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Pearcey system re-examined from the viewpoint of s-virtual turning points and non-hereditary turning points

Dedicated to Professor D.C. Struppa on his sixtieth birthday.

By

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§ 0. Introduction

The exact WKB analysis of the Pearcey system $\mathscr{M}$ given in (0.1) below was first proposed by T. Aoki ([A]) and later developed by one (S.H) of us ([H1]):

$$(0.1) \qquad \mathscr{M}: \begin{cases} \left(4\eta^{-3}\dfrac{\partial^3}{\partial x_1^3} + 2x_2\eta^{-1}\dfrac{\partial}{\partial x_1} + x_1\right)\psi = 0 & \text{(0.1.a)} \\ \left(\eta^{-1}\dfrac{\partial}{\partial x_2} - \eta^{-2}\dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0. & \text{(0.1.b)} \end{cases}$$

The system $\mathscr{M}$ is an over-determined system with a large parameter η, and the Pearcey integral

$$(0.2) \qquad \int \exp(\eta\varphi(x,t))dt,$$

where

$$(0.3) \qquad \varphi(x,t) = t^4 + x_2t^2 + x_1t,$$

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satisfies $\mathscr{M}$. One important feature of the Pearcey system is that when it is restricted to

(0.4) $\quad Y_1 = \{(x_1, x_2) \in \mathbb{C}^2; x_2 = A_2, \text{ where } A_2 \text{ is a non-zero constant } \}$

the Berk-Nevins-Robers equation (abbreviated as the BNR equation) appears, the description of whose Stokes geometry requires a new Stokes curve ([BNR]) and a virtual turning point ([AKT], where a "new turning point" is used to mean a virtual turning point). In view of the importance of the BNR equation in the development of the theory of virtual turning points (see [HKT]), it is reasonable to try to analyze the Pearcey system. In this report we show several interesting features of the system which have been found with the help of the theory of s-virtual turning points ([HKT, Section 1.8]). To begin with, we study in Section 1 the tangential system $\mathscr{N}_2$ of the Pearcey system $\mathscr{M}$ to the hyperplane

(0.5) $\quad Y_2 = \{(x_1, x_2) \in \mathbb{C}^2; x_1 = A_1, \text{ where } A_1 \text{ is a non-zero constant } \}.$

Contrary to the restriction of $\mathscr{M}$ to the hyperplane Y_1, no virtual turning point appears in the study of $\mathscr{N}_2$, but another intriguing feature is observed. In the Stokes geometry of $\mathscr{N}_2$ we find that three ordinary Stokes curves meet at a point and that this degeneration continues to be observed even when the parameter A_1 is changed. Then the following question naturally arises: What happens for the tangential system $\mathscr{N}(c)$ of $\mathscr{M}$ to the hyperplane

(0.6) $\quad Y(c) \underset{\text{def}}{=} \{(x_1, x_2) \in \mathbb{C}^2; x_2 = c(x_1 - 1), \text{ where } c \text{ is a non-zero constant } \}?$

In Section 2 we show that in the Stokes geometry of $\mathscr{N}(c)$ there exist not only an s-virtual turning point but also a non-hereditary turning point (abbreviated as NHTP). As is emphasized in [H3] a NHTP is an ordinary (versus virtual) turning point which is generically innocent in the Stokes geometry in the sense that it does not cause any Stokes phenomena of WKB solutions. Although a NHTP is introduced in [H3] by microlocal analysis applied to the tangential system $\mathscr{N}(c)$, we introduce the same notion by restricting the integral representation (0.2) of a solution of the Pearcey system to the hyperplane $Y(c)$.

§ 1. The Stokes geometry of the tangential system $\mathscr{N}_2$

§ 1.1. A review of the definition of an s-virtual turning point

For the sake of the convenience of the reader, we first recall the definition of an s-virtual turning point, which we usually abbreviate as s-VTP. See [HKT, Section 1.8] for the details. We also explain how it is theoretically related to the notion of a virtual turning point. (Cf. [HKT, Section 1.4].)

An s-VTP is defined in a very restricted situation when the Borel transform $\psi_B(x,y)$ of an integral $\psi(x,\eta)$ with a large parameter η given by

$$\int \exp(\eta\varphi(x,t))dt \tag{1.1.1}$$

satisfies a maximally over-determined system $\mathscr{L}$ of linear differential equations, which we abbreviate as MOS in what follows. (See [SKK, Chap. II] for the definition and basic properties of a MOS.) To guarantee the existence of $\mathscr{L}$, we assume the following:

(1.1.2) $\varphi(x,t)$ is a polynomial of (x,t) in $\mathbb{C}^n_x \times \mathbb{C}^p_t$,

(1.1.3) $\varpi : \{(x,t) \in \mathbb{C}^n_x \times \mathbb{C}^p_t;\ \partial\varphi/\partial t_1 = \cdots = \partial\varphi/\partial t_p = 0\} \to \mathbb{C}^n_x$ is a finite proper mapping.

We note that for the Pearcey integral (0.2) the condition (1.1.3) is satisfied. It is also satisfied for $\varphi|_{Y_1}, \varphi|_{Y_2}$ and $\varphi|_{Y(c)}$.

When the condition (1.1.3) is satisfied, the general theory of a MOS guarantees that the Borel transform $\psi_B(x,y)$ of $\psi(x,\eta)$, i.e.,

$$\psi_B(x,y) = \int \delta(y + \varphi(x,t))dt \tag{1.1.4}$$

satisfies a MOS whose characteristic variety is contained (outside the zero-section) by

$$V = \{(x,y;\xi,\eta) \in T^*\mathbb{C}^{n+1}_{(x,y)};\ y + \varphi(x,t) = 0 \text{ and } (\xi,\eta) = \alpha(\mathrm{grad}_x\varphi(x,t),1)\ \ (\alpha \neq 0) \text{ with } \mathrm{grad}_t\varphi(x,t) = 0\}. \tag{1.1.5}$$

Since all singularities of a solution of a MOS are cognate in (a connected component of) its characteristic variety, a self-intersection point of its projection $\pi(V)$ to the base manifold $\mathbb{C}^{n+1}_{(x,y)}$ plays the same role as a self-intersection point of the projection $\pi(\mathscr{b})$ of a bicharacteristic strip $\mathscr{b}$, the carrier of singularities of a solution. Thus we are led to introduce the following

Definition 1.1.1. Let ψ be the integral given by (1.1.1), and assume that the conditions (1.1.2) and (1.1.3) are satisfied. Then a point x_0 is said to be an s-virtual turning point (abbreviated as s-VTP) of the system of differential equations that ψ satisfies if there exist t' and t'' $(t' \neq t'')$ in $\mathbb{C}^p_t$ for which the following conditions are satisfied:

$$\varphi(x_0,t') = \varphi(x_0,t''), \tag{1.1.6}$$

$$(\mathrm{grad}_t\varphi)(x_0,t') = (\mathrm{grad}_t\varphi)(x_0,t'') = 0. \tag{1.1.7}$$

Remark 1.1.1. We are using here $\exp(\eta y)$, not $\exp(-\eta y)$, in the definition of $\psi_B(x,y)$ (cf. [HKT, (1.8.1)]) and this is the reason for the change of S in [HKT, (1.8.12)] to $-\varphi$ in the above discussion. Although no change is necessary in the actual computation, we note this to avoid the possible confusion of the reader.

Remark 1.1.2. In a neighborhood ω of an s-VTP x_0, it follows from the assumption (1.1.3) that we can find continuous functions $t^{(j)}(x)$ and $t^{(k)}(x)$ for which

$$(\mathrm{grad}_t\varphi)(x,t^{(j)}(x)) = (\mathrm{grad}_t\varphi)(x,t^{(k)}(x)) = 0 \tag{1.1.8}$$

and

$$t^{(j)}(x_0) = t' \ \text{ and } \ t^{(k)}(x_0) = t'' \tag{1.1.9}$$

hold. Hence a (new) Stokes surface emanating from x_0 is given, by definition, by

$$\mathrm{Im}\big(\varphi(x,t^{(j)}(x))\big) = \mathrm{Im}\big(\varphi(x,t^{(k)}(x))\big) \tag{1.1.10}$$

near x_0.

Remark 1.1.3. A counterpart of an s-VTP in the real category is known as a Maxwell set among geometers (e.g. [PS]), and it is imagined to be somehow related to WKB analysis ([W]). The theory of a MOS applied to the Borel transform $\psi_B(x,y)$ of the integral ψ given by (1.1.1) has thus elucidated how the notion of a Maxwell set is related to the asymptotic analysis.

§ 1.2. Non-existence of an s-virtual turning point for $\mathscr{N}_2$

We first show that $\mathscr{N}_2$ does not admit an s-VTP. In this subsection we let x denote x_2 and we denote by $\varphi(x,t)$ the polynomial

$$t^4 + xt^2 + A_1 t. \tag{1.2.1}$$

Let (α,β,γ) denote the roots of

$$\varphi_t = 4t^3 + 2xt + A_1 = 0. \tag{1.2.2}$$

Suppose that (α,β) determines an s-VTP x_0, that is, we have

$$4\alpha^3 + 2x_0\alpha + A_1 = 0, \tag{1.2.3}$$

$$4\beta^3 + 2x_0\beta + A_1 = 0, \tag{1.2.4}$$

$$\varphi(x_0,\alpha) = \varphi(x_0,\beta). \tag{1.2.5}$$

Then, by assuming $\alpha \neq \beta$, we find, by (1.2.3) and (1.2.4), that

$$4(\alpha^2 + \alpha\beta + \beta^2) + 2x_0 = 0, \tag{1.2.6}$$

and, by (1.2.5), that

$$(\alpha^2 + \beta^2)(\alpha + \beta) + x_0(\alpha + \beta) + A_1 = 0. \tag{1.2.7}$$

On the other hand, we find by (1.2.2)

$$\alpha + \beta = -\gamma \quad \text{and} \quad \alpha\beta\gamma = -\mathrm{A}_1/4, \tag{1.2.8}$$

and hence

$$4\alpha\beta(\alpha + \beta) = A_1. \tag{1.2.9}$$

Since $A_1 \neq 0$ by the assumption, we have

$$\alpha + \beta \neq 0. \tag{1.2.10}$$

By combining (1.2.7) and (1.2.9), we obtain

$$(\alpha + \beta)[(\alpha^2 + \beta^2) + x_0 + 4\alpha\beta] = 0. \tag{1.2.11}$$

Then by (1.2.10) we have

$$\alpha^2 + \beta^2 + x_0 + 4\alpha\beta = 0. \tag{1.2.12}$$

Substituting (1.2.6) into (1.2.12), we obtain

$$0 = \alpha^2 + \beta^2 + 4\alpha\beta - 2(\alpha^2 + \alpha\beta + \beta^2) = -(\alpha - \beta)^2. \tag{1.2.13}$$

Since α should be different from β at an s-VTP, we conclude that $\mathscr{N}_2$ does not admit an s-VTP.

§ 1.3. Non-existence of a virtual turning point for $\mathscr{N}_2$

As the above reasoning based on the new notion "s-VTP" might sound somewhat tricky, we confirm the non-existence of a (traditional (i.e., defined by [AKT])) virtual turning point for $\mathscr{N}_2$. Since reasoning in this subsection is very concrete, it will manifest some peculiar feature of $\mathscr{N}_2$ from the viewpoint of microlocal analysis.

To begin with, let us describe the characteristic equation of the Borel transform of $\mathscr{N}_2$, which is found by setting $x_1 = A_1$ and eliminating ξ_1 in the characteristic equation of the Borel transform of the Pearcey system $\mathscr{M}$, that is,

$$\begin{cases} 4(\xi_1/\eta)^3 + 2x_2\xi_1/\eta + x_1 = 0 \\ (\xi_2/\eta) - (\xi_1/\eta)^2 = 0. \end{cases} \tag{1.3.1}$$

Here η is identified with the principal symbol $\sigma_1(\partial/\partial y)$, where y denotes the variable dual to η through the Borel transformation. In what follows we always assume η to be different from 0. Hence the required characteristic equation is given by

(1.3.2) $$Q = q_+(x_2, \xi_2)q_-(x_2, \xi_2) = 0,$$

where

(1.3.3) $$q_\pm(x_2, \xi_2) = \sqrt{\xi_2/\eta}\left(\xi_2/\eta + \frac{1}{2}x_2\right) + \frac{1}{4}A_1.$$

Here we note that, if q_+ vanishes, then q_- is different from 0 as $A_1 \neq 0$. Hence in what follows we compute the bicharacteristic strip for q_+. For the sake of simplicity of the notation we use $(z, \zeta; A)$ to denote $(x_2, \xi_2; A_1/4)$ and we let q denote

(1.3.4) $$\sqrt{\zeta}\left(\zeta + \frac{\eta}{2}z\right) + A\eta^{3/2}.$$

Thus the bicharacteristic equation we have in mind is the following:

(1.3.5) $$\begin{cases} \dfrac{dz}{dr} = \dfrac{3}{2}\zeta^{1/2} + \dfrac{\eta}{4}z\zeta^{-1/2} & \text{(1.3.5.a)} \\[2ex] \dfrac{dy}{dr} = \dfrac{1}{2}z\zeta^{1/2} + \dfrac{3}{2}A\eta^{1/2} & \text{(1.3.5.b)} \\[2ex] \dfrac{d\zeta}{dr} = -\dfrac{1}{2}\zeta^{1/2}\eta & \text{(1.3.5.c)} \\[2ex] \dfrac{d\eta}{dr} = 0. & \text{(1.3.5.d)} \end{cases}$$

In view of (1.3.5.d) we set

(1.3.6) $$\eta = 1.$$

Then we find by (1.3.5.c)

(1.3.7) $$\zeta^{1/2} = -\frac{1}{4}(r + \alpha),$$

where

(1.3.8) $$\zeta(r = 0) \underset{\text{def}}{=} \zeta_0 = \alpha^2/16.$$

Combining (1.3.7) and (1.3.5.a), we have

(1.3.9) $$\frac{dz}{dr} = \frac{3}{2}\zeta^{1/2} + \frac{1}{4}z\zeta^{-1/2} = -\frac{3}{8}(r + \alpha) - (r + \alpha)^{-1}z,$$

that is,

$$\Big[(r+\alpha)\frac{d}{dr}+1\Big]z=-\frac{3}{8}(r+\alpha)^2. \tag{1.3.10}$$

Hence we find

$$z(r)=-\frac{1}{8}(r+\alpha)^2+C(r+\alpha)^{-1} \tag{1.3.11}$$

for some constant C. Then substitution of (1.3.7) and (1.3.11) into $q=0$ entails

$$0=-\frac{1}{4}(r+\alpha)\Big[\frac{1}{16}(r+\alpha)^2-\frac{1}{16}(r+\alpha)^2+\frac{C}{2}(r+\alpha)^{-1}\Big]+A=-\frac{C}{8}+A, \tag{1.3.12}$$

that is,

$$C=8A. \tag{1.3.13}$$

Assuming that the bicharacteristic strip in question starts from a turning point $z(r=0)=z_0$ with the characteristic value $\zeta(r=0)=\zeta_0$, that is,

$$q_\zeta(z_0,\zeta_0)\Big|_{\eta=1}=0, \tag{1.3.14}$$

we find the following relation between A and α:

$$-\frac{3}{8}\alpha+\frac{1}{4}(-4\alpha^{-1})\Big(-\frac{\alpha^2}{8}+\frac{8A}{\alpha}\Big)=-\frac{1}{4}\alpha-\frac{8A}{\alpha^2}=0, \tag{1.3.15}$$

that is,

$$\alpha^3=-32A. \tag{1.3.16}$$

Further, by substituting (1.3.7) and (1.3.11) with (1.3.13) into (1.3.5.b), we obtain

$$\begin{aligned}\frac{dy}{dr}&=\frac{1}{2}\Big[-\frac{1}{8}(r+\alpha)^2+8A(r+\alpha)^{-1}\Big]\Big(-\frac{1}{4}\Big)(r+\alpha)+\frac{3}{2}A\\&=\frac{1}{64}(r+\alpha)^3+\frac{1}{2}A.\end{aligned} \tag{1.3.17}$$

Hence, by assuming

$$y(r=0)=0, \tag{1.3.18}$$

we find

$$y(r)=\frac{1}{256}(r+\alpha)^4-\frac{\alpha^3}{64}(r+\alpha)+\frac{3\alpha^4}{256}. \tag{1.3.19}$$

Let us now try to find a virtual turning point, i.e., try to find a self-intersection point of the projection to the base manifold $\mathbb{C}^2_{(y,z)}$ of the above bicharacteristic strip. We let ρ (resp., $\tilde{\rho}$) denote $r+\alpha$ (resp., $\tilde{r}+\alpha$). Then

$$z(r) = z(\tilde{r}) \qquad (r \neq \tilde{r}) \tag{1.3.20}$$

reads as

$$\rho^2 + 2\alpha^3\rho^{-1} = \tilde{\rho}^{\,2} + 2\alpha^3\tilde{\rho}^{\,-1}, \tag{1.3.21}$$

which entails

$$\rho\tilde{\rho}\,(\rho+\tilde{\rho}) = 2\alpha^3. \tag{1.3.22}$$

Similarly

$$y(r) = y(\tilde{r}) \qquad (r \neq \tilde{r}) \tag{1.3.23}$$

implies

$$(\rho+\tilde{\rho})(\rho^2+\tilde{\rho}^{\,2}) = 4\alpha^3. \tag{1.3.24}$$

It then follows from (1.3.22) and (1.3.24) that we have

$$(\rho+\tilde{\rho})^3 = 8\alpha^3. \tag{1.3.25}$$

Denoting $\exp(2\pi i/3)$ by ω, we then find

$$\rho+\tilde{\rho} = 2\omega^j\alpha \qquad (j=0,1,2). \tag{1.3.26}$$

Then (1.3.22) implies

$$\rho\tilde{\rho} = \omega^{-j}\alpha^2 = \omega^{2j}\alpha^2. \tag{1.3.27}$$

Hence we find

$$\rho = \tilde{\rho} = \omega^j\alpha. \tag{1.3.28}$$

This contradicts the assumption $\rho \neq \tilde{\rho}$. Therefore we conclude that there exists no virtual turning point for $\mathscr{N}_2$.

Remark 1.3.1. We note the reasoning and the conclusion are exactly the same if we use q_- instead of q in the above discussion.

§ 1.4. The concrete figure of the Stokes geometry of $\mathscr{N}_2$ and its relevance to the behavior of the bicharacteristic strips

We begin our discussion in this subsection by showing the concrete figure of the Stokes geometry of $\mathscr{N}_2$, which is described with the help of a computer.

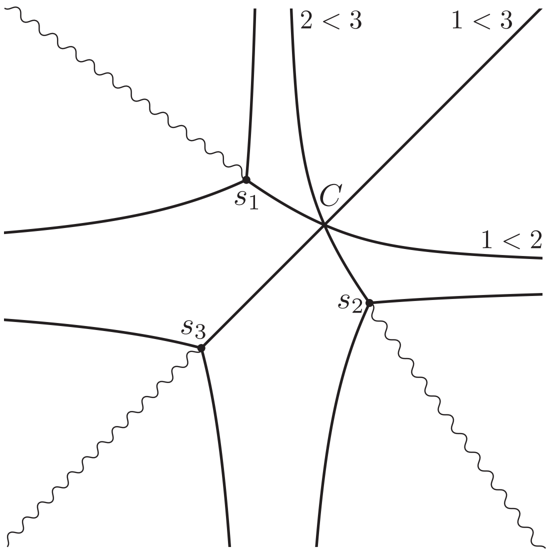

Figure 1.4.1 : The Stokes geometry of $\mathcal{N}_2$ with $A_1 = \exp(3\pi\sqrt{-1}/8)$. The wiggly lines designate the cut, and the label ($j < k$) indicates the order relation of the Stokes curve to which the label is attached; the indices in the labels are fixed by the appropriate cuts shown by the wiggly lines, and s_j ($j = 1, 2, 3$) designates a simple turning point.

In Figure 1.4.1 one observes three ordinary Stokes curves meet at a point C, and this is an inevitable situation in view of the non-existence of a virtual turning point; if the Stokes curve emanating from s_3 should fail to pass through the crossing point of the Stokes curve emanating from s_1 and that from s_2, we should be embarrassed how to deal with the ordered crossing point (in the sense of [HKT, Definition 1.4.2]) which is formed by the Stokes curves emanating from s_1 and s_2. Furthermore the explicit description of the bicharacteristic strip $\mathscr{b}$ emanating from a turning point (z_0, ζ_0), say s_1, shows the following

Fact A. $(y(r), z(r), \zeta(r))$ in the above description is single-valued. (Cf. (1.3.7), (1.3.11) [together with (1.3.13) and (1.3.16)] and (1.3.19).)

Fact B. The projection $b = \pi(\mathscr{b})$ of the bicharacteristic strip $\mathscr{b}$ to the base manifold, that is, the bicharacteristic curve passing through s_1 also passes through s_2 and s_3, and then comes back to s_1.

Fact C. In parallel with **Fact B** the bicharacteristic strip $\mathscr{b}$ emanating from (y_0, z_0, ζ_0) (with $\eta = 1$) comes back to the same point when its projection $\pi(\mathscr{b})$ makes a journey passing through s_2 and s_3 and coming back to s_1.

To confirm **Fact B** and **Fact C**, it suffices to choose the parameter r to be

$$r_1 = \omega\alpha - \alpha \tag{1.4.1}$$

and

$$r_2 = \omega^2\alpha - \alpha, \tag{1.4.2}$$

where $\omega = \exp(2\pi i/3)$ and α is chosen so that $z_0 = z(r = r_0 = 0)$ may be given by

$$-3\alpha^2/8. \tag{1.4.3}$$

We note

$$z(r = r_1) = -3\omega^2\alpha^2/8, \tag{1.4.4}$$

$$z(r = r_2) = -3\omega\alpha^2/8. \tag{1.4.5}$$

By those facts, we find

$$\int_{s_1}^{C} (\zeta_1 - \zeta_2)dz + \int_{s_2}^{C} (\zeta_2 - \zeta_3)dz = \int_{s_3}^{C} (\zeta_1 - \zeta_3)dz, \tag{1.4.6}$$

where $\big(\zeta_1(z), \zeta_2(z), \zeta_3(z)\big)$ are characteristic roots of $q(z,\zeta)|_{\eta=1} = 0$ labeled in accordance with the cuts shown in Figure 1.4.1 and the paths of integration in (1.4.6) are chosen (rather freely thanks to **Fact A**) as in Figure 1.4.2 below.

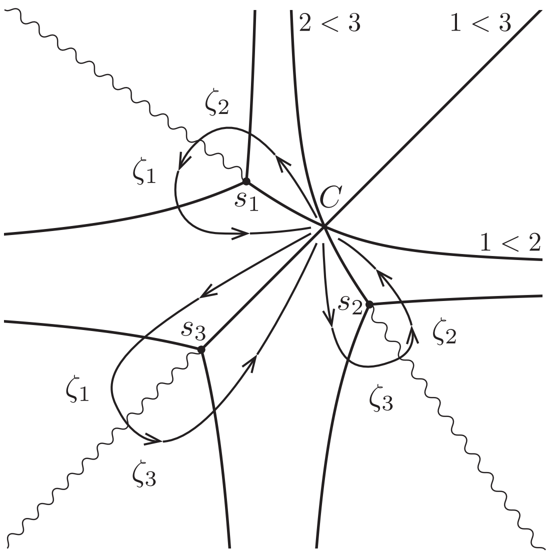

Figure 1.4.2

By comparing (1.4.6) with [HKT, (1.4.32) and (1.6.4)] we find that s_3 plays the same role, so to speak, as a virtual turning point in resolving the trouble caused by an ordered crossing point of Stokes curves. Actually, in the current situation, not only the bicharacteristic curve but also the bicharacteristic strip itself have self-intersection points, whose projection is s_3. We note that clearly s_3 is not an s-VTP by its definition requiring $t' \neq t''$ (cf. Definition 1.1.1). We also note that in the definition of a (traditional) virtual (versus ordinary) turning point we implicitly assume the relevant characteristic roots are mutually distinct at a virtual turning point. With this implicit understanding, s_3 is not a virtual turning point.

Remark 1.4.1. The stability (with respect to the parameter A) of the degeneration of the Stokes geometry shown in Figure 1.4.1 is evident from the fact that no virtual turning point appears there. Interestingly enough, however, such a degeneration with local stability can be found, as we will discuss in our subsequent article ([HiKT1]).

Remark 1.4.2. As we will show in [HiKT1] the Stokes curve emanating from s_3 actually turns out to be inert after it passes over the point C. To confirm this we analyze solutions of the non-linear equations associated with the Pearcey system (i.e., the counterpart of the Riccati equation associated with the Schrödinger equation) together with the connection formula formulated by Sasaki ([S, Section 2]).

§ 2. The Stokes geometry of the tangential system $\mathscr{N}(c)$

§ 2.1. An s-virtual turning point of $\mathscr{N}(c)$

To locate the s-VTP of $\mathscr{N}(c)$, the tangential system of $\mathscr{M}$ to the hyperplane $Y(c)$ given by (0.6), we consider the restriction $\varphi(c; x_1, t)$ of $\varphi(x, t)$ to $Y(c)$, where

$$\varphi(x, t) = t^4 + x_2 t^2 + x_1 t, \tag{2.1.1}$$

that is

$$\varphi(c; x_1, t) = t^4 + c(x_1 - 1)t^2 + x_1 t. \tag{2.1.2}$$

In this subsection we abbreviate x_1 to x, that is,

$$\varphi(c; x, t) = t^4 + c(x - 1)t^2 + xt. \tag{2.1.3}$$

Then

$$\psi(c; x, \eta) = \int \exp\big(\eta\varphi(c; x, t)\big)dt \tag{2.1.4}$$

satisfies the tangential system $\mathscr{N}(c)$, and we use $\varphi(c; x, t)$ to locate the s-VTP of $\mathscr{N}(c)$. Let us first consider the equation

$$\varphi_t(c; x, t) = 4t^3 + 2c(x - 1)t + x = 0, \tag{2.1.5}$$

and let (α, β, γ) denote the solutions of (2.1.5). Suppose that the pair of solutions (α, β) determines an s-VTP, that is,

$$\varphi(c; x, \alpha(x)) = \varphi(c; x, \beta(x)), \tag{2.1.6}$$

where (α, β) satisfy (2.1.5), that is,

$$4\alpha^3 + 2c(x-1)\alpha + x = 0 \tag{2.1.7}$$

and

$$4\beta^3 + 2c(x-1)\beta + x = 0, \tag{2.1.8}$$

with the additional condition

$$\alpha \neq \beta. \tag{2.1.9}$$

Combining (2.1.7) and (2.1.8) with (2.1.9), we find

$$4(\alpha^2 + \alpha\beta + \beta^2) + 2c(x-1) = 0. \tag{2.1.10}$$

Similarly, by using (2.1.6) and (2.1.9), we obtain

$$(\alpha^2 + \beta^2)(\alpha + \beta) + c(x-1)(\alpha + \beta) + x = 0. \tag{2.1.11}$$

We also note that, since (α, β, γ) satisfy (2.1.5), we have

$$\alpha + \beta = -\gamma, \quad \alpha\beta\gamma = -x/4. \tag{2.1.12}$$

Hence we find

$$x = 4\alpha\beta(\alpha + \beta). \tag{2.1.13}$$

It then follows from (2.1.11) and (2.1.13) that

$$(\alpha^2 + \beta^2)(\alpha + \beta) + 4[c(\alpha + \beta) + 1]\alpha\beta(\alpha + \beta) - c(\alpha + \beta) = 0. \tag{2.1.14}$$

Let M denote

$$\alpha^2 + \beta^2 + 4c\alpha\beta(\alpha + \beta) + 4\alpha\beta - c. \tag{2.1.15}$$

Then, again by (2.1.13), we obtain

$$M = \alpha^2 + \beta^2 + 4\alpha\beta + c(x-1). \tag{2.1.16}$$

It then follows from (2.1.16) and (2.1.10) that we have

$$M = \alpha^2 + \beta^2 + 4\alpha\beta - 2(\alpha^2 + \alpha\beta + \beta^2) = -(\alpha - \beta)^2. \tag{2.1.17}$$

Since $M \neq 0$ by (2.1.9), (2.1.14) and (2.1.15) imply

$$\alpha + \beta = 0. \tag{2.1.18}$$

Hence (2.1.13) entails that the s-VTP in question is given by

$$x = 0. \tag{2.1.19}$$

Remark 2.1.1. In order to help the understanding of the role of (2.1.9) in the definition of an s-VTP, we present the following discussion: Suppose $\alpha(x)$ and $\beta(x)$ merge at $x_\sharp$. Then, in view of (1.1.5) applied to $\varphi = \varphi(c; x_1, t)$, we find that $\xi = c\alpha(x)^2 + \alpha(x)$ and $\xi' = c\beta(x)^2 + \beta(x)$ are two characteristic vectors the MOS involved, i.e., $\mathscr{N}(c)$, which is actually an ordinary differential equation; that is, two characteristic roots merge at $x = x_\sharp$. Thus $x_\sharp$ is an ordinary turning point. It is clear that

$$(\alpha, \beta) = (\alpha(x_\sharp), \beta(x_\sharp)) \tag{2.1.20}$$

satisfies (1.1.6), (1.1.7) and (1.1.8), but it does not satisfy (1.1.9). Hence it is not an s-VTP by its definition.

Remark 2.1.2. It is evident $Y(c)$ approaches to a hypersurface parallel to $Y_2 = \{x_1 = A_1\}$ as c tends to infinity. Then a natural question is: "What will be observed concerning the behavior of the s-VTP $\{x = 0\}$ in $Y(c)$ as c tends to infinity?" To answer this question, let us compute the value

$$\varphi(c; 0, \alpha(0)) = \varphi(c; 0, \beta(0)). \tag{2.1.21}$$

Since $\gamma(0)$ is equal to 0, $\alpha(0)$ and $\beta(0)$ are different from 0. Hence (2.1.7) and (2.1.8) imply

$$4\alpha(0)^2 - 2c = 4\beta(0)^2 - 2c = 0. \tag{2.1.22}$$

Therefore we have

$$\varphi(c; 0, \alpha(0)) = \varphi(c; 0, \beta(0)) = \left(\frac{c}{2}\right)^2 - \frac{c^2}{2} = -\frac{c^2}{4}. \tag{2.1.23}$$

Hence in the variety V given by (1.1.5) the y-component associated with the s-VTP $x = 0$ tends to infinity. Thus by considering not only the x-component but also the y-component in V we find the result in this section is consistent with the result in Section 1.2, i.e., the non-existence of an s-VTP in $\mathscr{N}_2$.

§ 2.2. A non-hereditary turning point of $\mathscr{N}(c)$

As is studied in [H3], a non-hereditary turning point (abbreviated as NHTP) is an ordinary turning point which appears in a tangential system. Here we give another explanation of the origin of a NHTP, which is, from the logical viewpoint, different from the reasoning in [H3]; we make a straightforward use of the characteristic variety of $\mathscr{N}(c)$, which is given by (1.1.5) with $\varphi(x,t)$ being given by $\varphi(c;x,t)$ given in (2.1.3), that is,

$$\varphi(c;x,t) = t^4 + c(x-1)t^2 + xt. \tag{2.2.1}$$

Making use of (1.1.5) we immediately find a point x_* is an ordinary turning point if two cotangent vectors $(\xi, 1) = \big(\mathrm{grad}_x \varphi(x, t(x)), 1\big)$ and $(\xi', 1) = \big(\mathrm{grad}_x \varphi(x, t'(x)), 1\big)$ given in (1.1.5) merge at $x = x_*$, that is, if there exist continuous functions $t(x)$ and $t'(x)$ which satisfy

$$\xi = \mathrm{grad}_x \varphi(x_*, t(x_*)) = \mathrm{grad}_x \varphi(x_*, t'(x_*)) = \xi', \tag{2.2.2}$$

with

$$(\mathrm{grad}_t \varphi)(x, t(x)) = (\mathrm{grad}_t \varphi)(x, t'(x)) = 0. \tag{2.2.3}$$

Now, a non-hereditary turning point x_* is, by definition, supposed to further satisfy the following additional condition:

$$t(x_*) \neq t'(x_*). \tag{2.2.4}$$

Remark 2.2.1. We will confirm in Remark 2.3.1 in the next subsection that the above definition coincides with the definition which is given ([H3]) through the relation of the characteristic variety of the Pearcey system and that of its tangential system $\mathscr{N}(c)$ on $Y(c)$. Here we note that the condition (2.2.4) implies that the saddle points $(x_*, t(x_*))$ and $(x_*, t'(x_*))$ of the integral (2.1.4) are distinct.

Now, let us concretely locate the NHTP in this case. The relation (2.2.2) entails

$$ct(x_*)^2 + t(x_*) = ct'(x_*)^2 + t'(x_*). \tag{2.2.5}$$

Then, by (2.2.4), we find

$$c\big(t(x_*) + t'(x_*)\big) + 1 = 0. \tag{2.2.6}$$

Since both $t = t(x_*)$ and $t = t'(x_*)$ satisfy

$$4t^3 + 2c(x_* - 1)t + x_* = 0, \tag{2.2.7}$$

$$t = \tilde{t}(x_*) \underset{\mathrm{def}}{=} -\big(t(x_*) + t'(x_*)\big) = c^{-1} \tag{2.2.8}$$

should satisfy (2.2.7). Hence we find

$$4c^{-3}+2(x_*-1)+x_*=0. \tag{2.2.9}$$

Hence the required NHTP x_* is given by

$$(2-4c^{-3})/3. \tag{2.2.10}$$

Remark 2.2.2. The same reasoning as above shows that no NHTP exists for $\mathscr{N}_2$ because of the assumption $A_1 \neq 0$ in (0.5). Then, in parallel with Remark 2.1.2, it is an interesting problem to study the behavior, as c tends to infinity, of the cotangent vector $\xi = \mathrm{grad}_x \varphi(c; x, t(x))$ with $\mathrm{grad}_t \varphi(c; x, t(x)) = 0$, which is used to define the NHTP x_*. First we note

$$t(x_*)+t'(x_*)=-c^{-1}. \tag{2.2.6$'$}$$

Further we find by (2.2.7), (2.2.8) and (2.2.10) the following:

$$t(x_*)t'(x_*)\tilde{t}(x_*)=-x_*/4 \tag{2.2.11}$$

and

$$t(x_*)t'(x_*)=-cx_*/4. \tag{2.2.12}$$

Hence it follows from (2.2.10) that we have

$$t(x_*)t'(x_*)=(2c^{-2}-c)/6. \tag{2.2.13}$$

Thus we find that $t(x_*)$ and $t'(x_*)$ are solutions of the following equation:

$$T^2+c^{-1}T+(2c^{-2}-c)/6=0. \tag{2.2.14}$$

Then, as $\xi = ct(x)^2 + t(x)$, we find at the NHTP x_* the following:

$$\xi=(-t(x_*)+(c^2-2c^{-1})/6)+t(x_*)=(c^2-2c^{-1})/6. \tag{2.2.15}$$

Hence ξ tends to infinity as c tends to infinity. This behavior of the cotangent vector associated with the NHTP x_* is clearly consistent with the observation that no NHTP appears in $\mathscr{N}_2$.

§ 2.3. A computation of a bicharacteristic strip of $\mathscr{N}(c)$ with $c = 1$

In this subsection we present a computation of a bicharacteristic strip of the tangential system $\mathscr{N}(c)$ of $\mathscr{M}$ on $Y(c)$.

To find the explicit form of the characteristic equation of the tangential system of the Borel transform of $\mathscr{M}$ on $Y(c)$, we first introduce the coordinate change from x to z:

$$z_1 = x_1, \tag{2.3.1}$$

$$z_2 = x_2 - c(x_1 - 1). \tag{2.3.2}$$

Then we have

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z_1} - c\,\frac{\partial}{\partial z_2}, \tag{2.3.3}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial z_2}. \tag{2.3.4}$$

In what follows we let θ_j $(j = 1, 2)$ respectively denote

$$\sigma_0\big((\partial/\partial z_j)/(\partial/\partial y)\big) \quad (j = 1, 2). \tag{2.3.5}$$

Then in this coordinate system the characteristic variety of the Borel transform of $\mathscr{M}$ is given by

$$(2.3.6)\quad \begin{cases} 4(\theta_1 - c\theta_2)^3 + 2(z_2 + c(z_1 - 1))(\theta_1 - c\theta_2) + z_1 = 0 & (2.3.6.a) \\ \theta_2 - (\theta_1 - c\theta_2)^2 = 0. & (2.3.6.b) \end{cases}$$

Hence on $Y(c)$ we have

$$(2.3.7)\quad \begin{cases} 4(\theta_1 - c\theta_2)^3 + 2c(z_1 - 1)(\theta_1 - c\theta_2) + z_1 = 0 & (2.3.7.a) \\ \theta_2 - (\theta_1 - c\theta_2)^2 = 0. & (2.3.7.b) \end{cases}$$

Remark 2.3.1. For the sake of the convenience of the reader, we note the content in the current situation of the definition of a non-hereditary turning point $z_1^{(0)}$ given in [H3]: If there exist two solutions $(\theta_1(z_1), \theta_2(z_1))$ and $(\theta_1'(z_1), \theta_2'(z_1))$ of (2.3.7) for which

$$\theta_1(z_1^{(0)}) = \theta_1'(z_1^{(0)}) \tag{2.3.8}$$

and

$$\theta_2(z_1^{(0)}) \neq \theta_2'(z_1^{(0)}) \tag{2.3.9}$$

are satisfied, then $z_1^{(0)}$ is, by definition, a non-hereditary turning point. To see the relation of this notion and the definition of NHTP in Section 2.2, we first note that the condition (2.2.3) allows us to assume without loss of generality that, with the identification $x = z_1$, we have

$$t(x) = \theta_1(z_1) - c\theta_2(z_1), \tag{2.3.10}$$

$$t'(x) = \theta_1'(z_1) - c\theta_2(z_1). \tag{2.3.11}$$

Then (2.3.7.b) entails

$$\theta_2(z_1) = t(x)^2 \quad \text{and} \quad \theta_2'(z_1) = t'(x)^2. \tag{2.3.12}$$

Hence (2.3.8) can be rewritten as

$$t(z_1^{(0)}) + ct(z_1^{(0)})^2 = t'(z_1^{(0)}) + ct'(z_1^{(0)})^2. \tag{2.3.13}$$

This relation coincides with the characterization (2.2.5) of a NHTP x_*. Further (2.3.8) and (2.3.9), together with (2.3.10) and (2.3.11), entail

$$t(z_1^{(0)}) \neq t'(z_1^{(0)}). \tag{2.3.14}$$

This coincides with (2.2.4). Thus we have seen that the notion of a NHTP introduced in Section 2.2 and that given in [H3] are the same.

For the sake of simplicity of presentation, we assume $c = 1$ in what follows. Then our first task is to eliminate θ_2 in (2.3.7). By substituting (2.3.7.b) into (2.3.7.a) with $c = 1$, we find

$$4\theta_2(\theta_1 - \theta_2) + 2(z_1 - 1)(\theta_1 - \theta_2) + z_1 = 0. \tag{2.3.15}$$

By rewriting (2.3.7.b) as

$$\theta_2^2 = (2\theta_1 + 1)\theta_2 - \theta_1^2 \tag{2.3.16}$$

we obtain from (2.3.15) and (2.3.16)

$$-4\theta_1\theta_2 - 4\theta_2 - 2(z_1 - 1)\theta_2 + 4\theta_1^2 + 2(z_1 - 1)\theta_1 + z_1 = 0, \tag{2.3.17}$$

and hence, by using

$$\theta_2 = \frac{1}{2}\left[(2\theta_1 + 1) \pm \sqrt{4\theta_1 + 1}\right], \tag{2.3.18}$$

we conclude

$$4\theta_1^2 + 2(z_1 - 1)\theta_1 + z_1 = (2\theta_1 + 1 + z_1)(2\theta_1 + 1 \pm 2\sqrt{4\theta_1 + 1}). \tag{2.3.19}$$

A straightforward computation then shows (2.3.19) results in

$$-6\theta_1 - 1 = \pm(2\theta_1 + 1 + z_1)\sqrt{4\theta_1 + 1}\,. \tag{2.3.20}$$

Thus we have eliminated θ_2 in (2.3.7). In what follows we abbreviate (z_1, θ_1) to (z, θ), and we fix the branch of $\sqrt{4\theta+1}$ by choosing

$$\sqrt{4\theta+1}\,\Big|_{\theta=0} = 1 \tag{2.3.21}$$

with the cut

$$\{\theta \in \mathbb{C};\ \operatorname{Re}\theta \le -\frac{1}{4}, \operatorname{Im}\theta = 0\}. \tag{2.3.22}$$

Then we let q_+ (resp., q_-) denote

$$6\theta + 1 + (2\theta + 1 + z)\sqrt{4\theta+1} \quad \big(\text{resp., } 6\theta + 1 - (2\theta + 1 + z)\sqrt{4\theta+1}\,\big), \tag{2.3.23}$$

and define

$$Q = q_+ q_-. \tag{2.3.24}$$

By our computation presented in the above we find that the characteristic variety of $\mathscr{N}(c=1)$ is given by

$$Q = 0. \tag{2.3.25}$$

Now, at $(z_0, \theta_0) \underset{\text{def}}{=} (-2/3, -1/6)$, both q_+ and q_- vanish, and it follows from the concrete expression (2.3.23) that each of them is with simple characteristics. Hence we can compute the bicharacteristic strip $\mathscr{b}$ of, say $q \underset{\text{def}}{=} q_-$, that emanates from (z_0, θ_0). The purpose of this subsection is to study the relation of a self-intersection point of $\mathscr{b}$ and the s-VTP of $\mathscr{N}(c)$ discussed in Section 2.1. The effect on the global structure of $\mathscr{N}(c)$ of the multiple characteristic character of Q at (z_0, θ_0) will be studied in our forthcoming article ([HiKT2]). Incidentally we note that $z_0 = -2/3$ is the NHTP given by (2.2.10) with $c = 1$.

Now, the differential equation that describes the bichracteristic strip $\mathscr{b}$ for $q = q_-$ is given by (2.3.26) given below. There we let (ζ, η) denote $(\sigma_1(\partial/\partial z),\ \sigma_1(\partial/\partial y))$ and hence $\theta = \zeta/\eta$. To avoid the possible confusion of the reader, we note that the degree of q with respect to the cotangent vector θ is kept to be 0. (In Section 1.3, the counterpart of q in the above is multiplied by $\eta^{3/2}$ to simplify the computation; this time it does not seem to be of much help.)

$$
(2.3.26)\quad \begin{cases} \dfrac{dz}{d\sigma} = \dfrac{\partial q}{\partial \zeta} = \eta^{-1}\big(6 - 2\sqrt{4\theta+1} - 2(2\theta+1+z)/\sqrt{4\theta+1}\,\big) & (2.3.26.a) \\[2ex] \dfrac{dy}{d\sigma} = \dfrac{\partial q}{\partial \eta} = -\zeta\eta^{-2}\big(6 - 2\sqrt{4\theta+1} - 2(2\theta+1+z)/\sqrt{4\theta+1}\,\big) & (2.3.26.b) \\[2ex] \dfrac{d\zeta}{d\sigma} = -\dfrac{\partial q}{\partial z} = \sqrt{4\theta+1} & (2.3.26.c) \\[2ex] \dfrac{d\eta}{d\sigma} = 0 & (2.3.26.d) \end{cases}
$$

with the initial data at $\sigma = 0$ is given by

$$
(2.3.27)\qquad \big(z(0), y(0);\, \zeta(0), \eta(0)\big) = (-2/3, 0;\, -1/6, 1).
$$

Then we find

$$
(2.3.28)\qquad \eta = 1
$$

and hence by (2.3.26.c)

$$
(2.3.29)\qquad \frac{d\theta}{d\sigma} = \sqrt{4\theta+1}.
$$

Let χ denote $\sqrt{4\theta+1}$, and then we have

$$
(2.3.30)\qquad \chi^2 = 4\theta + 1.
$$

Hence (2.3.29) entails

$$
(2.3.31)\qquad \chi\frac{d\chi}{d\sigma} = 2\frac{d\theta}{d\sigma} = 2\chi.
$$

Since $\chi(\sigma=0) = \sqrt{4\theta(0)+1} = 1/\sqrt{3}$, we may assume χ is different from 0 near $\sigma = 0$. Then we obtain

$$
(2.3.32)\qquad \frac{d\chi}{d\sigma} = 2,
$$

and hence

$$
(2.3.33)\qquad \chi = 2\sigma + (1/\sqrt{3}\,).
$$

Let α and $\tilde{\sigma}$ be respectively defined by

$$
(2.3.34)\qquad \alpha = 1/(2\sqrt{3}\,)
$$

and

$$\tilde{\sigma} = \sigma + \alpha. \tag{2.3.35}$$

Then (2.3.30) and (2.3.31) entail

$$\theta = \tilde{\sigma}^2 - (1/4). \tag{2.3.36}$$

It follows from (2.3.26.a) together with (2.3.28), (2.3.33), (2.3.35) and (2.3.36) that we find

$$\begin{aligned} \frac{dz}{d\sigma} = \frac{dz}{d\tilde{\sigma}} &= 6 - 2\chi - 2(2\theta + 1 + z)/\chi \\ &= 6 - 2\chi - (4\theta + 1 + 2z + 1)/\chi \\ &= 6 - 3\chi - (2z + 1)/\chi \\ &= 6 - 6\tilde{\sigma} - (z + (1/2))/\tilde{\sigma}. \end{aligned} \tag{2.3.37}$$

Thus we obtain

$$\tilde{\sigma}\frac{dz}{d\tilde{\sigma}} + z = -\frac{1}{2} + 6\tilde{\sigma} - 6\tilde{\sigma}^2. \tag{2.3.38}$$

Therefore $z(\tilde{\sigma})$ has the following form (2.3.39) with some constant C_{-1}:

$$z(\tilde{\sigma}) = C_{-1}\tilde{\sigma}^{-1} - \frac{1}{2} + 3\tilde{\sigma} - 2\tilde{\sigma}^2. \tag{2.3.39}$$

Since

$$z(\sigma = 0) = z(\tilde{\sigma} = \alpha) = -2/3, \tag{2.3.40}$$

we find

$$-\frac{2}{3} = C_{-1}\alpha^{-1} - \frac{1}{2} + 3\alpha - 2\alpha^2. \tag{2.3.41}$$

Thus we find

$$C_{-1} = -\frac{1}{4}. \tag{2.3.42}$$

Using this expression of $z(\sigma)$ together with (2.3.28), (2.3.33) and (2.3.36), we find by (2.3.26.b) the following:

$$\begin{aligned} \frac{dy}{d\sigma} = \frac{dy}{d\tilde{\sigma}} &= -6\theta + 2\theta\chi + \theta\big[(4\theta + 1) + (2z + 1)\big]/\chi \\ &= -6\theta + 3\theta\chi + \theta(2z + 1)/\chi \\ &= -6\tilde{\sigma}^2 + \frac{3}{2} + 6\tilde{\sigma}\big(\tilde{\sigma}^2 - \frac{1}{4}\big) + \big(\tilde{\sigma} - \frac{1}{4}\tilde{\sigma}^{-1}\big)\big(-\frac{1}{4}\tilde{\sigma}^{-1} + 3\tilde{\sigma} - 2\tilde{\sigma}^2\big) \\ &= 4\tilde{\sigma}^3 - 3\tilde{\sigma}^2 - \tilde{\sigma} + \frac{1}{2} + \frac{1}{16}\tilde{\sigma}^{-2}. \end{aligned} \tag{2.3.43}$$

Hence, by taking into account the initial condition

$$y(\sigma = 0) = y(\tilde{\sigma} = \alpha) = 0, \tag{2.3.44}$$

we have

$$y(\tilde{\sigma}) = \tilde{\sigma}^4 - \tilde{\sigma}^3 - \frac{1}{2}\tilde{\sigma}^2 + \frac{1}{2}\tilde{\sigma} - \frac{1}{16}\tilde{\sigma}^{-1} + \mathscr{A}, \tag{2.3.45}$$

where

$$\mathscr{A} = \frac{\sqrt{3}}{18} + \frac{5}{144}. \tag{2.3.46}$$

Now, let us confirm by using (2.3.39) and (2.3.45) that the bicharacteristic curve of q emanating from $(z(\sigma = 0), y(0))$ forms a self-intersection at $z = 0$. To see this let us compute $\big(z(\tilde{\sigma}_\pm), y(\tilde{\sigma}_\pm)\big)$ for

$$\tilde{\sigma}_\pm \underset{\text{def}}{=} \frac{1 \pm \sqrt{2}}{2}. \tag{2.3.47}$$

Here, and in what follows, the sign $\pm$ should be understood to be chosen correspondingly, that is, (2.3.47) means

$$\tilde{\sigma}_+ = \frac{1 + \sqrt{2}}{2} \tag{2.3.47$'$}$$

and

$$\tilde{\sigma}_- = \frac{1 - \sqrt{2}}{2}. \tag{2.3.47$''$}$$

With this understanding (2.3.47) entails the following:

$$\tilde{\sigma}_\pm^{-1} = -2(1 \mp \sqrt{2}\,), \tag{2.3.48}$$

$$\tilde{\sigma}_\pm^2 = \frac{3 \pm 2\sqrt{2}}{4}\,, \tag{2.3.49}$$

$$\tilde{\sigma}_\pm^3 = \frac{7 \pm 5\sqrt{2}}{8}\,, \tag{2.3.50}$$

and

$$\tilde{\sigma}_\pm^4 = \frac{17}{16} \pm \frac{3}{4}\sqrt{2}\,. \tag{2.3.51}$$

Hence we find

$$z(\tilde{\sigma}_\pm) = 0 \tag{2.3.52}$$

and

$$y(\tilde{\sigma}_\pm) = \mathscr{A} + \frac{3}{16}\,. \tag{2.3.53}$$

To stay on the safer side, we further note

$$\theta(\tilde{\sigma}_\pm) = \frac{1 \pm \sqrt{2}}{2}\,. \tag{2.3.54}$$

Thus we find that $\{z = 0\}$ is not an ordinary turning point but a (traditional) turning point given by the self-intersection point

$$(z, y) = \big(0,\, \mathscr{A} + \frac{3}{16}\big), \tag{2.3.55}$$

which is formed by a bicharacteristic curve emanating from $(z, y) = (-2/3, 0)$. Thus we have seen the s-VTP detected in Section 2.1 is also a virtual turning point in the traditional sense.

§ 3. A simple example of the change of the redundancy and the non-redundancy of a virtual turning point

As was first observed in [H2], it sometimes occurs that a redundant virtual turning point turns out to be a non-redundant one with the change of a parameter contained in the equation in question. In this section we show that the same phenomenon occurs in a much simpler situation, i.e., when we change the parameter c in the tangential system $\mathscr{N}(c)$ of $\mathscr{M}$ to $Y(c)$. Furthermore the finiteness of the number (actually, the uniqueness) of its s-virtual turning points implies that the redundancy observed here is not an experimental one (in the sense that within the limited screen shown by a computer) but a rather theoretical one (in the sense that the Stokes curve studied here cannot become active even out of the scope of the figure). We hope the following figures will also help the reader to imagine the Stokes geometry of $\mathscr{N}(c)$ in general.

Now, Figures 3.1, 3.3, 3.5 and 3.7 indicate the Stokes geometry of the system $\mathscr{N}(c)$ for $c = \exp(\pi j\sqrt{-1}/32)$ with $j = 24, 25, 26$ and 27, respectively. (Figures 3.2, 3.4, 3.6 and 3.8 are their enlarged versions.) In Figures 3.1 and 3.3 a new Stokes curve written in red (please see the colored figures in the preprint version of this paper appearing on `http://www.kurims.kyoto-u.ac.jp/preprint/preprint_y2017.html`; in the printed version the new Stokes curve in question is written in black) emanating from the unique virtual turning point of $\mathscr{N}(c)$ has a solid portion. This means that some Stokes phenomena for WKB solutions of $\mathscr{N}(c)$ are observed on this portion and the virtual turning point in question is non-redundant. On the other hand, in Figures 3.5 and 3.7 the whole portion of the same new Stokes curve is dotted and hence the

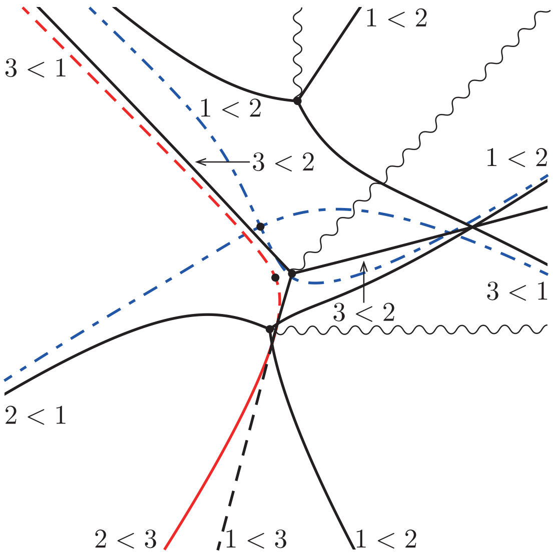

Figure 3.1 : Stokes geometry of $\mathscr{N}(c)$ for $c = \exp(24\pi\sqrt{-1}/32)$.

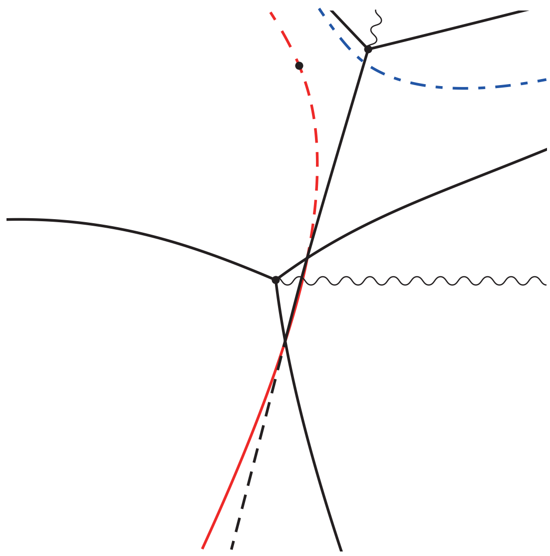

Figure 3.2 : Figure 3.1 enlarged.

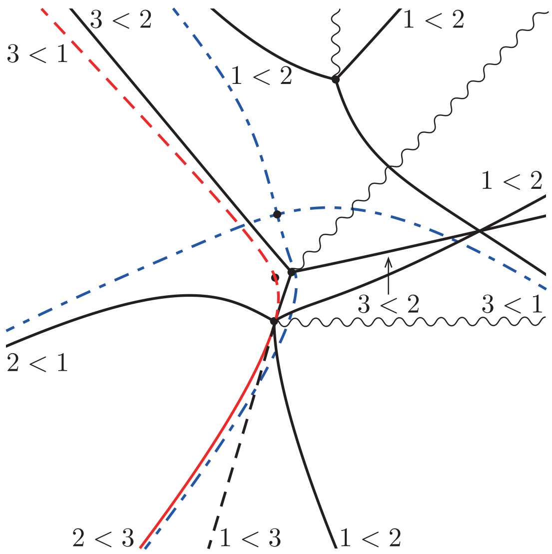

Figure 3.3 : Stokes geometry of $\mathscr{N}(c)$ for $c = \exp(25\pi\sqrt{-1}/32)$.

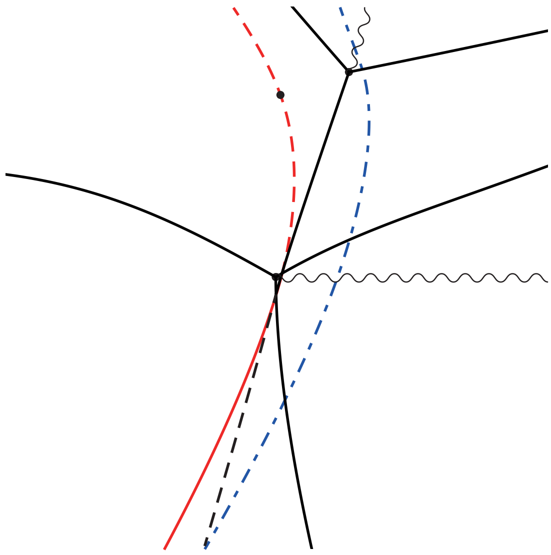

Figure 3.4 : Figure 3.3 enlarged.

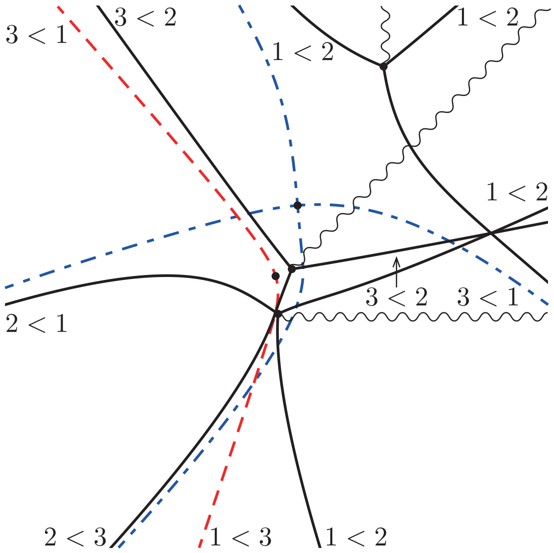

Figure 3.5 : Stokes geometry of $\mathscr{N}(c)$ for $c = \exp(26\pi\sqrt{-1}/32)$.

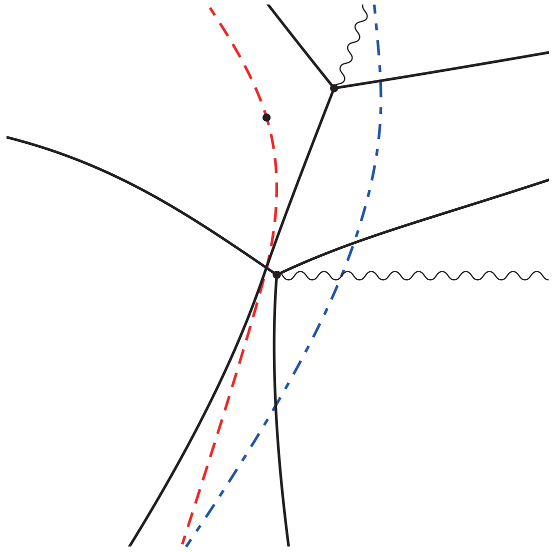

Figure 3.6 : Figure 3.5 enlarged.

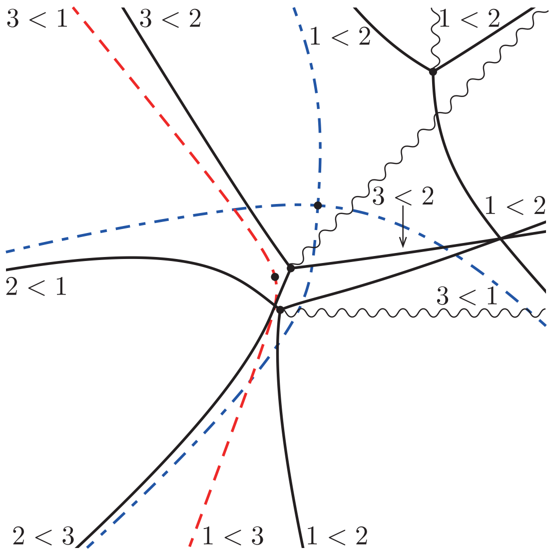

Figure 3.7 : Stokes geometry of $\mathscr{N}(c)$ for $c = \exp(27\pi\sqrt{-1}/32)$.

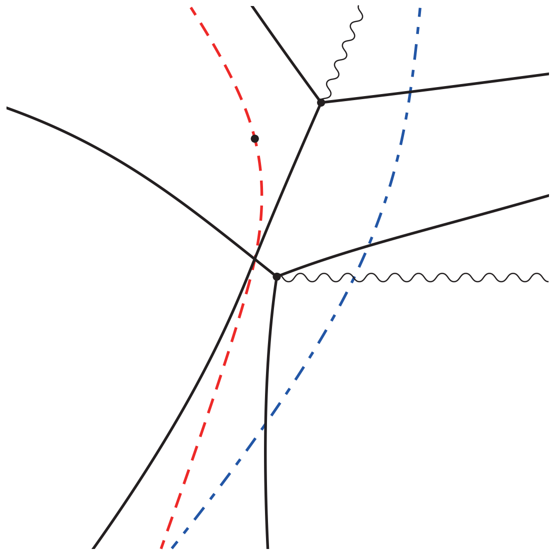

Figure 3.8 : Figure 3.7 enlarged.

virtual turning point is redundant. Thus these figures clearly show that the redundancy of the virtual turning point of $\mathscr{N}(c)$ varies with the change of c.

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