On a Weak Hom-version of the Grothendieck Conjecture for Curves of Type 
\((0, n)\) over Algebraically Closed Fields of Characteristic \(p > 0\)

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Sep 2017

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ON A WEAK HOM-VERSION OF THE
GROTHENDIECK CONJECTURE FOR CURVES OF
TYPE \((0, n)\) OVER ALGEBRAICALLY CLOSED
FIELDS OF CHARACTERISTIC \(p > 0\)

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Abstract
In the present paper, we study the anabelian geometry of curves over algebraically closed fields of characteristic \(p > 0\). Let \(X_1^\bullet := (X_1, D_{X_1})\) and \(X_2^\bullet := (X_2, D_{X_2})\) be smooth pointed stable curves of type \((g, n)\) over algebraically closed fields \(k_1\) and \(k_2\) of characteristic \(p > 0\), respectively. We prove that, if \(g = 0\) and 
\[ k_1 = k_2 = \mathbb{F}_p, \]
then \(X_1 \setminus D_{X_1}\) is isomorphic to \(X_2 \setminus D_{X_2}\) as schemes if and only if the set of open continuous homomorphisms between the tame fundamental groups of \(X_1 \setminus D_{X_1}\) and \(X_2 \setminus D_{X_2}\) is not empty. This result can be regarded as a weak Hom-version of the Grothendieck conjecture for curves of type \((0, n)\) over \(\mathbb{F}_p\). Moreover, this result is a generalization of the weak Isom-version of the Grothendieck conjecture for curves of type \((0, n)\) over \(\mathbb{F}_p\) which was proved by A. Tamagawa. On the other hand, for arbitrary \((g, n)\), we formulate a certain weak Hom-version of the Grothendieck conjecture for curves of type \((g, n)\) over arbitrary algebraically closed fields of characteristic \(p > 0\).

Keywords: hyperbolic curve, smooth pointed stable curve, tame fundamental group, Grothendieck’s anabelian conjecture, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 11G20.

Introduction
In the present paper, we study the anabelian geometry of curves over algebraically closed fields of characteristic \(p > 0\).

Before we explain the main theorem of the present paper, let us recall some general facts concerning anabelian geometry. Let \(k\) be a field and \(Z\) a geometrically connected and \(k\)-scheme of finite type. Then we have the following fundamental exact sequence of étale fundamental groups (for suitable choices of base point):
\[ 1 \rightarrow \pi_1(Z_{k^{\text{sep}}}) \rightarrow \pi_1(Z) \overset{pr}{\rightarrow} G_k \rightarrow 1. \]
Here, \(Z_{k^{\text{sep}}}\) denotes \(Z \times_k k^{\text{sep}}\), \(k^{\text{sep}}\) denotes a separable closure of \(k\) in an algebraically closed field which contains \(k\), and \(G_k\) denotes the absolute Galois group \(\text{Gal}(k^{\text{sep}}/k)\) of \(k\).

A. Grothendieck proposed the following philosophy (cf. [G1], [G2]):
if $Z$ is anabelian, then the group-theoretic data $(\pi_1(Z), \text{pr}_Z)$ functorially determines the isomorphism class of the $k$-scheme $Z$.

Although we do not have any general definition of the term “anabelian”, if $\dim(Z) = 1$, hyperbolic curves have been regarded as typical examples of anabelian schemes. Here, a smooth, geometrically connected curve $Z$ over $k$ is called hyperbolic if it may be obtained as the complement of the divisor of marked points in a smooth pointed stable curve over $k$.

Let $Z_1$ and $Z_2$ be hyperbolic curves over $k$. Suppose that $k$ is of characteristic $0$. Relative to the notational conventions introduced above for étale fundamental groups, write

$$\text{Isom}_{\text{pro-gps}}(-, -) \quad \text{(resp. } \text{Hom}_{\text{pro-gps}}(-, -)\text{)}$$

for the set of continuous isomorphisms (resp. continuous homomorphisms) of profinite groups between the two profinite groups in parentheses,

$$\text{Isom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2)) := \{ \Phi \in \text{Isom}_{\text{pro-gps}}(\pi_1(Z_1), \pi_1(Z_2)) \mid \text{pr}_{Z_1} = \text{pr}_{Z_2} \circ \Phi \}$$

(resp. $\text{Hom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2)) := \{ \Phi \in \text{Hom}_{\text{pro-gps}}(\pi_1(Z_1), \pi_1(Z_2)) \mid \text{pr}_{Z_1} = \text{pr}_{Z_2} \circ \Phi \}$).

Thus, by composing with inner automorphisms, we obtain a natural action of $\pi_1(Z_2 \times_k k^{\text{sep}})$ on $\text{Isom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2))$ (resp. $\text{Hom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2))$). Then, in this situation, the philosophy above can be formulated as follows (which is called Grothendieck’s anabelian conjecture or, simply, the Grothendieck conjecture, for short):

(weak $\text{Isom}_k$-version)

The set

$$\text{Isom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2)) \neq \emptyset$$

if and only if

$$Z_1 \cong Z_2$$

as $k$-schemes.

(Isom$_k$-version)

The natural morphism

$$\text{Isom}_{k\text{-schemes}}(Z_1, Z_2) \rightarrow \text{Isom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2))/\text{Inn}(\pi_1(Z_2 \times_k k^{\text{sep}}))$$

is a bijection.

The Grothendieck conjecture has been proven in many cases. For example, if $k$ is a number field, then the weak $\text{Isom}_k$-version was proved by H. Nakamura when the genera of $Z_1$ and $Z_2$ are 0 (cf. [N1], [N2]); the $\text{Isom}_k$-version was proved by A. Tamagawa in the case where $Z_1$ and $Z_2$ are affine (cf. [T1]) and proved by S. Mochizuki in full generality (cf. [M1]). In fact, Mochizuki proved a very general version when $k$ is sub-$p$-adic (i.e., a subfield of a finitely generated extension of a $p$-adic number field) as follows:
We denote by \( \text{Hom}_k^{\text{dom}}(Z_1, Z_2) \) the subset of the dominant morphisms of \( \text{Hom}_k \)-schemes \((Z_1, Z_2)\) and denote by \( \text{Hom}_G^G_{k}(\pi_1(Z_1), \pi_1(Z_2)) \) the subset of open homomorphisms of \( \text{Hom}_G^G_k \). Then the natural morphism \[ \text{Hom}_k^{\text{dom}}(Z_1, Z_2) \to \text{Hom}_G^G_{k}(\pi_1(Z_1), \pi_1(Z_2))/\text{Inn}(\pi_1(Z_2 \times_k k^{\text{sep}})) \] is a bijection.

Note that we have implications
\[ \text{Hom}_k\text{-version} \Rightarrow \text{Isom}_k\text{-version} \Rightarrow \text{weak Isom}_k\text{-version}. \]

Tamagawa also considered an analogue of the Grothendieck conjecture in positive characteristic and proved the Grothendieck conjecture (Isom-version) for affine hyperbolic curves over finite fields (cf. [T1]). Afterwards, Mochizuki generalized this result to the case of projective hyperbolic curves (cf. [M2]), and J. Stix generalized this result to the case where the base fields are finitely generated over \( F_p \) (cf. [Stil1], [Sti2]).

Unlike the characteristic 0 case, nothing is known about the Grothendieck conjecture for curves over local fields of positive characteristic. On the other hand, Tamagawa also considered the Grothendieck conjecture for curves over algebraically closed fields of characteristic \( p > 0 \). Note that all the proofs of the Grothendieck conjecture for curves over non-algebraically closed fields require the use of the highly non-trivial outer Galois representation induced by the fundamental exact sequence of étale fundamental groups reviewed above. In the case of algebraically closed fields, the Galois groups of the base fields are trivial, and the étale fundamental group coincides with the geometric fundamental group. As a result, the Grothendieck conjecture for curves over algebraically closed fields of characteristic \( p > 0 \) is quite different from that over non-algebraically closed fields.

In the remainder of this introduction, let \( X_i^\bullet := (X_i, D_{X_i}) \) and \( X_2^\bullet := (X_2, D_{X_2}) \) be smooth pointed stable curves of type \((g_X, n_X)\) over algebraically closed fields \( k_1 \) and \( k_2 \) of characteristic \( p > 0 \) (i.e., \( X_1 \setminus D_{X_1} \) and \( X_2 \setminus D_{X_2} \) are hyperbolic curves of type \((g_X, n_X)\) over \( k_1 \) and \( k_2 \), respectively). For \( i = 1, 2 \), write \( k_i^{\text{min}} \) for the minimal algebraically closed subfield of \( k_i \) over which \( X_i^\bullet \) is defined; thus, by considering the function field of \( X_i \), one verifies immediately that there exists a “natural” smooth pointed stable curve
\[ X_i^{\bullet, \text{min}} := (X_i^{\text{min}}, D_{X_i^{\text{min}}}), \]
where the function field of \( X_i^{\text{min}} \) is a subfield of the function field of \( X_i \), such that \( X_i \setminus D_{X_i} \) may be identified with \((X_i^{\text{min}} \setminus D_{X_i^{\text{min}}}) \times_{k_i^{\text{min}}} k_i \). In this situation, Tamagawa formulated the Grothendieck conjecture as follows (we only focus on the tame version in the present paper):
Conjecture 0.1. (weak Isom-version)

The set of continuous isomorphisms of profinite groups

\[ \text{Isom}_{\text{pro-gps}}(\pi_1^{\text{tame}}(X_1 \setminus D_{X_1}), \pi_1^{\text{tame}}(X_2 \setminus D_{X_2})) \neq \emptyset \]

if and only if

\[ X_1^{\text{min}} \setminus D_{X_1^{\text{min}}} \cong X_2^{\text{min}} \setminus D_{X_2^{\text{min}}} \]

as schemes.

(Isom-version)

The natural morphism

\[ \text{Isom}_{\text{schemes}}(X_1^{\text{min}} \setminus D_{X_1^{\text{min}}}, X_2^{\text{min}} \setminus D_{X_2^{\text{min}}}) \]

\[ \rightarrow \text{Isom}_{\text{pro-gps}}(\pi_1^{\text{tame}}(X_1 \setminus D_{X_1}), \pi_1^{\text{tame}}(X_2 \setminus D_{X_2}))/\text{Inn}(\pi_1^{\text{tame}}(X_2 \setminus D_{X_2})) \]

is a bijection.

Remark 0.1.1. Note that the existence of the specialization map of tame fundamental groups constitutes a counterexample to the “Hom-version” of the Grothendieck conjecture for tame fundamental groups obtained by simply replacing \( \text{Isom}(-,-) \) by \( \text{Hom}^{\text{open}}(-,-) \). Indeed, at the time of writing, we do not know how to give a formulation of a suitable Hom-version of the Grothendieck conjecture for tame fundamental groups that takes into account this counterexample (cf. [T3, Remark 1.34]). On the other hand, there exists a Hom-version of the Grothendieck conjecture for étale fundamental groups (cf. [T3, Conjecture 1.8]).

At present, no result is known about the Isom-version of Conjecture 0.1. On the other hand, Tamagawa proved the weak Isom-version of Conjecture 0.1 when \( g_X = 0 \) and \( k_1 = k_2 \) is an algebraic closure of \( \mathbb{F}_p \). More precisely, Tamagawa proved the following theorem (cf. [T4, Theorem 5.8]).

Theorem 0.2. Suppose that \( g_X = 0 \). Then we can detect whether \( X_1^* \) (resp. \( X_2^* \)) can be defined over the algebraic closure of \( \mathbb{F}_p \) in \( k_1 \) (resp. \( k_2 \)) or not, group-theoretically from the tame fundamental group \( \pi_1^{\text{tame}}(X_1 \setminus D_{X_1}) \) (resp. \( \pi_1^{\text{tame}}(X_2 \setminus D_{X_2}) \)). Moreover, suppose that \( X_1^* \) can be defined over the algebraic closure of \( \mathbb{F}_p \) in \( k_1 \). Then the set of continuous isomorphisms of profinite groups

\[ \text{Isom}_{\text{pro-gps}}(\pi_1^{\text{tame}}(X_1 \setminus D_{X_1}), \pi_1^{\text{tame}}(X_2 \setminus D_{X_2})) \neq \emptyset \]

if and only if

\[ X_1^{\text{min}} \setminus D_{X_1^{\text{min}}} \cong X_2^{\text{min}} \setminus D_{X_2^{\text{min}}} \]

as schemes.

Remark 0.2.1. Tamagawa also obtained an étale fundamental group version of Theorem 0.2 in a completely different way (by using wildly ramified coverings) (cf. [T2, Theorem 3.5]). Note that since the tame fundamental group can be reconstructed group-theoretically from the étale fundamental group (cf. [T2, Corollary 1.10]), the tame fundamental group version is stronger than the étale fundamental group version. Recently,
by using Tamagawa’s idea, A. Sarashina (a student of Tamagawa) proved a similar result of [T2, Theorem 3.5] for curves of type \((1, 1)\)(cf. [Sar], [T6, Theorem 6 (i)]). Moreover, by applying the theory of Tamagawa developed in [T4], Sarashina’s result also holds in the case of tame fundamental groups (cf. [T6, Theorem 6 (ii)]).

**Remark 0.2.2.** We do not know whether the weak Isom-version of Conjecture 0.1 for \(g_X > 0\) holds or not. On the other hand, we have the following finiteness theorem which was proved by M. Raynaud, F. Pop, M. Saïdi, and Tamagawa (cf. [R], [PS], [T5]):

over an algebraic closure of \(\mathbb{F}_p\), only finitely many isomorphism classes of hyperbolic curves have the same tame fundamental groups.

Moreover, the finiteness theorem also holds for (possibly singular) pointed stable curves (cf. [Y]).

In the present paper, we consider a weak Hom-version of Grothendieck conjecture over algebraically closed field of characteristic \(p > 0\). Our main theorem, which generalizes Tamagawa’s theorem above, is as follows (see also Theorem 4.2).

**Theorem 0.3.** Suppose that \(g_X = 0\). Then we can detect whether \(X_1^*\) can be defined over the algebraically closure of \(\mathbb{F}_p\) in \(k_1\) or not, group-theoretically from \(\pi_1^{tame}(X_1 \setminus D_{X_1})\). Moreover, suppose that \(X_1^*\) can be defined over the algebraic closure of \(\mathbb{F}_p\) in \(k_1\). Then the set of open homomorphisms

\[
\text{Hom}^{\text{open}}(\pi_1^{tame}(X_1 \setminus D_{X_1}), \pi_1^{tame}(X_2 \setminus D_{X_2})) \neq \emptyset
\]

if and only if

\[
X_1^{\text{min}} \setminus D_{X_1^{\text{min}}} \cong X_2^{\text{min}} \setminus D_{X_2^{\text{min}}}
\]

as schemes. In particular, if this is the case, \(X_2^*\) can be defined over the algebraic closure of \(\mathbb{F}_p\) in \(k_2\).

**Remark 0.3.1.** Similar arguments to the arguments developed in the present paper and [Sar], one may prove a similar result of Theorem 4.2 for curves of type \((1, 1)\) (see Remark 4.2.4 for a precise form).

Theorem 0.3 can be regarded as a weak Hom-version of the Grothendieck conjecture for curves of type \((0, n)\) over an algebraic closure of \(\mathbb{F}_p\). Moreover, although we do not know how to formulate “Hom-version” of the Grothendieck conjecture over algebraically closed fields of characteristic \(p > 0\), we can formulate a certain “weak Hom-version” of the Grothendieck conjecture for curves of type \((g, n)\) over algebraically closed fields of characteristic \(p > 0\) (cf. Conjecture 6.2). Then Theorem 0.3 implies that Conjecture 6.2 holds in a special case. Conjecture 6.2 also implies Tamagawa’s essential dimension conjecture (cf. Remark 4.2.3 and Remark 6.2.2).

The present paper is organized as follows. In Section 1, we give some definitions and propositions which will be used in the next sections. In Section 2, we construct a correspondence between the set of marked points of smooth pointed stable curves and line bundles. In Section 3, by applying the theory developed in Section 2, we reconstruct the inertia subgroups of marked points and their additive structures from a surjection of tame
fundamental groups. In Section 4, by applying the results obtained in previous sections, we prove our main theorem. In Section 5, we apply the main theorem to a question concerning moduli spaces of curves which is originally posed by K. Stevenson. Finally, in Section 6, we formulate a certain weak Hom-version of the Grothendieck conjecture for curves of type \((g,n)\) over algebraically closed fields of characteristic \(p > 0\).

**Acknowledgements**

The author would like to thank Yuichiro Hoshi, Akio Tamagawa, and Kazuki Tokimoto for helpful discussions. This research was supported by JSPS KAKENHI Grant Number 16J08847.

1 Preliminaries

Let \(k\) be an algebraically closed field of characteristic \(p > 0\), and let \(X := (X,D_X)\) be a smooth pointed stable curve of type \((g_X,n_X)\) over \(k\). Here, \(X\) denotes the underlying curve of \(X\), \(D_X\) denotes the set of marked points of \(X\), \(g_X\) denotes the genus of \(X\), and \(n_X\) denotes the cardinality of \(D_X\). By choosing a base point of \(x \in X \setminus D_X\), we obtain the tame fundamental group \(\pi^\text{tame}_1(X \setminus D_X, x)\) of \(X\) and the étale fundamental group \(\pi_1(X,x)\) of \(X\). Write \(\pi^\text{tame}_1(X \setminus D_X, x)\) and \(\pi_1(X,x)\) for the maximal pro-solvable quotients of \(\pi^\text{tame}_1(X \setminus D_X, x)\) and \(\pi_1(X,x)\), respectively. Note that, by the definition of tame coverings, there is a natural surjection

\[ \pi^\text{tame}_1(X \setminus D_X, x) \rightarrow \pi_1(X,x). \]

For simplicity of notation, we omit the base point and denote by \(X\) (resp. \(\Pi_X\)) the maximal pro-solvable quotient of the tame (resp. étale) fundamental group of \(X\).

**Definition 1.1.** Let \(\ell\) be a prime number, and let \(f^\bullet : Y^\bullet \rightarrow X^\bullet\) be a connected tame Galois covering (i.e., \(f^\bullet\) is a Galois covering and is at most tamely ramified over \(D_X\)) over \(k\) of degree \(\ell\). For any \(e \in D_X\), we set

\[ \text{Ram}_{f^\bullet} := \{ e \in D_X \mid f^\bullet \text{ is ramified over } e \}. \]

**Definition 1.2.** Let \(\Pi\) be a profinite group, \(n\) a natural number, and \(\ell\) a prime number.

(a) We denote by \(\Pi(n)\) the topological closure of the subgroup \([\Pi,\Pi]\Pi^n\) of \(\Pi\). Note that \(\Pi/\Pi(n) = \Pi^{ab} \otimes (\mathbb{Z}/n\mathbb{Z})\).

(b) We set \(\gamma_\ell := \dim_F(\Pi/\Pi(\ell)) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}\).

(c) Let \(n\) be a natural number such that \([\Pi : \Pi(n)] < \infty\). We define \(\ell\)-average of \(\Pi\) to be

\[ \gamma_\ell^\text{av}(n)(\Pi) := \gamma_\ell(\Pi(n))/[\Pi : \Pi(n)] \in \mathbb{Q}_{\geq 0} \cup \{\infty\}. \]

(d) We denote by \(\text{Sub(}\Pi)\) the set of closed subgroups of \(\Pi\).

The following highly non-trivial result concerning \(p\)-average of \(\Pi_X\) was proved by Tamagawa (cf. [T4, Theorem 0.5]).
Proposition 1.3. For any natural number \( r \in \mathbb{N} \), we set
\[
\gamma_p^{av}(p^r - 1)(X^\bullet) := \gamma_p^{av}(p^r - 1)(\Pi_X^\bullet).
\]
Then we have
\[
\lim_{r \to \infty} \gamma_p^{av}(p^r - 1)(X^\bullet) = g_X',
\]
where \( g_X' = g_X - 1 \) if \( n_X \leq 1 \) and \( g_X' = g_X \) if \( n_X > 1 \).

Remark 1.3.1. Tamagawa proved Proposition 1.3 as a main theorem of [T4] by developing a general theory of Raynaud’s theta divisor.

Let \( K \) be the function field of \( X^\bullet \), and define \( K_{sol}^\bullet \) to be the maximal pro-solvable Galois extension of \( K \) in a fixed separable closure of \( K \), unramified over \( X \setminus D_X \) and at most tamely ramified over \( D_X \). Then we may identify \( X^\bullet \) with \( \text{Gal}(K_{sol}^\bullet/K) \). We set
\[
\tilde{X}_{\bullet,sol}^\bullet := (\tilde{X}_{sol}^\bullet, D_{\tilde{X}_{sol}^\bullet}),
\]
where \( \tilde{X}_{sol}^\bullet \) denotes the normalization of \( X \) in \( \tilde{K}_{sol}^\bullet \) and \( D_{\tilde{X}_{sol}^\bullet} \) denotes the inverse image of \( D_X \) in \( \tilde{X}_{sol}^\bullet \). For each \( \tilde{e} \in D_{\tilde{X}_{sol}^\bullet} \), we denote by \( I_{\tilde{e}} \) the inertia subgroup of \( \tilde{X}_{\bullet,sol}^\bullet \) associated to \( \tilde{e} \) (i.e., the stabilizer of \( \tilde{e} \)). Note that we have \( I_{\tilde{e}} \cong \hat{\mathbb{Z}}(1)^p' \), where \( \hat{\mathbb{Z}}(1)^p' \) denotes the prime-to-\( p \) part of \( \hat{\mathbb{Z}}(1) \).

Lemma 1.4. Let \( m_1, m_2 \) be two positive numbers and \( G \) a finite solvable group of order \( m_1m_2 \). Let \( f_1^\bullet : Z^\bullet \to Y^\bullet \) and \( f_2^\bullet : Y^\bullet \to X^\bullet \) be connected cyclic tame Galois coverings of degrees \( m_1 \) and \( m_2 \) over \( k \), respectively. Suppose that the composition \( f_2^\bullet \circ f_1^\bullet : Z^\bullet \to X^\bullet \) is a connected tame Galois covering over \( k \) whose Galois group is isomorphic to \( G \). Then there exists two connected tame Galois coverings \( f_1^\bullet : Z^{*,\bullet} \to Y^{*,\bullet} \) and \( f_2^\bullet : Y^{*,\bullet} \to X^\bullet \) over \( k \) such that the following conditions hold:

(a) the Galois group of \( f_1^{*,\bullet} \) and \( f_2^{*,\bullet} \) are isomorphic to \( \Pi_{Y^{*,\bullet}}/\Pi_{Y^{*,\bullet}}(m_1) \) and \( \Pi_{X^{*,\bullet}}/\Pi_{X^{*,\bullet}}(m_2) \), respectively, where \( \Pi_{Y^{*,\bullet}} \) denotes the maximal pro-solvable quotient of the tame fundamental group of \( Y^{*,\bullet} \);

(b) there exist two morphisms \( g_1^\bullet : Z^{*,\bullet} \to Z^* \) and \( g_2^\bullet : Y^{*,\bullet} \to Y^* \) over \( k \) which fit into the following commutative diagram:

\[
\begin{array}{ccc}
Z^{*,\bullet} & \xrightarrow{g_1^\bullet} & Z^* \\
\downarrow f_1^{*,\bullet} & & \downarrow f_1^\bullet \\
Y^{*,\bullet} & \xrightarrow{g_2^\bullet} & Y^* \\
\downarrow f_2^{*,\bullet} & & \downarrow f_2^\bullet \\
X^* & \xrightarrow{f_2^\bullet} & X^*. \\
\end{array}
\]

Proof. Trivial. \( \square \)

Remark 1.4.1. Let \( C_{X^\bullet} := \{ H_i \}_{i \in \mathbb{Z}_{\geq 0}} \) be a set of open subgroups of \( \Pi_{X^\bullet} \) such that the following conditions:
(a) $H_0 = \Pi_{X^*}$ and $H_{i+1}$ is an open normal subgroup of $H_i$ for each $i \in \mathbb{Z}_{\geq 0}$;
(b) $\lim_{i \to \infty} \Pi_{X^*}/H_i \cong \Pi_{X^*}$.

Let $\bar{e} \in D_{X^*}$. For each $i \in \mathbb{Z}_{\geq 0}$, we write $X_{H_i}^* := (X_{H_i}, D_{H_i})$ for the smooth pointed stable curve corresponding to $H_i$ and $e_{H_i} \in D_{H_i}$ for the image of $\bar{e}$ in $X_{H_i}^*$. Then we obtain a sequence of marked points

$$\mathcal{T}_{\bar{e}}^{C_{X^*}} : \cdots \mapsto e_{H_2} \mapsto e_{H_1} \mapsto e_{H_0}$$

induced by $C_{X^*}$. Note that the sequence $\mathcal{T}_{\bar{e}}^{C_{X^*}}$ admits a natural action of $\Pi_{X^*}$.

We may identify the inertia subgroup $I_{\bar{e}}$ associated to $\bar{e}$ with the stabilizer of $\mathcal{T}_{\bar{e}}^{C_{X^*}}$. Moreover, Lemma 1.4 implies that we may assume that, for each $i \in \mathbb{Z}_{\geq 0}$,

$$H_{i+1} = H_i(n_i)$$

for some $n_i \in \mathbb{Z}_{\geq 0}$.

Next, we recall some well-known results concerning the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

**Definition 1.5.** (a) Given an object $\text{Ob}(X^*)$ (e.g., an invariant of $X^*$, the set of marked points of $X^*$) associated to $X^*$ depending on the isomorphism class of $X^*$ (as scheme), we shall say that $\text{Ob}(X^*)$ **can be reconstructed group-theoretically from** $\Pi_{X^*}$ if there exists a group-theoretically algorithm for reconstructing $\text{Ob}(X^*)$ from $\Pi_{X^*}$.

(b) Given an additional structure $\text{Add}(X^*)$ (e.g., a family of subgroups, a family of quotient groups) on the profinite group $\Pi_{X^*}$ depending functorially on $X^*$; then we shall say that $\text{Add}(X^*)$ **can be reconstructed group-theoretically from** $\Pi_{X^*}$ if there exists a group-theoretically algorithm for reconstructing $\text{Add}(X^*)$ from $\Pi_{X^*}$.

(c) Let $X^*$ and $Y^* := (Y, D_Y)$ be smooth pointed stable curves over algebraically closed fields of characteristic $p > 0$, $\Pi_{X^*}$ and $\Pi_{Y^*}$ the maximal pro-solvable quotient of the tame fundamental groups of $X^*$ and $Y^*$, respectively. Suppose that we are given $\text{Ob}(X^*)$ and $\text{Ob}(Y^*)$ (resp. $\text{Add}(X^*)$ and $\text{Add}(Y^*)$), and that a continuous homomorphism (in the category of profinite groups) $\Pi_{X^*} \to \Pi_{Y^*}$. We shall say that a map $\text{Ob}(X^*) \to \text{Ob}(Y^*)$ (resp. $\text{Add}(X^*) \to \text{Add}(Y^*)$) **can be reconstructed group-theoretically from** the morphism $\Pi_{X^*} \to \Pi_{Y^*}$ if there exists a group-theoretically algorithm for reconstructing the map $\text{Ob}(X^*) \to \text{Ob}(Y^*)$ (resp. $\text{Add}(X^*) \to \text{Add}(Y^*)$) from the morphism $\Pi_{X^*} \to \Pi_{Y^*}$.

**Proposition 1.6.** (a) The genus $g_X$ of $X^*$ and the cardinality of the set of the marked points $n_X$ of $X^*$ can be reconstructed group-theoretically from $\Pi_{X^*}$.

(b-i) Let $\bar{e}$ and $\bar{e}'$ be two points of $D_X$ distinct from each other. Then the intersection of $I_{\bar{e}}$ and $I_{\bar{e}'}$ is trivial in $\Pi_{X^*}$. (b-ii) The map

$$D_X \to \text{Sub}(\Pi_{X^*})$$

that maps $\bar{e} \mapsto I_{\bar{e}}$ is an injection.

(c) Write $\text{Ine}(\Pi_{X^*})$ for the set of inertia subgroups in $\Pi_{X^*}$, namely the image of the map $D_X \to \text{Sub}(\Pi_{X^*})$. Then $\text{Ine}(\Pi_{X^*})$ can be reconstructed group-theoretically from $\Pi_{X^*}$. In particular, the set of marked points $D_X$ and $\Pi_{X^*}$ can be reconstructed group-theoretically from $\Pi_{X^*}$.
Proof. (a) follows immediately from Proposition 1.3. The tame fundamental group version
of (b-i) and (b-ii) was proved by Tamagawa (cf. [T4, Lemma 5.1]). Moreover, it is easy
to see that Tamagawa’s proof holds for \( X \). (c) is a special case of a result of the author
(cf. [Y, Theorem 0.2, Remark 0.2.1, and Remark 0.2.2]).

Corollary 1.7. Let \( H \subseteq \Pi_X \) be an open normal subgroup and \( f^\bullet : Y^\bullet := (Y, D_Y) \to X^\bullet \)
the connected tame Galois covering over \( k \) corresponding to \( H \). Then \( D_X, D_Y, \) and the
natural morphism
\[
\gamma_f : D_Y \to D_X
\]
induced by \( f^\bullet \) can be reconstructed group-theoretically from \( \Pi_X, H, \) and the natural in-
clusion \( H \hookrightarrow \Pi_X \), respectively.

Proof. The corollary follows immediately from Proposition 1.6 (b-i), Proposition 1.6 (b-ii),
and Proposition 1.6 (c). \( \square \)

2 The set of marked points and line bundles

We maintain the notations introduced in Section 1. Moreover, in this section, we suppose
that \( g_X \geq 2 \) and \( n_X > 0 \). Let
\[
(\ell, d, f^\bullet : Y^\bullet := (Y, D_Y) \to X^\bullet)
\]
be a data satisfying the following conditions:

(a) \( \ell \) and \( d \) are prime numbers distinct from each other and from \( p \) such that
\( \ell \equiv 1 \pmod{d} \); then all \( d \)th roots of unity are contained in \( \mathbb{F}_\ell \);

(b) \( f^\bullet : Y^\bullet \to X^\bullet \) is an étale Galois covering over \( k \) whose Galois group is
isomorphic to \( G_d \), where \( G_d \subseteq \mathbb{F}_\ell \) denotes the subgroup of \( d \)th roots of unity.

Write \( M^\text{et}_Y \) and \( M_Y \) for \( H^1_{\text{ét}}(Y^\bullet, \mathbb{F}_\ell) \) and \( \text{Hom}(\Pi_Y^\bullet, \mathbb{F}_\ell) \), respectively, where \( \Pi_Y^\bullet \) denotes
the maximal pro-solvable quotient of the tame fundamental group of \( Y^\bullet \). Note that there
is a natural injection
\[
M^\text{et}_Y \hookrightarrow M_Y
\]
induced by the natural surjection \( \Pi_Y^\bullet \twoheadrightarrow \Pi^\text{et}_Y \). Then we obtain an exact sequence
\[
0 \to M^\text{et}_Y \to M_Y \to M^\text{et}_Y := \text{coker}(M^\text{et}_Y \hookrightarrow M_Y) \to 0
\]
with a natural action of \( G_d \).

Let
\[
M^\text{et}_{Y, G_d} \subseteq M^\text{et}_Y
\]
be the subset of elements on which \( G_d \) acts via the character \( G_d \hookrightarrow \mathbb{F}_\ell^\times \) and
\[
U^\bullet_Y \subseteq M_Y
\]
the subset of elements that map to nonzero elements of \( M^\text{et}_{Y, G_d} \). For each \( \alpha \in U^\bullet_Y \), write
\[
g^\bullet_\alpha : Y^\bullet = (Y_\alpha, D_{Y_\alpha}) \to Y^\bullet
\]
for the tame covering corresponding to $\alpha$. Then we obtain a morphism

$$\epsilon : U^*_Y \to \mathbb{Z}$$

that maps $\alpha$ to $\#D_Y$, where $\#(-)$ denotes the cardinality of $(-)$. We define a subset of $U^*_Y$ to be

$$U^*_{Y\mp} := \{ \alpha \in U^*_Y \mid \#\text{Ram}_{g\alpha} = d \} = \{ \alpha \in U^*_Y \mid \epsilon(\alpha) = \ell(dn_X - d) + d \}.$$ 

Note that $U^*_{Y\mp}$ is not empty. For each $\alpha \in U^*_{Y\mp}$, since the image of $\alpha$ is contained in $M^n_{Y\bullet \cdot, G_d}$, we obtain that the action of $G_d$ on the set $\text{Ram}_{g\alpha} \subseteq D_Y$ is transitive. Thus, there exists a unique marked point $e_\alpha$ of $X^\bullet$ such that $f^*(y) = e_\alpha$ for each $y \in \text{Ram}_{g\bullet}$. We define a pre-equivalence relation $\sim$ on $U^*_{Y\mp}$ as follows:

if $\alpha \sim \beta \in U^*_{Y\mp}$, then $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}^\times$ for which $\lambda\alpha + \mu\beta \in U^*_Y$, we have $\lambda\alpha + \mu\beta \in U^*_{Y\mp}$.

Then we have the following proposition.

**Proposition 2.1.** The pre-equivalence relation $\sim$ on $U^*_{Y\mp}$ is an equivalence relation, and, moreover, the quotient set $U^*_{Y\mp} / \sim$ is naturally isomorphic to $D_X$ that maps $[\alpha] \mapsto e_\alpha$.

**Proof.** Let $\beta, \gamma \in U^*_{Y\mp}$. If $\text{Ram}_{g\beta} = \text{Ram}_{g\gamma}$, then, for each $\lambda, \mu \in \mathbb{F}^\times$ for which $\lambda\beta + \mu\gamma \neq 0$, we have $\text{Ram}_{g\lambda\beta + \mu\gamma} = \text{Ram}_{g\beta} = \text{Ram}_{g\gamma}$. Thus, $\beta \sim \gamma$. On the other hand, if $\beta \sim \gamma$, we have $\text{Ram}_{g\beta} = \text{Ram}_{g\gamma}$. Otherwise, we have $\#\text{Ram}_{g\beta + \gamma} = 2d$. Thus, $\beta \sim \gamma$ if and only if $\text{Ram}_{g\beta} = \text{Ram}_{g\gamma}$. Then $\sim$ is an equivalence relation on $U^*_{Y\mp}$.

We define a map

$$\vartheta : U^*_{Y\mp} / \sim \to D_X$$

that maps $\alpha \mapsto e_\alpha$. Let us prove that $\vartheta$ is a bijection. It is easy to see that $\vartheta$ is an injection. On the other hand, for each $\epsilon \in D_X$, the structure of the maximal pro-$\ell$ tame fundamental groups implies that we may construct a connected tame Galois covering of $h^* : Z^\bullet \to Y^\bullet$ such that the line bundle corresponding to $h^*$ is contained in $U^*_{Y\mp}$. Then $\vartheta$ is a surjection. This completes the proof of the lemma.

**Remark 2.1.1.** We claim that the set $U^*_{Y\mp} / \sim$ does not depend on the choices of $\ell, d$, and the étale covering $f^* : Y^\bullet \to X^\bullet$.

Let

$$(\ell^*, d^*, f^{*,*} : Y^{*,*} \to X^*)$$

be a data. Hence we obtain a resulting $U^*_{Y^{*,*}} / \sim$ and a naturally isomorphism

$$\vartheta^* : U^*_{Y^{*,*}} / \sim \to D_X.$$ 

First, suppose that $\ell \neq \ell^*$, and that $d \neq d^*$. Then there exists a natural isomorphism

$$U^*_{Y^{*,*}} / \sim \cong U^*_{Y\mp} / \sim.$$
isomorphism which compatible with the isomorphism \( \vartheta \) and \( \vartheta^* \) as follows. Let \( \alpha \in U^\mathrm{mp}_Y \) and \( \alpha^* \in U^\mathrm{mp}_{Y^*} \). Write \( Y_\alpha \to Y^* \) and \( Y^*_{\alpha^*} \to Y^* \) for the tame coverings corresponding to \( \alpha \) and \( \alpha^* \), respectively. Let us consider

\[ Y^* \times_X Y^* \]

Thus, we have a connected tame Galois covering \( Y^* \times_X Y^* \to X^* \) of degree \( dd^* \ell \ell^* \). Then it is easy to check that \( \alpha \) and \( \alpha^* \) correspond to same marked points if and only if the cardinality of the set of marked points of \( Y^* \times_X Y^* \) is equal to \( dd^* (\ell \ell^* n_X - 1) + 1 \).

In general case, we may choose a data

\[ (\ell^{**}, d^{**}, f^{***} : Y^{***} \to X^*) \]

such that \( \ell^{**} \neq \ell, \ell \neq \ell^*, d^{**} \neq d, \) and \( d^{**} \neq d^* \). Hence we obtain a resulting \( U^\mathrm{mp}_{Y^{***}} / \sim \) and a naturally isomorphism \( \vartheta^{**} : U^\mathrm{mp}_{Y^{***}} / \sim \to D_X \). Then we obtain two natural isomorphisms \( U^\mathrm{mp}_{Y^{***}} / \sim \cong U^\mathrm{mp}_{Y^{**}} / \sim \) and \( U^\mathrm{mp}_{Y^{***}} / \sim \cong U^\mathrm{mp}_{Y^{*}} / \sim \). Thus, we have \( U^\mathrm{mp}_{Y^{***}} / \sim \cong U^\mathrm{mp}_{Y^{*}} / \sim \).

**Remark 2.1.2.** Note that \( U^\mathrm{mp}_{Y^*} \) can be reconstructed group-theoretically from \( \Pi_{Y^*} \) and \( \Pi_X \). Since an element \( \alpha \in U^\mathrm{mp}_{Y^*} \) is contained in \( U^\mathrm{mp}_{Y^*} \) if and only if

\[ \#D_{Y^*} = \ell (dn_X - d) + d, \]

Proposition 1.6 (a) implies that \( U^\mathrm{mp}_{Y^*} \) can be reconstructed group-theoretically from \( \Pi_{Y^*} \) and \( \Pi_X \). Moreover, Remark 2.1.1 implies that \( D_X \) can be reconstructed group-theoretically from \( \Pi_X \).

Next, we calculate \( \#U^\mathrm{mp}_{Y^*} \). For each \( e \in D_X \), we define

\[ U^\mathrm{mp}_{Y^*, e} := \{ \alpha \in U^\mathrm{mp}_{Y^*} \mid d_\alpha^* \text{ is ramified over } (f^*)^{-1}(e) \}. \]

Then, for any two marked points \( e, e' \in D_X \) distinct from each other, we have

\[ U^\mathrm{mp}_{Y^*, e} \cap U^\mathrm{mp}_{Y^*, e'} = \emptyset. \]

Moreover, we have

\[ U^\mathrm{mp}_{Y^*} = \bigcup_{e \in D_X} U^\mathrm{mp}_{Y^*, e}. \]

**Lemma 2.2.** Write \( g_Y \) for the genus of \( Y^* \). We have

\[ \#U^\mathrm{mp}_{Y^*, e} = \ell^{2g_Y + 1} - \ell^{2g_Y}. \]

Moreover, we have

\[ \#U^\mathrm{mp}_{Y^*} = n_X (\ell^{2g_Y + 1} - \ell^{2g_Y}). \]

**Proof.** Write \( E_e \subseteq D_Y \) for the set \( (f^*)^{-1}(e) \). Then \( U^\mathrm{mp}_{Y^*, e} \) can be naturally regarded as a subset of \( H^1_{\text{ét}}(Y \setminus E_e, \mathbb{F}_\ell) \) via the natural open immersion \( Y \setminus E_e \to Y \). Write \( L_e \) for the \( \mathbb{F}_\ell \)-vector space generated by \( U^\mathrm{mp}_{Y^*, e} \) in \( H^1_{\text{ét}}(Y \setminus E_e, \mathbb{F}_\ell) \). Then we have

\[ U^\mathrm{mp}_{Y^*, e} = L_e \setminus H^1_{\text{ét}}(Y, \mathbb{F}_\ell). \]
Write $H_e$ for the quotient $L_e/H^1_{\text{et}}(Y, \mathbb{F}_\ell)$. We have an exact sequence as follows:

$$0 \to H^1_{\text{et}}(Y, \mathbb{F}_\ell) \to L_e \to H_e \to 0.$$  

Since the action of $G_d$ on $(f^*)^{-1}(e)$ is translatable, we have

$$\dim_{\mathbb{F}_\ell} H_e = 1.$$  

On the other hand, since $\dim_{\mathbb{F}_\ell} H^1_{\text{et}}(Y, \mathbb{F}_\ell) = 2g_Y$, we obtain

$$\#U^\text{imp}_{Y^*} = \ell^{2g_Y+1} - \ell^{2g_Y}.$$  

Thus, we have

$$\#U^\text{imp}_{Y^*} = n_X(\ell^{2g_Y+1} - \ell^{2g_Y}).$$  

This completes the proof of the lemma. 

\section{Reconstruction of the inertia groups of marked points and their additive structures}

Let $k_1$ and $k_2$ be algebraically closed fields of characteristic $p > 0$, and let $X_1^\bullet$ and $X_2^\bullet$ be smooth pointed stable curves of type $(g_X, n_X)$ over $k_1$ and $k_2$, respectively. Write $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$ for the maximal pro-solvable quotient of the tame fundamental groups of $X_1^\bullet$ and $X_2^\bullet$, respectively. Suppose that $n_X > 0$, and that there is a continuous surjective morphism of profinite groups

$$\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}.$$  

Note that, since $X_1^\bullet$ and $X_2^\bullet$ are smooth pointed stable curves of type $(g_X, n_X)$, $\phi$ induces a natural isomorphism

$$\phi' : \Pi_{X_1^\bullet}^{\text{pr}} \simeq \Pi_{X_2^\bullet}^{\text{pr}},$$  

where $(-)^\text{pr}$ denotes the maximal prime-to-$p$ quotient of $(-)$.  

\textbf{Lemma 3.1.} Let $\ell$ be a prime number distinct from $p$. Then the isomorphism $(\phi')^{-1}$ induces an isomorphism

$$\psi_X^\ell : \mathbb{H}^1_{\text{et}}(X_1, \mathbb{F}_\ell) \simeq \mathbb{H}^1_{\text{et}}(X_2, \mathbb{F}_\ell).$$  

Moreover, $\psi_X^\ell$ can be reconstructed group-theoretically from the surjection $\phi$.

\textit{Proof.} Let

$$f_1^\bullet : Y_1^\bullet \to X_1^\bullet$$  

be an étale covering of degree $\ell$ over $k_1$. Write

$$f_2^\bullet : Y_2^\bullet \to X_2^\bullet$$  

for the connected tame Galois covering of degree $\ell$ over $k_2$ induced by $\phi'$. Then we claim that $f_2^\bullet$ is an étale covering over $k_2$. 

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Write $g_{Y_1}$ and $g_{Y_2}$ for the genera of $Y_1$ and $Y_2$. Since $f_1^\bullet$ is an étale covering of degree $\ell$, we have
\[ g_{Y_1} = \ell(g_{X_1} - 1) + 1. \]
On the other hand, by Riemann-Hurwitz formula, we have
\[ g_{Y_2} = \ell(g_{X_2} - 1) + 1 + \frac{1}{2}(\ell - 1)\#\text{Ram}_{f_2^\bullet}. \]
Since $\phi$ is a surjection, Proposition 1.3 implies that
\[ g_{Y_1} \geq g_{Y_2}. \]
Thus, we obtain that $\#\text{Ram}_{f_2^\bullet} = 0$. This means that $f_2^\bullet$ is an étale covering over $k_2$. Then we have that the morphism $(\phi')^{-1}$ induces an injection
\[ \psi_X : H^1_{\text{et}}(X_1, F_\ell) \hookrightarrow H^1_{\text{et}}(X_2, F_\ell). \]
Furthermore, since $\dim(H^1_{\text{et}}(X_1, F_\ell)) = \dim(H^1_{\text{et}}(X_2, F_\ell))$, $\psi_X$ is a bijection. We complete the proof of the lemma.

**Lemma 3.2.** Suppose that $g_X \geq 2$. Then the surjection $\phi : \Pi_{X_1^*} \to \Pi_{X_2^*}$ induces a bijection
\[ \rho_\phi : D_{X_1} \xrightarrow{\sim} D_{X_2} \]
between the sets of marked points of $X_1^*$ and $X_2^*$. Moreover, the bijection $\rho_\phi$ can be reconstructed group-theoretically from the surjection $\phi$.

**Proof.** Let $\ell$ and $d$ be prime numbers distinct from each other and from $p$. Suppose that
\[ \ell \equiv 1 \pmod{d}. \]
Then we have that all $d^{\text{th}}$ roots of unity are contained in $F_\ell$. Write $G_d \subseteq F_\ell^\times$ for the subgroup of $d^{\text{th}}$ roots of unity.

Let
\[ f_2^\bullet : Y_2^\bullet := (Y_2, D_{Y_2}) \to X_2^\bullet \]
be an étale covering of degree $d$ over $k_2$. Then $\phi$ induces a tame covering
\[ f_1^\bullet : Y_1^\bullet := (Y_1, D_{Y_1}) \to X_1^\bullet \]
of degree $d$ over $k_1$. Then Lemma 3.1 implies that $f_1^\bullet$ is an étale covering over $k_1$. Note that $Y_1^\bullet$ and $Y_2^\bullet$ are same type.

Write $\Pi_{Y_1^\bullet}$ and $\Pi_{Y_2^\bullet}$ for the open normal subgroups of $\Pi_{X_1^*}$ and $\Pi_{X_2^*}$ corresponding to $Y_1^\bullet$ and $Y_2^\bullet$, respectively. Write $M_{Y_1^\bullet}$, $M_{Y_1^\bullet}$, $M_{Y_1^\bullet}$, $M_{Y_2^\bullet}$, $M_{Y_2^\bullet}$, and $M_{Y_2^\bullet}$ for $\text{Hom}(\Pi_{Y_1^\bullet}, F_\ell)$, $H^1_{\text{et}}(Y_1, F_\ell)$, $M_{Y_1^\bullet}/M_{Y_1^\bullet}$, $\text{Hom}(\Pi_{Y_2^\bullet}, F_\ell)$, $H^1_{\text{et}}(Y_2, F_\ell)$, and $M_{Y_2^\bullet}/M_{Y_2^\bullet}$. Then Lemma 3.1 implies that $(\phi')^{-1}$ induces a commutative diagram as follows:
\[
\begin{array}{cccccc}
0 & \longrightarrow & M_{Y_1^\bullet} & \longrightarrow & M_{Y_1^\bullet} & \longrightarrow & M_{Y_1^\bullet} & \longrightarrow & 0 \\
& & \downarrow & \psi_1 & \downarrow & \Psi & \downarrow & \Psi & \downarrow \\
0 & \longrightarrow & M_{Y_2^\bullet} & \longrightarrow & M_{Y_2^\bullet} & \longrightarrow & M_{Y_2^\bullet} & \longrightarrow & 0,
\end{array}
\]
where all the vertical arrows are isomorphisms. Write $U_{1}^{\circ}$ and $U_{2}^{\circ}$ for the subsets of $M_{Y_{1}}^{\circ}$ and $M_{Y_{2}}^{\circ}$ defined as in Section 2, respectively. Since the actions of $G_{d}$ on the exact sequences are compatible with the isomorphisms appeared in the commutative diagram above, we have

$$\psi_{Y_{1}}(U_{1}^{\circ}) = U_{2}^{\circ}.$$  

Let $\alpha_{1} \in U_{1}^{\text{mp}}$ and

$$g_{\alpha_{1}}^{\bullet} : Y_{\alpha_{1}}^{\bullet} \rightarrow Y_{1}^{\bullet}$$

the tame covering of degree $\ell$ over $k_{1}$ corresponding to $\alpha_{1}$. Write

$$g_{\alpha_{2}}^{\bullet} : Y_{\alpha_{2}}^{\bullet} \rightarrow Y_{2}^{\bullet}$$

for the tame covering of degree $\ell$ over $k_{2}$ corresponding to $\alpha_{2} := \psi_{Y_{1}}^{\ell}(\alpha_{1})$. Write $g_{\alpha_{1}}^{\bullet}$ and $g_{\alpha_{2}}^{\bullet}$ for the genera of $Y_{\alpha_{1}}^{\bullet}$ and $Y_{\alpha_{2}}^{\bullet}$. Then Proposition 1.3 and Riemann-Hurwitz formula implies that

$$g_{\alpha_{1}}^{\bullet} - g_{\alpha_{2}}^{\bullet} = \frac{1}{2}(d - \#\text{Ram}_{g_{\alpha_{2}}^{\bullet}})(\ell - 1) \geq 0.$$  

This means that

$$d - \#\text{Ram}_{g_{\alpha_{2}}^{\bullet}} \geq 0.$$  

Since $\alpha_{2} \in U_{2}^{\circ}$, we have $d|\#\text{Ram}_{g_{\alpha_{2}}^{\bullet}}$. Thus, either $\#\text{Ram}_{g_{\alpha_{2}}^{\bullet}} = 0$ or $\#\text{Ram}_{g_{\alpha_{2}}^{\bullet}} = d$ holds. If $\#\text{Ram}_{g_{\alpha_{2}}^{\bullet}} = 0$, then $g_{\alpha_{2}}^{\bullet}$ is an étale covering over $k_{2}$. Then Lemma 3.1 implies that $g_{\alpha_{1}}^{\bullet}$ is an étale covering over $k_{1}$. This contradicts to $\alpha_{1} \in U_{1}^{\text{mp}}$. Then we have $\#\text{Ram}_{g_{\alpha_{2}}^{\bullet}} = d$. This means that $\alpha_{2} \in U_{2}^{\text{mp}}$. Thus, we obtain

$$\psi_{Y_{1}}^{\ell}(U_{1}^{\text{mp}}) \leq U_{2}^{\text{mp}}.$$  

On the other hand, Lemma 2.2 implies that $\#U_{1}^{\text{mp}} = \#U_{2}^{\text{mp}}$. We have

$$\psi_{Y_{1}}^{\ell}(U_{1}^{\text{mp}}) = U_{2}^{\text{mp}}.$$  

Then Proposition 2.1 implies that $\psi_{Y_{1}}^{\ell}$ induces a bijection

$$\rho_{\phi} : D_{X_{1}} \sim D_{X_{2}}.$$  

Remark 2.1.1 implies that $\rho_{\phi}$ does not depend on $\psi_{Y_{1}}^{\ell}$. Then Remark 2.1.2 implies that the bijection $\rho_{\phi}$ can be reconstructed group-theoretically from $\phi$. This completes the proof of the lemma.

Let $m$ be a natural number and $U_{2} := \Pi_{X_{2}^{\circ}}^{\circ}(m)$. We set $U_{1} := \phi^{-1}(U_{2}) \subseteq \Pi_{X_{1}^{\bullet}}$. Write $Y_{U_{1}}^{\bullet} := (Y_{U_{1}}, D_{Y_{U_{1}}})$ for the smooth pointed stable curve of type $(g_{Y_{U_{1}}}, n_{Y_{U_{1}}})$ over $k_{1}$ corresponding to $U_{1}$, $Y_{U_{2}}^{\bullet} := (Y_{U_{2}}, D_{Y_{U_{2}}})$ for the smooth pointed stable curve of type $(g_{Y_{U_{2}}}, n_{Y_{U_{2}}})$ over $k_{2}$ corresponding to $U_{2}$. Then we obtain two connected tame Galois coverings

$$f_{U_{1}}^{\bullet} : Y_{U_{1}}^{\bullet} \rightarrow X_{1}^{\bullet}$$

over $k_{1}$ and

$$f_{U_{2}}^{\bullet} : Y_{U_{2}}^{\bullet} \rightarrow X_{2}^{\bullet}$$
over $k_2$. Note that we have

$$(g_{Y_{U_1}^1}, n_{Y_{U_1}^1}) = (g_{Y_{U_2}^1}, n_{Y_{U_2}^1}).$$

Moreover, $\phi$ induces a commutative diagram as follows:

$$
\begin{array}{ccc}
U_1 & \xrightarrow{\phi|_{U_1}} & U_2 \\
\downarrow & & \downarrow \\
\Pi_{X_1^1} & \xrightarrow{\phi} & \Pi_{X_2^1} \\
\downarrow & & \downarrow \\
\Pi_{X_1^1}/U_1 & = & \Pi_{X_2^1}/U_2 = \Pi_{X_2^1}^{ab} \otimes \mathbb{Z}/m\mathbb{Z},
\end{array}
$$

where $\phi|_{U_1}$ is a surjection and the bottom arrow is an isomorphism. Note that, if $(m, p) = 1$, we have $U_1 = \Pi_{X_1^1}(m)$ and $\Pi_{X_1^1}/U_1 = \Pi_{X_2^1}^{ab} \otimes \mathbb{Z}/m\mathbb{Z}$.

Suppose that $g_X \geq 2$. Lemma 2.3 implies that $\phi|_{U_1}$ induces a bijection

$$\rho_{\phi|_{U_1}} : D_{Y_{U_1}} \xrightarrow{\sim} D_{Y_{U_2}}.$$

Then Corollary 1.7 implies a diagram as follows:

$$
\begin{array}{ccc}
D_{Y_{U_1}} & \xrightarrow{\rho_{\phi|_{U_1}}} & D_{Y_{U_2}} \\
\gamma_{U_1} \downarrow & & \gamma_{U_2} \downarrow \\
D_{X_1} & \xrightarrow{\rho_\phi} & D_{X_2}.
\end{array}
$$

We have the following lemma.

**Lemma 3.3.** Suppose that $g_X \geq 2$. The diagram obtained above

$$
\begin{array}{ccc}
D_{Y_{U_1}} & \xrightarrow{\rho_{\phi|_{U_1}}} & D_{Y_{U_2}} \\
\gamma_{U_1} \downarrow & & \gamma_{U_2} \downarrow \\
D_{X_1} & \xrightarrow{\rho_\phi} & D_{X_2}
\end{array}
$$

is a commutative diagram. Moreover, the commutative diagram can be reconstructed group-theoretically from the commutative diagram of profinite groups

$$
\begin{array}{ccc}
U_1 & \xrightarrow{\phi|_{U_1}} & U_2 \\
\downarrow & & \downarrow \\
\Pi_{X_1^1} & \xrightarrow{\phi} & \Pi_{X_2^1}.
\end{array}
$$

**Proof.** By applying Corollary 1.7 and Lemma 3.1, to verify the lemma, we only need to check that the diagram is commutative.
Let \( e_{U_{1}} \in D_{Y_{U_{1}}}, e_{U_{2}} := \rho_{\phi|U_{1}}(e_{U_{1}}) \in D_{Y_{U_{2}}}, e_{1} := \gamma_{f_{U_{1}}}(e_{U_{1}}) \in D_{X_{1}}, e_{2} = (\gamma_{f_{U_{2}}} \circ \rho_{\phi|U_{2}})(e_{U_{2}}) \in D_{X_{2}} \). Let us prove that \( e_{1} = e'_{1} \). Write \( S_{U_{1}} \) and \( S_{U_{2}} \) for the sets \( (f_{U_{1}})^{-1}(e_{1}) \) and \( (f_{U_{2}})^{-1}(e_{2}) \), respectively. Note that \( e_{U_{2}} \in S_{U_{2}} \). To verify \( e_{1} = e'_{1} \), we only need to prove that \( e_{U_{1}} \in S_{U_{1}} \).

Let \((\ell, d, f_{2}^{\bullet} : Y_{2}^{\bullet} \to X_{2}^{\bullet})\) be a data defined as in Section 2. Suppose that \((\ell, m) = 1\) and \((d, m) = 1\). By lemma 3.1, we obtain a data

\[(\ell, d, f_{1}^{\bullet} : Y_{1}^{\bullet} \to X_{1}^{\bullet})\]

induced by \( \phi \) and \((\ell, d, f_{2}^{\bullet} : Y_{2}^{\bullet} \to X_{2}^{\bullet})\). On the other hand, we have a data

\[(\ell, d, g_{2}^{\bullet} : Z_{2}^{\bullet} := Y_{2}^{\bullet} \times_{X_{2}^{\bullet}} Y_{U_{2}}^{\bullet} \to Y_{U_{2}}^{\bullet})\]

induced by the natural inclusion \( U_{2} \hookrightarrow \Pi_{X_{2}^{\bullet}} \) and \((\ell, d, f_{2}^{\bullet} : Y_{2}^{\bullet} \to X_{2}^{\bullet})\). Again, by lemma 3.1, we obtain a data

\[(\ell, d, g_{1}^{\bullet} : Z_{1}^{\bullet} := Y_{1}^{\bullet} \times_{X_{1}^{\bullet}} Y_{U_{1}}^{\bullet} \to Y_{U_{1}}^{\bullet})\]

induced by \( \phi|_{U_{1}} \) and \((\ell, d, g_{2}^{\bullet} : Z_{2}^{\bullet} \to Y_{U_{2}}^{\bullet})\).

Let \( \alpha_{2} \in U_{Y_{1}^{\bullet},e_{1}'}^{mp} \), where \( U_{(-)}^{mp} \) is defined as in Section 2. Then the proof of lemma 3.2 implies that \( \alpha_{2} \) induces an element

\[\alpha_{1} \in U_{Y_{2}^{\bullet},e_{2}'}^{mp}.\]

Write \( Y_{\alpha_{1}^{\bullet}} \) and \( Y_{\alpha_{2}^{\bullet}} \) for the smooth pointed stable curves over \( k_{1} \) and \( k_{2} \) corresponding to \( \alpha_{1} \) and \( \alpha_{2} \), respectively. Consider the connected tame Galois covering

\[Y_{\alpha_{2}^{\bullet}} \times_{X_{2}^{\bullet}} Y_{U_{2}}^{\bullet} \to Z_{2}^{\bullet}\]

degree \( \ell \) over \( k_{2} \), and write \( \beta_{2} \) for the element of \( U_{Z_{2}^{\bullet}}^{mp} \) corresponding to this connected tame Galois covering, where \( U_{(-)}^{mp} \) is defined as in Section 2. Then we have

\[\beta_{2} = \sum_{c_{2} \in S_{U_{2}}} t_{c_{2}} \beta_{c_{2}},\]

where \( t_{c_{2}} \in (\mathbb{Z}/\ell\mathbb{Z})^{\times} \) and \( \beta_{c_{2}} \in U_{Z_{2}^{\bullet},e_{2}'}^{mp} \). On the other hand, the proof of Lemma 3.2 implies that \( \beta_{2} \) induces an element

\[\beta_{1} = \sum_{c_{1} \in S_{U_{1}}} t_{c_{1}} \beta_{c_{1}} \in U_{Z_{1}^{\bullet},e_{1}'}^{mp},\]

where \( t_{c_{1}} \in (\mathbb{Z}/\ell\mathbb{Z})^{\times} \) and \( \beta_{c_{1}} \in U_{Z_{1}^{\bullet},e_{1}'}^{mp} \). Note that since \( \beta_{1} \) corresponds to the connected tame Galois covering \( Y_{\alpha_{2}^{\bullet}} \times_{X_{2}^{\bullet}} Y_{U_{2}}^{\bullet} \to Z_{2}^{\bullet} \), we have the composition of the connected tame Galois covering \( Y_{\alpha_{2}^{\bullet}} \times_{X_{2}^{\bullet}} Y_{U_{2}}^{\bullet} \to Z_{2}^{\bullet} \) and the étale Galois covering \( g_{1}^{\bullet} : Z_{1}^{\bullet} \to Y_{U_{1}}^{\bullet} \) is tamely ramified over \( e_{1}' \). This means that \( e_{U_{1}} \) is contained in \( S_{U_{1}} \). This completes the proof of the lemma.

\[\square\]

Remark 3.3.1. We maintain the notations introduced in the proof of Lemma 3.3. Let \( A_{U} := \Pi_{X_{1}^{\bullet},U_{1}} = \Pi_{X_{2}^{\bullet},U_{2}} = \Pi_{X_{2}^{\bullet},e_{2}'}^{ab} \otimes \mathbb{Z}/m\mathbb{Z} \). The sets of line bundles

\[\bigcup_{c_{1} \in S_{U_{1}}} U_{Z_{1}^{\bullet},c_{1}}^{mp} \quad \text{and} \quad \bigcup_{c_{2} \in S_{U_{2}}} U_{Z_{2}^{\bullet},c_{2}}^{mp}\]

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admit natural actions of $A_U$ which are induced by the surjections $\Pi_{X^*_1} \to A_U$ and $\Pi_{X^*_2} \to A_U$, respectively. Then $D_{\gamma_{U_1}}$ and $D_{\gamma_{U_2}}$ admit actions of $A_U$ which are induced by the actions of $A_U$ on the sets of line bundles above, respectively. Note that it is easy to see that the actions of $A_U$ on $D_{\gamma_{U_1}}$ and $D_{\gamma_{U_2}}$ above identify with the natural actions of $A_U$ on $D_{\gamma_{U_1}}$ and $D_{\gamma_{U_2}}$ respectively. Moreover, we have the commutative diagram

\[
\begin{array}{ccc}
D_{\gamma_{U_1}} & \xrightarrow[\rho_{\gamma_{U_1}}]{} & D_{\gamma_{U_2}} \\
\downarrow_{\gamma_{U_1}} & & \downarrow_{\gamma_{U_2}} \\
D_{X_1} & \xrightarrow[\rho_{\phi}]{} & D_{X_2}
\end{array}
\]

is compatible with the actions of $A_U$ on the sets of marked points.

Next, we prove the main theorem of this section.

**Theorem 3.4.** Let $X^{\ast,\text{sol}}_1 := (X^1_{\text{sol}}, D_{X^1_{\text{sol}}})$ and $X^{\ast,\text{sol}}_2 := (X^2_{\text{sol}}, D_{X^2_{\text{sol}}})$ be the pairs (see Section 1 for the definition) associated to $X^*_1$ and $X^*_2$, respectively. Let $\bar{e}_2 \in D_{X^2_{\text{sol}}}$ and $I_{\bar{e}_2} \in \text{Ine}(\Pi_{X^*_2})$ the inertia subgroup associated to $e_2$. Then there exists an inertia subgroup $I_{\bar{e}_1} \in \text{Ine}(\Pi_{X^*_1})$ associated to a point $\bar{e}_1 \in D_{X^1_{\text{sol}}}$ such that $\phi(I_{\bar{e}_1}) = I_{\bar{e}_2}$.

Moreover, the restriction homomorphism

$\phi|_{I_{\bar{e}_1}} : I_{\bar{e}_1} \to I_{\bar{e}_2}$

is an isomorphism.

**Proof.** Let $N >> 0$ be an integer number such that $(N, p) = 1$. We set $F_1 := \Pi_{X^*_1}(N)$ and $F_2 := \Pi_{X^*_2}(N)$. Then $\phi$ induces a commutative diagram as follows.

\[
\begin{array}{ccc}
F_1 & \longrightarrow & F_2 \\
\downarrow & & \downarrow \\
\Pi_{X^*_1} & \xrightarrow[\phi]{} & \Pi_{X^*_2} \\
\downarrow & & \downarrow \\
\Pi_{X^*_1}^{ab} \otimes \mathbb{Z}/N\mathbb{Z} & \longrightarrow & \Pi_{X^*_2}^{ab} \otimes \mathbb{Z}/N\mathbb{Z},
\end{array}
\]

where the top arrow is a surjection induced by $\phi$. Then we obtain a smooth pointed stable curve $Y^*_1$ of type $(g_{Y^*_1}, n_{Y^*_1})$ over $k_1$ corresponding to $F_1$ and a smooth pointed stable curve $Y^*_2$ of type $(g_{Y^*_2}, n_{Y^*_2})$ over $k_2$ corresponding to $F_2$. Since $X^*_1$ and $X^*_2$ are smooth pointed stable curves of type $(g_X, n_X)$, we obtain $g_{Y^*_1} = g_{Y^*_2} \geq 2$ and $n_{Y^*_1} = n_{Y^*_2}$. To verify the theorem, by replacing $X^*_1$ and $X^*_2$ by $Y^*_1$ and $Y^*_2$, respectively, we may assume that $g_X \geq 2$.

Let $C_{X^*_2} := \{H_{2,i}\}_{i \in \mathbb{Z}_{>0}}$ be a set of open subgroups of $\Pi_{X^*_1}$ satisfying the following conditions:
Let obtain a sequence of marked points $k$ for each homomorphism $\varphi$.

For each $i \in \mathbb{Z}_{\geq 0}$, we write $X^*_{H^2,i} := (X_{H^2,i}, D_{X_{H^2,i}})$ for the smooth pointed stable curve over $k_2$ corresponding to $H^2_i$, $e_{H^2,i} \in D_{X_{H^2,i}}$ for the image of $\bar{e}_2$ in $X^*_{H^2,i}$. Then we obtain a sequence of marked points

$$\mathcal{C}X^*_2 : \cdots \mapsto e_{H^2,2} \mapsto e_{H^2,1} \mapsto e_{H^2,0}.$$ 

Write $\{H_{1,i} := \phi^{-1}(H^2,i)\}_{i \in \mathbb{Z}_{\geq 0}}$ for the set of open subgroups of $\Pi X^2_1$ induced by $\phi$. For each $i \in \mathbb{Z}_{\geq 0}$, we write $X^*_{H^1,i} := (X_{H^1,i}, D_{X_{H^1,i}})$ for the smooth pointed stable curve over $k_1$ corresponding to $H_{1,i}$. Then, for each $i \in \mathbb{Z}_{\geq 0}$, Lemma 3.2 implies that the restriction homomorphism $\phi|_{H_{1,i}} : H_{1,i} \to H^2_i$ induces a natural bijection of the set of marked points $\rho_{\phi|_{H_i}} : D_{X_{1,i}} \xrightarrow{\sim} D_{X_{2,i}}$,

moreover, that $\rho_{\phi|_{H_i}}$ can be reconstructed group-theoretically from $\phi|_{H_{1,i}}$. We set

$$e_{H_{1,i}} := \rho_{\phi|_{H_i}}^{-1}(e_{H^2,i})$$

for each $i \in \mathbb{Z}_{\geq 0}$. Then, by applying Lemma 3.3, $\mathcal{C}X^*_2$ induces a sequence of marked points as follows:

$$\cdots \mapsto e_{H^2,2} \in D_{X_{H^2,2}} \mapsto e_{H^2,1} \in D_{X_{1,1}} \mapsto e_{H^1,0} \in D_{X_{1,0}} = D_{X_1}.$$ 

Let $K_{\ker(\phi)}$ be the subfield of $\overline{K}^\text{sol}$ corresponding to the closed subgroup $\ker(\phi)$ of $\Pi X^2_1$. We set

$$\overline{X}_{\ker(\phi)} := (\overline{X}_{\ker(\phi)}, D_{\overline{X}_{\ker(\phi)}}),$$

where $\overline{X}_{\ker(\phi)}$ denotes the normalization of $X$ in $K_{\ker(\phi)}$ and $D_{\overline{X}_{\ker(\phi)}}$ denotes the inverse image of $D_X$ in $\overline{X}_{\ker(\phi)}$. Then the sequence

$$\cdots \mapsto e_{H^2,2} \mapsto e_{H^2,1} \mapsto e_{H^1,0}.$$ 

determines a point $\overline{e}_{\ker(\phi)} \in D_{\overline{X}_{\ker(\phi)}}$. We choose a point of $\overline{e}_1 \in D_{\overline{X}_{1}}$ such that the image of $\overline{e}_1$ in $D_{\overline{X}_{\ker(\phi)}}$ is $\overline{e}_{\ker(\phi)}$. Then we have $\phi(I_{\overline{e}_1}) = I_{\overline{e}_2}$.

Moreover, since $I_{\overline{e}_1}$ and $I_{\overline{e}_2}$ are isomorphic to $\overline{\mathbb{Z}}(1)^{\delta'}$, the restriction homomorphism $\phi|_{I_{\overline{e}_1}}$ is an isomorphism. This completes the proof of the theorem.

In the remainder of this section, we reconstruct “additive structures” of inertia groups. Let $\overline{e}_2$ be any point of $D_{\overline{X}_{2}}$ and $\overline{e}_1$ a point of $D_{\overline{X}_{1}}$ such that $\phi(I_{\overline{e}_1}) = \overline{e}_2$. Write $\overline{F}_1$ (resp. $\overline{F}_2$) for the algebraic closure of $\mathbb{F}_p$ in $k_1$ (resp. $k_2$). We set

$$\overline{F}_{\overline{e}_1} := (I_{\overline{e}_1} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_{1}^{\delta'}) \coprod \{*_{\overline{e}_1}\}$$ (resp. $\overline{F}_{\overline{e}_2} := (I_{\overline{e}_2} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_{2}^{\delta'}) \coprod \{*_{\overline{e}_2}\}$).
where \{e_1\} (resp. \{e_2\}) is an one-point set, and \((\mathbb{Q}/\mathbb{Z})'_1\) (resp. \((\mathbb{Q}/\mathbb{Z})'_2\)) denotes the prime-to-\(p\) part of \((\mathbb{Q}/\mathbb{Z})_1\) (resp. \((\mathbb{Q}/\mathbb{Z})_2\)), can be canonically identified with

\[
\bigcup_{(p,m)=1} \mu_m(k_1) \quad \text{(resp.} \quad \bigcup_{(p,m)=1} \mu_m(k_2))
\]

Moreover, \(F_{e_1}\) (resp. \(F_{e_2}\)) can be identified with \(\overline{F}_1\) (resp. \(\overline{F}_2\)) as set, hence, carries a structure of field, whose multiplicative group is \(I_{e_1} \otimes \mathbb{Z}(\mathbb{Q}/\mathbb{Z})'_1\) (resp. \(I_{e_2} \otimes \mathbb{Z}(\mathbb{Q}/\mathbb{Z})'_2\)) and whose zero element is \(e_1\) (resp. \(e_2\)). Then we have the following proposition.

**Corollary 3.5.** The field structures of \(F_{e_1}\) and \(F_{e_2}\) can be reconstructed group-theoretically from \(\Pi_{X^*_1}\) and \(\Pi_{X^*_2}\), respectively. Moreover, \(\phi\) induces a field isomorphism

\[
\theta_{\phi,e_1,e_2} : F_{e_1} \rightarrow F_{e_2},
\]

and \(\theta_{\phi,e_1,e_2}\) can be reconstructed group-theoretically from \(\phi\).

**Proof.** To verify the theorem, similar arguments to the arguments given in the proof of [T4, Proposition 5.3] imply that we may assume that \(n_X = 3\).

For each natural number \(r\), we denote by \(F_{p^r,e_1}\) (resp. \(F_{p^r,e_2}\)) the unique subfield of \(F_{e_1}\) (resp. \(F_{e_2}\)). We fix any finite field \(F_{p^r}\) of cardinality \(p^r\) and an algebraic closure \(\overline{F}\) of \(F_{p^r}\). By Proposition 1.6 (c), we have \(F_{p^r,e_1} = I_{e_1}/(p^r - 1)\) (resp. \(F_{p^r,e_2} = I_{e_2}/(p^r - 1)\)) can be reconstructed group-theoretically from \(\Pi_{X^*_1}\) (resp. \(\Pi_{X^*_2}\)). Then the set

\[
\text{Hom}_{\text{group}}(F_{p^r,e_1}, F_{p^r}) \quad \text{(resp.} \quad \text{Hom}_{\text{group}}(F_{p^r,e_2}, F_{p^r}))
\]

is group-theoretically, and reconstructing the field structure of \(F_{p^r,e_1}\) (resp. \(F_{p^r,e_2}\)) is equivalent to reconstructing

\[
\text{Hom}_{\text{fields}}(F_{p^r,e_1}, F_{p^r}) \quad \text{(resp.} \quad \text{Hom}_{\text{fields}}(F_{p^r,e_2}, F_{p^r}))
\]

as a subset of \(\text{Hom}_{\text{group}}(F_{p^r,e_1}, F_{p^r})\) (resp. \(\text{Hom}_{\text{group}}(F_{p^r,e_2}, F_{p^r}))\). Note that, to reconstruct the field structure of \(F_{e_1}\) (resp. \(F_{e_2}\)), it is sufficient to reconstruct the subset \(\text{Hom}_{\text{fields}}(F_{p^r,e_1}, F_{p^r})\) (resp. \(\text{Hom}_{\text{fields}}(F_{p^r,e_2}, F_{p^r}))\) for \(r\) in a cofinal subset of \(N\) with respect to division.

Let

\[
\chi_1 \in \text{Hom}_{\text{groups}}(\Pi_{X^*_1} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}, F_{p^r})
\]

(resp. \(\chi_2 \in \text{Hom}_{\text{groups}}(\Pi_{X^*_2} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}, F_{p^r})\)).

Write \(H_{\chi_1}\) (resp. \(H_{\chi_2}\)) for \(\Pi_{X^*_1}(p^r - 1)\) (resp. \(\Pi_{X^*_2}(p^r - 1)\)), \(M_{\chi_1}\) for \(H_{\chi_1} \otimes F_p\) (resp. \(M_{\chi_2}\) for \(H_{\chi_2} \otimes F_p\)), and

\[
X^*_1 := (X_{H_{\chi_1}}, D_{X_{H_{\chi_1}}}) \quad \text{(resp.} \quad X^*_2 := (X_{H_{\chi_2}}, D_{X_{H_{\chi_2}}}))
\]

for the smooth pointed stable curve over \(k_1\) (resp. \(k_2\)) corresponding to \(H_{\chi_1}\) (resp. \(H_{\chi_2}\)). Note that \(M_{\chi_1}\) (resp. \(M_{\chi_2}\)) is a \(F_{p}[\Pi_{X^*_1} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}]-\text{module}\) (resp. \(F_{p}[\Pi_{X^*_2} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}]-\text{module}\) via conjugation. We define

\[
M_{\chi_1}[\chi_1] := \{a \in M_{\chi_1} \otimes F_p \mid \sigma(a) = \chi_1(\sigma)a \text{ for all } \sigma \in \Pi_{X^*_1} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}\}
\]
Thus, we obtain that the field structures of \( F \). Suppose that \( \phi \) is an isomorphism. On the other hand, by Theorem 3.4, we have an isomorphism \( \Gamma_1, r : \text{Hom}_{\text{groups}}(\Pi_{X^1}^{ab} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}, \mathbb{F}_p^\times) \rightarrow \text{Hom}(\mathbb{F}_p^\times, F_p^\times) \)

(resp. \( \Gamma_2, r : \text{Hom}_{\text{groups}}(\Pi_{X^2}^{ab} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}, \mathbb{F}_p^\times) \rightarrow \mathbb{Z}_{\geq 0} \))

where the map \( \text{Res}_{1, r} \) (resp. \( \text{Res}_{2, r} \)) is the restriction with respect to the natural inclusion \( F_{p^r, e_1} \leftarrow \Pi_{X^1}^{ab} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z} \) (resp. \( F_{p^r, e_2} \leftarrow \Pi_{X^2}^{ab} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z} \)), and the map \( \Gamma_1, r \) (resp. \( \Gamma_2, r \)) is the map that maps \( \chi_1 \mapsto \gamma_1(M_{X^1}) \) (resp. \( \chi_2 \mapsto \gamma_2(M_{X^2}) \)).

Let \( m_0 \) be the product of all prime numbers \( \leq p - 2 \) if \( p \neq 2, 3 \) and \( m_0 = 1 \) if \( p = 2, 3 \). Let \( r_0 \) be the order of \( p \) in the multiplicative group \( (\mathbb{Z}/m_0\mathbb{Z})^\times \). Then [T4, Claim 5.4] implies the following result:

there exists a constant \( C(g_X) \) which only depends on \( g_X \) such that, for each \( r > \log_p(C(g_X) + 1) \) divisible by \( r_0 \), we have

\[
\text{Hom}_{\text{fields}}(F_{p^r, e_1}, F_{p^r}) = \text{Hom}_{\text{groups}}(\mathbb{F}_p^\times, F_{p^r}^\times) \setminus \text{Res}_{1, r}(\Gamma_1^{-1}(\{g_X + 1\}))
\]

(resp. \( \text{Hom}_{\text{fields}}(F_{p^r, e_2}, F_{p^r}) = \text{Hom}_{\text{groups}}(\mathbb{F}_p^\times, F_{p^r}^\times) \setminus \text{Res}_{2, r}(\Gamma_2^{-1}(\{g_X + 1\})) \)),

where \( \text{Hom}_{\text{groups}}(-, -) \) denotes the set of surjections of \( \text{Hom}_{\text{groups}}(-, -) \). Thus, we obtain that the field structures of \( F_{e_1} \) and \( F_{e_2} \) can be reconstructed group-theoretically from \( \Pi_{X^1}^{ab} \) and \( \Pi_{X^2}^{ab} \), respectively.

Next, we prove the "moreover" part of the proposition. Let

\[
\kappa_2 \in \text{Hom}_{\text{groups}}(\Pi_{X^2}^{ab} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}, \mathbb{F}_p^\times).
\]

Then \( \phi \) induced a character

\[
\kappa_1 \in \text{Hom}_{\text{groups}}(\Pi_{X^1}^{ab} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}, \mathbb{F}_p^\times).
\]

Moreover, \( \phi|_{H_{e_1}} \) induces a surjection

\[
M_{\kappa_1}[\kappa_1] \rightarrow M_{\kappa_2}[\kappa_2].
\]

Suppose that \( \kappa_2 \in \Gamma_{2, r}^{-1}(\{g_X + 1\}) \). Then we obtain that the surjection \( M_{\kappa_1}[\kappa_1] \rightarrow M_{\kappa_2}[\kappa_2] \) is an isomorphism. On the other hand, by Theorem 3.4, we have an isomorphism \( \phi|_{I_{e_1}} : I_{e_1} \rightarrow I_{e_1} \). Then the isomorphism \( \phi|_{I_{e_1}} \) induces an injective

\[
\text{Res}_{2, r}(\Gamma_{2, r}^{-1}(\{g_X + 1\})) \hookrightarrow \text{Res}_{1, r}(\Gamma_{1, r}^{-1}(\{g_X + 1\})).
\]
Since \( \#\text{Hom}_{\text{fields}}(F, X_1, F) = \#\text{Hom}_{\text{fields}}(F, X_2, F) \), \( \phi|_{X_1} \) induces a bijection

\[
\text{Hom}_{\text{fields}}(F, X_1, F) \cong \text{Hom}_{\text{fields}}(F, X_2, F).
\]

Thus, \( \phi|_{X_1} \) induces a bijection

\[
\text{Hom}_{\text{fields}}(F, X_1, F) \cong \text{Hom}_{\text{fields}}(F, X_2, F).
\]

If we choose \( F = F_{\bar{e}_1} \), then the bijection above induces a field isomorphism

\[
\theta_{\phi, \bar{e}_1, \bar{e}_2} : F_{\bar{e}_1} \cong F_{\bar{e}_2}.
\]

This completes the proof of the proposition.

\[
\square
\]

4 A weak Hom-version of the Grothendieck conjecture for curves of type \((0, n)\)

We maintain the notations introduced in Section 3. Moreover, in this section, we suppose that \((g_X, n_X) = (0, n)\).

Fix two marked points \( e_{1,\infty}, e_{1,0} \in D_X \) distinct from each other. We choose any field \( k_1' \) that is isomorphic to \( k_1 \), and choose any isomorphism \( \varphi_1 : X_1 \cong \mathbb{P}_{k_1}^1 \) as schemes such that \( \varphi_1(e_{1,\infty}) = \infty \) and \( \varphi_1(e_{1,0}) = 0 \). Then the set of \( k_1 \)-rational points \( X_1(k_1) \setminus \{ e_{1,\infty} \} \) is equipped with a structure of \( F_p \)-module via the bijection \( \varphi_1 \). Note that since any \( k_1' \)-isomorphism of \( \mathbb{P}_{k_1}^1 \) fixing \( \infty \) and \( 0 \) is a scalar multiplication, the \( F_p \)-module structure of \( X_1(k_1) \setminus \{ e_{1,\infty} \} \) does not depend on the choices of \( k_1' \) and \( \varphi_1 \) but depends only on the choices of \( e_{1,\infty} \) and \( e_{1,0} \). Then we shall call \( X_1(k_1) \setminus \{ e_{1,\infty} \} \) equipped with a structure of \( F_p \)-module with respect to \( e_{1,\infty} \) and \( e_{1,0} \). On the other hand, by Lemma 3.2, \( \phi \) induces a bijection \( \rho_\phi : D_X \cong D_X \). We write \( e_{2,\infty} \) and \( e_{2,0} \) for \( \rho_\phi(e_{1,\infty}) \) and \( \rho_\phi(e_{1,0}) \), respectively.

**Lemma 4.1.** Consider the following linear condition:

\[
\sum_{e_1 \in D_X \setminus \{ e_{1,\infty}, e_{1,0} \}} b_{e_1} e_1 = e_{1,0}, \text{ with respect to } e_{1,\infty}, e_{1,0}
\]

on \( X_1^* \), where \( b_{e_1} \in F_p \) for each \( e_1 \in D_X \setminus \{ e_{1,\infty}, e_{1,0} \} \). Then we can detect, group-theoretically from \( \Pi_X \), whether the linear condition defined above holds or not. Moreover, if the linear condition defined above holds, then the linear condition

\[
\sum_{e_2 \in D_X \setminus \{ e_{2,\infty}, e_{2,0} \}} b_{e_1} \rho_\phi(e_1) = e_{1,0}, \text{ with respect to } e_{2,\infty}, e_{2,0}
\]

on \( X_2^* \) also holds.

**Proof.** Let \( \bar{e}_{2,\infty} \in D_{X_2} \) be a point over \( e_{2,\infty} \). Then the set

\[
F_{e_{2,\infty}} := (I_{\bar{e}_{2,\infty}} \otimes \mathbb{Q}/\mathbb{Z}) \prod \{ *_{e_{2,\infty}} \}
\]

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carries a structure of field, and Corollary 3.5 implies that the field structure can be reconstructed group-theoretically from $\Pi X^\bullet_1$. Theorem 3.4 implies that there exists a point $\tilde{e}_{1,\infty} \in D_{\tilde{X}^1_{pol}}$, and that $\phi(I_{\tilde{e}_{1,\infty}}) = \tilde{e}_{2,\infty}$. Moreover, Corollary 3.5 implies that the set

$$F_{\tilde{e}_{1,\infty}} := (I_{\tilde{e}_{1,\infty}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_1') \prod \{ *_{e_{1,\infty}} \}$$

carries a structure of field, that the field structure can be reconstructed group-theoretically from $\Pi X^\bullet_1$, and that $\phi$ induces a field isomorphism

$$\theta_{\phi, \tilde{e}_{1,\infty}, \tilde{e}_{2,\infty}} : F_{\tilde{e}_{1,\infty}} \cong F_{\tilde{e}_{2,\infty}}.$$

For each $e_1 \in D_X$, we choose $b'_{e_1} \in \mathbb{Z}_{\geq 0}$ such that

$$b'_{e_1} = b_{e_1} \pmod{p}, \quad \text{and that} \quad \sum_{e_1 \in D_X \setminus \{ e_{1,\infty}, e_{1,0} \}} b'_{e_1} \geq 2.$$ 

Let $r \geq 1$ such that

$$p^r - 2 \geq \sum_{e_1 \in D_{X_1} \setminus \{ e_{1,\infty}, e_{1,0} \}} b'_{e_1}.$$

For each $e_1 \in D_{X_1}$ and each $\tilde{e}_1 \in D_{\tilde{X}_{pol}}$ over $e_1$, write $I_{\tilde{e}_1, ab}$ for the image of the composition of the natural morphisms $I_{\tilde{e}_1} \hookrightarrow \Pi_{X^\bullet_{1}} \twoheadrightarrow \Pi_{X^\bullet_{1}}^{ab}$. Moreover, since the image of $I_{\tilde{e}_1, ab}$ does not depend on the choice of $\tilde{e}_1$, we may write $I_{e_1}$ for $I_{\tilde{e}_1, ab}$. We define

$$I_{e_{1,\infty}} \twoheadrightarrow \mathbb{Z}/(p^r - 1)\mathbb{Z} \quad \text{that maps} \ 1 \mapsto 1,$$

$$I_{e_{1,0}} \twoheadrightarrow \mathbb{Z}/(p^r - 1)\mathbb{Z} \quad \text{that maps} \ 1 \mapsto (\sum_{e_1 \in D_{X_1} \setminus \{ e_{1,\infty}, e_{1,0} \}} b'_{e_1}) - 1,$$

and

$$I_{e_1} \twoheadrightarrow \mathbb{Z}/(p^r - 1)\mathbb{Z} \quad \text{that maps} \ 1 \mapsto -b'_{e_1} \quad \text{for each} \ e_1 \in D_{X_1} \setminus \{ e_{1,\infty}, e_{1,0} \}.$$

Then the surjections of inertia groups defined above induces a sujection

$$\delta_1 : \Pi_{X^\bullet_{1}} \twoheadrightarrow \mathbb{Z}/(p^r - 1)\mathbb{Z}.$$

Write $H_{\delta_1}$ for the kernel of $\delta_1$, $M_{\delta_1}$ for $H_{\delta_1}^{ab} \otimes \mathbb{F}_p$, and $X^{\bullet}_{H_{\delta_1}} := (X_{H_{\delta_1}}, D_{X_{H_{\delta_1}}})$ for the smooth pointed stable curve over $k_1$ corresponding to $H_{\delta_1}$. Note that $M_{\delta_1}$ admits a natural action of $\Pi_{X^\bullet_{1}}$ via conjugation. Then $M_{\delta_1}$ admits a natural action of $I_{\tilde{e}_{1,\infty}}$ via a character

$$\chi_{I_{\tilde{e}_{1,\infty}, r}} : I_{\tilde{e}_{1,\infty}} \hookrightarrow \Pi_{X^\bullet_{1}} \twoheadrightarrow \mathbb{Z}/(p^r - 1)\mathbb{Z} = I_{\tilde{e}_{1,\infty}}/(p^r - 1) \hookrightarrow \mathbb{F}_{\tilde{e}_{1,\infty}}^\times,$$

where the middle morphism is $\delta_1$. We set

$$M_{\delta_1}[X_{I_{\tilde{e}_{1,\infty}, r}}] := \{ a \in M_{\delta_1} \otimes \mathbb{F}_p F_{\tilde{e}_{1,\infty}} \mid \sigma(a) = \chi_{I_{\tilde{e}_{1,\infty}, r}}(\sigma)a \text{ for all } \sigma \in I_{\tilde{e}_{1,\infty}} \}.$$ 

Then the proof of [T2, Lemma 3.3] implies that the linear condition

$$\sum_{e_1 \in D_{X_1} \setminus \{ e_{1,\infty}, e_{1,0} \}} b_{e_1} e_1 = e_{1,0}, \quad \text{with respect to } e_{1,\infty}, e_{1,0}.$$
on $X_1^\bullet$ holds if and only if $M_{\delta_1}[\chi_{\bar{e}_1,\infty,r}] = 0$. This completes the proof of the first part of the lemma.

Next, let us prove the “moreover” part. Since $(p, p^r - 1) = 1$, the surjection $\phi$ induces a surjection

$$\delta_2 : \Pi X_2^\bullet \to \mathbb{Z}/(p^r - 1)\mathbb{Z}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc}
H_{\delta_1} & \xrightarrow{\phi|_{H_{\delta_1}}} & H_{\delta_2} \\
\downarrow & & \downarrow \\
\Pi X_1^\bullet & \xrightarrow{\phi} & \Pi X_2^\bullet \\
\delta_1 \downarrow & & \delta_2 \downarrow \\
\mathbb{Z}/(p^r - 1)\mathbb{Z} & \xrightarrow{=} & \mathbb{Z}/(p^r - 1)\mathbb{Z},
\end{array}$$

where $H_{\delta_2}$ denotes the kernel of $\delta_2$. Write $M_{\delta_2}$ for $H_{\delta_2}^0 \otimes \mathbb{F}_p$ and $X_{H_{\delta_2}}^\bullet := (X_{H_{\delta_1}}, DX_{H_{\delta_2}})$ for the smooth pointed stable curve over $k_2$ corresponding to $H_{\delta_2}$. Similar arguments to the arguments given above imply that $M_{\delta_2}$ admits a natural action of $I_{e_2,\infty}$ via a character $\chi_{\bar{e}_2,\infty,r} : I_{\bar{e}_2,\infty} \hookrightarrow \Pi X_2^\bullet \to \mathbb{Z}/(p^r - 1)\mathbb{Z} = I_{\bar{e}_2,\infty}/(p^r - 1) \hookrightarrow \mathbb{F}_{\bar{e}_2,\infty}^\times$,

where the middle morphism is $\delta_2$. We set

$$M_{\delta_2}[\chi_{\bar{e}_2,\infty,r}] := \{ a \in M_{\delta_2} \otimes \mathbb{F}_p \mathbb{F}_{\bar{e}_2,\infty} \mid \sigma(a) = \chi_{\bar{e}_2,\infty,r}(\sigma)a \text{ for all } \sigma \in I_{\bar{e}_2,\infty} \}.$$ 

Then we obtain a surjection

$$M_{\delta_1}[\chi_{\bar{e}_1,\infty,r}] \twoheadrightarrow M_{\delta_2}[\chi_{\bar{e}_2,\infty,r}]$$

induced by $\phi|_{H_{\delta_1}}$ and $\theta_{\phi, \bar{e}_1,\infty, \bar{e}_2,\infty}$.

Since the linear condition

$$\sum_{e_1 \in D_{X_1}(e_1, e_1, e_{1,0})} b_{e_1}e_1 = e_{1,0} \text{ with respect to } e_{1,\infty}, e_{1,0}$$

on $X_1^\bullet$ holds, we have $M_{\delta_1}[\chi_{\bar{e}_1,\infty,r}] = 0$. Thus, we obtain $M_{\delta_2}[\chi_{\bar{e}_2,\infty,r}] = 0$. Then, by applying the first part of the lemma to $X_2^\bullet$, we have the linear condition

$$\sum_{e_2 \in D_{X_2}(e_2, e_{2,0})} b_{e_1}e_1 = e_{1,0}, \text{ with respect to } e_{2,\infty}, e_{2,0}$$

on $X_2^\bullet$ holds. \hfill \Box

**Remark 4.1.1.** Note that, if $X_1 = \mathbb{P}_k^1$, then the linear condition is the follows:

$$\sum_{e_1 \in D_{X_1}(\infty,0)} b_{e_1}e_1 = 0 \text{ with respect to } \{\infty,0\}.$$
Next, we prove the main theorem of the present paper.

**Theorem 4.2.** Let \( X^*_1 := (X_1, D_{X_1}) \) and \( X^*_2 := (X_2, D_{X_2}) \) be smooth pointed stable curves of type \((0, n)\) over algebraically closed fields \(k_1\) and \(k_2\) of characteristic \(p > 0\), respectively. Write \( \Pi_{X^*_1} \) and \( \Pi_{X^*_2} \) for the maximal pro-solvable quotients of the tame fundamental groups of \( X^*_1 \) and \( X^*_2 \), respectively. Let \( k_1^{\text{min}} \) and \( k_2^{\text{min}} \) be the minimal algebraically closed subfields of \( k_1 \) and \( k_2 \) over which \( X^*_1 \) and \( X^*_2 \) are defined, respectively; thus, by considering the function fields of \( X_1 \) and \( X_2 \), we obtain smooth pointed stable curves

\[
X_1^{\ast, \text{min}} := (X_1^{\text{min}}, D_{X_1}^{\text{min}}) \quad \text{and} \quad X_2^{\ast, \text{min}} := (X_2^{\text{min}}, D_{X_2}^{\text{min}})
\]

such that \( X_1 \setminus D_{X_1} \cong (X_1^{\text{min}} \setminus D_{X_1}^{\text{min}}) \times_{k_1^{\text{min}}} k_1 \) and \( X_2 \setminus D_{X_2} \cong (X_2^{\text{min}} \setminus D_{X_2}^{\text{min}}) \times_{k_2^{\text{min}}} k_2 \) as \( k_1 \)-schemes and \( k_2 \)-schemes, respectively.

Then we can detect whether \( X^*_1 \) can be defined over the algebraic closure \( \overline{F}_1 \) of \( F_p \) in \( k_1 \) or not, group-theoretically from \( \Pi_{X^*_1} \). Moreover, suppose that \( X^*_1 \) can be defined over the algebraic closure \( \overline{F}_1 \) of \( F_p \) in \( k_1 \). Then the set of open homomorphisms

\[
\text{Hom}^{\text{open}}(\Pi_{X^*_1}, \Pi_{X^*_2}) \neq \emptyset
\]

if and only if

\[
X_1^{\text{min}} \setminus D_{X_1}^{\text{min}} \cong X_2^{\text{min}} \setminus D_{X_2}^{\text{min}}
\]

as schemes. In particular, if this is the case, \( X^*_2 \) can be defined over the algebraic closure \( \overline{F}_2 \) of \( F_p \) in \( k_2 \).

**Proof.** Note that, it is easy to see that the proof of [T2, Theorem 3.5] also holds for \( \Pi_{X^*_1} \), then first part of the theorem follows from [T4, Theorem 5.8] and the proof of [T2, Theorem 3.5]. Let us prove the “moreover” part of the theorem.

The “if” part of the theorem is trivial. We only prove the “only if” part of the theorem. Suppose that

\[
\text{Hom}^{\text{open}}(\Pi_{X^*_1}, \Pi_{X^*_2}) \neq \emptyset.
\]

Let \( \phi \in \text{Hom}^{\text{open}}(\Pi_{X^*_1}, \Pi_{X^*_2}) \). Since \( X^*_1 \) and \( X^*_2 \) are type \((0, n)\), we have that \( \phi \) is a surjection.

Let \( \bar{e}_{2,0} \in D_{X^*_2}^{\text{sol}} \) be a point over \( e_{2,0} \). Then

\[
\overline{F}_{\bar{e}_{2,0}} := (I_{\bar{e}_{2,0}} \otimes \mathbb{Q}/\mathbb{Z})^p \coprod \{*_{e_{2,0}}\}
\]

carries a structure of field, and Corollary 3.5 implies that the field structure can be reconstructed group-theoretically from \( \Pi_{X^*_1} \). Theorem 3.4 implies that there exists a point \( \bar{e}_{1,0} \in D_{X^*_1}^{\text{sol}} \), and that \( \phi(I_{\bar{e}_{1,0}}) = \bar{e}_{2,0} \). Moreover, Corollary 3.5 implies that

\[
\overline{F}_{\bar{e}_{1,0}} := (I_{\bar{e}_{1,0}} \otimes \mathbb{Q}/\mathbb{Z})^p \coprod \{*_{e_{1,0}}\}
\]

carries a structure of field, that the field structure can be reconstructed group-theoretically from \( \Pi_{X^*_1} \), and that \( \phi \) induces a field isomorphism

\[
\theta_{\phi, \bar{e}_{1,0}, \bar{e}_{2,0}} : \overline{F}_{\bar{e}_{1,0}} \cong \overline{F}_{\bar{e}_{2,0}}.
\]
Proposition 1.6 (a) implies that $n$ can be reconstructed group-theoretically from $\Pi_{X_1^\bullet}$ or $\Pi_{X_2^\bullet}$. If $n = 3$, then the theorem is trivial, so we may assume that $n \geq 4$. Moreover, since $X_1^\bullet$ can be defined over $\mathbb{F}_1$, without loss of generality, we may assume that $k_1 = \mathbb{F}_1 = \mathbb{F}_{e_1,0}$, that $X_1 = \mathbb{P}_{\mathbb{F}_1}$, and that

$$D_{X_1} := \{e_{1,\infty} = \infty, e_{1,0} = 0, e_{1,1} = 1, e_{1,2}, \ldots, e_{1,n-2}\}.$$  

Here, $e_{1,2}, \ldots, e_{1,n-2} \in \mathbb{F}_1 \setminus \{e_{1,0}, e_{1,1}\}$ distinct from each other. By [T2, Lemma 3.4], there exists a natural number $r$ prime to $p$ such that $\mathbb{F}_p(\zeta_r)$ contains $r^{th}$ roots of $e_{1,2}, \ldots, e_{1,n-2}$, where $\zeta_r$ denotes a fixed primitive $r^{th}$ root of unity in $\mathbb{F}_1$. Let $s := [\mathbb{F}_p(\zeta_r), \mathbb{F}_p]$. For each $e_{1,i} \in \{e_{1,2}, \ldots, e_{1,n-2}\}$, we fix an $r^{th}$ root $e_{1,i}^{1/r}$ in $\mathbb{F}_1$. Then we have

$$e_{1,i}^{1/r} = \sum_{j=0}^{s-1} b_{1,ij} \zeta_r^j \text{ for each } i \in \{2, \ldots, n-2\},$$  

where $b_{1,ij} \in \mathbb{F}_p$ for each $i = 2, \ldots, n-2, j = 0, \ldots, s-1$.

Let $X_1 \setminus \{\infty\} = \text{Spec} \mathbb{F}_1[x_1]$ and

$$X_{H_1}^\bullet := (X_{H_1}, \text{D}_{X_{H_1}}) \rightarrow X_1^\bullet$$

the tame covering over $\mathbb{F}_1$ determined by the equation $y_1^r = x_1$. Write $H_1$ for the maximal pro-solvable quotient of the tame fundamental group of $X_{H_1}^\bullet$. The tame covering $X_{H_1}^\bullet \rightarrow X_1^\bullet$ is totally ramified over $e_{1,\infty}, e_{1,0}$ and is étale over $D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}$. Note that $X_{H_1} = \mathbb{P}_{\mathbb{F}_1}$, and that the unique points of $D_{X_{H_1}}$ over $e_{1,\infty} \in D_{X_1}$ and $e_{1,0} \in D_{X_1}$ are $e_{H_1,\infty} := \infty$ and $e_{H_1,0} := 0$, respectively. We set

$$e_{H_1,i} := e_{1,i}^{1/r} \in D_{X_{H_1}}$$

for each $i \in \{2, \ldots, n-2\}$ and

$$e_{H_1,1}^j := \zeta_r^j \in D_{X_{H_1}}$$

for each $j = 0, \ldots, s-1$. Thus, we obtain a linear condition on $X_{H_1}^\bullet$ as follows:

$$e_{1,H_i} = \sum_{j=0}^{s-1} b_{1,ij} e_{H_1,1}^j \text{ with respect to } \{e_{H_1,\infty}, e_{H_1,0}\}$$

for each $i \in \{2, \ldots, n-2\}$.

Since the order $\#(\Pi_{X_1^\bullet}/H_1)$ is prime to $p$, then we have the following commutative diagram:

$$
\begin{array}{ccc}
H_1 & \xrightarrow{\phi_{H_1}} & H_2 \\
\downarrow & & \downarrow \\
\Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet} \\
\downarrow & & \downarrow \\
\mathbb{Z}/(p^r - 1)\mathbb{Z} & \xrightarrow{\phi} & \mathbb{Z}/(p^r - 1)\mathbb{Z}
\end{array}
$$
Write $X_{H_2}^\bullet := (X_{H_2}, D_{X_2})$ for the smooth pointed stable curve over $k_2$ corresponding to $H_2$. Lemma 3.2 implies that the following commutative diagram of the sets of marked points can be reconstructed group-theoretically from the commutative diagram of profinite groups above:

\[
\begin{array}{c}
D_{H_1} \xrightarrow{\rho \phi H_1} D_{H_2} \\
\downarrow \quad \downarrow \\
D_{X_1} \xrightarrow{\rho \phi} D_{X_2}.
\end{array}
\]

Write
\[e_{2,\infty}, e_{2,0}, e_{2,i}, i = 1, \ldots, n - 2,\]
for $\rho \phi(e_{1,\infty}), \rho \phi(e_{1,0}), \rho \phi(e_{1,i}), i = 1, \ldots, n - 2,$
\[e_{H_2,\infty}, e_{H_2,0}, e_{H_2,i}, i = 2, \ldots, n - 2,\]
for $\rho \phi(e_{H_1,\infty}), \rho \phi(e_{H_1,0}), \rho \phi(e_{H_1,i}), i = 2, \ldots, n - 2,$
and
\[e^j_{H_2,1}, j \in \{0, \ldots, s - 1\}\]
for $\rho \phi(e^j_{H_2,1}), j \in \{0, \ldots, s - 1\}$.

We may assume that $X_2 = \mathbb{F}^1_{k_2}$, and that $e_{2,\infty} = \infty, e_{2,0} = 0, e_{2,1} = 1$. Note that
\[e^j_{H_2,1} = \xi_r^j,\]
where $\xi_r := \theta_{\phi, e_{1,0}, e_{2,0}}(\zeta_r)$ is an $r$th root of unity in $\mathbb{F}_{e_{2,0}}$. Then
\[e_{1,2}, \ldots, e_{2,n-2} \in k_2 \setminus \{e_{2,\infty}, e_{2,0}\}\]
distinct from each other.

Lemma 4.1 implies that the following linear condition
\[e_{2,H_2} = \sum_{j=0}^{s-1} b_{1,ij} e^j_{H_2,1} = e_{2,i}^{1/r}\]
with respect to $\{e_{H_2,\infty}, e_{H_1,0}\}$
on $X_{H_2}^\bullet$ holds for each $i \in \{2, \ldots, n - 2\}$. Thus, we obtain
\[e_{2,i} = (\sum_{j=0}^{s-1} b_{1,ij} e^j_{H_2,1})^r = (\sum_{j=0}^{s-1} b_{1,ij} \xi_r)^r\]
for each $i \in \{2, \ldots, n - 2\}$. This means that $X_{2,\min}^\bullet$ can be defined over $\mathbb{F}_2$. Moreover, we obtain
\[X_1 \setminus D_{X_1} \cong X_{2,\min} \setminus D_{X_{2,\min}}\]
as schemes. We complete the proof of the main theorem.

\[\square\]

**Remark 4.2.1.** Since $\Pi_X^\bullet$ and $\Pi_{X_2}^\bullet$ are topologically finitely generated, by Theorem 4.2, we obtain that
\[\text{Hom}^{\text{open}}(\Pi_X^\bullet, \Pi_{X_2}^\bullet) = \text{Isom}(\Pi_X^\bullet, \Pi_{X_2}^\bullet),\]
where $\text{Isom}(-, -)$ denotes the set of continuous isomorphisms of profinite groups.

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Remark 4.2.2. Let \( C_1^* := (C_1, D_{C_1}) \) and \( C_2^* := (C_2, D_{C_2}) \) be pointed stable smooth curves of type \((g_C, n_C)\) over algebraically closed fields \( l_1 \) and \( l_2 \) of characteristic \( p > 0 \). Suppose that \( \overline{\mathbb{F}}_p := l_1 = l_2 \) is an algebraic closure of \( \mathbb{F}_p \). Write \( \pi_1^{tame}(C_1 \setminus D_{C_1}) \) and \( \pi_1^{tame}(C_2 \setminus D_{C_2}) \) for the tame fundamental groups of \( C_1^* \) and \( C_2^* \), respectively. Then the weak Isom-version of the Grothendieck conjecture for curves of type \((g_C, n_C)\) over \( \overline{\mathbb{F}}_p \) can be formulated as follows:

The set of continuous isomorphisms

\[
\text{Isom}(\pi_1^{tame}(C_1 \setminus D_{C_1}), \pi_1^{tame}(C_2 \setminus D_{C_2})) \neq \emptyset
\]

if and only if \( C_1 \setminus D_{C_1} \cong C_2 \setminus D_{C_2} \) as schemes.

This conjecture was proved by Tamagawa when \( g_C = 0 \) (cf. [T4, Theorem 5.8]). Theorem 4.2 extends Tamagawa’s result to the case of open continuous homomorphisms. Moreover, Theorem 4.2 can be regarded as a weak Hom-version of the Grothendieck conjecture for curves of type \((0, n_C)\) over \( \overline{\mathbb{F}}_p \).

Remark 4.2.3. Let \( C^* := (C, D_C) \) be a pointed stable smooth curve of type \((g_C, n_C)\) over algebraically closed fields \( l \) of characteristic \( p > 0 \). We denote by \( \text{td}(l) \) the transcendence degree of \( l \) over \( \overline{\mathbb{F}}_p \subseteq l \). We define the essential dimension \( \text{ed}(C^*) \) of \( C^* \) to be the minimum of \( \text{td}(l') \), where \( l' \) runs over the algebraically closed subfields of \( l \) over which there exists a smooth curve \( C'^{'*} \) such that \( C^* \) is \( l \)-isomorphic to \( C'^{'*} \times_{l'} l \). Tamagawa posed a conjecture concerning the essential dimensions as follows (cf. [T3, Conjecture 5.3 (ii)]):

Let \( C''^{'*} := (C'', D_{C''}) \) be a smooth pointed stable curve over an algebraically closed field \( l'' \) of characteristic \( p > 0 \). Suppose that \( \pi_1^{tame}(C \setminus D_C) \) is isomorphic to \( \pi_1^{tame}(C'' \setminus D_{C''}) \) as profinite groups. Then we have

\[
\text{ed}(C^*) = \text{ed}(C''^{'*}).
\]

Tamagawa proved the essential dimension conjecture above in the case where \( \text{ed}(C^*) = 1 \) and \( g_C = 0 \) (cf. [T4, Theorem 5.8]). Moreover, the author extended Tamagawa’s result to the case of (possibly singular) pointed stable curves (cf. [Y, Theorem 6.6 (i-b)]).

On the other hand, let \( C_1^* := (C_1, D_{C_1}) \) and \( C_2^* := (C_2, D_{C_2}) \) be pointed stable smooth curves of type \((g_C, n_C)\) over algebraically closed fields \( l_1 \) and \( l_2 \) of characteristic \( p > 0 \). Then Theorem 4.2 implies that, if \( g_C = 0 \) and there exists a continuous surjective morphism \( \pi_1^{tame}(C_1 \setminus D_{C_1}) \to \pi_1^{tame}(C_2 \setminus D_{C_2}) \), we have

\[
\text{ed}(C_1^*) = \text{ed}(C_2^*).
\]

Moreover, we posed the following question:

**Question.** Suppose that there exists a continuous surjective morphism \( \pi_1^{tame}(C_1 \setminus D_{C_1}) \to \pi_1^{tame}(C_2 \setminus D_{C_2}) \). Does

\[
\text{ed}(C_1^*) \geq \text{ed}(C_2^*)
\]

hold?
Remark 4.2.4. Before Tamagawa proved [T4, Theorem 5.8], he also obtained an étale fundamental group version of [T4, Theorem 5.8] in a completely different way (by using wildly ramified coverings) (cf. [T2, Theorem 3.5]). Note that, for any nonsingular pointed stable curve over an algebraically closed field of positive characteristic, since the tame fundamental group can be reconstructed group-theoretically from the étale fundamental group (cf. [T2, Corollary 1.10]), the tame fundamental group version is stronger than the étale fundamental group version. Recently, by using Tamagawa’s idea, A. Sarashina proved a similar result of [T2, Theorem 3.5] for curves of type (1,1) (cf. [Sar], [T6, Theorem 6 (i)]). Moreover, by applying the theory of Tamagawa developed in [T4], Sarashina’s result also holds in the case of tame fundamental groups.

Moreover, similar arguments to the arguments developed in the present paper and [Sar], one may prove a similar result of Theorem 4.2 for curves of type (1,1) as follows:

Let $C_1^\bullet := (C_1, D_{C_1})$ and $C_2^\bullet := (C_2, D_{C_2})$ be smooth pointed stable curves of type (1,1) over algebraically closed fields $l_1$ and $l_2$ of characteristic $p > 0$, respectively. Write $\Pi C_1^\bullet$ and $\Pi C_2^\bullet$ for the maximal pro-solvable quotients of the tame fundamental groups of $C_1^\bullet$ and $C_2^\bullet$, respectively. Let $l_1^{\min}$ and $l_2^{\min}$ be the minimal algebraically closed subfields of $l_1$ and $l_2$ over which $C_1^\bullet$ and $C_2^\bullet$ are defined, respectively; thus, by considering the function fields of $C_1$ and $C_2$, we obtain smooth pointed stable curves

$C_1^{\min} := (C_1^{\min}, D_{C_1^{\min}})$ and $C_2^{\min} := (C_2^{\min}, D_{C_2^{\min}})$

such that $C_1 \setminus D_{C_1} \cong (C_1^{\min} \setminus D_{C_1^{\min}}) \times_{l_1^{\min}} l_1$ and $C_2 \setminus D_{C_2} \cong (C_2^{\min} \setminus D_{C_2^{\min}}) \times_{l_2^{\min}} l_2$ as $l_1$-schemes and $l_2$-schemes, respectively.

Then we can detect whether $C_1^\bullet$ can be defined over the algebraic closure of $\overline{F}_p$ or not, group-theoretically from $\Pi C_1^\bullet$. Moreover, suppose that $C_1^\bullet$ can be defined over the algebraic closure of $\overline{F}_p$ in $l_1$. Then the set of open continuous homomorphisms

$\text{Hom}^{\text{open}}(\Pi C_1^\bullet, \Pi C_2^\bullet) \neq \emptyset$

if and only if

$C_1^{\min} \setminus D_{C_1^{\min}} \cong C_2^{\min} \setminus D_{C_2^{\min}}$

as schemes. In particular, if this is the case, $C_2^\bullet$ can be defined over the algebraic closure of $\overline{F}_p$ in $l_2$.

5 An application to moduli spaces of curves

Let $\overline{F}_p$ be an algebraic closure of $F_p$, and let $M_{g,n}$ be the moduli stack over $\overline{F}_p$ parameterizing smooth pointed stable curves of type $(g,n)$ and $\mathcal{M}_{g,n}$ the coarse moduli space of $M_{g,n}$. Let $X^\bullet$ be a pointed stable smooth curve of type $(g,n)$ over an algebraically closed field $k \supseteq \overline{F}_p$. Then there exists a unique composition of morphisms

$c_{X^\bullet} : \text{Spec} k \to M_{g,n} \to M_{g,n}$
determined by $X^\bullet \to \text{Spec } k$ and the natural morphism $\mathcal{M}_{g,n} \to M_{g,n}$. We write

$$q_{\cdot} X^\bullet \in M_{g,n}$$

for the image of $c_{X^\bullet}$. Moreover, for any $q \in M_{g,n}$, let $k_q$ be an algebraically closed field which contains the residue field $k(q)$ of $q$. Then the natural morphisms $\text{Spec } k_q \to \text{Spec } k(q) \to M_{g,n}$ determine a smooth pointed stable curve $X^\bullet_q := (X_q, D_{X_q})$ of type $(g, n)$ over $k_q$. We write

$$\pi_{1,\text{tame}}^q(q)$$

for the tame fundamental group of $X_q \setminus D_{X_q}$ of $X^\bullet_q$ and

$$\pi_{\text{A, tame}}^q(q)$$

for the set of finite quotients of $\pi_{1,\text{tame}}^q(q)$. Note that $\pi_{1,\text{tame}}^q(q)$ and $\pi_{\text{A, tame}}^q(q)$ do not depend on the choice of $k_q$ but depend only on $q$. Moreover, for two points $q_1, q_2 \in M_{g,n}$, we have $\pi_{1,\text{tame}}^q(q_1) \cong \pi_{1,\text{tame}}^q(q_2)$ as profinite groups if and only if $\pi_{\text{A, tame}}^q(q_1) = \pi_{\text{A, tame}}^q(q_2)$ as sets.

K. Stevenson proved the following result (cf. [Ste, Proposition 4.2]).

**Proposition 5.1.** Suppose that $n = 0$. Let $q$ be a closed point of $M_g := M_{g,0}$ and $G \in \pi_{\text{A, tame}}^q(q)$ a finite group. Then there exists an open neighborhood $q \in U \subseteq M_g$ such that, for each $q' \in U$, $G \in \pi_{\text{A, tame}}^{q'}(q')$.

Similar arguments to the arguments given in the proof of [Ste, Proposition 4.2] imply Proposition 5.1 also holds for $n \geq 0$. Then we obtain the following result.

**Proposition 5.2.** Let $q$ be a closed point of $M_{g,n}$ and $G \in \pi_{\text{A, tame}}^q(q)$ a finite group. Then there exists an open neighborhood $q \in U \subseteq M_{g,n}$ such that, for each $q' \in U$, $G \in \pi_{\text{A, tame}}^{q'}(q')$.

**Remark 5.2.1.** Proposition 5.2 means that, for any finite group $H$, either $H$ is not a quotient of the tame fundamental group of any smooth pointed stable curves of type $(g, n)$ over algebraically closed fields fields of characteristic $p > 0$, or is a quotient of the tame fundamental group of almost each such curve.

Suppose that $H$ is any finite quotient of the tame fundamental group of a smooth pointed stable curves of type $(g, n)$ over algebraically closed fields fields of characteristic $p > 0$. We define

$$U_H \subseteq M_{g,n}$$

for the maximal open subset such that, for each $q' \in U_H$, $H \in \pi_{\text{A, tame}}^{q'}(q')$. Stevenson posed a question as follows (cf. [Question 4.3] for $n = 0$ case):

is the intersection of all the $U_H$’s contains any $\overline{\mathbb{F}}_p$-rational points?

Let $q_{\text{gen}}$ be the generic point of $M_{g,n}$ and $q''$ any closed point of $M_{g,n}$. Then by [T5, Theorem 0.3], $\pi_{\text{A, tame}}^{q_{\text{gen}}}(q_{\text{gen}})$ is not equal to $\pi_{\text{A, tame}}^{q''}(q'')$. This means that the answer of Stevenson’s question above is “No”. Moreover, we may refine Stevenson’s question above as follows:
let \( q \) be any closed point of \( M_{g,n} \); what is the set

\[( \bigcap_{H \in \pi_A^{tame}(q)} U_H )^{\text{cl}}, \]

where \((-)^{\text{cl}}\) denotes the set of closed points of \((-)\)?

For this question, we have the following result.

**Theorem 5.3.** Let \( q \) be any closed point of \( M_{0,n} \) and \( X_q^\bullet \) the smooth pointed stable curve over \( \mathbb{F}_p = k(q) \) determined by the natural morphism \( \text{Spec} \ k(q) \to M_{0,n} \). For each \( m \in \mathbb{Z} \), write \( q^{(m)} \) for \( q(X_q^\bullet)^{(m)} \), where \( (X_q^\bullet)^{(m)} \) denotes the \( m \)th Frobenius twist of \( X_q^\bullet \). Then we have

\[( \bigcap_{H \in \pi_A^{tame}(q)} U_H )^{\text{cl}} = \{ q^{(m)} \}_{m \in \mathbb{Z}}. \]

Note that since \( X_q^\bullet \) can be defined over a finite field, \( \{ q^{(m)} \}_{m \in \mathbb{Z}} \) is a finite set.

**Proof.** Since “\( \supseteq \)" is trivial, we only need to prove that “\( \subseteq \)" holds. Let \( q' \) be any closed point of \( \bigcap_{H \in \pi_A^{tame}(q)} U_H \). Then we have that, for each \( H \in \pi_A^{tame}(q) \),

\[ \text{Hom}^{\text{surj}}(\pi_1^{tame}(q'), H) \neq \emptyset, \]

where \( \text{Hom}^{\text{surj}}(-, -) \) denotes the set of surjections of \( \text{Hom}(-, -) \). Since \( \pi_1^{tame}(q') \) is topologically finitely generated, the set \( \text{Hom}^{\text{surj}}(\pi_1^{tame}(q'), H) \) is finite. Then the set of open continuous homomorphisms

\[ \lim_{H \in \pi_A^{tame}(q)} \text{Hom}^{\text{surj}}(\pi_1^{tame}(q'), H) = \text{Hom}^{\text{open}}(\pi_1^{tame}(q'), \pi_1^{tame}(q)) \neq \emptyset. \]

Thus, Theorem 4.2 implies that \( q' \in \{ q^{(m)} \}_{m \in \mathbb{Z}} \). This completes the proof of the theorem. \( \square \)

The author is very interested in the following question.

**Question 5.4.** Does

\[ | \bigcap_{H \in \pi_A^{tame}(q)} U_H | = \bigcup_{m \in \mathbb{Z}} | \text{Spec} \ O_{M_0,n,q^{(m)}} | \]

hold? Here, \(|(-)|\) denotes the underlying topological space of \((-)\).

6 Formulation of a weak Hom-version of the Grothendieck conjecture for curves of type \((g, n)\)

We maintain the notations introduced in Section 5. Let \( X_1^\bullet \) and \( X_2^\bullet \) be smooth pointed stable curve of type \((g, n)\) over algebraically closed fields \( k_1 \) and \( k_2 \), respectively. Write \( q_1 \) and \( q_2 \) for \( q_{X_1^\bullet} \) and \( q_{X_2^\bullet} \), \( V_1 \) and \( V_2 \) for the topological closure of \( q_1 \) and \( q_2 \) in \( M_{g,n} \), respectively.
Definition 6.1. We shall say that \( V_2 \) is essentially contained in \( V_1 \) if, for each \( q \in V_2 \), there exists \( m \in \mathbb{Z} \) such that \( q^{(m)} \in V_1 \). We denote by \( V_2 \subset \text{ec} V_1 \) if \( V_2 \) is essentially contained in \( V_1 \).

Then we formulate a certain weak Hom-version of the Grothendieck conjecture for curves of type \((g,n)\) over algebraically closed fields of characteristic \( p > 0 \) as follows.

**Conjecture 6.2.** (weak Hom-version for curves of type \((g,n)\))

*The set of open continuous homomorphisms*

\[
\text{Hom}^{\text{open}}(\pi_1^{\text{tame}}(q_1), \pi_1^{\text{tame}}(q_2)) \neq \emptyset
\]

if and only if

\( V_2 \subset \text{ec} V_1 \).

*Moreover,*

\[
\text{Hom}^{\text{open}}(\pi_1^{\text{tame}}(q_1), \pi_1^{\text{tame}}(q_2)) = \text{Isom}(\pi_1^{\text{tame}}(q_1), \pi_1^{\text{tame}}(q_2)) \neq \emptyset
\]

if and only if

\( V_2 \subset \text{ec} V_1 \) and \( V_1 \subset \text{ec} V_2 \).

**Remark 6.2.1.** Theorem 4.2 implies that Conjecture 6.2 holds in the case where \( q_1 \) is a closed point of \( M_{0,n} \). Moreover, we have

weak Hom-version for curves of type \((g,n)\) \( \Rightarrow \) weak Isom-version.

**Remark 6.2.2.** We note that \( \dim(V_1) = \text{ed}(X_1^*) \) and \( \dim(V_2) = \text{ed}(X_2^*) \). Thus, we obtain that Conjecture 6.2 implies Tamagawa’s essential dimension conjecture, Remark 4.2.3 Question, and Question 5.4.
References


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