RIMS-1879

On a Weak Hom-version of the Grothendieck Conjecture for Curves of Type $(0, n)$ over Algebraically Closed Fields of Characteristic $p > 0$

By

Yu YANG

Sep 2017

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

ON A WEAK HOM-VERSION OF THE GROTHENDIECK CONJECTURE FOR CURVES OF TYPE $(0, n)$ OVER ALGEBRAICALLY CLOSED FIELDS OF CHARACTERISTIC $p > 0$

Yu Yang

Abstract

In the present paper, we study the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$. Let $X_1^\bullet := (X_1, D_{X_1})$ and $X_2^\bullet := (X_2, D_{X_2})$ be smooth pointed stable curves of type (g, n) over algebraically closed fields k_1 and k_2 of characteristic $p > 0$, respectively. We prove that, if $g = 0$ and $k_1 = k_2 = \overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$, then $X_1 \setminus D_{X_1}$ is isomorphic to $X_2 \setminus D_{X_2}$ as schemes if and only if the set of open continuous homomorphisms between the tame fundamental groups of $X_1 \setminus D_{X_1}$ and $X_2 \setminus D_{X_2}$ is not empty. This result can be regarded as a weak Hom-version of the Grothendieck conjecture for curves of type $(0, n)$ over $\overline{\mathbb{F}}_p$. Moreover, this result is a generalization of the weak Isom-version of the Grothendieck conjecture for curves of type $(0, n)$ over $\overline{\mathbb{F}}_p$ which was proved by A. Tamagawa. On the other hand, for arbitrary (g, n), we formulate a certain weak Hom-version of the Grothendieck conjecture for curves of type (g, n) over arbitrary algebraically closed fields of characteristic $p > 0$.

Keywords: hyperbolic curve, smooth pointed stable curve, tame fundamental group, Grothendieck's anabelian conjecture, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 11G20.

Introduction

In the present paper, we study the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

Before we explain the main theorem of the present paper, let us recall some general facts concerning anabelian geometry. Let k be a field and Z a geometrically connected and k-scheme of finite type. Then we have the following fundamental exact sequence of étale fundamental groups (for suitable choices of base point):

$$1 \to \pi_1(Z_{k^{\mathrm{sep}}}) \to \pi_1(Z) \overset{\mathrm{pr}_Z}{\to} G_k \to 1.$$

Here, $Z_{k^{\mathrm{sep}}}$ denotes $Z \times_k k^{\mathrm{sep}}$, k^{sep} denotes a separable closure of k in an algebraically closed field which contains k, and G_k denotes the absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ of k. A. Grothendieck proposed the following philosophy (cf. [G1], [G2]):

if Z is anabelian, then the group-theoretic data $(\pi_1(Z), \mathrm{pr}_Z)$ functorially determines the isomorphism class of the k-scheme Z.

Although we do not have any general definition of the term "anabelian", if $\dim(Z) = 1$, hyperbolic curves have been regarded as typical examples of anabelian schemes. Here, a smooth, geometrically connected curve Z over k is called hyperbolic if it may be obtained as the complement of the divisor of marked points in a smooth pointed stable curve over k.

Let Z_1 and Z_2 be hyperbolic curves over k. Suppose that k is of characteristic 0. Relative to the notational conventions introduced above for étale fundamental groups, write

$$\mathrm{Isom}_{\text{pro-gps}}(-,-) \text{ (resp. } \mathrm{Hom}_{\text{pro-gps}}(-,-))$$

for the set of continuous isomorphisms (resp. continuous homomorphisms) of profinite groups between the two profinite groups in parentheses,

$$\mathrm{Isom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2)) := \{\Phi \in \mathrm{Isom}_{\text{pro-gps}}(\pi_1(Z_1), \pi_1(Z_2)) \mid \mathrm{pr}_{Z_1} = \mathrm{pr}_{Z_2} \circ \Phi\}$$

$$(\text{resp. } \mathrm{Hom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2)) := \{\Phi \in \mathrm{Hom}_{\text{pro-gps}}(\pi_1(Z_1), \pi_1(Z_2)) \mid \mathrm{pr}_{Z_1} = \mathrm{pr}_{Z_2} \circ \Phi\}).$$

Thus, by composing with inner automorphisms, we obtain a natural action of $\pi_1(Z_2 \times_k k^{\mathrm{sep}})$ on $\mathrm{Isom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2))$ (resp. $\mathrm{Hom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2))$). Then, in this situation, the philosophy above can be formulated as follows (which is called **Grothendieck's anabelian conjecture** or, simply, the Grothendieck conjecture, for short):

(weak Isom_k-version)

The set

$$\mathrm{Isom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2)) \neq \emptyset$$

if and only if

$$Z_1 \cong Z_2$$

as k-schemes.

(Isom_k-version)

The natural morphism

$$\mathrm{Isom}_{k\text{-schemes}}(Z_1, Z_2) \to \mathrm{Isom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2))/\mathrm{Inn}(\pi_1(Z_2 \times_k k^{\mathrm{sep}}))$$

is a bijection.

The Grothendieck conjecture has been proven in many cases. For example, if k is a number field, then the weak Isom_k-version was proved by H. Nakamura when the genera of Z_1 and Z_2 are 0 (cf. [N1], [N2]); the Isom_k-version was proved by A. Tamagawa in the case where Z_1 and Z_2 are affine (cf. [T1]) and proved by S. Mochizuki in full generality (cf. [M1]). In fact, Mochizuki proved a very general version when k is sub-p-adic (i.e., a subfield of a finitely generated extension of a p-adic number field) as follows:

(Hom_k-version)

We denote by

$$\mathrm{Hom}_k^{\mathrm{dom}}(Z_1, Z_2)$$

the subset of the dominant morphisms of $\mathrm{Hom}_{k\text{-schemes}}(Z_1, Z_2)$ *and denote by*

$$\mathrm{Hom}_{G_k}^{\mathrm{open}}(\pi_1(Z_1), \pi_1(Z_2))$$

the subset of open homomorphisms of $\mathrm{Hom}_{G_k}(\pi_1(Z_1), \pi_1(Z_2))$*. Then the natural morphism*

$$\mathrm{Hom}_k^{\mathrm{dom}}(Z_1, Z_2) \to \mathrm{Hom}_{G_k}^{\mathrm{open}}(\pi_1(Z_1), \pi_1(Z_2))/\mathrm{Inn}(\pi_1(Z_2 \times_k k^{\mathrm{sep}}))$$

is a bijection.

Note that we have implications

$$\mathrm{Hom}_k\text{-version} \Rightarrow \mathrm{Isom}_k\text{-version} \Rightarrow \text{weak } \mathrm{Isom}_k\text{-version}.$$

Tamagawa also considered an analogue of the Grothendieck conjecture in positive characteristic and proved the Grothendieck conjecture (Isom-version) for affine hyperbolic curves over finite fields (cf. [T1]). Afterwards, Mochizuki generalized this result to the case of projective hyperbolic curves (cf. [M2]), and J. Stix generalized this result to the case where the base fields are finitely generated over $\mathbb{F}_p$ (cf. [Sti1], [Sti2]).

Unlike the characteristic 0 case, nothing is known about the Grothendieck conjecture for curves over local fields of positive characteristic. On the other hand, Tamagawa also considered the Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$. Note that all the proofs of the Grothendieck conjecture for curves over non-algebraically closed fields require the use of the highly non-trivial outer Galois representation induced by the fundamental exact sequence of étale fundamental groups reviewed above. In the case of algebraically closed fields, the Galois groups of the base fields are trivial, and the étale fundamental group coincides with the geometric fundamental group. As a result, the Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$ is quite different from that over non-algebraically closed fields.

In the remainder of this introduction, let $X_1^\bullet := (X_1, D_{X_1})$ and $X_2^\bullet := (X_2, D_{X_2})$ be smooth pointed stable curves of type (g_X, n_X) over algebraically closed fields k_1 and k_2 of characteristic $p > 0$ (i.e., $X_1 \setminus D_{X_1}$ and $X_2 \setminus D_{X_2}$ are hyperbolic curves of type (g_X, n_X) over k_1 and k_2, respectively). For $i = 1, 2$, write k_i^{min} for the minimal algebraically closed subfield of k_i over which $X_i^\bullet$ is defined; thus, by considering the function field of X_i, one verifies immediately that there exists a "natural" smooth pointed stable curve

$$X_i^{\bullet,\mathrm{min}} := (X_i^{\mathrm{min}}, D_{X_i^{\mathrm{min}}}),$$

where the function field of X_i^{min} is a subfield of the function field of X_i, such that $X_i \setminus D_{X_i}$ may be identified with $(X_i^{\mathrm{min}} \setminus D_{X_i^{\mathrm{min}}}) \times_{k_i^{\mathrm{min}}} k_i$. In this situation, Tamagawa formulated the Grothendieck conjecture as follows (we only focus on the tame version in the present paper):

Conjecture 0.1. (weak Isom-version)

The set of continuous isomorphisms of profinite groups

$$\mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{tame}}(X_1 \setminus D_{X_1}), \pi_1^{\mathrm{tame}}(X_2 \setminus D_{X_2})) \neq \emptyset$$

if and only if

$$X_1^{\min} \setminus D_{X_1^{\min}} \cong X_2^{\min} \setminus D_{X_2^{\min}}$$

as schemes.

(Isom-version)

The natural morphism

$$\mathrm{Isom}_{\text{schemes}}(X_1^{\min} \setminus D_{X_1^{\min}}, X_2^{\min} \setminus D_{X_2^{\min}})$$

$$\longrightarrow \mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{tame}}(X_1 \setminus D_{X_1}), \pi_1^{\mathrm{tame}}(X_2 \setminus D_{X_2}))/\mathrm{Inn}(\pi_1^{\mathrm{tame}}(X_2 \setminus D_{X_2}))$$

is a bijection.

Remark 0.1.1. Note that the existence of the specialization map of tame fundamental groups constitutes a counterexample to the "Hom-version" of the Grothendieck conjecture for tame fundamental groups obtained by simply replacing $\mathrm{Isom}(-,-)$ by $\mathrm{Hom}^{\mathrm{open}}(-,-)$. Indeed, at the time of writing, we do not know how to give a formulation of a suitable Hom-version of the Grothendieck conjecture for tame fundamental groups that takes into account this counterexample (cf. [T3, Remark 1.34]). On the other hand, there exists a Hom-version of the Grothendieck conjecture for étale fundamental groups (cf. [T3, Conjecture 1.8]).

At present, no result is known about the Isom-version of Conjecture 0.1. On the other hand, Tamagawa proved the **weak Isom-version** of Conjecture 0.1 when $g_X = 0$ and $k_1 = k_2$ is an algebraic closure of $\mathbb{F}_p$. More precisely, Tamagawa proved the following theorem (cf. [T4, Theorem 5.8]).

Theorem 0.2. *Suppose that $g_X = 0$. Then we can detect whether $X_1^\bullet$ (resp. $X_2^\bullet$) can be defined over the algebraic closure of $\mathbb{F}_p$ in k_1 (resp. k_2) or not, group-theoretically from the tame fundamental group $\pi_1^{\mathrm{tame}}(X_1 \setminus D_{X_1})$ (resp. $\pi_1^{\mathrm{tame}}(X_2 \setminus D_{X_2})$). Moreover, suppose that $X_1^\bullet$ can be defined over the algebraic closure of $\mathbb{F}_p$ in k_1. Then the set of continuous isomorphisms of profinite groups*

$$\mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{tame}}(X_1 \setminus D_{X_1}), \pi_1^{\mathrm{tame}}(X_2 \setminus D_{X_2})) \neq \emptyset$$

if and only if

$$X_1^{\min} \setminus D_{X_1^{\min}} \cong X_2^{\min} \setminus D_{X_2^{\min}}$$

as schemes.

Remark 0.2.1. Tamagawa also obtained an étale fundamental group version of Theorem 0.2 in a completely different way (by using wildly ramified coverings) (cf. [T2, Theorem 3.5]). Note that since the tame fundamental group can be reconstructed group-theoretically from the étale fundamental group (cf. [T2, Corollary 1.10]), the tame fundamental group version is stronger than the étale fundamental group version. Recently, by using Tamagawa's idea, A. Sarashina (a student of Tamagawa) proved a similar result of [T2, Theorem 3.5] for curves of type $(1, 1)$(cf. [Sar], [T6, Theorem 6 (i)]). Moreover, by applying the theory of Tamagawa developed in [T4], Sarashina's result also holds in the case of tame fundamental groups (cf. [T6, Theorem 6 (ii)]).

Remark 0.2.2. We do not know whether the weak Isom-version of Conjecture 0.1 for $g_X > 0$ holds or not. On the other hand, we have the following finiteness theorem which was proved by M. Raynaud, F. Pop, M. Saïdi, and Tamagawa (cf. [R], [PS], [T5]):

> over an algebraic closure of $\mathbb{F}_p$, only finitely many isomorphism classes of hyperbolic curves have the same tame fundamental groups.

Moreover, the finiteness theorem also holds for (possibly singular) pointed stable curves (cf. [Y]).

In the present paper, we consider a **weak Hom-version** of Grothendieck conjecture over algebraically closed field of characteristic $p > 0$. Our main theorem, which generalizes Tamagawa's theorem above, is as follows (see also Theorem 4.2).

Theorem 0.3. *Suppose that $g_X = 0$. Then we can detect whether $X_1^\bullet$ can be defined over the algebraically closure of $\mathbb{F}_p$ in k_1 or not, group-theoretically from $\pi_1^{\mathrm{tame}}(X_1 \setminus D_{X_1})$. Moreover, suppose that $X_1^\bullet$ can be defined over the algebraic closure of $\mathbb{F}_p$ in k_1. Then the set of open homomorphisms*

$$\mathrm{Hom}^{\mathrm{open}}(\pi_1^{\mathrm{tame}}(X_1 \setminus D_{X_1}), \pi_1^{\mathrm{tame}}(X_2 \setminus D_{X_2})) \neq \emptyset$$

if and only if

$$X_1^{\min} \setminus D_{X_1^{\min}} \cong X_2^{\min} \setminus D_{X_2^{\min}}$$

as schemes. In particular, if this is the case, $X_2^\bullet$ can be defined over the algebraic closure of $\mathbb{F}_p$ in k_2.

Remark 0.3.1. Similar arguments to the arguments developed in the present paper and [Sar], one may prove a similar result of Theorem 4.2 for curves of type $(1, 1)$ (see Remark 4.2.4 for a precise form).

Theorem 0.3 can be regarded as a weak Hom-version of the Grothendieck conjecture for curves of type $(0, n)$ over an algebraic closure of $\mathbb{F}_p$. Moreover, although we do not know how to formulate "Hom-version" of the Grothendieck conjecture over algebraically closed fields of characteristic $p > 0$, we can formulate a certain **"weak Hom-version" of the Grothendieck conjecture for curves of type (g, n) over algebraically closed fields of characteristic $p > 0$** (cf. Conjecture 6.2). Then Theorem 0.3 implies that Conjecture 6.2 holds in a special case. Conjecture 6.2 also implies Tamagawa's essential dimension conjecture (cf. Remark 4.2.3 and Remark 6.2.2).

The present paper is organized as follows. In Section 1, we give some definitions and propositions which will be used in the next sections. In Section 2, we construct a correspondence between the set of marked points of smooth pointed stable curves and line bundles. In Section 3, by applying the theory developed in Section 2, we reconstruct the inertia subgroups of marked points and their additive structures from a surjection of tame fundamental groups. In Section 4, by applying the results obtained in previous sections, we prove our main theorem. In Section 5, we apply the main theorem to a question concerning moduli spaces of curves which is originally posed by K. Stevenson. Finally, in Section 6, we formulate a certain weak Hom-version of the Grothendieck conjecture for curves of type (g, n) over algebraically closed fields of characteristic $p > 0$.

ACKNOWLEDGEMENTS

The author would like to thank Yuichiro Hoshi, Akio Tamagawa, and Kazuki Tokimoto for helpful discussions. This research was supported by JSPS KAKENHI Grant Number 16J08847.

1 Preliminaries

Let k be an algebraically closed field of characteristic $p > 0$, and let $X^\bullet := (X, D_X)$ be a smooth pointed stable curve of type (g_X, n_X) over k. Here, X denotes the underlying curve of $X^\bullet$, D_X denotes the set of marked points of $X^\bullet$, g_X denotes the genus of $X^\bullet$, and n_X denotes the cardinality of D_X. By choosing a base point of $x \in X \setminus D_X$, we obtain the tame fundamental group $\pi_1^{\mathrm{tame}}(X \setminus D_X, x)$ of $X^\bullet$ and the étale fundamental group $\pi_1(X, x)$ of $X^\bullet$. Write $\pi_1^{\mathrm{tame}}(X \setminus D_X, x)^{\mathrm{sol}}$ and $\pi_1(X, x)^{\mathrm{sol}}$ for the maximal pro-solvable quotients of $\pi_1^{\mathrm{tame}}(X \setminus D_X, x)$ and $\pi_1(X, x)$, respectively. Note that, by the definition of tame coverings, there is a natural surjection

$$\pi_1^{\mathrm{tame}}(X \setminus D_X, x)^{\mathrm{sol}} \twoheadrightarrow \pi_1(X, x)^{\mathrm{sol}}.$$

For simplicity of notation, we omit the base point and denote by $\Pi_{X^\bullet}$ (resp. $\Pi_{X^\bullet}^{\text{ét}}$) the maximal pro-solvable quotient of the tame (resp. étale) fundamental group of $X^\bullet$.

Definition 1.1. Let ℓ be a prime number, and let $f^\bullet : Y^\bullet \to X^\bullet$ be a connected tame Galois covering (i.e., $f^\bullet$ is a Galois covering and is at most tamely ramified over D_X) over k of degree ℓ. For any $e \in D_X$, we set

$$\mathrm{Ram}_{f^\bullet} := \{e \in D_X \mid f^\bullet \text{ is ramified over } e\}.$$

Definition 1.2. Let Π be a profinite group, n a natural number, and ℓ a prime number.

(a) We denote by $\Pi(n)$ the topological closure of the subgroup $[\Pi, \Pi]\Pi^n$ of Π. Note that $\Pi/\Pi(n) = \Pi^{\mathrm{ab}} \otimes (\mathbb{Z}/n\mathbb{Z})$.

(b) We set $\gamma_\ell := \dim_{\mathbb{F}_\ell}(\Pi/\Pi(\ell)) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

(c) Let n be a natural number such that $[\Pi : \Pi(n)] < \infty$. We define ℓ-average of Π to be

$$\gamma_\ell^{\mathrm{av}}(n)(\Pi) := \gamma_\ell(\Pi(n))/[\Pi : \Pi(n)] \in \mathbb{Q}_{\geq 0} \cup \{\infty\}.$$

(d) We denote by $\mathrm{Sub}(\Pi)$ the set of closed subgroups of Π.

The following highly non-trivial result concerning p-average of $\Pi_{X^\bullet}$ was proved by Tamagawa (cf. [T4, Theorem 0.5]).

Proposition 1.3. *For any natural number $r \in \mathbb{N}$, we set*

$$\gamma_p^{\mathrm{av}}(p^r-1)(X^\bullet) := \gamma_p^{\mathrm{av}}(p^r-1)(\Pi_{X^\bullet}).$$

Then we have

$$\lim_{r\to\infty} \gamma_p^{\mathrm{av}}(p^r-1)(X^\bullet) = g'_X,$$

where $g'_X = g_X - 1$ if $n_X \leq 1$ and $g'_X = g_X$ if $n_X > 1$.

Remark 1.3.1. Tamagawa proved Proposition 1.3 as a main theorem of [T4] by developing a general theory of Raynaud's theta divisor.

Let K be the function field of $X^\bullet$, and define $\widetilde{K}^{\mathrm{sol}}$ to be the maximal pro-solvable Galois extension of K in a fixed separable closure of K, unramified over $X \setminus D_X$ and at most tamely ramified over D_X. Then we may identify $\Pi_{X^\bullet}$ with $\mathrm{Gal}(\widetilde{K}^{\mathrm{sol}}/K)$. We set

$$\widetilde{X}^{\bullet,\mathrm{sol}} := (\widetilde{X}^{\mathrm{sol}}, D_{\widetilde{X}^{\mathrm{sol}}}),$$

where $\widetilde{X}^{\mathrm{sol}}$ denotes the normalization of X in $\widetilde{K}^{\mathrm{sol}}$ and $D_{\widetilde{X}^{\mathrm{sol}}}$ denotes the inverse image of D_X in $\widetilde{X}^{\mathrm{sol}}$. For each $\widetilde{e} \in D_{\widetilde{X}^{\mathrm{sol}}}$, we denote by $I_{\widetilde{e}}$ the inertia subgroup of $\Pi_{X^\bullet}$ associated to $\widetilde{e}$ (i.e., the stabilizer of $\widetilde{e}$). Note that we have $I_{\widetilde{e}} \cong \widehat{\mathbb{Z}}(1)^{p'}$, where $\widehat{\mathbb{Z}}(1)^{p'}$ denotes the prime-to-p part of $\widehat{\mathbb{Z}}(1)$.

Lemma 1.4. *Let m_1, m_2 be two positive numbers and G a finite solvable group of order m_1m_2. Let $f_1^\bullet : Z^\bullet \to Y^\bullet$ and $f_2^\bullet : Y^\bullet \to X^\bullet$ be connected cyclic tame Galois coverings of degrees m_1 and m_2 over k, respectively. Suppose that the composition $f_2^\bullet \circ f_1^\bullet : Z^\bullet \to X^\bullet$ is a connected tame Galois covering over k whose Galois group is isomorphic to G. Then there exists two connected tame Galois coverings $f_1^{*,\bullet} : Z^{*,\bullet} \to Y^{*,\bullet}$ and $f_2^{*,\bullet} : Y^{*,\bullet} \to X^\bullet$ over k such that the following conditions hold:*

(a) the Galois group of $f_1^{*,\bullet}$ and $f_2^{*,\bullet}$ are isomorphic to $\Pi_{Y^{*,\bullet}}/\Pi_{Y^{*,\bullet}}(m_1)$ and $\Pi_{X^\bullet}/\Pi_{X^\bullet}(m_2)$, respectively, where $\Pi_{Y^{*,\bullet}}$ denotes the maximal pro-solvable quotient of the tame fundamental group of $Y^{*,\bullet}$;

(b) there exist two morphisms $g_1^\bullet : Z^{*,\bullet} \to Z^\bullet$ and $g_2^\bullet : Y^{*,\bullet} \to Y^\bullet$ over k which fit into the following commutative diagram:

$$\begin{array}{ccc}
Z^{*,\bullet} & \xrightarrow{g_1^\bullet} & Z^\bullet \\
{\scriptstyle f_1^{*,\bullet}}\downarrow & & {\scriptstyle f_1^\bullet}\downarrow \\
Y^{*,\bullet} & \xrightarrow{g_2^\bullet} & Y^\bullet \\
{\scriptstyle f_2^{*,\bullet}}\downarrow & & {\scriptstyle f_2^\bullet}\downarrow \\
X^\bullet & = & X^\bullet.
\end{array}$$

Proof. Trivial. □

Remark 1.4.1. Let $C_{X^\bullet} := \{H_i\}_{i\in\mathbb{Z}_{\geq 0}}$ be a set of open subgroups of $\Pi_{X^\bullet}$ such that the following conditions:

(a) $H_0 = \Pi_{X^\bullet}$ and H_{i+1} is an open normal subgroup of H_i for each $i \in \mathbb{Z}_{\geq 0}$;

(b) $\varprojlim_i \Pi_{X^\bullet}/H_i \cong \Pi_{X^\bullet}$.

Let $\widetilde{e} \in D_{\widetilde{X}^{\mathrm{sol}}}$. For each $i \in \mathbb{Z}_{\geq 0}$, we write $X^\bullet_{H_i} := (X_{H_i}, D_{X_{H_i}})$ for the smooth pointed stable curve corresponding to H_i and $e_{H_i} \in D_{X_{H_i}}$ for the image of $\widetilde{e}$ in $X^\bullet_{H_i}$. Then we obtain a sequence of marked points

$$\mathcal{I}^{C_{X^\bullet}}_{\widetilde{e}} : \cdots \mapsto e_{H_2} \mapsto e_{H_1} \mapsto e_{H_0}$$

induced by $C_{X^\bullet}$. Note that the sequence $\mathcal{I}^{C_{X^\bullet}}_{\widetilde{e}}$ admits a natural action of $\Pi_{X^\bullet}$.

We may identify the inertia subgroup $I_{\widetilde{e}}$ associated to $\widetilde{e}$ with the stabilizer of $\mathcal{I}^{C_{X^\bullet}}_{\widetilde{e}}$. Moreover, Lemma 1.4 implies that we may assume that, for each $i \in \mathbb{Z}_{\geq 0}$,

$$H_{i+1} = H_i(n_i)$$

for some $n_i \in \mathbb{Z}_{>0}$.

Next, we recall some well-known results concerning the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

Definition 1.5. (a) Given an object $\mathrm{Ob}(X^\bullet)$ (e.g., an invariant of $X^\bullet$, the set of marked points of $X^\bullet$) associated to $X^\bullet$ depending on the isomorphism class of $X^\bullet$ (as scheme), we shall say that $\mathrm{Ob}(X^\bullet)$ **can be reconstructed group-theoretically from** $\Pi_{X^\bullet}$ if there exists a group-theoretically algorithm for reconstructing $\mathrm{Ob}(X^\bullet)$ from $\Pi_{X^\bullet}$.

(b) Given an additional structure $\mathrm{Add}(X^\bullet)$ (e.g., a family of subgroups, a family of quotient groups) on the profinite group $\Pi_{X^\bullet}$ depending functorially on $X^\bullet$; then we shall say that $\mathrm{Add}(X^\bullet)$ **can be reconstructed group-theoretically from** $\Pi_{X^\bullet}$ if there exists a group-theoretically algorithm for reconstructing $\mathrm{Add}(X^\bullet)$ from $\Pi_{X^\bullet}$.

(c) Let $X^\bullet$ and $Y^\bullet := (Y, D_Y)$ be smooth pointed stable curves over algebraically closed fields of characteristic $p > 0$, $\Pi_{X^\bullet}$ and $\Pi_{Y^\bullet}$ the maximal pro-solvable quotient of the tame fundamental groups of $X^\bullet$ and $Y^\bullet$, respectively. Suppose that we are given $\mathrm{Ob}(X^\bullet)$ and $\mathrm{Ob}(Y^\bullet)$ (resp. $\mathrm{Add}(X^\bullet)$ and $\mathrm{Add}(Y^\bullet)$), and that a continuous homomorphism (in the category of profinite groups) $\Pi_{X^\bullet} \to \Pi_{Y^\bullet}$. We shall say that a map $\mathrm{Ob}(X^\bullet) \to \mathrm{Ob}(Y^\bullet)$ (resp. $\mathrm{Add}(X^\bullet) \to \mathrm{Add}(Y^\bullet)$) **can be reconstructed group-theoretically from** the morphism $\Pi_{X^\bullet} \to \Pi_{Y^\bullet}$ if there exists a group-theoretically algorithm for reconstructing the map $\mathrm{Ob}(X^\bullet) \to \mathrm{Ob}(Y^\bullet)$ (resp. $\mathrm{Add}(X^\bullet) \to \mathrm{Add}(Y^\bullet)$) from the morphism $\Pi_{X^\bullet} \to \Pi_{Y^\bullet}$.

Proposition 1.6. *(a) The genus g_X of $X^\bullet$ and the cardinality of the set of the marked points n_X of $X^\bullet$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.*

(b-i) Let $\widetilde{e}$ and $\widetilde{e}'$ be two points of $D_{\widetilde{X}}$ distinct from each other. Then the intersection of $I_{\widetilde{e}}$ and $I_{\widetilde{e}'}$ is trivial in $\Pi_{X^\bullet}$. (b-ii) The map

$$D_{\widetilde{X}} \to \mathrm{Sub}(\Pi_{X^\bullet})$$

that maps $\widetilde{e} \mapsto I_{\widetilde{e}}$ is an injection.

(c) Write $\mathrm{Ine}(\Pi_{X^\bullet})$ for the set of inertia subgroups in $\Pi_{X^\bullet}$, namely the image of the map $D_{\widetilde{X}} \to \mathrm{Sub}(\Pi_{X^\bullet})$. Then $\mathrm{Ine}(\Pi_{X^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. In particular, the set of marked points D_X and $\Pi^{\text{ét}}_{X^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.

Proof. (a) follows immediately from Proposition 1.3. The tame fundamental group version of (b-i) and (b-ii) was proved by Tamagawa (cf. [T4, Lemma 5.1]). Moreover, it is easy to see that Tamagawa's proof holds for $\Pi_{X^\bullet}$. (c) is a special case of a result of the author (cf. [Y, Theorem 0.2, Remark 0.2.1, and Remark 0.2.2]). $\square$

Corollary 1.7. *Let $H \subseteq \Pi_{X^\bullet}$ be an open normal subgroup and $f^\bullet : Y^\bullet := (Y, D_Y) \to X^\bullet$ the connected tame Galois covering over k corresponding to H. Then D_X, D_Y, and the natural morphism*

$$\gamma_f : D_Y \to D_X$$

induced by $f^\bullet$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$, H, and the natural inclusion $H \hookrightarrow \Pi_{X^\bullet}$, respectively.

Proof. The corollary follows immediately from Proposition 1.6 (b-i), Proposition 1.6 (b-ii), and Proposition 1.6 (c). $\square$

2 The set of marked points and line bundles

We maintain the notations introduced in Section 1. Moreover, in this section, we suppose that $g_X \geq 2$ and $n_X > 0$. Let

$$(\ell, d, f^\bullet : Y^\bullet := (Y, D_Y) \to X^\bullet)$$

be a data satisfying the following conditions:

(a) ℓ and d are prime numbers distinct from each other and from p such that $\ell \equiv 1 \pmod{d}$; then all d^{th} roots of unity are contained in $\mathbb{F}_\ell$;

(b) $f^\bullet : Y^\bullet \to X^\bullet$ is an **étale** Galois covering over k whose Galois group is isomorphic to G_d, where $G_d \subseteq \mathbb{F}_\ell^\times$ denotes the subgroup of d^{th} roots of unity.

Write $M^{\text{ét}}_{Y^\bullet}$ and $M_{Y^\bullet}$ for $\mathrm{H}^1_{\text{ét}}(Y^\bullet, \mathbb{F}_\ell)$ and $\mathrm{Hom}(\Pi_{Y^\bullet}, \mathbb{F}_\ell)$, respectively, where $\Pi_{Y^\bullet}$ denotes the maximal pro-solvable quotient of the tame fundamental group of $Y^\bullet$. Note that there is a natural injection

$$M^{\text{ét}}_{Y^\bullet} \hookrightarrow M_{Y^\bullet}$$

induced by the natural surjection $\Pi_{Y^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{Y^\bullet}$. Then we obtain an exact sequence

$$0 \to M^{\text{ét}}_{Y^\bullet} \to M_{Y^\bullet} \to M^{\text{ra}}_{Y^\bullet} := \mathrm{coker}(M^{\text{ét}}_{Y^\bullet} \hookrightarrow M_{Y^\bullet}) \to 0$$

with a natural action of G_d.

Let

$$M^{\text{ra}}_{Y^\bullet, G_d} \subseteq M^{\text{ra}}_{Y^\bullet}$$

be the subset of elements on which G_d acts via the character $G_d \hookrightarrow \mathbb{F}_\ell^\times$ and

$$U^*_{Y^\bullet} \subseteq M_{Y^\bullet}$$

the subset of elements that map to nonzero elements of $M^{\text{ra}}_{Y^\bullet, G_n}$. For each $\alpha \in U^*_{Y^\bullet}$, write

$$g^\bullet_\alpha : Y^\bullet_\alpha := (Y_\alpha, D_{Y_\alpha}) \to Y^\bullet$$

for the tame covering corresponding to α. Then we obtain a morphism

$$\epsilon : U^*_{Y^\bullet} \to \mathbb{Z}$$

that maps α to $\#D_{Y_\alpha}$, where $\#(-)$ denotes the cardinality of $(-)$. We define a subset of $U^*_{Y^\bullet}$ to be

$$U^{\mathrm{mp}}_{Y^\bullet} := \{\alpha \in U^*_{Y^\bullet} \mid \#\mathrm{Ram}_{g^\bullet_\alpha} = d\} = \{\alpha \in U^*_{Y^\bullet} \mid \epsilon(\alpha) = \ell(dn_X - d) + d\}.$$

Note that $U^{\mathrm{mp}}_{Y^\bullet}$ is not empty. For each $\alpha \in U^{\mathrm{mp}}_{Y^\bullet}$, since the image of α is contained in $M^{\mathrm{ra}}_{Y^\bullet, G_d}$, we obtain that the action of G_d on the set $\mathrm{Ram}_{g^\bullet_\alpha} \subseteq D_{Y^\bullet}$ is transitive. Thus, there exists a unique marked point e_α of $X^\bullet$ such that $f^\bullet(y) = e_\alpha$ for each $y \in \mathrm{Ram}_{g^\bullet_\alpha}$.

We define a pre-equivalence relation $\sim$ on $U^{\mathrm{mp}}_{Y^\bullet}$ as follows:

> if $\alpha \sim \beta \in U^{\mathrm{mp}}_{Y^\bullet}$, then $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in U^*_{Y^\bullet}$, we have $\lambda\alpha + \mu\beta \in U^{\mathrm{mp}}_{Y^\bullet}$.

Then we have the following proposition.

Proposition 2.1. *The pre-equivalence relation $\sim$ on $U^{\mathrm{mp}}_{Y^\bullet}$ is an equivalence relation, and, moreover, the quotient set $U^{\mathrm{mp}}_{Y^\bullet}/\sim$ is naturally isomorphic to D_X that maps $[\alpha] \mapsto e_\alpha$.*

Proof. Let $\beta, \gamma \in U^{\mathrm{mp}}_{Y^\bullet}$. If $\mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$, then, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\beta+\mu\gamma \neq 0$, we have $\mathrm{Ram}_{g^\bullet_{\lambda\beta+\mu\gamma}} = \mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$. Thus, $\beta \sim \gamma$. On the other hand, if $\beta \sim \gamma$, we have $\mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$. Otherwise, we have $\#\mathrm{Ram}_{g^\bullet_{\beta+\gamma}} = 2d$. Thus, $\beta \sim \gamma$ if and only if $\mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$. Then $\sim$ is an equivalence relation on $U^{\mathrm{mp}}_{Y^\bullet}$.

We define a map

$$\vartheta : U^{\mathrm{mp}}_{Y^\bullet}/\sim \to D_X$$

that maps $\alpha \mapsto e_\alpha$. Let us prove that ϑ is a bijection. It is easy to see that ϑ is an injection. On the other hand, for each $e \in D_X$, the structure of the maximal pro-ℓ tame fundamental groups implies that we may construct a connected tame Galois covering of $h^\bullet : Z^\bullet \to Y^\bullet$ such that the line bundle corresponding to $h^\bullet$ is contained in $U^{\mathrm{mp}}_{Y^\bullet}$. Then ϑ is a surjection. This completes the proof of the lemma. $\square$

Remark 2.1.1. We claim that the set $U^{\mathrm{mp}}_{Y^\bullet}/\sim$ does not depend on the choices of ℓ, d, and the étale covering $f^\bullet : Y^\bullet \to X^\bullet$.

Let

$$(\ell^*, d^*, f^{\bullet,*} : Y^{\bullet,*} \to X^\bullet)$$

be a data. Hence we obtain a resulting $U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim$ and a naturally isomorphism

$$\vartheta^* : U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim \to D_X.$$

First, suppose that $\ell \neq \ell^*$, and that $d \neq d^*$. Then there exists a natural isomorphism

$$U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim \cong U^{\mathrm{mp}}_{Y^\bullet}/\sim$$

isomorphism which compatible with the isomorphism ϑ and ϑ^* as follows. Let $\alpha \in U^{\mathrm{mp}}_{Y^\bullet}$ and $\alpha^* \in U^{\mathrm{mp}}_{Y^{\bullet,*}}$. Write $Y^\bullet_\alpha \to Y^\bullet$ and $Y^{\bullet^*}_{\alpha^*} \to Y^{\bullet,*}$ for the tame coverings corresponding to α and α^*, respectively. Let us consider

$$Y^\bullet \times_{X^\bullet} Y^{\bullet,*}.$$

Thus, we have a connected tame Galois covering $Y^\bullet \times_{X^\bullet} Y^{\bullet,*} \to X^\bullet$ of degree $dd^*\ell\ell^*$. Then it is easy to check that α and α^* correspond to same marked points if and only if the cardinality of the set of marked points of $Y^\bullet \times_{X^\bullet} Y^{\bullet,*}$ is equal to $dd^*(\ell\ell^* n_X - 1) + 1)$.

In general case, we may choose a data

$$(\ell^{**}, d^{**}, f^{\bullet,**} : Y^{\bullet,**} \to X^\bullet)$$

such that $\ell^{**} \neq \ell$, $\ell \neq \ell^*$, $d^{**} \neq d$, and $d^{**} \neq d^*$. Hence we obtain a resulting $U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim$ and a naturally isomorphism $\vartheta^{**} : U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim \to D_X$. Then we obtain two natural isomorphisms $U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim \cong U^{\mathrm{mp}}_{Y^\bullet}/\sim$ and $U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim \cong U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim$. Thus, we have $U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim \cong U^{\mathrm{mp}}_{Y^\bullet}/\sim$.

Remark 2.1.2. Note that $U^*_{Y^\bullet}$ can be reconstructed group-theoretically from $\Pi_{Y^\bullet}$ and $\Pi_{X^\bullet}$. Since an element $\alpha \in U^*_{Y^\bullet}$ is contained in $U^{\mathrm{mp}}_{Y^\bullet}$ if and only if

$$\#D_{Y_\alpha} = \ell(dn_X - d) + d,$$

Proposition 1.6 (a) implies that $U^{\mathrm{mp}}_{Y^\bullet}$ can be reconstructed group-theoretically from $\Pi_{Y^\bullet}$ and $\Pi_{X^\bullet}$. Moreover, Remark 2.1.1 implies that D_X can be reconstructed group-theoretically from $\Pi_{X^\bullet}$.

Next, we calculate $\#U^{\mathrm{mp}}_{Y^\bullet}$. For each $e \in D_X$, we define

$$U^{\mathrm{mp}}_{Y^\bullet,e} := \{\alpha \in U^{\mathrm{mp}}_{Y^\bullet} \mid g^\bullet_\alpha \text{ is ramified over } (f^\bullet)^{-1}(e)\}.$$

Then, for any two marked points $e, e' \in D_X$ distinct from each other, we have

$$U^{\mathrm{mp}}_{Y^\bullet,e} \cap U^{\mathrm{mp}}_{Y^\bullet,e'} = \emptyset.$$

Moreover, we have

$$U^{\mathrm{mp}}_{Y^\bullet} = \bigcup_{e \in D_X} U^{\mathrm{mp}}_{Y^\bullet,e}.$$

Lemma 2.2. *Write g_Y for the genus of $Y^\bullet$. We have*

$$\#U^{\mathrm{mp}}_{Y^\bullet,e} = \ell^{2g_Y+1} - \ell^{2g_Y}.$$

Moreover, we have

$$\#U^{\mathrm{mp}}_{Y^\bullet} = n_X(\ell^{2g_Y+1} - \ell^{2g_Y}).$$

Proof. Write $E_e \subseteq D_Y$ for the set $(f^\bullet)^{-1}(e)$. Then $U^{\mathrm{mp}}_{Y^\bullet,e}$ can be naturally regarded as a subset of $\mathrm{H}^1_{\text{ét}}(Y \setminus E_e, \mathbb{F}_\ell)$ via the natural open immersion $Y \setminus E_e \hookrightarrow Y$. Write L_e for the $\mathbb{F}_\ell$-vector space generated by $U^{\mathrm{mp}}_{Y^\bullet,e}$ in $\mathrm{H}^1_{\text{ét}}(Y \setminus E_e, \mathbb{F}_\ell)$. Then we have

$$U^{\mathrm{mp}}_{Y^\bullet,e} = L_e \setminus \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell).$$

Write H_e for the quotient $L_e/\mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell)$. We have an exact sequence as follows:

$$0 \to \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell) \to L_e \to H_e \to 0.$$

Since the action of G_d on $(f^\bullet)^{-1}(e)$ is translative, we have

$$\dim_{\mathbb{F}_\ell} H_e = 1.$$

On the other hand, since $\dim_{\mathbb{F}_\ell} \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell) = 2g_Y$, we obtain

$$\# U^{\mathrm{mp}}_{Y^\bullet, e} = \ell^{2g_Y+1} - \ell^{2g_Y}.$$

Thus, we have

$$\# U^{\mathrm{mp}}_{Y^\bullet} = n_X(\ell^{2g_Y+1} - \ell^{2g_Y}).$$

This completes the proof of the lemma. □

3 Reconstruction of the inertia groups of marked points and their additive structures

Let k_1 and k_2 be algebraically closed fields of characteristic $p > 0$, and let $X_1^\bullet$ and $X_2^\bullet$ be smooth pointed stable curves of type (g_X, n_X) over k_1 and k_2, respectively. Write $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$ for the maximal pro-solvable quotient of the tame fundamental groups of $X_1^\bullet$ and $X_2^\bullet$, respectively. Suppose that $n_X > 0$, and that there is a continuous surjective morphism of profinite groups

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}.$$

Note that, since $X_1^\bullet$ and $X_2^\bullet$ are smooth pointed stable curves of type (g_X, n_X), ϕ induces a natural isomorphism

$$\phi^{p'} : \Pi^{p'}_{X_1^\bullet} \xrightarrow{\sim} \Pi^{p'}_{X_2^\bullet},$$

where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$.

Lemma 3.1. *Let ℓ be a prime number distinct from p. Then the isomorphism $(\phi^{p'})^{-1}$ induces an isomorphism*

$$\psi^\ell_X : \mathrm{H}^1_{\text{ét}}(X_1, \mathbb{F}_\ell) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X_2, \mathbb{F}_\ell).$$

Moreover, ψ^ℓ_X can be reconstructed group-theoretically from the surjection ϕ.

Proof. Let

$$f_1^\bullet : Y_1^\bullet \to X_1^\bullet$$

be an étale covering of degree ℓ over k_1. Write

$$f_2^\bullet : Y_2^\bullet \to X_2^\bullet$$

for the connected tame Galois covering of degree ℓ over k_2 induced by $\phi^{p'}$. Then we claim that $f_2^\bullet$ is an étale covering over k_2.

Write g_{Y_1} and g_{Y_2} for the genera of Y_1 and Y_2. Since $f_1^\bullet$ is an étale covering of degree ℓ, we have

$$g_{Y_1} = \ell(g_{X_1} - 1) + 1.$$

On the other hand, by Riemann-Hurwitz formula, we have

$$g_{Y_2} = \ell(g_{X_2} - 1) + 1 + \frac{1}{2}(\ell - 1)\#\mathrm{Ram}_{f_2^\bullet}.$$

Since ϕ is a surjection, Proposition 1.3 implies that

$$g_{Y_1} \geq g_{Y_2}.$$

Thus, we obtain that $\#\mathrm{Ram}_{f_2^\bullet} = 0$. This means that $f_2^\bullet$ is an étale covering over k_2. Then we have that the morphism $(\phi^{p'})^{-1}$ induces an injection

$$\psi_X^\ell : \mathrm{H}^1_{\text{ét}}(X_1, \mathbb{F}_\ell) \hookrightarrow \mathrm{H}^1_{\text{ét}}(X_2, \mathbb{F}_\ell).$$

Furthermore, since $\dim_{\mathbb{F}_\ell}\mathrm{H}^1_{\text{ét}}(X_1, \mathbb{F}_\ell) = \dim_{\mathbb{F}_\ell}\mathrm{H}^1_{\text{ét}}(X_2, \mathbb{F}_\ell)$, ψ_X^ℓ is a bijection. We complete the proof of the lemma. $\square$

Lemma 3.2. *Suppose that $g_X \geq 2$. Then the surjection $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces a bijection*

$$\rho_\phi : D_{X_1} \xrightarrow{\sim} D_{X_2}$$

between the sets of marked points of $X_1^\bullet$ and $X_2^\bullet$. Moreover, the bijection ρ_ϕ can be reconstructed group-theoretically from the surjection ϕ.

Proof. Let ℓ and d be prime numbers distinct from each other and from p. Suppose that

$$\ell \equiv 1 \pmod d.$$

Then we have that all d^{th} roots of unity are contained in $\mathbb{F}_\ell$. Write $G_d \subseteq \mathbb{F}_\ell^\times$ for the subgroup of d^{th} roots of unity.

Let

$$f_2^\bullet : Y_2^\bullet := (Y_2, D_{Y_2}) \to X_2^\bullet$$

be an étale covering of degree d over k_2. Then ϕ induces a tame covering

$$f_1^\bullet : Y_1^\bullet := (Y_1, D_{Y_1}) \to X_1^\bullet$$

of degree d over k_1. Then Lemma 3.1 implies that $f_1^\bullet$ is an étale covering over k_1. Note that $Y_1^\bullet$ and $Y_2^\bullet$ are same type.

Write $\Pi_{Y_1^\bullet}$ and $\Pi_{Y_2^\bullet}$ for the open normal subgroups of $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$ corresponding to $Y_1^\bullet$ and $Y_2^\bullet$, respectively. Write $M_{Y_1^\bullet}$, $M^{\text{ét}}_{Y_1^\bullet}$, $M^{\text{ra}}_{Y_1^\bullet}$, $M_{Y_2^\bullet}$, $M^{\text{ét}}_{Y_2^\bullet}$, and $M^{\text{ra}}_{Y_2^\bullet}$ for $\mathrm{Hom}(\Pi_{Y_1^\bullet}, \mathbb{F}_\ell)$, $\mathrm{H}^1_{\text{ét}}(Y_1, \mathbb{F}_\ell)$, $M_{Y_1^\bullet}/M^{\text{ét}}_{Y_1^\bullet}$, $\mathrm{Hom}(\Pi_{Y_2^\bullet}, \mathbb{F}_\ell)$, $\mathrm{H}^1_{\text{ét}}(Y_2, \mathbb{F}_\ell)$, and $M_{Y_2^\bullet}/M^{\text{ét}}_{Y_2^\bullet}$. Then Lemma 3.1 implies that $(\phi^{p'})^{-1}$ induces a commutative diagram as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^{\text{ét}}_{Y_1^\bullet} & \longrightarrow & M_{Y_1^\bullet} & \longrightarrow & M^{\text{ra}}_{Y_1^\bullet} & \longrightarrow & 0 \\
 & & \big\downarrow & & \psi_Y^\ell \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & M^{\text{ét}}_{Y_2^\bullet} & \longrightarrow & M_{Y_2^\bullet} & \longrightarrow & M^{\text{ra}}_{Y_2^\bullet} & \longrightarrow & 0,
\end{array}$$

where all the vertical arrows are isomorphisms. Write $U^*_{Y_1^\bullet}$ and $U^*_{Y_2^\bullet}$ for the subsets of $M_{Y_1^\bullet}$ and $M_{Y_2^\bullet}$ defined as in Section 2, respectively. Since the actions of G_d on the exact sequences are compatible with the isomorphisms appeared in the commutative diagram above, we have

$$\psi^\ell_Y(U^*_{Y_1^\bullet}) = U^*_{Y_2^\bullet}.$$

Let $\alpha_1 \in U^{\mathrm{mp}}_{Y_1^\bullet}$ and

$$g^\bullet_{\alpha_1} : Y^\bullet_{\alpha_1} \to Y^\bullet_1$$

the tame covering of degree ℓ over k_1 corresponding to α_1. Write

$$g^\bullet_{\alpha_2} : Y^\bullet_{\alpha_2} \to Y^\bullet_2$$

for the tame covering of degree ℓ over k_2 corresponding to $\alpha_2 := \psi^\ell_Y(\alpha_1)$. Write $g_{Y_{\alpha_1}}$ and $g_{Y_{\alpha_2}}$ for the genera of $Y^\bullet_{\alpha_1}$ and $Y^\bullet_{\alpha_2}$. Then Proposition 1.3 and Riemann-Hurwitz formula implies that

$$g_{Y_{\alpha_1}} - g_{Y_{\alpha_2}} = \frac{1}{2}(d - \#\mathrm{Ram}_{g^\bullet_{\alpha_2}})(\ell - 1) \geq 0.$$

This means that

$$d - \#\mathrm{Ram}_{g^\bullet_{\alpha_2}} \geq 0.$$

Since $\alpha_2 \in U^*_{Y_2^\bullet}$, we have $d | \#\mathrm{Ram}_{g^\bullet_{\alpha_2}}$. Thus, either $\#\mathrm{Ram}_{g^\bullet_{\alpha_2}} = 0$ or $\#\mathrm{Ram}_{g^\bullet_{\alpha_2}} = d$ holds.

If $\#\mathrm{Ram}_{g^\bullet_{\alpha_2}} = 0$, then $g^\bullet_{\alpha_2}$ is an étale covering over k_2. Then Lemma 3.1 implies that $g^\bullet_{\alpha_1}$ is an étale covering over k_1. This contradicts to $\alpha_1 \in U^{\mathrm{mp}}_{Y_1^\bullet}$. Then we have $\#\mathrm{Ram}_{g^\bullet_{\alpha_2}} = d$. This means that $\alpha_2 \in U^{\mathrm{mp}}_{Y_2^\bullet}$. Thus, we obtain

$$\psi^\ell_Y(U^{\mathrm{mp}}_{Y_1^\bullet}) \subseteq U^{\mathrm{mp}}_{Y_2^\bullet}.$$

On the other hand, Lemma 2.2 implies that $\#U^{\mathrm{mp}}_{Y_1^\bullet} = \#U^{\mathrm{mp}}_{Y_2^\bullet}$. We have

$$\psi^\ell_Y(U^{\mathrm{mp}}_{Y_1^\bullet}) = U^{\mathrm{mp}}_{Y_2^\bullet}.$$

Then Proposition 2.1 implies that ψ^ℓ_Y induces a bijection

$$\rho_\phi : D_{X_1} \overset{\sim}{\to} D_{X_2}.$$

Remark 2.1.1 implies that ρ_ϕ does not depend on ψ^ℓ_Y. Then Remark 2.1.2 implies that the bijection ρ_ϕ can be reconstructed group-theoretically from ϕ. This completes the proof of the lemma. $\square$

Let m be a natural number and $U_2 := \Pi_{X_2^\bullet}(m)$. We set $U_1 := \phi^{-1}(U_2) \subseteq \Pi_{X_1^\bullet}$. Write $Y^\bullet_{U_1} := (Y_{U_1}, D_{Y_{U_1}})$ for the smooth pointed stable curve of type $(g_{Y_{U_1}}, n_{Y_{U_1}})$ over k_1 corresponding to U_1, $Y^\bullet_{U_2} := (Y_{U_2}, D_{Y_{U_2}})$ for the smooth pointed stable curve of type $(g_{Y_{U_2}}, n_{Y_{U_2}})$ over k_2 corresponding to U_2. Then we obtain two connected tame Galois coverings

$$f^\bullet_{U_1} : Y^\bullet_{U_1} \to X^\bullet_1$$

over k_1 and

$$f^\bullet_{U_2} : Y^\bullet_{U_2} \to X^\bullet_2$$

over k_2. Note that we have

$$(g_{Y_{U_1}}, n_{Y_{U_1}}) = (g_{Y_{U_2}}, n_{Y_{U_2}}).$$

Moreover, ϕ induces a commutative diagram as follows:

$$\begin{tikzcd}
U_1 \arrow[r, "\phi|_{U_1}"] \arrow[d] & U_2 \arrow[d] \\
\Pi_{X_1^\bullet} \arrow[r, "\phi"] \arrow[d] & \Pi_{X_2^\bullet} \arrow[d] \\
\Pi_{X_1^\bullet}/U_1 \arrow[r, equal] & \Pi_{X_2^\bullet}/U_2 = \Pi_{X_2^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/m\mathbb{Z},
\end{tikzcd}$$

where $\phi|_{U_1}$ is a surjection and the bottom arrow is an isomorphism. Note that, if $(m, p) = 1$, we have $U_1 = \Pi_{X_1^\bullet}(m)$ and $\Pi_{X_1^\bullet}/U_1 = \Pi_{X_1^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/m\mathbb{Z}$.

Suppose that $g_X \geq 2$. Lemma 2.3 implies that $\phi|_{U_1}$ induces a bijection

$$\rho_{\phi|_{U_1}} : D_{Y_{U_1}} \xrightarrow{\sim} D_{Y_{U_2}}.$$

Then Corollary 1.7 implies a diagram as follows:

$$\begin{tikzcd}
D_{Y_{U_1}} \arrow[r, "\rho_{\phi|_{U_1}}"] \arrow[d, "\gamma_{f_{U_1}}"'] & D_{Y_{U_2}} \arrow[d, "\gamma_{f_{U_2}}"'] \\
D_{X_1} \arrow[r, "\rho_\phi"] & D_{X_2}.
\end{tikzcd}$$

We have the following lemma.

Lemma 3.3. *Suppose that $g_X \geq 2$. The diagram obtained above*

$$\begin{tikzcd}
D_{Y_{U_1}} \arrow[r, "\rho_{\phi|_{U_1}}"] \arrow[d, "\gamma_{f_{U_1}}"'] & D_{Y_{U_2}} \arrow[d, "\gamma_{f_{U_2}}"'] \\
D_{X_1} \arrow[r, "\rho_\phi"] & D_{X_2}
\end{tikzcd}$$

is a commutative diagram. Moreover, the commutative diagram can be reconstructed group-theoretically from the commutative diagram of profinite groups

$$\begin{tikzcd}
U_1 \arrow[r, "\phi|_{U_1}"] \arrow[d] & U_2 \arrow[d] \\
\Pi_{X_1^\bullet} \arrow[r, "\phi"] & \Pi_{X_2^\bullet}.
\end{tikzcd}$$

Proof. By applying Corollary 1.7 and Lemma 3.1, to verify the lemma, we only need to check that the diagram is commutative.

Let $e_{U_1} \in D_{Y_{U_1}}$, $e_{U_2} := \rho_{\phi|_{U_1}}(e_{U_1}) \in D_{Y_{U_2}}$, $e_1 := \gamma_{f_{U_1}}(e_{U_1}) \in D_{X_1}$, $e_2 = (\gamma_{f_{U_2}} \circ \rho_{\phi|_{U_1}})(e_{U_1}) \in D_{X_2}$, and $e_1' := \rho_\phi^{-1}(e_2) \in D_{X_1}$. Let us prove that $e_1 = e_1'$. Write S_{U_1} and S_{U_2} for the sets $(f_{U_1}^\bullet)^{-1}(e_1')$ and $(f_{U_2}^\bullet)^{-1}(e_2)$, respectively. Note that $e_{U_2} \in S_{U_2}$. To verify $e_1 = e_1'$, we only need to prove that $e_{U_1} \in S_{U_1}$

Let $(\ell, d, f_2^\bullet : Y_2^\bullet \to X_2^\bullet)$ be a data defined as in Section 2. Suppose that $(\ell, m) = 1$ and $(d, m) = 1$. By lemma 3.1, we obtain a data

$$(\ell, d, f_1^\bullet : Y_1^\bullet \to X_1^\bullet)$$

induced by ϕ and $(\ell, d, f_2^\bullet : Y_2^\bullet \to X_2^\bullet)$. On the other hand, we have a data

$$(\ell, d, g_2^\bullet : Z_2^\bullet := Y_2^\bullet \times_{X_2^\bullet} Y_{U_2}^\bullet \to Y_{U_2}^\bullet)$$

induced by the natural inclusion $U_2 \hookrightarrow \Pi_{X_2^\bullet}$ and $(\ell, d, f_2^\bullet : Y_2^\bullet \to X_2^\bullet)$. Again, by lemma 3.1, we obtain a data

$$(\ell, d, g_1^\bullet : Z_1^\bullet := Y_1^\bullet \times_{X_1^\bullet} Y_{U_1}^\bullet \to Y_{U_1}^\bullet)$$

induced by $\phi|_{U_1}$ and $(\ell, d, g_2^\bullet : Z_2^\bullet \to Y_{U_2}^\bullet)$.

Let $\alpha_2 \in U^{\mathrm{mp}}_{Y_2^\bullet, e_2}$, where $U^{\mathrm{mp}}_{(-)}$ is defined as in Section 2. Then the proof of lemma 3.2 implies that α_2 induces an element

$$\alpha_1 \in U^{\mathrm{mp}}_{Y_1^\bullet, e_1'}.$$

Write $Y_{\alpha_1}^\bullet$ and $Y_{\alpha_2}^\bullet$ for the smooth pointed stable curves over k_1 and k_2 corresponding to α_1 and α_2, respectively. Consider the connected tame Galois covering

$$Y_{\alpha_2}^\bullet \times_{X_2^\bullet} Y_{U_2}^\bullet \to Z_2^\bullet$$

of degree ℓ over k_2, and write β_2 for the element of $U^*_{Z_2^\bullet}$ corresponding to this connected tame Galois covering, where $U^*_{(-)}$ is defined as in Section 2. Then we have

$$\beta_2 = \sum_{c_2 \in S_{U_2}} t_{c_2} \beta_{c_2},$$

where $t_{c_2} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and $\beta_{c_2} \in U^{\mathrm{mp}}_{Z_2^\bullet, c_2}$. On the other hand, the proof of Lemma 3.2 implies that β_2 induces an element

$$\beta_1 = \sum_{c_1 \in S_{U_1}} t_{c_1} \beta_{c_1} \in U^*_{Z_1^\bullet},$$

where $t_{c_1} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and $\beta_{c_1} \in U^{\mathrm{mp}}_{Z_1^\bullet, c_1}$. Note that since β_1 corresponds to the connected tame Galois covering $Y_{\alpha_2}^\bullet \times_{X_2^\bullet} Y_{U_2}^\bullet \to Z_2^\bullet$, we have the composition of the connected tame Galois covering $Y_{\alpha_2}^\bullet \times_{X_2^\bullet} Y_{U_2}^\bullet \to Z_2^\bullet$ and the étale Galois covering $g_1^\bullet : Z_1^\bullet \to Y_{U_1}^\bullet$ is tamely ramified over e_1'. This means that e_{U_1} is contained in S_{U_1}. This completes the proof of the lemma. $\square$

Remark 3.3.1. We maintain the notations introduced in the proof of Lemma 3.3. Let $A_U := \Pi_{X_1^\bullet}/U_1 = \Pi_{X_2^\bullet}/U_2 = \Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/m\mathbb{Z}$. The sets of line bundles

$$\bigcup_{c_1 \in S_{U_1}} U^{\mathrm{mp}}_{Z_1^\bullet, c_1} \text{ and } \bigcup_{c_2 \in S_{U_2}} U^{\mathrm{mp}}_{Z_2^\bullet, c_2}$$

admit natural actions of A_U which are induced by the surjections $\Pi_{X_1^\bullet} \twoheadrightarrow A_U$ and $\Pi_{X_2^\bullet} \twoheadrightarrow A_U$, respectively. Then $D_{Y_{U_1}}$ and $D_{Y_{U_2}}$ admit actions of A_U which are induced by the actions of A_U on the sets of line bundles above, respectively. Note that it is easy to see that the actions of A_U on $D_{Y_{U_1}}$ and $D_{Y_{U_2}}$ above identify with the natural actions of A_U on $D_{Y_{U_1}}$ and $D_{Y_{U_2}}$ induced by the connected tame Galois coverings $f^\bullet_{U_1}$ and $f^\bullet_{U_2}$, respectively. Moreover, we have the commutative diagram

$$\begin{array}{ccc} D_{Y_{U_1}} & \xrightarrow{\rho_\phi|_{U_1}} & D_{Y_{U_2}} \\ {\scriptstyle \gamma_{f_{U_1}}}\downarrow & & {\scriptstyle \gamma_{f_{U_2}}}\downarrow \\ D_{X_1} & \xrightarrow{\rho_\phi} & D_{X_2} \end{array}$$

is compatible with the actions of A_U on the sets of marked points.

Next, we prove the main theorem of this section.

Theorem 3.4. *Let $\widetilde{X}_1^{\bullet,\mathrm{sol}} := (\widetilde{X}_1^{\mathrm{sol}}, D_{\widetilde{X}_1^{\mathrm{sol}}})$ and $\widetilde{X}_2^{\bullet,\mathrm{sol}} := (\widetilde{X}_2^{\mathrm{sol}}, D_{\widetilde{X}_2^{\mathrm{sol}}})$ be the pairs (see Section 1 for the definition) associated to $X_1^\bullet$ and $X_2^\bullet$, respectively. Let $\widetilde{e}_2 \in D_{\widetilde{X}_2^{\mathrm{sol}}}$ and $I_{\widetilde{e}_2} \in \mathrm{Ine}(\Pi_{X_2^\bullet})$ the inertia subgroup associated to $\widetilde{e}_2$. Then there exists an inertia subgroup $I_{\widetilde{e}_1} \in \mathrm{Ine}(\Pi_{X_1^\bullet})$ associated to a point $\widetilde{e}_1 \in D_{\widetilde{X}_1^{\mathrm{sol}}}$ such that*

$$\phi(I_{\widetilde{e}_1}) = I_{\widetilde{e}_2}.$$

Moreover, the restriction homomorphism

$$\phi|_{I_{\widetilde{e}_1}} : I_{\widetilde{e}_1} \twoheadrightarrow I_{\widetilde{e}_2}$$

is an isomorphism.

Proof. Let $N >> 0$ be an integer number such that $(N, p) = 1$. We set $F_1 := \Pi_{X_1^\bullet}(N)$ and $F_2 := \Pi_{X_2^\bullet}(N)$. Then ϕ induces a commutative diagram as follows.

$$\begin{array}{ccc} F_1 & \longrightarrow & F_2 \\ \downarrow & & \downarrow \\ \Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet} \\ \downarrow & & \downarrow \\ \Pi^{\mathrm{ab}}_{X_1^\bullet} \otimes \mathbb{Z}/N\mathbb{Z} & = & \Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/N\mathbb{Z}, \end{array}$$

where the top arrow is a surjection induced by ϕ. Then we obtain a smooth pointed stable curve $Y^\bullet_{F_1}$ of type $(g_{Y_{F_1}}, n_{Y_{F_1}})$ over k_1 corresponding to F_1 and a smooth pointed stable curve $Y^\bullet_{F_2}$ of type $(g_{Y_{F_2}}, n_{Y_{F_2}})$ over k_2 corresponding to F_2. Since $X_1^\bullet$ and $X_2^\bullet$ are smooth pointed stable curves of type (g_X, n_X), we obtain $g_{Y_{F_1}} = g_{Y_{F_2}} \geq 2$ and $n_{Y_{F_1}} = n_{Y_{F_2}}$. To verify the theorem, by replacing $X_1^\bullet$ and $X_2^\bullet$ by $Y^\bullet_{F_1}$ and $Y^\bullet_{F_2}$, respectively, we may assume that

$$g_X \geq 2.$$

Let $C_{X_2^\bullet} := \{H_{2,i}\}_{i \in \mathbb{Z}_{\geq 0}}$ be a set of open subgroups of $\Pi_{X^\bullet}$ satisfying the following conditions:

(a) $H_{2,0} = \Pi_{X_2^\bullet}$ and $H_{2,i+1}$ is an open normal subgroup of $H_{2,i}$ for each $i \in \mathbb{Z}_{\geq 0}$;

(b) $\varprojlim_i \Pi_{X_2^\bullet}/H_{2,i} \cong \Pi_{X_2^\bullet}$;

(c) for each $i \in \mathbb{Z}_{\geq 0}$, $H_{2,i+1} = H_{2,i}(n_{2,i})$ for some $n_i \in \mathbb{Z}_{>0}$.

For each $i \in \mathbb{Z}_{\geq 0}$, we write $X^\bullet_{H_{2,i}} := (X_{H_{2,i}}, D_{X_{H_{2,i}}})$ for the smooth pointed stable curve over k_2 corresponding to $H_{2,i}$ and $e_{H_{2,i}} \in D_{X_{H_{2,i}}}$ for the image of $\widetilde{e}_2$ in $X^\bullet_{H_{2,i}}$. Then we obtain a sequence of marked points

$$\mathcal{I}_{\widetilde{e}_2}^{C_{X_2^\bullet}} : \cdots \mapsto e_{H_{2,2}} \mapsto e_{H_{2,1}} \mapsto e_{H_{2,0}}.$$

Write $\{H_{1,i} := \phi^{-1}(H_{2,i})\}_{i \in \mathbb{Z}_{\geq 0}}$ for the set of open subgroups of $\Pi_{X_1^\bullet}$ induced by ϕ. For each $i \in \mathbb{Z}_{\geq 0}$, we write $X^\bullet_{H_{1,i}} := (X_{H_{1,i}}, D_{X_{H_{1,i}}})$ for the smooth pointed stable curve over k_1 corresponding to $H_{1,i}$. Then, for each $i \in \mathbb{Z}_{\geq 0}$, Lemma 3.2 implies that the restriction homomorphism $\phi|_{H_{1,i}} : H_{1,i} \to H_{2,i}$ induces a natural bijection of the set of marked points

$$\rho_{\phi|_{H_i}} : D_{X_{1,i}} \xrightarrow{\sim} D_{X_{2,i}},$$

moreover, that $\rho_{\phi|_{H_i}}$ can be reconstructed group-theoretically from $\phi|_{H_{1,i}}$. We set

$$e_{H_{1,i}} := \rho^{-1}_{\phi|_{H_i}}(e_{H_{2,i}})$$

for each $i \in \mathbb{Z}_{\geq 0}$. Then, by applying Lemma 3.3, $\mathcal{I}_{\widetilde{e}_2}^{C_{X_2^\bullet}}$ induces a sequence of marked points as follows:

$$\cdots \mapsto e_{H_{1,2}} \in D_{X_{H_{1,2}}} \mapsto e_{H_{1,1}} \in D_{X_{1,1}} \mapsto e_{H_{1,0}} \in D_{X_{1,0}} = D_{X_1}.$$

Let $K_{\ker(\phi)}$ be the subfield of $\widetilde{K}^{\mathrm{sol}}$ corresponding to the closed subgroup $\ker(\phi)$ of $\Pi_{X_1^\bullet}$. We set

$$\widetilde{X}^\bullet_{\ker(\phi)} := (\widetilde{X}_{\ker(\phi)}, D_{\widetilde{X}_{\ker(\phi)}}),$$

where $\widetilde{X}_{\ker(\phi)}$ denotes the normalization of X in $K_{\ker(\phi)}$ and $D_{\widetilde{X}_{\ker(\phi)}}$ denotes the inverse image of D_X in $\widetilde{X}_{\ker(\phi)}$. Then the sequence

$$\cdots \mapsto e_{H_{1,2}} \mapsto e_{H_{1,1}} \mapsto e_{H_{1,0}}.$$

determines a point $\widetilde{e}_{\ker(\phi)} \in D_{\widetilde{X}_{\ker(\phi)}}$. We choose a point of $\widetilde{e}_1 \in D_{\widetilde{X}_1^{\mathrm{sol}}}$ such that the image of $\widetilde{e}_1$ in $D_{\widetilde{X}_{\ker(\phi)}}$ is $\widetilde{e}_{\ker(\phi)}$. Then we have $\phi(I_{\widetilde{e}_1}) = I_{\widetilde{e}_2}$.

Moreover, since $I_{\widetilde{e}_1}$ and $I_{\widetilde{e}_2}$ are isomorphic to $\widehat{\mathbb{Z}}(1)^{p'}$, the restriction homomorphism $\phi|_{I_{\widetilde{e}_1}}$ is an isomorphism. This completes the proof of the theorem. $\square$

In the remainder of this section, we reconstruct "additive structures" of inertia groups. Let $\widetilde{e}_2$ be any point of $D_{\widetilde{X}_2^{\mathrm{sol}}}$ and $\widetilde{e}_1$ a point of $D_{\widetilde{X}_1^{\mathrm{sol}}}$ such that $\phi(I_{\widetilde{e}_1}) = \widetilde{e}_2$. Write $\overline{\mathbb{F}}_1$ (resp. $\overline{\mathbb{F}}_2$) for the algebraic closure of $\mathbb{F}_p$ in k_1 (resp. k_2). We set

$$\mathbb{F}_{\widetilde{e}_1} := (I_{\widetilde{e}_1} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_1^{p'}) \coprod \{*_{e_1}\} \text{ (resp. } \mathbb{F}_{\widetilde{e}_2} := (I_{\widetilde{e}_2} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_2^{p'}) \coprod \{*_{e_2}\}),$$

where $\{*_{e_1}\}$ (resp. $\{*_{e_2}\}$) is an one-point set, and $(\mathbb{Q}/\mathbb{Z})_1^{p'}$ (resp. $(\mathbb{Q}/\mathbb{Z})_2^{p'}$) denotes the prime-to-$p$ part of $(\mathbb{Q}/\mathbb{Z})_1$ (resp. $(\mathbb{Q}/\mathbb{Z})_2$), can be canonically identified with

$$\bigcup_{(p,m)=1} \mu_m(k_1) \text{ (resp. } \bigcup_{(p,m)=1} \mu_m(k_2)).$$

Moreover, $\mathbb{F}_{\widetilde{e}_1}$ (resp. $\mathbb{F}_{\widetilde{e}_2}$) can be identified with $\overline{\mathbb{F}}_1$ (resp. $\overline{\mathbb{F}}_2$) as set, hence, carries a structure of field, whose multiplicative group is $I_{\widetilde{e}_1} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_1^{p'}$ (resp. $I_{\widetilde{e}_2} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_2^{p'}$) and whose zero element is $*_{e_1}$ (resp. $*_{e_2}$). Then we have the following proposition.

Corollary 3.5. *The field structures of $\mathbb{F}_{\widetilde{e}_1}$ and $\mathbb{F}_{\widetilde{e}_2}$ can be reconstructed group-theoretically from $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$, respectively. Moreover, ϕ induces a field isomorphism*

$$\theta_{\phi,\widetilde{e}_1,\widetilde{e}_2} : \mathbb{F}_{\widetilde{e}_1} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_2},$$

and $\theta_{\phi,\widetilde{e}_1,\widetilde{e}_2}$ can be reconstructed group-theoretically from ϕ.

Proof. To verify the theorem, similar arguments to the arguments given in the proof of [T4, Proposition 5.3] imply that we may assume that $n_X = 3$.

For each natural number r, we denote by $\mathbb{F}_{p^r,\widetilde{e}_1}$ (resp. $\mathbb{F}_{p^r,\widetilde{e}_2}$) the unique subfield of $\mathbb{F}_{\widetilde{e}_1}$ (resp. $\mathbb{F}_{\widetilde{e}_2}$). We fix any finite field $\mathbb{F}_{p^r}$ of cardinality p^r and an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}_p$. By Proposition 1.6 (c), we have $\mathbb{F}^\times_{p^r,\widetilde{e}_1} = I_{\widetilde{e}_1}/(p^r-1)$ (resp. $\mathbb{F}^\times_{p^r,\widetilde{e}_2} = I_{\widetilde{e}_2}/(p^r-1)$) can be reconstructed group-theoretically from $\Pi_{X_1^\bullet}$ (resp. $\Pi_{X_2^\bullet}$). Then the set

$$\mathrm{Hom}_{\mathrm{group}}(\mathbb{F}^\times_{p^r,\widetilde{e}_1}, \mathbb{F}^\times_{p^r}) \text{ (resp. } \mathrm{Hom}_{\mathrm{group}}(\mathbb{F}^\times_{p^r,\widetilde{e}_2}, \mathbb{F}^\times_{p^r}))$$

is group-theoretically, and reconstructing the field structure of $\mathbb{F}_{p^r,\widetilde{e}_1}$ (resp. $\mathbb{F}_{p^r,\widetilde{e}_2}$) is equivalent to reconstructing

$$\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_1}, \mathbb{F}_{p^r}) \text{ (resp. } \mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_2}, \mathbb{F}_{p^r}))$$

as a subset of $\mathrm{Hom}_{\mathrm{group}}(\mathbb{F}^\times_{p^r,\widetilde{e}_1}, \mathbb{F}^\times_{p^r})$ (resp. $\mathrm{Hom}_{\mathrm{group}}(\mathbb{F}^\times_{p^r,\widetilde{e}_2}, \mathbb{F}^\times_{p^r})$). Note that, to reconstruct the field structure of $\mathbb{F}_{\widetilde{e}_1}$ (resp. $\mathbb{F}_{\widetilde{e}_2}$), it is sufficient to reconstruct the subset $\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_1}, \mathbb{F}_{p^r})$ (resp. $\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_2}, \mathbb{F}_{p^r})$) for r in a cofinal subset of $\mathbb{N}$ with respect to division.

Let

$$\chi_1 \in \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_1^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}^\times_{p^r})$$
$$\text{(resp. } \chi_2 \in \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}^\times_{p^r})).$$

Write H_{χ_1} (resp. H_{χ_2}) for $\Pi_{X_1^\bullet}(p^r-1)$ (resp. $\Pi_{X_2^\bullet}(p^r-1)$), M_{χ_1} for $H^{\mathrm{ab}}_{\chi_1} \otimes \mathbb{F}_p$ (resp. M_{χ_2} for $H^{\mathrm{ab}}_{\chi_2} \otimes \mathbb{F}_p$), and

$$X^\bullet_{H_{\chi_1}} := (X_{H_{\chi_1}}, D_{X_{H_{\chi_1}}}) \text{ (resp. } X^\bullet_{H_{\chi_2}} := (X_{H_{\chi_2}}, D_{X_{H_{\chi_2}}}))$$

for the smooth pointed stable curve over k_1 (resp. k_2) corresponding to H_{χ_1} (resp. H_{χ_2}). Note that M_{χ_1} (resp. M_{χ_2}) is a $\mathbb{F}_p[\Pi^{\mathrm{ab}}_{X_1^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}]$-module (resp. $\mathbb{F}_p[\Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}]$-module) via conjugation. We define

$$M_{\chi_1}[\chi_1] := \{a \in M_{\chi_1} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}} \mid \sigma(a) = \chi_1(\sigma)a \text{ for all } \sigma \in \Pi^{\mathrm{ab}}_{X_1^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}\}$$

(resp. $M_{\chi_2}[\chi_2] := \{a \in M_{\chi_2} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}} \mid \sigma(a) = \chi_2(\sigma)a \text{ for all } \sigma \in \Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}\}$)
and

$$\gamma_{\chi_1}(M_{\chi_1}) := \dim_{\overline{\mathbb{F}}}(M_{\chi_1}[\chi_1]) \text{ (resp.} \gamma_{\chi_2}(M_{\chi_2}) := \dim_{\overline{\mathbb{F}}}(M_{\chi_2}[\chi_2])).$$

[T4, Remark 3.7] implies that $\gamma_{\chi_1}(M_{\chi_1}) \leq g_X + 1$ (resp. $\gamma_{\chi_2}(M_{\chi_2}) \leq g_X + 1$).

We define two maps

$$\mathrm{Res}_{1,r} : \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_1^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}^\times_{p^r}) \to \mathrm{Hom}(\mathbb{F}^\times_{p^r,\widetilde{e}_1}, \mathbb{F}^\times_{p^r})$$

$$\text{(resp. } \mathrm{Res}_{2,r} : \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}^\times_{p^r}) \to \mathrm{Hom}(\mathbb{F}^\times_{p^r,\widetilde{e}_2}, \mathbb{F}^\times_{p^r}))$$

and

$$\Gamma_{1,r} : \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_1^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}^\times_{p^r}) \to \mathbb{Z}_{\geq 0}$$

$$\text{(resp. } \Gamma_{2,r} : \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}^\times_{p^r}) \to \mathbb{Z}_{\geq 0}),$$

where the map $\mathrm{Res}_{1,r}$ (resp. $\mathrm{Res}_{2,r}$) is the restriction with respect to the natural inclusion $\mathbb{F}^\times_{p^r,\widetilde{e}_1} \hookrightarrow \Pi^{\mathrm{ab}}_{X_1^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}$ (resp. $\mathbb{F}^\times_{p^r,\widetilde{e}_2} \hookrightarrow \Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}$), and the map $\Gamma_{1,r}$ (resp. $\Gamma_{2,r}$) is the map that maps $\chi_1 \mapsto \gamma_{\chi_1}(M_{\chi_1})$ (resp. $\chi_2 \mapsto \gamma_{\chi_2}(M_{\chi_2})$).

Let m_0 be the product of all prime numbers $\leq p-2$ if $p \neq 2,3$ and $m_0 = 1$ if $p = 2,3$. Let r_0 be the order of p in the multiplicative group $(\mathbb{Z}/m_0\mathbb{Z})^\times$. Then [T4, Claim 5.4] implies the following result:

> there exists a constant $C(g_X)$ which only depends on g_X such that, for each $r > \log_p(C(g_X)+1)$ divisible by r_0, we have
>
> $$\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_1}, \mathbb{F}_{p^r}) = \mathrm{Hom}^{\mathrm{surj}}_{\mathrm{groups}}(\mathbb{F}^\times_{p^r,\widetilde{e}_1}, \mathbb{F}^\times_{p^r}) \setminus \mathrm{Res}_{1,r}(\Gamma^{-1}_{1,r}(\{g_X+1\}))$$
>
> $$\text{(resp. } \mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_2}, \mathbb{F}_{p^r}) = \mathrm{Hom}^{\mathrm{surj}}_{\mathrm{groups}}(\mathbb{F}^\times_{p^r,\widetilde{e}_2}, \mathbb{F}^\times_{p^r}) \setminus \mathrm{Res}_{2,r}(\Gamma^{-1}_{2,r}(\{g_X+1\}))),$$
>
> where $\mathrm{Hom}^{\mathrm{surj}}_{\mathrm{groups}}(-,-)$ denotes the set of surjections of $\mathrm{Hom}^{\mathrm{surj}}_{\mathrm{groups}}(-,-)$.

Thus, we obtain that the field structures of $\mathbb{F}_{\widetilde{e}_1}$ and $\mathbb{F}_{\widetilde{e}_2}$ can be reconstructed group-theoretically from $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$, respectively.

Next, we prove the "moreover" part of the proposition. Let

$$\kappa_2 \in \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}^\times_{p^r}).$$

Then ϕ induced a character

$$\kappa_1 \in \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}^\times_{p^r}).$$

Moreover, $\phi|_{H_{\kappa_1}}$ induces a surjection

$$M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2].$$

Suppose that $\kappa_2 \in \Gamma^{-1}_{2,r}(\{g_X+1\})$. Then we obtain that the surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$ is an isomorphism. On the other hand, by Theorem 3.4, we have an isomorphism $\phi|_{I_{\widetilde{e}_1}} : I_{\widetilde{e}_1} \xrightarrow{\sim} I_{\widetilde{e}_1}$. Then the isomorphism $\phi|_{I_{\widetilde{e}_1}}$ induces an injective

$$\mathrm{Res}_{2,r}(\Gamma^{-1}_{2,r}(\{g_X+1\})) \hookrightarrow \mathrm{Res}_{1,r}(\Gamma^{-1}_{1,r}(\{g_X+1\})).$$

Since $\#\mathrm{Hom}_{\text{fields}}(\mathbb{F}_{p^r,\widetilde{e}_1}, \mathbb{F}_{p^r}) = \#\mathrm{Hom}_{\text{fields}}(\mathbb{F}_{p^r,\widetilde{e}_2}, \mathbb{F}_{p^r})$, $\phi|_{I_{\widetilde{e}_1}}$ induces a bijection

$$\mathrm{Hom}_{\text{fields}}(\mathbb{F}_{p^r,\widetilde{e}_2}, \mathbb{F}_{p^r}) \xrightarrow{\sim} \mathrm{Hom}_{\text{fields}}(\mathbb{F}_{p^r,\widetilde{e}_1}, \mathbb{F}_{p^r}).$$

Thus, $\phi|_{I_{\widetilde{e}_1}}$ induces a bijection

$$\mathrm{Hom}_{\text{fields}}(\mathbb{F}_{\widetilde{e}_2}, \overline{\mathbb{F}}) \xrightarrow{\sim} \mathrm{Hom}_{\text{fields}}(\mathbb{F}_{\widetilde{e}_1}, \overline{\mathbb{F}}).$$

If we choose $\overline{\mathbb{F}} = \mathbb{F}_{\widetilde{e}_1}$, then the bijection above induces a field isomorphism

$$\theta_{\phi,\widetilde{e}_1,\widetilde{e}_2} : \mathbb{F}_{\widetilde{e}_1} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_2}.$$

This completes the proof of the proposition. □

4 A weak Hom-version of the Grothendieck conjecture for curves of type $(0, n)$

We maintain the notations introduced in Section 3. Moreover, in this section, we suppose that $(g_X, n_X) = (0, n)$.

Fix two marked points $e_{1,\infty}, e_{1,0} \in D_{X_1}$ distinct from each other. We choose any field k_1' that is isomorphic to k_1, and choose any isomorphism $\varphi_1 : X_1 \xrightarrow{\sim} \mathbb{P}^1_{k_1'}$ as schemes such that $\varphi_1(e_{1,\infty}) = \infty$ and $\varphi_1(e_{1,0}) = 0$. Then the set of k_1-rational points $X_1(k_1) \setminus \{e_{1,\infty}\}$ is equipped with a structure of $\mathbb{F}_p$-module via the bijection φ_1. Note that since any k_1'-isomorphism of $\mathbb{P}^1_{k_1'}$ fixing ∞ and 0 is a scalar multiplication, the $\mathbb{F}_p$-module structure of $X_1(k_1) \setminus \{e_{1,\infty}\}$ does not depend on the choices of k_1' and ϕ_1 but depends only on the choices of $e_{1,\infty}$ and $e_{1,0}$. Then we shall call $X_1(k_1) \setminus \{e_{1,\infty}\}$ is equipped with a structure of $\mathbb{F}_p$-module with respect to $e_{1,\infty}$ and $e_{1,0}$. On the other hand, by Lemma 3.2, ϕ induces a bijection $\rho_\phi : D_{X_1} \xrightarrow{\sim} D_{X_2}$. We write $e_{2,\infty}$ and $e_{2,0}$ for $\rho_\phi(e_{1,\infty})$ and $\rho_\phi(e_{1,0})$, respectively.

Lemma 4.1. *Consider the following linear condition :*

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} e_1 = e_{1,0}, \ \textit{with respect to } e_{1,\infty}, e_{1,0}$$

on $X_1^\bullet$, where $b_{e_1} \in \mathbb{F}_p$ for each $e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}$. Then we can detect, group-theoretically from $\Pi_{X_1^\bullet}$, whether the linear condition defined above holds or not. Moreover, if the linear condition defined above holds, then the linear condition

$$\sum_{e_2 \in D_{X_2} \setminus \{e_{2,\infty}, e_{2,0}\}} b_{e_1} \rho_\phi(e_1) = e_{1,0}, \ \textit{with respect to } e_{2,\infty}, e_{2,0}$$

on $X_2^\bullet$ also holds.

Proof. Let $\widetilde{e}_{2,\infty} \in D_{\widetilde{X}_2^{\text{sol}}}$ be a point over $e_{2,\infty}$. Then the set

$$\mathbb{F}_{\widetilde{e}_{2,\infty}} := (I_{\widetilde{e}_{2,\infty}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_2^{p'}) \coprod \{*_{e_{2,\infty}}\}$$

carries a structure of field, and Corollary 3.5 implies that the field structure can be reconstructed group-theoretically from $\Pi_{X_1^\bullet}$. Theorem 3.4 implies that there exists a point $\widetilde{e}_{1,\infty} \in D_{\widetilde{X}_1^{\rm sol}}$, and that $\phi(I_{\widetilde{e}_{1,\infty}}) = \widetilde{e}_{2,\infty}$. Moreover, Corollary 3.5 implies that the set

$$\mathbb{F}_{\widetilde{e}_{1,\infty}} := (I_{\widetilde{e}_{1,\infty}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_1^{p'}) \coprod \{*_{e_{1,\infty}}\}$$

carries a structure of field, that the field structure can be reconstructed group-theoretically from $\Pi_{X_1^\bullet}$, and that ϕ induces a field isomorphism

$$\theta_{\phi,\widetilde{e}_{1,\infty},\widetilde{e}_{2,\infty}} : \mathbb{F}_{\widetilde{e}_{1,\infty}} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_{2,\infty}}.$$

For each $e_1 \in D_{X_1}$, we choose $b'_{e_1} \in \mathbb{Z}_{\geq 0}$ such that

$$b'_{e_1} \equiv b_{e_1} \pmod p, \text{ and that } \sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b'_{e_1} \geq 2.$$

Let $r \geq 1$ such that

$$p^r - 2 \geq \sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b'_{e_1}.$$

For each $e_1 \in D_{X_1}$ and each $\widetilde{e}_1 \in D_{\widetilde{X}_1^{\rm sol}}$ over e_1, write $I_{\widetilde{e}_1,\rm ab}$ for the image of the composition of the natural morphisms $I_{\widetilde{e}_1} \hookrightarrow \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_1^\bullet}^{\rm ab}$. Moreover, since the image of $I_{\widetilde{e}_1,\rm ab}$ does not depend on the choice of $\widetilde{e}_1$, we may write I_{e_1} for $I_{\widetilde{e}_1,\rm ab}$. We define

$$I_{e_{1,\infty}} \twoheadrightarrow \mathbb{Z}/(p^r-1)\mathbb{Z} \text{ that maps } 1 \mapsto 1,$$

$$I_{e_{1,0}} \twoheadrightarrow \mathbb{Z}/(p^r-1)\mathbb{Z} \text{ that maps } 1 \mapsto (\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b'_{e_1}) - 1,$$

and

$$I_{e_1} \twoheadrightarrow \mathbb{Z}/(p^r-1)\mathbb{Z} \text{ that maps } 1 \mapsto -b'_{e_1} \text{ for each } e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}.$$

Then the surjections of inertia groups defined above induces a sujection

$$\delta_1 : \Pi_{X_1^\bullet} \twoheadrightarrow \mathbb{Z}/(p^r-1)\mathbb{Z}.$$

Write H_{δ_1} for the kernel of δ_1, M_{δ_1} for $H_{\delta_1}^{\rm ab} \otimes \mathbb{F}_p$, and $X_{H_{\delta_1}}^\bullet := (X_{H_{\delta_1}}, D_{X_{H_{\delta_1}}})$ for the smooth pointed stable curve over k_1 corresponding to H_{δ_1}. Note that M_{δ_1} admits a natural action of $\Pi_{X_1^\bullet}$ via conjugation. Then M_{δ_1} admits a natural action of $I_{\widetilde{e}_{1,\infty}}$ via a character

$$\chi_{I_{\widetilde{e}_{1,\infty}},r} : I_{\widetilde{e}_{1,\infty}} \hookrightarrow \Pi_{X_1^\bullet} \twoheadrightarrow \mathbb{Z}/(p^r-1)\mathbb{Z} = I_{\widetilde{e}_{1,\infty}}/(p^r-1) \hookrightarrow \mathbb{F}_{\widetilde{e}_{1,\infty}}^\times,$$

where the middle morphism is δ_1. We set

$$M_{\delta_1}[\chi_{I_{\widetilde{e}_{1,\infty}},r}] := \{a \in M_{\delta_1} \otimes_{\mathbb{F}_p} \mathbb{F}_{\widetilde{e}_{1,\infty}} \mid \sigma(a) = \chi_{I_{\widetilde{e}_{1,\infty}},r}(\sigma)a \text{ for all } \sigma \in I_{\widetilde{e}_{1,\infty}}\}.$$

Then the proof of [T2, Lemma 3.3] implies that the linear condition

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} e_1 = e_{1,0}, \text{ with respect to } e_{1,\infty}, e_{1,0}$$

on $X_1^\bullet$ holds if and only if $M_{\delta_1}[\chi_{I_{\tilde{e}_{1,\infty}},r}] = 0$. This completes the proof of the first part of the lemma.

Next, let us prove the "moreover" part. Since $(p, p^r - 1) = 1$, the surjection ϕ induces a surjection

$$\delta_2 : \Pi_{X_2^\bullet} \to \mathbb{Z}/(p^r - 1)\mathbb{Z}$$

which fits into the following commutative diagram:

$$\begin{tikzcd}
H_{\delta_1} \arrow[r, "\phi|_{H_{\delta_1}}"] \arrow[d] & H_{\delta_2} \arrow[d] \\
\Pi_{X_1^\bullet} \arrow[r, "\phi"] \arrow[d, "\delta_1"'] & \Pi_{X_2^\bullet} \arrow[d, "\delta_2"'] \\
\mathbb{Z}/(p^r - 1)\mathbb{Z} \arrow[r, equal] & \mathbb{Z}/(p^r - 1)\mathbb{Z},
\end{tikzcd}$$

where H_{δ_2} denotes the kernel of δ_2. Write M_{δ_2} for $H_{\delta_2}^{\mathrm{ab}} \otimes \mathbb{F}_p$ and $X_{H_{\delta_2}}^\bullet := (X_{H_{\delta_1}}, D_{X_{H_{\delta_2}}})$ for the smooth pointed stable curve over k_2 corresponding to H_{δ_2}. Similar arguments to the arguments given above imply that M_{δ_2} admits a natural action of $I_{\tilde{e}_{2,\infty}}$ via a character

$$\chi_{I_{\tilde{e}_{2,\infty}},r} : I_{\tilde{e}_{2,\infty}} \hookrightarrow \Pi_{X_2^\bullet} \twoheadrightarrow \mathbb{Z}/(p^r - 1)\mathbb{Z} = I_{\tilde{e}_{2,\infty}}/(p^r - 1) \hookrightarrow \mathbb{F}_{\tilde{e}_{2,\infty}}^\times,$$

where the middle morphism is δ_2. We set

$$M_{\delta_2}[\chi_{I_{\tilde{e}_{2,\infty}},r}] := \{a \in M_{\delta_2} \otimes_{\mathbb{F}_p} \mathbb{F}_{\tilde{e}_{2,\infty}} \mid \sigma(a) = \chi_{I_{\tilde{e}_{2,\infty}},r}(\sigma)a \text{ for all } \sigma \in I_{\tilde{e}_{2,\infty}}\}.$$

Then we obtain a surjection

$$M_{\delta_1}[\chi_{I_{\tilde{e}_{1,\infty}},r}] \twoheadrightarrow M_{\delta_2}[\chi_{I_{\tilde{e}_{2,\infty}},r}]$$

induced by $\phi|_{H_\delta}$ and $\theta_{\phi,\tilde{e}_{1,\infty},\tilde{e}_{2,\infty}}$.

Since the linear condition

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} e_1 = e_{1,0} \text{ with respect to } e_{1,\infty}, e_{1,0}$$

on $X_1^\bullet$ holds, we have $M_{\delta_1}[\chi_{I_{\tilde{e}_{1,\infty}},r}] = 0$. Thus, we obtain $M_{\delta_2}[\chi_{I_{\tilde{e}_{2,\infty}},r}] = 0$. Then, by applying the first part of the lemma to $X_2^\bullet$, we have the linear condition

$$\sum_{e_2 \in D_{X_2} \setminus \{e_{2,\infty}, e_{2,0}\}} b_{e_1} \rho_\phi(e_1) = e_{1,0}, \text{ with respect to } e_{2,\infty}, e_{2,0}$$

on $X_2^\bullet$ holds. $\square$

Remark 4.1.1. Note that, if $X_1 = \mathbb{P}_k^1$, then the linear condition is the follows:

$$\sum_{e_1 \in D_{X_1} \setminus \{\infty, 0\}} b_{e_1} e_1 = 0 \text{ with respect to } \{\infty, 0\}.$$

Next, we prove the main theorem of the present paper.

Theorem 4.2. *Let $X_1^\bullet := (X_1, D_{X_1})$ and $X_2^\bullet := (X_2, D_{X_2})$ be smooth pointed stable curves of type $(0,n)$ over algebraically closed fields k_1 and k_2 of characteristic $p>0$, respectively. Write $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$ for the maximal pro-solvable quotients of the tame fundamental groups of $X_1^\bullet$ and $X_2^\bullet$, respectively. Let k_1^{min} and k_2^{min} be the minimal algebraically closed subfields of k_1 and k_2 over which $X_1^\bullet$ and $X_2^\bullet$ are defined, respectively; thus, by considering the function fields of X_1 and X_2, we obtain smooth pointed stable curves*

$$X_1^{\bullet,\mathrm{min}} := (X_1^{\mathrm{min}}, D_{X_1^{\mathrm{min}}}) \text{ and } X_2^{\bullet,\mathrm{min}} := (X_2^{\mathrm{min}}, D_{X_2^{\mathrm{min}}})$$

such that $X_1 \setminus D_{X_1} \cong (X_1^{\mathrm{min}} \setminus D_{X_1^{\mathrm{min}}}) \times_{k_1^{\mathrm{min}}} k_1$ and $X_2 \setminus D_{X_2} \cong (X_2^{\mathrm{min}} \setminus D_{X_2^{\mathrm{min}}}) \times_{k_2^{\mathrm{min}}} k_2$ as k_1-schemes and k_2-schemes, respectively.

Then we can detect whether $X_1^\bullet$ can be defined over the algebraic closure $\overline{\mathbb{F}}_1$ of $\mathbb{F}_p$ in k_1 or not, group-theoretically from $\Pi_{X_1^\bullet}$. Moreover, suppose that $X_1^\bullet$ can be defined over the algebraic closure $\overline{\mathbb{F}}_1$ of $\mathbb{F}_p$ in k_1. Then the set of open homomorphisms

$$\mathrm{Hom}^{\mathrm{open}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) \neq \emptyset$$

if and only if

$$X_1^{\mathrm{min}} \setminus D_{X_1^{\mathrm{min}}} \cong X_2^{\mathrm{min}} \setminus D_{X_2^{\mathrm{min}}}$$

as schemes. In particular, if this is the case, $X_2^\bullet$ can be defined over the algebraic closure $\overline{\mathbb{F}}_2$ of $\mathbb{F}_p$ in k_2.

Proof. Note that, it is easy to see that the proof of [T2, Theorem 3.5] also holds for $\Pi_{X_1^\bullet}$, then first part of the theorem follows from [T4, Theorem 5.8] and the proof of [T2, Theorem 3.5]. Let us prove the "moreover" part of the theorem.

The "if" part of the theorem is trivial. We only prove the "only if" part of the theorem. Suppose that

$$\mathrm{Hom}^{\mathrm{open}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) \neq \emptyset.$$

Let $\phi \in \mathrm{Hom}^{\mathrm{open}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$. Since $X_1^\bullet$ and $X_2^\bullet$ are type $(0,n)$, we have that ϕ is a surjection.

Let $\widetilde{e}_{2,0} \in D_{\widetilde{X}_2^{\mathrm{sol}}}$ be a point over $e_{2,0}$. Then

$$\mathbb{F}_{\widetilde{e}_{2,0}} := (I_{\widetilde{e}_{2,0}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_2^{p'}) \coprod \{*_{e_{2,0}}\}$$

carries a structure of field, and Corollary 3.5 implies that the field structure can be reconstructed group-theoretically from $\Pi_{X_1^\bullet}$. Theorem 3.4 implies that there exists a point $\widetilde{e}_{1,0} \in D_{\widetilde{X}_1^{\mathrm{sol}}}$, and that $\phi(I_{\widetilde{e}_{1,0}}) = \widetilde{e}_{2,0}$. Moreover, Corollary 3.5 implies that

$$\mathbb{F}_{\widetilde{e}_{1,0}} := (I_{\widetilde{e}_{1,0}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_1^{p'}) \coprod \{*_{e_{1,0}}\}$$

carries a structure of field, that the field structure can be reconstructed group-theoretically from $\Pi_{X_1^\bullet}$, and that ϕ induces a field isomorphism

$$\theta_{\phi,\widetilde{e}_{1,0},\widetilde{e}_{2,0}} : \mathbb{F}_{\widetilde{e}_{1,0}} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_{2,0}}.$$

Proposition 1.6 (a) implies that n can be reconstructed gorup-theoretically from $\Pi_{X_1^\bullet}$ or $\Pi_{X_2^\bullet}$. If $n=3$, then the theorem is trivial, so we may assume that $n \geq 4$. Moreover, since $X_1^\bullet$ can be defined over $\overline{\mathbb{F}}_1$, without loss of generality, we may assume that $k_1 = \overline{\mathbb{F}}_1 = \mathbb{F}_{\widetilde{e}_{1,0}}$, that $X_1 = \mathbb{P}^1_{\overline{\mathbb{F}}_1}$, and that

$$D_{X_1} := \{e_{1,\infty} = \infty, e_{1,0} = 0, e_{1,1} = 1, e_{1,2}, \dots, e_{1,n-2}\}.$$

Here, $e_{1,2}, \dots, e_{1,n-2} \in \overline{\mathbb{F}}_1 \setminus \{e_{1,0}, e_{1,1}\}$ distinct from each other. By [T2, Lemma 3.4], there exists a natural number r prime to p such that $\mathbb{F}_p(\zeta_r)$ contains r^{th} roots of $e_{1,2}, \dots, e_{1,n-2}$, where ζ_r denotes a fixed primitive r^{th} root of unity in $\overline{\mathbb{F}}_1$. Let $s := [\mathbb{F}_p(\zeta_r), \mathbb{F}_p]$. For each $e_{1,i} \in \{e_{1,2}, \dots, e_{1,n-2}\}$, we fix an r^{th} root $e_{1,i}^{1/r}$ in $\overline{\mathbb{F}}_1$. Then we have

$$e_{1,i}^{1/r} = \sum_{j=0}^{s-1} b_{1,ij}\zeta_r^j \quad \text{for each } i \in \{2, \dots, n-2\},$$

where $b_{1,ij} \in \mathbb{F}_p$ for each $i = 2, \dots, n-2, j = 0, \dots, s-1$.

Let $X_1 \setminus \{\infty\} = \operatorname{Spec} \overline{\mathbb{F}}_1[x_1]$ and

$$X_{H_1}^\bullet := (X_{H_1}, D_{X_{H_1}}) \to X_1^\bullet$$

the tame covering over $\overline{\mathbb{F}}_1$ determined by the equation $y_1^r = x_1$. Write H_1 for the maximal pro-solvable quotient of the tame fundamental group of $X_{H_1}^\bullet$. The tame covering $X_{H_1}^\bullet \to X_1^\bullet$ is totally ramified over $e_{1,\infty}, e_{1,0}$ and is étale over $D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}$. Note that $X_{H_1} = \mathbb{P}^1_{\overline{\mathbb{F}}_1}$, and that the unique points of $D_{X_{H_1}}$ over $e_{1,\infty} \in D_{X_1}$ and $e_{1,0} \in D_{X_1}$ are $e_{H_1,\infty} := \infty$ and $e_{H_1,0} := 0$, respectively. We set

$$e_{H_1,i} := e_{1,i}^{1/r} \in D_{X_{H_1}}$$

for each $i \in \{2, \dots, n-2\}$ and

$$e_{H_1,1}^j := \zeta_r^j \in D_{X_{H_1}}$$

for each $j = 0, \dots, s-1$. Thus, we obtain a linear condition on $X_{H_1}^\bullet$ as follows:

$$e_{1,H_i} = \sum_{j=0}^{s-1} b_{1,ij} e_{H_1,1}^j \text{ with respect to } \{e_{H_1,\infty}, e_{H_1,0}\}$$

for each $i \in \{2, \dots, n-2\}$.

Since the order $\#(\Pi_{X_1^\bullet}/H_1)$ is prime to p, then we have the following commutative diagram:

$$\begin{array}{ccc}
H_1 & \xrightarrow{\phi|_{H_1}} & H_2 \\
\downarrow & & \downarrow \\
\Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet} \\
\downarrow & & \downarrow \\
\mathbb{Z}/(p^r-1)\mathbb{Z} & = & \mathbb{Z}/(p^r-1)\mathbb{Z}.
\end{array}$$

Write $X^{\bullet}_{H_2} := (X_{H_2}, D_{X_2})$ for the smooth pointed stable curve over k_2 corresponding to H_2. Lemma 3.2 implies that the following commutative diagram of the sets of marked points can be reconstructed group-theoretically from the commutative diagram of profinite groups above:

$$\begin{array}{ccc} D_{H_1} & \xrightarrow{\rho_{\phi|H_1}} & D_{H_2} \\ \downarrow & & \downarrow \\ D_{X_1} & \xrightarrow{\rho_\phi} & D_{X_2}. \end{array}$$

Write

$$e_{2,\infty}, e_{2,0}, e_{2,i}, i = 1, \dots, n-2, \text{ for } \rho_\phi(e_{1,\infty}), \rho_\phi(e_{1,0}), \rho_\phi(e_{1,i}), i = 1, \dots, n-2,$$

$$e_{H_2,\infty}, e_{H_2,0}, e_{H_2,i}, i = 2, \dots, n-2, \text{ for } \rho_\phi(e_{H_1,\infty}), \rho_\phi(e_{H_1,0}), \rho_\phi(e_{H_1,i}), i = 2, \dots, n-2,$$

and

$$e^j_{H_2,1}, j \in \{0, \dots, s-1\} \text{ for } \rho_\phi(e^j_{H_2,1}), j \in \{0, \dots, s-1\}.$$

We may assume that $X_2 = \mathbb{P}^1_{k_2}$, and that $e_{2,\infty} = \infty, e_{2,0} = 0, e_{2,1} = 1$. Note that

$$e^j_{H_2,1} = \xi^j_r,$$

where $\xi_r := \theta_{\phi,\widetilde{e}_{1,0},\widetilde{e}_{2,0}}(\zeta_r)$ is an r^{th} root of unity in $\mathbb{F}_{\widetilde{e}_{2,0}}$. Then

$$e_{1,2}, \dots, e_{2,n-2} \in k_2 \setminus \{e_{2,\infty}, e_{2,0}\}$$

distinct from each other.

Lemma 4.1 implies that the following linear condition

$$e_{2,H_i} = \sum_{j=0}^{s-1} b_{1,ij} e^j_{H_2,1} = e^{1/r}_{2,i} \text{ with respect to } \{e_{H_2,\infty}, e_{H_1,0}\}$$

on $X^{\bullet}_{H_2}$ holds for each $i \in \{2, \dots, n-2\}$. Thus, we obtain

$$e_{2,i} = (\sum_{j=0}^{s-1} b_{1,ij} e^j_{H_2,1})^r = (\sum_{j=0}^{s-1} b_{1,ij} \xi_r)^r$$

for each $i \in \{2, \dots, n-2\}$. This means that $X^{\min,\bullet}_2$ can be defined over $\overline{\mathbb{F}}_2$. Moreover, we obtain

$$X_1 \setminus D_{X_1} \cong X^{\min}_2 \setminus D_{X^{\min}_2}$$

as schemes. We complete the proof of the main theorem. □

Remark 4.2.1. Since $\Pi_{X^{\bullet}_1}$ and $\Pi_{X^{\bullet}_2}$ are topologically finitely generated, by Theorem 4.2, we obtain that

$$\mathrm{Hom}^{\mathrm{open}}(\Pi_{X^{\bullet}_1}, \Pi_{X^{\bullet}_2}) = \mathrm{Isom}(\Pi_{X^{\bullet}_1}, \Pi_{X^{\bullet}_2}),$$

where $\mathrm{Isom}(-,-)$ denotes the set of continuous isomorphisms of profinite groups.

Remark 4.2.2. Let $C_1^\bullet := (C_1, D_{C_1})$ and $C_2^\bullet := (C_2, D_{C_2})$ be pointed stable smooth curves of type (g_C, n_C) over algebraically closed fields l_1 and l_2 of characteristic $p > 0$. Suppose that $\overline{\mathbb{F}}_p := l_1 = l_2$ is an algebraic closure of $\mathbb{F}_p$. Write $\pi_1^{\mathrm{tame}}(C_1 \setminus D_{C_1})$ and $\pi_1^{\mathrm{tame}}(C_2 \setminus D_{C_2})$ for the tame fundamental groups of $C_1^\bullet$ and $C_2^\bullet$, respectively. Then the **weak Isom-version of the Grothendieck conjecture for curves of type** (g_C, n_C) **over** $\overline{\mathbb{F}}_p$ can be formulated as follows:

> The set of continuous isomorphisms
>
> $$\mathrm{Isom}(\pi_1^{\mathrm{tame}}(C_1 \setminus D_{C_1}), \pi_1^{\mathrm{tame}}(C_2 \setminus D_{C_2})) \neq \emptyset$$
>
> if and only if $C_1 \setminus D_{C_1} \cong C_2 \setminus D_{C_2}$ as schemes.

This conjecture was proved by Tamagawa when $g_C = 0$ (cf. [T4, Theorem 5.8]). Theorem 4.2 extends Tamagawa's result to the case of open continuous homomorphisms. Moreover, Theorem 4.2 can be regarded as a **weak Hom-version of the Grothendieck conjecture for curves of type** $(0, n_C)$ **over** $\overline{\mathbb{F}}_p$.

Remark 4.2.3. Let $C^\bullet := (C, D_C)$ be a pointed stable smooth curve of type (g_C, n_C) over algebraically closed fields l of characteristic $p > 0$. We denote by $\mathrm{td}(l)$ the transcendence degree of l over $\overline{\mathbb{F}}_p \subseteq l$. We define the essential dimension $\mathrm{ed}(C^\bullet)$ of $C^\bullet$ to be the minimum of $\mathrm{td}(l')$, where l' runs over the algebraically closed subfields of l over which there exists a smooth curve $C^{',\bullet}$ such that $C^\bullet$ is l-isomorphic to $C^{',\bullet} \times_{l'} l$. Tamagawa posed a conjecture concerning the essential dimensions as follows (cf. [T3, Conjecture 5.3 (ii)]):

> Let $C^{'',\bullet} := (C'', D_{C''})$ be a smooth pointed stable curve over an algebraically closed field l'' of characteristic $p > 0$. Suppose that $\pi_1^{\mathrm{tame}}(C \setminus D_C)$ is isomorphic to $\pi_1^{\mathrm{tame}}(C'' \setminus D_{C''})$ as profinite groups. Then we have
>
> $$\mathrm{ed}(C^\bullet) = \mathrm{ed}(C^{'',\bullet}).$$

Tamagawa proved the **essential dimension conjecture** above in the case where $\mathrm{ed}(C^\bullet) = 1$ and $g_C = 0$ (cf. [T4, Theorem 5.8]). Moreover, the author extended Tamagawa's result to the case of (possibly singular) pointed stable curves (cf. [Y, Theorem 6.6 (i-b)]).

On the other hand, let $C_1^\bullet := (C_1, D_{C_1})$ and $C_2^\bullet := (C_2, D_{C_2})$ be pointed stable smooth curves of type (g_C, n_C) over algebraically closed fields l_1 and l_2 of characteristic $p > 0$. Then Theorem 4.2 implies that, if $g_C = 0$ and there exists a continuous surjective morphism $\pi_1^{\mathrm{tame}}(C_1 \setminus D_{C_1}) \twoheadrightarrow \pi_1^{\mathrm{tame}}(C_2 \setminus D_{C_2})$, we have

$$\mathrm{ed}(C_1^\bullet) = \mathrm{ed}(C_2^\bullet).$$

Moreover, we posed the following question:

> **Question.** Suppose that there exists a continuous surjective morphism $\pi_1^{\mathrm{tame}}(C_1 \setminus D_{C_1}) \twoheadrightarrow \pi_1^{\mathrm{tame}}(C_2 \setminus D_{C_2})$. Does
>
> $$\mathrm{ed}(C_1^\bullet) \geq \mathrm{ed}(C_2^\bullet)$$
>
> hold?

Remark 4.2.4. Before Tamagawa proved [T4, Theorem 5.8], he also obtained an étale fundamental group version of [T4, Theorem 5.8] in a completely different way (by using wildly ramified coverings) (cf. [T2, Theorem 3.5]). Note that, for any nonsingular pointed stable curve over an algebraically closed field of positive characteristic, since the tame fundamental group can be reconstructed group-theoretically from the étale fundamental group (cf. [T2, Corollary 1.10]), the tame fundamental group version is stronger than the étale fundamental group version. Recently, by using Tamagawa's idea, A. Sarashina proved a similar result of [T2, Theorem 3.5] for curves of type $(1,1)$ (cf. [Sar], [T6, Theorem 6 (i)]). Moreover, by applying the theory of Tamagawa developed in [T4], Sarashina's result also holds in the case of tame fundamental groups.

Moreover, similar arguments to the arguments developed in the present paper and [Sar], one may prove a similar result of Theorem 4.2 for curves of type $(1,1)$ as follows:

> *Let $C_1^\bullet := (C_1, D_{C_1})$ and $C_2^\bullet := (C_2, D_{C_2})$ be smooth pointed stable curves of type $(1,1)$ over algebraically closed fields l_1 and l_2 of characteristic $p > 0$, respectively. Write $\Pi_{C_1^\bullet}$ and $\Pi_{C_2^\bullet}$ for the maximal pro-solvable quotients of the tame fundamental groups of $C_1^\bullet$ and $C_2^\bullet$, respectively. Let $l_1^{\min}$ and $l_2^{\min}$ be the minimal algebraically closed subfields of l_1 and l_2 over which $C_1^\bullet$ and $C_2^\bullet$ are defined, respectively; thus, by considering the function fields of C_1 and C_2, we obtain smooth pointed stable curves*
>
> $$C_1^{\bullet,\min} := (C_1^{\min}, D_{C_1^{\min}}) \text{ and } C_2^{\bullet,\min} := (C_2^{\min}, D_{C_2^{\min}})$$
>
> *such that $C_1 \setminus D_{C_1} \cong (C_1^{\min} \setminus D_{C_1^{\min}}) \times_{l_1^{\min}} l_1$ and $C_2 \setminus D_{C_2} \cong (C_2^{\min} \setminus D_{C_2^{\min}}) \times_{l_2^{\min}} l_2$ as l_1-schemes and l_2-schemes, respectively.*
>
> *Then we can detect whether $C_1^\bullet$ can be defined over the algebraic closure of $\mathbb{F}_p$ or not, group-theoretically from $\Pi_{C_1^\bullet}$. Moreover, suppose that $C_1^\bullet$ can be defined over the algebraic closure of $\mathbb{F}_p$ in l_1. Then the set of open continuous homomorphisms*
>
> $$\mathrm{Hom}^{\mathrm{open}}(\Pi_{C_1^\bullet}, \Pi_{C_2^\bullet}) \neq \emptyset$$
>
> *if and only if*
>
> $$C_1^{\min} \setminus D_{C_1^{\min}} \cong C_2^{\min} \setminus D_{C_2^{\min}}$$
>
> *as schemes. In particular, if this is the case, $C_2^\bullet$ can be defined over the algebraic closure of $\mathbb{F}_p$ in l_2.*

5 An application to moduli spaces of curves

Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, and let $\mathcal{M}_{g,n}$ be the moduli stack over $\overline{\mathbb{F}}_p$ parameterizing smooth pointed stable curves of type (g,n) and $M_{g,n}$ the coarse moduli space of $\mathcal{M}_{g,n}$. Let $X^\bullet$ be a pointed stable smooth curve of type (g,n) over an algebraically closed field $k \supseteq \overline{\mathbb{F}}_p$. Then there exists a unique composition of morphisms

$$c_{X^\bullet} : \mathrm{Spec}\, k \to \mathcal{M}_{g,n} \to M_{g,n}$$

determined by $X^\bullet \to \operatorname{Spec} k$ and the natural morphism $\mathcal{M}_{g,n} \to M_{g,n}$. We write

$$q_{X^\bullet} \in M_{g,n}$$

for the image of $c_{X^\bullet}$. Moreover, for any $q \in M_{g,n}$, let k_q be an algebraically closed field which contains the residue field $k(q)$ of q. Then the natural morphisms $\operatorname{Spec} k_q \to \operatorname{Spec} k(q) \to M_{g,n}$ determine a smooth pointed stable curve $X_q^\bullet := (X_q, D_{X_q})$ of type (g,n) over k_q. We write

$$\pi_1^{\mathrm{tame}}(q)$$

for the tame fundamental group $\pi_1^{\mathrm{tame}}(X_q \setminus D_{X_q})$ of $X_q^\bullet$ and

$$\pi_A^{\mathrm{tame}}(q)$$

for the set of finite quotients of $\pi_1^{\mathrm{tame}}(q)$. Note that $\pi_1^{\mathrm{tame}}(q)$ and $\pi_A^{\mathrm{tame}}(q)$ do not depend on the choice of k_q but depend only on q. Moreover, for two points $q_1, q_2 \in M_{g,n}$, we have $\pi_1^{\mathrm{tame}}(q_1) \cong \pi_1^{\mathrm{tame}}(q_2)$ as profinite groups if and only if $\pi_A^{\mathrm{tame}}(q_1) = \pi_A^{\mathrm{tame}}(q_2)$ as sets.

K. Stevenson proved the following result (cf. [Ste, Proposition 4.2]).

Proposition 5.1. *Suppose that $n = 0$. Let q be a closed point of $M_g := M_{g,0}$ and $G \in \pi_A^{\mathrm{tame}}(q)$ a finite group. Then there exists an open neighborhood $q \in U \subseteq M_g$ such that, for each $q' \in U$, $G \in \pi_A^{\mathrm{tame}}(q')$.*

Similar arguments to the arguments given in the proof of [Ste, Proposition 4.2] imply Proposition 5.1 also holds for $n \geq 0$. Then we obtain the following result.

Proposition 5.2. *Let q be a closed point of $M_{g,n}$ and $G \in \pi_A^{\mathrm{tame}}(q)$ a finite group. Then there exists an open neighborhood $q \in U \subseteq M_g$ such that, for each $q' \in U$, $G \in \pi_A^{\mathrm{tame}}(q')$.*

Remark 5.2.1. Proposition 5.2 means that, for any finite group H, either H is not a quotient of the tame fundamental group of any smooth pointed stable curves of type (g,n) over algebraically closed fields fields of characteristic $p > 0$, or is a quotient of the tame fundamental group of almost each such curve.

Suppose that H is any finite quotient of the tame fundamental group of a smooth pointed stable curves of type (g,n) over algebraically closed fields fields of characteristic $p > 0$. We define

$$U_H \subseteq M_{g,n}$$

for the maximal open subset such that, for each $q' \in U_H$, $H \in \pi_A^{\mathrm{tame}}(q')$. Stevenson posed a question as follows (cf. [Question 4.3] for $n = 0$ case):

> is the intersection of all the U_H's contains any $\overline{\mathbb{F}}_p$-rational points?

Let q_{gen} be the generic point of $M_{g,n}$ and q'' any closed point of $M_{g,n}$. Then by [T5, Theorem 0.3], $\pi_A^{\mathrm{tame}}(q_{\mathrm{gen}})$ is not equal to $\pi_A^{\mathrm{tame}}(q'')$. This means that the answer of Stevenson's question above is "No". Moreover, we may refine Stevenson's question above as follows:

let q be any closed point of $M_{g,n}$; what is the set

$$(\bigcap_{H\in\pi_A^{\mathrm{tame}}(q)} U_H)^{\mathrm{cl}},$$

where $(-)^{\mathrm{cl}}$ denotes the set of closed points of $(-)$?

For this question, we have the following result.

Theorem 5.3. *Let q be any closed point of $M_{0,n}$ and $X_q^\bullet$ the smooth pointed stable curve over $\overline{\mathbb{F}}_p = k(q)$ determined by the natural morphism $\operatorname{Spec} k(q) \to M_{0,n}$. For each $m \in \mathbb{Z}$, write $q^{(m)}$ for $q_{(X^\bullet)^{(m)}}$, where $(X^\bullet)^{(m)}$ denotes the m^{th} Frobenius twist of $X^\bullet$. Then we have*

$$(\bigcap_{H\in\pi_A^{\mathrm{tame}}(q)} U_H)^{\mathrm{cl}} = \{q^{(m)}\}_{m\in\mathbb{Z}}.$$

Note that since $X^\bullet$ can be defined over a finite field, $\{q^{(m)}\}_{m\in\mathbb{Z}}$ is a finite set.

Proof. Since " $\supseteq$ " is trivial, we only need to prove that " $\subseteq$ " holds. Let q' be any closed point of $\bigcap_{H\in\pi_A^{\mathrm{tame}}(q)} U_H$. Then we have that, for each $H \in \pi_A^{\mathrm{tame}}(q)$,

$$\mathrm{Hom}^{\mathrm{surj}}(\pi_1^{\mathrm{tame}}(q'), H) \neq \emptyset,$$

where $\mathrm{Hom}^{\mathrm{surj}}(-,-)$ denotes the set of surjections of $\mathrm{Hom}(-,-)$. Since $\pi_1^{\mathrm{tame}}(q')$ is topologically finitely generated, the set $\mathrm{Hom}^{\mathrm{surj}}(\pi_1^{\mathrm{tame}}(q'), H)$ is finite. Then the set of open continuous homomorphisms

$$\varprojlim_{H\in\pi_A^{\mathrm{tame}}(q)} \mathrm{Hom}^{\mathrm{surj}}(\pi_1^{\mathrm{tame}}(q'), H) = \mathrm{Hom}^{\mathrm{open}}(\pi_1^{\mathrm{tame}}(q'), \pi_1^{\mathrm{tame}}(q)) \neq \emptyset.$$

Thus, Theorem 4.2 implies that $q' \in \{q^{(m)}\}_{m\in\mathbb{Z}}$. This completes the proof of the theorem. $\square$

The author is very interested in the following question.

Question 5.4. *Does*

$$|\bigcap_{H\in\pi_A^{\mathrm{tame}}(q)} U_H| = \bigcup_{m\in\mathbb{Z}} |\operatorname{Spec} \mathcal{O}_{M_{0,n},q^{(m)}}|$$

holds? Here, $|(-)|$ denotes the underlying topological space of $(-)$.

6 Formulation of a weak Hom-version of the Grothendieck conjecture for curves of type (g, n)

We maintain the notations introduced in Section 5. Let $X_1^\bullet$ and $X_2^\bullet$ be smooth pointed stable curve of type (g,n) over algebraically closed fields k_1 and k_2, respectively. Write q_1 and q_2 for $q_{X_1^\bullet}$ and $q_{X_2^\bullet}$, V_1 and V_2 for the topological closure of q_1 and q_2 in $M_{g,n}$, respectively.

Definition 6.1. We shall say that V_2 **is essentially contained in** V_1 if, for each $q \in V_2$, there exists $m \in \mathbb{Z}$ such that $q^{(m)} \in V_1$. We denote by

$$V_2 \subseteq_{\text{ec}} V_1$$

if V_2 is essentially contained in V_1.

Then we formulate a certain **weak Hom-version of the Grothendieck conjecture for curves of type** (g, n) **over algebraically closed fields of charcteristic** $p > 0$ as follows.

Conjecture 6.2. (weak Hom-version for curves of type (g, n))
The set of open continuous homomorphisms

$$\text{Hom}^{\text{open}}(\pi_1^{\text{tame}}(q_1), \pi_1^{\text{tame}}(q_2)) \neq \emptyset$$

if and only if

$$V_2 \subseteq_{\text{ec}} V_1.$$

Moreover,

$$\text{Hom}^{\text{open}}(\pi_1^{\text{tame}}(q_1), \pi_1^{\text{tame}}(q_2)) = \text{Isom}(\pi_1^{\text{tame}}(q_1), \pi_1^{\text{tame}}(q_2)) \neq \emptyset$$

if and only if

$$V_2 \subseteq_{\text{ec}} V_1 \textit{ and } V_1 \subseteq_{\text{ec}} V_2.$$

Remark 6.2.1. Theorem 4.2 implies that Conjecture 6.2 holds in the case where q_1 is a closed point of $M_{0,n}$. Moreover, we have

$$\text{weak Hom-version for curves of type } (g, n) \Rightarrow \text{weak Isom-version.}$$

Remark 6.2.2. We note that $\dim(V_1) = \text{ed}(X_1^\bullet)$ and $\dim(V_2) = \text{ed}(X_2^\bullet)$. Thus, we obtain that Conjecture 6.2 implies Tamagawa's essential dimension conjecture, Remark 4.2.3 Question, and Question 5.4.

References

[G1] A. Grothendieck, La longue marche à travers la théorie de Galois, manuscript (1981).

[G2] A. Grothendieck, Letter to G. Faltings (German), manuscript (1984).

[M1] S. Mochizuki, The local pro-p anabelian geometry of curves. *Invent. Math.* **138** (1999), 319-423.

[M2] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves. *J. Math. Kyoto Univ.* **47** (2007), 451-539.

[N1] H. Nakamura, Rigidity of the arithmetic fundamental group of a punctured projective line. *J. Reine Angew. Math.* **405** (1990), 117-130.

[N2] H. Nakamura, Galois rigidity of the étale fundamental groups of punctured projective lines. *J. Reine Angew. Math.* **411** (1990), 205-216.

[PS] F. Pop, M. Saïdi, On the specialization homomorphism of fundamental groups of curves in positive characteristic. *Galois groups and fundamental groups,* 107-118, *Math. Sci. Res. Inst. Publ.,* **41**, *Cambridge Univ. Press, Cambridge,* 2003.

[R] M. Raynaud, Sur le groupe fondamental d'une courbe complète en caractéristique $p > 0$. *Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999)*, 335-351, *Proc. Sympos. Pure Math.,* **70**, *Amer. Math. Soc., Providence, RI,* 2002.

[Sar] A. Sarashina, Reconstruction of one-punctured elliptic curves in positive characteristic by their geometric fundamental groups, RIMS Preprint 1876.

[Ste] K. Stevenson, Galois groups of unramified covers of projective curves in characteristic p. *Journal of Algebra* **182** (1996), 770-804.

[Sti1] J. Stix, Affine anabelian curves in positive characteristic. *Compositio Math.* **134** (2002), 75-85.

[Sti2] J. Stix, Projective anabelian curves in positive characteristic and descent theory for log-étale covers. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2002. *Bonner Mathematische Schriften*, **354**. Universität Bonn, Mathematisches Institut, Bonn, 2002.

[T1] A. Tamagawa, The Grothendieck conjecture for affine curves. *Compositio Math.* **109** (1997), 135-194.

[T2] A. Tamagawa, On the fundamental groups of curves over algebraically closed fields of characteristic > 0. *Internat. Math. Res. Notices* (1999), 853-873.

[T3] A. Tamagawa, Fundamental groups and geometry of curves in positive characteristic. *Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999)*, 297-333, *Proc. Sympos. Pure Math.*, **70**, *Amer. Math. Soc., Providence, RI,* 2002.

[T4] A. Tamagawa, On the tame fundamental groups of curves over algebraically closed fields of characteristic > 0. *Galois groups and fundamental groups,* 47-105, *Math. Sci. Res. Inst. Publ.,* **41**, *Cambridge Univ. Press, Cambridge,* 2003.

[T5] A. Tamagawa, Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups. *J. Algebraic Geom.* **13** (2004), 675-724.

[T6] A. Tamagawa, Fundamental groups and moduli of curves in positive characteristic: On the occasion of Professor Oort's "sanju". https://www.math.ru.nl/~bmoonen/FO80/Tamagawa.pdf

[Y] Y. Yang, On the admissible fundamental groups of curves over algebraically closed fields of characteristic $p > 0$, RIMS Preprint 1869. Submitted. See also http://www.kurims.kyoto-u.ac.jp/~yuyang/