RIMS-1884

Finite Quotients of Fundamental Groups and Moduli Spaces of Curves in Positive Characteristic

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Apr 2018

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Abstract

In the present paper, we study finite quotients of admissible fundamental groups of pointed stable curves over algebraically closed fields of characteristic $p > 0$. Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack over an algebraically closed field k of characteristic $p > 0$ classifying pointed stable curves of type (g, n) and $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$. For each point $q \in \overline{M}_{g,n}$, we denote by Π_q^{adm} the admissible fundamental group of the pointed stable curves determined by q over an algebraically closed field which contains the residue field of q, and denote by $\pi_A^{\mathrm{adm}}(q)$ the set of finite quotients of Π_q^{adm}. For each $G \in \pi_A^{\mathrm{adm}}(q)$, we take $U_G := \{q' \in \overline{M}_{g,n} \mid G \in \pi_A^{\mathrm{adm}}(q')\}$. We prove that U_G is an open subset of $\overline{M}_{g,n}$. By applying this result, we give an alternative proof of a finiteness result for pointed stable curves over $\overline{\mathbb{F}}_p$ which has been proven by the author in a completely different way. Moreover, by using the intersection of certain elements of $\{U_G\}_{G \in \pi_A^{\mathrm{adm}}(q)}$, we formulate the pointed collection conjecture for arbitrary pointed stable curves which is a generalization of the weak Isom-version of the Grothendieck conjecture of pointed stable curves over algebraically closed fields of characteristic $p > 0$.

Keywords: pointed stable curve, moduli space of curve, fundamental group, positive characteristic, anabelian geometry.

Mathematics Subject Classification: Primary 14H30; Secondary 14H32.

Contents

Introduction

Let k be an algebraically closed field of characteristic $p > 0$, $\overline{\mathcal{M}}_{g,n}$ the moduli stack over k classifying pointed stable curves of type (g, n), and $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ the open substack parametrizing smooth pointed stable curves. Write $M_{g,n}$ and $\overline{M}_{g,n}$ for the coarse moduli spaces of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, respectively. Let q be an arbitrary point of $\overline{M}_{g,n}$, $k(q)$ the residue field of q, and l_q an algebraically closed field which contains $k(q)$. Then the natural morphism

$$\mathrm{Spec}\, l_q \to \mathrm{Spec}\, k(q) \to \overline{M}_{g,n}$$

determines a pointed stable curve

$$X^\bullet_{l_q} := (X_{l_q}, D_{X_{l_q}})$$

of type (g, n) over l_q. Here, X_{l_q} denotes the underlying curve of $X^\bullet_{l_q}$, and $D_{X_{l_q}}$ denotes the set of marked points of $X^\bullet_{l_q}$. By choosing a base point of $X^\bullet_{l_q}$, we obtain the admissible fundamental group (which is a generalization of the tame fundamental group of a smooth pointed stable curve to an arbitrary pointed stable curve (cf. Definition 1.2))

$$\Pi^{\mathrm{adm}}_q$$

which only depends on q. The global properties and the structure concerning the admissible fundamental group Π^{adm}_q are very mysterious (e.g. anabelian phenomenons are exist), only a few results are known.

On the other hand, since Π^{adm}_q is a topologically finitely generated profinite group, the isomorphism class of Π^{adm}_q is determined completely by the set of finite quotients of Π^{adm}_q. We denote by

$$\pi^{\mathrm{adm}}_A(q)$$

the set of finite quotients of Π^{adm}_q. Moreover, for each finite group $G \in \Pi^{\mathrm{adm}}_q$, we define a subset of $\overline{M}_{g,n}$ to be

$$U_G := \{q' \in \overline{M}_{g,n} \mid G \in \pi^{\mathrm{adm}}_A(q')\},$$

and take $U^{\mathrm{sm}}_G := U_G \cap M_{g,n}$ when $q \in M_{g,n}$. In the present paper, we are interested in the following question:

Question 0.1. *What is U_G?*

Remark 0.1.1. The specialization theorem of admissible fundamental groups implies that U_G is a dense subset of $\overline{M}_{g,n}$. Moreover, when $n = 0$ and q is a closed point of $M_{g,0}$, K. Stevenson proved that U^{sm}_G contains an open subset of $M_{g,0}$ (cf. [S, Proposition 4.2]).

Before we show our main theorem, let us explain some motivations of the theory developed in the present paper. Some developments of M. Raynaud, F. Pop, M. Saïdi, and A. Tamagawa (cf. [R], [PS], [T1], [T2]) from the 1990's showed evidence for very strong anabelian phenomena for smooth pointed stable curves over algebraically closed fields of characteristic $p > 0$. In this situation, the Galois group of the base field is trivial, and the tame fundamental group coincides with the geometric fundamental group, thus in a total absence of a Galois action of the base field. Note that, in the case of algebraically closed fields of characteristic 0, since the tame fundamental groups of curves depend only on the genera and the cardinality of the sets of cusps, the anabelian geometry of curves does not exist in this situation.

Suppose that $k_q := l_q$ is an algebraic closure of $k(q)$. One of the main problems of the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ is the following conjecture which is called the weak Isom-version of the Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$ (=weak Isom-version).

Conjecture 0.2. *The isomorphism class of $X^{\bullet}_{k_q}$ as a scheme can be determined completely from the isomorphism class of the admissible fundamental group Π^{adm}_q as a profinite group.*

Conjecture 0.2 has only been proven in some special cases (cf. [T1, Theorem 0.2] for the case of smooth pointed stable curves and [Y2, Theorem 0.3 (a)] for the case of pointed stable curves). On the other hand, at the present, almost all of the results concerning Conjecture 0.2 are proved only in the case where $k = k_q = \overline{\mathbb{F}}_p$ is an algebraic closure of the finite field $\mathbb{F}_p$. When $q \in M_{g,n}$, the author reformulated Conjecture 0.2 from the point of view of moduli spaces (cf. [Y2, Conjecture 0.5]), and posed a conjecture (i.e., pointed collection conjecture (cf. [Y2, Conjecture 0.9])) which is a generalization of (the weak Isom-version), and which makes clear the relationship between (weak Isom-version) over $\overline{\mathbb{F}}_p$ and (weak Isom-version) over arbitrary algebraically closed fields of characteristic $p > 0$. The set $\{U^{\mathrm{sm}}_G\}_{G \in \pi^{\mathrm{adm}}_A(q)}$ plays a key role in the formulation of the pointed collection conjecture for smooth pointed stable curves. Moreover, when $g = 0$, the pointed collection conjecture for smooth pointed stable curves holds if one can prove that, for each closed point $t \in M_{0,n}$, $\{U^{\mathrm{sm}}_G\}_{G \in \pi^{\mathrm{adm}}_A(t)}$ is a neighbourhood base of the set

$$\{t' \in M_{0,n} \mid t \sim t'\},$$

where $t \sim t'$ if $X^{\bullet}_{k_t}$ is isomorphic to $X^{\bullet}_{k_{t'}}$ as schemes; then Conjecture 0.2 holds when $g = 0$ and $q \in M_{0,n}$.

In the present paper, we study the set U_G. The main theorem of the present paper is as follows (cf. Theorem 3.6):

Theorem 0.3. *Let $q \in \overline{M}_{g,n}$ be an arbitrary point and $G \in \pi^{\mathrm{adm}}_A(q)$ an arbitrary finite quotient of Π^{adm}_q. Then U_G is an open subset of $\overline{M}_{g,n}$.*

As an application, we obtain an alternative proof of the following finiteness theorem.

Theorem 0.4. *Suppose that $k = \overline{\mathbb{F}}_p$ and q is a closed point. Then there are only finitely many k-isomorphism classes of pointed stable curves over k whose admissible fundamental groups are isomorphic to Π^{adm}_q.*

Remark 0.4.1. Suppose that $q \in M_{g,n}$. Then Theorem 0.4 was proved by Raynaud (cf. [R]) and Pop-Saidi (cf. [PS]) under certain assumptions of Jacobian, and by Tamagawa in the fully general case (cf. [T2]).

Remark 0.4.2. In [Y2, Theorem 0.3 (b)], the author proved Theorem 0.4 in a completely different way (i.e., by using [T2, Theorem 0.3] and the combinatorial Grothendieck conjecture in positive characteristic obtained by the author).

Moreover, by using $\{U_G\}_{G\in\pi_A^{\mathrm{adm}}(q)}$, we formulate the pointed collection conjecture for arbitrary pointed stable curves (cf. Conjecture 4.8) which is a generalization of the pointed collection conjecture for smooth pointed stable curves.

The present paper is organized as follows. In Section 1, we fix some notations and review some definitions which will be used in the present paper. In Section 2 and Section 3, we study the set $\pi_A^{\mathrm{adm}}(q)$ and prove our main theorem. In Section 4, we prove Theorem 0.4 by using Theorem 0.3, and formulate the pointed collection conjecture for arbitrary pointed stable curves.

Acknowledgements

This research was supported by JSPS KAKENHI Grant Number 16J08847.

1 Preliminaries

In this section, we fix some notations and recall some definitions.

Definition 1.1. Let $\mathbb{G} := (v(\mathbb{G}), e(\mathbb{G}), \{\zeta_e^{\mathbb{G}}\}_{e\in e(\mathbb{G})})$ be a semi-graph (cf. [Y1, Section 2]). Here, $v(\mathbb{G})$, $e(\mathbb{G})$, and $\{\zeta_e^{\mathbb{G}}\}_{e\in e(\mathbb{G})}$ denote the set of vertices of $\mathbb{G}$, the set of edges of $\mathbb{G}$, and the set of coincidence maps of $\mathbb{G}$, respectively.

(a) We write $e^{\mathrm{op}}(\mathbb{G}) \subseteq e(\mathbb{G})$ and $e^{\mathrm{cl}}(\mathbb{G}) \subseteq e(\mathbb{G})$ for the set of **open** edges and the set of **closed** edges of $\mathbb{G}$, respectively.

(b) We shall call that $\mathbb{G}$ is **2-connected at** v if $\mathbb{G} \setminus \{v\}$ is either empty or connected for each $v \in v(\mathbb{G})$.

(c) We define an **one-point compactification** $\mathbb{G}^{\mathrm{cpt}}$ of $\mathbb{G}$ as follows: if $e^{\mathrm{op}}(\mathbb{G}) = \emptyset$, we set $\mathbb{G}^{\mathrm{cpt}} = \mathbb{G}$; otherwise, the set of vertices of $\mathbb{G}^{\mathrm{cpt}}$ is $v(\mathbb{G}^{\mathrm{cpt}}) := v(\mathbb{G}) \coprod \{v_\infty\}$, the set of edges of $\mathbb{G}^{\mathrm{cpt}}$ is $e(\mathbb{G}^{\mathrm{cpt}}) := e(\mathbb{G})$, and each edge $e \in e^{\mathrm{op}}(\mathbb{G}) \subseteq e(\mathbb{G}^{\mathrm{cpt}})$ connects v_∞ with the vertex that is abutted by e.

(d) For each $v \in v(\mathbb{G})$, we set

$$b(v) := \sum_{e\in e(\mathbb{G})} b_e(v),$$

where $b_e(v) \in \{0, 1, 2\}$ denotes the number of times that e meets v. Moreover, we set

$$v(\mathbb{G}^{\mathrm{cpt}})^{b\leq 1} := \{v \in v(\mathbb{G}) \subseteq v(\mathbb{G}^{\mathrm{cpt}}) \mid b(v) \leq 1\}.$$

Let D be a scheme, and let

$$X_D^\bullet := (X_D, D_{X_D})$$

be a pointed stable curve of type (g, n) over D. Here, X_D denotes the underlying curve of $X_D^\bullet$ over D, and D_{X_D} denotes the set of marked points of $X_D^\bullet$. Let D' be a scheme and $D' \to D$ a morphism of schemes. We denote by

$$X_{D'}^\bullet := X_D^\bullet \times_D D'$$

the pointed stable curve over D' induced by $X_D^\bullet$ and the morphism $D' \to D$.

Definition 1.2. Let d be an algebraically closed field, $X_d^\bullet$ a pointed stable curve of type (g,n) over d, and

$$f_d^\bullet : Y_d^\bullet \to X_d^\bullet$$

a morphism of pointed stable curves over $\mathrm{Spec}\, d$. We shall call $f_d^\bullet$ a **Galois admissible covering** over $\mathrm{Spec}\, d$ if the following conditions hold:

(i) there exists a finite group $G \subseteq \mathrm{Aut}_d(Y_d^\bullet)$ such that $Y_d^\bullet/G = X_d^\bullet$, and $f_d^\bullet$ is equal to the quotient morphism $Y_d^\bullet \to Y_d^\bullet/G$;

(ii) for each $y \in \mathrm{Sm}(Y_d) \setminus D_{Y_d}$, $f_d^\bullet$ is étale at y, where $\mathrm{Sm}(-)$ denotes the smooth locus of $(-)$;

(iii) for any $y \in \mathrm{Sing}(Y_d)$, the image $f_d^\bullet(y)$ is contained in $\mathrm{Sing}(X_d)$, where $\mathrm{Sing}(-)$ denotes the singular locus of $(-)$;

(iv) for each $y \in \mathrm{Sing}(Y_d)$, the local morphism between two nodes induced by $f_d^\bullet$ may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X_d, f_d^\bullet(y)} \cong d[[u,v]]/uv & \to & \widehat{\mathcal{O}}_{Y_d,y} \cong d[[s,t]]/st \\ u & \mapsto & s^n \\ v & \mapsto & t^n, \end{array}$$

where $(n, \mathrm{char}(d)) = 1$ if $\mathrm{char}(d) > 0$; moreover, write $D_y \subseteq G$ for the decomposition group of y and $\#D_y$ for the cardinality of D_y; then

$$\tau(s) = \zeta_{\#D_y} s \text{ and } \tau(t) = \zeta_{\#D_y}^{-1} t$$

for each $\tau \in D_y$, where $\zeta_{\#D_y}$ is a primitive $\#D_y$-th root of unit;

(v) the local morphism between two marked points induced by $f_d^\bullet$ may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X_d, f_d^\bullet(y)} \cong d[[a]] & \to & \widehat{\mathcal{O}}_{Y_d,y} \cong d[[b]] \\ a & \mapsto & b^m, \end{array}$$

where $(m, \mathrm{char}(d)) = 1$ if $\mathrm{char}(d) > 0$ (i.e., a tamely ramified extension).

Moreover, we shall call $f_d^\bullet$ an **admissible covering** over $\mathrm{Spec}\, d$ if there exists a morphism of pointed stable curves $(f_d^\bullet)' : (Y_d^\bullet)' \to Y_d^\bullet$ over $\mathrm{Spec}\, d$ such that the composite morphism $f_d^\bullet \circ (f_d^\bullet)' : (Y_d^\bullet)' \to X_d^\bullet$ is a Galois admissible covering over $\mathrm{Spec}\, d$. Let $Z_d^\bullet$ be the disjoint union of finitely many pointed stable curves over $\mathrm{Spec}\, d$. We shall call a morphism

$$Z_d^\bullet \to X_d^\bullet$$

over $\mathrm{Spec}\, d$ **multi-admissible covering** over $\mathrm{Spec}\, d$ if the restriction of $Z_d^\bullet \to X_d^\bullet$ to each connected component of $Z_d^\bullet$ is an admissible covering over $\mathrm{Spec}\, d$.

We define a category $\mathrm{Cov}^{\mathrm{adm}}(X_d^\bullet)$ as follows:

(i) the objects of $\mathrm{Cov}^{\mathrm{adm}}(X_d^\bullet)$ are either empty object or the multi-admissible coverings of $X_d^\bullet$ over $\mathrm{Spec}\, d$;

(ii) for any $A, B \in \mathrm{Cov}^{\mathrm{adm}}(X_d^\bullet)$, $\mathrm{Hom}(A, B)$ consists of all the morphisms whose restriction to each connected component of B is a multi-admissible covering over $\mathrm{Spec}\, d$.

It is well-known that $\mathrm{Cov}^{\mathrm{adm}}(X_d^\bullet)$ is a Galois category. Thus, by choosing a base point $x \in \mathrm{Sm}(X_d) \setminus D_{X_d}$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(X_d^\bullet, x)$ which is called the **admissible fundamental group** of $X_d^\bullet$. For simplicity of notation, we omit the base point and denote by

$$\Pi_{X_d^\bullet}$$

the admissible fundamental group of $X_d^\bullet$.

Let d' be an arbitrary field, $\overline{d}'$ an algebraically closure of d', $f_{d'}^\bullet : Y_{d'}^\bullet \to X_{d'}^\bullet$ a morphism of pointed stable curves over d'. We shall call $f_{d'}^\bullet$ an admissible covering (resp. a Galois admissible covering) over d' if the natural morphism

$$f_{\overline{d}'}^\bullet : Y_{\overline{d}'}^\bullet \to X_{\overline{d}'}^\bullet$$

induced by $f_{d'}^\bullet$ is an admissible covering (resp. a Galois admissible covering) over $\overline{d}'$. Let D' be an arbitrary scheme and $f_{D'}^\bullet : Y_{D'}^\bullet \to X_{D'}^\bullet$ a morphism of pointed stable curves over D'. We shall call $f_{D'}^\bullet$ a Galois admissible covering over D' if, for each $d' \in D'$,

$$f_{d'}^\bullet : Y_{d'}^\bullet \to X_{d'}^\bullet$$

is a Galois admissible covering over each d'.

For more details on admissible coverings and the admissible fundamental groups for pointed stable curves, see [M1, Section 3], and [M2, Section 2].

Remark 1.2.1. If $X_d^\bullet$ is smooth over d, by the definition of admissible fundamental groups, then the admissible fundamental group of $X_d^\bullet$ is naturally isomorphic to the tame fundamental group of $X_d \setminus D_{X_d}$.

Remark 1.2.2. Let $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ be the moduli stack over $\mathbb{Z}$ classifying pointed stable curves of type (g, n) and $\mathcal{M}_{g,n,\mathbb{Z}}$ the open substack of $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ parametrizing smooth pointed stable curves. Write $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}^{\log}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ with the natural log structure associated to the divisor with normal crossings

$$\overline{\mathcal{M}}_{g,n,\mathbb{Z}} \setminus \mathcal{M}_{g,n,\mathbb{Z}} \subset \overline{\mathcal{M}}_{g,n,\mathbb{Z}}$$

relative to $\mathrm{Spec}\, \mathbb{Z}$. The pointed stable curve $X_d^\bullet \to \mathrm{Spec}\, d$ induces a morphism $\mathrm{Spec}\, d \to \overline{\mathcal{M}}_{g,n,\mathbb{Z}}$. Write $s_{X_d}^{\log}$ for the log scheme whose underlying scheme is $\mathrm{Spec}\, d$, and whose log structure is the pulling-back log structure induced by the morphism $\mathrm{Spec}\, d \to \overline{\mathcal{M}}_{g,n,\mathbb{Z}}$. We obtain a natural morphism $s_{X_d}^{\log} \to \overline{\mathcal{M}}_{g,n,\mathbb{Z}}^{\log}$ induced by the morphism $\mathrm{Spec}\, d \to \overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ and a stable log curve

$$X_d^{\log} := s_{X_d}^{\log} \times_{\overline{\mathcal{M}}_{g,n,\mathbb{Z}}^{\log}} \overline{\mathcal{M}}_{g,n+1,\mathbb{Z}}^{\log}$$

over $s_{X_d}^{\log}$ whose underlying scheme is X_d. Then the admissible fundamental group $\Pi_{X_d^\bullet}$ of $X_d^\bullet$ is naturally isomorphic to the geometric log étale fundamental group of $X_d^{\log}$ (i.e., $\mathrm{Ker}(\pi_1(X_d^{\log}) \to \pi_1(s_{X_d}^{\log}))$).

From now on, let k be an algebraically closed field of characteristic $p > 0$. Let

$$\overline{\mathcal{M}}_{g,n} := \overline{\mathcal{M}}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} k$$

be the moduli stack over k classifying pointed stable curves of type (g, n) and

$$\mathcal{M}_{g,n} := \mathcal{M}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} k$$

the open substack of $\overline{\mathcal{M}}_{g,n}$ parameterizing smooth pointed stable curves. We denote by $\overline{M}_{g,n}$ and $M_{g,n}$ for the coarse moduli spaces of $\overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}$, $\overline{\pi}_{g,n} : \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ and $\pi_{g,n} : \mathcal{M}_{g,n} \to M_{g,n}$ for the natural morphism, respectively.

If $g = 0$, then $\mathcal{M}_{0,n}$ is a scheme over k. Thus, we have $\mathcal{M}_{0,n} = M_{0,n}$. Moreover, $M_{0,n}$ is a quasi-variety over k. In general, the coarse moduli space is not a fine moduli space. In order to build the family of curves over schemes in general case, we use the level structure. Let $m \geq 3$ be an integer number distinct from p.

Fist, we treat the case where $g = 1$. We denotes by $\mathcal{M}^{(m)}_{1,1,\mathbb{F}_p}$ the moduli stack over $\mathbb{F}_p$ classifying smooth pointed stable curves of type $(1, 1)$ with level m-structure (i.e., the moduli stack of elliptic curve in characteristic p with level m-structure). Moreover, we set

$$M^{(m)}_{1,1} := \mathcal{M}^{(m)}_{1,1} \times_{\mathbb{F}_p} k.$$

There exists a natural covering morphism $\pi^{(m)}_{1,1} : M^{(m)}_{1,1} \to M_{1,1}$. We set

$$M^{(m)}_{1,n} := M^{(m)}_{1,1} \times_{M_{1,1}} M_{1,n}.$$

Then we obtain a natural covering morphism

$$\pi^{(m)}_{1,g} : M^{(m)}_{1,n} \to M_{1,n}$$

determined by the second projection morphism of $M^{(m)}_{1,1} \times_{M_{1,1}} M_{1,n} \to M_{1,n}$. Note that $M^{(m)}_{1,1,\mathbb{F}_p}$ is a quasi-projective varieties over k. For each k-scheme S, $M^{(m)}_{1,n}(S)$ is the set of S-isomorphism classes of smooth pointed stable curves of type $(1, n)$ over S such that the smooth pointed stable curves of type $(1, 1)$ over S obtained by forgetting the last $n - 1$ marked points of the smooth pointed stable curves of type $(1, n)$ are elliptic curves over S with level m-structure.

Next, we suppose that $g \geq 2$. Let $\mathcal{M}^{(m)}_{g,0,\mathbb{F}_p}$ be the moduli stack over $\mathbb{F}_p$ classifying smooth pointed stable curves of type $(g, 0)$ with level m-structure. Moreover, we set

$$M^{(m)}_{g,0} := \mathcal{M}^{(m)}_{g,0} \times_{\mathbb{F}_p} k,$$

and there exists a natural covering morphism $\pi^{(m)}_{g,0} : M^{(m)}_{g,0} \to M_{g,0}$. We set

$$M^{(m)}_{g,n} := M^{(m)}_{g,0} \times_{M_{g,0}} M_{g,n}.$$

Then we obtain a covering morphism

$$\pi^{(m)}_{g,n} : M^{(m)}_{g,n} \to M_{g,n}$$

determined by the second projection of $M_{g,0}^{(m)} \times_{M_{g,0}} M_{g,n} \to M_{g,n}$. Note that $M_{g,n}^{(m)}$ is a quasi-projective variety over k. For each k-scheme S, $M_{g,n}^{(m)}(S)$ is the set of S-isomorphism classes of smooth pointed stable curves of type (g, n) over S whose underlying curve is a curve of genus g over S with level m-structure.

We shall write

$$H_{g,n}$$

for $M_{g,n}^{(m)}$ when $g \geq 1$, and for $M_{0,n}$ when $g = 0$. We use the notation $\pi_{g,n}^{(m)}$ to denote the morphism $\pi_{g,n}^{(m)} : H_{g,n} = M_{g,n}^{(m)} \to M_{g,n}$ when $g \geq 1$, and $\mathrm{id}_{M_{0,n}} : M_{0,n} \to M_{0,n}$ when $g = 0$. Moreover, we shall write

$$X_{H_{g,n}}^{\bullet}$$

for the universal smooth pointed stable curve over $H_{g,n}$ with a level m-structure $\sigma_{H_{g,n}} := \sigma_{H_{g,0}} \times_{H_{g,0}} H_{g,n}$ induced by the level m-structure

$$\sigma_{H_{g,0}} : \mathrm{Pic}^0_{X^{\bullet}_{H_{g,0}}/H_{g,0}}[m] \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^{2g}_{H_{g,0}}$$

when $g \geq 2$, with a level m-structure $\sigma_{H_{1,n}} := \sigma_{H_{1,1}} \times_{H_{1,1}} H_{g,n}$ induced by the level m-structure

$$\sigma_{H_{1,1}} : \mathrm{Pic}^0_{X^{\bullet}_{H_{1,1}}/H_{1,1}}[m] \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^{2g}_{H_{1,1}}$$

when $g = 1$, and with the trivial level m-structure when $g = 0$.

2 The set of finite quotients of admissible fundamental groups

We maintain the notations introduced in Section 1. Let $q \in \overline{M}_{g,n}$ be an arbitrary point, $k(q)$ the residue field of q, and l_q an algebraically closed field which contains $k(q)$. Then the natural morphism

$$\mathrm{Spec}\, l_q \to \mathrm{Spec}\, k(q) \to \overline{M}_{g,n}$$

determines a pointed stable curve

$$X_{l_q}^{\bullet}$$

over l_q. We shall write Γ_q for the dual semi-graph of $X_{l_q}^{\bullet}$ which only depends on q. Since the admissible fundamental group $\Pi_{X_{l_q}^{\bullet}}$ depends only on q (i.e., does not depend on the choices of l_q), we denote by

$$\Pi_q^{\mathrm{adm}}$$

the admissible fundamental group of $X_{l_q}^{\bullet}$. Moreover, we write

$$\pi_A^{\mathrm{adm}}(q)$$

for the set of finite quotients of Π_q^{adm}. Since Π_q^{adm} is a topologically finitely generated profinite group, the isomorphism class of Π_q^{adm} is determined completely by the set of finite quotients $\pi_A^{\mathrm{adm}}(q)$. First, we have the following proposition.

Proposition 2.1. *Let $q_1, q_2 \in \overline{M}_{g,n}$ be arbitrary points such that $q_2 \in \overline{\{q_1\}}$. Then we have*

$$\pi_A^{\mathrm{adm}}(q_2) \subseteq \pi_A^{\mathrm{adm}}(q_1).$$

Proof. The proposition follows immediately from the specialization theorem of admissible fundamental groups of pointed stable curves. □

Lemma 2.2. *Let S be a smooth variety over k, η_S the generic point of S, and $X_S^\bullet$ a smooth pointed stable curve over S. Let $Y_{\eta_S}^\bullet$ be a smooth pointed stable curve over η_S and*

$$f_{\eta_S}^\bullet : Y_{\eta_S}^\bullet \to X_{\eta_S}^\bullet$$

a Galois admissible covering over η_S. Then there exist an open subset $U \subseteq S$ and a morphism

$$f_U^\bullet : Y_U^\bullet \to X_U^\bullet$$

of smooth pointed stable curves over U such that the restriction of $f_U^\bullet$ on η_S is isomorphic to $f_{\eta_S}^\bullet$ over η_S, and $f_U^\bullet$ is a Galois admissible covering over U.

Proof. Write Y_S for the normalization of X_S in the function field of Y_{η_S}, and D_{Y_S} for the set of the topological closures of the elements of $D_{Y_{\eta_S}}$ in Y_S. Furthermore, [Har, Proposition 5] implies that, by replacing S by an open subset of S, we may assume that the fiber $Y_s := Y_S \times_S s$ is geometrically irreducible over each **closed** point $s \in S$.

The normalization $f_S : Y_S \to X_S$ induces a morphism

$$g_S := f_S|_{Y_S \setminus D_{Y_S}} : Y_S \setminus D_{Y_S} \to X_S \setminus D_{X_S}$$

over S. Since the restriction of g_S on the generic fiber η_S is étale, there exists a open subset $U \subseteq S$ such that

$$g_u : Y_S \setminus D_{Y_S} \times_S u \to X_S \setminus D_{X_S} \times_S u$$

is étale at each $u \in U$. Thus, by replacing S by the open subset U, we may assume that g_S is étale. Since the fiber $Y_s := Y_S \times_S s$ is generically smooth over each $s \in S$, Y_s is geometrically irreducible over each point $s \in S$.

Let $X_S^{\log}$ be the log scheme over S whose underlying scheme is X_S, and whose log structure is determined by the marked points of D_{X_S}. Since S is smooth over k, we may check that $X_S^{\log}$ is log regular. Note that f_S is tamely ramified over the generic points of D_{X_S}. Then the log purity (cf. [M3, Theorem B]) implies that g_S extends uniquely to a Galois log étale morphism

$$f_S^{\log} : Y_S^{\log} \to X_S^{\log}$$

over S. We take

$$Y_S^\bullet := (Y_S, D_S),$$

which is a smooth pointed stable curve over S. Then $f_S^{\log}$ induces a morphism

$$f_S^\bullet : Y_S^\bullet \to X_S^\bullet$$

such that the restriction of $f_S^\bullet$ on η_S is equal to $f_{\eta_S}^\bullet$, and

$$f_s : Y_s^\bullet \to X_s^\bullet$$

induced by $f_S^\bullet$ is a connected Galois admissible covering over each $s \in S$. □

Proposition 2.3. *Let $q \in M_{g,n}$ be an arbitrary point, V_q^{sm} the topological closure of q in $M_{g,n}$, and $C \subseteq V_q^{\mathrm{sm,cl}}$ a set of closed points of V_q^{sm}, where $(-)^{\mathrm{cl}}$ denotes the set of closed points of $(-)$. Suppose that C is dense in V_q^{sm}. Then we have*

$$\pi_A^{\mathrm{adm}}(q) = \bigcup_{c \in C} \pi_A^{\mathrm{adm}}(c).$$

Proof. If q is a closed point, then the proposition is trivial. Then we may assume that q is not a closed point. Proposition 2.1 implies that, to verify the proposition, it is sufficient to prove that, for each $G \in \pi_A^{\mathrm{adm}}(q)$, there exists a closed point $c \in C$ such that $G \in \pi_A^{\mathrm{adm}}(c)$.

Let $q^{(m)} \in (\pi_{g,n}^{(m)})^{-1}(q)$ be a point of $H_{g,n}$, $V_{q^{(m)}}$ the topological closure of $q^{(m)}$ in $H_{g,n}$, and $k(q^{(m)})$ the residue field of $q^{(m)}$ which is the function field of $V_{q^{(m)}}$. Write M' for the normalization of $V_{q^{(m)}}$ in $k(q^{(m)})$. Then there exists an open subset of $M \subseteq M'$ such that M is smooth over k. Moreover , the natural morphism

$$M \hookrightarrow M' \to V_{q^{(m)}} \hookrightarrow H_{g,n}$$

determines a smooth pointed stable curve

$$X_M^\bullet := X_{H_{g,n}}^\bullet \times_{H_{g,n}} M$$

over M.

Let k_q be an algebraic closure of $k(q^{(m)})$. By the construction, k_q is also an algebraic closure of $k(q)$, where $k(q)$ denotes the residue field of q. Let

$$Y_{k_q}^\bullet \to X_{k_q}^\bullet$$

be a G-Galois admissible covering (i.e., a Galois admissible covering with Galois group G) over k_q. By replacing $k(q^{(m)})$ by a finite extension l of $k(q^{(m)})$, the G-Galois admissible covering can be descended to a G-Galois admissible covering

$$Y_l^\bullet \to X_l^\bullet$$

over l. Write N for the normalization of M in l, $X_N^\bullet$ for $X_N^\bullet := X_M^\bullet \times_M N$, and $Y_N^\bullet$ for the normalization of $X_N^\bullet$ in the function field of $Y_l^\bullet$. Then we obtain a natural G-Galois covering

$$Y_N^\bullet \to X_N^\bullet$$

such the restriction on generic fibers is isomorphic to the G-Galois admissible covering $Y_l^\bullet \to X_l^\bullet$ over l. Since N is generically smooth over k, by replacing N by an open subset of N, we may assume that N is smooth over k. Thus, Lemma 2.2 implies that there exists an open subset $U \subseteq N$ such that the morphism

$$Y_U^\bullet \to X_U^\bullet$$

is a connected G-admissible covering over each $u \in U$.

We denote by $U_q \subseteq V_q^{\mathrm{sm}}$ the image of U in V_q^{sm}, which is a dense constructible set of V_q^{sm}. Then U_q contains an open subset W_q of V_q^{sm}. Since C is dense in V_q^{sm}, $U_q \cap C \neq \emptyset$. This means that, there exists a closed point $c \in C$ such that $G \in \pi_A^{\mathrm{adm}}(c)$. This completes the proof of the proposition. $\square$

The proof of Proposition 2.3 implies the following corollary.

Corollary 2.4. *We maintain the notations introduced in the proof of Proposition 2.3. Let $f^\bullet_{k_q} : Y^\bullet_{k_q} \to X^\bullet_{k_q}$ be a G-admissible covering over k_q. Then there exist a smooth k-variety U_{q_v} and a finite morphism $U_q \to H_{g,n}$ (not necessary a surjection) such that*

(i) the image of U_q of the composition of the morphisms $U_q \to H_{g,n} \overset{\pi^{(m)}_{g,n}}{\to} M_{g,n}$ is open in V^{sm}_q;

(ii) the morphism $U_q \to H_{g,n}$ induces a smooth pointed stable curve

$$X^\bullet_{U_q} := X^\bullet_{H_{g,n}} \times_{H_{g,n}} U_q$$

over U_q with a level m-structure $\sigma_{U_q} := \sigma_{H_{g,n}} \times_{H_{g,n}} U_q$;

(iii) there exists a G-Galois covering $f^\bullet_{U_q} : Y^\bullet_{U_q} \to X^\bullet_{U_q}$ of smooth pointed stable curves over U_q such that $f^\bullet_{U_q} \times_{U_q} \operatorname{Spec} k_q$ is isomorphic to $f^\bullet_{k_q}$ over k_q, and $f^\bullet_{U_q}$ is a G-admissible covering over U_q.

In the remainder of this section, we extend Proposition 2.3 to the case where $q \in \overline{M}_{g,n}$.

Lemma 2.5. *Let S be a k-variety and $s_1, s_2 \in S$ two points such that $s_1 \neq s_2$ and $s_2 \in \overline{\{s_1\}}$. Then there exist a complete discrete valuation ring R and a morphism $\operatorname{Spec} R \to S$ such that the image of the morphism (as a set) is $\{s_1, s_2\}$.*

Proof. It is easy to see that we may assume that s_1 is the generic point of S, and s_2 is a closed point of S. If $\dim(S) = 1$, then the lemma is trivial. We may assume that $\dim(S) \geq 2$.

Let $\overline{s}_1$ be a geometric point over s_1. Write $\overline{S}$ for $S \times_S \overline{s}_1$. Then the natural morphism $\overline{s}_1 \to s_1 \to S$ and $s_2 \to S$ induces a morphism $f_1 : \overline{s}_1 \to \overline{S}$ and $f_2 : s_2 \to \overline{S}$, respectively. We denote by s'_1 the image (as as set) of f_1, and denote by s'_2 the image (as a set) of f_2. Note that s'_1, s'_2 are closed points of $\overline{S}$ and $s'_1 \neq s'_2$. Then there exists a curve $C \subseteq \overline{S}$ which contains s'_1, s'_2. Write η_C for the generic point of C. Thus, the image (as a set) of the composition of the morphisms $\eta_C \hookrightarrow C \hookrightarrow \overline{S} \to S$ is s_1.

There is a complete discrete valuation ring R and a morphism $\operatorname{Spec} R \to C$ such that the image of the morphism (as a set) is $\{\eta_C, s'_2\}$. Then the desired morphism is the composition of the morphisms

$$\operatorname{Spec} R \to C \hookrightarrow \overline{S} \to S.$$

This completes the proof of the lemma. □

Lemma 2.6. *Let R be a complete discrete valuation, K_R the quotient field of R, and k_R the residue field of R such that k_R is an algebraically closed field. Let*

$$f^\bullet_{K_R} : Y^\bullet_{K_R} \to X^\bullet_{K_R}$$

be a morphism of pointed stable curves over K_R. Write $\Gamma_{X^\bullet_{K_R}}$ for the dual semi-graph of $X^\bullet_{K_R}$,

$$\mathrm{nl}_v : X_{K_R,v} \to X'_{K_R,v}$$

for the normalization of the irreducible component $X'_{K_R,v}$ of X_{K_R} corresponding to each $v \in v(\Gamma_{X^\bullet_{K_R}})$, $\Gamma_{Y^\bullet_{K_R}}$ for the dual semi-graph of $Y^\bullet_{K_R}$, and

$$\mathrm{nl}_w : Y_{K_R,w} \to Y'_{K_R,w}$$

for the normalization of the irreducible component $Y'_{K_R,w}$ of Y_{K_R} corresponding to each $w \in v(\Gamma_{Y^\bullet_{K_R}})$. Suppose that

$$D_{X_{K_R,v}} := (D_{X_{K_R}} \cap X_{K_R,v}) \cup (X_{K_R,v} \cap (\mathrm{Sing}(X_{K_R}) \setminus \mathrm{Sing}(X'_{K_R}))) \cup (\mathrm{nl}_v)^{-1}(\mathrm{Sing}(X'_{K_R}))$$

of X_{K_R} is a set of K_R-rational points of $X_{K_R,v}$, and that

$$D_{Y_{K_R,w}} := (D_{Y_{K_R}} \cap Y_{K_R,w}) \cup (Y_{K_R,w} \cap (\mathrm{Sing}(Y_{K_R}) \setminus \mathrm{Sing}(Y'_{K_R}))) \cup (\mathrm{nl}_w)^{-1}(\mathrm{Sing}(Y'_{K_R}))$$

of Y_{K_R} is a set of K_R-rational points of $Y_{K_R,w}$ for each $w \in v(\Gamma_{Y_{K_R}})$. We define two smooth pointed stable curve

$$X^\bullet_{K_R,v} := (X_{K_R,v}, D_{X_{K_R,v}}) \text{ and } Y^\bullet_{K_R,w} := (Y_{K_R,w}, D_{Y_{K_R,w}})$$

of type (g_v, n_v) and (g_w, n_w) for each $v \in v(\Gamma_{X^\bullet_{K_R}})$ and each $w \in v(\Gamma_{Y^\bullet_{K_R}})$ over K_R, respectively. Moreover, suppose that, for each $v \in v(\Gamma_{X^\bullet_{K_R}})$ and each $w \in v(\Gamma_{Y^\bullet_{K_R}})$, $X^\bullet_{K_R,v}$ and $Y^\bullet_{K_{R,w}}$ have good reduction over R, and that $f^\bullet_{K_R}$ is a G-admissible covering over K_R. Then there exists a morphism

$$f^\bullet_R : Y^\bullet_R \to X^\bullet_R$$

of pointed stable curves over R such that $f^\bullet_R$ is a G-admissible covering over R, and that the restriction $f^\bullet_{k_R} := f^\bullet_R \times_R k_R$ of $f^\bullet_R$ on the special fibers is a G-admissible covering over k_R.

Proof. For each $w \in v(\Gamma_{Y^\bullet_{K_R}})$, the smooth pointed stable curve $Y^\bullet_{K_R}$ over K_R determines a morphism

$$c_{Y^\bullet_{K_R,w}} : \mathrm{Spec}\, K_R \to \mathcal{M}_{g_w,n_w,\mathbb{Z}}.$$

Suppose that $Y^\bullet_{K_R}$ is a pointed stable curve of type (g_Y, n_Y) over K_R. Write $c_{Y^\bullet_{K_R}}$: $\mathrm{Spec}\, K_R \to \overline{\mathcal{M}}_{g_Y,n_Y,\mathbb{Z}}$ for the morphism determined by $Y^\bullet_{K_R}$ over K_R. Then the pointed stable curve $Y^\bullet_{K_R}$ determines a clutching morphism

$$\kappa_{Y^\bullet_{K_R}} : \coprod_{w \in v(\Gamma_{Y^\bullet_{K_R}})} \mathcal{M}_{g_w,n_w,\mathbb{Z}} \to \overline{\mathcal{M}}_{g_Y.n_Y,\mathbb{Z}}$$

such that the composition of morphisms $\kappa_{Y^\bullet_{K_R}} \circ (\bigtimes_{w \in v(\Gamma_{Y^\bullet_{K_R}})} c_{Y^\bullet_{K_R,w}}) = c_{Y^\bullet_{K_R}}$. For each $w \in v(\Gamma_{Y^\bullet_{K_R}})$, we denote by $Y^\bullet_{R,w}$ the smooth pointed stable curve of type (g_w, n_w) over R induced by $Y^\bullet_{K_R,w}$. Then, by using the clutching morphism $\kappa_{Y^\bullet_{K_R}}$, we may glue the pointed stable curves $\{Y^\bullet_{R,w}\}_{w \in v(\Gamma_{Y^\bullet_{K_R}})}$ and obtain a pointed stable curve $Y^\bullet_R$ over R.

Since $Y^\bullet_{K_R}$ admits an action of G, this action induces an action of G on the pointed stable curve $Y^\bullet_R$. Let $Z^\bullet_R := Y^\bullet_R/G$, $f^\bullet_R : Y^\bullet_R \to Z^\bullet_R$ the quotient morphism, $Z^\bullet_{K_R}$ the generic fiber over K_R, and $Z^\bullet_{k_R}$ the special fiber over k_R. [L, Proposition 10.3.48] implies $Z^\bullet_R$ is a pointed semi-stable curve over R. Since $f^\bullet_{K_R}$ is a G-admissible covering over K_R, $Z^\bullet_{K_R}$ is isomorphic to $X^\bullet_{K_R}$ over K_R.

On the other hand, write $\gamma_{f^\bullet_{K_R}} : \Gamma_{Y^\bullet_{K_R}} \to \Gamma_{X^\bullet_{K_R}}$ for the morphism of dual semi-graphs induced by $f^\bullet_{K_R}$. Note that, for each $v \in v(\Gamma_{X^\bullet_{K_R}})$ and each $w \in \gamma^{-1}_{f^\bullet_{K_R}}(v)$, $f^\bullet_{K_R}$ induces a G-admissible covering $f^\bullet_{R,w} : Y^\bullet_{R,w} \to X^\bullet_{R,v}$ of smooth pointed stable curves over R. Then we obtain $Y^\bullet_{R,w}/G \cong X^\bullet_{R,v}$ over R. This implies that $Z^\bullet_{k_R}$ is a pointed stable curve over k_R. Then we have $X^\bullet_R \cong Z^\bullet_R$ over R. We complete the proof of the lemma. $\square$

Proposition 2.7. *Let $q \in \overline{M}_{g,n}$ be an arbitrary point, V_q the topological closure of q in $\overline{M}_{g,n}$, and $G \in \pi_A^{\mathrm{adm}}(q)$ a finite group. Then there exists a closed point $c \in V_q^{\mathrm{cl}}$ such that Γ_q is isomorphic to Γ_c, and that $G \in \pi_A^{\mathrm{adm}}(c)$.*

Proof. If q is a closed point, then the proposition is trivial. Then we may assume that q is not a closed point. If $q \in M_{g,n}$, then the proposition follows form Proposition 2.3. Then we may assume that $q \in \overline{M}_{g,n} \setminus M_{g,n}$.

The natural morphism

$$\mathrm{Spec}\, k_q \to \mathrm{Spec}\, k(q) \to \overline{M}_{g,n}$$

determines a pointed stable curve

$$X^\bullet_{k_q}$$

over k_q. For each $v \in v(\Gamma_q)$, write

$$\mathrm{nl}_v : X_{k_q,v} \to X'_{k_q,v}$$

for the normalization of the irreducible component $X'_{k_q,v}$ of X_{k_q} corresponding to v. Let $D_{X_{k_q,v}}$ be a set of closed points

$$(D_{X_{k_q}} \cap X_{k_q,v}) \cup (X_{k_q,v} \cap (\mathrm{Sing}(X_{k_q}) \setminus \mathrm{Sing}(X'_{k_q}))) \cup (\mathrm{nl}_v)^{-1}(\mathrm{Sing}(X'_{k_q}))$$

for each $v \in v(\Gamma_q)$, where $\mathrm{Sing}(-)$ denotes the set of singular points of $(-)$. We define a smooth pointed stable curve

$$X^\bullet_{k_q,v} := (X_{k_q,v}, D_{X_{k_q,v}})$$

of type (g_v, n_v) over k_q for each $v \in v(\Gamma_q)$.

Let $Y^\bullet_{k_q}$ be a pointed stable curve of type (g_Y, n_Y) over k_q,

$$f^\bullet_{k_q} : Y^\bullet_{k_q} \to X^\bullet_{k_q}$$

a G-admissible covering over k_q, $\Gamma_{Y^\bullet_{k_q}}$ the dual semi-graph of $Y^\bullet_{k_q}$, and $\gamma_{f^\bullet_{k_q}} : \Gamma_{Y^\bullet_{k_q}} \to \Gamma_q$ the morphism of dual semi-graphs induced by $f^\bullet_{k_q}$. Note that $\gamma_{f^\bullet_{k_q}}$ does not depends on the choices of k_q. For each $v \in v(\Gamma_q)$, write I_v for the set $\gamma^{-1}_{f^\bullet_{k_q}}(v)$. Then $f^\bullet_{k_q}$ and the natural morphism of underlying curves $X_{k_q,v} \to X_{k_q}$ induce a Galois multi-admissible covering

$$f^\bullet_{k_q,v} : \coprod_{w \in I_v} Y^\bullet_{k_q,w} \to X^\bullet_{k_q,v}$$

over k_q with Galois group G, where $Y^\bullet_{k_q,w}$, $w \in I_v$, is a smooth pointed stable curve of type $(g_{Y,w}, n_{Y,w})$ over k_q whose underlying curve is a normalization of the irreducible component of Y_{k_q} corresponding to w. Note that $\coprod_{w\in I_v} Y^\bullet_{k_q,w}$ admits an action of G induced by the action of G on $Y^\bullet_{k_q}$. This action induces an action of G on the set I_v. For each $w \in I_v$, write G_w for the inertia subgroup of w. Then we obtain a G_w-admissible covering

$$f^\bullet_{k_q,w} : Y^\bullet_{k_q,w} \to X^\bullet_{k_q,v}$$

over k_q.

The pointed stable curves $X^\bullet_{k_q}$, $\{X^\bullet_{k_q,v}\}_{v\in v(\Gamma_q)}$, $Y^\bullet_{k_q}$, and $\{Y^\bullet_{k_q,w}\}_{w\in v(\Gamma_{Y^\bullet_{k_q}})}$ over k_q determine morphisms $c_{X^\bullet_{k_q}} : \operatorname{Spec} k_q \to \overline{\mathcal{M}}_{g,n}$, $\{c_{X^\bullet_{k_q,v}} : \operatorname{Spec} k_q \to \mathcal{M}_{g_v,n_v}\}_{v\in v(\Gamma_q)}$, $c_{Y^\bullet_{k_q}}$: $\operatorname{Spec} k_q \to \overline{\mathcal{M}}_{g_Y,n_Y}$, and $\{c_{Y^\bullet_{k_q,w}} : \operatorname{Spec} k_q \to \mathcal{M}_{g_{Y,w},n_{Y,w}}\}_{w\in v(\Gamma_{Y^\bullet_{k_q}})}$, respectively. Then the pointed stable curves $X^\bullet_{k_q}$ and $Y^\bullet_{k_q}$ induce two clutching morphisms as follows:

$$\kappa_{X^\bullet_{k_q}} : \prod_{v\in v(\Gamma_q)} \mathcal{M}_{g_v,n_v} \to \overline{\mathcal{M}}_{g,n}$$

and

$$\kappa_{Y^\bullet_{k_q}} : \prod_{w\in v(\Gamma_{Y^\bullet_{k_q}})} \mathcal{M}_{g_w,n_w} \to \overline{\mathcal{M}}_{g,n}$$

such that $\kappa_{X^\bullet_{k_q}} \circ (\bigtimes_{v\in v(\Gamma_q)} c_{X^\bullet_{k_q,v}}) = c_{X^\bullet_{k_q}}$ and $\kappa_{Y^\bullet_{k_q}} \circ (\bigtimes_{w\in v(\Gamma_{Y^\bullet_{k_q}})} c_{Y^\bullet_{k_q,w}}) = c_{Y^\bullet_{k_q}}$.

On the other hand, the smooth pointed stable curve $X^\bullet_{k_q}$, $v \in v(\Gamma_q)$, over k_q determines a morphism

$$\operatorname{Spec} k_q \to M_{g_v,n_v},$$

and we denote by $q_v \in M_{g_v,n_v}$ for the image of the morphism. Write $V^{\mathrm{sm}}_{q_v}$ for the topological closure of q_v in M_{g_v,n_v}. Let k_{q_v} be an algebraically closure of the residue field $k(q_v)$ of q_v. Since the admissible coverings over algebraically closed fields do not depends on the choices of base fields, $f^\bullet_{k_q,w}$ induces a G_w-admissible covering

$$f^\bullet_{k_{q_v},w} : Y^\bullet_{k_{q_v},w} \to X^\bullet_{k_{q_v},v}$$

over k_{q_v}. Then Corollary 2.4 implies that there exist a smooth k-variety U_{q_v} and a finite morphism $U_{q_v} \to H_{g_v,n_v}$ (not necessary a surjection) such that

(i) the image of U_{q_v} of the composition of the morphisms $U_{q_v} \to H_{g_v,n_v} \overset{\pi^{(m)}_{g_v,n_v}}{\to} M_{g_v,n_v}$ is open in $V^{\mathrm{sm}}_{q_v}$;

(ii) the morphism $U_{q_v} \to H_{g_v,n_v}$ induces a smooth pointed stable curve

$$X^\bullet_{U_{q_v},v} := X^\bullet_{H_{g_v,n_v}} \times_{H_{g_v,n_v}} U_{q_v}$$

over U_{q_v} with a level m-structure $\sigma_{U_{q_v}} := \sigma_{H_{g_v,n_v}} \times_{H_{g_v,n_v}} U_{q_v}$;

(iii) for each $w \in I_v$, there exists a G-Galois covering $f^\bullet_{U_{q_v},w} : Y^\bullet_{U_{q_v},w} \to X^\bullet_{U_{q_v},v}$ of smooth pointed stable curves over U_{q_v} such that $f^\bullet_{U_{q_v},w} \times_{U_{q_v}} \operatorname{Spec} k_{q_v}$ is $f^\bullet_{k_{q_v},w}$, and $f^\bullet_{U_{q_v},w}$ is a G_w-admissible covering over U_{q_v}.

The clutching morphism induces a morphism

$$\kappa : \prod_{v \in v(\Gamma_q)} U_{q_v} \to \prod_{v \in v(\Gamma_q)} H_{g_v, n_v} \to \prod_{v \in v(\Gamma_q)} \mathcal{M}_{g_v, n_v} \overset{\kappa_{X^\bullet_{k_q}}}{\to} \overline{\mathcal{M}}_{g,n} \overset{\overline{\pi}_{g,n}}{\to} \overline{M}_{g,n}$$

over k. Since the image of κ is a dense constructible subset of V_q, the image of κ contains an open subset U_q of V_q.

Let c be a closed point of U_q. Then Lemma 2.5 implies that there exist a complete discrete valuation ring R, whose residue field is an algebraically closed field, and a morphism

$$\operatorname{Spec} R \to V_q$$

such that the image of the morphism (as a set) is $\{q, c\}$. By replacing R by a finite extension of R, there is a pointed stable curve

$$X^\bullet_R$$

over R. Write K_R for the quotient field of R, $\overline{K}_R$ for an algebraically closure of K_R, and k_R for the residue field of R. We may assume that $\overline{K}_R$ contains k_q. For each $v \in v(\Gamma_q)$, the smooth pointed stable curve

$$X^\bullet_{\overline{K}_R, v} := X^\bullet_{k_q, v} \times_{k_q} \overline{K}_R$$

of type (g_v, n_v) over $\overline{K}_R$ determines a morphism $\operatorname{Spec} \overline{K}_R \to \mathcal{M}_{g_v, n_v}$. Thus, we choose a morphism

$$\operatorname{Spec} \overline{K}_R \to H_{g_v, n_v}$$

induced by the morphism $\operatorname{Spec} \overline{K}_R \hookrightarrow \coprod \operatorname{Spec} \overline{K}_R = \operatorname{Spec} \overline{K}_R \times_{\mathcal{M}_{g_v, n_v}} H_{g_v, n_v} \to H_{g_v, n_v}$. The morphism $\operatorname{Spec} \overline{K}_R \to H_{g_v, n_v}$ above induces a level m-structure

$$\sigma_{\overline{K}_R} := \sigma_{H_{g_v, n_v}} \times_{H_{g_v, n_v}} \operatorname{Spec} \overline{K}_R.$$

By replacing R by a finite extension of R, $X^\bullet_{\overline{K}_R, v}$ descents to a smooth pointed stable curve $X^\bullet_{K_R, v}$ over K_R, and the level m-structure $\sigma_{\overline{K}_R}$ descents to a level m-structure σ_{K_R} on the smooth pointed stable curve $X^\bullet_{K_R, v}$ over K_R. Write

$$X^\bullet_{R, v}$$

for the pointed stable model over R. Note that, by the construction, $X^\bullet_{R, v}$ is smooth over R. Then the level m-structure σ_{K_R} extends to a level m-structure σ_R. Thus, for each $v \in v(\Gamma_q)$, the smooth pointed stable curve $X^\bullet_{R, v}$ over R with the level m-structure σ_R determines a morphism

$$\operatorname{Spec} R \to H_{g_v, n_v}$$

such that the image (as a set) of the composition morphism

$$\operatorname{Spec} R \to \prod_{v \in v(\Gamma_q)} H_{g_v, n_v} \to \prod_{v \in v(\Gamma_q)} \mathcal{M}_{g_v, n_v} \overset{\kappa_{X^\bullet_{k_q}}}{\to} \overline{\mathcal{M}}_{g,n} \overset{\overline{\pi}_{g,n}}{\to} \overline{M}_{g,n}$$

is $\{q, c\}$. Moreover, by choosing a suitable level m-structure (or the morphism $\operatorname{Spec} \overline{K}_R \to H_{g_v,n_v}$), we may assume that the image (as a set) of $\operatorname{Spec} R \to \prod_{v \in v(\Gamma_q)} H_{g_v,n_v}$ is contained in the image (as a set) of $\prod_{v \in v(\Gamma_q)} U_{q_v} \to \prod_{v \in v(\Gamma_q)} H_{g_v,n_v}$. Since the morphism $\prod_{v \in v(\Gamma_q)} U_{q_v} \to \prod_{v \in v(\Gamma_q)} H_{g_v,n_v}$ is finite, by replacing R by a finite extension of R, we may assume that the morphism $\operatorname{Spec} R \to \prod_{v \in v(\Gamma_q)} H_{g_v,n_v}$ obtained above is a composition of a morphism

$$\operatorname{Spec} R \to \prod_{v \in v(\Gamma_q)} U_{q_v}$$

and the natural morphism

$$\prod_{v \in v(\Gamma_q)} U_{q_v} \to \prod_{v \in v(\Gamma_q)} H_{g_v,n_v}.$$

Thus, for each $v \in v(\Gamma_q)$ and each $w \in I_v$, we obtain a G_w-Galois covering

$$f^\bullet_{R,w} := f^\bullet_{U_{q_v},w} \times_{U_{q_v}} \operatorname{Spec} R : Y^\bullet_{R,w} := Y^\bullet_{U_{q_v},w} \times_{U_{q_v}} \operatorname{Spec} R \to X^\bullet_{R,v} := X^\bullet_{U_{q_v},w} \times_{U_{q_v}} \operatorname{Spec} R$$

of smooth pointed stable curves over R such that $f^\bullet_{R,w}$ is G_w-admissible covering over $\operatorname{Spec} R$. Moreover, the clutching morphism $\kappa_{Y^\bullet_{k_q}}$ implies that we may glue $\{Y^\bullet_{R,w}\}_w$ along the marked points and obtain a pointed stable curve

$$Y^\bullet_R$$

over R such that

(i) $Y^\bullet_R \times_{K_R} \overline{K}_R \cong Y^\bullet_{k_q} \times_{k_q} \overline{K}_R$ over $\overline{K}_R$;

(ii) there exists a morphism $f^\bullet_{K_R} : Y^\bullet_{K_R} \to X^\bullet_{K_R}$ of pointed stable curves over K_R which is a G-admissible covering over K_R such that $f^\bullet_{K_R} \times_{K_R} \overline{K}_R$ isomorphic to $f^\bullet_{k_q} \times_{k_q} \overline{K}_R$.

Then Lemma 2.6 implies that there exists a G-admissible covering $f^\bullet_R : Y^\bullet_R \to X^\bullet_R$ such that the restriction of $f^\bullet_R$ on the special fibers is a connected G-admissible covering over k_R. This means that $G \in \pi^{\mathrm{adm}}_A(c)$. We completes the proof of the proposition. $\square$

Definition 2.8. Let $q \in \overline{M}_{g,n}$ be an arbitrary point. For each $G \in \pi^{\mathrm{adm}}_A(q)$, we define

$$U_G := \{q' \in \overline{M}_{g,n} \mid G \in \pi^{\mathrm{adm}}_A(q')\}.$$

Moreover, we take

$$U^{\mathrm{sm}}_G := U_G \cap M_{g,n}.$$

3 The openness of U_G in $\overline{M}_{g,n}$

We maintain the notations introduced in the previous sections. In this section, we prove U_G is an open subset of $\overline{M}_{g,n}$.

3.1 $M_{g,n}$ case

First, let us prove that U_G^{sm} is an open subset of $M_{g,n}$.

Lemma 3.1. *Let v be a closed point of $H_{g,n}$, $\widehat{\mathcal{O}}_{H_{g,n},v}$ the completion of the local ring $\mathcal{O}_{H_{g,n},v}$, $\widehat{V} = \operatorname{Spec}\widehat{\mathcal{O}}_{H_{g,n},v}$ with the natural morphism $\widehat{V} \to H_{g,n}$, and $X^\bullet_{\widehat{V}}$ the smooth pointed stable curve $X^\bullet_{H_{g,n}} \times_{H_{g,n}} \widehat{V}$ over $\widehat{V}$ with a level m-structure $\sigma_{\widehat{V}} := \sigma_{H_{g,n}} \times_{H_{g,n}} \widehat{V}$ induced by $\sigma_{H_{g,n}}$. Let $Y^\bullet_{\widehat{V}}$ be a smooth pointed stable curve over $\widehat{V}$ and*

$$f^\bullet_{\widehat{V}} : Y^\bullet_{\widehat{V}} \to X^\bullet_{\widehat{V}}$$

be G-Galois covering such that $f^\bullet_{\widehat{V}}$ is a G-Galois admissible covering over $\widehat{V}$. Then there exists a subring $A \subseteq \widehat{\mathcal{O}}_{H,v}$, a morphism $\alpha_E : E := \operatorname{Spec} A \to H$, and a G-Galois covering $f^\bullet_E : Y^\bullet_E \to X^\bullet_E := X^\bullet_H \times_H E$ such that the following conditions hold:

(a) $X^\bullet_E \times_E \widehat{V}$ is isomorphic to $X^\bullet_{\widehat{V}}$ over $\widehat{V}$, and the pulling-back of $f^\bullet_E \times_E \widehat{V}^\bullet$ via the natural morphism $\widehat{V} \to E$ is isomorphic to $f^\bullet_{\widehat{V}}$ over $\widehat{V}$;

(b) $f^\bullet_E$ is a connected G-admissible covering over each $e \in E$.

Proof. By applying [V, Proposition 4.3 (1)], there exists a subring $A' \subseteq \mathcal{O}_{H,v}$ which is of finite type over k such that the Galois covering $f^\bullet_{\widehat{V}}$ descents to a Galois covering

$$f^\bullet_{E'} : Y^\bullet_{E'} \to X^\bullet_{E'}$$

over $E' := \operatorname{Spec} A'$ with a level m-structure $\sigma_{\widehat{V}}$ on $X^\bullet_E$, and that the restriction of $f^\bullet_{E'}$ on each $e' \in E'$ is a G-admissible covering over e'. Moreover, by the construction, the pulling-back $f^\bullet_{E'} \times_{E'} \widehat{V}$ via $\widehat{V} \to E'$ is isomorphic to $f^\bullet_{\widehat{V}}$ over $\widehat{V}$. The smooth pointed stable curve $X^\bullet_{E'}$ over E' determines a morphism $\alpha_{E'} : E' \to H_{g,n}$.

We denote by $v_{E'} \in E'$ the image of $v \in \widehat{V}$ via the natural morphism $\widehat{V} \to E'$ which is a closed point of E'. [Har, Proposition 5] implies that, there exists by replacing E' by an affine open subset

$$v_{E'} \in E := \operatorname{Spec} A \subseteq E',$$

the fiber $Y^\bullet_e := Y^\bullet_E \times_E e$ is geometrically irreducible over each **closed** point $e \in E$, where $A \subseteq \widehat{\mathcal{O}}_{H,v}$. Moreover, since the underlying curve of the fiber $Y^\bullet_e := Y^\bullet_E \times_E e$ is smooth over each e, we have that $Y^\bullet_e$ is geometrically irreducible over each point $e \in E$. Thus, for each point $e \in E$, the restriction of $f^\bullet_E := f^\bullet_{E'}|_E$ on e is a connected G-admissible covering over e. We define $\alpha_E := \alpha_{E'}|_E : E \to H_{g,n}$. Then we obtain the desired curve and complete the proof of the proposition. □

Theorem 3.2. *Let q be an arbitrary point of $M_{g,n}$ and $G \in \pi_A^{\mathrm{adm}}(q)$. Then U_G^{sm} is an open subset of $M_{g,n}$.*

Proof. To verify the theorem, Proposition 2.3 (or Proposition 2.7) implies that it is sufficient to prove that, for each **closed point** $c \in U_G^{\mathrm{sm}}$, there exists an open subset $c \in U_c \subseteq M_{g,n}$ which is contained in U_G^{sm}.

Let $v \in H_{g,n}$ be a closed point such that $\pi_{g,n}^{(m)}(v) = c$. We maintain the notations introduced in Lemma 3.1. Then we obtain an affine k-variety E and a morphism

$$\alpha_E : E \to H_{g,n}$$

over k such that $(\pi_{g,n}^{(m)} \circ \alpha_E)(v_{E'}) = c$. Moreover, since the image $\widehat{W}$ of the composition of morphisms

$$\widehat{V} \to E \stackrel{\alpha_E}{\to} H_{g,n} \stackrel{\pi_{g,n}^{(m)}}{\to} M_{g,n}$$

is dense in $M_{g,n}$, the image of the composition of morphisms

$$E \stackrel{\alpha_E}{\to} H_{g,n} \stackrel{\pi_{g,n}^{(m)}}{\to} M_{g,n}$$

is a dense constructible subset of $M_{g,n}$.

Write W for the image of E in $M_{g,n}$. Since W is constructible subset, we have

$$W = \bigcup_{i=1}^{r} W_i$$

is a finite disjoint union of local closed subsets W_i, $i = 1, \dots, r$, of $M_{g,n}$. Without loss of generality, we may assume that $c \in W_1$. Since W_1 contains the image of $\widehat{W}$, we obtain that W_1 is an open subset of $M_{g,n}$. This completes the proof of the theorem. $\square$

Remark 3.2.1. In [S, Section 4], Stevenson proved that U_G^{sm} contains an open subset of $M_{g,n}$ when $n = 0$.

3.2 $\overline{M}_{g,n}$ case

In this subsection, we generalizes Theorem 3.2 to the case of an arbitrary point $q \in \overline{M}_{g,n}$ and U_G.

Lemma 3.3. *Let R be a complete discrete valuation ring, K_R the quotient field of R of characteristic $p > 0$, and k_R the residue field of R such that k_R is an algebraically closed field. Let $X_R^\bullet$ be a pointed stable curve of type (g, n) over R and*

$$f_{k_R}^\bullet : Y_{k_R}^\bullet \to X_{k_R}^\bullet$$

a G-admissible covering over k_R. Then, by replacing R by a finite extension of R, there exist a pointed stable curve $Y_R^\bullet$ over R and a G-admissible covering

$$f_R^\bullet : Y_R^\bullet \to X_R^\bullet$$

over R such that the restriction of $f_R^\bullet$ on the special fibers $f_R^\bullet \times_R k_R$ is isomorphic to $f_{k_R}^\bullet$ over k_R.

Proof. Let $X^\bullet_{\mathcal{M}'}$ be the versal formal deformation of the special fiber $X^\bullet_{k_R}$ of $X^\bullet_R$ over

$$\mathcal{M}' = \operatorname{Spec} \mathcal{O}_k[[t_1, \ldots, t_{3g-3+n}]],$$

where $\mathcal{O}_k$ is a regular local ring with maximal ideal $p\mathcal{O}_k$ and residue field k_R (cf. [DM, p79]). The pointed stable curve $X^\bullet_R$ over R determines a morphism

$$\operatorname{Spec} R \to \mathcal{M}'$$

such that $X^\bullet_{\mathcal{M}'} \times_{\mathcal{M}'} \operatorname{Spec} R$ is isomorphic to $X^\bullet_R$ over R. Moreover, since $R \cong k_R[[t]]$, the morphism $\operatorname{Spec} R \to \mathcal{M}'$ induces a morphism

$$\operatorname{Spec} R \to \mathcal{M} = \operatorname{Spec} k_R[[t_1, \ldots, t_{3g-3+n}]],$$

and the natural morphism $\mathcal{M} \to \mathcal{M}'$ induces a pointed stable curve $X^\bullet_{\mathcal{M}}$ over $\mathcal{M}$.

Let $\overline{\mathcal{M}}^{\log}_{g,n}$ be the log stack obtained by equipping $\overline{\mathcal{M}}_{g,n}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$. Then we obtain a log scheme $\mathcal{M}^{\log}$ whose underlying scheme is $\mathcal{M}$, and whose log structure is the pulling-back log structure induced by the natural morphism $\mathcal{M} \to \mathcal{M}' \to \overline{\mathcal{M}}_{g,n}$. Moreover, we obtain a stable log curve

$$X^{\log}_{\mathcal{M}^{\log}} := \overline{\mathcal{M}}^{\log}_{g,n+1} \times_{\overline{\mathcal{M}}^{\log}_{g,n}} \mathcal{M}^{\log}$$

over $\mathcal{M}^{\log}$ whose underlying curve is $X_{\mathcal{M}}$. Note that $X^{\log}_{\mathcal{M}^{\log}}$ is log regular.

By replacing $\mathcal{M}^{\log}$ by a finite log étale covering $\mathcal{N}^{\log}$, and replacing R by a finite extension of R, we obtain a morphism $\operatorname{Spec} R \to \mathcal{N}$ induced by the morphism $\operatorname{Spec} R \to \mathcal{M}$, we obtain a log scheme $s^{\log}_{k_R}$ whose underlying scheme is $\operatorname{Spec} k_R$, and whose log structure is the pulling-back log structure induced by $s_{k_R} \to \operatorname{Spec} R \to \mathcal{N}$; moreover, the G-admissible covering $f^\bullet_{k_R}$ determines a log étale covering

$$f^{\log}_{k_R} : Y^{\log}_{k_R} \to X^{\log}_{k_R}$$

over $s^{\log}_{k_R}$ such that the underlying morphism of $f^{\log}_{k_R}$ is $f^\bullet_{k_R}$. Moreover, [Hos, Corollary 1] implies that there exist a Galois log étale covering

$$f^{\log}_{\mathcal{N}^{\log}} : Y^{\log}_{\mathcal{N}^{\log}} \to X^{\log}_{\mathcal{N}^{\log}} := X^{\log}_{\mathcal{M}^{\log}} \times_{\mathcal{M}^{\log}} \mathcal{N}^{\log}$$

with Galois group G over $\mathcal{N}^{\log}$ such that

$$f^{\log}_{\mathcal{N}^{\log}} \times_{\mathcal{N}^{\log}} s^{\log}_{k_R} : Y^{\log}_{\mathcal{N}^{\log}} \times_{\mathcal{N}^{\log}} s^{\log}_{k_R} \to X^{\log}_{\mathcal{N}^{\log}} \times_{\mathcal{N}^{\log}} s^{\log}_{k_R}$$

is isomorphic to $f^{\log}_{k_R}$ over $s^{\log}_{k_R}$. Furthermore, by replacing $\mathcal{N}^{\log}$ by a finite log étale covering of $\mathcal{N}^{\log}$, we may assume that the underlying morphism of $f^{\log}_{\mathcal{N}^{\log}}$ is a morphism of pointed stable curves over $\mathcal{N}$.

Let $s^{\log}_R$ be the log scheme whose underlying scheme is $\operatorname{Spec} R$, and whose log structure is the pulling-back log structure induced by the morphism $\operatorname{Spec} R \to \mathcal{N}$. Then we obtain a log étale covering

$$f^{\log}_{\mathcal{N}^{\log}} \times_{\mathcal{N}^{\log}} s^{\log}_R : Y^{\log}_{\mathcal{N}^{\log}} \times_{\mathcal{N}^{\log}} s^{\log}_R \to X^{\log}_{\mathcal{N}^{\log}} \times_{\mathcal{N}^{\log}} s^{\log}_R$$

over $s_R^{\log}$. We denote by

$$f_R^\bullet : Y_R^\bullet \to X_R^\bullet$$

the underlying morphism $f_{\mathcal{N}^{\log}}^{\log} \times_{\mathcal{N}^{\log}} s_R^{\log}$ over R. Note that, since the special fiber $Y_R^\bullet$ is connected, the Zariski main theorem implies that $Y_R^\bullet \times_R R'$ is connected for each finite extension R' of R. Thus, the generic fiber of $Y_R^\bullet$ is geometrically connected.

Let us prove that $f_R^\bullet$ is a G-admissible covering over R. We have a log scheme $s_{K_R}^{\log}$ whose underlying scheme is $s_{K_R} := \operatorname{Spec} K_R$, and whose log structure is the pulling-back log structure induced by the morphism $s_{K_R} \to \operatorname{Spec} R \to \mathcal{N}$. Then we see that

$$f_{\mathcal{N}^{\log}}^{\log} \times_{\mathcal{N}^{\log}} s_{K_R}^{\log} : Y_{\mathcal{N}^{\log}}^{\log} \times_{\mathcal{N}^{\log}} s_{K_R}^{\log} \to X_{\mathcal{N}^{\log}}^{\log} \times_{\mathcal{N}^{\log}} s_{K_R}^{\log}$$

is geometrically connected Galois log étale covering over $s_{K_R}^{\log}$. This means that the underlying morphism of $f_{\mathcal{N}^{\log}}^{\log} \times_{\mathcal{N}^{\log}} s_{K_R}^{\log}$ is a G-admissible covering over K_R. This completes the proof of the lemma. $\square$

Let $c \in \overline{M}_{g,n}$ be a closed point and $k_c = k$ the residue field of c. Then c determines a pointed stable curve

$$X_{k_c}^\bullet$$

over k. For each $v \in v(\Gamma_c)$, write

$$\mathrm{nl}_v : X_{k_c,v} \to X'_{k_c,v}$$

for the normalization of the irreducible component $X'_{k_c,v}$ of X_{k_c} corresponding to v. Let $D_{X_{k_c,v}}$ be a set of closed points

$$(D_{X_{k_c}} \cap X_{k_c,v}) \cup (X_{k_c,v} \cap (\operatorname{Sing}(X_{k_c}) \setminus \operatorname{Sing}(X'_{k_c}))) \cup (\mathrm{nl}_v)^{-1}(\operatorname{Sing}(X'_{k_c})),$$

where $\operatorname{Sing}(-)$ denotes the set of singular points of $(-)$. We define a smooth pointed stable curve

$$X_{k_c,v}^\bullet := (X_{k_c,v}, D_{X_{k_c,v}})$$

of type (g_v, n_v) over k which determines a morphism $c_{X_{k_c,v}^\bullet} : \operatorname{Spec} k_c \to \mathcal{M}_{g_v,n_v}$ for each $v \in v(\Gamma_c)$. Write $c_{X_{k_c}^\bullet} : \operatorname{Spec} k_c \to \overline{\mathcal{M}}_{g,n}$ for the morphism induced by $X_{k_c}^\bullet$ over k. Moreover, $X_{k_c}^\bullet$ induces a clutching morphism

$$\kappa_{X_{k_c}^\bullet} : \prod_{v \in v(\Gamma_c)} \mathcal{M}_{g_v,n_v} \to \overline{\mathcal{M}}_{g,n}$$

such that $\kappa_{X_{k_c}^\bullet} \circ (\bigtimes_{v \in v(\Gamma_c)} c_{X_{k_c,v}^\bullet}) = c_{X_{k_c}^\bullet}$. We denote by M_c the image of the composition of the morphisms

$$\prod_{v \in v(\Gamma_c)} \mathcal{M}_{g_v,n_v} \overset{\kappa_{X_{k_c}^\bullet}}{\to} \overline{\mathcal{M}}_{g,n} \overset{\pi_{g,n}}{\to} \overline{M}_{g,n}.$$

Lemma 3.4. *We maintain the notations introduced above. Let $G \in \pi_A^{\mathrm{adm}}(c)$ be a finite group. Then*

$$U_G \cap M_c$$

contains an open subset of M_c which contains c.

Proof. Let $Y^\bullet_{k_c}$ be a pointed stable curve of type (g_Y, n_Y) over k and

$$f^\bullet_{k_c} : Y^\bullet_{k_c} \to X^\bullet_{k_c}$$

a G-admissible covering over k. Write $\Gamma_{Y^\bullet_{k_c}}$ for the dual semi-graph of $Y^\bullet_{k_c}$, and $\gamma_{f^\bullet_{k_c}} : \Gamma_{Y^\bullet_{k_c}} \to \Gamma_c$ for the morphism of dual semi-graphs induced by $f^\bullet_{k_c}$. For each $v \in v(\Gamma_c)$, write I_v for the set $\gamma^{-1}_{f^\bullet_{k_c}}(v)$. Then $f^\bullet_{k_c}$ and the natural morphism $X_{k_c,v} \to X_{k_c}$ induce a multi-admissible covering

$$f^\bullet_{k_c,v} : \coprod_{w \in I_v} Y^\bullet_{k_c,w} \to X^\bullet_{k_c,v}$$

over k, where $Y^\bullet_{k_c,w}$, $w \in I_v$, is a smooth pointed stable curve of type $(g_{Y,w}, n_{Y,w})$ over k whose underlying curve is a normalization of the irreducible component of Y_{k_c} corresponding to w. Note that $\coprod_{w \in I_v} Y^\bullet_{k_c,w}$ admits an action of G induced by the action of G on $Y^\bullet_{k_c}$. This action induces an action of G on the set I_v. For each $w \in I_v$, write G_w for the inertia subgroup of w. Then we obtain a G_w-admissible covering

$$f^\bullet_{k_c,w} : Y^\bullet_{k_c,w} \to X^\bullet_{k_c,v}$$

over k. Write $c_{Y^\bullet_{k_c}} : \operatorname{Spec} k_c \to \overline{\mathcal{M}}_{g_Y,n_Y}$ for the morphism determined by $Y^\bullet_{k_c}$ over k_c, and $c_{Y^\bullet_{k_c,w}} : \operatorname{Spec} k_c \to \overline{\mathcal{M}}_{g_w,n_w}$ for the morphism determined by $Y^\bullet_{k_c,w}$ over k_c for each $w \in v(\Gamma_{Y^\bullet_{k_c}})$. Then the pointed stable curve $Y^\bullet_{k_c}$ over k_c induces a clutching morphism as follows:

$$\kappa_{Y^\bullet_{k_c}} : \prod_{w \in v(\Gamma_{Y^\bullet_{k_c}})} \mathcal{M}_{g_w,n_w} \to \overline{\mathcal{M}}_{g,n}$$

such that the composition of morphisms $\kappa_{Y^\bullet_{k_c}} \circ (\bigtimes_{w \in v(\Gamma_{Y^\bullet_{k_c}})} c_{Y^\bullet_{k_c,w}}) = c_{Y^\bullet_{k_c}}$.

For each $v \in v(\Gamma_c)$, the smooth pointed stable curve $X^\bullet_{k_c,v}$ of type (g_v, n_v) over k determines a natural morphism

$$\operatorname{Spec} k \to M_{g_v,n_v},$$

and write $c_v \in M_{g_v,n_v}$ for the image. Then the proof of Theorem 3.2 implies that, for each $v \in v(\Gamma_c)$, there exist an affine k-variety E_{c_v} and a morphism $\alpha_{E_{c_v}} : E_{c_v} \to H_{g_v,n_v}$ such that

(i) the image of $\alpha_{E_{c_v}}$ contains an open subset U_{c_v} of H_{g_v,n_v} whose image $\pi^{(m)}_{g_v,n_v}(U_{c_v})$ in M_{g_v,n_v} contains c_v;

(ii) there exists a smooth pointed stable curve $X^\bullet_{E_{c_v}}$ with a level m-structure $\sigma_{E_{c_v}} := \sigma_{H_{g_v,n_v}} \times_{H_{g_v,n_v}} E_{c_v}$;

(iii) for each $w \in I_v$, there exists a G_w-Galois covering of smooth pointed stable curves

$$f^\bullet_{E_{c_v},w} : Y^\bullet_{E_{c_v},w} \to X^\bullet_{E_{c_v},v}$$

over E_{c_v} such that $f^\bullet_{E_{c_v},w}$ is a G_w-admissible covering over E_{c_v}, and that the restriction of $f^\bullet_{E_{c_v},w}$ on each point of $(\pi^{(m)}_{g_v,n_v} \circ \alpha_{E_{c_v}})^{-1}(c_v)$ is isomorphic to the G_w-admissible covering $f^\bullet_{k_c,w}$ over k.

Then the image of

$$\prod_{v\in v(\Gamma_c)} U_{c_v} \hookrightarrow \prod_{v\in v(\Gamma_c)} H_{g_v,n_v} \to \prod_{v\in v(\Gamma_c)} \mathcal{M}_{g_v,n_v} \overset{\kappa_{X^\bullet_{k_c}}}{\to} \overline{\mathcal{M}}_{g,n} \overset{\overline{\pi}_{g,n}}{\to} \overline{M}_{g,n}$$

contains an open subset $c \in W_c$ of M_c. To verify the lemma, it is sufficient to prove that $G \in \pi_A^{\mathrm{adm}}(c')$ for each $c' \in W_c$.

Since W_c is a k-variety, there exists a k-curve $C' \subseteq W_c$ which contains c and c'. Write C for the normalization of C', c_1 for a closed point of C over c, and c_2 for a closed point of C over c'. Let R_i, $i \in \{1,2\}$ be a complete discrete valuation ring which is a finite extension of $\widehat{\mathcal{O}}_{C,c_i}$, K_{R_i} the quotient field of R_i, $\overline{K}_{R_i}$ an algebraic closure of K_{R_i}, and $k_{R_i} = k$ the residue field of R_i.

By replacing R_1 by a finite extension of R_1, there is a smooth pointed stable curve

$$X^\bullet_{R_1}$$

over R_1 whose special fiber $X^\bullet_{k_{R_1}}$ over the residue field $k_{R_1} = k$ of R_1 is isomorphic to $X^\bullet_{k_c}$ over k. Lemma 3.3 implies that the G-admissible covering $f^\bullet_{k_c}$ over k can be lifted to a G-admissible covering

$$f^\bullet_{R_1} : Y^\bullet_{R_1} \to X^\bullet_{R_1}$$

over R_1. Moreover, for each $v \in v(\Gamma_c)$ and each $w \in I_v$, the G_w-admissible covering over k can be lifted to a G_w-admissible covering

$$f^\bullet_{R_1,w} : Y^\bullet_{R_1,w} \to X^\bullet_{R_1,v}$$

over R_1. Write $c_v^{(m)} \in U_{c_v} \subseteq H_{g_v,n_v}$ for a closed point over c_v. The level m-structure

$$\sigma_{H_{g_v,n_v}} \times_{H_{g_1,n_1}} c_v$$

on the special fiber of $X^\bullet_{R_1,v}$ extends to a level m-structure $\sigma_{R_1,v}$ on $X^\bullet_{R_1,v}$. Then, for $v \in v(\Gamma_c)$, the pointed stable curve $X^\bullet_{R_1,v}$ with the level m-structure $\sigma_{R_1,v}$ determines a morphism

$$l_{R_1,v} : \operatorname{Spec} R_1 \to H_{g_v,n_v}.$$

Thus, $X^\bullet_{R_1,v}$ is isomorphic to $X^\bullet_{H_{g_v,n_v}} \times_{H_{g_v,n_v}} \operatorname{Spec} R_1$ over R_1. Moreover, for each $v \in v(\Gamma_c)$ and each $w \in I_v$, we have a G_w-admissible covering

$$f_{\overline{K}_{R_1},w} : Y^\bullet_{\overline{K}_{R_1},w} \to X^\bullet_{\overline{K}_{R_1},v}$$

over $\overline{K}_{R_1}$.

Let η_v be a closed point over $E_{c_v} \times_{H_{g_v,n_v}} \operatorname{Spec} K_{R_1}$ and $s_{1,v} \in E_{c_v}$ a closed point contained in $V_{\eta_v} := \overline{\{\eta_v\}}$ such that $\alpha_{E_{c_v}}(s_{1,v})$ is equal to the image (as a set) of

$$\operatorname{Spec} k_{R_1} \hookrightarrow \operatorname{Spec} R_1 \overset{l_{R_1,v}}{\to} H_{g_v,n_v}.$$

Note that since $R_1 \cong k[[t]]$, the scheme-theoretic image of $l_{R_1,v}$ is a local ring of dimension one. Moreover, since the residue field of η_v is a finite extension of K_{R_1}, V_{η_v} is an one dimensional k-subscheme of E_{c_v}. Write $A_{1,v}$ for the normalization of $\widehat{\mathcal{O}}_{V_{\eta_v},s_{1,v}}$. Note that $A_{1,v}$ is a complete discrete valuation ring, and the natural morphism $\mathrm{Spec}\,A_{1,v} \to \mathrm{Spec}\,R_1$ is finite. Then we may assume that $\overline{K}_{R_1}$ contains $A_{1,v}$. Thus, the geometric generic fiber of the G_w-admissible covering

$$f^\bullet_{A_{1,v},w} := f^\bullet_{E_{c_v},w} \times_{E_{c_v}} \mathrm{Spec}\,A_{1,v} : Y^\bullet_{A_{1,v},w} \to X^\bullet_{A_{1,v},v}$$

of smooth pointed stable curves over $A_{1,v}$ is isomorphic to $f^\bullet_{\overline{K}_{R_1},w}$ over $\overline{K}_{R_1}$ as G_w-admissible coverings.

On the other hand, by replacing R_2 by a finite extension of R_2, there is a smooth pointed stable curve

$$X^\bullet_{R_2}$$

over R_2, and the G-admissible covering $f^\bullet_{R_1} \times_{R_1} \overline{K}_{R_1}$ over $\overline{K}_{R_1}$ induces a G-admissible covering

$$f^\bullet_{K_{R_2}} : Y^\bullet_{K_{R_2}} \to X^\bullet_{K_{R_2}}$$

of pointed stable curves over K_{R_2}; moreover, for each $v \in v(\Gamma_c)$ and each $w \in I_v$, $f^\bullet_{K_{R_2}}$ induces a G_w-admissible covering

$$f^\bullet_{K_{R_2},w} : Y^\bullet_{K_{R_2},w} \to X^\bullet_{K_{R_2},v}$$

of smooth pointed stable curves over K_{R_2}. For $v \in v(\Gamma_c)$, the level m-structure $\sigma_{R_1,v} \times_{R_1} \overline{K}_{R_1}$ induces a level m-structure $\sigma_{R_2,v}$ on $X^\bullet_{R_2,v}$ such that the pointed stable curve $X^\bullet_{R_2,v}$ with the level m-structure $\sigma_{R_2,v}$ determines a morphism

$$l_{R_2,v} : \mathrm{Spec}\,R_2 \to H_{g_v,n_v}$$

whose image (as a set) is contained in U_{c_v}.

By replacing R_2 by a finite extension of R_2, we may assume that $Y^\bullet_{K_{R_2},w}$ has pointed stable reduction over R_2. Next, let us prove that $Y^\bullet_{K_{R_2},w}$ has good reduction for each $w \in v(\Gamma_{Y^\bullet_{k_c}})$.

If the image of $l_{R_2,v}$ is a constant morphism, then $Y^\bullet_{K_{R_2},w}$ has good reduction over R_2. Then we may assume that $l_{R_2,v}$ is not a constant morphism. Since $R_2 \cong k[[t]]$, the scheme-theoretic image of $l_{R_2,v}$ is a local ring of dimension one. Let $s_{2,v} \in E_{c_v}$ a closed point contained in $V_{\eta_v} := \overline{\{\eta_v\}}$ such that $\alpha_{E_{c_v}}(s_{2,v})$ is equal to the image (as a set) of

$$\mathrm{Spec}\,k_{R_2} \hookrightarrow \mathrm{Spec}\,R_2 \overset{l_{R_2,v}}{\to} H_{g_v,n_v}.$$

Write $A_{2,v}$ for the normalization of $\widehat{\mathcal{O}}_{V_{\eta_v},s_{2,v}}$. Note that $A_{2,v}$ is a complete discrete valuation ring, and the natural morphism $\mathrm{Spec}\,A_{2,v} \to \mathrm{Spec}\,R_2$ is finite. We assume that $\overline{K}_{R_2} = \overline{K}_{R_1}$. Thus, we obtain a G_w-admissible covering

$$f^\bullet_{A_{2,v},w} := f^\bullet_{E_{c_v},w} \times_{E_{c_v}} \mathrm{Spec}\,A_{2,v} : Y^\bullet_{A_{2,v},w} \to X^\bullet_{A_{2,v},v}$$

of smooth pointed stable curves over $A_{2,v}$ such that the geometric generic fiber $f^\bullet_{A_{2,v},w} \times_{A_{2,v}} \overline{K}_{R_2}$ is isomorphic to $f^\bullet_{K_{R_2},w} \times_{K_{R_2}} \overline{K}_{R_2} = f^\bullet_{\overline{K}_{R_1},w}$ over $\overline{K}_{R_2}$ as G_w-admissible coverings. This implies that $Y^\bullet_{K_{R_2},w}$ has good reduction.

The clutching morphism $\kappa_{Y^\bullet_{k_c}}$ implies that we may glue $\{Y^\bullet_{R,w}\}_{w\in v(\Gamma_{Y^\bullet_{k_c}})}$ and obtain a pointed stable curve

$$Y^\bullet_{R_2}$$

over R_2 such that

(i) $Y^\bullet_{R_2} \times_{R_2} K_{R_2} \cong Y^\bullet_{K_{R_2}}$ over K_{R_2};

(ii) there exists a morphism $f^\bullet_{K_{R_2}} : Y^\bullet_{K_{R_2}} \to X^\bullet_{K_{R_2}}$ of pointed stable curves over K_{R_2} which is a G-admissible covering over K_{R_2} such that $f^\bullet_{K_{R_2}} \times_{K_{R_2}} \overline{K}_{R_2}$ isomorphic to $f^\bullet_{R_1} \times_{R_1} \overline{K}_{R_1}$.

Then Lemma 2.6 implies that there exists a G-admissible covering $f^\bullet_{R_2} : Y^\bullet_{R_2} \to X^\bullet_{R_2}$ such that the restriction of $f^\bullet_{R_2}$ on the special fibers is a connected G-admissible covering over k_{R_2}. This means that $G \in \pi_A^{\mathrm{adm}}(c')$. We completes the proof of the lemma. □

Corollary 3.5. *We maintain the notations introduced in Lemma 3.4. Let $G \in \pi_A^{\mathrm{adm}}(c)$ be a finite group. Then*

$$U_G \cap M_c$$

is an open subset of M_c.

Proof. The corollary follows immediately from Proposition 2.7 and Lemma 3.4. □

For each $j \in \mathbb{Z}_{\geq 0}$, we take

$$M_j := \{q' \in \overline{M}_{g,n} \mid \#e^{\mathrm{cl}}(\Gamma_{q'}) = j\},$$

and denote by $\mathrm{Gen}(M_j)$ the set of generic points of M_j. Note that $M_0 = M_{g,n}$. Write $\overline{M}_{\eta_j}$ for the topological closure of M_{η_j} in $\overline{M}_{g,n}$ for each $j \in \mathbb{Z}_{\geq 0}$. Note that $M_j = \emptyset$ if $j >> 0$. Then we have

$$\overline{M}_{g,n} = \bigcup_{j\in\mathbb{Z}_{\geq 0}} M_j,$$

and $M_{j'} \cap M_{j''} = \emptyset$ if $j' \neq j''$. Moreover, for each $\eta_j \in \mathrm{Gen}(M_j)$, we set

$$M_{\eta_j} := V_{\eta_j} \cap M_j.$$

Then we obtain that, for each $j \in \mathbb{Z}_{\geq 0}$,

$$M_j = \bigcup_{\eta_j \in \mathrm{Gen}(M_j)} M_{\eta_j},$$

and $M_{\eta'_j} \cap M_{\eta''_j} = \emptyset$ if $\eta'_j \neq \eta''_j$. Thus, we obtain

$$\overline{M}_{g,n} = \bigcup_{j\in\mathbb{Z}_{\geq 0}} \bigcup_{\eta_j \in \mathrm{Gen}(M_j)} M_{\eta_j}$$

which is a finite disjoint union.

Next, we prove our main theorem of the present paper.

Theorem 3.6. *Let q be an arbitrary point of $\overline{M}_{g,n}$ and $G \in \pi_A^{\mathrm{adm}}(q)$. Then U_G is an open subset of $\overline{M}_{g,n}$.*

Proof. We have

$$U_G = \bigcup_{j \in \mathbb{Z}_{\geq 0}} \bigcup_{\eta_j \in \mathrm{Gen}(M_j)} M_{\eta_j} \cap U_G.$$

Corollary 3.5 implies that $M_{\eta_j} \cap U_G$ is an open subset of M_{η_j}. This means that $M_{\eta_j} \cap U_G$ is a constructible set and $M_{\eta_j} \cap U_G$ is stable under generation in $M_{\eta_j} \cap U_G$. Then U_G is a constructible set.

Let $j' \geq j''$. If $M_{\eta_{j'}} \subseteq \overline{M}_{\eta_{j''}}$ and $M_{\eta_{j'}} \cap U_G \neq \emptyset$, then Proposition 2.1 implies that $M_{\eta_{j''}} \cap U_G \neq \emptyset$. Then U_G is stable under the generation in $\overline{M}_{g,n}$. Thus, U_G is an open subset of $\overline{M}_{g,n}$. This completes the proof of the theorem. □

4 Anabelian geometry of pointed stable curves over algebraically closed fields of characteristic $p > 0$

In this section, we study the anabelian geometry of pointed stable curves over algebraically closed fields of characteristic $p > 0$. We maintain the notations introduced in the previous sections and suppose that $k = \overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$.

4.1 An alternative proof of a finiteness result for pointed stable curves

Let $c \in \overline{M}_{g,n}^{\mathrm{cl}}$ be an arbitrary **closed point**. We take

$$S_c := \{c' \in \overline{M}_{g,n}^{\mathrm{cl}} \mid \Pi_{c'}^{\mathrm{adm}} \cong \Pi_c^{\mathrm{adm}}\}.$$

Theorem 4.1. *We have $\#S_c < \infty$.*

Proof. If S_c is not a finite set, then the topological closure $\overline{S}_c$ in $\overline{M}_{g,n}$ contains a $\overline{\mathbb{F}}_p$-curve $C \subseteq \overline{S}_c$ such that $C \cap S_c \neq \emptyset$. Write η_C for the generic point of C. Then for any $G_1, G_2 \in \pi_A^{\mathrm{adm}}(\eta_C)$, Theorem 3.6 and the definition of S_c imply that $U_{G_1} \cap S_c \neq \emptyset$, $U_{G_2} \cap S_c \neq \emptyset$, and $U_{G_1} \cap S_c = U_{G_2} \cap S_c$. This means that there exists a closed point $c' \in S_c \cap C$ such that $\pi_A^{\mathrm{adm}}(\eta_C) \subseteq \pi_A^{\mathrm{adm}}(c')$. Moreover, Proposition 2.1 implies that

$$\pi_A^{\mathrm{adm}}(\eta_C) = \pi_A^{\mathrm{adm}}(c').$$

Thus, $\Pi_{\eta_C}^{\mathrm{adm}}$ is isomorphic to $\Pi_{c'}^{\mathrm{adm}}$ as profinite groups.

Let $R' := \widehat{\mathcal{O}}_{C,c'}$. By replacing R' by a finite extension R of R', we have a pointed stable curve

$$X_R^{\bullet}$$

over R. Then we obtain a specialization map

$$sp_{\eta_C, c'} : \Pi_{\eta_C}^{\mathrm{adm}} \twoheadrightarrow \Pi_{c'}^{\mathrm{adm}}$$

which is a surjection. Since $\Pi^{\mathrm{adm}}_{\eta_C}$ and $\Pi^{\mathrm{adm}}_{c'}$ are topologically finitely generated, the specialization map $sp_{\eta_C,c'}$ is an isomorphism.

On the other hand, let $\eta_C \in M_{j_1}$ and $c' \in M_{j_2}$ (cf. Section 3 for the definitions of M_{j_1} and M_{j_2}). If $j_1 = j_2$, we have that Γ_{η_C} is isomorphic to $\Gamma_{c'}$. Then [T2, Theorem 0.3] implies that $sp_{\eta_C,c'}$ is not an isomorphism. This is a contradiction. Thus, to verify the theorem, we may assume that $j_1 \neq j_2$.

Let k_{η_C} and $k_{c'}$ be algebraic closures of the residue fields of η_C and c', $X^\bullet_{k_{\eta_C}}$ and $X^\bullet_{k_{c'}}$ the pointed stable curves corresponding to the natural morphisms

$$\operatorname{Spec} k_{\eta_C} \to \overline{M}_{g,n} \text{ and } \operatorname{Spec} k_{c'} \to \overline{M}_{g,n},$$

respectively. Let $\ell >> 0$ be a prime number distinct from p. We denote by $H_{c'}$ the kernel of the morphism

$$\Pi^{\mathrm{adm,ab}}_{c'} \twoheadrightarrow G := \Pi^{\mathrm{adm,ab}}_{c'} \otimes \mathbb{F}_\ell,$$

where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$, and write $Y^\bullet_{k_{c'}} \to X^\bullet_{k_{c'}}$ for the G-admissible covering over $k_{c'}$ induced by the surjection. Moreover, $H_{\eta_C} := sp^{-1}_{\eta_C,c'}(H_{c'})$ determines a G-admissible covering $Y^\bullet_{k_{\eta_C}} \to X^\bullet_{k_{\eta_C}}$ over k_{η_C}.

The specialization map implies that, by replacing R by a finite extension of R, we have a pointed stable curve

$$Y^\bullet_R$$

of type (g_Y, n_Y) over R whose geometric generic fiber is isomorphic to $Y^\bullet_{k_{\eta_C}}$ over k_{η_C}, and whose special fiber is isomorphic to $Y^\bullet_{k'_c}$ over $k_{c'}$. Then the types of $Y^\bullet_{k_{\eta_C}}$ and $Y^\bullet_{k'_c}$ are (g_Y, n_Y).

Write $\Gamma_{Y^\bullet_{\eta_C}}$ and $\Gamma_{Y^\bullet_{k'_c}}$ for the dual semi-graphs of $Y^\bullet_{\eta_C}$ and $Y^\bullet_{k'_c}$, respectively. It is easy to check that $\Gamma_{Y^\bullet_{\eta_C}}$ and $\Gamma_{Y^\bullet_{k'_c}}$ are 2-connected, and that $v(\Gamma_{Y^\bullet_{\eta_C}})^{b\leq 1} = v(\Gamma_{Y^\bullet_{k'_c}})^{b\leq 1} = 0$. Moreover, since $j_1 \neq j_2$, one sees that the Betti number of $\Gamma_{Y^\bullet_{\eta_C}}$ is **strictly less than** the Betti number of $\Gamma_{Y^\bullet_{k'_c}}$. Since $sp_{\eta_C,c'}$ is an isomorphism, $sp_{\eta_C,c'}|_{H_{\eta_C}} : H_{\eta_C} \overset{\sim}{\to} H_{c'}$ is an isomorphism. This contradict to [T3, Theorem 3.10]. We complete the proof of the theorem. $\square$

Remark 4.1.1. Suppose that $c \in M^{\mathrm{cl}}_{g,n}$. Then $\#(S_c \cap M^{\mathrm{cl}}_{g,n}) < \infty$ was proved by Raynaud (cf. [R]) and Pop-Saidi (cf. [PS]) under certain assumptions of Jacobian, and by Tamagawa in the fully general case (cf. [T2]).

Remark 4.1.2. In [Y2, Theorem 0.3 (b)], the author proved Theorem 4.1 in a completely different way (i.e., by using [T2, Theorem 0.3] and the combinatorial Grothendieck conjecture in positive characteristic (cf. [Y2, Theorem 0.2])). Moreover, we suppose that $c \in M_j$. Then there exists a unique generic point $\eta_{c,j} \in \mathrm{Gen}(M_j)$ such that $c \in V_{\eta_{c,j}} \cap M_j$. Thus, [Y2, Theorem 0.2] implies that $c' \in V_{\eta_{g,j}} \cap M_j$ for each $c' \in S_c$. In particular, if $c \in M_{g,n}$, then $S_c \cap M_{g,n} = S_c$.

4.2 Pointed collection conjecture for pointed stable curves

We denote by

$$\overline{\mathcal{M}}_{g,[n]} := [\overline{\mathcal{M}}_{g,n}/S_n]$$

the quotient stack, and denote by

$$\overline{M}_{g,[n]}$$

the coarse moduli space of $\overline{\mathcal{M}}_{g,[n]}$, where S_n denotes the n-symmetric group. Note that we obtain a morphism

$$\pi : \overline{M}_{g,n} \to \overline{M}_{g,[n]}$$

induced by the natural quotient morphism $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,[n]}$. We define an equivalence relation on the set of closed points $\overline{M}^{\mathrm{cl}}_{g,n}$ as follows:

> for any closed points $c_1, c_2 \in \overline{M}^{\mathrm{cl}}_{g,n}$, $c_1 \sim c_2$ if there exists $m \in \mathbb{Z}$ such that $\pi(c_2) = \pi(c_1^{(m)})$, where $c_1^{(m)}$ denotes the closed point corresponding to the m^{th} Frobenius twist of the pointed stable curve corresponding to c_1.

Let $q \in \overline{M}_{g,n}$ be an arbitrary point and $q \in M_{\eta_{j_q}}$, where $\eta_{j_q} \in \mathrm{Gen}(M_{j_q})$. We denote by

$$W_q := M_{\eta_{j_q}} \cap V_q.$$

Note that, by the definition, we have $W_{\eta_j} = M_{\eta_{j_{\eta_j}}} = M_{\eta_j}$ for each $\eta_j \in \mathrm{Gen}(M_j)$.

Definition 4.2. Let $q_1, q_2 \in \overline{M}_{g,n}$ be arbitrary points. We denote by

$$W_{q_1} \supseteq_{\mathrm{ec}} W_{q_2}$$

if, for each closed point $c_2 \in W^{\mathrm{cl}}_{q_2}$, there exists a closed point $c_1 \in W^{\mathrm{cl}}_{q_1}$ such that $c_1 \sim c_2$. Moreover, we denote by

$$W_{q_1} =_{\mathrm{ec}} W_{q_2}$$

if $W_{q_1} \supseteq_{\mathrm{ec}} W_{q_2}$ and $W_{q_1} \subseteq_{\mathrm{ec}} W_{q_2}$. We shall call that W_{q_1} **essentially contains** W_{q_2} if $W_{q_1} \supseteq_{\mathrm{ec}} W_{q_2}$ and shall call that W_{q_1} is **essentially equal** to W_{q_2} if $W_{q_1} =_{\mathrm{ec}} V_{q_2}$.

First, we have the following proposition.

Proposition 4.3. *Let $q_1, q_2 \in \overline{M}_{g,n}$ be arbitrary points. Suppose that $\Pi^{\mathrm{adm}}_{q_1} \cong \Pi^{\mathrm{adm}}_{q_2}$. Then $M_{\eta_{j_{q_1}}} =_{\mathrm{ec}} M_{\eta_{j_{q_2}}}$. Moreover, $M_{\eta_{j_{q_1}}} = M_{\eta_{j_{q_2}}}$.*

Proof. The proposition follows immediately from [Y2, Theorem 0.2]. □

Remark 4.3.1. The proposition means that, for any $q \in \overline{M}_{g,n}$, $M_{\eta_{j_q}}$ can be reconstructed group-theoretically from Π^{adm}_q.

The weak Hom-version of the Grothendieck conjecture of curves over algebraically closed fields of characteristic $p > 0$ can be formulated as follows:

Conjecture 4.4. *Let $q_1, q_2 \in \overline{M}_{g,n}$ be arbitrary points. The set of continuous open homomorphisms of profinite groups*

$$\mathrm{Hom}_{\text{pro-gps}}(\Pi^{\mathrm{adm}}_{q_1}, \Pi^{\mathrm{adm}}_{q_2}) \neq \emptyset$$

if and only if $W_{q_1} \supseteq_{\mathrm{ec}} W_{q_2}$. In particular, the set of continuous isomorphisms of profinite groups

$$\mathrm{Isom}_{\text{pro-gps}}(\Pi^{\mathrm{adm}}_{q_1}, \Pi^{\mathrm{adm}}_{q_2}) \neq \emptyset$$

if and only if $W_{q_1} =_{\mathrm{ec}} W_{q_2}$.

Remark 4.4.1. The "in particular" part of the conjecture is called the weak Isom-version of the Grothendieck conjecture of curves over algebraically closed fields of characteristic $p > 0$.

Remark 4.4.2. At the present, only a few cases concerning Conjecture 4.4 have been proven (cf. [T1, Theorem 0.2], [T2, Theorem 0.3], [Y2, Theorem 0.3], [Y3, Theorem 0.7], and [Y4, Theorem 0.6]).

Almost all of the results concerning Conjecture 4.4 are proved only in the case where q_1 and q_2 are closed points. One of the main goals of the anabelian geometry of curves in positive characteristic is to extend [T1, Theorem 0.2], [T2, Theorem 0.3], and [Y2, Theorem 0.3] to the case where q_1 and q_2 are arbitrary points of $\overline{M}_{g,n}$.

Let $q \in \overline{M}_{g,n}$ be an arbitrary point. The main difficulty is that we do not know how to reconstruct the admissible fundamental groups of the closed points of W_q group-theoretically from Π_q^{adm}. Once the admissible fundamental groups of the closed points of W_q are reconstructed group-theoretically from Π_q^{adm}, then, by applying the results concerning Conjecture 4.4 for closed points, the set of closed points of W_q can be reconstructed from Π_q^{adm}. Thus, Conjecture 4.4 for non-closed points follows from Conjecture 4.4 for closed points. On the other hand, since the isomorphism class of Π_q^{adm} as profinite group is determined completely by the set $\pi_A^{\mathrm{adm}}(q)$. In order overcome the difficulty mentioned above, we consider the following question:

Question 4.5. *(i) For each closed point t of W_q, which collection of finite groups, whose elements are contained in $\pi_A^{\mathrm{adm}}(q)$, coincides with $\pi_A^{\mathrm{adm}}(t)$?*

(ii) For each closed point t of $\overline{M}_{\eta_{jq}}$, if $\pi_A^{\mathrm{tame}}(t) \subseteq \pi_A^{\mathrm{tame}}(q)$, then is t a closed point of W_q?

Let $t \in \overline{M}_{\eta_{jq}}^{\mathrm{cl}}$ be arbitrary closed point. Let $k_t = \overline{\mathbb{F}}_p$ be the residue field of t and $X_{k_t}^{\bullet}$ the pointed stable curve over k_t determined by the natural morphism $\operatorname{Spec} k_t \to \overline{M}_{g,n}$. We take

$$F_t := \left(\bigcap_{G \in \pi_A^{\mathrm{adm}}(t)} U_G\right) \cap M_{\eta_{jq}}^{\mathrm{cl}}$$

which is a set of closed points of $\overline{M}_{g,n}$.

Proposition 4.6. *Suppose that the genus of the normalization of each irreducible component of the underlying curve of $X_{k_t}^{\bullet}$ is 0. Then $\#F_t < \infty$. Moreover, suppose that $X_{k_t}^{\bullet}$ is irreducible. Then $\#(F_t/\sim) = 1$. In particular, $\#F_t < \infty$ for each $q \in \overline{M}_{0,n}$ and each $t \in M_{\eta_{jq}}^{\mathrm{cl}}$.*

Proof. Let t' be any closed point of F_t. Then we have that, for each $G \in \pi_A^{\mathrm{tame}}(t)$,

$$\mathrm{Hom}_{\text{pro-gps}}^{\mathrm{surj}}(\Pi_{t'}^{\mathrm{adm}}, G) \neq \emptyset,$$

where $\mathrm{Hom}_{\text{pro-gps}}^{\mathrm{surj}}(-,-)$ denotes the set of surjections of $\mathrm{Hom}_{\text{pro-gps}}(-,-)$. Since $\Pi_{t'}^{\mathrm{adm}}$ is topologically finitely generated, the set $\mathrm{Hom}_{\text{pro-gps}}^{\mathrm{surj}}(\Pi_{t'}^{\mathrm{adm}}, G)$ is finite. Then the set of open continuous homomorphisms

$$\varprojlim_{G \in \pi_A^{\mathrm{tame}}(t)} \mathrm{Hom}_{\text{pro-gps}}^{\mathrm{surj}}(\Pi_{t'}^{\mathrm{adm}}, G) = \mathrm{Hom}_{\text{pro-gps}}(\Pi_{t'}^{\mathrm{adm}}, \Pi_t^{\mathrm{adm}}) \neq \emptyset.$$

Thus, the proposition follows from [Y4, Theorem 0.6]. $\square$

Let

$$\mathcal{C} \subseteq \pi_A^{\mathrm{tame}}(\eta_{j_q}) = \bigcup_{t \in M^{\mathrm{cl}}_{\eta_{j_q}}} \pi_A^{\mathrm{tame}}(t)$$

be the a collection of finite groups contained in $\pi_A^{\mathrm{tame}}(\eta_{j_q})$.

Definition 4.7. We shall call that $\mathcal{C}$ is a **pointed collection** if the following conditions are satisfied:

(i) $(\bigcap_{G\in\mathcal{C}} U_G) \cap M^{\mathrm{cl}}_{\eta_{j_q}} \neq \emptyset$;

(ii) $\#(((\bigcap_{G\in\mathcal{C}} U_G) \cap M^{\mathrm{cl}}_{\eta_{j_q}})/\sim)=1$;

(iii) $U_{G'} \cap (\bigcap_{G\in\mathcal{C}} U_G) \cap M^{\mathrm{cl}}_{\eta_{j_q}} = \emptyset$ for each $G' \in \pi_A^{\mathrm{tame}}(\eta_{j_q})$ such that $G' \notin \mathcal{C}$.

On the other hand, for each closed point $t \in M^{\mathrm{cl}}_{\eta_{j_q}}$, we may define a collection associated to t as follows:

$$\mathcal{C}_t := \{G \in \pi_A^{\mathrm{adm}}(\eta_{j_q}) \mid t \in U_G\}.$$

Note that, if $t \in W_q^{\mathrm{cl}}$, then $\mathcal{C}_t \subseteq \pi_A^{\mathrm{adm}}(q)$. Moreover, we denote by

$$\mathscr{C}_q := \{\mathcal{C} \text{ pointed collection} \mid \mathcal{C} \subseteq \pi_A^{\mathrm{adm}}(q)\}$$

the set of pointed collections which are contained in $\pi_A^{\mathrm{adm}}(q)$.

We conjectured the set of closed points W_q^{cl} can be reconstructed from $\pi_A^{\mathrm{adm}}(q)$ (or Π_q^{adm}) as follows:

Conjecture 4.8. *For each $t \in M^{\mathrm{cl}}_{\eta_{j_q}}$, the collection $\mathcal{C}_t$ associated to t is a pointed collection. Moreover, the natural map*

$$\theta_q : W_q^{\mathrm{cl}}/\sim \rightarrow \mathscr{C}_q$$

that $[t] \mapsto \mathcal{C}_t$ is a bijection, where $[t]$ denotes the image of t in $W_q^{\mathrm{cl}}/\sim$.

Remark 4.8.1. By the similar arguments to the arguments given in the proof of [Y3, Proposition 7.2] imply that

$$\text{Conjecture 4.4} \Leftrightarrow \text{Conjecture 4.8.}$$

Remark 4.8.2. Let k_q and $k_{\eta_{j_q}}$ be algebraic closures of the residue field of q and η_{j_q}, $X^\bullet_{k_q}$ and $X^\bullet_{k_{\eta_{j_q}}}$ the pointed stable curves over k_q and $k_{\eta_{j_q}}$ determined by the natural morphisms $\mathrm{Spec}\, k_q \rightarrow \overline{M}_{g,n}$ and $\mathrm{Spec}\, k_{\eta_{j_q}} \rightarrow \overline{M}_{g,n}$, respectively.

Suppose that $X^\bullet_{k_q}$ is irreducible and the genus of the normalization of the underlying curve of $X^\bullet_{k_q}$ is 0 (then $X^\bullet_{k_{\eta_{j_q}}}$ is irreducible and the genus of the normalization of the underlying curve of $X^\bullet_{k_{\eta_{j_q}}}$ is 0). Proposition 4.6 implies that the collection $\mathcal{C}_t$ associated to t is a pointed collection for each $t \in M^{\mathrm{cl}}_{\eta_{j_q}}$, and that θ_q is an injection. Moreover, if q is closed point, θ_q is a surjection.

Remark 4.8.3. Conjecture 4.8 generalizes the pointed collection conjecture for **smooth** pointed stable curves (cf. [Y3, Conjecture 0.9]) to the case of arbitrary pointed stable curves.

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