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On a Hom-version of the Grothendieck Conjecture for Almost Open Immersions of Curves

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ON A HOM-VERSION OF THE GROTHENDIECK CONJECTURE FOR ALMOST OPEN IMMERSIONS OF CURVES

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Abstract

Let p be a prime number and k either a finite field of characteristic p or a generalized sub-p-adic field. Let X_1 and X_2 be hyperbolic curves over k. We shall call a separable k-morphism $f : X_1 \to X_2$ almost open immersion if f is a composition of an open immersion and a finite étale morphism. In the present paper, we give some group-theoretic characterizations for the set of almost open immersions between X_1 and X_2 via their arithmetic fundamental groups. This result can be regarded as a certain Hom-version of the Grothendieck conjecture for almost open immersions of curves over k.

Keywords: hyperbolic curve, tame fundamental group, Grothendieck conjecture, anabelian geometry.

Mathematics Subject Classification: Primary 14G32; Secondary 11G20, 14H30.

Introduction

In the present paper, we study the anabelian geometry of curves. Let k be a field, $\overline{k}$ an algebraic closure of k, and G_k the absolute Galois group of k. Let $X_i, i \in \{1, 2\}$, be hyperbolic curves over k and $\overline{X}_i$ the curve $X_i \times_k \overline{k}$ over $\overline{k}$. Then, for suitable choices of base point, we have the following exact sequence of tame fundamental groups:

$$1 \to \pi_1^{\mathrm{t}}(\overline{X}_i) \to \pi_1^{\mathrm{t}}(X_i) \overset{\mathrm{pr}_{X_i}}{\to} G_k \to 1.$$

Note that if $\mathrm{char}(k) = 0$, then the tame fundamental groups of X_i coincides with the étale fundamental groups of X_i.

Let $\mathfrak{Primes}$ be the set of prime numbers and Σ a non-empty subset of $\mathfrak{Primes}$. We denote by Δ_{X_i} either the maximal pro-Σ quotient of $\pi_1^{\mathrm{t}}(\overline{X}_i)$ or the maximal pro-solvable quotient of $\pi_1^{\mathrm{t}}(\overline{X}_i)$. Then the kernel of the natural surjection $\pi_1^{\mathrm{t}}(\overline{X}_i) \twoheadrightarrow \Delta_{X_i}$ is a closed normal subgroup of $\pi_1^{\mathrm{t}}(X_i)$. Moreover, we denote by

$$\Pi_{X_i} := \pi_1^{\mathrm{t}}(X_i)/(\mathrm{Ker}(\pi_1^{\mathrm{t}}(\overline{X}_i) \twoheadrightarrow \Delta_{X_i})).$$

Thus, we obtain the following exact sequence of fundamental groups:

$$1 \to \Delta_{X_i} \to \Pi_{X_i} \overset{\mathrm{pr}^{\Sigma}_{X_i}}{\to} G_k \to 1.$$

We define

$$\mathrm{Isom}_{\text{pro-gps}}(-,-) \text{ and } \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(-,-)$$

to be the set of continuous isomorphisms and the set of open continuous homomorphisms of profinite groups between the two profinite groups in parentheses, respectively, and define

$$\mathrm{Isom}_{G_k}(\Pi_{X_1},\Pi_{X_2}) := \{\Phi \in \mathrm{Isom}_{\text{pro-gps}}(\Pi_{X_1},\Pi_{X_2}) \mid \mathrm{pr}^{\Sigma}_{X_1} = \mathrm{pr}^{\Sigma}_{X_2} \circ \Phi\},$$

$$\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1},\Pi_{X_2}) := \{\Phi \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_{X_1},\Pi_{X_2}) \mid \mathrm{pr}^{\Sigma}_{X_1} = \mathrm{pr}^{\Sigma}_{X_2} \circ \Phi\}.$$

Thus, by composing with inner automorphisms, we obtain a natural action of Δ_{X_2} on $\mathrm{Isom}_{G_k}(\Pi_{X_1},\Pi_{X_2})$ and a natural action of Δ_{X_2} on $\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1},\Pi_{X_2})$.

Consider the category $\mathcal{C}_k$ of smooth k-curves and dominant k-morphisms. If $\mathrm{char}(k) = p > 0$, we denote by $\mathcal{FC}_k$ the localization of $\mathcal{C}_k$ at geometric k-Frobenius maps between curves (cf. [S1, Section 3]). The ultimate aim of Grothendieck's anabelian conjectures (or, simply, the Grothendieck conjectures, for short) for curves over suitable k is to reconstruct the curves from their fundamental groups. Moreover precisely, these conjectures can be formulated as follows:

Conjecture 0.1. (Isom-version): *The natural maps*

$$\text{Isom-}\pi_1^{\Sigma} : \mathrm{Isom}_{\mathcal{C}_k}(X_1,X_2) \to \mathrm{Isom}_{G_k}(\Pi_{X_1},\Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = 0$ *and*

$$\text{Isom-}\pi_1^{\mathrm{t},\Sigma} : \mathrm{Isom}_{\mathcal{FC}_k}(X_1,X_2) \to \mathrm{Isom}_{G_k}(\Pi_{X_1},\Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = p > 0$ *are bijections.*

Conjecture 0.2. (Hom-version): *The natural maps*

$$\pi_1^{\Sigma} : \mathrm{Hom}_{\mathcal{C}_k}(X_1,X_2) \to \mathrm{Hom}_{G_k}(\Pi_{X_1},\Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = 0$ *and*

$$\pi_1^{\mathrm{t},\Sigma} : \mathrm{Hom}_{\mathcal{FC}_k}(X_1,X_2) \to \mathrm{Hom}_{G_k}(\Pi_{X_1},\Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = p > 0$ *are bijections.*

Moreover, we have the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{Isom}_{\mathcal{C}_k}(X_1,X_2) & \xrightarrow{\text{Isom-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1},\Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}_k}(X_1,X_2) & \xrightarrow{\pi_1^{\Sigma}} & \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1},\Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}), \end{array}$$

if $\mathrm{char}(k) = 0$ and

$$\begin{array}{ccc} \mathrm{Isom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\mathrm{Isom}\text{-}\pi_1^{\mathrm{t},\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\mathrm{t},\Sigma}} & \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \end{array}$$

if $\mathrm{char}(k) = p > 0$. Since all the vertical arrows appeared in the commutative diagrams above are injections, we have that

$$\text{Hom-version} \Rightarrow \text{Isom-version}.$$

Suppose that $\mathrm{char}(k) = 0$. If $X_i, i \in \{1, 2\}$, is affine, $\Sigma = \mathfrak{Primes}$, and k is a number field, then Conjecture 0.1 was proved by H. Nakamura (cf. [N1], [N2]) when the genus of $X_i, i \in \{1, 2\}$, is 0, and was proved by A. Tamagawa (cf. [T1]) in general. Later, S. Mochizuki (cf. [M1], [M2]) generalized their results to the case where k is a generalized sub-p-adic field (i.e., a field which can be embedded as a subfield of a finitely generated extension of the quotient field of the ring of Witt vectors with coefficients in an algebraic closed field of $\mathbb{F}_p$), Σ is a set which contains p, and $X_i, i \in \{1, 2\}$, is an arbitrary hyperbolic curve over k.

Suppose that $\mathrm{char}(k) = p > 0$. If $\Sigma = \mathfrak{Primes}$ and k is a finite field, then Conjecture 0.1 was proved by Tamagawa (cf. [T1]) when $X_i, i \in \{1, 2\}$, is affine, and was proved by Mochizuki (cf. [M3]) when $X_i, i \in \{1, 2\}$, is projective. Recently, M. Saïdi and Tamagawa (cf. [ST1], [ST3]) generalized their results to the case where $p \notin \Sigma$ is a complement of a finite subset of $\mathfrak{Primes}$. On the other hand, J. Stix (cf. [S1], [S2]) proved Conjecture 0.1 when $\Sigma = \mathfrak{Primes}$ and k is a field that is finitely generated over $\mathbb{F}_p$.

For Conjecture 0.2, if $\mathrm{char}(k) = 0$, by applying p-adic Hodge theory, Mochizuki (cf. [M1]) proved Conjecture 0.2 when k is a sub-p-adic fields (i.e., a field which can be embedded as a subfield of a finitely generated extension of $\mathbb{Q}_p$), and Σ is a set which contains p. If $\mathrm{char}(k) = p > 0$, a birational version of Conjecture 0.2 for function fields of curves over finite fields was proved by Saïdi and Tamagawa (cf. [ST2]), but at the time of writing, nothing is known about Conjecture 0.2. It is not clear how to adapt Mochizuki's method to the case of positive characteristic. On the other hand, although the Isom-version of the Grothendieck conjecture for curves over sub-p-adic fields obtained by Mochizuki in [M1] can be generalized to the case of generalized sub-p-adic fields (cf. [M2]), since the method used in [M1] can not work well in the case of generalized sub-p-adic fields, we do not know whether or not Conjecture 0.2 holds if k is a generalized sub-p-adic field. Thus, it is worth finding a new approach to Conjecture 0.2 without using p-adic Hodge theory.

In the present, we investigate Conjecture 0.2 for a certain kind of morphisms of curves which are called almost open immersions. For simplicity, in the remainder of this introduction, we assume that k is either a finite field of characteristic p or a generalized sub-p-adic field, and that Σ is either $\mathfrak{Primes} \setminus \{p\}$ when $\mathrm{char}(k) = p > 0$ or $\mathfrak{Primes}$ when $\mathrm{char}(k) = 0$.

Let $f \in \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$ be a separable k-morphism. We shall call $f : X_1 \to X_2$ almost open immersion if f is a composition of an open immersion and a finite étale morphism.

Suppose that $\mathrm{char}(k) = p > 0$. Let $\phi \in \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$. We shall call $\phi : X_1 \to X_2$ an almost open immersion if ϕ can be represented by the following k-morphisms

$$X_1 \cong_k Y(m) \leftarrow Y \to X_2,$$

where $Y(m)$ denotes the m^{th}-Frobenius twist of Y, and $\cong_k$ is a k-isomorphism. Then we define

$$\mathrm{Hom}^{\text{al-op-im}}_{\mathcal{C}_k}(X_1, X_2) \subseteq \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$$

if $\mathrm{char}(k) = 0$ and

$$\mathrm{Hom}^{\text{al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) \subseteq \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$$

if $\mathrm{char}(k) = p > 0$ to be the sets of the almost open immersions between X_1 and X_2. Moreover, we introduce a purely group-theoretic condition (Σ-gnc) for the open continuous homomorphisms of Π_{X_1} and Π_{X_2} (cf. Proposition 1.2). We denote by

$$\mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$$

for the elements of $\mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$ satisfying the condition (Σ-gnc). Then the natural maps π_1^{Σ} and $\pi_1^{\mathrm{t},\Sigma}$ induce the following natural maps:

$$\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}^{\text{al-op-im}}_{\mathcal{C}_k}(X_1, X_2) \to \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = 0$ and

$$\pi_1^{\mathrm{t},\Sigma\text{-gnc}} : \mathrm{Hom}^{\text{al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) \to \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = p > 0$ which fit into the following commutative diagrams:

$$\begin{array}{ccc}
\mathrm{Isom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\text{Isom-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}^{\text{al-op-im}}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\Sigma\text{-gnc}}} & \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\Sigma}} & \mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}),
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Isom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\text{Isom-}\pi_1^{\mathrm{t},\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}^{\text{al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\mathrm{t},\Sigma\text{-gnc}}} & \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\mathrm{t},\Sigma}} & \mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}),
\end{array}$$

respectively. Here, all the vertical arrows appeared in the commutative diagrams above are injections. Now, our main theorem of the present paper is as follows (cf. Theorem 3.2).

Theorem 0.3. (Hom-version for almost open immersions): *The natural maps*

$$\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}^{\text{al-op-im}}_{\mathcal{C}_k}(X_1, X_2) \xrightarrow{\sim} \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = 0$ *and*

$$\pi_1^{\mathrm{t},\Sigma\text{-gnc}} : \mathrm{Hom}^{\text{al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) \xrightarrow{\sim} \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = p > 0$ *are bijections.*

Remark 0.3.1. Note that we have

Hom-version ⇒ Hom-version for almost open immersions ⇒ Isom-version.

Our method of proving Theorem 0.3 is as follows. The main difficult is proving the surjectivity of $\pi_1^{\Sigma\text{-gnc}}$ and $\pi_1^{\mathrm{t},\Sigma\text{-gnc}}$. Let $\Phi \in \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$. To verify that the image of Φ in $\mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$ comes from a morphism of curves, it is easy to see that we may assume that Φ is a surjection. By using the condition (Σ-gnc), we prove that the kernel of the surjection $\Delta_{X_1} \twoheadrightarrow \Delta_{X_2}$ induced by Φ is generated by inertia subgroups of Δ_{X_1} associated to cups of X_1. Then we can reduce Theorem 0.3 to the Isom-version of the Grothendieck conjecture for curves over k which has been proven by Mochizuki when k is a generalized sub-p-adic field (cf. [M2]), and by Saïdi and Tamagawa when k is a finite field (cf. [ST3]).

Finally, let us come back to Conjecture 0.2. Note that, for any ϕ which is either an element of $\mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$ or an element of $\mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$, there exist an open sub-curve $U_i \subseteq X_i, i \in \{1, 2\}$ such that the restriction of ϕ on U_1 is an almost open immersion. Let Φ be an arbitrary element of $\mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$. If one can develop a suitable theory of anabelian cuspidalizations for surjections (i.e., group-theoretic reconstructions of the fundamental groups of open sub-curves of given curves from the fundamental group of given curves which has already been established by Mochizuki in the case of isomorphisms (cf. [M3])), then one may obtain a homomorphism $\Phi^{\text{cusp}} : \Pi_{U'_1} \to \Pi_{U'_2}$ group-theoretically from Φ such that the condition (Σ-gnc). Here, $U'_i, i \in \{1, 2\}$, is an open sub-curve of X_i, and $\Pi_{U'_i}$ is $\pi^{\mathrm{t}}(U'_i)/(\mathrm{Ker}(\pi^{\mathrm{t}}(U'_i \times_k \overline{k}) \twoheadrightarrow \Delta_{U'_i}))$, where $\Delta_{U'_i}$ denotes the maximal pro-Σ quotient of the geometric tame fundamental group $\pi_1^{\mathrm{t}}(U'_i \times_k \overline{k})$. Then the Conjecture 0.2 follows from Theorem 0.3.

The present paper is organized as follows. In Section 1, we review well-known facts concerning the Isom-version of the Grothendieck conjecture for curves, introduce a purely group-theoretic condition (Σ-gnc), and give a group-theoretic characterization of the sets of cusps of hyperbolic curves. In Section 2, we study the kernels of surjections of geometric fundamental groups, and prove that the kernels are generated by inertia subgroups under the condition (Σ-gnc). In Section 3, by applying the Isom-version of the Grothendieck conjecture for curves and the result obtained in Section 2, we prove our main theorem.

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1 Preliminaries

Let p be a prime number, $\overline{\mathbb{F}}_p$ a finite field of characteristic p, and $\overline{\mathbb{F}}_p$ an algebraic closure of $\mathbb{F}_p$. We shall call that a field is generalized sub-p-adic if the field may be embedded as a subfield of a finitely generated extension of the quotient field of $W(\overline{\mathbb{F}}_p)$ (i.e., the ring of Witt vectors with coefficients in $\overline{\mathbb{F}}_p$). Let k be either a **finite field** of characteristic p or a **generalized sub-p-adic field**, $\overline{k}$ an algebraic closure of k, and $X_i, i \in \{1, 2\}$, hyperbolic curves of type (g_{X_i}, n_{X_i}) over k. Then we have the following fundamental exact sequence of tame fundamental groups (for suitable choices of base point):

$$1 \to \pi_1^{\mathrm{t}}(\overline{X}_i) \to \pi_1^{\mathrm{t}}(X_i) \stackrel{\mathrm{pr}_{X_i}}{\to} G_k \to 1,$$

where $\overline{X}_i$ denotes the curve $X_i \times_k \overline{k}$, and G_k denotes the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. Note that, if $\mathrm{char}(k) = 0$, then the tame fundamental groups of X_i coincides with the étale fundamental groups of X_i.

Let $\mathfrak{Primes}$ be the set of prime numbers, $p \in \Sigma_1 \subseteq \mathfrak{Primes}$ a **finite** subset, $p \notin \Sigma_2 \subseteq \mathfrak{Primes}$ a **finite** subset,

$$\Sigma \in \{\mathfrak{Primes}, \mathfrak{Primes} \setminus \Sigma_1, \mathrm{sol}\} \text{ if } \mathrm{char}(k) = p,$$

and

$$p \in \Sigma := \mathfrak{Primes} \setminus \Sigma_2 \text{ if } \mathrm{char}(k) = 0.$$

Write Δ_{X_i} for the maximal pro-Σ quotient of $\pi_1^{\mathrm{t}}(\overline{X}_i)$, respectively. Here, if $\Sigma = \mathrm{sol}$, $\Delta_{X_i}^{\mathrm{sol}}$ is the maximal pro-solvable quotient of $\pi_1^{\mathrm{t}}(\overline{X}_i)$. Note that

$$\mathrm{Ker}(\pi_1^{\mathrm{t}}(\overline{X}_i) \twoheadrightarrow \Delta_{X_i})$$

is also a normal closed subgroup of $\pi_1^{\mathrm{t}}(X_i)$. We set

$$\Pi_{X_i} := \pi_1^{\mathrm{t}}(X_i)/\mathrm{Ker}(\pi_1^{\mathrm{t}}(\overline{X}_i) \twoheadrightarrow \Delta_{X_i}).$$

Then we obtain a commutative diagram as follows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1^{\mathrm{t}}(\overline{X}_i) & \longrightarrow & \pi_1^{\mathrm{t}}(X_i) & \xrightarrow{\mathrm{pr}_{X_i}} & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_{X_i} & \longrightarrow & \Pi_{X_i} & \xrightarrow{\mathrm{pr}_{X_i}^{\Sigma}} & G_k & \longrightarrow & 1,
\end{array}$$

where all the vertical arrows are surjections.

We define

$$\mathrm{Isom}_{\text{pro-gps}}(-,-) \text{ and } \mathrm{Hom}_{\text{pro-gps}}^{\mathrm{open}}(-,-)$$

to be the set of continuous isomorphisms and the set of open continuous homomorphisms of profinite groups between the two profinite groups in parentheses, respectively, and define

$$\mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) := \{\Phi \in \mathrm{Isom}_{\text{pro-gps}}(\Pi_{X_1}, \Pi_{X_2}) \mid \mathrm{pr}_{X_1}^{\Sigma} = \mathrm{pr}_{X_2}^{\Sigma} \circ \Phi\},$$

$$\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) := \{\Phi \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_{X_1}, \Pi_{X_2}) \mid \mathrm{pr}^{\Sigma}_{X_1} = \mathrm{pr}^{\Sigma}_{X_2} \circ \Phi\}.$$

Thus, by composing with inner automorphisms, we obtain a natural action of Δ_{X_2} on $\mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$ and a natural action of Δ_{X_2} on $\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$.

Consider the category $\mathcal{C}_k$ of smooth k-curves and dominant k-morphisms. If $\mathrm{char}(k) = p$, we denote by $\mathcal{FC}_k$ the localization of $\mathcal{C}_k$ at geometric k-Frobenius maps between curves. Then we obtain the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{Isom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\mathrm{Isom}\text{-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\Sigma}} & \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}), \end{array}$$

if $\mathrm{char}(k) = 0$ and

$$\begin{array}{ccc} \mathrm{Isom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\mathrm{Isom}\text{-}\pi_1^{\mathrm{t},\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\mathrm{t},\Sigma}} & \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \end{array}$$

if $\mathrm{char}(k) = p$, where all the vertical arrows are injections. Moreover, we have the following Isom-version of the Grothendieck conjecture for curves over k:

Theorem 1.1. *The natural maps*

$$\mathrm{Isom}\text{-}\pi_1^{\Sigma} : \mathrm{Isom}_{\mathcal{C}_k}(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = 0$ *and*

$$\mathrm{Isom}\text{-}\pi_1^{\mathrm{t},\Sigma} : \mathrm{Isom}_{\mathcal{FC}_k}(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = p$ *are bijections.*

Proof. The theorem follows from [M2, Theorem 4.12] and [ST3, Theorem 4.22]. □

Let $\Phi \in \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$. Then Φ induces a homomorphism

$$\overline{\Phi} : \Delta_{X_1} \to \Delta_{X_2}.$$

We denote by

$$\Pi_{\Phi} := \mathrm{Im}(\Phi) \subseteq \Pi_{X_2}$$

the images of Φ. Write Δ_{Φ} for $\Pi_{\Phi} \cap \Delta_{X_2}$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{X_1} & \longrightarrow & \Pi_{X_1} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_{\Phi} & \longrightarrow & \Pi_{\Phi} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_{X_2} & \longrightarrow & \Pi_{X_2} & \longrightarrow & G_k & \longrightarrow & 1. \end{array}$$

We introduce a genus condition as follows:

(Σ-gnc): For each open subgroup $\overline{H}_2 \subseteq \Delta_\Phi$, write $\overline{H}_1$ for the inverse image $\overline{\Phi}^{-1}(\overline{H}_2)$. We denote by g_{H_1} and g_{H_2} the genera of the curves over $\overline{k}$ corresponding to $\overline{H}_1$ and $\overline{H}_2$, respectively. We shall say that Φ satisfies (Σ-gnc) if $g_{H_1} = g_{H_2}$ for each open subgroup $\overline{H}_2 \subseteq \Delta_\Phi$.

Note that if Φ satisfies (Σ-gnc), then, for each open subgroup $Q_2 \subseteq \Pi_{X_2}$, the morphism $\Phi^{-1}(Q_2) \to Q_2$ induced by Φ also satisfies (Σ-gnc).

Proposition 1.2. *The condition* (Σ-gnc) *is a purely group-theoretic condition.*

Proof. Let $H_1 \subseteq \Pi_{X_1}$ and $H_2 \subseteq \Pi_\Phi$ be open subgroups such that $H_1 \cap \Delta_{X_1} = \overline{H}_1$ and $H_2 \cap \Delta_\Phi = \overline{H}_2$.

Suppose that $\mathrm{char}(k) = 0$. To verify the proposition, we may reduce immediately to the case where k is finite over the quotient field of $W(\overline{\mathbb{F}}_p)$. Let $\ell \in \Sigma$ distinct from p. Then we obtain that the genera g_{H_1} and g_{H_2} are reconstructed by the monodromy filtrations of the abelianization of the maximal pro-ℓ quotient of $\overline{H}_1$ and $\overline{H}_2$, respectively. Moreover, the genera g_{H_1} and g_{H_2} are also equal to the dimension of the weight 0 part of the Hodge-Tate decomposition of the abelianization of the maximal pro-p quotient of $\overline{H}_1$ and $\overline{H}_2$, respectively.

Suppose that $\mathrm{char}(k) = p$. Let ℓ be a prime number distinct from p. Then the genera g_{H_1} and g_{H_2} are equal to the dimension of the Frobenius weight 1 part of the abelianization of the maximal pro-ℓ quotient of $\overline{H}_1$ and $\overline{H}_2$, respectively. Moreover, if $\Sigma = \mathfrak{Primes}$ or $\Sigma = \mathrm{sol}$, by Tamagawa's p-average theorem (cf. [T2, Theorem 0.5]), g_{H_1} and g_{H_2} can be also reconstructed group-theoretically from $\overline{H}_1$ and $\overline{H}_2$.

Then g_{H_1} and g_{H_2} can be reconstructed group-theoretically from H_1 and H_2, respectively. This completes the proof of the proposition. □

In the remainder of this section, let X be a hyperbolic curve of type (g_X, n_X) over $\overline{k}$. Write X^{cpt} for the smooth compactification of X over $\overline{k}$. We define a pointed smooth stable curve

$$X^\bullet := (X^{\mathrm{cpt}}, D_X := X^{\mathrm{cpt}} \setminus X).$$

Here, X^{cpt} denotes the underlying curve of $X^\bullet$, and D_X denotes the set of marked points of $X^\bullet$.

Let K_X be the function field of X, and define K_X^Σ to be the maximal pro-Σ Galois extension of K_X in a fixed separable closure of K_X, unramified over X and at most tamely ramified over D_X. Then we may identify the maximal pro-Σ quotient Δ_X of the tame fundamental group $\pi_1^{\mathrm{t}}(X)$ of X with $\mathrm{Gal}(K_X^\Sigma/K_X)$. We set

$$X^{\bullet,\Sigma} := (X^\Sigma, D_{X^\Sigma}),$$

where X^Σ denotes the normalization of X^{cpt} in K_X^Σ, and D_{X^Σ} denotes the inverse image of D_X in X^Σ. For each $e^\Sigma \in D_{X^\Sigma}$, we denote by I_{e^Σ} the inertia subgroup of Δ_X associated to e^Σ (i.e., the stabilizer of e^Σ). Note that we have $I_{e^\Sigma} \cong \widehat{\mathbb{Z}}(1)^\Sigma$, where $\widehat{\mathbb{Z}}(1)^\Sigma$ denotes the pro-Σ part of $\widehat{\mathbb{Z}}(1)$. Let $C_{X^\bullet} := \{H_i\}_{i \in \mathbb{Z}_{\geq 0}}$ be a set of open normal subgroups of Δ_X such that $H_0 = \Delta_X$, that H_{i+1} is a proper subgroup of H_i for each $i \in \mathbb{Z}_{\geq 0}$, and that

$$\varprojlim_i \Delta_X/H_i \cong \Delta_X.$$

Let $e^\Sigma \in D_{X^\Sigma}$. For each $i \in \mathbb{Z}_{\geq 0}$, we write $X^\bullet_{H_i} := (X_{H_i}, D_{X_{H_i}})$ for the smooth pointed stable curve corresponding to H_i and $e_{H_i} \in D_{X_{H_i}}$ for the image of e^Σ in $X^\bullet_{H_i}$. Then we obtain a sequence of marked points

$$\mathcal{I}^{C_{X^\bullet}}_{e^\Sigma} : \cdots \mapsto e_{H_2} \mapsto e_{H_1} \mapsto e_{H_0}$$

induced by $C_{X^\bullet}$. We may identify the inertia subgroup I_{e^Σ} associated to e^Σ with the stabilizer of $\mathcal{I}^{C_{X^\bullet}}_{e^\Sigma}$.

Definition 1.3. Let ℓ be a prime number, and let $f^\bullet : Y^\bullet \to X^\bullet$ be a connected tame Galois covering (i.e., $f^\bullet$ is a Galois covering and is at most tamely ramified over D_X) over k of degree ℓ. For any $e \in D_X$, we set

$$\mathrm{Ram}_{f^\bullet} := \{e \in D_X \mid f^\bullet \text{ is ramified over } e\}.$$

In the remainder of this section, we suppose that $g_X \geq 2$, and that $n_X > 0$. We define

$$(\ell, d, f^\bullet : Y^\bullet := (Y, D_Y) \to X^\bullet)$$

to be a data satisfying the following conditions:

(a) $\ell, d \in \Sigma$ are prime numbers distinct from each other and from p such that $\ell \equiv 1 \pmod{d}$; then all d^{th} roots of unity are contained in $\mathbb{F}_\ell$;

(b) $f^\bullet : Y^\bullet \to X^\bullet$ is an **étale** Galois covering (i.e., the morphism of underlying curves induced by $f^\bullet$ is an étale Galois covering) over $\overline{k}$ whose Galois group is isomorphic to G_d, where $G_d \subseteq \mathbb{F}_\ell^\times$ denotes the subgroup of d^{th} roots of unity.

Write $M^{\text{ét}}_{Y^\bullet}$ and $M_{Y^\bullet}$ for $\mathrm{H}^1_{\text{ét}}(Y^\bullet, \mathbb{F}_\ell)$ and $\mathrm{Hom}(\Delta_Y, \mathbb{F}_\ell)$, respectively, where Δ_Y denotes the maximal pro-Σ quotient of the tame fundamental group of $Y \setminus D_Y$. Note that there is a natural injection

$$M^{\text{ét}}_{Y^\bullet} \hookrightarrow M_{Y^\bullet}$$

induced by the natural surjection $\Delta_Y \twoheadrightarrow \Delta^{\text{ét}}_Y$, where $\Delta^{\text{ét}}_Y$ denotes the maximal pro-Σ quotient of Δ_Y. Then we obtain an exact sequence

$$0 \to M^{\text{ét}}_{Y^\bullet} \to M_{Y^\bullet} \to M^{\text{ra}}_{Y^\bullet} := \mathrm{coker}(M^{\text{ét}}_{Y^\bullet} \hookrightarrow M_{Y^\bullet}) \to 0$$

with a natural action of G_d.

Let

$$M^{\text{ra}}_{Y^\bullet, G_d} \subseteq M^{\text{ra}}_{Y^\bullet}$$

be the subset of elements on which G_d acts via the character $G_d \hookrightarrow \mathbb{F}_\ell^\times$ and

$$U^*_{Y^\bullet} \subseteq M_{Y^\bullet}$$

the subset of elements that map to nonzero elements of $M^{\mathrm{ra}}_{Y^\bullet, G_n}$. For each $\alpha \in U^*_{Y^\bullet}$, write

$$g^\bullet_\alpha : Y^\bullet_\alpha := (Y_\alpha, D_{Y_\alpha}) \to Y^\bullet$$

for the tame covering over $\overline{k}$ corresponding to α. Then we obtain a morphism

$$\epsilon : U^*_{Y^\bullet} \to \mathbb{Z}$$

that maps α to $\#D_{Y_\alpha}$, where $\#(-)$ denotes the cardinality of $(-)$. We define a subset of $U^*_{Y^\bullet}$ to be

$$U^{\mathrm{mp}}_{Y^\bullet} := \{\alpha \in U^*_{Y^\bullet} \mid \#\mathrm{Ram}_{g^\bullet_\alpha} = d\} = \{\alpha \in U^*_{Y^\bullet} \mid \epsilon(\alpha) = \ell(dn_X - d) + d\}.$$

Note that $U^{\mathrm{mp}}_{Y^\bullet}$ is not empty. For each $\alpha \in U^{\mathrm{mp}}_{Y^\bullet}$, since the image of α is contained in $M^{\mathrm{ra}}_{Y^\bullet, G_d}$, we obtain that the action of G_d on the set $\mathrm{Ram}_{g^\bullet_\alpha} \subseteq D_{Y^\bullet}$ is transitive. Thus, there exists a unique marked point e_α of $X^\bullet$ such that $f^\bullet(y) = e_\alpha$ for each $y \in \mathrm{Ram}_{g^\bullet_\alpha}$. We define a pre-equivalence relation $\sim$ on $U^{\mathrm{mp}}_{Y^\bullet}$ as follows:

if $\alpha \sim \beta \in U^{\mathrm{mp}}_{Y^\bullet}$, then $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}^\times_\ell$ for which $\lambda\alpha + \mu\beta \in U^*_{Y^\bullet}$, we have $\lambda\alpha + \mu\beta \in U^{\mathrm{mp}}_{Y^\bullet}$.

On the other hand, for each $e \in D_X$, we define

$$U^{\mathrm{mp}}_{Y^\bullet, e} := \{\alpha \in U^{\mathrm{mp}}_{Y^\bullet} \mid g^\bullet_\alpha \text{ is ramified over } (f^\bullet)^{-1}(e)\}.$$

Then, for any two marked points $e, e' \in D_X$ distinct from each other, we have

$$U^{\mathrm{mp}}_{Y^\bullet, e} \cap U^{\mathrm{mp}}_{Y^\bullet, e'} = \emptyset.$$

Moreover, we have

$$U^{\mathrm{mp}}_{Y^\bullet} = \bigcup_{e \in D_X} U^{\mathrm{mp}}_{Y^\bullet, e}.$$

Then we have the following proposition.

Proposition 1.4. *(i) The pre-equivalence relation $\sim$ on $U^{\mathrm{mp}}_{Y^\bullet}$ is an equivalence relation, and, moreover, the quotient set $U^{\mathrm{mp}}_{Y^\bullet}/\sim$ is naturally isomorphic to D_X that maps $[\alpha] \mapsto e_\alpha$. Moreover, the set $U^{\mathrm{mp}}_{Y^\bullet}/\sim$ does not depend on the choices of ℓ, d, and the étale covering $f^\bullet : Y^\bullet \to X^\bullet$.*

(ii) Write g_Y for the genus of $Y^\bullet$. We have, for each $e \in D_X$,

$$\#U^{\mathrm{mp}}_{Y^\bullet} = \ell^{2g_Y+1} - \ell^{2g_Y}.$$

Proof. First, let us prove (i). Let $\beta, \gamma \in U^{\mathrm{mp}}_{Y^\bullet}$. If $\mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$, then, for each $\lambda, \mu \in \mathbb{F}^\times_\ell$ for which $\lambda\beta + \mu\gamma \neq 0$, we have $\mathrm{Ram}_{g^\bullet_{\lambda\beta+\mu\gamma}} = \mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$. Thus, $\beta \sim \gamma$. On the other hand, if $\beta \sim \gamma$, we have $\mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$. Otherwise, we have $\#\mathrm{Ram}_{g^\bullet_{\beta+\gamma}} = 2d$. Thus, $\beta \sim \gamma$ if and only if $\mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$. Then $\sim$ is an equivalence relation on $U^{\mathrm{mp}}_{Y^\bullet}$.

We define a map

$$\vartheta : U^{\mathrm{mp}}_{Y^\bullet}/\sim \to D_X$$

that maps $\alpha \mapsto e_\alpha$. Let us prove that ϑ is a bijection. It is easy to see that ϑ is an injection. On the other hand, for each $e \in D_X$, the structure of the maximal pro-ℓ tame fundamental groups implies that we may construct a connected tame Galois covering of $h^\bullet : Z^\bullet \to Y^\bullet$ such that the line bundle corresponding to $h^\bullet$ is contained in $U^{\mathrm{mp}}_{Y^\bullet}$. Then ϑ is a surjection.

Let

$$(\ell^*, d^*, f^{\bullet,*} : Y^{\bullet,*} \to X^\bullet)$$

be a data. Hence we obtain a resulting $U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim$ and a naturally isomorphism

$$\vartheta^* : U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim \to D_X.$$

First, suppose that $\ell \neq \ell^*$, and that $d \neq d^*$. Then there exists a natural isomorphism

$$U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim \cong U^{\mathrm{mp}}_{Y^\bullet}/\sim$$

isomorphism which compatible with the isomorphism ϑ and ϑ^* as follows. Let $\alpha \in U^{\mathrm{mp}}_{Y^\bullet}$ and $\alpha^* \in U^{\mathrm{mp}}_{Y^{\bullet,*}}$. Write $Y^\bullet_\alpha \to Y^\bullet$ and $Y^{\bullet,*}_{\alpha^*} \to Y^{\bullet,*}$ for the tame coverings corresponding to α and α^*, respectively. Let us consider

$$Y^\bullet \times_{X^\bullet} Y^{\bullet,*}.$$

Thus, we have a connected tame Galois covering $Y^\bullet \times_{X^\bullet} Y^{\bullet,*} \to X^\bullet$ of degree $dd^*\ell\ell^*$. Then it is easy to check that α and α^* correspond to same marked points if and only if the cardinality of the set of marked points of $Y^\bullet \times_{X^\bullet} Y^{\bullet,*}$ is equal to $dd^*(\ell\ell^* n_X - 1) + 1)$. In general case, we may choose a data

$$(\ell^{**}, d^{**}, f^{\bullet,**} : Y^{\bullet,**} \to X^\bullet)$$

such that $\ell^{**} \neq \ell$, $\ell^{**} \neq \ell^*$, $d^{**} \neq d$, and $d^{**} \neq d^*$. Hence we obtain a resulting $U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim$ and a naturally isomorphism $\vartheta^{**} : U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim \to D_X$. Then we obtain two natural isomorphisms $U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim \cong U^{\mathrm{mp}}_{Y^\bullet}/\sim$ and $U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim \cong U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim$. Thus, we have $U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim \cong U^{\mathrm{mp}}_{Y^\bullet}/\sim$. This completes the proof of (i).

Next, let us prove (ii). Write $E_e \subseteq D_Y$ for the set $(f^\bullet)^{-1}(e)$. Then $U^{\mathrm{mp}}_{Y^\bullet,e}$ can be naturally regarded as a subset of $\mathrm{H}^1_{\text{ét}}(Y \setminus E_e, \mathbb{F}_\ell)$ via the natural open immersion $Y \setminus E_e \hookrightarrow Y$. Write L_e for the $\mathbb{F}_\ell$-vector space generated by $U^{\mathrm{mp}}_{Y^\bullet,e}$ in $\mathrm{H}^1_{\text{ét}}(Y \setminus E_e, \mathbb{F}_\ell)$. Then we have

$$U^{\mathrm{mp}}_{Y^\bullet,e} = L_e \setminus \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell).$$

Write H_e for the quotient $L_e/\mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell)$. We have an exact sequence as follows:

$$0 \to \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell) \to L_e \to H_e \to 0.$$

Since the action of G_d on $(f^\bullet)^{-1}(e)$ is translative, we have

$$\dim_{\mathbb{F}_\ell} H_e = 1.$$

On the other hand, since $\dim_{\mathbb{F}_\ell} \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell) = 2g_Y$, we obtain

$$\# U^{\mathrm{mp}}_{Y^\bullet,e} = \ell^{2g_Y+1} - \ell^{2g_Y}.$$

Thus, we have

$$\# U^{\mathrm{mp}}_{Y^\bullet} = n_X(\ell^{2g_Y+1} - \ell^{2g_Y}).$$

This completes the proof of the lemma. $\square$

2 The kernels of surjections of geometric fundamental groups

We maintain the notations introduced in Section 1. Let $\overline{X}_i^{\rm cpt}, i \in \{1, 2\}$, be the smooth compactification of $\overline{X}_i$ over $\overline{k}$. We define a pointed smooth stable curve over $\overline{k}$ to be

$$\overline{X}_i^{\bullet} := (\overline{X}_i^{\rm cpt}, D_{\overline{X}_i} := \overline{X}^{\rm cpt} \setminus \overline{X}_i),\ i \in \{1, 2\}.$$

Let $\Phi \in \mathrm{Hom}^{\rm open}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$ and $\overline{\Phi} : \Delta_{X_1} \to \Delta_{X_2}$ the homomorphism induced by Φ. In this section, we suppose that $n_{X_2} > 0$, that Φ is a surjection, and that Φ satisfies (Σ-gnc). Then $\overline{\Phi}$ is also a surjection.

Lemma 2.1. *Suppose that $g_{X_2} \geq 2$. Then the surjection $\overline{\Phi}$ induces an injection*

$$\lambda_{\overline{\Phi}} : D_{\overline{X}_2} \hookrightarrow D_{\overline{X}_1}.$$

Proof. Let

$$(\ell, d, f_2^{\bullet} : Y_2^{\bullet} := (Y_2, D_{Y_2}) \to \overline{X}_2^{\bullet})$$

be a data defined as in Section 1. For each $i \in \{1, 2\}$, write $M_{\overline{X}_i^{\bullet}}$ for $\mathrm{Hom}(\Delta_{X_i}, \mathbb{F}_d)$ and $M^{\text{ét}}_{\overline{X}_i^{\bullet}}$ for $\mathrm{H}^1_{\text{ét}}(\overline{X}_i^{\bullet}, \mathbb{F}_d) = \mathrm{H}^1_{\text{ét}}(\overline{X}_i^{\rm cpt}, \mathbb{F}_d)$. Then we have the following claim.

Claim: Write $\Delta^{\text{ét}}_{X_i}, i \in \{1, 2\}$, for the étale fundamental group of $\overline{X}_i^{\bullet}$ (i.e., $\pi_1(\overline{X}_i^{\rm cpt})$). Then $\overline{\Phi}$ induces an isomorphism

$$\Delta^{\text{ét}}_{X_1} \xrightarrow{\sim} \Delta^{\text{ét}}_{X_2}.$$

In particular, $\overline{\Phi}$ induces an isomorphism $M^{\text{ét}}_{\overline{X}_2^{\bullet}} \xrightarrow{\sim} M^{\text{ét}}_{\overline{X}_1^{\bullet}}$.

Let us prove the claim. For each $N_2 \subseteq \Delta_{X_2}$ open subgroup such that the covering $X_{N_2} \to \overline{X}_2$ corresponding to N_2 is étale. Write N_1 for the inverse image $\overline{\Phi}^{-1}(N_2)$. Then the condition ($\Sigma$-gcn) and the Riemann-Hurwitz formula imply that the covering $X_{N_1} \to \overline{X}_1$ corresponding to N_1 is étale. Thus, $\overline{\Phi}$ induces a surjection

$$\Delta^{\text{ét}}_{X_1} \twoheadrightarrow \Delta^{\text{ét}}_{X_2}.$$

On the other hand, $\Delta^{\text{ét}}_{X_1}$ is isomorphic to $\Delta^{\text{ét}}_{X_2}$ as abstract profinite groups. Indeed, $\Delta^{\text{ét}}_{X_i}, i \in \{1, 2\}$, is a free profinite group of rank $2g_{X_i}$. Then $\Delta^{\text{ét}}_{X_1}$ and $\Delta^{\text{ét}}_{X_2}$ are topologically finitely generated. Thus, the surjection $\Delta^{\text{ét}}_{X_1} \twoheadrightarrow \Delta^{\text{ét}}_{X_2}$ is an isomorphism. This completes the proof of the claim.

The claim implies the data $(\ell, d, f_2^{\bullet} : Y_2^{\bullet} \to \overline{X}_2^{\bullet})$ induces a data

$$(\ell, d, f_1^{\bullet} : Y_1^{\bullet} := (Y_1, D_{Y_1}) \to \overline{X}_1^{\bullet}),$$

where the étale Galois covering $f_1^{\bullet}$ is induced by $f_2^{\bullet}$ via the isomorphism $M^{\text{ét}}_{\overline{X}_2^{\bullet}} \xrightarrow{\sim} M^{\text{ét}}_{\overline{X}_1^{\bullet}}$.

Write $\Delta_{Y_1} \subseteq \Delta_{X_1}$ and $\Delta_{Y_2} \subseteq \Delta_{X_2}$ for the maximal pro-Σ quotients of the tame fundamental groups of $Y_1^\bullet$ and $Y_2^\bullet$, respectively. Write $M_{Y_1^\bullet}$, $M_{Y_1^\bullet}^{\text{ét}}$, $M_{Y_1^\bullet}^{\text{ra}}$, $M_{Y_2^\bullet}$, $M_{Y_2^\bullet}^{\text{ét}}$, and $M_{Y_2^\bullet}^{\text{ra}}$ for $\text{Hom}(\Delta_{Y_1}, \mathbb{F}_\ell)$, $\text{H}^1_{\text{ét}}(Y_1, \mathbb{F}_\ell)$, $M_{Y_1^\bullet}/M_{Y_1^\bullet}^{\text{ét}}$, $\text{Hom}(\Delta_{Y_2}, \mathbb{F}_\ell)$, $\text{H}^1_{\text{ét}}(Y_2, \mathbb{F}_\ell)$, and $M_{Y_2^\bullet}/M_{Y_2^\bullet}^{\text{ét}}$. Similar arguments to the arguments given in the proof of the claim imply that the surjection $\overline{\Phi}_Y := \overline{\Phi}|_{\Delta_{Y_1}} : \Delta_{Y_1} \twoheadrightarrow \Delta_{Y_2}$ induces a commutative diagram as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{Y_1^\bullet}^{\text{ét}} & \longrightarrow & M_{Y_1^\bullet} & \longrightarrow & M_{Y_1^\bullet}^{\text{ra}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow \scriptstyle{\overline{\Psi}_{Y,\ell}^{\text{ab}}} & & \uparrow & & \\ 0 & \longrightarrow & M_{Y_2^\bullet}^{\text{ét}} & \longrightarrow & M_{Y_2^\bullet} & \longrightarrow & M_{Y_2^\bullet}^{\text{ra}} & \longrightarrow & 0, \end{array}$$

where the vertical arrows on the right-hand side and the middle side are injections, and the vertical arrows on the left-hand side is an isomorphism. Write $U^*_{Y_1^\bullet}$ and $U^*_{Y_2^\bullet}$ for the subsets of $M_{Y_1^\bullet}$ and $M_{Y_2^\bullet}$ defined as in Section 1, respectively. Since the actions of G_d on the exact sequences are compatible with the morphisms appeared in the commutative diagram above, we have

$$\overline{\Psi}_{Y,\ell}^{\text{ab}}(U^*_{Y_2^\bullet}) \subseteq U^*_{Y_1^\bullet}.$$

Let $e_2 \in D_{\overline{X}_2}$, $\alpha_2 \in U^{\text{mp}}_{Y_2^\bullet, e_2}$, and $g^\bullet_{\alpha_2} : Y^\bullet_{\alpha_2} \to Y_2^\bullet$ the tame covering of degree ℓ over $\overline{k}$ corresponding to α_2. Write

$$g^\bullet_{\alpha_1} : Y^\bullet_{\alpha_1} \to Y_1^\bullet$$

for the tame covering of degree ℓ over $\overline{k}$ corresponding to $\alpha_1 := \overline{\Psi}_{Y,\ell}^{\text{ab}}(\alpha_2)$. Write $g_{Y_{\alpha_1}}$ and $g_{Y_{\alpha_2}}$ for the genera of $Y^\bullet_{\alpha_1}$ and $Y^\bullet_{\alpha_2}$. Then the condition (Σ-gnc) and the Riemann-Hurwitz formula imply that

$$g_{Y_{\alpha_1}} - g_{Y_{\alpha_2}} = \frac{1}{2}(d - \#\text{Ram}_{g^\bullet_{\alpha_1}})(\ell - 1) = 0.$$

Then we have $d = \#\text{Ram}_{g^\bullet_{\alpha_1}}$. This means that $\alpha_1 \in U^{\text{mp}}_{Y_1^\bullet}$. Moreover, there exists $e_1 \in D_{\overline{X}_1}$ such that $\alpha_1 \in U^{\text{mp}}_{Y_1^\bullet, e_1}$.

Let $\alpha_2' \in U^{\text{mp}}_{Y_2^\bullet, e_2}$ distinct from α_2. Since, for each $a\alpha_2 + b\alpha_2' \neq 0$, $a, b \in \mathbb{F}_\ell^\times$, $a\alpha_2 + b\alpha_2' \in U^{\text{mp}}_{Y_2^\bullet, e_2}$, we have $\overline{\Psi}_{Y,\ell}^{\text{ab}}(a\alpha_2 + b\alpha_2') \in U^{\text{mp}}_{Y_1^\bullet}$. Moreover, we have $\overline{\Psi}_{Y,\ell}^{\text{ab}}(a\alpha_2 + b\alpha_2') \in U^{\text{mp}}_{Y_1^\bullet, e_1}$. This implies that $\overline{\Psi}_{Y,\ell}^{\text{ab}}(\alpha_2') \in U^{\text{mp}}_{Y_1^\bullet, e_1}$. Thus, we obtain

$$\overline{\Psi}_{Y,\ell}^{\text{ab}}(U^{\text{mp}}_{Y_2^\bullet, e_2}) \subseteq U^{\text{mp}}_{Y_1^\bullet, e_1}.$$

On the other hand, Proposition 1.4 (ii) implies that $\#U^{\text{mp}}_{Y_1^\bullet, e_1} = \#U^{\text{mp}}_{Y_2^\bullet, e_2}$. We have

$$\overline{\Psi}_{Y,\ell}^{\text{ab}}(U^{\text{mp}}_{Y_2^\bullet, e_2}) = U^{\text{mp}}_{Y_1^\bullet, e_1}.$$

Then Proposition 1.4 (i) implies that $\overline{\Psi}_{Y,\ell}^{\text{ab}}$ induces an injection

$$\lambda_{\overline{\Phi}} : D_{\overline{X}_2} \hookrightarrow D_{\overline{X}_1}.$$

This completes the proof of the lemma. $\square$

Let $H_2 \subseteq \Pi_{X_2}$ be an open subgroup and $P_1 \subseteq \Delta_{X_1}$ the inverse image $\overline{\Phi}^{-1}(P_2)$. Write

$$Z_1^\bullet := (Z_1, D_{Z_1}) \text{ and } Z_2^\bullet := (Z_2, D_{Z_2})$$

for the pointed smooth stable curves of over $\overline{k}$ corresponding to P_1 and P_2, respectively. The surjection $\overline{\Phi}$ induces a surjection

$$\overline{\Phi}_Z := \overline{\Phi}|_{P_1} : P_1 \twoheadrightarrow P_2.$$

Then Lemma 2.1 implies an injective map

$$\lambda_{\overline{\Phi}_Z} : D_{Z_2} \hookrightarrow D_{Z_1}.$$

On the other hand, the tame coverings $f_{P_1}^\bullet : Z_1^\bullet \to \overline{X}_1^\bullet$ and $f_{P_2}^\bullet : Z_2^\bullet \to \overline{X}_2^\bullet$ induced by P_1 and P_2 determine the surjective maps of the set of marked points

$$\gamma_{f_{P_1}} : D_{Z_1} \twoheadrightarrow D_{\overline{X}_1} \text{ and } \gamma_{f_{P_2}} : D_{Z_2} \twoheadrightarrow D_{\overline{X}_2},$$

respectively. Furthermore, we have the following lemma.

Lemma 2.2. *Suppose that $g_{X_2} \geq 2$. Then the natural diagram*

$$\begin{array}{ccc} D_{Z_2} & \xrightarrow{\lambda_{\overline{\Phi}_Z}} & D_{Z_1} \\ \gamma_{f_{P_2}} \downarrow & & \gamma_{f_{P_1}} \downarrow \\ D_{\overline{X}_2} & \xrightarrow{\lambda_{\overline{\Phi}}} & D_{\overline{X}_1} \end{array}$$

is commutative.

Proof. Let $e_{Z_2} \in D_{Z_2}$, $e_{Z_1} := \lambda_{\overline{\Phi}_Z}(e_{Z_2}) \in D_{Z_1}$, $e_2 := \gamma_{f_{P_2}}(e_{Z_2}) \in D_{\overline{X}_2}$, $e_1 := (\gamma_{f_{P_1}} \circ \lambda_{\overline{\Phi}_Z})(e_{Z_2}) \in D_{\overline{X}_2}$, and $e_1' := \lambda_{\overline{\Phi}}(e_2) \in D_{\overline{X}_1}$. Let us prove that $e_1 = e_1'$. Write S_{Z_1} and S_{Z_2} for the sets $(\gamma_{f_{P_1}})^{-1}(e_1')$ and $(\gamma_{f_{P_2}})^{-1}(e_2)$, respectively. Note that $e_{Z_2} \in S_{Z_2}$. To verify $e_1 = e_1'$, it is sufficient to prove that $e_{Z_1} \in S_{Z_1}$

Let $(\ell, d, f_2^\bullet : Y_2^\bullet \to \overline{X}_2^\bullet)$ be a data defined as in Section 1 such that $(\ell, \#(\Delta_{X_2}/P_2)) = 1$ and $(d, \#(\Delta_{X_2}/P_2)) = 1$. By the proof of Lemma 2.1, we obtain a data

$$(\ell, d, f_1^\bullet : Y_1^\bullet \to \overline{X}_1^\bullet)$$

induced by $\overline{\Phi}$ and $(\ell, d, f_2^\bullet : Y_2^\bullet \to X_2^\bullet)$. On the other hand, we have a data

$$(\ell, d, g_2^\bullet : W_2^\bullet := Y_2^\bullet \times_{\overline{X}_2^\bullet} Z_2^\bullet \to Z_2^\bullet).$$

Again, by the proof of Lemma 2.1, we obtain a data

$$(\ell, d, g_1^\bullet : W_1^\bullet := Y_1^\bullet \times_{\overline{X}_1^\bullet} Z_1^\bullet \to Z_1^\bullet)$$

induced by $\overline{\Phi}_Z$ and $(\ell, d, g_2^\bullet : W_2^\bullet \to Z_2^\bullet)$.

Let $\alpha_2 \in U^{\mathrm{mp}}_{Y_2^\bullet, e_2}$, where $U^{\mathrm{mp}}_{(-)}$ is defined as in Section 1. Then the proof of Lemma 2.1 implies that α_2 induces an element

$$\alpha_1 \in U^{\mathrm{mp}}_{Y_1^\bullet, e_1'}.$$

Write $Y^\bullet_{\alpha_1}$ and $Y^\bullet_{\alpha_2}$ for the smooth pointed stable curves over $\overline{k}$ corresponding to α_1 and α_2, respectively. Consider the connected tame Galois covering

$$Y^\bullet_{\alpha_2} \times_{\overline{X}_2^\bullet} Z_2^\bullet \to W_2^\bullet$$

of degree ℓ over $\overline{k}$, and write β_2 for the element of $U^*_{W_2^\bullet}$ corresponding to this connected tame Galois covering, where $U^*_{(-)}$ is defined as in Section 1. Then we have

$$\beta_2 = \sum_{c_2 \in S_{Z_2}} t_{c_2} \beta_{c_2},$$

where $t_{c_2} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and $\beta_{c_2} \in U^{\mathrm{mp}}_{W_2^\bullet, c_2}$. On the other hand, the proof of Lemma 2.1 implies that β_{c_2} induced an element $\beta_{\lambda_{\overline{\Phi}_Z}(c_2)} \in U^{\mathrm{mp}}_{W_1^\bullet, \lambda_{\overline{\Phi}_Z}(c_2)}$. Then β_2 induces an element

$$\beta_1 := \sum_{c_2 \in S_{Z_2}} t_{c_2} \beta_{\lambda_{\overline{\Phi}_Z}(c_2)} \in U^*_{W_1^\bullet}.$$

Note that since β_1 corresponds to the connected tame Galois covering $Y^\bullet_{\alpha_1} \times_{\overline{X}_1^\bullet} Z_1^\bullet \to W_1^\bullet$, we have the composition of the connected tame Galois covering $Y^\bullet_{\alpha_1} \times_{\overline{X}_1^\bullet} Z_1^\bullet \to W_1^\bullet$ and the étale Galois covering $g_1^\bullet : W_1^\bullet \to Z_1^\bullet$ is tamely ramified over e_1'. This means that e_{Z_1} is contained in S_{Z_1}. This completes the proof of the lemma. □

Let $K_{\overline{X}_i}, i \in \{1, 2\}$, be the function field of $\overline{X}_i$, and define $K^\Sigma_{\overline{X}_i}$ to be the maximal pro-Σ Galois extension of $K_{\overline{X}_i}$ in a fixed separable closure of $K_{\overline{X}_i}$, unramified over $\overline{X}_i$ and at most tamely ramified over $D_{\overline{X}_i}$. We set

$$\overline{X}_i^{\bullet, \Sigma} := (\overline{X}_i^\Sigma, D_{\overline{X}_i^\Sigma}),\ i \in \{1, 2\},$$

where $\overline{X}_i^\Sigma$ denotes the normalization of $\overline{X}_i^{\mathrm{cpt}}$ in $K^\Sigma_{\overline{X}_i}$, and $D_{\overline{X}_i^\Sigma}$ denotes the inverse image of D_X in $\overline{X}_i^\Sigma$. We have the following proposition.

Proposition 2.3. *Let $e_2^\Sigma \in D_{\overline{X}_2^\Sigma}$, $e_2 \in D_{\overline{X}_2}$ the image of e_2^Σ, and $I_{e_2^\Sigma}$ the inertia subgroup of Δ_{X_2} associated to e_2^Σ. Then the following results hold.*

(i) There exists a point $e_1^\Sigma \in D_{\overline{X}_1^\Sigma}$ such that $\overline{\Phi}$ induces an isomorphism

$$\overline{\Phi}|_{I_{e_1^\Sigma}} : I_{e_1^\Sigma} \xrightarrow{\sim} I_{e_2^\Sigma},$$

where $I_{e_1^\Sigma}$ denotes the inertia subgroup associated to e_1^Σ.

(ii) Let $e^\Sigma_{1,1}, e^\Sigma_{1,2} \in D_{\overline{X}_1^\Sigma}$. Write $I_{e^\Sigma_{1,1}}$ and $I_{e^\Sigma_{1,2}}$ for the inertia subgroups of Δ_{X_1} associated to $e^\Sigma_{1,1}$ and $e^\Sigma_{1,2}$, and write $e_{1,1}$ and $e_{1,2}$ for the images of $e^\Sigma_{1,1}$ and $e^\Sigma_{1,2}$ in $D_{\overline{X}_1}$, respectively. Suppose that $\overline{\Phi}|_{I_{e^\Sigma_{1,2}}} : I_{e_1^\Sigma} \xrightarrow{\sim} I_{e_2^\Sigma}$ and $\overline{\Phi}|_{I_{e^\Sigma_{1,2}}} : I_{e_1^\Sigma} \xrightarrow{\sim} I_{e_2^\Sigma}$ are isomorphism. Then we have $\lambda_{\overline{\Phi}}(e_2) = e_{1,1} = e_{1,2}$.

Proof. First, we have the following claim.

> **Claim:** Suppose that the proposition holds when $g_{X_2} \geq 2$. Then the proposition holds when $g_{X_2} \geq 0$.

Let us prove the claim. Let $\ell' \in \Sigma$ be a prime number distinct from p. We choose an open normal subgroup $Q_1 \subseteq \Pi_{X_1}$ such that

$$\overline{Q}_2 := Q_2 \cap \Delta_{X_2} = \mathrm{Ker}(\Delta_{X_2} \twoheadrightarrow \mathbb{Z}/\ell'\mathbb{Z}),$$

and that the tame covering

$$g_2^\bullet : X^\bullet_{\overline{Q}_2} := (X_{\overline{Q}_2}, D_{X_{\overline{Q}_2}}) \to \overline{X}^\bullet_2$$

over $\overline{k}$ corresponding to $\overline{Q}_2$ is totally ramified over $D_{\overline{X}_2}$. Write Q_1 for $\Phi^{-1}(Q_1)$, $\overline{Q}_1$ for $\overline{\Phi}^{-1}(\overline{Q}_1)$, and

$$g_1^\bullet : X^\bullet_{\overline{Q}_1} := (X_{\overline{Q}_1}, D_{X_{\overline{Q}_1}}) \to \overline{X}^\bullet_1$$

for the tame covering over $\overline{k}$ corresponding to $\overline{Q}_1$. Note that the genera of $X^\bullet_{\overline{Q}_1}$ and $X^\bullet_{\overline{Q}_2}$ are ≥ 2. Write $e_{\overline{Q}_2}$ for the image of e_2^Σ in $D_{X_{\overline{Q}_2}}$. By Lemma 2.1, we have $e_{\overline{Q}_1} := \lambda_{\overline{\Phi}|_{\overline{Q}_2}}(e_{\overline{Q}_2}) \in D_{\overline{X}_{Q_1}}$. Moreover, by (i) of the proposition, there exists $e_1^\Sigma \in D_{\overline{X}_1^\Sigma}$ such that the inertia subgroup $I_{e_{\overline{Q}_1}}$ of $\overline{Q}_1$ associated to e_1^Σ is equal to $I_{e_1^\Sigma} \cap \overline{Q}_1$, and that $\overline{\Phi}|_{I_{e_{\overline{Q}_1}}} : I_{e_{\overline{Q}_1}} \xrightarrow{\sim} I_{e_{\overline{Q}_2}}$, where $I_{e_1^\Sigma}$ denotes the inertia subgroup of Δ_{X_1} associated to e_1^Σ. Then [T2, Lemma 5.1] or [M4, Proposition 1.2] implies that $\overline{\Phi}(I_{e_1^\Sigma}) \subseteq I_{e_2^\Sigma}$. On the other hand, (ii) of the proposition implies that $g_1^\bullet$ is totally ramified at $e_{\overline{Q}_1}$. Thus, $I_{e_{\overline{Q}_1}} \neq I_{e_1^\Sigma}$. This means that

$$\overline{\Phi}(I_{e_1^\Sigma}) = I_{e_2^\Sigma}.$$

Let $e_{1,1}^\Sigma, e_{1,2}^\Sigma \in D_{\overline{X}_1^\Sigma}$ satisfying (i) of the proposition. Then we have the images of $e_{1,1}^\Sigma$ and $e_{1,2}^\Sigma$ in $D_{X_{\overline{Q}_1}}$ are equal to $e_{\overline{Q}_1}$. Thus, the images of $e_{1,1}^\Sigma$ and $e_{1,2}^\Sigma$ in D_{X_1} are equal. This completes the proof of the claim.

By the claim above, we may assume that $g_{X_2} \geq 2$. Let us prove (i). Let $C_{\overline{X}_2^\bullet} := \{\overline{H}_{2,i}\}_{i \in \mathbb{Z}_{\geq 0}}$ be a set of open normal subgroups of Δ_{X_2} such that $\overline{H}_{2,0} = \Delta_{X_2}$, that $\overline{H}_{2,i+1}$ is a proper subgroup of $\overline{H}_{2,i}$ for each $i \in \mathbb{Z}_{\geq 0}$, and that

$$\varprojlim_i \Delta_{X_2}/\overline{H}_{2,i} \cong \Delta_{X_2}.$$

For each $i \in \mathbb{Z}_{\geq 0}$, we write $X^\bullet_{\overline{H}_{2,i}} := (X_{\overline{H}_{2,i}}, D_{X_{\overline{H}_{2,i}}})$ for the smooth pointed stable curve corresponding to $\overline{H}_{2,i}$ and $e_{\overline{H}_{2,i}} \in D_{X_{\overline{H}_{2,i}}}$ for the image of e_2^Σ in $X^\bullet_{\overline{H}_{2,i}}$. Then we obtain a sequence of marked points

$$\mathcal{I}_{e_2^\Sigma}^{C_{\overline{X}_2^\bullet}} : \cdots \mapsto e_{\overline{H}_{2,2}} \mapsto e_{\overline{H}_{2,1}} \mapsto e_{\overline{H}_{2,0}} = e_2$$

induced by $C_{\overline{X}_2^\bullet}$. Then we may identify the inertia subgroup $I_{e_2^\Sigma}$ associated to e_2^Σ with the stabilizer of $\mathcal{I}_{e_2^\Sigma}^{C_{\overline{X}_2^\bullet}}$.

On the other hand, write $\overline{H}_{1,i}, i \in \mathbb{Z}_{\geq 0}$ for $\overline{\Phi}^{-1}(\overline{H}_{2,i})$, and $X_{\overline{H}_{1,i}}^\bullet := (X_{\overline{H}_{1,i}}, D_{X_{\overline{H}_{1,i}}})$ for the smooth pointed stable curve corresponding to $\overline{H}_{1,i}$. Write $\overline{\Phi}_{X_{\overline{H}_i}}$ for $\overline{\Phi}|_{\overline{H}_{1,i}} : \overline{H}_{1,i} \twoheadrightarrow \overline{H}_{2,i}$. Then Lemma 2.1 implies that, for each $i \in \mathbb{Z}_{\geq 0}$, we have an injection

$$\lambda_{\overline{\Phi}_{X_{\overline{H}_i}}} : D_{X_{\overline{H}_{2,i}}} \hookrightarrow D_{X_{\overline{H}_{1,i}}}.$$

We denote by $e_{\overline{H}_{1,i}} := \lambda_{\overline{\Phi}_{X_{\overline{H}_i}}}(e_{\overline{H}_{2,i}})$ for each $i \in \mathbb{Z}_{\geq 0}$. Then Lemma 2.2 induces a sequence of marked points

$$\cdots \mapsto e_{\overline{H}_{1,2}} \mapsto e_{\overline{H}_{1,1}} \mapsto e_{\overline{H}_{1,0}}.$$

Write $L_{\overline{\Phi}}$ for the kernel of $\overline{\Phi} : \Delta_{X_1} \twoheadrightarrow \Delta_{X_2}$. We denote by $K_{\overline{X}_1}^{L_{\overline{\Phi}}} \subseteq K_{\overline{X}_1}^{\Sigma}$ the subfield corresponding to $L_{\overline{\Phi}}$. We set

$$X_{1,L_{\overline{\Phi}}}^\bullet := (X_{1,L_{\overline{\Phi}}}, D_{X_{1,L_{\overline{\Phi}}}}),$$

where $X_{1,L_{\overline{\Phi}}}$ denotes the normalization of $\overline{X}_1^{\mathrm{cpt}}$ in $K_{\overline{X}_1}^{L_{\overline{\Phi}}}$, $D_{X_{1,L_{\overline{\Phi}}}}$ denotes the inverse image of $D_{\overline{X}_1}$ in $X_{1,L_{\overline{\Phi}}}$. Then the sequence of marked points above $\cdots \mapsto e_{\overline{H}_{1,2}} \mapsto e_{\overline{H}_{1,1}} \mapsto e_{\overline{H}_{1,0}}$ determines a point $e_{1,L_{\overline{\Phi}}} \in D_{X_{1,L_{\overline{\Phi}}}}$. We choose a point $e_1^\Sigma \in D_{\overline{X}_1^\Sigma}$ such that the image of e_1^Σ in $D_{X_{1,L_{\overline{\Phi}}}}$ is $e_{1,L_{\overline{\Phi}}}$. Write $I_{e_1^\Sigma}$ for the inertia subgroup of Δ_{X_1} associated to e_1^Σ. Then $\overline{\Phi}$ induces a surjection

$$\overline{\Phi}|_{I_{e_1^\Sigma}} : I_{e_1^\Sigma} \twoheadrightarrow I_{e_2^\Sigma}.$$

Moreover, since $I_{e_1^\Sigma}$ and $I_{e_2^\Sigma}$ are isomorphic to $\widehat{\mathbb{Z}}(1)^\Sigma$, we have $\overline{\Phi}|_{I_{e_1^\Sigma}}$ is an isomorphism. This completes the proof of (i).

Let us prove (ii). Write e_1 for the image of e_1^Σ in $D_{\overline{X}_1}$. Let $(\ell, d, f_2^\bullet : Y_2^\bullet \to \overline{X}_2^\bullet)$ be a data defined as in Section 1. Then $\overline{\Phi}$ induces a data

$$(\ell, d, f_1^\bullet : Y_1^\bullet \to \overline{X}_1^\bullet).$$

Write

$$\Delta_{Y_2} \subseteq \Delta_{X_2} \text{ and } \Delta_{Y_1} \subseteq \Delta_{X_1}$$

for the normal open subgroup corresponding to $Y_1^\bullet$. Since $f_2^\bullet$ and $f_1^\bullet$ are étale, we have

$$I_{e_2^\Sigma} \subseteq \Delta_{Y_2}, I_{e_{1,1}^\Sigma} \subseteq \Delta_{Y_1}, \text{ and } I_{e_{1,2}^\Sigma} \subseteq \Delta_{Y_1}.$$

Let $\alpha_2 \in U_{Y_2^\bullet, e_2}^{\mathrm{mp}}$ be a line bundle such that the composition of the natural injection $I_{e_2^\Sigma} \hookrightarrow \Delta_{Y_2}$ and the morphism $\Delta_{Y_2} \to \mathbb{Z}/\ell\mathbb{Z}$ induced by α_2 is nontrivial. Then, by the proof of Lemma 2.1, we obtain a line bundle $\alpha_1 \in U_{Y_1^\bullet, e_1}^{\mathrm{mp}}$. Moreover, the composition of the natural injection $I_{e_{1,1}^\Sigma} \hookrightarrow \Delta_{Y_1}$ (resp. $I_{e_{1,2}^\Sigma} \hookrightarrow \Delta_{Y_1}$) and the morphism $\Delta_{Y_1} \to \mathbb{Z}/\ell\mathbb{Z}$ induced by α_1 is nontrivial. This means that $e_1 = e_{1,1} = e_{1,2}$. Moreover, the proof of (i) implies that $\lambda_{\overline{\Phi}}(e_2) = e_1 = e_{1,1} = e_{1,2}$. This completes the proof of the proposition. $\square$

We set

$$D_{\overline{X}_1\setminus\overline{X}_2} := D_{\overline{X}_1} \setminus \lambda_{\overline{\Phi}}(D_{\overline{X}_2}).$$

Next, we prove the main theorem of the present section.

Theorem 2.4. *Let $L_{\overline{\Phi}}$ be the kernel of $\overline{\Phi} : \Delta_{X_1} \twoheadrightarrow \Delta_{X_2}$. For each $e \in D_{\overline{X}_1\setminus\overline{X}_2}$ and each $e^{\Sigma} \in D_{X_1^{\Sigma}}$ over e, we denote by $I_{e^{\Sigma}}$ the inertia subgroup of Δ_{X_2} associated to e^{Σ}. Then we have*

$$I_{e^{\Sigma}} \subseteq L_{\overline{\Phi}}.$$

Proof. We denote by $I \subseteq \Delta_{X_2}$ the image $\overline{\Phi}(I_{e^{\Sigma}})$. To verify the theorem, we may assume that I is not trivial. Then I is a pro-cyclic subgroup of Δ_{X_2}.

Let $\overline{H}_2$ be any open subgroup of Δ_{X_2} and $\overline{H}_1 := \overline{\Phi}^{-1}(\overline{H}_2)$. Write $\overline{H}_1^{\text{ét}}$ and $\overline{H}_2^{\text{ét}}$ for the étale fundamental groups of the smooth compactifications of the curves over $\overline{k}$ corresponding to $\overline{H}_1$ and $\overline{H}_2$, respectively. Then we obtain natural surjections

$$\overline{H}_1 \twoheadrightarrow \overline{H}_1^{\text{ét}} \text{ and } \overline{H}_2 \twoheadrightarrow \overline{H}_2^{\text{ét}}.$$

By the claim in the proof of Lemma 2.1, $\overline{\Phi}$ induces an isomorphism $\overline{H}_1^{\text{ét}} \xrightarrow{\sim} \overline{H}_2^{\text{ét}}$. Moreover, since $I_{e^{\Sigma}} \cap \overline{H}_1$ is contained in the kernel of the surjection $\overline{H}_1 \twoheadrightarrow \overline{H}_1^{\text{ét}}$, the natural morphism

$$I \cap \overline{H}_2 \hookrightarrow \overline{H}_2 \twoheadrightarrow \overline{H}_2^{\text{ét,ab}}$$

is trivial. Thus, by applying [N3, Lemma 2.1.4], we have I is contained in an inertia subgroups $J \subseteq \Delta_{X_2}$. Proposition 2.3 implies that $e \in \lambda_{\Phi}(D_{\overline{X}_2})$. This contradicts to $e \in D_{\overline{X}_1\setminus\overline{X}_2}$. We complete the proof of the theorem. □

3 Group-theoretic characterizations of almost open immersions

We maintain the notations introduced in Section 1 and Section 2.

Definition 3.1. Let $f \in \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$ be a separable morphism over k. We shall call $f : X_1 \to X_2$ an **almost open immersion** if f is a composition of an open immersion and a finite étale morphism. Note that the open immersion and the finite étale morphism are unique.

Suppose that $\mathrm{char}(k) = p$. Let $\phi \in \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$. We shall call $\phi : X_1 \to X_2$ an **almost open immersion** if ϕ can be represented by the following k-morphisms

$$X_1 \cong_k Y(m) \leftarrow Y \to X_2,$$

where $Y(m)$ denotes the m^{th}-Frobenius twist of Y, and $\cong_k$ is a k-isomorphism.

Remark 3.1.1. Let $f : X_1 \to X_2$ be a separable morphism over k, $K_{X_i}, i \in \{1, 2\}$, the function fields of X_i, and X_2^{sep} the normalization of X_2 in K_{X_1}. Then f is an almost open immersion if and only if the natural finite morphism of $X_2^{\text{sep}} \to X_1$ is étale, and the natural morphism $X_1 \to X_2^{\text{sep}}$ induced by f is an open immersion.

On the other hand, if X_1 and X_2 are projective, then f is an almost open immersion if and only if f is an finite étale morphism.

We define

$$\mathrm{Hom}^{\text{al-op-im}}_{\mathcal{C}_k}(X_1, X_2) \subseteq \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$$

if $\mathrm{char}(k) = 0$ and

$$\mathrm{Hom}^{\text{al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) \subseteq \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$$

if $\mathrm{char}(k) = p$ to be the sets of the almost open immersions between X_1 and X_2. On the other hand, we set

$$\mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) := \{\Phi \in \mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) \mid \Phi \text{ satisfies } (\Sigma\text{-gnc})\}.$$

Thus, the natural maps

$$\pi_1^{\Sigma} : \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2) \to \mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = 0$ and

$$\pi_1^{\text{t},\Sigma} : \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2) \to \mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = p$ induce the following natural maps:

$$\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}^{\text{al-op-im}}_{\mathcal{C}_k}(X_1, X_2) \to \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

and

$$\pi_1^{\text{t},\Sigma\text{-gnc}} : \mathrm{Hom}^{\text{al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) \to \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

which fit into the following commutative diagrams:

$$\begin{array}{ccc}
\mathrm{Isom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\text{Isom-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}^{\text{al-op-im}}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\Sigma\text{-gnc}}} & \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\Sigma}} & \mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}),
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Isom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\text{Isom-}\pi_1^{\text{t},\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}^{\text{al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\text{t},\Sigma\text{-gnc}}} & \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\pi_1^{\text{t},\Sigma}} & \mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}),
\end{array}$$

respectively, where all the vertical arrows are injections. Moreover, our main theorem of the present paper is as follows:

Theorem 3.2. *The natural maps*

$$\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}_{\mathcal{C}_k}^{\text{al-op-im}}(X_1, X_2) \xrightarrow{\sim} \mathrm{Hom}_{G_k}^{\text{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = 0$ *and*

$$\pi_1^{\mathrm{t},\Sigma\text{-gnc}} : \mathrm{Hom}_{\mathcal{FC}_k}^{\text{al-op-im}}(X_1, X_2) \xrightarrow{\sim} \mathrm{Hom}_{G_k}^{\text{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = p$ *are bijections.*

Proof. We prove the theorem when $\mathrm{char}(k) = p$. Let us prove $\pi_1^{\mathrm{t},\Sigma\text{-gnc}}$ is a bijection. If $n_{X_2} = 0$, then the theorem follows from Theorem 1.1. Thus, we may assume that $n_{X_2} \geq 0$. First, let us prove that $\pi_1^{\mathrm{t},\Sigma\text{-gnc}}$ is a surjection. Let

$$\Phi \in \mathrm{Hom}_{G_k}^{\text{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2}).$$

To verify the surjectivity, it is sufficient to prove that the image of Φ in

$$\mathrm{Hom}_{G_k}^{\text{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

is induced by an almost immersion of X_1 and X_2. Moreover, Φ is a composite of a open surjection and an open injection. Since any open injection is induced by a finite étale covering, to verify the surjectivity, we may assume that Φ is a surjection. Note that (Σ-gnc) implies that $g := g_{X_1} = g_{X_2}$.

Let $\Sigma_3 \subseteq \mathfrak{Primes}$ be a finite set which contains $\Sigma_1 \cup \{2\}$ and $\Lambda := \mathfrak{Primes} \setminus \Sigma_3$. Then, for each $i \in \{1, 2\}$, we have a surjection

$$\Delta_{X_i} \twoheadrightarrow \Delta_{X_i}^{\Lambda},$$

where $\Delta_{X_i}^{\Lambda}$ denotes the maximal pro-Λ quotient of Δ_{X_i}. We denote by

$$\Pi_{X_i}^{\Lambda} := \Pi_{X_i}/(\mathrm{Ker}(\Delta_{X_i} \twoheadrightarrow \Delta_{X_i}^{\Lambda}))$$

for each $i \in \{1, 2\}$. Then the surjection Φ induces the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{X_1}^{\Lambda} & \longrightarrow & \Pi_{X_1}^{\Lambda} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & {\scriptstyle \overline{\Phi}^{\Lambda}}\big\downarrow & & {\scriptstyle \Phi^{\Lambda}}\big\downarrow & & \big\| & & \\ 1 & \longrightarrow & \Delta_{X_2}^{\Lambda} & \longrightarrow & \Pi_{X_2}^{\Lambda} & \longrightarrow & G_k & \longrightarrow & 1. \end{array}$$

We define a pointed smooth curve over $\overline{k}$ to be

$$\overline{X}_1^{*,\bullet} := (\overline{X}_1^{\mathrm{cpt}}, D_{\overline{X}_1^*} := \lambda_{\overline{\Phi}}(D_{\overline{X}_2})).$$

Let X_1^{cpt} be the smooth compactification of X_1 over k and $D_{X_1^*}$ the image of $D_{\overline{X}_1^*}$ in X_1^{cpt}. We set

$$X_1^* := X_1^{\mathrm{cpt}} \setminus D_{X_1^*}.$$

Note that X_1^* is a hyperbolic curve of type (g, n_{X_2}) over k. Write $\Delta_{X_1^*}$ for the maximal pro-Σ quotient of $\pi_1^{\mathrm{t}}(X_1^* \times_k \overline{k})$, $\Delta_{X_1^*}^{\Lambda}$ for the maximal pro-Λ quotient of $\Delta_{X_1^*}$, $\Pi_{X_1^*}$ for

$\pi_1^{\mathrm{t}}(X_1^*)/(\mathrm{Ker}(\pi_1^{\mathrm{t}}(X_1^* \times_k \overline{k}) \twoheadrightarrow \Delta_{X_1^*})$, and $\Pi_{X_1^*}^{\Lambda}$ for $\Pi_{X_1^*}/(\mathrm{Ker}(\Delta_{X_1^*} \times_k \overline{k}) \twoheadrightarrow \Delta_{X_1^*}^{\Lambda})$. Since X_1 is an open subcurve of X_1^*, we have a natural surjection $\Pi_{X_1} \to \Pi_{X_1^*}$.

Write $D_{\overline{X}_1 \setminus \overline{X}_2}$ for $D_{\overline{X}_1} \setminus \lambda_{\overline{\Phi}}(D_{\overline{X}_2})$, and write E for the subgroup generated by the inertia subgroups associated the inverse images of the elements of $D_{\overline{X}_1 \setminus \overline{X}_2}$ in $D_{\overline{X}_1^{\Sigma}}$. Then the kernel of $\Pi_{X_1} \to \Pi_{X_1^*}$ and $\Delta_{X_1} \twoheadrightarrow \Delta_{X_1^*}$ are E. Moreover, Theorem 2.4 implies that Φ induces the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X_1} & \longrightarrow & \Pi_{X_1} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \Vert & & \\
1 & \longrightarrow & \Delta_{X_1^*} & \longrightarrow & \Pi_{X_1^*} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & {\scriptstyle \overline{\Phi}^*}\downarrow & & \downarrow{\scriptstyle \Phi^*} & & \Vert & & \\
1 & \longrightarrow & \Delta_{X_2} & \longrightarrow & \Pi_{X_2} & \longrightarrow & G_k & \longrightarrow & 1,
\end{array}$$

where all the vertical arrows are surjections. Thus, to verify the surjectivity, it is sufficient to prove that the image of Φ^* in $\mathrm{Hom}_{G_k}(\Pi_{X_1^*}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$ is induced by an isomorphism of $\mathrm{Hom}_{\mathcal{FC}_k}(X_1^*, X_2)$.

On the other hand, Φ^* induces the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X_1^*}^{\Lambda} & \longrightarrow & \Pi_{X_1^*}^{\Lambda} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & {\scriptstyle \overline{\Phi}^{*,\Lambda}}\downarrow & & {\scriptstyle \Phi^{*,\Lambda}}\downarrow & & \Vert & & \\
1 & \longrightarrow & \Delta_{X_2}^{\Lambda} & \longrightarrow & \Pi_{X_2}^{\Lambda} & \longrightarrow & G_k & \longrightarrow & 1,
\end{array}$$

where all the vertical arrows are surjections. Since X_1^* and X_2 are hyperbolic curves of type (g, n_{X_2}), and $p \notin \Lambda$, we obtain that $\overline{\Phi}^{*,\Lambda}$ is an isomorphism. Thus, $\Phi^{*,\Lambda}$ is also an isomorphism. Then Theorem 1.1 implies that the image of $\Phi^{*,\Lambda}$ in $\mathrm{Isom}_{G_k}(\Pi_{X_1^*}^{\Lambda}, \Pi_{X_2}^{\Lambda})/\mathrm{Inn}(\Delta_{X_2}^{\Lambda})$ is induced by an isomorphism of $\mathrm{Isom}_{\mathcal{FC}_k}(X_1^*, X_2)$. This means that Φ^* is an isomorphism. Again, by applying Theorem 1.1, we obtain that the image of Φ^* in

$$\mathrm{Isom}_{G_k}(\Pi_{X_1^*}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

is induced by an isomorphism of $\mathrm{Isom}_{\mathcal{FC}_k}(X_1^*, X_2)$.

Next, let us prove $\pi_1^{\mathrm{t},\Sigma\text{-gnc}}$ is an injection. Let

$$\phi_1, \phi_2 \in \mathrm{Hom}_{\mathcal{FC}_k}^{\text{al-op-im}}(X_1, X_2) \text{ such that } [\Phi'] := \pi_1^{\mathrm{t},\Sigma\text{-gnc}}(\phi_1) = \pi_1^{\mathrm{t},\Sigma\text{-gnc}}(\phi_2).$$

We may assume that ϕ_1 and ϕ_2 are separable k-morphisms. Note that, if ϕ_1 and ϕ_2 are finite étale morphisms, then we obtain immediately $\phi_1 = \phi_2$. Since ϕ_1 and ϕ_2 are compositions of a unique open immersion and a unique finite étale morphism, to verify the injectivity, we may assume that ϕ_1 and ϕ_2 are open immersions. Let $\Phi' \in \mathrm{Hom}_{G_k}^{\text{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})$ such that the image of Φ in $\mathrm{Hom}_{G_k}^{\text{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$ is $[\Phi']$. Then the kernel of Φ' is generated by the inertia subgroups associated to the inverse images of the elements of $D_{\overline{X}_1 \setminus \overline{X}_2}$ in $D_{\overline{X}_1^{\Sigma}}$. Then the injectivity follows immediately from [T2, Lemma 5.1] or [M4, Proposition 1.2].

On the other hand, suppose that $\mathrm{char}(k) = 0$. By replacing Λ and $\mathrm{Hom}^{\text{al-op-im}}_{\mathcal{FC}_k}(X_1, X_2)$ by Σ and $\mathrm{Hom}^{\text{al-op-im}}_{\mathcal{C}_k}(X_1, X_2)$, respectively, similar arguments to the arguments given in the proof of the case where k is a finite field imply that $\pi_1^{\Sigma\text{-gnc}}$ is a bijection. This completes the proof of the theorem. $\square$

Remark 3.2.1. Theorem 3.2 can be regarded as a certain Hom-version of the Grothendieck conjecture for almost open immersion of curves over finite fields.

Remark 3.2.2. We maintain the notations introduced in the proof of Theorem 3.2. Then, if $\mathrm{char}(k) = p$, Theorem 3.2 implies that

$$\mathrm{Hom}^{\text{al-op-im}}_{k}(X_1, X_2) \xrightarrow{\sim} \mathrm{Hom}^{\text{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi^{\mathrm{sol}}_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

$$\xrightarrow{\sim} \mathrm{Hom}^{\text{open},\Lambda\text{-gnc}}_{G_k}(\Pi^{\Lambda}_{X_1}, \Pi^{\Lambda}_{X_2})/\mathrm{Inn}(\Delta^{\Lambda}_{X_2})$$

are bijections.

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