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Abstract. For any given hyperplane arrangment, we introduce Q-deformations of its external zonotopal algebras. Furthermore, we introduce Hecke deformations of all three types of unimodular zonotopal algebras, which gives a new definition of non-deformed central and internal zonotopal algebras.

1. Introduction

The main objective of the present note is to continue the study of deformations of certain class of commutative algebras associated with a collection of real n-vectors which span the space $\mathbb{R}^n$. This class algebras includes the so-called *zonotopal algebras* introduced and studied in depth by O. Holtz, A. Ron [10] and by F. Ardila and A. Postnikov [2], and later on studied by M. Lenz [12, 13] and A. Berget [4, 5], among others. The class of zonotopal algebras includes graphical, unimodal zonotopal and many other interesting algebras. These algebras have drawn considerable attention of the wide mathematical community, since it turned out that these algebras have many interesting and deep algebraic and combinatorial properties, as well as many applications in pure and applied mathematics.

The first motivation to study these objects comes from the box-splines theory (see [1, 8]) and from the study of Dahmen-Micchelli spaces (see [7]); the central case is more natural in this motivation (see [9] by N. Dyn and A. Ron). The second reason to study them comes from homology theory and it was motivated by a problem posed by V. I. Arnold [3]. In papers [20] and [19] A. Postnikov, B. Shapiro, and M. Shapiro described the algebra $\mathcal{C}_n$ generated by the curvature forms of the tautological Hermitian linear bundles over the type A complete flag variety, where the authors actually define external zonotopal algebras.

In the present paper we study certain deformations of zonotopal type algebras, that is, we are going to deform the underlying algebraic structure of an algebra we are looking for, but in a such way that we keep some important properties of the latter; for example, we want to keep its dimension. A deformed algebra may contain lot of additional information about the algebra we are started with.

Key words and phrases. Commutative algebra, zonotopes, Spanning trees and forests, Score vectors.

1.1. **Definition of zonotopal algebras.** Let $A \in \mathbb{R}^{n\times m}$ be a matrix of rank n. Denote by $y_1, \dots, y_m \in \mathbb{R}^n$ its columns and by $t_1, \dots, t_n \in \mathbb{R}^m$ its rows. For a matrix A, we define the zonotope

$$Z_A := \bigoplus_{i\in[m]} [0, y_i] \subset \mathbb{R}^n$$

as the Minkovskii sum of the intervals $[0, y_i]$, $i \in [m]$. By $\mathcal{F}(A)$ we denote the set of facets of Z_A. For any facet $H \in \mathcal{F}(A)$, we define $m(H)$ as the number of non-zero cordinates of the vector $\eta_H A \in \mathbb{R}^m$, where $\eta_H \in \mathbb{R}^n$ is a normal for H.

Let $\mathcal{C}_A^{(k)}$ be the quotient algebra

$$\mathcal{C}_A^{(k)} := \mathbb{R}[x_1, \dots, x_n]/\mathcal{I}_A^{(k)},$$

where $\mathcal{I}_A^{(k)}$ is the zonotopal ideal generated by the polynomials

$$p_H = (\eta_h \cdot (x_1, \dots, x_n))^{m(H)+k}, \quad H \in \mathcal{F}(A).$$

There are 3 main cases, where $k = \pm 1$ and 0; they were considered in [2, 10].

- $k = 1: \quad \mathcal{C}_A^{\mathcal{E}x} = \mathcal{C}_A^{(1)}$ is the *external zonotopal algebra* for A;
- $k = 0: \quad \mathcal{C}_A^{C} = \mathcal{C}_A^{(0)}$ is the *central zonotopal algebra* for A;
- $k = -1: \quad \mathcal{C}_A^{\mathcal{I}n} = \mathcal{C}_A^{(-1)}$ is the *internal zonotopal algebra* for A.

Theorem 1 (cf. [2, 4, 10, 13], External [19])**.** *For a matrix $A \in \mathbb{R}^{n\times m}$, the Hilbert series of zonotopal algebras are given by*

- $\mathcal{H}_{\mathcal{C}_A^{\mathcal{E}x}}(t) = t^{m-n} T_A(1 + t, \frac{1}{t})$;
- $\mathcal{H}_{\mathcal{C}_A^{C}}(t) = t^{m-n} T_A(1, \frac{1}{t})$;
- $\mathcal{H}_{\mathcal{C}_A^{\mathcal{I}n}}(t) = t^{m-n} T_A(0, \frac{1}{t})$,

where T_A is the Tutte polynomial of the vector configuration of columns of A (i.e., vectors $y_1, \dots, y_m$).

In the paper [16] the second author classified all external zonotopal algebras up to isomorphism.

The main interesting examples of zonotopal algebras are defined for totally unimodular matrices and graphs. The matrix A is *totally unimodular* if any of its minor is equal to ± 1 or 0. In this case the total dimensions of the algebras have a nice interpretation.

Theorem 2 (cf. [10])**.** *Let $A \in \mathbb{R}^{n\times m}$ be a totally unimodular matrix of rank n. Then the total dimension*

- *$dim(\mathcal{C}_A^{\mathcal{E}x})$ is equal to the number of lattice points of Z_A;*
- *$dim(\mathcal{C}_A^{C})$ is equal to the volume of Z_A;*
- *$dim(\mathcal{C}_A^{\mathcal{I}n})$ is equal to the number of interior lattice points of Z_A.*

Graphical algebras are particular cases of unimodular zonotopal algebras, it is easy to define them without the corresponding zonotopes, see [18].

Theorem 3 (cf. [18]). *Let G be a connected graph. Then the total dimension*

- $dim(\mathcal{C}_G^{\mathcal{E}x})$ *is equal to the number of forests in G;*
- $dim(\mathcal{C}_G^{C})$ *is equal to the number of trees in G.*

In fact external graphical and unimodular zonotopal algebras depend only on corresponding matroids.

Theorem 4 (cf. [16, 15]). *Given two totally unimodular matrices A_1 and A_2, the following are equivalent*

- *the algebras $\mathcal{C}_{A_1}^{\mathcal{E}x}$ and $\mathcal{C}_{A_2}^{\mathcal{E}x}$ are isomorphic as non-graded algebras;*
- *the algebras $\mathcal{C}_{A_1}^{\mathcal{E}x}$ and $\mathcal{C}_{A_2}^{\mathcal{E}x}$ are isomorphic as graded algebras;*
- *the matroids M_{A_1} and M_{A_2} are isomorphic.*

Furthermore, in paper [14] the K-theoretic filtration of external graphical algebra was presented, which remembers the whole graph.

There is another useful definition of external algberas from [19]. Let Φ_m be the square free commutative algebra generated by $\phi_i,\ i \in [m]$, i.e., with relations

$$\phi_i\phi_j = \phi_j\phi_i,\ i,j \in [m] \quad \text{and} \quad \phi_i^2 = 0,\ i \in [m].$$

Theorem 5 (cf. [19]). *The external algebra $\mathcal{C}_A^{\mathcal{E}x}$ is isomorphic to the subalgebra of $\Phi_A^{\mathcal{E}x} := \Phi_m$ generated by*

$$X_i := t_i \cdot (\phi_1, \dots, \phi_m),\ i \in [n].$$

1.2. **Main results.** In this paper, we will consider the Hecke deformations of square-free relations, namely

$$u_i^2 = q_i u_i \text{ for some } q_i \in \mathbb{R}.$$

The total dimension of these algebras has an interpretation in terms of score-vectors, see Theorem 14. The graphical case of these deformations was considered in [11].

In the case of totally unimodular matrices and when all parameters are the same, we can describe all relations between generators, see Theorem 17. We did it for all three cases: external (see § 4), central and internal (see § 5). We obtain another definition of original unimodular zonotopal algebras, which generalizes the result of [18].

Define Φ_A^C as the quotient algebra

$$\Phi_A^C := \Phi_A^{\mathcal{E}x}/\langle \prod_{e\in C} \phi_e,\ s \text{ is a cut}\rangle,$$

where $C \subset [m]$ is *a cut* if the dimension of $span{<}y_i,\ i \in [m] \setminus C{>}$ is less than n.

For the internal case we should define a derivative δ of the cut. We call $s \in \{\pm 1, 0\}^n$ a *cut-vector* if $supp(s \cdot A)$ is a minimal cut. We will use the following important well known lemma.

Lemma 6. *For a totally unimodular A and any of its facets $H \in \mathcal{F}(Z_a)$, there is a vector $s \in \{\pm 1, 0\}^n$ such that $\eta_H = s \cdot A$. Furthermore, $s \cdot A \in \{\pm 1, 0\}^m$ and $supp(s \cdot A)$ is a minimal cut.*

It is clear that, for any minimal cut, there is corresponding facet and, thus, there is a cut-vector.

Let s be a cut-vector, then

$$\delta(s) := \left(\prod_{i \in supp(s \cdot A)} \phi_i \right) \left(\sum_{i \in supp(s \cdot A)} \frac{1}{(s \cdot A)_i \phi_i} \right),$$

note that $(s \cdot A)_i = \pm 1,\ i \in supp(s \cdot A)$.

Define $\Phi_A^{\mathcal{I}n}$ as the quotient algebra

$$\Phi_A^{\mathcal{I}n} := \Phi_A^{\mathcal{E}x} / \langle \delta(s),\ s \text{ is a cut-vector} \rangle,$$

i.e., $\Phi_A^{\mathcal{I}n}$ is an arithmetic Orlik-Terao algebra, see [17].

Theorem 7. *Given a totally unimodular matrix $A \in \mathbb{R}^{n \times m}$ of rank n, the central algebra $\mathcal{C}_A^C$ and the internal algebra $\mathcal{C}_A^{\mathcal{I}n}$ are isomorphic to the subalgebras of Φ_A^C and of $\Phi_A^{\mathcal{I}n}$, respectively, generated by*

$$X_i := t_i \cdot (\phi_1, \ldots, \phi_m),\ i \in [n].$$

The structure of our paper is as follows: In section 2, we study bases of deformations; In section 3, we start studying Q-deformations of external zonotopal algebras. In sections 4 and 5 we study Hecke deformations of unimodular zonotopal algebras.

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2. The algebra $\Phi_{m,Q}$ and its basis

Let $Q \in (\mathbb{R} \setminus \{0\})^m$ be a vector of non-zero parameters. Define the algebra $\Phi_{m,Q}$ as the free commutative algebra generated by $u_i,\ i \in [m]$ with relations

$$(u_i)^2 = q_i u_i,\ i \in [m].$$

For a subset $E \subset [m]$, we define a monomial

$$\alpha_E := \prod_{e \in E} \frac{u_e}{q_e} \in \Phi_{m,Q}.$$

Since $q_e \neq 0$ this basis is well defined. For an element

$$z = \sum_{E \subset [m]} z_E \alpha_E \in \Phi_{m,Q},$$

we define the vector $\widetilde{z} = [\widetilde{z}_E]_{E \subset [m]} \in \mathbb{R}^{2^{e(G)}}$, where

$$\widetilde{z}_E = \sum_{E' \subset E} z_{E'}.$$

It is clear that from this vector we can reconstruct z; also it is easy to describe the product on these coordinates. Furthermore the unit element $\mathbb{1}$ is given by

$$\mathbb{1} := [1]_{E \subset [m]}.$$

Proposition 8. *Let $y, z \in \Phi_{m,Q}$ be two elements. Then the sum in tilde coordinates of elements is the sum by coordinates*

$$\widetilde{(y+z)} = \widetilde{y} + \widetilde{z},$$

and the product is the Hadamard product of coordinates

$$\widetilde{(yz)} = \widetilde{y} \circ \widetilde{z}.$$

Proof. The case of the summation is clear.

It is easy to check that for $E_1, E_2 \subset [m]$, we have

$$\alpha_{E_1} \alpha_{E_2} = \alpha_{E_1 \cup E_2}.$$

Then we obtain

$$(yz)_E = \sum_{\substack{E_1, E_2: \\ E_1 \cup E_2 = E}} y_{E_1} z_{E_2}.$$

After the change of coordinates, we get

$$\begin{aligned}\widetilde{(yz)}_E = \sum_{E' \subset E} \sum_{\substack{E_1, E_2: \\ E_1 \cup E_2 = E'}} y_{E_1} z_{E_2} &= \sum_{\substack{E_1, E_2: \\ E_1 \cup E_2 \subset E}} y_{E_1} z_{E_2} \\ &= \left(\sum_{E_1 \subset E} y_{E_1} \right) \left(\sum_{E_2 \subset E} z_{E_2} \right) = \widetilde{y}_E \widetilde{z}_E.\end{aligned}$$

Then our product in these coordinates coincides with the Hadamard product. $\square$

Corollary 9. *The elements*

$$\chi_E := [\delta_{E,E'}]_{E' \subset [m]}, \ E \subset [m]$$

form a linear basis of $\Phi_{m,Q}$. This basis has the following property

$$\chi_{E_1} \chi_{E_2} = \delta_{E_1, E_2} \cdot \chi_{E_1}.$$

Proof. For any vector we can use the Möbius inversion formula and get its element in the algebra $\Phi_{m,Q}$. Since the dimension of the space of our vectors is 2^m, this is also the dimension of the algebra $\Phi_{m,q}$. Hence, the elements $\chi_E,\ E \subset [m]$ form a linear basis. $\square$

Proposition 10. *For an element $R \in \Phi_{m,Q}$ the minimal anihilating polynomial is given by*

$$f(x) = \prod_{\beta \in \mathcal{R}} (x - \beta),$$

where $\mathcal{R}$ is the set of all coordinates of $\widetilde{R}$.

Proof. $f(x)$ is an annihilating polynomial of R, because by Proposition 8 we have

$$\widetilde{f(R)} = \prod_{\beta \in \mathcal{M}} (\widetilde{R} - \beta \cdot \mathbb{1}).$$

Checking the coordinate $E \subset [m]$: $\widetilde{R}_E \in \mathcal{R}$, there is factor of the product which has zero E-coordinate. Hence, the product has zero E-coordinate. Then all coordinates are zeroes, i.e., f is an annihilating polynomial of R.

Let $g(x)$ be the minimal unitary annihilating polynomial of R, it is clear that

$$\deg(g) = \dim(\text{span}{<}1, R, R^2, \ldots{>}).$$

We have $g|f$. If $g \neq f$, then there is $\alpha \in \mathcal{R}$ such that $g|\frac{f}{(x-\alpha)}$. Hence,

$$\prod_{\beta \in \mathcal{R} \setminus \{\alpha\}} (R - \beta) = 0.$$

Consider the coordinate $E \subset [m]$ such that $\widetilde{R}_E = \alpha$. We have $(\widetilde{R - \beta})_E \neq 0,\ \beta \in \mathcal{R}\setminus\{\alpha\}$; hence, the previous product has non-zero E-coordinate, which is impossible. We must then have $g = f$, which finishes our proof. $\square$

We have a trivial corollary:

Corollary 11. *For an element $R \in \Phi_{m,Q}$, the dimension of the subalgebra generated by R (i.e, span$<1, R, R^2, \ldots>$) is equal to the number of different coordinates of the vector $\widetilde{R}$.*

Proposition 12. *Given a set of elements $R_i \in \Phi_{m,Q},\ i \in [k]$, the dimension of the subalgebra generated by $R_i,\ i \in [k]$ is equal to the number of distinct vectors*

$$[(\widetilde{R_i})_E]_{i \in [k]} \in \mathbb{R}^k,\ E \subset [m].$$

Proof. By Proposition 8 if $[(\widetilde{R_i})_{E_1}]_{i \in [k]} = [(\widetilde{R_i})_{E_2}]_{i \in [k]}$, then the coordinates E_1 and E_2 are always the same for all elements in the subalgebra generated by $R_i,\ i \in [k]$. Hence, the dimension of the subalgebra is at most the number of distinct vectors $[(\widetilde{R_i})_E]_{i \in [k]} \in \mathbb{R}^k,\ E \subset [m]$.

For the converse, we consider an element

$$T = \sum_{i\in[k]}, a_i R_i \in \Phi_{m,Q}$$

where the $a_i \in \mathbb{R}$ are generic.

Since the $a_i \in \mathbb{R}$ are generic, the coordinates $\widetilde{T}$ are non-zeroes and $\widetilde{T}_{E_1} = \widetilde{T}_{E_2}$ if and only if $[(\widetilde{R_i})_{E_1}]_{i\in[k]} = [(\widetilde{R_i})_{E_1}]_{i\in[k]}$. Then, by Corollary 11, the dimension of the subalgebra generated by T is equal to number of distinct vectors. □

3. Q-deformations of external zonotopal algebra

Given a matrix $A \in \mathbb{R}^{n\times m}$ and a vector of parameters $Q := \{q_i \in \mathbb{R},\ i \in [m]\}$, let $\Phi^{\mathcal{E}x}_{A,Q} = \Phi_{m,Q}$ be the algebra generated by $u_i,\ i \in [m]$ with relations

$$u_i u_j = u_j u_i,\ i, j \in [m] \quad \text{and} \quad u_i^2 = q_i u_i,\ i \in [m].$$

Let $\Psi^{\mathcal{E}x}_{A,Q}$ be the filtered subalgebra of $\Phi^{\mathcal{E}x}_{A,Q}$ generated by

$$X_i := t_i \cdot (u_1, \ldots, u_m),\ i \in [n].$$

The filtered structure on $\Psi^{\mathcal{E}x}_{A,Q}$ is induced by the elements $X_i,\ i \in [n]$. More precisely, the filtered structure is an increasing sequence

$$\mathbb{R} = F_0 \subset F_1 \subset F_2 \ldots \subset F_l = \Psi^{\mathcal{E}x}_{A,Q}$$

of subspaces of $\Psi^{\mathcal{E}x}_{A,Q}$, where F_k is the linear span of all monomials $X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ such that $\alpha_1+\ldots+\alpha_n \le k$. Note that the algebra $\Phi^{\mathcal{E}x}_{A,Q}$ has a finite dimension, thus $\Psi^{\mathcal{E}x}_{A,Q}$ has also a finite dimension, which implies that the length of the filtration is finite. The Hilbert polynomial of a filtered algebra is the Hilbert polynomial of the associated graded algebra, and it has the following formula

$$\mathcal{H}(t) = 1 + \sum_{i=1} (\mathcal{HF}(i) - \mathcal{HF}(i-1))t^i,$$

where $\mathcal{HF}(i) = dim(F_i)$ is the dimension of the i-th filtered component.

We call $\Psi^{\mathcal{E}x}_{A,Q}$ the *Q-deformation* of the external zonotopal algebra of $\Psi^{\mathcal{E}x}_{A}$. In the case when all parameters coincide, i.e., $q_e = q$, $\forall e \in [m]$, we denote the corresponding algebras by $\Psi^{\mathcal{E}x}_{A,q}$ and $\Phi^{\mathcal{E}x}_{A,q}$, respectively. We refer to $\Psi^{\mathcal{E}x}_{A,q}$ as the *Hecke deformation* of external zonotopal algebra of $\Psi^{\mathcal{E}x}_{A,Q}$.

Proposition 13. *Consider two vectors of parameters Q and Q' and a non-zero number $0 \neq c \in \mathbb{R}$ such that*

$$q'_i = cq_i \text{ or } (-cq_i).$$

Then these two deformations are isomorphic, i.e.,

$$\Psi^{\mathcal{E}x}_{A,Q'} \cong \Psi^{\mathcal{E}x}_{A,Q}.$$

Proof. Denote by ϕ_i the elements and by X_i the generators from $\Phi^{\mathcal{E}x}_{A,Q}$; by ϕ'_i the elements and by X'_i the generators from $\Phi^{\mathcal{E}x}_{A,Q'}$. Consider a map ζ from the algebra $\Phi^{\mathcal{E}x}_{A,Q}$ to $\Phi'^{\mathcal{E}x}_{A,Q}$, such that

$$\zeta(\phi_i) = \frac{1}{c}\phi'_i, \text{if } q'_i = cq_i,$$

and

$$\zeta(\phi_i) = \frac{1}{c}\phi'_i + q_i, \text{if } q'_i = -cq_i.$$

Since

$$(\zeta(\phi_i) - q_i)\zeta(\phi_i) = \frac{1}{c^2}\phi'_i(\phi'_i - q'_i) = 0,$$

this map is well defined, and the inverse map is also well defined.

Then $\zeta(X_i) = \frac{1}{c}X'_i + b_i$, where $b_i \in \mathbb{R}$; hence, $\zeta(\Psi^{\mathcal{E}x}_{A,Q}) = \Psi^{\mathcal{E}x}_{A,Q'}$, and there is an inverse map, i.e., the algebras are isomorphic. □

When Q consists of non-zero elements, it is possible to describe the total dimension of the Q-deformation.

Theorem 14. *Given a matrix $A \in \mathbb{R}^{n\times m}$ and a vector of non-zero parameters $Q \in (\mathbb{R} \setminus \{0\})^m$, then the dimension of $\Psi^{\mathcal{E}x}_{A,Q}$ is equal to the number of different sums of $q_i y_i$, i.e.,*

$$dim\left(\Psi^{\mathcal{E}x}_{A,Q}\right) = \#\{A \cdot ((q_1, \dots, q_m) \circ \chi(E))^t \in \mathbb{R}^n : \; E \subset [m]\},$$

where $\circ$ is the Hadamard product and $\chi(E) \in \mathbb{R}^m$ is the characteristic vector of $E \subset [m]$.

Proof. see Proposition 12. □

Our proof works only for non-zero parameters.

Problem 1. *Describe the total dimension of Q-deformations of external zonotopal algebra, when Q allows zero numbers.*

Our method gives the total dimension, but it does not say anything about the Hilbert series.

Problem 2. *Describe the Hilbert series of the Q-deformations of the external zonotopal algebra.*

The last problem is to define Q-deformations as quotient algebras.

Problem 3. *Describe all relations between the elements $X_1, \dots, X_n \in \Psi^{\mathcal{E}x}_{A,Q}$.*

We solved all these problems for Hecke deformations of unimodular zonotopal algebras, see § 4.

4. External unimodular zonotopal algebra

In this section we will work with Hecke deformations (i.e., $q_i = q,\ i \in [m]$) of unimodular zonotopal algebras. For the case of usual unimodular zonotopal algebras, we can define the zonotopal ideals as follows:

Proposition 15. *Given a totally unimodular matrix $A \in \mathbb{R}^{n\times m}$ of rank n, then its zonotopal ideal $\mathcal{I}_A^{(k)},\ k \in \{0, \pm 1\}$ is generated by cut-vectors*

$$p_s^{(k)} = (s \cdot (x_1, \dots, x_n))^{d_s+k}, \quad s \in \{0, \pm 1\}^n,$$

where

$$d_s := |supp(s \cdot A)|.$$

Proof. Let H be a facet of Z_A; then $supp(\eta_H)$ is a minimal cut. Hence, by Lemma 6, there is a cut-vector s such that $supp(s \cdot A) = supp(\eta_H)$. □

First we describe the total dimension of the Hecke deformation $\Psi^{\mathcal{E}x}_{A,q}$, and later we describe all its relations and the Hilbert series.

Lemma 16. *Given a totally unimodular matrix $A \in \mathbb{R}^{n\times m}$ of rank n and $0 \neq q \in \mathbb{R}$, the total dimensions of the algebra $\Psi^{\mathcal{E}x}_{A,q}$ is equal to the number of lattice points of Z_A.*

Proof. By Proposition 12, we know that the dimension is equal to the number of distinct vectors $\sum_{i\in I} y_i,\ I \subset [n]$. It is well known that, for a totally unimodular matrix, the lattice points of Z_A are exactly the set $\sum_{i\in I} y_i,\ I \subset [n]$. □

We split d_s into two numbers

$$d_s^+ := |supp^+(s \cdot A)|$$

and

$$d_s^- := -|supp^-(s \cdot A)|,$$

where $supp^+(s \cdot A) \subset supp(s \cdot A)$ is the set of positive coordinates of $s \cdot A$ (i.e., which are equal to $+1$) and $supp^-(s \cdot A) \subset supp(s \cdot A)$ is the set of negative coordinates of $s \cdot A$ (i.e., which are equal to -1)

Theorem 17. *Given a totally unimodular matrix $A \in \mathbb{R}^{n\times m}$ of rank n, then the Hecke deformation $\Psi^{\mathcal{E}x}_{A,q}$ is isomorphic to the quotient algebra $\mathbb{R}[x_1, \dots, x_n]/\mathcal{I}^{\mathcal{E}x}_{A,q}$, where the ideal $\mathcal{I}^{\mathcal{E}x}_{A,q}$ is generated by cut-vectors*

$$p^{\mathcal{E}x}_{s,q} = \prod_{i=d_s^-}^{d_s^+} (s \cdot (x_1, \dots, x_n) - qi), \quad s \in \{0, \pm 1\}^n.$$

Furthermore, the Hilbert series of $\Psi^{\mathcal{E}x}_{A,q}$ is given by

$$\mathcal{H}_{\Psi^{\mathcal{E}x}_{A,Q}}(t) = t^{m-n} T_A(1 + t, \frac{1}{t}).$$

Proof. Since $\mathcal{I}^{\mathcal{E}x}_{A,q}$ is not homogeneous and, for any of its generators, the part of maximal degree coincides with the generator of $\mathcal{I}^{\mathcal{E}x}_{A}$, we have

$$\mathcal{HF}_{\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}^{\mathcal{E}x}_{A,q}}(h) \leq \mathcal{HF}_{\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}^{\mathcal{E}x}_{A}}(h) = \mathcal{HF}_{\Psi^{\mathcal{E}x}_{A,0}}(h),\ h \in \mathbb{Z}_{\geq 0}.$$

For the same reasons, we have

$$\mathcal{HF}_{\Psi^{\mathcal{E}x}_{A,0}}(h) \leq \mathcal{HF}_{\Psi^{\mathcal{E}x}_{A,q}}(h),\ h \in \mathbb{Z}_{\geq 0}.$$

Since by Proposition 10 we have relations $p^{\mathcal{E}x}_{s,q}(X_1,\ldots,X_n) = 0$ in $\Psi^{\mathcal{E}x}_{A,q}$, we obtain

$$\mathcal{HF}_{\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}^{\mathcal{E}x}_{A,q}}(h) \geq \mathcal{HF}_{\Psi^{\mathcal{E}x}_{A,q}}(h),\ h \in \mathbb{Z}_{\geq 0},$$

and if for some h we had a strict inequality, then $dim(\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}^{\mathcal{E}x}_{A,q}) < dim(\Psi^{\mathcal{E}x}_{A,q})$, which is impossible. Hence,

$$\mathcal{HF}_{\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}^{\mathcal{E}x}_{A,q}}(h) = \mathcal{HF}_{\Psi^{\mathcal{E}x}_{A,q}}(h) = \mathcal{HF}_{\Psi^{\mathcal{E}x}_{A,0}}(h),\ h \in \mathbb{Z}_{\geq 0}.$$

Then, for the algebra $\Psi^{\mathcal{E}x}_{A,q}$ we have only these relations, i.e., it is isomorphic to $\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}^{\mathcal{E}x}_{A,q}$. □

Remark 1. *In fact, we have proved that, for a totally unimodular matrix, the filtrations of its Hecke deformation $\Psi^{\mathcal{E}x}_{A,q}$ induced by X_i and, thus, induced from the algebra $\Phi^{\mathcal{E}x}_{A,q}$, are the same. In general, this is not true.*

5. Central and internal unimodular zonotopal algebras

We want to find $\Psi^{C}_{A,q}$ and $\Psi^{\mathcal{I}n}_{A,q}$ in a similar way; first we should find relations for $\Phi^{C}_{A,q}$ and for $\Phi^{\mathcal{I}n}_{A,q}$. In both cases we will have relations for cut-vectors. Let s be a cut-vector, define its relations as

$$\sigma(s,q) := \left(\prod_{i\in supp^+(s\cdot A)} u_i\right)\left(\prod_{i\in supp^-(s\cdot A)} (u_i - q)\right) \in \Phi^{\mathcal{E}x}_{A,q}.$$

Note that $\sigma(s,q) \neq \sigma(-s,q)$.

Proposition 18. *Consider a totally unimodular matrix A and $q \neq 0$. Let s be its cut-vector. Then*

$$deg(\sigma(s,q)) = |supp(s\cdot A)|;$$
$$deg(\sigma(s,q) - \sigma(-s,q)) = |supp(s\cdot A)| - 1;$$
$$(\sigma(s,q) - \sigma(-s,q))\cdot u_i = q\cdot\sigma(s,q),\ \mathit{if}\ i \in supp^+(s\cdot A);$$
$$(\sigma(s,q) - \sigma(-s,q))\cdot u_i = -q\cdot\sigma(-s,q),\ \mathit{if}\ i \in supp^-(s\cdot A);$$
$$\widetilde{\sigma(s,q)}_E = \begin{cases} (-1)^{supp^-(s\cdot A)} q^{|supp(s\cdot A)|}, & \mathit{if}\ E \cap supp(s\cdot A) = supp^+(s\cdot A); \\ 0, & \mathit{otherwise}. \end{cases}$$

Proof. The first two properties are clear. Let us check the third one. It is easy to see that if $i \in supp^+(s \cdot A)$, then $u_i\sigma(s,q) = q_i\sigma(s,q)$ and $u_i\sigma(-s,q) = 0$. The proof of the fourth property is similar.

It remains to check the last one. Note that

$$\widetilde{(u_i)}_E = \begin{cases} q, & i \in E; \\ 0, & i \notin E. \end{cases}$$

and

$$\widetilde{(u_i - q)}_E = \begin{cases} 0, & i \in E; \\ -q, & i \notin E. \end{cases}$$

Since, the E-coordinate of a product is the product of the E-coordinates of factors, we get that the E-coordinate is zero if $E \cap supp(s \cdot A) \neq supp^+(s \cdot A)$, otherwise it is equal to $(-1)^{supp^-(s\cdot A)} q^{|supp(s\cdot A)|}$. $\square$

Define the algebras $\Phi^C_{A,q}$ and $\Phi^{\mathcal{I}n}_{A,q}$ as the quotient algebras

$$\Phi^C_{A,q} := \Phi^{\mathcal{E}x}_{A,q}/\langle \sigma(s),\ s \text{ is a cut-vector s.t. } (s\cdot A)(1,2,\ldots,2^m) > 0\rangle;$$

$$\Phi^{\mathcal{I}n}_{A,q} := \Phi^{\mathcal{E}x}_{A,q}/\langle \sigma(s),\ s \text{ is a cut-vector}\rangle.$$

Define the Hecke deformations $\Psi^C_{A,q}$ and $\Psi^{\mathcal{I}n}_{A,q}$ as the subalgebras of $\Phi^C_{A,q}$ and of $\Phi^{\mathcal{I}n}_{A,q}$, respectively, generated by

$$X_i := t_i \cdot (u_1, \ldots, u_m),\ i \in [n].$$

Define $\Psi^C_{A,0}$ and $\Psi^{\mathcal{I}n}_{A,0}$ as the subalgebras of Φ^C_A and $\Phi^{\mathcal{I}n}_A$, respectively (see definition in the Introduction). Note that

$$\sigma(s,0) = \prod_{i \in supp(s\cdot A)} \phi_i$$

and

$$\frac{\sigma(s,q) - \sigma(-s,q)}{q}|_{q=0} = \delta(s).$$

Proposition 19. *Given a totally unimodular matrix $A \in \mathbb{R}^{n\times m}$ of rank n and $0 \neq q \in \mathbb{R}$. The total dimensions of the algebras $\Psi^C_{A,q}$ and $\Psi^{\mathcal{I}n}_{A,q}$ are equal to the volume and to the number of interior lattice points of Z_A, respectively.*

Now we present only a proof of lower bounds. We shall present a proof of upper bounds together with Theorem 21.

Lemma 20. *Given a totally unimodular matrix $A \in \mathbb{R}^{n\times m}$ of rank n and $0 \neq q \in \mathbb{R}$, the total dimensions of the algebras $\Psi^C_{A,q}$ and $\Psi^{\mathcal{I}n}_{A,q}$ are at least the volume and the number of interior lattice points of Z_A, respectively.*

Proof. First we will prove it for the internal case. For a cut vector, we have a relation $\sigma(s,q)=0\in\Phi^{\mathcal{I}n}_{A,q}$ and it is the same as forgetting all E-cordinates for the algebra $\Phi^{\mathcal{I}n}_{A,q}$ such that $E\cap supp(s\cdot A)=supp^+(s\cdot A)$. Then tilde basis for $\Phi^{\mathcal{I}n}_{A,q}$ consists of score vectors corresponding to subsets E such that there is no cut vector $s:\ supp(s\cdot A)\cup E=supp^+(s\cdot A)$.

Note that, for a cut vector s and the point $\sum_{i\in E} y_i\in Z_A$, the scalar product is $(\sum_{i\in E} y_i)(s\cdot A)=|E\cap supp^+(s\cdot A)|-|E\cap supp^-(s\cdot A)|$, which is maximal exactly when E is such that $supp(s\cdot A)\cup E=supp^+(s\cdot A)$. Then we should forget exactly the sums corresponding to the boundary. Using the above and Proposition 12, we get that dimension of $\Psi^{\mathcal{I}n}_{A,q}$ is exactly the number of interior lattice points.

For the central case, we should forget only for $(s\cdot A)(1,2,\ldots,2^m)>0$. Let $\mathcal{S}$ be the set of all such cut vectors. Consider the following lattice grid

$$Ł:=\{\sum_{s\in\mathcal{S}} a_s s:\ a_s\in\mathbb{Z}\}.$$

We claim that Z_A is a tiling of $\mathbb{R}^n$ for Ł, i.e.,

$$Volume((Z_A+r_1)\cap(Z_A+r_2))=0,\ r_1\neq r_2\in Ł.$$

It follows from Proposition 2.2.10 in [6].

We say that lattice points $w_1,w_2\in Z_A\cap\mathbb{Z}^m$ are equivalent if there is $r\in Ł$ such that $w_1=w_2+r$. Since Z_A is a tiling, the number of equivalence classes is the volume of Z_A. We want to prove that we do not forget the score vector corresponding to exactly one point from each class. Let χ be a equivalence class. Choose the $E_0\subset[m]$ such that

$$w:=\sum_{i\in E_0} y_i\in\chi \text{ and } \sum_{i\in E_0} 2^i \text{ is the minimal.}$$

Let us check that we do not need forget the E_0-coordinate. Assume the contrary; then there is a cut vector s such that $E_0\cup supp(s\cdot A)=supp^+(s\cdot A)$ and $(s\cdot A)(1,2,\ldots,2^m)>0$. Consider the point

$$w':=w-\sum_{i\in supp(s\cdot A)}(s\cdot A)_i y_i=\sum_{i\in E_1} y_i,$$

where $E_1=(E_0\setminus supp^+(s\cdot A))\cup supp^-(s\cdot A)$. Thus, $w'\in Z_A$ and, hence, $w'\in\chi$. Furthermore,

$$\sum_{i\in E_1} 2^i=\sum_{i\in E_0} 2^i-\left(\sum_{i\in supp^+(s\cdot A)} 2^i-\sum_{i\in supp^-(s\cdot A)} 2^i\right)=$$
$$\sum_{i\in E_0} 2^i-(s\cdot A)(1,2,\ldots,2^m)<\sum_{i\in E_0} 2^i,$$

i.e., E_0 is not the minimal, contradiction. Hence, the dimension of $\Psi^{\mathcal{I}n}_{A,q}$ is at least the volume of Z_A. □

We can describe all the relations between generators of $\Psi^C_{A,q}$ and of $\Psi^{\mathcal{I}n}_{A,q}$.

Theorem 21. *Given a totally unimodular matrix $A \in \mathbb{R}^{n\times m}$ of rank n and $0 \neq q \in \mathbb{R}$. Then the Hecke deformation $\Psi^C_{A,q}$ and $\Psi^{\mathcal{I}n}_{A,q}$ are isomorphic to the quotient algebra $\mathbb{R}[x_1,\dots,x_n]/\mathcal{I}^C_{A,q}$ and the quotient algebra $\mathbb{R}[x_1,\dots,x_n]/\mathcal{I}^{\mathcal{I}n}_{A,q}$, respectively, where the ideal $\mathcal{I}^C_{A,q}$ is generated by*

$$p^C_{s,q} = \prod_{i=d_s^- - sign(s)+1}^{d_s^+ - sign(s)} (s\cdot(x_1,\dots,x_n) - qi), \quad s \in \{0,\pm 1\}^n,$$

where $sign(s)$ is equal to $+1$ if the scalar product is positive and to 0 if negative;

the ideal $\mathcal{I}^{\mathcal{I}n}_{A,q}$ is generated by

$$p^{\mathcal{I}n}_{s,q} = \prod_{i=d_s^- +1}^{d_s^+ -1} (s\cdot(x_1,\dots,x_n) - qi), \quad s \in \{0,\pm 1\}^n.$$

Furthermore, the Hilbert series of the Hecke deformations $\Psi^C_{A,q}$ and $\Psi^{\mathcal{I}n}_{A,q}$ are given by

- $\mathcal{H}_{\Psi^C_{A,q}}(t) = t^{m-n}T_A(1,\frac{1}{t})$;
- $\mathcal{H}_{\Psi^{\mathcal{I}n}_{A,q}}(t) = t^{m-n}T_A(0,\frac{1}{t})$.

Proof of Theorem 21 and Proposition 19. Let $k = 0$ or -1.

We do not forget about the E-coordinate if and only if there is no a cut vector s such that $E \cup supp(s\cdot A) = supp^+(s\cdot A)$ (for $k=0$ additionally $(s\cdot A)(1,2,\dots,2^m) > 0$). Denote the subset of reasonable E-coordinates by $\mathcal{E}^{(k)} \subset 2^{[m]}$. Let us check that $p^{(k)}_{s,q}(X_1,\dots,X_n) \in \Psi^{(k)}_{A,Q}$ vanishes. Consider a cut vector s; then we should prove that for any $E \in \mathcal{E}^{(k)}$ there is a factor such that its E-coordinate is zero. We know that for any $E \subset [m]$ there is such a factor in $p^{\mathcal{E}x}_{s,q}(X_1,\dots,X_n)$.

Let us check the central case (to check the internal case everything is the same, but for one factor we use s and for another $-s$). It is enough to prove that there is no $E \in \mathcal{E}^{(C)}$ such that the E-coordinate is a zero of the factor $(s\cdot(X_1,\dots,X_n) - qd^*_s)$, where $d^*_s = d^+_s$ if $(s\cdot A)(1,2,\dots,2^m)$ is positive and $d^*_s = d^-_s$ if $(s\cdot A)(1,2,\dots,2^m)$ is negative. Note that the E-coordinate of $(s\cdot(X_1,\dots,X_n) - qd^+_s) = ((s\cdot A)(u_1,\dots,u_m) - qd^+_s)$ is equal to

$$q(|E\cap supp^+(s\cdot A)| - |E \cap supp^-(s\cdot A)| - d^+_s).$$

Since $|supp^+(s\cdot A)| = d_s^+$, we obtain that the E-coordinate of that factor being zero is equiivalent to $E\cap supp(s\cdot A) = supp^+(s\cdot A)$, then $E \notin \mathcal{E}^{(k)}$.

We know that all $p_{s,q}^{(k)}(X_1,\ldots,X_n) \in \Psi_{A,Q}^{(k)}$, for cut vectors vanish in $\Phi_{A,Q}^{(k)}$. We obtain

$$dim(\Psi_{A,q}^{(k)}) \le dim(\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}_{A,q}^{(k)}) \le \\ dim(\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}_{A,0}^{(k)}) = dim(\mathcal{C}_A^{(k)}).$$

Using Lemma 20, we get that all these dimensions are equal and, then, $\Psi_{A,q}^{(k)}$ and $\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}_{A,q}^{(k)}$ are isomorphic. Hence, the Hilbert series of $\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}_{A,q}^{(k)}$ and that of $\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}_{A,0}^{(k)}$ coincide. We have proved Theorem 21 and Proposition 19. $\square$

Proof of Theorem 7. Consider $0 \neq q \in \mathbb{R}$. By Theorem 21 we know that

$$\Psi_{A,q}^{(k)} = \Psi_{A,q}^{\mathcal{E}x}/\mathcal{J}_{A,q}^{(k)},$$

where $\mathcal{J}_{A,q}^{(k)} = \langle p_{s,q}^{(k)}(X_1,\ldots,X_n),\ s \text{ is a cut-vector}\rangle$ is an ideal in $\Phi_{A,q}^{(k)}$. Since the associated graded ring for $\Phi_{A,q}^{\mathcal{E}x}$ is $\Phi_{A,0}^{\mathcal{E}x}$, we get, for subalgebras of their quotients algebras $\Phi_{A,q}^{(k)}$ is $\Phi_{A,0}^{(k)}$,

$$\mathcal{HF}_{\Psi_{A,q}^{(k)}}(h) \le \mathcal{HF}_{\Psi_{A,0}^{(k)}}(h),\ h \in \mathbb{Z}_{\ge 0}.$$

On the other hand, we already know that

$$\mathcal{HF}_{\Psi_{A,0}^{(k)}}(h) \le \mathcal{HF}_{\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}_{A,0}^{(k)}}(h) = \mathcal{HF}_{\mathbb{R}[x_1,\ldots,x_n]/\mathcal{I}_{A,q}^{(k)}}(h) = \mathcal{HF}_{\Psi_{A,q}^{(k)}}(h).$$

Hence, $\mathcal{HF}_{\Psi_{A,q}^{(k)}}(h) = \mathcal{HF}_{\Psi_{A,0}^{(k)}}(h),\ h \in \mathbb{Z}_{\ge 0}$ and, thus, $\Psi_{A,0}^{(k)}$ and $\mathcal{C}_A^{(k)}$ are isomorphic. $\square$

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