\frac{\delta}{p}y^2}\, e^{\frac{4k}{p}y}}{(e^y+e^{-y})(e^{Ny}-1)}\,,\end{aligned}$$

where we put $C' = \sqrt{-1}\,C$. Furthermore, we consider to change the contour C' to $C'_\ell = C' - \frac{\pi\sqrt{-1}\,p\ell}{\delta}$ in order that the new contour C'_ℓ goes through the saddle point of the integrand. Then, we have that

$$S = \frac{N}{2\pi\sqrt{-1}} \sum_{\ell=0}^{2\delta-1} \int_{C'_\ell} \frac{e^{\frac{4k}{p}y}}{e^y + e^{-y}} \, e^{N(\frac{\delta}{2\pi\sqrt{-1}\,p}y^2 + \ell y)} \, dy$$
$$+ N \sum_{\ell=0}^{2\delta-1} \sum_{\substack{0<c<\frac{2p\ell}{\delta} \\ c \text{ is odd}}} \operatorname*{Res}_{y=-\frac{\pi\sqrt{-1}\,c}{2}} \frac{e^{\frac{4k}{p}y}}{e^y + e^{-y}} \, e^{N(\frac{\delta}{2\pi\sqrt{-1}\,p}y^2 + \ell y)}$$
$$- N \sum_{\substack{0<c<4p \\ c \text{ is odd}}} \operatorname*{Res}_{y=\frac{\pi\sqrt{-1}\,c}{2}} \frac{e^{\frac{N}{2\pi\sqrt{-1}}\frac{\delta}{p}y^2} \, e^{\frac{4k}{p}y}}{(e^y + e^{-y})(e^{Ny} - 1)} .$$

Hence, by Lemma 3.9 below, $\frac{S}{N}$ is approximated up to $O\big(\frac{1}{N^{1/2}}\big)$ as

$$\frac{S}{N} \sim \sum_{\ell=0}^{2\delta-1} \sum_{\substack{0<c<\frac{2p\ell}{\delta} \\ c \text{ is odd}}} \operatorname*{Res}_{y=-\frac{\pi\sqrt{-1}\,c}{2}} \frac{e^{\frac{4k}{p}y}}{e^y + e^{-y}} \, e^{N(\frac{\delta}{2\pi\sqrt{-1}\,p}y^2 + \ell y)}$$
$$- \sum_{\substack{0<c<4p \\ c \text{ is odd}}} \operatorname*{Res}_{y=\frac{\pi\sqrt{-1}\,c}{2}} \frac{e^{\frac{N}{2\pi\sqrt{-1}}\frac{\delta}{p}y^2} \, e^{\frac{4k}{p}y}}{(e^y + e^{-y})(e^{Ny} - 1)} .$$

By calculating these residues, we have that

$$\frac{S}{N} \sim \frac{1}{2} \sum_{\ell=0}^{2\delta-1} \sum_{\substack{0<c<\frac{2p\ell}{\delta} \\ c \text{ is odd}}} \Big(\frac{e^{\frac{4k}{p}y}}{e^y} \, e^{N(\frac{\delta}{2\pi\sqrt{-1}\,p}y^2 + \ell y)} \Big)\Big|_{y=-\frac{\pi\sqrt{-1}\,c}{2}}$$
$$- \frac{1}{2} \sum_{\substack{0<c<4p \\ c \text{ is odd}}} \Big(\frac{e^{\frac{N}{2\pi\sqrt{-1}}\frac{\delta}{p}y^2} \, e^{\frac{4k}{p}y}}{e^y \, (e^{Ny} - 1)} \Big)\Big|_{y=\frac{\pi\sqrt{-1}\,c}{2}} .$$

In the following of this proof, we consider to simplify the above formula. Let d be an odd integer with $0 < d < 16p$. We consider a subsequence of N such that $N \equiv d$ modulo $16p$; we note that, for such a subsequence, $\frac{S}{N}$ converges to the value of the right-hand side of the above formula. In fact, the above formula is calculated as

$$\frac{S}{N} \sim \frac{\sqrt{-1}}{2} \sum_{\ell=0}^{2\delta-1} \sum_{\substack{0<c<\frac{2p\ell}{\delta} \\ c \text{ is odd}}} (-1)^{\frac{c-1}{2}} \, e^{\frac{\delta\pi\sqrt{-1}}{8p}c^2 d} \, e^{-\frac{\pi\sqrt{-1}}{2}cd\ell} \, e^{-\frac{2\pi\sqrt{-1}}{p}ck}$$
$$+ \frac{\sqrt{-1}}{2} \sum_{\substack{0<c<4p \\ c \text{ is odd}}} \frac{(-1)^{\frac{c-1}{2}}}{e^{\frac{\pi\sqrt{-1}}{2}cd} - 1} \, e^{\frac{\delta\pi\sqrt{-1}}{8p}c^2 d} \, e^{\frac{2\pi\sqrt{-1}}{p}ck}.$$

By exchanging the order of the two sums in the first line, we have that

$$\begin{aligned}\frac{S}{N} \sim\ & \frac{\sqrt{-1}}{2} \sum_{\substack{0<c<4p \\ c \text{ is odd}}} (-1)^{\frac{c-1}{2}}\, e^{\frac{\delta\pi\sqrt{-1}}{8p}c^2 d}\, e^{-\frac{2\pi\sqrt{-1}}{p}ck} \sum_{\frac{c\delta}{2p}<\ell<2\delta} \sqrt{-1}^{\,-cd\ell} \\ & + \frac{\sqrt{-1}}{2} \sum_{\substack{0<c<4p \\ c \text{ is odd}}} \frac{(-1)^{\frac{c-1}{2}}}{\sqrt{-1}^{\,cd}-1}\, e^{\frac{\delta\pi\sqrt{-1}}{8p}c^2 d}\, e^{\frac{2\pi\sqrt{-1}}{p}ck}.\end{aligned}$$

Further, we replace c with $4p-c$, and replace ℓ with $2\delta-\ell$ in the first line. Then, since $\frac{1}{\sqrt{-1}^{cd}-1} = -\frac{1+\sqrt{-1}^{cd}}{2}$, we obtain that

$$\frac{S}{N} \sim -\frac{\sqrt{-1}}{2} \sum_{\substack{0<c<4p \\ c \text{ is odd}}} (-1)^{\frac{c-1}{2}}\, e^{\frac{\delta\pi\sqrt{-1}}{8p}c^2 d}\, e^{\frac{2\pi\sqrt{-1}}{p}ck} \Big(\frac{1+\sqrt{-1}^{\,cd}}{2} + \sum_{0<\ell<\frac{c\delta}{2p}} \sqrt{-1}^{\,-cd\ell} \Big).$$

The last factor is calculated as

$$\begin{aligned}\frac{1+\sqrt{-1}^{\,cd}}{2} + \sum_{0<\ell<\frac{c\delta}{2p}} \sqrt{-1}^{\,-cd\ell} &= \begin{cases} \frac{1+\sqrt{-1}^{\,cd}}{2} & \text{if } \lfloor \frac{c\delta}{2p} \rfloor \equiv 0 \text{ modulo } 4, \\ \frac{1-\sqrt{-1}^{\,cd}}{2} & \text{if } \lfloor \frac{c\delta}{2p} \rfloor \equiv 1 \text{ modulo } 4, \\ \frac{-1-\sqrt{-1}^{\,cd}}{2} & \text{if } \lfloor \frac{c\delta}{2p} \rfloor \equiv 2 \text{ modulo } 4, \\ \frac{-1+\sqrt{-1}^{\,cd}}{2} & \text{if } \lfloor \frac{c\delta}{2p} \rfloor \equiv 3 \text{ modulo } 4 \end{cases} \\ &= \frac{1+\sqrt{-1}^{\,cd}}{2} \sqrt{-1}^{\,-cd\lfloor \frac{c\delta}{2p} \rfloor},\end{aligned}$$

where $\lfloor \cdot \rfloor$ is the floor function, *i.e.*, for a real number x, $\lfloor x \rfloor$ denotes the greatest integer which is less than or equal to x. Hence,

$$\frac{S}{N} \sim -\frac{\sqrt{-1}}{2} \sum_{\substack{c\in\mathbb{Z}/4p\mathbb{Z} \\ c \text{ is odd}}} (-1)^{\frac{c-1}{2}}\, e^{\frac{\delta\pi\sqrt{-1}}{8p}c^2 d}\, e^{\frac{2\pi\sqrt{-1}}{p}ck}\, \frac{1+\sqrt{-1}^{\,cd}}{2} \sqrt{-1}^{\,-cd\lfloor \frac{c\delta}{2p} \rfloor}.$$

Let $\overline{\delta}$ be the inverse of δ in $\mathbb{Z}/16p\mathbb{Z}$, *i.e.*, $\overline{\delta}$ is an integer such that $\delta\,\overline{\delta} \equiv 1$ modulo $16p$. By replacing c with $\overline{\delta}\,c$, we have that

$$\begin{aligned}\frac{S}{N} \sim\ & -\frac{\sqrt{-1}}{2} \sum_{\substack{c\in\mathbb{Z}/4p\mathbb{Z} \\ c \text{ is odd}}} (-1)^{\frac{\overline{\delta}\,c-1}{2}}\, e^{\frac{\pi\sqrt{-1}}{8p}\,\overline{\delta}\,c^2 d}\, e^{\frac{2\pi\sqrt{-1}}{p}\,\overline{\delta}\,ck}\, \frac{1+\sqrt{-1}^{\,\overline{\delta}\,cd}}{2} \sqrt{-1}^{\,-\overline{\delta}\,cd\lfloor \frac{c}{2p} \rfloor} \\ =\ & -\frac{\sqrt{-1}}{2} \sum_{\substack{c\in\mathbb{Z}/4p\mathbb{Z} \\ c \text{ is odd}}} (-1)^{\frac{c-\delta}{2}}\, e^{\frac{\pi\sqrt{-1}}{8p}\,\overline{\delta}\,c^2 d}\, e^{\frac{2\pi\sqrt{-1}}{p}\,\overline{\delta}\,ck}\, \frac{1+\sqrt{-1}^{\,\delta cd}}{2} \sqrt{-1}^{\,-\delta cd\lfloor \frac{c}{2p} \rfloor}\end{aligned}$$

$$= -\frac{\sqrt{-1}}{2} \sum_{\substack{0<c<2p \\ c \text{ is odd}}} (-1)^{\frac{c-\delta}{2}} e^{\frac{\pi\sqrt{-1}}{8p}\bar{\delta}\, c^2 d} e^{\frac{2\pi\sqrt{-1}}{p}\bar{\delta}\, ck} \frac{1+\sqrt{-1}^{\,\delta cd}}{2}$$

$$- \frac{\sqrt{-1}}{2} \sum_{\substack{-2p<c<0 \\ c \text{ is odd}}} (-1)^{\frac{c-\delta}{2}} e^{\frac{\pi\sqrt{-1}}{8p}\bar{\delta}\, c^2 d} e^{\frac{2\pi\sqrt{-1}}{p}\bar{\delta}\, ck} \frac{\sqrt{-1}^{\,\delta cd}-1}{2}$$

$$= -\frac{\sqrt{-1}}{2} \sum_{\substack{0<c<2p \\ c \text{ is odd}}} (-1)^{\frac{c-\delta}{2}} e^{\frac{\pi\sqrt{-1}}{8p}\bar{\delta}\, c^2 d} \big(e^{\frac{2\pi\sqrt{-1}}{p}\bar{\delta}\, ck} + e^{-\frac{2\pi\sqrt{-1}}{p}\bar{\delta}\, ck} \big) \frac{1+\sqrt{-1}^{\,\delta cd}}{2}$$

$$= -\frac{\sqrt{-1}}{2} (-1)^{\frac{\delta-1}{2}} \sum_{\substack{0<c<2p \\ c \text{ is odd}}} \frac{(-1)^{\frac{c-1}{2}}+\sqrt{-1}^{\,\delta d}}{2} e^{\frac{\pi\sqrt{-1}}{8p}\bar{\delta}\, c^2 d} \big(e^{\frac{2\pi\sqrt{-1}}{p}\bar{\delta}\, ck} + e^{-\frac{2\pi\sqrt{-1}}{p}\bar{\delta}\, ck} \big).$$

Further, by replacing c with $2c+p$, we have that

$$\frac{S}{N} \sim -\frac{\sqrt{-1}}{2} (-1)^{\frac{\delta-1}{2}} e^{\frac{\pi\sqrt{-1}}{8}\bar{\delta}\, pd}$$

$$\times \sum_{-\frac{p}{2}<c<\frac{p}{2}} \frac{(-1)^c(-1)^{\frac{p-1}{2}}+\sqrt{-1}^{\,\delta d}}{2} \sqrt{-1}^{\,\delta cd} e^{\frac{\pi\sqrt{-1}}{2p}\bar{\delta}\, c^2 d} \big(e^{\frac{4\pi\sqrt{-1}}{p}\bar{\delta}\, ck} + e^{-\frac{4\pi\sqrt{-1}}{p}\bar{\delta}\, ck} \big).$$

We note that, when we replace c with $-c$, the summand becomes $(-1)^c$-multiple. Hence, we can restrict the sum to even c. By replacing c with $2c$, we have that

$$\frac{S}{N} \sim -\sqrt{-1}\,(-1)^{\frac{\delta-1}{2}} e^{\frac{\pi\sqrt{-1}}{8}\bar{\delta}\, pd} \frac{(-1)^{\frac{p-1}{2}}+\sqrt{-1}^{\,\delta d}}{4}$$

$$\times \sum_{-\frac{p}{4}<c<\frac{p}{4}} (-1)^c e^{\frac{2\pi\sqrt{-1}}{p}\bar{\delta}\, c^2 d} \big(e^{\frac{8\pi\sqrt{-1}}{p}\bar{\delta}\, ck} + e^{-\frac{8\pi\sqrt{-1}}{p}\bar{\delta}\, ck} \big).$$

Therefore, we obtain the required formula of the lemma. □

Lemma 3.9. *For the notation in the proof of Lemma 3.8,*

$$\int_{C'_\ell} \frac{e^{\frac{4k}{p}y}}{e^y+e^{-y}} e^{N(\frac{\delta}{2\pi\sqrt{-1}\,p}y^2+\ell y)}\, dy = O\big(\frac{1}{N^{1/2}}\big).$$

Proof. We put $\alpha = \frac{p\ell}{\delta}$, noting that this is not a half-integer. By changing the variable putting $y = -\pi\sqrt{-1}\,\alpha + (-1+\sqrt{-1})\,t$, the integral of the lemma is rewritten as

$$(-1+\sqrt{-1})\, e^{-\frac{\pi\sqrt{-1}}{2\delta}p\ell^2 N} \int_{\mathbb{R}} g(t)\, e^{-N\frac{\delta}{\pi p}t^2}\, dt,$$

where we put

$$g(t) = \frac{e^{\frac{4k}{p}(-\pi\sqrt{-1}\,\alpha+(-1+\sqrt{-1})\,t)}}{e^{-\pi\sqrt{-1}\,\alpha+(-1+\sqrt{-1})\,t}+e^{\pi\sqrt{-1}\,\alpha-(-1+\sqrt{-1})\,t}}.$$

Since the absolute value of $(-1+\sqrt{-1})\, e^{-\frac{\pi\sqrt{-1}}{2\delta} p\ell^2 N}$ is bounded by $\sqrt{2}$, it is sufficient to show that

$$\int_{\mathbb{R}} g(t)\, e^{-N\frac{\delta}{\pi p}t^2}\, dt \;=\; O\big(\frac{1}{N^{1/2}}\big). \tag{24}$$

We show (24), as follows. We consider to estimate $|g(t)|$. We have that

$$\begin{aligned}
&\big|\, e^{-\pi\sqrt{-1}\,\alpha+(-1+\sqrt{-1})\,t} + e^{\pi\sqrt{-1}\,\alpha-(-1+\sqrt{-1})\,t}\,\big|^2 \\
&= \big|\, e^{-\pi\sqrt{-1}\,\alpha+\sqrt{-1}\,t} e^{-t} + e^{\pi\sqrt{-1}\,\alpha-\sqrt{-1}\,t} e^{t}\,\big|^2 \\
&= \big(e^{-\pi\sqrt{-1}\,\alpha+\sqrt{-1}\,t} e^{-t} + e^{\pi\sqrt{-1}\,\alpha-\sqrt{-1}\,t} e^{t}\big)\big(e^{\pi\sqrt{-1}\,\alpha-\sqrt{-1}\,t} e^{-t} + e^{-\pi\sqrt{-1}\,\alpha+\sqrt{-1}\,t} e^{t}\big) \\
&= e^{2t} + e^{-2t} + 2\cos\big(2\pi\alpha - 2t\big).
\end{aligned}$$

Since $e^{2t}+e^{-2t} \geq 2$ and $2\cos\big(2\pi\alpha-2t\big) \geq -2$, the above formula is non-negative, and the above formula would be 0 only if $t=0$ and α was a half-integer. Actually, α is not a half-integer (the difference of α and a half-integer is at least $\frac{1}{2\delta}$), and there exists a constant $\varepsilon > 0$ (depending on δ) such that

$$e^{2t} + e^{-2t} + 2\cos\big(2\pi\alpha - 2t\big) \;\geq\; \varepsilon.$$

Hence,

$$\big|g(t)\big| \;\leq\; \frac{e^{-\frac{4k}{p}t}}{\sqrt{\varepsilon}}$$

Therefore,

$$\Big|\int_{\mathbb{R}} g(t)\, e^{-N\frac{\delta}{\pi p}t^2}\, dt\Big| \;\leq\; \frac{1}{\sqrt{\varepsilon}}\int_{\mathbb{R}} e^{-\frac{4k}{p}t}\, e^{-N\frac{\delta}{\pi p}t^2}\, dt \;=\; O\big(\frac{1}{N^{1/2}}\big).$$

Hence, we obtain (24), as required. □

Remark 3.10. By the saddle point method (see *e.g.* [46]), the value of the integral of Lemma 3.9 is expanded in the following form,

$$e^{-\frac{\pi\sqrt{-1}}{2\delta}p\ell^2 N}\, \frac{\lambda}{N^{1/2}}\Big(1+\frac{\lambda_1}{N}+\frac{\lambda_2}{N^2}+\cdots\Big),$$

with some complex constants λ and λ_i. The higher order part of the asymptotic expansion of the formula of Proposition 3.11 below can be obtained from such expansion.

By using Lemma 3.8, we calculate the semi-classical limit of the quantum invariants of M_{p_1,p_2} in the following proposition.

Proposition 3.11. *Let p_1 and p_2 be coprime positive odd integers. Then, the semi-classical limit of the quantum invariants of M_{p_1,p_2} as $N\to\infty$ (where N is odd) is given by*

$$\begin{aligned}
\hat{\tau}_N(M_{p_1,p_2}) \;=\; &\frac{-e^{\frac{\pi\sqrt{-1}}{4}}}{\pi\sqrt{2p_1p_2}}\, N^{3/2}\,(-1)^{\frac{N-1}{2}}\, e^{\frac{\pi\sqrt{-1}}{8}N} \\
&\times \sum_{1\leq c<\frac{1}{4}p_1p_2} (-1)^c\, e^{\frac{2\pi\sqrt{-1}}{p_1p_2}c^2\bar{\delta}N}\, \sin\frac{2\pi\,\overline{p_2}\,c}{p_1}\, \sin\frac{2\pi\,\overline{p_1}\,c}{p_2} \;+O\big(N\big),
\end{aligned}$$

where we put $\delta = p_1p_2 - 2p_1 - 2p_2$, $\overline{\delta}$ *denotes the inverse of* δ *in* $\mathbb{Z}/p_1p_2\mathbb{Z}$, $\overline{p_1}$ *denotes the inverse of* p_1 *in* $\mathbb{Z}/p_2\mathbb{Z}$, *and* $\overline{p_2}$ *denotes the inverse of* p_2 *in* $\mathbb{Z}/p_1\mathbb{Z}$.

Proof. We put p_1p_2. In this proof, we denote by $\overline{\delta}$ the inverse of δ in $\mathbb{Z}/16p\mathbb{Z}$. By Lemma 3.8, we have the following approximation up to $O\big(\frac{1}{N^{1/2}}\big)$,

$$\begin{aligned}
&\frac{1}{N}\sum_{n\in\mathbb{Z}/pN\mathbb{Z}} b^{\delta n^2}\,\frac{b^{2(p_1+p_2)n}-b^{2(p_1-p_2)n}-b^{2(-p_1+p_2)n}+b^{2(-p_1-p_2)n}}{b^{pn}+b^{-pn}}\\
&\sim\ -\sqrt{-1}\,(-1)^{\frac{\delta-1}{2}}\,e^{\frac{\pi\sqrt{-1}}{8}\overline{\delta}\,pd}\ \frac{(-1)^{\frac{p-1}{2}}+\sqrt{-1}^{\,\delta d}}{2}\\
&\qquad\times\sum_{-\frac{p}{4}<c<\frac{p}{4}}(-1)^c\,e^{\frac{2\pi\sqrt{-1}}{p}\overline{\delta}\,c^2d}\Big(e^{\frac{4\pi\sqrt{-1}}{p}\overline{\delta}\,c(p_1+p_2)}-e^{\frac{4\pi\sqrt{-1}}{p}\overline{\delta}\,c(p_1-p_2)}\\
&\qquad\qquad -e^{\frac{4\pi\sqrt{-1}}{p}\overline{\delta}\,c(-p_1+p_2)}+e^{\frac{4\pi\sqrt{-1}}{p}\overline{\delta}\,c(-p_1-p_2)}\Big)\\
&=\ -\sqrt{-1}\,(-1)^{\frac{\delta-1}{2}}\,e^{\frac{\pi\sqrt{-1}}{8}\overline{\delta}\,pd}\ \Big((-1)^{\frac{p-1}{2}}+\sqrt{-1}^{\,\delta d}\Big)\\
&\qquad\times\sum_{1\le c<\frac{p}{4}}(-1)^c\,e^{\frac{2\pi\sqrt{-1}}{p}\overline{\delta}\,c^2d}\Big(e^{\frac{4\pi\sqrt{-1}}{p_1}\overline{\delta}\,c}-e^{-\frac{4\pi\sqrt{-1}}{p_1}\overline{\delta}\,c}\Big)\Big(e^{\frac{4\pi\sqrt{-1}}{p_2}\overline{\delta}\,c}-e^{-\frac{4\pi\sqrt{-1}}{p_2}\overline{\delta}\,c}\Big).
\end{aligned}$$

Since $\delta\equiv -2p_2$ modulo p_1, $\delta\equiv -2p_1$ modulo p_2 and $\delta\equiv p$ modulo 4, the above formula is equal to

$$\begin{aligned}
&-\sqrt{-1}\,(-1)^{\frac{p-1}{2}}\,e^{\frac{\pi\sqrt{-1}}{8}\overline{\delta}\,pd}\ \Big((-1)^{\frac{p-1}{2}}+\sqrt{-1}^{\,\delta d}\Big)\\
&\qquad\times\sum_{1\le c<\frac{p}{4}}(-1)^c\,e^{\frac{2\pi\sqrt{-1}}{p}\overline{\delta}\,c^2d}\Big(e^{\frac{2\pi\sqrt{-1}}{p_1}\overline{p_2}\,c}-e^{-\frac{2\pi\sqrt{-1}}{p_1}\overline{p_2}\,c}\Big)\Big(e^{\frac{2\pi\sqrt{-1}}{p_2}\overline{p_1}\,c}-e^{-\frac{2\pi\sqrt{-1}}{p_2}\overline{p_1}\,c}\Big)\\
&=\ 4\sqrt{-1}\,(-1)^{\frac{p-1}{2}}\,e^{\frac{\pi\sqrt{-1}}{8}\overline{\delta}\,pd}\ \Big((-1)^{\frac{p-1}{2}}+\sqrt{-1}^{\,\delta d}\Big)\\
&\qquad\times\sum_{1\le c<\frac{p}{4}}(-1)^c\,e^{\frac{2\pi\sqrt{-1}}{p}\overline{\delta}\,c^2d}\,\sin\frac{2\pi\,\overline{p_2}\,c}{p_1}\,\sin\frac{2\pi\,\overline{p_1}\,c}{p_2}\,.
\end{aligned}$$

Hence, since $a-a^{-1}\ \sim\ \frac{4\pi\sqrt{-1}}{N}$, we have the following approximation up to $O\big(N\big)$ by Proposition 3.3,

$$\begin{aligned}
\hat{\tau}_N(M_{p_1,p_2})\ \sim\ &\frac{(-1)^{\frac{d^2-1}{8}}\,N^{1/2}}{8\pi\sqrt{-1}\sqrt{p}}\,e^{\frac{3\pi\sqrt{-1}}{4}}\,e^{\frac{\pi\sqrt{-1}}{4}(1-p_1-p_2)d}\\
&\times N\cdot 4\sqrt{-1}\,(-1)^{\frac{p-1}{2}}\,e^{\frac{\pi\sqrt{-1}}{8}\overline{\delta}\,pd}\ \Big((-1)^{\frac{p-1}{2}}+\sqrt{-1}^{\,\delta d}\Big)\\
&\times\sum_{1\le c<\frac{p}{4}}(-1)^c\,e^{\frac{2\pi\sqrt{-1}}{p}\overline{\delta}\,c^2d}\,\sin\frac{2\pi\,\overline{p_2}\,c}{p_1}\,\sin\frac{2\pi\,\overline{p_1}\,c}{p_2}\,.
\end{aligned}$$

Therefore, in order to obtain the proposition, it is sufficient to show that

$$\begin{aligned}
&\frac{(-1)^{\frac{d^2-1}{8}}\,N^{3/2}}{8\pi\sqrt{-1}\sqrt{p}}\,e^{\frac{3\pi\sqrt{-1}}{4}}\,e^{\frac{\pi\sqrt{-1}}{4}(1-p_1-p_2)d}\cdot 4\sqrt{-1}\,(-1)^{\frac{p-1}{2}}\,e^{\frac{\pi\sqrt{-1}}{8}\overline{\delta}\,pd}\ \Big((-1)^{\frac{p-1}{2}}+\sqrt{-1}^{\,\delta d}\Big)\\
&=\frac{-e^{\frac{\pi\sqrt{-1}}{4}}\,N^{3/2}}{\pi\sqrt{2p}}\,(-1)^{\frac{d-1}{2}}\,e^{\frac{\pi\sqrt{-1}}{8}d}.
\end{aligned}$$

This formula is rewritten as

$$(-1)^{\frac{d^2-1}{8}}\, e^{\frac{\pi\sqrt{-1}}{2}}\, e^{\frac{\pi\sqrt{-1}}{4}(1-p_1-p_2)d}\, (-1)^{\frac{p-1}{2}}\, e^{\frac{\pi\sqrt{-1}}{8}\,\overline{\delta}\, pd}\, \frac{(-1)^{\frac{p-1}{2}}+\sqrt{-1}^{\,\delta d}}{\sqrt{2}} \;=\; -(-1)^{\frac{d-1}{2}}\, e^{\frac{\pi\sqrt{-1}}{8}d}. \tag{25}$$

Hence, it is sufficient to show this formula (25).

We show (25) in the following of this proof. We have that

$$\sqrt{-1}^{\,\delta d} \;=\; \sqrt{-1}^{\,pd} \;=\; (-1)^{\frac{pd-1}{2}}\sqrt{-1} \;=\; (-1)^{\frac{p-d}{2}}\sqrt{-1},$$

where we obtain the first equality since $\delta \equiv p$ modulo 4, and obtain the last equality since $\frac{pd-1}{2} - \frac{p-d}{2} = \frac{(p-1)(d+1)}{2}$, which is even. Further, we can obtain the following equality by verifying it for each $d \equiv \pm 1, \pm 3$ modulo 8,

$$\frac{1+(-1)^{\frac{d-1}{2}}\sqrt{-1}}{\sqrt{2}} \;=\; (-1)^{\frac{d^2-1}{8}}\, e^{\frac{\pi\sqrt{-1}}{4}d}.$$

Hence, by using the above two formulas, we have that

$$\begin{aligned}\frac{(-1)^{\frac{p-1}{2}}+\sqrt{-1}^{\,\delta d}}{\sqrt{2}} \;&=\; \frac{(-1)^{\frac{p-1}{2}}+(-1)^{\frac{p-d}{2}}\sqrt{-1}}{\sqrt{2}} \;=\; (-1)^{\frac{p-1}{2}}\,\frac{1+(-1)^{\frac{d-1}{2}}\sqrt{-1}}{\sqrt{2}}\\ &=\; (-1)^{\frac{p-1}{2}}\,(-1)^{\frac{d^2-1}{8}}\, e^{\frac{\pi\sqrt{-1}}{4}d}.\end{aligned}$$

Therefore, the left-hand side of (25) is equal to

$$e^{\frac{\pi\sqrt{-1}}{2}}\, e^{\frac{\pi\sqrt{-1}}{2}d}\, e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)d}\, e^{\frac{\pi\sqrt{-1}}{8}\,\overline{\delta}\, pd}.$$

Further, since $e^{\frac{\pi\sqrt{-1}}{2}d} = (-1)^{\frac{d-1}{2}} e^{\frac{\pi\sqrt{-1}}{2}}$, the left-hand side of (25) is equal to

$$-(-1)^{\frac{d-1}{2}}\, e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)d}\, e^{\frac{\pi\sqrt{-1}}{8}\,\overline{\delta}\, pd}.$$

We calculate the last factor $e^{\frac{\pi\sqrt{-1}}{8}\,\overline{\delta}\, pd}$, as follows. We have that $\delta \equiv \overline{\delta}$ modulo 8, since $\delta^2 \equiv 1$ modulo 8, noting that the square of an odd integer is equivalent to 1 modulo 8. Hence, since $(-1)^{\frac{\delta-\overline{\delta}}{8}} = \big((-1)^{\delta}\big)^{\frac{\delta-\overline{\delta}}{8}} = (-1)^{\frac{\delta^2-1}{8}}$, we have that

$$\begin{aligned}e^{\frac{\pi\sqrt{-1}}{8}\,\overline{\delta}\, pd} \;&=\; (-1)^{\frac{\delta^2-1}{8}}\cdot e^{\frac{\pi\sqrt{-1}}{8}\,\delta\, pd} \;=\; (-1)^{\frac{p^2-1}{8}}\,(-1)^{\frac{p_1+p_2}{2}}\cdot e^{\frac{\pi\sqrt{-1}}{8}p^2 d}\, e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)pd}\\ &=\; (-1)^{\frac{p_1+p_2}{2}}\, e^{\frac{\pi\sqrt{-1}}{8}d}\, e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)d} \;=\; e^{\frac{\pi\sqrt{-1}}{8}d}\, e^{\frac{\pi\sqrt{-1}}{4}(p_1+p_2)d},\end{aligned}$$

where we obtain the second equality since $\delta^2 \equiv p^2 - 4p(p_1+p_2)$ modulo 16, and obtain the third equality since $e^{\frac{\pi\sqrt{-1}}{8}p^2 d} = (-1)^{\frac{p^2-1}{8}} e^{\frac{\pi\sqrt{-1}}{8}d}$ and $e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)pd} = e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)d}$, noting that the square of an odd integer is equivalent to 1 modulo 8. Therefore, the left-hand side of (25) is equal to

$$-(-1)^{\frac{d-1}{2}}\, e^{\frac{\pi\sqrt{-1}}{8}d},$$

which is equal to the right-hand side of (25). Hence, we obtain (25), as required. □

As a particular case of the above proposition, we obtain the following proposition.

Proposition 3.12. *Let p be an odd integer > 6. We put $\delta = p-6$, and we denote by $\overline{\delta}$ the inverse of δ in $\mathbb{Z}/3p\mathbb{Z}$. Then, the semi-classical limit of the quantum invariant of $M_{3,p}$ as $N \to \infty$ (where N is odd) is given as follows.*
(1) *When $p \equiv 1$ modulo 3,*

$$\hat{\tau}_N(M_{3,p}) \sim \frac{e^{\frac{\pi\sqrt{-1}}{4}}}{2\pi\sqrt{2p}} N^{3/2} (-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{8}N} \sum_{\substack{1 \le c_1 < \frac{3}{4}p \\ 3 \not| c_1}} (\pm 1)\, e^{\frac{2\pi\sqrt{-1}}{3p} c_1^2 \overline{\delta} N} \sin \frac{2\pi\, \overline{3}\, c_1}{p},$$

where we put $\overline{3} = \frac{1-p}{3}$, and the sign in the sum is given by

$$\begin{cases} +1 & \text{if } c_1 \equiv 1,2 \text{ modulo } 6, \\ -1 & \text{if } c_1 \equiv 4,5 \text{ modulo } 6. \end{cases}$$

(2) *When $p \equiv -1$ modulo 3,*

$$\hat{\tau}_N(M_{3,p}) \sim \frac{-e^{\frac{\pi\sqrt{-1}}{4}}}{2\pi\sqrt{2p}} N^{3/2} (-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{8}N} \sum_{\substack{1 \le c_1 < \frac{3}{4}p \\ 3 \not| c_1}} (\pm 1)\, e^{\frac{2\pi\sqrt{-1}}{3p} c_1^2 \overline{\delta} N} \sin \frac{2\pi\, \overline{3}\, c_1}{p},$$

where we put $\overline{3} = \frac{p+1}{3}$, and the sign in the sum is given as mentioned above.

We show some examples of the proposition below.

Example 3.13. The semi-classical limit of the quantum invariant of $M_{3,7}$ is given by

$$\hat{\tau}_N(M_{3,7}) \sim \frac{e^{\frac{\pi\sqrt{-1}}{4}}}{2\pi\sqrt{14}} N^{3/2} (-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{8}N} \times \Big(- e^{\frac{2\pi\sqrt{-1}}{21}N} \sin\frac{3\pi}{7} + e^{\frac{2\pi\sqrt{-1}}{21}4N}\, 2\sin\frac{\pi}{7} + e^{\frac{2\pi\sqrt{-1}}{21}16N} \sin\frac{2\pi}{7}\Big).$$

Example 3.14. The semi-classical limit of the quantum invariant of $M_{3,11}$ is given by

$$\hat{\tau}_N(M_{3,11}) \sim \frac{-e^{\frac{\pi\sqrt{-1}}{4}}}{2\pi\sqrt{22}} N^{3/2} (-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{8}N} \Big(e^{\frac{2\pi\sqrt{-1}}{33}20N} \sin\frac{3\pi}{11} - e^{\frac{2\pi\sqrt{-1}}{33}14N} \sin\frac{5\pi}{11} - e^{\frac{2\pi\sqrt{-1}}{33}23N}\, 2\sin\frac{\pi}{11} + e^{\frac{2\pi\sqrt{-1}}{33}5N} \sin\frac{4\pi}{11} - e^{\frac{2\pi\sqrt{-1}}{33}26N} \sin\frac{2\pi}{11}\Big).$$

Example 3.15. The semi-classical limit of the quantum invariant of $M_{3,5}$ is given by

$$\hat{\tau}_N(M_{3,5}) \sim \frac{e^{\frac{\pi\sqrt{-1}}{4}}}{2\pi\sqrt{10}} N^{3/2} e^{\frac{\pi\sqrt{-1}}{8}N} \Big(e^{-\frac{2\pi\sqrt{-1}}{15}N} \sin\frac{\pi}{5} - e^{-\frac{2\pi\sqrt{-1}}{15}4N} \sin\frac{2\pi}{5}\Big).$$

We note that, as mentioned in Example 3.7, we obtain the above formula as $\frac{c_+}{c_-}$ times the formula of Proposition 3.12, where we note that the asymptotic behavior of $\frac{c_+}{c_-}$ is given by $\frac{c_+}{c_-} \sim (-1)^{\frac{N+1}{2}}$.

3.4 $\mathrm{SL}_2\mathbb{C}$ representations of the fundamental group of M_{p_1,p_2}

In this section, we give a classification of conjugacy classes of irreducible representations of $\pi_1(M_{p_1,p_2})$ to $\mathrm{SL}_2\mathbb{C}$. As we show in Proposition 3.16, any irreducible representation is conjugate to a $\mathrm{SL}_2\mathbb{R}$ representation or a $\mathrm{SU}(2)$ representation.

We recall that p_1 and p_2 are coprime odd integers ≥ 3. The fundamental group of M_{p_1,p_2} is presented by

$$\pi_1(M_{p_1,p_2}) \;=\; \big\langle\, x,y,z,h \;\big|\; h \text{ is central},\;\; x^{p_1}h=1,\;\; y^{p_2}h=1,\;\; z^2h=1,\;\; xyzh=1 \,\big\rangle.$$

We consider an irreducible representation $\rho : \pi_1(M_{p_1,p_2}) \to \mathrm{SL}_2\mathbb{C}$. Since h is central, $\rho(h) = \pm E$, where E denotes the unit matrix. If $\rho(h)$ was equal to E, then $\rho(z) = \pm E$ and ρ would be reducible. Hence, $\rho(h) = -E$. Therefore, the eigenvalues of $\rho(x)$ are equal to $e^{\pm\pi\sqrt{-1}\,\frac{k_1}{p_1}}$ for some odd integer k_1 with $0 < k_1 < p_1$, and the eigenvalues of $\rho(y)$ are equal to $e^{\pm\pi\sqrt{-1}\,\frac{k_2}{p_2}}$ for some odd integer k_2 with $0 < k_2 < p_2$. Therefore, an irreducible representation ρ determines (k_1,k_2). We show that this correspondence is bijective in the following proposition.

Proposition 3.16. *There is a bijective correspondence between the set of conjugacy classes of irreducible representations of* $\pi_1(M_{p_1,p_2})$ *to* $\mathrm{SL}_2\mathbb{C}$ *and the following set,*

$$\big\{(k_1,k_2) \;\big|\; k_1 \text{ and } k_2 \text{ are odd integers with } 0<k_1<p_1 \text{ and } 0<k_2<p_2\big\}. \tag{26}$$

We denote by ρ_{k_1,k_2} *an irreducible representation of the conjugacy class corresponding to* (k_1,k_2) *by this correspondence.*
(1) ρ_{k_1,k_2} *is conjugate to a* $\mathrm{SL}_2\mathbb{R}$ *representation if and only if*

$$\frac{k_1}{p_1}+\frac{k_2}{p_2} < \frac{1}{2} \quad \text{or} \quad \frac{k_1}{p_1}+\frac{k_2}{p_2} > \frac{3}{2} \quad \text{or} \quad \Big|\frac{k_1}{p_1}-\frac{k_2}{p_2}\Big| > \frac{1}{2}\,. \tag{27}$$

(2) ρ_{k_1,k_2} *is conjugate to a* $\mathrm{SU}(2)$ *representation if and only if*

$$\Big|\frac{k_1}{p_1}-\frac{k_2}{p_2}\Big| \;<\; \frac{1}{2} \;<\; \frac{k_1}{p_1}+\frac{k_2}{p_2} \;<\; \frac{3}{2}\,. \tag{28}$$

We give a proof of the proposition at the end of this section. In order to give a proof of the proposition, we show some lemmas below.

Lemma 3.17. *Let* k_1 *and* k_2 *be odd integers with* $0<k_1<p_1$ *and* $0<k_2<p_2$. *If* (27) *holds, then there exists a representation* $\rho : \pi_1(M_{p_1,p_2}) \to \mathrm{SL}_2\mathbb{R}$ *such that the eigenvalues of* $\rho(x)$ *is equal to* $e^{\pm\pi\sqrt{-1}\,\frac{k_1}{p_1}}$ *and the eigenvalues of* $\rho(y)$ *is equal to* $e^{\pm\pi\sqrt{-1}\,\frac{k_2}{p_2}}$.

Proof. If $\frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}$, we consider a hyperbolic triangle with angles $\frac{k_1}{p_1}\pi$, $\frac{k_2}{p_2}\pi$, $\frac{1}{2}\pi$ in the hyperbolic plane $\mathbb{H}^2$, on which $\mathrm{SL}_2\mathbb{R}$ acts. We can construct a representation ρ in such a way that $\rho(x)$ is a $\frac{2k_1}{p_1}\pi$ rotation around the vertex of the angle $\frac{k_1}{p_1}\pi$ and $\rho(y)$ is a $\frac{2k_2}{p_2}\pi$ rotation around the vertex of the angle $\frac{k_2}{p_2}\pi$ and $\rho(z)$ is a π rotation around the vertex of the angle $\frac{1}{2}\pi$. Hence, we obtain the lemma in this case.

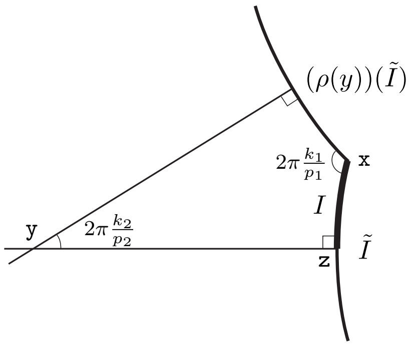
If $\frac{k_1}{p_1} + \frac{k_2}{p_2} > \frac{3}{2}$, we consider a hyperbolic triangle with angles $\frac{p_1-k_1}{p_1}\pi$, $\frac{p_2-k_2}{p_2}\pi$, $\frac{1}{2}\pi$. Then, in the same as above, we can obtain the lemma in this case.

If $\frac{k_1}{p_1} - \frac{k_2}{p_2} > \frac{1}{2}$, we consider a hyperbolic triangle with angles $\frac{p_1-k_1}{p_1}\pi$, $\frac{k_2}{p_2}\pi$, $\frac{1}{2}\pi$. Then, in a similar was as above, we can obtain the lemma in this case.

If $\frac{k_2}{p_2} - \frac{k_1}{p_1} > \frac{1}{2}$, we can obtain the lemma in the same way as above. □

Lemma 3.18. *Under the assumption of Lemma 3.17, for a given* (k_1, k_2)*, the existence of such a representation is unique up to conjugation of* $\mathrm{SL}_2\mathbb{C}$*.*

Proof. We consider the case where $\frac{k_1}{p_1} + \frac{k_2}{p_2} < \frac{1}{2}$. Since $\rho(x)$ is a $\frac{2k_1}{p_1}\pi$ rotation of $\mathbb{H}^2$, $\rho(x)$ has a unique fixed point $\mathtt{x}$ in $\mathbb{H}^2$. Similarly, $\rho(y)$ has a unique fixed point $\mathtt{y}$ in $\mathbb{H}^2$. Further, since $\rho(z)$ is an orientation-preserving π rotation of $\mathbb{H}^2$, $\rho(z)$ has a unique fixed point $\mathtt{z}$ in $\mathbb{H}^2$. We consider the geodesic interval I between $\mathtt{x}$ and $\mathtt{z}$. Since $\rho(z)$ is a π rotation, the union $\tilde{I}$ of I and $(\rho(z))(I)$ is a geodesic interval. We consider a geodesic perpendicular to $\tilde{I}$ at $\mathtt{z}$. Further, we consider $(\rho(y))(\tilde{I})$, which is equal to $(\rho(x^{-1}z^{-1}))(\tilde{I}) = (\rho(x^{-1}))(\tilde{I})$.



Hence, as shown above, $\tilde{I}$ and $(\rho(y))(\tilde{I})$ makes an angle $2\pi\frac{k_1}{p_1}$ at $\mathtt{x}$. Therefore, the hyperbolic triangle whose vertices are $\mathtt{x}$, $\mathtt{y}$, $\mathtt{z}$ is a triangle with angles $\frac{k_1}{p_1}\pi$, $\frac{k_2}{p_2}\pi$, $\frac{1}{2}\pi$. We note that such a triangle in $\mathbb{H}^3$ is unique up to the action of $\mathrm{SL}_2\mathbb{C}$. Hence, ρ is conjugate (by conjugation of $\mathrm{SL}_2\mathbb{C}$) to the representation obtained in Lemma 3.17, and we obtain the lemma in this case.

In the other cases, we obtain the lemma in the same way as above. □

Lemma 3.19. *Let* k_1 *and* k_2 *be odd integers with* $0 < k_1 < p_1$ *and* $0 < k_2 < p_2$*. If* (28) *holds, then there exists a representation* $\rho : \pi_1(M_{p_1,p_2}) \to \mathrm{SU}(2)$ *such that the eigenvalues of* $\rho(x)$ *is equal to* $e^{\pm\pi\sqrt{-1}\,\frac{k_1}{p_1}}$ *and the eigenvalues of* $\rho(y)$ *is equal to* $e^{\pm\pi\sqrt{-1}\,\frac{k_2}{p_2}}$*.*

Proof. We consider a spherical triangle with angles θ_1, θ_2, θ_3 in the 2-sphere S^2, and consider the condition of θ_1, θ_2, θ_3 so that such a triangle exists. We consider three great circles on S^2, and suppose that one of the regions bounded by such circles is a triangle with angles θ_1, θ_2, θ_3. We note that there are four pairs of triangles bounded by such circles, *i.e.*, triangles whose angles are $(\theta_1, \theta_2, \theta_3)$, $(\theta_1, \pi-\theta_2, \pi-\theta_3)$, $(\pi-\theta_1, \theta_2, \pi-\theta_3)$, $(\pi-\theta_1, \pi-\theta_2, \theta_3)$. Then, since these triangles are spherical,

$$\theta_1 + \theta_2 + \theta_3 > \pi,\ 2\pi + \theta_1 - \theta_2 - \theta_3 > \pi,\ 2\pi - \theta_1 + \theta_2 - \theta_3 > \pi,\ 2\pi - \theta_1 - \theta_2 + \theta_3 > \pi.$$

When $\theta_1 = \frac{k_1}{p_1}\pi$, $\theta_2 = \frac{k_2}{p_2}\pi$, $\theta_3 = \frac{1}{2}\pi$, these conditions are equivalent to (28).

Conversely, we can verify that, when (28) holds, there exists a spherical triangle with angles $\frac{k_1}{p_1}\pi$, $\frac{k_2}{p_2}\pi$, $\frac{1}{2}\pi$. Hence, in a similar way as in the proof of Lemma 3.17, we can construct a representation $\rho : \pi_1(M_{p_1,p_2}) \to \mathrm{SU}(2)$, as required. □

Lemma 3.20. *Under the assumption of Lemma 3.19, for a given* (k_1, k_2)*, the existence of such a representation is unique up to conjugation of* $\mathrm{SL}_2\mathbb{C}$*.*

Proof. Since $\rho(x)$ is a $\frac{2k_1}{p_1}\pi$ rotation of S^2, $\rho(x)$ has two fixed points in S^2. We denote them by $\mathtt{x}$, $\mathtt{x}'$ in such a way that $\rho(x)$ is a couter-clockwise $\frac{2k_1}{p_1}\pi$ rotation around $\mathtt{x}$ and a clockwise $\frac{2k_1}{p_1}\pi$ rotation around $\mathtt{x}'$. We denote by $\mathtt{y}$, $\mathtt{y}'$ two fixed points of $\rho(y)$ in the same way. Further, we denote by $\mathtt{z}$, $\mathtt{z}'$ two fixed points of $\rho(z)$ in such a way that the cyclic order of $\mathtt{x}$, $\mathtt{y}$, $\mathtt{z}$ is counter-clockwise around the spherical triangle whose vertices are $\mathtt{x}$, $\mathtt{y}$, $\mathtt{z}$. Then, in a similar way as in the proof of Lemma 3.18, this triangle is a triangle with angles $\frac{k_1}{p_1}\pi$, $\frac{k_2}{p_2}\pi$, $\frac{1}{2}\pi$. We note that such a triangle (such that the cyclic order of $\mathtt{x}$, $\mathtt{y}$, $\mathtt{z}$ is counter-clockwise) is unique up to the action of $\mathrm{SU}(2)$. Hence, ρ is conjugate to the representation obtained in Lemma 3.19, and we obtain the lemma. □

By using the above lemmas, we now give a proof of Proposition 3.16.

Proof of Proposition 3.16. By Lemmas 3.17 and 3.18, there is a bijective correspondence between the set of conjugacy classes of irreducible representations of $\pi_1(M_{p_1,p_2})$ to $\mathrm{SL}_2\mathbb{R}$ and the following set

$$\big\{(k_1, k_2) \in (26) \ \big|\ (k_1, k_2) \text{ satisfies } (27)\big\},$$

where we consider conjugacy classes with respect to conjugation of $\mathrm{SL}_2\mathbb{C}$. Further, by Lemmas 3.19 and 3.20, there is a bijective correspondence between the set of conjugacy classes of irreducible representations of $\pi_1(M_{p_1,p_2})$ to $\mathrm{SU}(2)$ and the following set

$$\big\{(k_1, k_2) \in (26) \ \big|\ (k_1, k_2) \text{ satisfies } (28)\big\}.$$

We note that the set (26) is the disjoint union of the above two sets. Hence, it is sufficient to show that any irreducible representation of $\pi_1(M_{p_1,p_2})$ to $\mathrm{SL}_2\mathbb{C}$ is conjugate to a $\mathrm{SL}_2\mathbb{R}$ representation or a $\mathrm{SU}(2)$ representation.

We show that any irreducible representation $\rho : \pi_1(M_{p_1,p_2}) \to \mathrm{SL}_2\mathbb{C}$ is conjugate to a $\mathrm{SL}_2\mathbb{R}$ representation or a $\mathrm{SU}(2)$ representation, in the following of this proof. As mentioned at the beginning of this section, the eigenvalues of $\rho(x)$ are equal to $e^{\pm\pi\sqrt{-1}\,\frac{k_1}{p_1}}$ and the eigenvalues of $\rho(y)$ are equal to $e^{\pm\pi\sqrt{-1}\,\frac{k_2}{p_2}}$ and the eigenvalues of $\rho(z)$ are equal to $\pm\sqrt{-1}$. There is a unique geodesic $L_{\rho(x)}$ in $\mathbb{H}^3$ fixed by the action of $\rho(x)$. Similarly, there is a unique geodesic $L_{\rho(z)}$ in $\mathbb{H}^3$ fixed by the action of $\rho(z)$.

If $L_{\rho(x)}$ and $L_{\rho(z)}$ intersect, $\rho(x)$ and $\rho(z)$ fix the crossing point, and the representation ρ fixes this crossing point. Hence, ρ is conjugate to a $\mathrm{SU}(2)$ representation.

If $L_{\rho(x)}$ and $L_{\rho(z)}$ do not intersect, there exists a unique geodesic interval I which is perpendicular to $L_{\rho(x)}$ at $\mathtt{x}$ and perpendicular to $L_{\rho(z)}$ at $\mathtt{z}$. Further, in the same way as in the proof of Lemma 3.18, we can show that the union $\tilde{I}$ of I and $(\rho(z))(I)$ is a geodesic interval. We consider the union of $(\rho(y^n))(\tilde{I})$ for $n = 0, 1, 2, \cdots$. Similarly as in the proof of Lemma 3.18, we can see that it is a piecewise geodesic path. Since $\rho(y)$ is of finite order, this path must be a closed path, and this closed path is included in a totally geodesic plane in $\mathbb{H}^3$. The representation ρ preserves this totally geodesic plane in such a way that the action of ρ on this plane is orientation-preserving. Hence, ρ is conjugate to a $\mathrm{SL}_2\mathbb{R}$ representation, as required. □

3.5 The Chern–Simons invariant and the Reidemeister torsion of M_{p_1,p_2}

In this section, we give formulas for the Chern–Simons invariant of $(M_{p_1,p_2}, \mathrm{ad}\circ\rho_{k_1,k_2})$ and the twisted Reidemeister torsion of $(M_{p_1,p_2}, \mathrm{ad}\circ\rho_{k_1,k_2})$. The formula for the Chern–Simons invariant (Proposition 3.21) is a natural extension of a formula of [5]. The formula for the twisted Reidemeister torsion (Proposition 3.22) is essentially due to [9]. We also give a formula of the spectral flow of ρ_{k_1,k_2} in Remark 3.23.

Proposition 3.21. *Let ρ_{k_1,k_2} be the irreducible representation of $\pi_1(M_{p_1,p_2})$ to $\mathrm{SL}_2\mathbb{C}$ given in Proposition 3.16. We suppose that ρ_{k_1,k_2} is conjugate to a $\mathrm{SU}(2)$ representation or a $\mathrm{SL}_2\mathbb{R}$ representation with $\frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}$. Then, the Chern–Simons invariant of $(M_{p_1,p_2}, \mathrm{ad}\circ\rho_{k_1,k_2})$ is presented by*

$$\mathrm{CS}\big(M_{p_1,p_2};\, \mathrm{ad}\circ\rho_{k_1,k_2}\big) \;=\; \frac{1}{4}\Big(\frac{1}{2}-\frac{k_1^2}{p_1}-\frac{k_2^2}{p_2}\Big) \qquad \text{(modulo 1)}.$$

Proof. When ρ_{k_1,k_2} is a $\mathrm{SU}(2)$ representation, the Chern–Simons invariant of a Seifert 3-manifold is calculated in [5], which gives the formula of the proposition, noting that our formula is (-1) times the formula of [5] because of the difference of the convention of an orientation of a Seifert 3-manifold (see [10]). See also [42] for the formula of the Chern–Simons invariant of a Seifert 3-manifold, which exactly gives the formula of the proposition, noting that our M_{p_1,p_2} is $X(-2,p_1,p_2)$ of the notation of [42] as we show in Remark C.1 below.

When ρ_{k_1,k_2} be a $\mathrm{SL}_2\mathbb{C}$ representation in general, we can calculate the Chern–Simons invariant in the same way as in [5], as mentioned in [20]. In fact, we can calculate the Chern–Simons invariant by using a path of $\mathrm{SL}_2\mathbb{C}$ representations ρ_t (for $0\le t\le 1$) of $\pi_1(N)$, where N is the 3-manifold obtained from M_{p_1,p_2} by removing tubular neighborhoods of the three singular fibers. As shown in [5], the Chern–Simons invariant is calculated by using integrals of the following form,

$$\mathrm{CS}\big(M_{p_1,p_2};\, \mathrm{ad}\circ\rho_{k_1,k_2}\big)-\mathrm{CS}\big(M_{p_1,p_2};\, \mathrm{ad}\circ\rho_{k_1',k_2'}\big) \;=\; 2\sum_{i=1}^{3}\int_0^1 a_i'(t)\, b_i(t)\, dt,$$

where we fix $\rho_t(h)=-E$, and $\exp(\pm 2\pi\sqrt{-1}\,a_1(t))$, $\exp(\pm 2\pi\sqrt{-1}\,a_2(t))$, $\exp(\pm 2\pi\sqrt{-1}\,a_3(t))$ are eigenvalues of $\rho_t(x^{p_1}h)$, $\rho_t(y^{p_2}h)$, $\rho_t(z^2h)$, and $\exp(\pm 2\pi\sqrt{-1}\,b_1(t))$, $\exp(\pm 2\pi\sqrt{-1}\,b_2(t))$, $\exp(\pm 2\pi\sqrt{-1}\,b_3(t))$ are eigenvalues of $\rho_t(x^{-1})$, $\rho_t(y^{-1})$, $\rho_t(z^{-1})$. Hence, $a_i'(t)$ is a constant multiple of $b_i'(t)$, and the above integral is a constant multiple of $(b_i(t))^2$. Here, we note that, in the same way as in the proofs of Lemmas 3.17 and 3.19, we can construct a path between any pair of representations ρ_{k_1,k_2} and $\rho_{k_1',k_2'}$, by considering a triangle with angles $\theta_1, \theta_2, \frac{1}{2}\pi$, where θ_1 changes from $\frac{k_1'}{p_1}$ to $\frac{k_1}{p_1}$ and θ_2 changes from $\frac{k_2'}{p_2}$ to $\frac{k_2}{p_2}$. We further note that such a triangle is spherical if $\theta_1+\theta_2 > \frac{1}{2}\pi$, hyperbolic if $\theta_1+\theta_2 < \frac{1}{2}\pi$, and Euclidean if $\theta_1+\theta_2 = \frac{1}{2}\pi$, and we can construct a path between a pair of such triangles by changing θ_1 and θ_2 smoothly. Since the above integral is a constant multiple of $(b_i(t))^2$, the value of the above sum depends only on (k_1, k_2) and (k_1', k_2'). Therefore, we can extend the required formula to the case where ρ_{k_1,k_2} is conjugate to a $\mathrm{SL}_2\mathbb{R}$ representation, and we obtain the required formula in this general case. □

Proposition 3.22 (see [9]). *Let ρ_{k_1,k_2} be the irreducible representation of $\pi_1(M_{p_1,p_2})$ to $\mathrm{SL}_2\mathbb{C}$ given in Proposition 3.16. Then, the Reidemeister torsion of M_{p_1,p_2} with $\mathfrak{sl}_2$ coefficient twisted by the adjoint action of ρ_{k_1,k_2} is presented by*

$$\mathrm{Tor}\big(M_{p_1,p_2};\, \mathrm{ad}\circ\rho_{k_1,k_2}\big) \;=\; \pm\frac{32}{p_1p_2}\sin^2\frac{k_1\pi}{p_1}\sin^2\frac{k_2\pi}{p_2}\,.$$

Proof. Freed [9] calculates the Reidemeister torsion of a Brieskorn sphere M with $\mathfrak{sl}_2$ coefficient twisted by the adjoint action of any $\mathrm{SU}(2)$ representation of $\pi_1(M)$. We note that his calculation ingores the sign of the Reidemeister torsion. We can apply his calculation to a Seifert 3-manifold with three singular fibers. Further, we can naturally extend his calculation to the case of a $\mathrm{SL}_2\mathbb{C}$ representation. Hence, we can obtain the required formula from his formula. □

Remark 3.23. Let ρ_{k_1,k_2} be the irreducible representation of $\pi_1(M_{p_1,p_2})$ to $\mathrm{SL}_2\mathbb{C}$ given in Proposition 3.16. Then, the spectral flow of ρ_{k_1,k_2} is presented by

$$I(\rho_{k_1,k_2}) \;=\; -2\,(k_1+k_2).$$

Proof. By a formula of the spectral flow given in [12] (see also [42]), we have that

$$\begin{aligned} I(\rho_{k_1,k_2}) &= -3+8\,\mathrm{CS}(M_{p_1,p_2},\,\rho_{k_1,k_2}) - \sum_{i=1}^{2}\frac{2}{p_i}\sum_{n_i=1}^{p_i}\cot^2\frac{\pi n_i}{p_i}\sin^2\frac{\pi k_i n_i}{p_i} \\ &= -3+2\Big(\frac{1}{2}-\frac{k_1^2}{p_1}-\frac{k_2^2}{p_2}\Big) - \sum_{i=1}^{2}\frac{2}{p_i}\Big(-k_i^2+k_i\,p_i-\frac{p_i}{2}\Big) \\ &= -2\,(k_1+k_2), \end{aligned}$$

where we obtain the second equality by Proposition 3.21, and obtain the third equality by Lemma 3.24 below. Hence, we obtain the required formula. □

Lemma 3.24. *Let k and p be odd integers with $0<k<p$. Then,*

$$\sum_{n=1}^{p}\cot^2\frac{\pi n}{p}\sin^2\frac{\pi k n}{p} \;=\; -k^2+pk-\frac{p}{2}\,.$$

Proof. The lemma is shown in a similar way as the proof of [15, Proposition 5.2]. We put $\zeta = e^{\frac{2\pi\sqrt{-1}}{p}}$. Then, we have that

$$\cot\frac{\pi n}{p} \;=\; \frac{\cos\frac{\pi n}{p}}{\sin\frac{\pi n}{p}} \;=\; \frac{1}{\sqrt{-1}}\cdot\frac{\zeta^{\frac{n}{2}}+\zeta^{-\frac{n}{2}}}{\zeta^{\frac{n}{2}}-\zeta^{-\frac{n}{2}}}\,, \qquad \sin\frac{\pi k n}{p} \;=\; \frac{\zeta^{\frac{kn}{2}}-\zeta^{-\frac{kn}{2}}}{2\sqrt{-1}}\,,$$

and

$$\frac{\zeta^{\frac{kn}{2}} - \zeta^{-\frac{kn}{2}}}{\zeta^{\frac{n}{2}} - \zeta^{-\frac{n}{2}}} \;=\; \zeta^{\frac{k-1}{2}n} + \zeta^{\frac{k-2}{2}n} + \cdots + \zeta^{-\frac{k-1}{2}n}.$$

Hence, the left-hand side of the required formula is calculated as

$$\begin{aligned}
&\frac{1}{4}\sum_{n=1}^{p-1} \big(\zeta^{\frac{n}{2}} + \zeta^{-\frac{n}{2}}\big)^2 \big(\zeta^{\frac{k-1}{2}n} + \zeta^{\frac{k-3}{2}n} + \cdots + \zeta^{-\frac{k-1}{2}n}\big)^2 \\
&= -k^2 + \frac{1}{4}\sum_{n\,\in\,\mathbb{Z}/p\mathbb{Z}} \big(\zeta^n + 2 + \zeta^{-n}\big)\Big(\zeta^{(k-1)n} + 2\zeta^{(k-2)n} + 3\zeta^{(k-3)n} + \cdots \\
&\qquad\qquad + (k-1)\zeta^n + k + (k-1)\zeta^{-n} + \cdots + 2\zeta^{-(k-2)n} + \zeta^{-(k-1)n}\Big) \\
&= -k^2 + \frac{p}{4}\big((k-1) + 2k + (k-1)\big) \;=\; -k^2 + pk - \frac{p}{2}\,,
\end{aligned}$$

where we obtain the second equality, since

$$\sum_{n\,\in\,\mathbb{Z}/p\mathbb{Z}} \zeta^{nm} \;=\; \begin{cases} p & \text{if } m \text{ is divisible by } p, \\ 0 & \text{otherwise,} \end{cases}$$

for any integer m. Therefore, we obtain the lemma. □

3.6 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. In order to rewrite the formula of Proposition 3.11 in terms of $\mathrm{SL}_2\mathbb{C}$ representations, we show some lemmas below, before we give a proof of Theorem 1.2.

We recall that p_1 and p_2 are coprime odd integers ≥ 3. Putting

$$S \;=\; \big\{(k_1,k_2) \;\big|\; k_1 \text{ and } k_2 \text{ are odd integers with } \; 0<k_1<p_1, \;\; 0<k_2<p_2\big\},$$

we put

$$\begin{aligned}
S_1 &= \Big\{(k_1,k_2)\in S \;\Big|\; \frac{k_1}{p_1} + \frac{k_2}{p_2} < \frac{1}{2}\Big\}, \\
S_2 &= \Big\{(k_1,k_2)\in S \;\Big|\; \frac{k_1}{p_1} + \frac{k_2}{p_2} > \frac{3}{2}\Big\}, \\
S_3 &= \Big\{(k_1,k_2)\in S \;\Big|\; 0 < \frac{k_1}{p_1} - \frac{k_2}{p_2} < \frac{1}{2}\Big\}, \\
S_4 &= \Big\{(k_1,k_2)\in S \;\Big|\; 0 < \frac{k_2}{p_2} - \frac{k_1}{p_1} < \frac{1}{2}\Big\}.
\end{aligned}$$

Further, modifying the range of S_2, we put

$$\hat{S}_2 \;=\; \Big\{(k_1,k_2) \;\Big|\; k_1,\, k_2 \text{ are odd integers}, \;\; 0<k_1\leq p_1, \;\; 0<k_2\leq p_2, \;\; \frac{3}{2} < \frac{k_1}{p_1} + \frac{k_2}{p_2} < 2\Big\}.$$

Lemma 3.25. *We have a bijection*

$$\varphi : S_1 \sqcup \hat{S}_2 \sqcup S_3 \sqcup S_4 \ \longrightarrow \ \big\{c \in \mathbb{Z} \ \big|\ 0 < c < \frac{1}{4}\, p_1 p_2\big\},$$

where $\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \varphi_3 \sqcup \varphi_4$ *and* $\varphi_1, \cdots, \varphi_4$ *are given by*

$$\begin{aligned} \varphi_1(k_1,k_2) \ &= \ \frac{p_2k_1 + p_1k_2}{2}\,, \qquad & \varphi_2(k_1,k_2) \ &= \ p_1p_2 - \frac{p_2k_1 + p_1k_2}{2}\,, \\ \varphi_3(k_1,k_2) \ &= \ \frac{p_2k_1 - p_1k_2}{2}\,, \qquad & \varphi_4(k_1,k_2) \ &= \ \frac{p_1k_2 - p_2k_1}{2}\,. \end{aligned}$$

Proof. We identify $\hat{S}_2$ with

$$\hat{S}'_2 \ = \ \big\{(k_1,k_2) \ \big|\ k_1,\, k_2 \text{ are odd integers}, \quad p_1 \le k_1, \quad -p_2 \le k_2, \quad 0 < \frac{k_1}{p_1} + \frac{k_2}{p_2} < \frac{1}{2}\big\}$$

by replacing (k_1, k_2) with $(2p_1 - k_1, -k_2)$, and identify S_3 with

$$S'_3 \ = \ \big\{(k_1,k_2) \ \big|\ k_1,\, k_2 \text{ are odd integers}, \quad k_1 < p_1, \quad k_2 < 0, \quad 0 < \frac{k_1}{p_1} + \frac{k_2}{p_2} < \frac{1}{2}\big\}$$

by replacing (k_1, k_2) with $(k_1, -k_2)$, and identify S_4 with

$$S'_4 \ = \ \big\{(k_1,k_2) \ \big|\ k_1,\, k_2 \text{ are odd integers}, \quad k_1 < 0, \quad k_2 < p_2, \quad 0 < \frac{k_1}{p_1} + \frac{k_2}{p_2} < \frac{1}{2}\big\}$$

by replacing (k_1, k_2) with $(-k_1, k_2)$. Then, by putting

$$S' \ = \ S_1 \sqcup \hat{S}'_2 \sqcup S'_3 \sqcup S'_4 \ = \ \big\{(k_1,k_2) \ \big|\ -p_2 \le k_2 < p_2, \quad 0 < \frac{k_1}{p_1} + \frac{k_2}{p_2} < \frac{1}{2}\big\},$$

the map φ is rewritten

$$\varphi' : S' \ \longrightarrow \ \big\{c \in \mathbb{Z} \ \big|\ 0 < c < \frac{1}{4}\, p_1p_2\big\},$$

where we put $\varphi'(k_1, k_2) = \frac{p_2k_1+p_1k_2}{2}$. Further, we consider a bijection

$$\varphi'' : \big\{(k_1,k_2) \ \in \ \mathbb{Z}/2p_1\mathbb{Z} \times \mathbb{Z}/2p_2\mathbb{Z} \ \big|\ k_1 \text{ and } k_2 \text{ are odd}\big\} \ \longrightarrow \ \mathbb{Z}/p_1p_2\mathbb{Z}$$

defined by $\varphi''(k_1, k_2) = \frac{p_2k_1+p_1k_2}{2}$. Then, we can see that φ' is a restriction of φ'' such that the image of the map satisfies that

$$0 \ < \ c \ = \ \frac{p_2k_1 + p_1k_2}{2} \ < \ \frac{1}{4}\, p_1p_2, \qquad \text{which means that} \ \ 0 \ < \ \frac{k_1}{p_1} + \frac{k_2}{p_2} \ < \ \frac{1}{2}\,.$$

Hence, φ' is a bijection. Therefore, φ is a bijection, as required. □

Lemma 3.26. *Let $\varphi_1, \cdots, \varphi_4$ be the maps defined in Lemma 3.25.*
(1) *If $c = \varphi_1(k_1, k_2)$, $\varphi_3(k_1, k_2)$ or $\varphi_4(k_1, k_2)$, then*

$$
\begin{aligned}
&(-1)^c \, e^{\frac{2\pi\sqrt{-1}}{p_1p_2} c^2 \bar{\delta} N} \sin \frac{2\pi \, \overline{p_2} \, c}{p_1} \sin \frac{2\pi \, \overline{p_1} \, c}{p_2} \\
&= -(-1)^{\frac{k_1+k_2}{2}} \, e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N} \, e^{-\frac{\pi\sqrt{-1}}{4}(\frac{k_1^2}{p_1}+\frac{k_2^2}{p_2})N} \sin \frac{k_1\pi}{p_1} \sin \frac{k_2\pi}{p_2} .
\end{aligned}
$$

(2) *If $c = \varphi_2(k_1, k_2)$, then*

$$
\begin{aligned}
&(-1)^c \, e^{\frac{2\pi\sqrt{-1}}{p_1p_2} c^2 \bar{\delta} N} \sin \frac{2\pi \, \overline{p_2} \, c}{p_1} \sin \frac{2\pi \, \overline{p_1} \, c}{p_2} \\
&= (-1)^{\frac{k_1+k_2}{2}} \, e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N} \, e^{-\frac{\pi\sqrt{-1}}{4}(\frac{k_1^2}{p_1}+\frac{k_2^2}{p_2})N} \sin \frac{k_1\pi}{p_1} \sin \frac{k_2\pi}{p_2} .
\end{aligned}
$$

Proof. When $c = \varphi_1(k_1, k_2) = \frac{p_2k_1+p_1k_2}{2}$, we show the lemma, as follows. Since $c = p_2 \frac{k_1+p_1}{2} + p_1 \frac{k_2-p_2}{2}$,

$$
\begin{aligned}
(-1)^c &= (-1)^{\frac{k_1+p_1}{2}} (-1)^{\frac{k_2-p_2}{2}} = -(-1)^{\frac{p_1+p_2}{2}} (-1)^{\frac{k_1+k_2}{2}}, \\
\sin \frac{2\pi \, \overline{p_2} \, c}{p_1} &= \sin \big(\frac{2\pi}{p_1} \cdot \frac{k_1+p_1}{2} \big) = -\sin \frac{k_1\pi}{p_1} , \\
\sin \frac{2\pi \, \overline{p_1} \, c}{p_2} &= \sin \big(\frac{2\pi}{p_2} \cdot \frac{k_2-p_2}{2} \big) = -\sin \frac{k_2\pi}{p_2} .
\end{aligned}
$$

Further, since $c^2 \equiv p_2^2 \frac{(k_1+p_1)^2}{4} + p_1^2 \frac{(k_2-p_2)^2}{4}$ modulo p_1p_2,

$$
e^{\frac{2\pi\sqrt{-1}}{p_1p_2} c^2 \bar{\delta} N} = e^{\frac{2\pi\sqrt{-1}}{p_1} p_2 \bar{\delta} N \frac{(k_1+p_1)^2}{4}} \, e^{\frac{2\pi\sqrt{-1}}{p_2} p_1 \bar{\delta} N \frac{(k_2-p_2)^2}{4}} .
$$

Since $\delta \equiv -2p_2$ modulo p_1, the first factor of the right-hand side is calculated as

$$
\begin{aligned}
&e^{\frac{2\pi\sqrt{-1}}{p_1} p_2 \bar{\delta} N \frac{(k_1+p_1)^2}{4}} = e^{\frac{2\pi\sqrt{-1}}{p_1} p_2 (-\bar{2}\, \overline{p_2}) N \frac{(k_1+p_1)^2}{4}} = e^{-\frac{2\pi\sqrt{-1}}{p_1} \frac{1+p_1}{2} N \frac{(k_1+p_1)^2}{4}} \\
&= (-1)^{(\frac{k_1+p_1}{2})^2 N} e^{-\frac{\pi\sqrt{-1}}{p_1} \frac{(k_1+p_1)^2}{4} N} = (-1)^{\frac{k_1+p_1}{2}} e^{-\frac{\pi\sqrt{-1}}{4} \frac{k_1^2}{p_1} N} e^{-\frac{\pi\sqrt{-1}}{2} k_1 N} e^{-\frac{\pi\sqrt{-1}}{4} p_1 N} .
\end{aligned}
$$

Similarly, the second factor can be calculated as

$$
e^{\frac{2\pi\sqrt{-1}}{p_2} p_1 \bar{\delta} N \frac{(k_2-p_2)^2}{4}} = (-1)^{\frac{k_2-p_2}{2}} e^{-\frac{\pi\sqrt{-1}}{4} \frac{k_2^2}{p_2} N} e^{\frac{\pi\sqrt{-1}}{2} k_2 N} e^{-\frac{\pi\sqrt{-1}}{4} p_2 N} .
$$

Hence, we have that

$$
\begin{aligned}
e^{\frac{2\pi\sqrt{-1}}{p_1p_2} c^2 \bar{\delta} N} &= -(-1)^{\frac{p_1+p_2}{2}} (-1)^{\frac{k_1+k_2}{2}} e^{-\frac{\pi\sqrt{-1}}{4}(\frac{k_1^2}{p_1}+\frac{k_2^2}{p_2})N} (-1)^{\frac{k_2-k_1}{2} N} e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N} \\
&= (-1)^{\frac{p_1+p_2}{2}} e^{-\frac{\pi\sqrt{-1}}{4}(\frac{k_1^2}{p_1}+\frac{k_2^2}{p_2})N} e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N} .
\end{aligned}
$$

Therefore, from the above formulas, we obtain the lemma in this case.

When $c=\varphi_3(k_1,k_2)$, we obtain the lemma from the above case by replacing k_2 with $-k_2$.

When $c=\varphi_4(k_1,k_2)$, we obtain the lemma from the first case by replacing k_1 with $-k_1$.

When $c=\varphi_2(k_1,k_2)=\frac{p_2(2p_1-k_1)-p_1k_2}{2}$, we show the lemma, as follows. We obtain the formula of (1) of the lemma from the first case by replacing (k_1,k_2) with $(-k_1,-k_2)$. Further, by replacing $(-k_1,-k_2)$ with $(2p_1-k_1,-k_2)$, the formula becomes (-1)-multiple, and we obtain the formula of (2) of the lemma, as required. □

By using the above two lemmas, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. By Proposition 3.11, we have that

$$\hat{\tau}_N(M_{p_1,p_2}) \sim \frac{-e^{\frac{\pi\sqrt{-1}}{4}}}{\pi\sqrt{2p_1p_2}} N^{3/2}\,(-1)^{\frac{N-1}{2}}\, e^{\frac{\pi\sqrt{-1}}{8}N} \times \sum_{1\le c<\frac{1}{4}p_1p_2} (-1)^c\, e^{\frac{2\pi\sqrt{-1}}{p_1p_2}c^2\bar{\delta}N}\,\sin\frac{2\pi\,\overline{p_2}\,c}{p_1}\,\sin\frac{2\pi\,\overline{p_1}\,c}{p_2},$$

where we recall that $\delta=p_1p_2-2p_1-2p_2$, $\bar{\delta}$ denotes the inverse of δ in $\mathbb{Z}/p_1p_2\mathbb{Z}$, $\overline{p_1}$ denotes the inverse of p_1 in $\mathbb{Z}/p_2\mathbb{Z}$, and $\overline{p_2}$ denotes the inverse of p_2 in $\mathbb{Z}/p_1\mathbb{Z}$. We consider to rewrite the sum of this formula as a sum with respect to (k_1,k_2) by Lemma 3.25. We note that we can ignore the difference of S_2 and $\hat{S}_2$, because $\hat{S}_2-S_2$ consists of elements of the form (k_1,p_2) or (p_1,k_2), which do not contribute to the value of the sum. We also note that, by Lemma 3.26, the summand becomes (-1)-multiple on S_2. Hence, the sum is rewritten by a sum over $(S_1\sqcup S_3\sqcup S_4)-S_2$, as follows,

$$\begin{aligned}\hat{\tau}_N(M_{p_1,p_2}) &\sim \frac{-e^{\frac{\pi\sqrt{-1}}{4}}}{\pi\sqrt{2p_1p_2}} N^{3/2}\,(-1)^{\frac{N-1}{2}}\, e^{\frac{\pi\sqrt{-1}}{8}N} \\ &\quad\times \sum_{(k_1,k_2)\,\in\,(S_1\sqcup S_3\sqcup S_4)-S_2} (-1)\,(-1)^{\frac{k_1+k_2}{2}}\, e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N}\, e^{-\frac{\pi\sqrt{-1}}{4}(\frac{k_1^2}{p_1}+\frac{k_2^2}{p_2})N}\,\sin\frac{k_1\pi}{p_1}\,\sin\frac{k_2\pi}{p_2} \\ &\sim \frac{e^{\frac{\pi\sqrt{-1}}{4}}}{\pi\sqrt{2p_1p_2}} N^{3/2}\,(-1)^{\frac{N-1}{2}}\, e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N} \\ &\quad\times \sum_{(k_1,k_2)\,\in\,(S_1\sqcup S_3\sqcup S_4)-S_2} (-1)^{\frac{k_1+k_2}{2}}\, e^{\frac{\pi\sqrt{-1}}{4}(\frac{1}{2}-\frac{k_1^2}{p_1}-\frac{k_2^2}{p_2})N}\,\sin\frac{k_1\pi}{p_1}\,\sin\frac{k_2\pi}{p_2}.\end{aligned}$$

From the definition of S_i, we can see that $(S_1\sqcup S_3\sqcup S_4)-S_2$ is equal to the disjoint union of

$$\big\{(k_1,k_2)\;\big|\;(k_1,k_2)\text{ satisfies (28)}\big\}\quad\text{and}\quad\text{two copies of }\big\{(k_1,k_2)\in(26)\;\big|\;\frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}\big\}.$$

Further, by Proposition 3.16, these sets are rewritten

$$\big\{(k_1,k_2)\;\big|\;\rho_{k_1,k_2}\text{ is a SU(2) representation}\big\}\quad\text{and}$$

$$\text{two copies of }\big\{(k_1,k_2)\;\big|\;\rho_{k_1,k_2}\text{ is a }\mathrm{SL}_2\mathbb{R}\text{ representation, and }\frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}\big\}.$$

Hence, the formula of $\hat{\tau}_N(M_{p_1,p_2})$ is rewritten,

$$\begin{aligned}
\hat{\tau}_N(M_{p_1,p_2}) \;\sim\; & \frac{e^{\frac{\pi\sqrt{-1}}{4}}}{\pi\sqrt{2p_1p_2}} N^{3/2} (-1)^{\frac{N-1}{2}} e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N} \\
& \times \Bigg(\sum_{\substack{\mathrm{SU}(2)\text{ rep} \\ \rho_{k_1,k_2}}} +2 \sum_{\substack{\mathrm{SL}_2\mathbb{R}\text{ rep} \\ \rho_{k_1,k_2} \\ \frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}}} \Bigg) (-1)^{\frac{k_1+k_2}{2}} e^{\frac{\pi\sqrt{-1}}{4}(\frac{1}{2}-\frac{k_1^2}{p_1}-\frac{k_2^2}{p_2})N} \sin\frac{k_1\pi}{p_1} \sin\frac{k_2\pi}{p_2} \\
\sim\; & e^{\frac{\pi\sqrt{-1}}{4}} N^{3/2} (-1)^{\frac{N-1}{2}} e^{-\frac{\pi\sqrt{-1}}{4}(p_1+p_2)N} \\
& \times \Bigg(\frac{1}{2} \sum_{\substack{\mathrm{SU}(2)\text{ rep} \\ \rho_{k_1,k_2}}} + \sum_{\substack{\mathrm{SL}_2\mathbb{R}\text{ rep} \\ \rho_{k_1,k_2} \\ \frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}}} \Bigg) \Big(e^{\pi\sqrt{-1}\,\mathrm{CS}(M_{p_1,p_2};\,\mathrm{ad}\circ\rho_{k_1,k_2})\,N}\, \omega(M_{p_1,p_2};\,\rho_{k_1,k_2}) \Big),
\end{aligned} \tag{29}$$

where $\mathrm{CS}(M_{p_1,p_2};\,\mathrm{ad}\circ\rho_{k_1,k_2})$ is given in (3.21) (see also Remark 3.27 below), and we put

$$\omega(M_{p_1,p_2};\,\rho_{k_1,k_2}) \;=\; (-1)^{\frac{k_1+k_2}{2}} \frac{\sqrt{2}}{\pi\sqrt{p_1p_2}} \sin\frac{k_1\pi}{p_1} \sin\frac{k_2\pi}{p_2}\,, \tag{30}$$

noting that, by Proposition 3.22, we have that

$$\omega(M_{p_1,p_2};\,\rho_{k_1,k_2})^2 \;=\; \pm\frac{1}{16\pi^2}\,\mathrm{Tor}(M_{p_1,p_2};\,\mathrm{ad}\circ\rho_{k_1,k_2}). \tag{31}$$

Therefore, we obtain the theorem. □

Remark 3.27. The Chern–Simons invariant is defined in $\mathbb{C}/\mathbb{Z}$. When we substitute $\mathrm{CS}(M_{p_1,p_2};\,\mathrm{ad}\circ\rho_{k_1,k_2})$ into (29), we choose a lift of its value in $\mathbb{C}/2\mathbb{Z}$ by

$$\mathrm{CS}(M_{p_1,p_2};\,\mathrm{ad}\circ\rho_{k_1,k_2}) \;=\; \frac{1}{4}\big(\frac{1}{2}-\frac{k_1^2}{p_1}-\frac{k_2^2}{p_2}\big)$$

as the same form of the formula of Proposition 3.21. We note that the choice of this lift changes the sign of $\omega(M_{p_1,p_2};\,\rho_{k_1,k_2})$.

Remark 3.28. By (31), we can obtain $\omega(M_{p_1,p_2};\,\rho_{k_1,k_2})$ as a constant multiple of a square root of the Reidemeister torsion. We consider how we should choose the sign of this square root. As mentioned in Remark 3.27, this sign depends on the choice of a lift of the Chern–Simons invariant in $\mathbb{C}/2\mathbb{Z}$. Further, as in (30), there is a factor $(-1)^{\frac{k_1+k_2}{2}}$. When ρ_{k_1,k_2} is a SU(2) representation, we can regard this factor as a factor derived from the spectral flow by

$$(-1)^{\frac{k_1+k_2}{2}} \;=\; e^{-\frac{\pi}{4}I(\rho_{k_1,k_2})},$$

where we obtain this equality by Remark 3.23. That is, the Reidemeister torsion is equal to the Ray–Singer torsion, which is defined to be the product of eigenvalues of

Laplacian, and we can expect that we can choose an appropriate sign of the square root of the Ray–Singer torsion by using the spectral flow, since the spectral flow algebraically counts the number of eigenvalues which change the sign when we move SU(2) connections. When ρ_{k_1,k_2} is a $\mathrm{SL}_2\mathbb{C}$ representation in general, it is a problem how we should choose an appropriate sign of the square root of the Reidemeister torsion, because eigenvalues of Laplacian are not necessarily real numbers, and we do not have a spectral flow in the $\mathrm{SL}_2\mathbb{C}$ case. Instead of the spectral flow, it might be possible to choose an appropriate sign of the square root of the Ray–Singer torsion directly, which behaves well when we move $\mathrm{SL}_2\mathbb{C}$ connections.

We show some numerical experiments for Theorem 1.2, in the following of this section.

Example 3.29. We consider the case of $M_{3,7}$. There are one $\mathrm{SL}_2\mathbb{R}$ representation $\rho_{1,1}$ and two SU(2) representations $\rho_{1,3}$, $\rho_{1,5}$. The Chern–Simons invariant and $\omega(M_{3,7};\,\rho_{1,k})$ are given by

$$\begin{aligned}\mathrm{CS}\big(M_{3,7};\,\mathrm{ad}\circ\rho_{1,k}\big) &= \frac{1}{21\cdot 8},\quad -\frac{47}{21\cdot 8},\quad -\frac{143}{21\cdot 8} \qquad \text{for } k=1,3,5,\\ \omega(M_{3,7};\,\rho_{1,k}) &= \frac{1}{\pi\sqrt{7\cdot 2}}\,(-1)^{\frac{k+1}{2}}\sin\frac{k\pi}{7}\,.\end{aligned}$$

By Theorem 1.2, we have that

$$\begin{aligned}&\hat{\tau}_N(M_{3,7})\,e^{-\frac{\pi\sqrt{-1}}{4}}N^{-3/2}(-1)^{\frac{N-1}{2}}e^{\frac{\pi\sqrt{-1}}{4}(3+7)N}\\ &\sim\ \frac{1}{\pi\sqrt{7\cdot 2}}\Big(-e^{\frac{\pi\sqrt{-1}}{21\cdot 8}N}\sin\frac{\pi}{7}+\frac{1}{2}\,e^{-\frac{47\pi\sqrt{-1}}{21\cdot 8}N}\sin\frac{3\pi}{7}-\frac{1}{2}\,e^{-\frac{143\pi\sqrt{-1}}{21\cdot 8}N}\sin\frac{5\pi}{7}\Big).\end{aligned}$$

We fix an odd integer d, and consider a subsequence of N such that $N\equiv d$ modulo $21\cdot 16$. Then, for such a subsequence, the above formula converges to

$$\frac{1}{\pi\sqrt{7\cdot 2}}\Big(-e^{\frac{\pi\sqrt{-1}}{21\cdot 8}d}\sin\frac{\pi}{7}+\frac{1}{2}\,e^{-\frac{47\pi\sqrt{-1}}{21\cdot 8}d}\sin\frac{3\pi}{7}-\frac{1}{2}\,e^{-\frac{143\pi\sqrt{-1}}{21\cdot 8}d}\sin\frac{5\pi}{7}\Big). \tag{32}$$

When $d=1$, we show some values of this subsequence in the following table.

N	$\hat{\tau}_N(M_{3,7})\,e^{-\frac{\pi\sqrt{-1}}{4}}N^{-3/2}(-1)^{\frac{N-1}{2}}e^{\frac{\pi\sqrt{-1}}{4}(3+7)N}$
$21\cdot 16\cdot 10+1 = 3361$	$0.019211144024... - \sqrt{-1}\ 0.017668061351...$
$21\cdot 16\cdot 20+1 = 6721$	$0.019225170739... - \sqrt{-1}\ 0.017652718199...$
$21\cdot 16\cdot 30+1 = 10081$	$0.019230028250... - \sqrt{-1}\ 0.017647390950...$

We can numerically observe that this sequence tends to converge to

$$\big((32)\text{ for } d{=}1\big) \ = \ 0.019240198633... - \sqrt{-1}\ 0.017636247827...\ ,$$

as required.

When $d = 3$, we show some values of this subsequence in the following table.

N	$\hat{\tau}_N(M_{3,7})\, e^{-\frac{\pi\sqrt{-1}}{4}} N^{-3/2} (-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{4}(3+7)N}$
$21 \cdot 16 \cdot 10 + 3 \;=\; 3363$	$-0.067554438323... + \sqrt{-1}\ 0.010785310472...$
$21 \cdot 16 \cdot 20 + 3 \;=\; 6723$	$-0.067566600760... + \sqrt{-1}\ 0.010720563469...$
$21 \cdot 16 \cdot 30 + 3 \;=\; 10083$	$-0.067570465672... + \sqrt{-1}\ 0.010699176381...$

We can numerically observe that this sequence tends to converge to

$$\big((32) \text{ for } d{=}3\big) \;=\; -0.067577728940... + \sqrt{-1}\ 0.010656864668... ,$$

as required.

Example 3.30. We consider the case of $M_{3,11}$. In a similar way as above, by Theorem 1.2, we have that

$$\begin{aligned}&\hat{\tau}_N(M_{3,11})\, e^{-\frac{\pi\sqrt{-1}}{4}} N^{-3/2} (-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{4}(3+11)N} \\ &\sim \frac{1}{\pi\sqrt{11\cdot 2}}\Big(- e^{\frac{5\pi\sqrt{-1}}{33\cdot 8}N} \sin\frac{\pi}{11} + \frac{1}{2}\, e^{-\frac{43\pi\sqrt{-1}}{33\cdot 8}N} \sin\frac{3\pi}{11} - \frac{1}{2}\, e^{-\frac{139\pi\sqrt{-1}}{33\cdot 8}N} \sin\frac{5\pi}{11} \\ &\qquad + \frac{1}{2}\, e^{-\frac{283\pi\sqrt{-1}}{33\cdot 8}N} \sin\frac{7\pi}{11} - \frac{1}{2}\, e^{-\frac{475\pi\sqrt{-1}}{33\cdot 8}N} \sin\frac{9\pi}{11}\Big).\end{aligned}$$

We fix an odd integer d, and consider a subsequence of N such that $N \equiv d$ modulo $33\cdot 16$. Then, for such a subsequence, the above formula converges to

$$\begin{aligned}&\frac{1}{\pi\sqrt{11\cdot 2}}\Big(- e^{\frac{5\pi\sqrt{-1}}{33\cdot 8}d} \sin\frac{\pi}{11} + \frac{1}{2}\, e^{-\frac{43\pi\sqrt{-1}}{33\cdot 8}d} \sin\frac{3\pi}{11} - \frac{1}{2}\, e^{-\frac{139\pi\sqrt{-1}}{33\cdot 8}d} \sin\frac{5\pi}{11} \\ &\qquad + \frac{1}{2}\, e^{-\frac{283\pi\sqrt{-1}}{33\cdot 8}d} \sin\frac{7\pi}{11} - \frac{1}{2}\, e^{-\frac{475\pi\sqrt{-1}}{33\cdot 8}d} \sin\frac{9\pi}{11}\Big).\end{aligned} \tag{33}$$

When $d = 1$, we show some values of this subsequence in the following table.

N	$\hat{\tau}_N(M_{3,11})\, e^{-\frac{\pi\sqrt{-1}}{4}} N^{-3/2} (-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{4}(3+11)N}$
$33 \cdot 16 \cdot 10 + 1 \;=\; 5281$	$-0.036915658417... + \sqrt{-1}\ 0.015119457605...$
$33 \cdot 16 \cdot 20 + 1 \;=\; 10561$	$-0.037485993595... + \sqrt{-1}\ 0.015295341804...$
$33 \cdot 16 \cdot 30 + 1 \;=\; 15841$	$-0.037736409572... + \sqrt{-1}\ 0.015386990592...$

We can numerically observe that this sequence tends to converge to

$$\big((33) \text{ for } d{=}1\big) \;=\; -0.038827415505... + \sqrt{-1}\ 0.015877504237... ,$$

as required.

When $d = 3$, we show some values of this subsequence in the following table.

N	$\hat{\tau}_N(M_{3,11})\, e^{-\frac{\pi\sqrt{-1}}{4}} N^{-3/2} (-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{4}(3+11)N}$
$33 \cdot 16 \cdot 10 + 3 \;=\; 5283$	$-0.044503577075... - \sqrt{-1}\ 0.058157288474...$
$33 \cdot 16 \cdot 20 + 3 \;=\; 10563$	$-0.044426050815... - \sqrt{-1}\ 0.058619376426...$
$33 \cdot 16 \cdot 30 + 3 \;=\; 15843$	$-0.044409385039... - \sqrt{-1}\ 0.058813875897...$

We can numerically observe that this sequence tends to converge to

$$\big((33) \text{ for } d{=}3\big) \;=\; -0.044446449091... - \sqrt{-1}\; 0.059604727759... \,,$$

as required.

Remark 3.31. We note that $M_{3,7}$ is an integral homology 3-sphere, and there is no abelian representation of $\pi_1(M_{3,7})$. In this case, as mentioned in Remark 1.3, we can expect that the convergence of $\hat{\tau}_N(M_{3,7})\, e^{-\frac{\pi\sqrt{-1}}{4}} N^{-3/2}(-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{4}(3+7)N}$ is of order $\frac{1}{N}$. We can numerically observe it in Example 3.29.

On the other hand, $M_{3,11}$ is not an integral homology 3-sphere, and there are abelian representations of $\pi_1(M_{3,11})$. In this case, we can observe in Example 3.30 that the convergence of $\hat{\tau}_N(M_{3,11})\, e^{-\frac{\pi\sqrt{-1}}{4}} N^{-3/2}(-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{4}(3+11)N}$ is of order $\frac{1}{N^{1/2}}$.

Example 3.32. We consider the case of $M_{3,13}$. In a similar way as above, by Theorem 1.2, we have that

$$\begin{aligned}&\hat{\tau}_N(M_{3,13})\, e^{-\frac{\pi\sqrt{-1}}{4}} N^{-3/2}(-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{4}(3+13)N} \\ &\sim \frac{1}{\pi\sqrt{13\cdot 2}}\Big(- e^{\frac{7\pi\sqrt{-1}}{39\cdot 8}N} \sin\frac{\pi}{12} + \frac{1}{2}\, e^{-\frac{41\pi\sqrt{-1}}{39\cdot 8}N} \sin\frac{3\pi}{13} - \frac{1}{2}\, e^{-\frac{137\pi\sqrt{-1}}{39\cdot 8}N} \sin\frac{5\pi}{13} \\ &\qquad + \frac{1}{2}\, e^{-\frac{281\pi\sqrt{-1}}{39\cdot 8}N} \sin\frac{7\pi}{13} - \frac{1}{2}\, e^{-\frac{473\pi\sqrt{-1}}{39\cdot 8}N} \sin\frac{9\pi}{13}\Big).\end{aligned}$$

We fix an odd integer d, and consider a subsequence of N such that $N \equiv d$ modulo $39\cdot 16$. Then, for such a subsequence, the above formula converges to

$$\begin{aligned}&\frac{1}{\pi\sqrt{13\cdot 2}}\Big(- e^{\frac{7\pi\sqrt{-1}}{39\cdot 8}d} \sin\frac{\pi}{13} + \frac{1}{2}\, e^{-\frac{41\pi\sqrt{-1}}{39\cdot 8}d} \sin\frac{3\pi}{13} - \frac{1}{2}\, e^{-\frac{137\pi\sqrt{-1}}{39\cdot 8}d} \sin\frac{5\pi}{13} \\ &\qquad + \frac{1}{2}\, e^{-\frac{281\pi\sqrt{-1}}{39\cdot 8}d} \sin\frac{7\pi}{13} - \frac{1}{2}\, e^{-\frac{473\pi\sqrt{-1}}{39\cdot 8}d} \sin\frac{9\pi}{13}\Big).\end{aligned} \tag{34}$$

When $d = 1$, we show some values of this subsequence in the following table.

N	$\hat{\tau}_N(M_{3,13})\, e^{-\frac{\pi\sqrt{-1}}{4}} N^{-3/2}(-1)^{\frac{N-1}{2}} e^{\frac{\pi\sqrt{-1}}{4}(3+13)N}$
$39\cdot 16\cdot 10+1 \;=\; 6241$	$-0.032174499512... - \sqrt{-1}\; 0.013829506117...$
$39\cdot 16\cdot 20+1 \;=\; 12481$	$-0.032287936561... - \sqrt{-1}\; 0.015790659198...$
$39\cdot 16\cdot 30+1 \;=\; 18721$	$-0.033425481385... - \sqrt{-1}\; 0.015971768695...$

We can numerically observe that this sequence tends to converge to

$$\big((34) \text{ for } d{=}1\big) \;=\; -0.032273455128... - \sqrt{-1}\; 0.015875035932... \,,$$

as required.

By Proposition 3.16, there are $\mathrm{SL}_2\mathbb{R}$ representations $\rho_{1,1}$, $\rho_{1,11}$ and SU(2) representations $\rho_{1,3}$, $\rho_{1,5}$, $\rho_{1,7}$, $\rho_{1,9}$. We note that a contribution from $\rho_{1,11}$ does not appear in the above formula (34) (which would have the factor "$\sin\frac{11\pi}{13}$" if it appeared). It would be a problem how we choose necessary representations in general; see Remark 1.6.

A The quantum SU(2) and SO(3) invariants

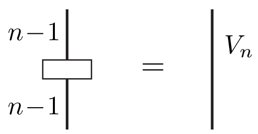
In this section, we review the definition and basic properties of the quantum SU(2) and SO(3) invariants. We recall that N is an odd integer ≥ 3, and we put $A = e^{\frac{\pi\sqrt{-1}}{N}}$, $a = e^{\frac{2\pi\sqrt{-1}}{N}}$ and $[n] = \frac{a^n - a^{-n}}{a - a^{-1}}$.

In Section A.1, we review the definition of the quantum SU(2) and SO(3) invariants, and gives a formula of the quantum SO(3) invariant of the Seifert 3-manifold M_{f_1,f_2,f_3}. In Section A.2, we review the equivalence between the quantum SU(2) and SO(3) invariants at $q = e^{\frac{4\pi\sqrt{-1}}{N}}$.

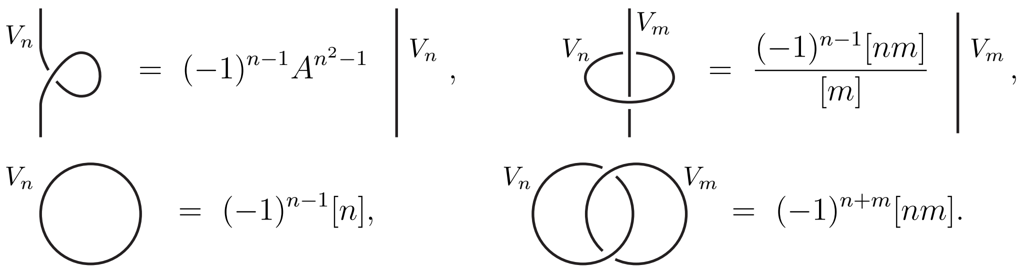
A.1 Review of the quantum SU(2) and SO(3) invariants

In this section, we briefly review the definition of the quantum SU(2) and SO(3) invariant of [40, 19] at $q = e^{\frac{4\pi\sqrt{-1}}{N}}$, following the construction of [25]; we note that these two invariants are equivalent as we see in Section A.2 later. Further, we give a formula of the quantum SO(3) invariant of the Seifert 3-manifold M_{f_1,f_2,f_3}, which we use in Section 3.2.

We briefly review the definition of the quantum SU(2) and SO(3) invariant at $q = e^{\frac{4\pi\sqrt{-1}}{N}}$ following the construction of Lickorish [25], noting that we put $A = e^{\frac{\pi\sqrt{-1}}{N}}$ in this paper unlike the usual case where A is a primitive $4N$-th root of unity. In order to state the symmetry principle of our case later in Section A.2, we use the notation of [40, 19], that is, the notation in the right-hand side of the following formula,



where the left-hand side denotes the box of the Jones-Wenzl idempotent in the Lickorish notation, and the right-hand side is a strand associated with the n-dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$. It is known, see [25, 17], that



(35)

For a ℓ-component framed link L whose components are associated with $V_{n_1}, \cdots, V_{n_\ell}$, we denote by $Q(L; V_{n_1}, \cdots, V_{n_\ell})$ the invariant of L defined by the linear skein of Lickorish [25]. Let M be the 3-manifold obtained from S^3 by surgery along L. Then, the *quantum* SU(2) *invariant* $\hat{\tau}_N^{\mathrm{SU}(2)}(M)$ and the *quantum* SO(3) *invariant* $\hat{\tau}_N(M)$ are defined by

$$\hat{\tau}_N^{\mathrm{SU}(2)}(M) = {c'_+}^{-\sigma_+} {c'_-}^{-\sigma_-} \sum_{1 \leq n_1, \cdots, n_\ell < N} \big((-1)^{n_1-1}[n_1]\big) \cdots \big((-1)^{n_\ell-1}[n_\ell]\big)\, Q(L; V_{n_1}, \cdots, V_{n_\ell}),$$

$$\hat{\tau}_{N}(M) \;=\; c_+^{-\sigma_+} c_-^{-\sigma_-} \sum_{\substack{1\le n_1,\cdots,n_\ell<N\\ n_1,\cdots,n_\ell \text{ are odd}}} [n_1]\cdots[n_\ell]\; Q(L;\, V_{n_1},\cdots,V_{n_\ell}),$$

where σ_+ and σ_- are the numbers of the positive and negative eigenvalues of the linking matrix of L, and $c'_\pm$ and $c_\pm$ are constants given by

$$c'_\pm \;=\; \sum_{1\le n<N} (-1)^{n-1} A^{\pm(n^2-1)}[n]^2, \qquad c_\pm \;=\; \sum_{\substack{1\le n<N\\ n \text{ is odd}}} A^{\pm(n^2-1)}[n]^2.$$

We calculate the value of the constant c_+, by using Lemma 3.1, as follows,

$$\begin{aligned} c_+ \;&=\; \sum_{\substack{1\le n<N\\ n \text{ is odd}}} A^{n^2-1}[n]^2 \;=\; \frac{-A^{-1}}{(a-a^{-1})^2} \sum_{j\in\mathbb{Z}/N\mathbb{Z}} a^{2j^2}\big(a^{4j}-1\big) \\ &=\; \frac{-A^{-1}(a^{-2}-1)}{(a-a^{-1})^2} \sum_{j\in\mathbb{Z}/N\mathbb{Z}} a^{2j^2} \;=\; \frac{A^{-3}}{a-a^{-1}} \sum_{j\in\mathbb{Z}/N\mathbb{Z}} a^{2j^2} \\ &=\; \frac{A^{-3}}{a-a^{-1}}\big(-\sqrt{-1}\big)^{\frac{N-1}{2}}\sqrt{N}, \end{aligned} \tag{36}$$

where we obtain the last equality by [35, Lemma A.2]. The constant c_- is the complex conjugate of c_+,

$$c_- \;=\; \frac{A^3}{a^{-1}-a}\sqrt{-1}^{\frac{N-1}{2}}\sqrt{N}.$$

Hence,

$$\frac{c_+}{c_-} \;=\; A^{-6}(-1)^{\frac{N+1}{2}}. \tag{37}$$

We give a formula of the quantum SO(3) invariant of the Seifert 3-manifold given in Section 3.2. Let L be the framed link in (23). When i,j,k,m are odd, it follows from (35) that the invariant of L is given by

$$Q(L;\, V_i,V_j,V_k,V_m) \;=\; A^{2(i^2-1)+p_1(j^2-1)+p_2(k^2-1)+(m^2-1)}\,\frac{[mi][mj][mk]}{[m]}\,[i][j][k].$$

Hence, the quantum SO(3) invariant of M_{p_1,p_2} is given by

$$\hat{\tau}_{N}(M_{p_1,p_2}) \;=\; c_+^{-\sigma_+} c_-^{-\sigma_-} \sum_{\substack{1\le i,j,k,m<N\\ i,j,k,m \text{ are odd}}} A^{2(i^2-1)+p_1(j^2-1)+p_2(k^2-1)+(m^2-1)}\,\frac{[mi][mj][mk]}{[m]}\,[i][j][k], \tag{38}$$

where σ_+ and σ_- are the numbers of the positive and negative eigenvalues of the linking matrix of L.

A.2 Equivalence of the quantum SU(2) and SO(3) invariants

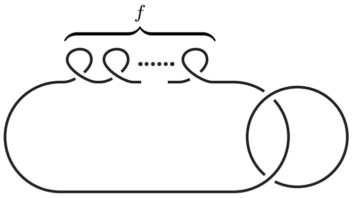
It is known that the quantum SU(2) and SO(3) invariants are equivalent at $q = e^{\frac{4\pi\sqrt{-1}}{N}}$ in the sense of Proposition A.2, unlike the usual case where $q = e^{\frac{2\pi\sqrt{-1}}{N}}$. This equivalence is mentioned in the first paragraph of Section 1 of [26]. In this section, we briefly review this equivalence.

Before we review this equivalence, we review the symmetry principle at $q = e^{\frac{4\pi\sqrt{-1}}{N}}$ in the following proposition.

Proposition A.1. *When $A = e^{\frac{\pi\sqrt{-1}}{N}}$, for an ℓ-component framed link L,*

$$Q(L;\, V_{n_1}, V_{n_2}, \cdots, V_{n_\ell}) \;=\; Q(L;\, V_{N-n_1}, V_{n_2}, \cdots, V_{n_\ell}).$$

Proof. The symmetry principle is shown in [24, Proposition 9] when A is a primitive $4N$-th root of unity. When $A = e^{\frac{\pi\sqrt{-1}}{N}}$, we can show the symmetry principle in the same way as in the proof of [24, Proposition 9]. As shown in the proof of [24, Proposition 9], we can reduce the proof to the case where L is a framed Hopf link of the following form.

$H_{f,0} \;=$ 

In this case, it follows from (35) that the invariants of $H_{f,0}$ are given by

$$\begin{aligned} Q(H_{f,0};\, V_n, V_m) \;&=\; \big((-1)^{n-1}A^{n^2-1}\big)^f\,(-1)^{n+m}[nm], \\ Q(H_{f,0};\, V_{N-n}, V_m) \;&=\; \big((-1)^{N-n-1}A^{(N-n)^2-1}\big)^f\,(-1)^{N-n+m}[(N-n)m], \end{aligned}$$

and it is sufficient to show that these values are equal.

In fact, these values are equal, since

$$(-1)^{n-1}A^{n^2-1} \;=\; (-1)^{N-n-1}A^{(N-n)^2-1} \quad \text{and} \quad (-1)^{n+m}[nm] \;=\; (-1)^{N-n+m}[(N-n)m],$$

when $A = e^{\frac{\pi\sqrt{-1}}{N}}$. Therefore, we obtain the proposition. □

By using the symmetry principle (the above proposition), we can obtain the following proposition.

Proposition A.2 (see [26])**.** *When $A = e^{\frac{\pi\sqrt{-1}}{N}}$, the quantum* SU(2) *and* SO(3) *invariants of a closed 3-manifold M are related by*

$$\hat{\tau}_N^{\mathrm{SU}(2)}(M) \;=\; 2^{b_1(M)}\,\hat{\tau}_N(M),$$

where $b_1(M)$ denotes the first Betti number of M.

Proof. We have that

$$\begin{aligned} c'_\pm \;&=\; \sum_{\substack{1\le n<N \\ n \text{ is odd}}} A^{\pm(n^2-1)}[n]^2 + \sum_{\substack{1\le m<N \\ m \text{ is even}}} (-1)A^{\pm(m^2-1)}[m]^2 \\ &=\; \sum_{\substack{1\le n<N \\ n \text{ is odd}}} A^{\pm(n^2-1)}[n]^2 + \sum_{\substack{1\le m<N \\ m \text{ is odd}}} A^{\pm(m^2-1)}[m]^2 \;=\; 2\,c_\pm\,, \end{aligned}$$

where we obtain the second equality by replacing m with $N-m$. In a similar way, by Proposition A.1, we can obtain that

$$\sum_{1\le n_1,\cdots,n_\ell<N} \big((-1)^{n_1-1}[n_1]\big)\cdots\big((-1)^{n_\ell-1}[n_\ell]\big)\, Q(L;\, V_{n_1},\cdots,V_{n_\ell}) \\ = 2^\ell \sum_{\substack{1\le n_1,\cdots,n_\ell<N \\ n_1,\cdots,n_\ell \text{ are odd}}} [n_1]\cdots[n_\ell]\; Q(L;\, V_{n_1},\cdots,V_{n_\ell}).$$

Therefore, since $\ell=\sigma_+ + \sigma_- + b_1(M)$, we obtain the proposition from the definitions of the quantum SU(2) and SO(3) invariants. □

B The case of lens spaces $L(p,1)$ for odd p

In this section, we calculate the semi-classical limit of the quantum invariant of the lens space $L(p,1)$ for odd $p\ge 3$. We recall that $A=e^{\frac{\pi\sqrt{-1}}{N}}$, $a=e^{\frac{2\pi\sqrt{-1}}{N}}$ and $[n]=\frac{a^n-a^{-n}}{a-a^{-1}}$.

Let p be an odd integer ≥ 3. We denote by $L(p,1)$ the lens space obtained from S^3 by p surgery along the trivial knot; we note that this $L(p,1)$ has the opposite orientation of $L(p,1)$ of [15] because of the difference of the convention of an orientation of a lens space.

By definition, the quantum invariant of $L(p,1)$ is presented by

$$\hat{\tau}_N\big(L(p,1)\big) = \frac{1}{c_+}\sum_{\substack{1\le m<N \\ m \text{ is odd}}} A^{p(m^2-1)}[m]^2 = \frac{-A^{-p}}{c_+\,(a-a^{-1})^2}\sum_{j\,\in\,\mathbb{Z}/N\mathbb{Z}} a^{2pj^2}(a^{4j}-1).$$

The last sum is calculated as

$$\sum_{j\,\in\,\mathbb{Z}/N\mathbb{Z}} a^{2pj^2}(a^{4j}-1) = \frac{\sqrt{N}}{2\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{4}} \sum_{m\,\in\,\mathbb{Z}/4p\mathbb{Z}} \big(e^{-\frac{\pi\sqrt{-1}}{4pN}(Nm+4)^2} - e^{-\frac{\pi\sqrt{-1}}{4pN}N^2m^2}\big) \\ \sim \frac{\sqrt{N}}{2\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{4}} \sum_{m\,\in\,\mathbb{Z}/4p\mathbb{Z}} e^{-\frac{\pi\sqrt{-1}}{4p}Nm^2}\big(e^{-\frac{2\pi\sqrt{-1}}{p}m}-1\big).$$

By replacing m with $m+2p$, we can show that the last sum satisfies that

$$\sum_{m\,\in\,\mathbb{Z}/4p\mathbb{Z}} e^{-\frac{\pi\sqrt{-1}}{4p}Nm^2}\big(e^{-\frac{2\pi\sqrt{-1}}{p}m}-1\big) = \sum_{m\,\in\,\mathbb{Z}/4p\mathbb{Z}} (-1)^{m+1} e^{-\frac{\pi\sqrt{-1}}{4p}Nm^2}\big(e^{-\frac{2\pi\sqrt{-1}}{p}m}-1\big).$$

Hence, we can restrict this sum for odd m. Therefore, by putting $m=2n+p$, we have that

$$\sum_{j\,\in\,\mathbb{Z}/N\mathbb{Z}} a^{2pj^2}(a^{4j}-1) \sim \frac{\sqrt{N}}{2\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{4}} \sum_{\substack{m\,\in\,\mathbb{Z}/4p\mathbb{Z} \\ m \text{ is odd}}} e^{-\frac{\pi\sqrt{-1}}{4p}Nm^2}\big(e^{-\frac{2\pi\sqrt{-1}}{p}m}-1\big) \\ = \frac{\sqrt{N}}{2\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{4}}\, e^{-\frac{\pi\sqrt{-1}}{4}pN} \sum_{n\,\in\,\mathbb{Z}/2p\mathbb{Z}} (-1)^n\, e^{-\frac{\pi\sqrt{-1}}{p}Nn^2}\big(e^{-\frac{4\pi\sqrt{-1}}{p}n}-1\big)$$

$$
\begin{aligned}
&= \frac{\sqrt{N}}{\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{4}}\, e^{-\frac{\pi\sqrt{-1}}{4}pN} \sum_{n\,\in\,\mathbb{Z}/p\mathbb{Z}} (-1)^n\, e^{-\frac{\pi\sqrt{-1}}{p}Nn^2}\big(e^{-\frac{4\pi\sqrt{-1}}{p}n}-1\big) \\
&= \frac{\sqrt{N}}{2\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{4}}\, e^{-\frac{\pi\sqrt{-1}}{4}pN} \sum_{k\,\in\,\mathbb{Z}/p\mathbb{Z}} (-1)^k\, e^{-\frac{\pi\sqrt{-1}}{p}Nk^2}\big(e^{\frac{4\pi\sqrt{-1}}{p}k}-2+e^{-\frac{4\pi\sqrt{-1}}{p}k}\big) \\
&= \frac{\sqrt{N}}{2\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{4}}\, e^{-\frac{\pi\sqrt{-1}}{4}pN} \sum_{k\,\in\,\mathbb{Z}/p\mathbb{Z}} (-1)^k\, e^{-\frac{\pi\sqrt{-1}}{p}Nk^2}\big(e^{\frac{2\pi\sqrt{-1}}{p}k}-e^{-\frac{2\pi\sqrt{-1}}{p}k}\big)^2 \\
&= \frac{-2\sqrt{N}}{\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{4}}\, e^{-\frac{\pi\sqrt{-1}}{4}pN} \sum_{k\,\in\,\mathbb{Z}/p\mathbb{Z}} (-1)^k\, e^{-\frac{\pi\sqrt{-1}}{p}Nk^2}\sin^2\frac{2\pi k}{p} \\
&= \frac{4\sqrt{N}}{\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{4}}\, e^{-\frac{\pi\sqrt{-1}}{4}pN} \sum_{\substack{1\le k<p \\ k \text{ is odd}}} e^{-\frac{\pi\sqrt{-1}}{p}Nk^2}\sin^2\frac{2\pi k}{p},
\end{aligned}
$$

where we obtain the fourth line by making the average of the third line for $k=\pm n$, and obtain the last equality by replacing even k with $p-k$. Therefore, by (36), we have that

$$
\begin{aligned}
\hat{\tau}_N\big(L(p,1)\big) &\sim \frac{-1}{a-a^{-1}}\, e^{\frac{\pi\sqrt{-1}}{2}\frac{N-1}{2}}\, \frac{4}{\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{4}}\, e^{-\frac{\pi\sqrt{-1}}{4}pN} \sum_{\substack{1\le k<p \\ k \text{ is odd}}} e^{-\frac{\pi\sqrt{-1}}{p}Nk^2}\sin^2\frac{2\pi k}{p} \\
&\sim \frac{N}{\pi\sqrt{p}}\, e^{\frac{\pi\sqrt{-1}}{2}}\, e^{-\frac{\pi\sqrt{-1}}{2}\frac{p-1}{2}N} \sum_{\substack{1\le k<p \\ k \text{ is odd}}} e^{-\frac{\pi\sqrt{-1}}{p}Nk^2}\sin^2\frac{2\pi k}{p} \\
&\sim \frac{-N}{\pi\sqrt{p}}\, e^{-\frac{\pi\sqrt{-1}}{2}\frac{p+1}{2}}\, (-1)^{\frac{p-1}{2}\frac{N-1}{2}} \sum_{\substack{1\le k<p \\ k \text{ is odd}}} e^{-\pi\sqrt{-1}\,\frac{k^2}{p}\,N}\sin^2\frac{2\pi k}{p}, \qquad (39)
\end{aligned}
$$

where we obtain the second approximation since $a-a^{-1}=2\sqrt{-1}\sin\frac{2\pi}{N}\sim\frac{4\pi\sqrt{-1}}{N}$.

Let ρ_k be a representation of $\pi_1\big(L(p,1)\big)=\mathbb{Z}/p\mathbb{Z}$ to $\mathrm{SL}_2\mathbb{C}$ such that the eigenvalues of ρ_k(generator) are $\exp(\pm 2\pi\sqrt{-1}\,\frac{k}{p})$. Then, we obtain the following proposition from the above formula.

Proposition B.1. *Let p be an odd integer ≥ 3. Then, the quantum invariant $\hat{\tau}_N\big(L(p,1)\big)$ of $L(p,1)$ for odd N is expanded as $N\to\infty$ in the following form,*

$$
\hat{\tau}_N\big(L(p,1)\big) \;\sim\; -\,e^{-\frac{\pi\sqrt{-1}}{2}\frac{p+1}{2}}\,(-1)^{\frac{p-1}{2}\frac{N-1}{2}}N \sum_{\substack{1\le k<p \\ k \text{ is odd}}} e^{\pi\sqrt{-1}\,\mathrm{CS}\,(L(p,1);\,\mathrm{ad}\circ\rho_k)\,N}\,\omega\big(L(p,1);\,\rho_k\big),
$$

where we put

$$
\omega\big(L(p,1);\,\rho_k\big) \;=\; \frac{1}{\pi\sqrt{p}}\sin^2\frac{2\pi k}{p}.
$$

Further, we have that

$$
\omega\big(L(p,1);\,\rho_k\big)^2 \;=\; \pm\frac{1}{32\pi^2}\,\mathrm{Tor}\big(L(p,1);\,\mathrm{ad}\circ\rho_k\big).
$$

Proof. It is known, see [15], that the Chern–Simons invariant and the twisted Reidemeister torsion of $L(p,1)$ are given by

$$\begin{aligned} \mathrm{CS}\big(L(p,1);\, \mathrm{ad}\circ\rho_k\big) &= -\frac{k^2}{p}\,, \\ \mathrm{Tor}\big(L(p,1);\, \mathrm{ad}\circ\rho_k\big) &= \pm\frac{32}{p}\,\sin^4\frac{2\pi k}{p}\,. \end{aligned}$$

Hence, by (39), we obtain the proposition. □

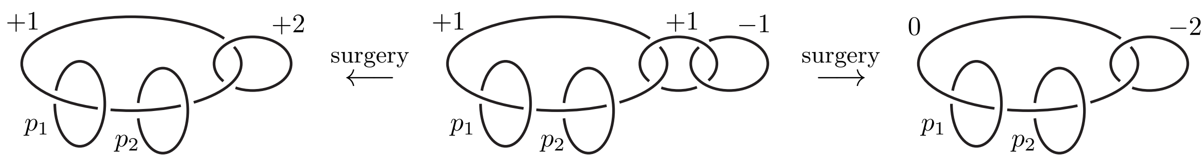
C Equivalences between some Seifert 3-manifolds

In this appendix, we review some homeomorphisms between some Seifert 3-manifolds in Remarks C.1 and C.2 below.

We recall that, for an integer p, we denote by M_p the 3-manifold obtained from S^3 by p surgery along the figure-eight knot. We also recall that, for coprime odd integers p_1, p_2, we denote by M_{p_1,p_2} the Seifert 3-manifold obtained from S^3 by surgery along the framed link (23).

Remark C.1. M_{p_1,p_2} is homeomorphic to $X(-2,p_1,p_2)$ of [42].

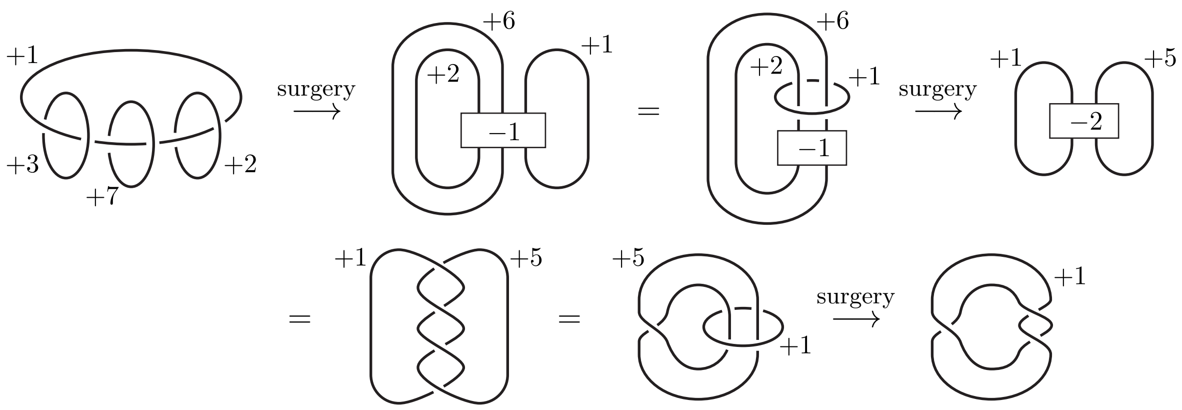
Proof. By definition, our M_{p_1,p_2} is given by a surgery presentation the left picture below. Further, as in [42], $X(-2,p_1,p_2)$ is given by a surgery presentation the right picture below. They are related by by Kirby moves, as follows.



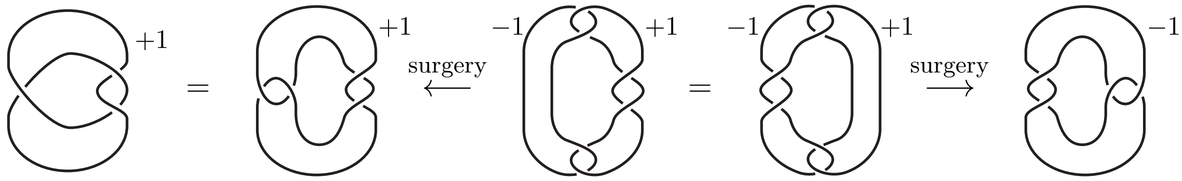
Hence, M_{p_1,p_2} is homeomorphic to $X(-2,p_1,p_2)$, as required. □

Remark C.2 (see *e.g.* [18, Remark (5) of Problem 1.77])**.** $M_{3,7}$ is homeomorphic to M_{-1}, which is homeomorphic to M_1 with opposite orientation.

Proof. We show a surgery presentation of $M_{3,7}$ is related to a surgery presentation of M_{-1} by Kirby moves, as follows. A surgery presentation of $M_{3,7}$ is given by the left picture below, which is calculated as follows, where "an integer n in a box" means n full twists.



Further, the last picture is calculated by Kirby moves, as follows.



The last picture is a surgery presentation of M_{-1}. Hence, $M_{3,7}$ is homeomorphic to M_{-1}.

Further, since the figure-eight knot is isotopic to its mirror image, M_{-1} is homeomorphic to M_1 with the opposite orientation. Therefore, we obtain the statement of the remark. □

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