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Abstract. — In the present paper, we discuss the hyperbolic ordinariness of hyperelliptic curves in characteristic three. In particular, we prove that every hyperelliptic projective hyperbolic curve of genus less than or equal to five in characteristic three is hyperbolically ordinary.

Contents

Introduction

In the present paper, we study the theory of *hyperbolically ordinary curves* established in [4]. Let us first recall that we shall say that a hyperbolic curve over a connected noetherian scheme of odd characteristic is *hyperbolically ordinary* [cf. [4], Chapter II, Definition 3.3] if, étale locally on the base noetherian scheme, the hyperbolic curve admits a nilpotent [cf. [4], Chapter II, Definition 2.4] ordinary [cf. [4], Chapter II, Definition 3.1] indigenous bundle [cf. [4], Chapter I, Definition 2.2]. In the present paper, we consider the following [weaker version of the] basic question in *p-adic Teichmüller theory* discussed in [5], Introduction, §2.1, (1):

Is every hyperbolic curve in odd characteristic *hyperbolically ordinary*?

Let us recall that one important result of the theory of hyperbolically ordinary curves is that every *sufficiently general* hyperbolic curve is *hyperbolically ordinary* [cf. [4], Chapter II, Corollary 3.8]. Moreover, it has already been proved that, for nonnegative integers g, r and a prime number p, if either

$$(g, r) \;=\; (0, 3),$$

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or

$$(g, r, p) \in \{(0, 4, 3),\ (1, 1, 3),\ (1, 1, 5),\ (1, 1, 7),\ (2, 0, 3)\},$$

then every hyperbolic curve of type (g, r) in characteristic p is *hyperbolically ordinary* [cf. [2], Theorem C; the remarks following [2], Theorem C]. The main result of the present paper gives an affirmative answer to the above question for *hyperelliptic curves of lower genus in characteristic three* [cf. Theorem A below].

Now let us recall some discussions of [1], §A. Let k be an algebraically closed field *of characteristic three* and X a *projective hyperbolic curve* over k. Write X^F for the projective hyperbolic curve over k obtained by base-changing X via the absolute Frobenius morphism of k and $\Phi\colon X \to X^F$ for the relative Frobenius morphism over k. Let $(\mathcal{L}, \Theta)$ be a square-trivialized invertible sheaf on X [cf. [1], Definition A.3], i.e., a pair consisting of an invertible sheaf $\mathcal{L}$ on X and an isomorphism $\Theta\colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$. Then the isomorphism Θ determines an isomorphism $\mathcal{L} \xrightarrow{\sim} \Phi^*\mathcal{L}^F$ [cf. the discussion following [1], Definition A.3] — where we write $\mathcal{L}^F$ for the invertible sheaf on X^F obtained by pulling back $\mathcal{L}$ via the absolute Frobenius morphism of k. Moreover, this isomorphism and the [usual] Cartier operator $\Phi_*\omega_{X/k} \to \omega_{X^F/k}$ — where we use the notation "ω" to denote the cotangent sheaf — determine a k-linear homomorphism

$$C_{(\mathcal{L},\Theta)}\colon\ \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/k}) \longrightarrow \Gamma(X^F, \mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/k}),$$

i.e., the *Cartier operator associated to* $(\mathcal{L}, \Theta)$ [cf. [1], Definition A.4]. We shall say that a nonzero global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/k}$ is a *normalized Cartier eigenform* associated to $(\mathcal{L}, \Theta)$ [cf. [1], Definition A.8, (i)] if $C_{(\mathcal{L},\Theta)}(u) = -u^F$ — where we write u^F for the global section of $\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/k}$ obtained by pulling back u via the absolute Frobenius morphism of k.

Under this preparation, an immediate consequence of [1], Theorem B, is as follows:

> In the above situation, it holds that the projective hyperbolic curve X over k is *hyperbolically ordinary* if and only if there exist a square-trivialized invertible sheaf $(\mathcal{L}, \Theta)$ on X and a nonzero global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/k}$ that satisfy the following three conditions:
>
> • The zero divisor of the nonzero global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/k}$ is *reduced*.
>
> • The nonzero global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/k}$ is a *normalized Cartier eigenform* associated to $(\mathcal{L}, \Theta)$.
>
> • If $\mathcal{L}$ is *trivial*, then the Jacobian variety of X is an *ordinary* abelian variety over k. If $\mathcal{L}$ is *nontrivial* [i.e., of order two], then the Prym variety associated to $\mathcal{L}$ is an *ordinary* abelian variety over k.
>
> [Now let us recall that we shall refer to an effective divisor on X obtained by forming the zero divisor of "u" as above as a divisor *of CEO-type* — cf. Definition 5.1, (iii).]

In the present paper, by applying this consequence to *hyperelliptic projective hyperbolic curves in characteristic three*, we prove the following result [cf. Corollary 4.3]:

THEOREM A. — *Every* **hyperelliptic** *projective hyperbolic curve* **of genus ≤ 5** *over a connected noetherian scheme of characteristic* 3 *is* **hyperbolically ordinary**.

The present paper is organized as follows: In §1, we prove some lemmas on polynomials in characteristic three. In §2, we recall some basic facts concerning hyperelliptic curves in characteristic three. In §3, we discuss divisors of CEO-type on hyperelliptic curves in characteristic three. In §4, we prove the main result of the present paper.

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1. SOME LEMMAS ON POLYNOMIALS

In the present §1, let us prove some lemmas on polynomials *in characteristic three.* In the present §1, let k be an algebraically closed field *of characteristic three.* Write

$$\mathbb{A}^1_k$$

for the affine line over k and

$$\mathbb{P}^1_k$$

for the smooth compactification of $\mathbb{A}^1_k$, i.e, the projective line over k. Thus, there exists a regular function $t \in \Gamma(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k})$ on $\mathbb{A}^1_k$ that determines an isomorphism of k-algebras

$$k[t] \xrightarrow{\sim} \Gamma(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}).$$

Moreover, one verifies easily that this regular function $t \in \Gamma(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k})$ on $\mathbb{A}^1_k$ determines a bijection between

- the set of closed points of $\mathbb{A}^1_k$ (respectively, $\mathbb{P}^1_k$)

and

- the set k (respectively, $k \cup \{\infty\}$).

In the remainder of the present paper, let us fix such a regular function $t \in \Gamma(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k})$ on $\mathbb{A}^1_k$ and identify the set of closed points of $\mathbb{A}^1_k$ (respectively, $\mathbb{P}^1_k$) with the set k (respectively, $k \cup \{\infty\}$) by means of the bijection determined by this fixed t.

If d is a nonnegative integer, then we shall write

$$k[t^d] \subseteq k[t]$$

for the k-subalgebra of $k[t]$ generated by $t^d \in k[t]$ and

$$k[t]^{\leq d} \subseteq k[t]$$

for the k-submodule of $k[t]$ generated by $1, t, \ldots, t^d \in k[t]$.

DEFINITION 1.1. — Let $f(t) \in k[t] \setminus k$ be an element of $k[t]$ of positive degree.

(i) We shall write

$$g_{f(t)}$$

for the uniquely determined nonnegative integer such that the polynomial $f(t)$ is of degree either $2g_{f(t)}+1$ or $2g_{f(t)}+2$.

(ii) Suppose that $g_{f(t)} \geq 1$. Then we shall write

$$C_{f(t)} \colon \ k[t]^{\leq g_{f(t)}-1} \ \longrightarrow \ k[t]$$

for the k-linear homomorphism given by

$$g(t) \ \mapsto \ -\frac{d^2}{dt^2}\big(f(t)\cdot g(t)\big).$$

Note that one verifies easily that the image of this homomorphism is contained in

$$k[t]^{\leq 3g_{f(t)}-3} \cap k[t^3] \ \subseteq \ k[t],$$

i.e., the k-submodule of $k[t]$ generated by $1, t^3, \ldots, t^{3g_{f(t)}-3} \in k[t]$.

(iii) Suppose that $g_{f(t)} \geq 1$. Then the composite

$$k[t]^{\leq g_{f(t)}-1} \ \stackrel{C_{f(t)}}{\longrightarrow} \ k[t]^{\leq 3g_{f(t)}-3} \cap k[t^3] \ \stackrel{\sim}{\longrightarrow} \ k[t]^{\leq g_{f(t)}-1}$$

— where the second arrow is the k-linear isomorphism given by "$g(t^3) \mapsto g(t)$" — is a k-linear endomorphism of $k[t]^{\leq g_{f(t)}-1}$. We shall write

$$D_{f(t)} \ \in \ k$$

for the determinant of this k-linear endomorphism.

(iv) Suppose that $g_{f(t)} = 0$. Then we shall write

$$D_{f(t)} \ = \ 1 \ \in \ k.$$

REMARK 1.1.1. — In the situation of Definition 1.1, suppose that $g_{f(t)} \geq 1$. Then it is immediate that it holds that the k-linear homomorphism $C_{f(t)}$ of Definition 1.1, (ii), is *injective* if and only if $D_{f(t)} \in k$ of Definition 1.1, (iii), is *nonzero*.

LEMMA 1.2. — *Let $f(t) = c_0 + c_1 t + \cdots + c_d t^d \in k[t] \setminus k$ be an element of $k[t]$ of positive degree. Then the following hold:*

(i) *Suppose that $g_{f(t)} = 1$. Then it holds that*

$$C_{f(t)}(1) \ = \ c_2.$$

(ii) *Suppose that $g_{f(t)} = 2$. Then it holds that*

$$C_{f(t)}(1,t) \ = \ (1,t^3)\begin{pmatrix} c_2 & c_1 \\ c_5 & c_4 \end{pmatrix}.$$

(iii) *Suppose that $g_{f(t)} = 3$. Then it holds that*

$$C_{f(t)}(1,t,t^2) \ = \ (1,t^3,t^6)\begin{pmatrix} c_2 & c_1 & c_0 \\ c_5 & c_4 & c_3 \\ c_8 & c_7 & c_6 \end{pmatrix}.$$

PROOF. — These assertions follow from straightforward calculations. $\square$

DEFINITION 1.3. — Let

$$A \subseteq \mathbb{P}^1_k$$

be a nonempty finite closed subset of $\mathbb{P}^1_k$ of even cardinality. [So A may be regarded as a finite subset of $k \cup \{\infty\}$ — cf. the discussion at the beginning of the present §1.]

(i) We shall write

$$f_A(t) \stackrel{\text{def}}{=} \prod_{a \in A \setminus (A \cap \{\infty\})} (t-a) \in k[t].$$

(ii) We shall write

$$g_A \stackrel{\text{def}}{=} g_{f_A(t)}, \quad D_A \stackrel{\text{def}}{=} D_{f_A(t)}$$

[cf. Definition 1.1, (i), (iii), (iv)]. If, moreover, $g_A \geq 1$, then we shall write

$$C_A \stackrel{\text{def}}{=} C_{f_A(t)}$$

[cf. Definition 1.1, (ii)].

(iii) We shall refer to a [nonordered] pair $\{A_1, A_2\}$ of two nonempty subsets $A_1, A_2 \subseteq A$ of A such that $A_1 \cup A_2 = A$, $A_1 \cap A_2 = \emptyset$, and A_1 — hence also A_2 — is of even cardinality as an *even decomposition* of A.

REMARK 1.3.1. — In the situation of Definition 1.3, one verifies easily that the following three conditions are equivalent:

(1) The polynomial $f_A(t)$ is *of odd degree*.

(2) The polynomial $f_A(t)$ is *of degree* $2g_A + 1$.

(3) It holds that $\infty \in A$.

LEMMA 1.4. — *Let a, b, $c \in k \setminus \{0\}$ be three distinct elements of $k \setminus \{0\}$. Then the following hold:*

(i) *The set*

$$\{ d \in k \setminus \{a, b, c\} \mid D_{\{a,b,c,d\}} = 0 \}$$

is **of cardinality ≤ 1**.

(ii) *The set*

$$\{ d \in k \setminus \{0, a, b, c\} \mid D_{\{0,\infty,a,b,c,d\}} = 0 \}$$

is **of cardinality ≤ 2**.

PROOF. — Write

$$f(t,x) \stackrel{\text{def}}{=} (t-a)(t-b)(t-c)(t-x)$$

$$= abcx - (abc+abx+acx+bcx)\cdot t + (ab+ac+ax+bc+bx+cx)\cdot t^2 - (a+b+c+x)\cdot t^3 + t^4 \in k[t,x]$$

— where x is an indeterminate —

$$\mathbb{D}_{f(t,x)} \stackrel{\text{def}}{=} ab + ac + ax + bc + bx + cx \in k[x],$$

$$\mathbb{D}_{t\cdot f(t,x)} \stackrel{\text{def}}{=} \det \begin{pmatrix} abc + abx + acx + bcx & abcx \\ 1 & a+b+c+x \end{pmatrix} \in k[x].$$

Then it follows from Lemma 1.2, (i) (respectively, (ii)), that, for each $d \in k \setminus \{a, b, c\}$ (respectively, $d \in k \setminus \{0, a, b, c\}$), it holds that

$$\mathbb{D}_{f(t,x)}|_{x=d} \;=\; D_{\{a,b,c,d\}}, \quad \text{(respectively, } \mathbb{D}_{t \cdot f(t,x)}|_{x=d} \;=\; D_{\{0,\infty,a,b,c,d\}}\text{)}.$$

Thus, since [one verifies easily that] $\mathbb{D}_{f(t,x)}$ (respectively, $\mathbb{D}_{t \cdot f(t,x)}$) is, as a polynomial of x, *of degree* ≤ 1 (respectively, ≤ 2), to verify assertion (i) (respectively, (ii)), it suffices to show that the element $\mathbb{D}_{f(t,x)}$ (respectively, $\mathbb{D}_{t \cdot f(t,x)}$) of $k[x]$ is *nonzero*.

To verify assertion (i), assume that $\mathbb{D}_{f(t,x)} = 0$, or, alternatively,

$$a + b + c \;=\; 0, \quad ab + ac + bc \;=\; 0.$$

Then we obtain that

$$0 \;=\; ab + (a + b)c \;=\; ab + (a + b)(-a - b) \;=\; -(a^2 + ab + b^2) \;=\; -(a - b)^2.$$

Thus, since [we have assumed that] $a \neq b$, we obtain a *contradiction*. This completes the proof of assertion (i).

Next, to verify assertion (ii), assume that $a + b + c \neq 0$, and that $\mathbb{D}_{t \cdot f(t,x)} = 0$. Now observe that it is immediate from the above description of $\mathbb{D}_{t \cdot f(t,x)}$ that

$$0 \;=\; \mathbb{D}_{t \cdot f(t,x)}|_{x=0} \;=\; abc(a + b + c).$$

Thus, since [we have assumed that] $0 \notin \{a, b, c, a + b + c\}$, we obtain a *contradiction*.

Finally, to complete the verification of assertion (ii), assume that $a + b + c = 0$, and that $\mathbb{D}_{t \cdot f(t,x)} = 0$. Now observe that it is immediate from the above description of $\mathbb{D}_{t \cdot f(t,x)}$ that

$$a + b + c \;=\; 0, \quad ab + ac + bc \;=\; 0.$$

Thus, it follows from a similar argument to the argument applied in the proof of assertion (i) that we obtain a *contradiction*. This completes the proof of assertion (ii), hence also of Lemma 1.4. $\square$

LEMMA 1.5. — *Let a, b, c, $d \in k \setminus \{0\}$ be four distinct elements of $k \setminus \{0\}$. Suppose that $ab + ac + ad + bc + bd + cd \neq 0$. Then the subset of $k \setminus \{0, a, b, c, d\}$ consisting of the elements $e \in k \setminus \{0, a, b, c, d\}$ that satisfy the following condition is* **of cardinality ≤ 2**: *The set*

$$\{\, f \in k \setminus \{0, a, b, c, d, e\} \mid D_{\{0,\infty,a,b,c,d,e,f\}} = 0 \,\}$$

is **of cardinality > 3**.

PROOF. — Write

$$g(t, x, y) \;\stackrel{\text{def}}{=}\; t(t - a)(t - b)(t - c)(t - d)(t - x)(t - y)$$

— where x and y are indeterminates. Thus, there exist elements $c_1(x, y), \ldots, c_6(x, y) \in k[x, y]$ such that

$$g(t, x, y) \;=\; c_1(x, y) \cdot t + c_2(x, y) \cdot t^2 + \cdots + c_6(x, y) \cdot t^6 + t^7.$$

Write, moreover,

$$\mathbb{D}_{g(t,x,y)} \;\stackrel{\text{def}}{=}\; \det \begin{pmatrix} c_2(x, y) & c_1(x, y) & 0 \\ c_5(x, y) & c_4(x, y) & c_3(x, y) \\ 0 & 1 & c_6(x, y) \end{pmatrix} \;\in\; k[x, y].$$

Then it follows from Lemma 1.2, (iii), that, for each pair (e, f) of two distinct elements of $k \setminus \{0, a, b, c, d\}$, it holds that

$$\mathbb{D}_{g(t,x,y)}|_{(x,y)=(e,f)} \;=\; D_{\{0,\infty,a,b,c,d,e,f\}}.$$

Thus, since [one verifies easily that], for each $e \in k \setminus \{0, a, b, c, d\}$, the element $\mathbb{D}_{g(t,x,y)}|_{x=e}$ is, as a polynomial of y, *of degree* ≤ 3, to verify Lemma 1.5, it suffices to show that the set

$$\{\, e \in k \setminus \{0, a, b, c, d\} \mid \mathbb{D}_{g(t,x,y)}|_{x=e} = 0 \,\}$$

is *of cardinality* ≤ 2. In particular, to verify Lemma 1.5, it suffices to show that the set

$$\{\, e \in k \setminus \{0, a, b, c, d\} \mid \mathbb{D}_{g(t,x,y)}|_{(x,y)=(e,0)} = 0 \,\}$$

is *of cardinality* ≤ 2.

Next, observe that one verifies easily that

$$c_1(x,0) \;=\; 0, \quad c_2(x,0) \;=\; -abcdx, \quad c_3(x,0) \;=\; abcd + abcx + abdx + acdx + bcdx,$$

$$c_4(x,0) \;=\; -abc - abd - abx - acd - acx - adx - bcd - bcx - bdx - cdx,$$

$$c_6(x,0) \;=\; -a - b - c - d - x.$$

Thus, we obtain that

$$\mathbb{D}_{g(t,x,y)}|_{y=0} \;=\; c_2(x,0) \cdot \big(c_4(x,0) \cdot c_6(x,0) - c_3(x,0)\big).$$

In particular, since [one verifies immediately that] the element $c_4(x,0) \cdot c_6(x,0) - c_3(x,0)$ is, as a polynomial of x, *of degree* ≤ 2, and [we have assumed that] $c_2(x,0) = -abcdx \neq 0$, to verify Lemma 1.5, it suffices to show that

$$c_4(x,0) \cdot c_6(x,0) - c_3(x,0) \;\neq\; 0.$$

On the other hand, this follows from our assumption that $ab+ac+ad+bc+bd+cd \neq 0$. This completes the proof of Lemma 1.5. $\square$

LEMMA 1.6. — *Let $A \subseteq \mathbb{P}^1_k$ be a nonempty finite subset of $\mathbb{P}^1_k$* **of cardinality 8**. *Suppose that $0, \infty \notin A$. Then there exists an* **even decomposition** *$\{A_1, A_2\}$ of A that satisfies the following two conditions:*

(1) *The subset A_1 is* **of cardinality 4** [*which thus implies that the subset A_2 is* **of cardinality 4**].

(2) *Both the elements D_{A_1}, $D_{A_2} \in k$ are* **nonzero**.

PROOF. — Take three distinct elements $a_1, a_2, a_3 \in A$. Then since the set $A \setminus \{a_1, a_2, a_3\}$ is *of cardinality* 5, it follows from Lemma 1.4, (i), that there exists a subset $A' \subseteq A \setminus \{a_1, a_2, a_3\}$ *of cardinality* 4 such that each element $a_4 \in A'$ satisfies the condition that

(a) $D_{\{a_1,a_2,a_3,a_4\}} \neq 0$.

Write $b_1 \in A \setminus (\{a_1, a_2, a_3\} \cup A')$ for the unique element. Let $b_2, b_3 \in A \setminus \{a_1, a_2, a_3, b_1\}$ be two distinct elements. Then since the set $A \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\}$ is *of cardinality* 2, it follows from Lemma 1.4, (i), that there exists an element $b_4 \in A \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\}$ such that

(b) $D_{\{b_1,b_2,b_3,b_4\}} \neq 0$.

Write $A_1 \stackrel{\text{def}}{=} \{b_1, b_2, b_3, b_4\}$ and $A_2 \stackrel{\text{def}}{=} A \setminus A_1$. Then one verifies immediately [cf. (a), (b)] that $\{A_1, A_2\}$ is an even decomposition of A of the desired type. This completes the proof of Lemma 1.6. $\square$

LEMMA 1.7. — *Let $A \subseteq \mathbb{P}^1_k$ be a nonempty finite subset of $\mathbb{P}^1_k$* **of cardinality 10**. *Suppose that $0,\ \infty \in A$. Then there exists an* **even decomposition** $\{A_1, A_2\}$ *of A that satisfies the following two conditions:*

(1) *The subset A_1 is* **of cardinality 4** [*which thus implies that the subset A_2 is* **of cardinality 6**].

(2) *Both the elements $D_{A_1},\ D_{A_2} \in k$ are* **nonzero**.

PROOF. — Take three distinct elements $a_1,\ a_2,\ a_3 \in A \setminus \{0, \infty\}$. Then since the set $A \setminus \{0, \infty, a_1, a_2, a_3\}$ is *of cardinality* 5, it follows from Lemma 1.4, (ii), that there exists a subset $A' \subseteq A \setminus \{0, \infty, a_1, a_2, a_3\}$ *of cardinality* 3 such that each element $a_4 \in A'$ satisfies the condition that

(a) $D_{\{0,\infty,a_1,a_2,a_3,a_4\}} \neq 0$.

Write $b_1,\ b_2 \in A \setminus (\{0, \infty, a_1, a_2, a_3\} \cup A')$ for the two distinct elements. Let $b_3 \in A \setminus \{0, \infty, a_1, a_2, a_3, b_1, b_2\}$ be an element. Then since the set $A \setminus \{0, \infty, a_1, a_2, a_3, b_1, b_2, b_3\}$ is *of cardinality* 2, it follows from Lemma 1.4, (i), that there exists an element $b_4 \in A \setminus \{0, \infty, a_1, a_2, a_3, b_1, b_2, b_3\}$ such that

(b) $D_{\{b_1,b_2,b_3,b_4\}} \neq 0$.

Write $A_1 \stackrel{\text{def}}{=} \{b_1, b_2, b_3, b_4\}$ and $A_2 \stackrel{\text{def}}{=} A \setminus A_1$. Then one verifies immediately [cf. (a), (b)] that $\{A_1, A_2\}$ is an even decomposition of A of the desired type. This completes the proof of Lemma 1.7. $\square$

LEMMA 1.8. — *Let $A \subseteq \mathbb{P}^1_k$ be a nonempty finite subset of $\mathbb{P}^1_k$* **of cardinality 12**. *Suppose that $0,\ \infty \in A$. Then there exists an* **even decomposition** $\{A_1, A_2\}$ *of A that satisfies the following two conditions:*

(1) *The subset A_1 is* **of cardinality 4** [*which thus implies that the subset A_2 is* **of cardinality 8**].

(2) *Both the elements $D_{A_1},\ D_{A_2} \in k$ are* **nonzero**.

PROOF. — Take three distinct elements $a_1,\ a_2,\ a_3 \in A \setminus \{0, \infty\}$. Observe that we may assume without loss of generality, by replacing a_3 by a suitable element of the set $A \setminus \{0, \infty, a_1, a_2\}$ *of cardinality* 8, that

(a) $a_1 + a_2 + a_3 \neq 0$.

Thus, since the set $A \setminus \{0, \infty, a_1, a_2, a_3\}$ is *of cardinality* 7, there exists an element $a_4 \in A \setminus \{0, \infty, a_1, a_2, a_3\}$ such that

(b) $a_4(a_1 + a_2 + a_3) + a_1a_2 + a_2a_3 + a_3a_1 \neq 0$.

In particular, since the set $A \setminus \{0, \infty, a_1, a_2, a_3, a_4\}$ is *of cardinality* 6, it follows from Lemma 1.5, together with (b), that there exists an element $a_5 \in A \setminus \{0, \infty, a_1, a_2, a_3, a_4\}$

and a subset $A' \subseteq A \setminus \{0, \infty, a_1, a_2, a_3, a_4, a_5\}$ *of cardinality* 2 such that each element $a_6 \in A'$ satisfies the condition that

(c) $D_{\{0,\infty,a_1,a_2,a_3,a_4,a_5,a_6\}} \neq 0$.

Write $b_1, b_2, b_3 \in A \setminus (\{0, \infty, a_1, a_2, a_3, a_4, a_5\} \cup A')$ for the three distinct elements. Then since the set $A \setminus \{0, \infty, a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3\}$ is *of cardinality* 2, it follows from Lemma 1.4, (i), that there exists an element $b_4 \in A \setminus \{0, \infty, a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3\}$ such that

(d) $D_{\{b_1,b_2,b_3,b_4\}} \neq 0$.

Write $A_1 \stackrel{\text{def}}{=} \{b_1, b_2, b_3, b_4\}$ and $A_2 \stackrel{\text{def}}{=} A \setminus A_1$. Then one verifies immediately [cf. (c), (d)] that $\{A_1, A_2\}$ is an even decomposition of A of the desired type. This completes the proof of Lemma 1.8. $\square$

2. BASIC FACTS CONCERNING HYPERELLIPTIC CURVES

In the present §2, let us recall some basic facts concerning *hyperelliptic curves in characteristic three*. In the present §2, we maintain the notational conventions introduced at the beginning of the preceding §1. Let

$$A \subseteq \mathbb{P}^1_k$$

be a nonempty finite closed subset of $\mathbb{P}^1_k$ *of even cardinality*. [So A may be regarded as a finite subset of $k \cup \{\infty\}$ — cf. the discussion at the beginning of §1.]

DEFINITION 2.1. — We shall write

$$\xi_A \colon X_A \longrightarrow \mathbb{P}^1_k$$

for the [uniquely determined] connected finite flat covering of degree two whose branch locus coincides with the reduced closed subscheme of $\mathbb{P}^1_k$ determined by A,

$$\iota_A \in \mathrm{Gal}(X_A/\mathbb{P}^1_k)$$

for the [uniquely determined] nontrivial automorphism of X_A over $\mathbb{P}^1_k$, and

$$\omega_A \stackrel{\text{def}}{=} \omega_{X_A/k}$$

for the cotangent sheaf of X_A/k.

REMARK 2.1.1. — It follows from the *Riemann-Hurwitz formula* that X_A is a projective smooth curve *of genus* g_A [cf. Definition 1.3, (ii)].

One verifies easily that there exists an open immersion over $\mathbb{P}^1_k$ from the affine scheme

$$\mathrm{Spec}\big(k[s,t]/(s^2 - f_A(t))\big)$$

[cf. Definition 1.3, (i)] into X_A. Let us fix such an open immersion by means of which we regard the above affine scheme as an open subscheme of X_A. Write

$$U_A \subseteq X_A$$

for the [uniquely determined] maximal open subscheme of the resulting affine open subscheme of X_A on which the function s is invertible. [So we have an identification

$$\mathrm{Spec}\big(k[s, s^{-1}, t]/(s^2 - f_A(t))\big) \;=\; U_A$$

of affine schemes.]

LEMMA 2.2. — *Suppose that $g_A \geq 1$. Then the restriction homomorphism*

$$\Gamma(X_A, \omega_A) \;\longrightarrow\; \Gamma(U_A, \omega_A)$$

determines an **isomorphism**

$$\Gamma(X_A, \omega_A) \;\xrightarrow{\sim}\; \Big\{ \frac{f(t)dt}{s} \in \Gamma(U_A, \omega_A) \;\Big|\; f(t) \in k[t]^{\leq g_A - 1} \Big\}.$$

PROOF. — This assertion follows immediately from a straightforward calculation. ☐

DEFINITION 2.3. — Suppose that $g_A \geq 1$. Then, for each $f(t) \in k[t]^{\leq g_A - 1}$, we shall write

$$\omega_{f(t)} \;\in\; \Gamma(X_A, \omega_A)$$

for the global section of ω_A corresponding, via the isomorphism of Lemma 2.2, to

$$\frac{f(t)dt}{s} \;\in\; \Gamma(U_A, \omega_A).$$

LEMMA 2.4. — *Suppose that $g_A \geq 1$. Let $f(t) \in k[t]^{\leq g_A - 1} \setminus \{0\}$ be a nonzero element of $k[t]^{\leq g_A - 1}$. Thus, there exist elements $a, a_1, \ldots, a_d \in k$ of k such that*

$$f(t) \;=\; a \cdot \prod_{i=1}^{d} (t - a_i).$$

Then the **zero divisor** *of the nonzero global section $\omega_{f(t)}$ of ω_A is given by the pull-back, via ξ_A, of the divisor on $\mathbb{P}^1_k$*

$$(g_A - 1 - d)[\infty] + \sum_{i=1}^{d} [a_i]$$

— where we write "$[-]$" for the prime divisor on $\mathbb{P}^1_k$ determined by the closed point of $\mathbb{P}^1_k$ corresponding to "$(-)$".

PROOF. — This assertion follows immediately from the definition of the global section $\omega_{f(t)}$. ☐

DEFINITION 2.5. — Let $\{A_1, A_2\}$ be an even decomposition of A [cf. Definition 1.3, (iii)]. Then we shall write

$$X_{A_1, A_2} \;\stackrel{\mathrm{def}}{=}\; X_{A_1} \times_{\mathbb{P}^1_k} X_{A_2}$$

for the fiber product of ξ_{A_1} and ξ_{A_2},

$$\iota_{A_1, A_2} \;\stackrel{\mathrm{def}}{=}\; (\iota_{A_1}, \iota_{A_2}) \;\in\; \mathrm{Gal}(X_{A_1, A_2}/\mathbb{P}^1_k)$$

for the automorphism of X_{A_1,A_2} over $\mathbb{P}^1_k$ [necessarily of order two] determined by the automorphisms ι_{A_1} and ι_{A_2}, and

$$\omega_{A_1,A_2} \stackrel{\text{def}}{=} \omega_{X_{A_1,A_2}/k}$$

for the cotangent sheaf of $X_{A_1,A_2}/k$.

LEMMA 2.6. — *Suppose that $g_A \geq 2$. Then the following hold:*

(i) *Let $\{A_1, A_2\}$ be an even decomposition of A. Then X_{A_1,A_2} is a* **projective hyperbolic curve of genus $2g_A - 1$**. *Moreover, there exists a* **finite étale morphism of degree two** *over $\mathbb{P}^1_k$*

$$\xi_{A_1,A_2} \colon \ X_{A_1,A_2} \longrightarrow X_A$$

such that the Galois group $\mathrm{Gal}(X_{A_1,A_2}/X_A)$ of ξ_{A_1,A_2} is generated by $\iota_{A_1,A_2} \in \mathrm{Gal}(X_{A_1,A_2}/\mathbb{P}^1_k)$.

(ii) *There exists a uniquely determined* **bijection** *between*

- *the set of isomorphism classes of invertible sheaves on X_A* **of order two**

and

- *the set of* **even decompositions** *of A*

that satisfies the following condition: Let $\mathcal{L}$ be an invertible sheaf on X_A of order two. Write $\{A_1, A_2\}$ for the even decomposition of A corresponding, via the bijection, to the isomorphism class of $\mathcal{L}$. Then the invertible sheaf $\xi^*_{A_1,A_2}\mathcal{L}$ [cf. (i)] on X_{A_1,A_2} is **trivial**.

PROOF. — These assertions follow immediately from the discussion given in [6], p.346. □

LEMMA 2.7. — *Suppose that $g_A \geq 2$. Let $\mathcal{L}$ be an invertible sheaf on X_A of order two. Write $\{A_1, A_2\}$ for the even decomposition of A corresponding, via the bijection of Lemma 2.6, (ii), to the isomorphism class of $\mathcal{L}$. Then the following hold:*

(i) *For each $1 \leq i \leq 2$, the natural morphism $X_{A_1,A_2} \to X_{A_i}$ determines a homomorphism*

$$\Gamma(X_{A_i}, \omega_{A_i}) \longrightarrow \Gamma(X_{A_1,A_2}, \omega_{A_1,A_2}).$$

Moreover, let us fix a trivialization Θ of $\xi^*_{A_1,A_2}\mathcal{L}$ [cf. Lemma 2.6, (ii)]. Thus, the finite étale morphism ξ_{A_1,A_2}, together with Θ, determines homomorphisms

$$\Gamma(X_A, \mathcal{L} \otimes_{\mathcal{O}_{X_A}} \omega_A) \longrightarrow \Gamma(X_{A_1,A_2}, (\xi^*_{A_1,A_2}\mathcal{L}) \otimes_{\mathcal{O}_{X_{A_1,A_2}}} \omega_{A_1,A_2}) \xrightarrow[\sim]{\Theta} \Gamma(X_{A_1,A_2}, \omega_{A_1,A_2}).$$

Then these homomorphisms determine an **isomorphism**

$$\Gamma(X_{A_1}, \omega_{A_1}) \oplus \Gamma(X_{A_2}, \omega_{A_2}) \xrightarrow{\sim} \Gamma(X_A, \mathcal{L} \otimes_{\mathcal{O}_{X_A}} \omega_A).$$

(ii) *Suppose that $g_{A_1} \geq 1$. Let $f(t) \in k[t]^{\leq g_{A_1}-1} \setminus \{0\}$ be a nonzero element of $k[t]^{\leq g_{A_1}-1}$. Thus, there exist elements $a, a_1, \ldots, a_d \in k$ of k such that*

$$f(t) = a \cdot \prod_{i=1}^{d} (t - a_i).$$

Then the **zero divisor** *of the nonzero global section of* ω_{A_1,A_2} *obtained by forming the image — via the homomorphism*

$$\Gamma(X_{A_1}, \omega_{A_1}) \longrightarrow \Gamma(X_{A_1,A_2}, \omega_{A_1,A_2})$$

induced by the natural morphism $X_{A_1,A_2} \to X_{A_1}$ *— of* $\omega_{f(t)} \in \Gamma(X_{A_1}, \omega_{A_1})$ *is given by the* **sum** *of*

- *the pull-back, via the composite of* $\xi_{A_1,A_2} \colon X_{A_1,A_2} \to X_A$ *and* $\xi_A \colon X_A \to \mathbb{P}^1_k$*, of the divisor on* $\mathbb{P}^1_k$

$$(g_{A_1} - 1 - d)[\infty] + \sum_{i=1}^{d} [a_i]$$

— where we write "$[-]$*" for the prime divisor on* $\mathbb{P}^1_k$ *determined by the closed point of* $\mathbb{P}^1_k$ *corresponding to "*$(-)$*" —*

and

- *the* **reduced** *effective divisor on* X_{A_1,A_2} *whose support is given by the pull-back, via the composite of* $\xi_{A_1,A_2} \colon X_{A_1,A_2} \to X_A$ *and* $\xi_A \colon X_A \to \mathbb{P}^1_k$*, of the closed subset* $A_2 \subseteq \mathbb{P}^1_k$*.*

PROOF. — Assertion (i) follows immediately from the discussion given in [6], p.346. Assertion (ii) follows immediately from the definition of the global section $\omega_{f(t)}$, together with the [easily verified] fact that the ramification divisor of the natural morphism $X_{A_1,A_2} \to X_{A_1}$ is given by the *reduced* effective divisor on X_{A_1,A_2} whose support is the pull-back, via the composite of $\xi_{A_1,A_2} \colon X_{A_1,A_2} \to X_A$ and $\xi_A \colon X_A \to \mathbb{P}^1_k$, of the closed subset $A_2 \subseteq \mathbb{P}^1_k$. This completes the proof of Lemma 2.7. □

3. DIVISORS OF CEO-TYPE ON HYPERELLIPTIC CURVES

In the present §3, let us discuss divisors *of CEO-type* [cf. [1], Definition 5.1, (iii)] on hyperelliptic curves in characteristic three. In the present §3, we maintain the notational conventions introduced at the beginning of the preceding §2. In particular, we are given a nonempty finite closed subset

$$A \subseteq \mathbb{P}^1_k$$

of $\mathbb{P}^1_k$ *of even cardinality.*

LEMMA 3.1. — *Suppose that* $g_A \geq 1$*. Then the following hold:*

(i) *The Cartier operator on* $\Gamma(X_A, \omega_A)$ *is, relative to the isomorphism*

$$\begin{array}{ccc} k[t]^{\leq g_A - 1} & \xrightarrow{\sim} & \Gamma(X_A, \omega_A) \\ f(t) & \mapsto & \omega_{f(t)} \end{array}$$

[*cf. Definition* 2.3], *given by* C_A [*cf. Definition* 1.3, (ii)].

(ii) *It holds that the Jacobian variety of* X_A *is* **ordinary** *if and only if the element* $D_A \in k$ [*cf. Definition* 1.3, (ii)] *is* **nonzero**.

(iii) *Suppose that* $g_A \geq 2$*. Let* $f(t) \in k[t]^{\leq g_A - 1} \setminus \{0\}$ *be a nonzero element of* $k[t]^{\leq g_A - 1}$*. Then it holds that the global section* $w_{f(t)} \in \Gamma(X_A, \omega_A)$ *of* ω_A *is a* **Cartier eigenform** *associated to* $\mathcal{O}_{X_A}$ [*cf.* [1], *Definition* A.8, (ii)] *if and only if*

$$C_A(f(t)) \in k^\times \cdot f(t)^3.$$

Proof. — First, we verify assertion (i). Let us first recall [cf., e.g., the discussion given in [3], §2.1 — especially, the equality (2.1.13) in [3], §2.1] that the restriction to the open subscheme $U_A \subseteq X_A$ [cf. the discussion preceding Lemma 2.2] of the Cartier operator on $\Gamma(X_A, \omega_A)$ is given by

$$\frac{f(t)dt}{s} \mapsto -\frac{d^2}{dt^2}\Big(\frac{f(t)}{s}\Big)\Big|_{(t^3,s^3)=(t^F,s^F)} \cdot dt^F$$

— where we write t^F, s^F for the global sections, determined by t, s, respectively, of the structure sheaf of the base-change of U_A via the absolute Frobenius morphism of k. Thus, since

$$-\frac{d^2}{dt^2}\Big(\frac{f(t)}{s}\Big) = -\frac{d^2}{dt^2}\Big(\frac{s^2 \cdot f(t)}{s^3}\Big) = -\frac{1}{s^3} \cdot \frac{d^2}{dt^2}\big(f_A(t) \cdot f(t)\big) = \frac{C_A(f(t))}{s^3},$$

assertion (i) holds. This completes the proof of assertion (i).

Assertion (ii) follows from assertion (i), together with Remark 1.1.1. Assertion (iii) follows — in light of [1], Remark A.4.1 — from assertion (i). This completes the proof of Lemma 3.1. □

Theorem 3.2. — *In the notational conventions introduced at the beginning of the present* §3, *suppose that* $g_A \geq 2$ [*cf. Definition* 1.3, (ii)]. *Let* D *be an effective divisor on the projective hyperbolic curve* X_A [*cf. Definition* 2.1, *Remark* 2.1.1]. *Then the following hold:*

(i) *Suppose that there exists a nonzero element*

$$f(t) = \prod_{i=1}^{d}(t - a_i) \in k[t]^{\leq g_A - 1} \setminus \{0\}$$

— *where* $a_1, \ldots, a_d \in k$ — *of* $k[t]^{\leq g_A-1}$ *that satisfies the following two conditions:*

(1) *The divisor* D *is given by the pull-back, via* $\xi_A \colon X_A \to \mathbb{P}^1_k$ [*cf. Definition* 2.1], *of the divisor on* $\mathbb{P}^1_k$

$$(g_A - 1 - d)[\infty] + \sum_{i=1}^{d}[a_i]$$

— *where we write "*$[-]$*" for the prime divisor on* $\mathbb{P}^1_k$ *determined by the closed point of* $\mathbb{P}^1_k$ *corresponding to "*$(-)$*".*

(2) *It holds that*

$$C_A(f(t)) \in k^\times \cdot f(t)^3$$

[*cf. Definition* 1.3, (ii)].

Then the divisor $2D$ *coincides with the zero divisor of the* **square Hasse invariant** [*cf.* [4], *Chapter* II, *Proposition* 2.6, (1)] *of a* **nilpotent** [*cf.* [4], *Chapter* II, *Definition* 2.4] **indigenous bundle** [*cf.* [4], *Chapter* I, *Definition* 2.2].

(ii) *It holds that the divisor D coincides with the* **supersingular divisor** [*cf.* [4], *Chapter* II, *Proposition* 2.6, (3)] *of a* **nilpotent admissible** [*cf.* [4], *Chapter* II, *Definition* 2.4] **indigenous bundle** *whose* **Hasse defect** [*cf.* [1], *Definition* B.2] *is* **trivial** [*which thus implies that the divisor D is* **of CE-type** — *cf.* [1], *Definition* 5.1, (iii)*;* [1], *Theorem* B] *if and only if there exists a nonzero element*

$$f(t) \;=\; \prod_{i=1}^{d}(t-a_i) \;\in\; k[t]^{\leq g_A-1}\setminus\{0\}$$

— *where $a_1,\ldots,a_d \in k$ — of $k[t]^{\leq g_A-1}$ that satisfies conditions* (1), (2) *of* (i) *and, moreover, the following condition:*

(3) *The a_i's are* **distinct**, *$a_i \notin A$ for every $1 \leq i \leq d$, and $g_A - 2 \leq d \leq g_A - 1$. Moreover, if $\infty \in A$, then $d = g_A - 1$.*

(iii) *It holds that the divisor D coincides with the* **supersingular divisor** *of a* **nilpotent ordinary** [*cf.* [4], *Chapter* II, *Definition* 3.1] **indigenous bundle** *whose* **Hasse defect** *is* **trivial** [*which thus implies that the divisor D is* **of CEO-type** — *cf.* [1], *Definition* 5.1, (iii)*;* [1], *Theorem* B] *if and only if*

$$D_A \;\neq\; 0$$

[*cf. Definition* 1.3, (ii)] *and, moreover, there exists a nonzero element*

$$f(t) \;=\; \prod_{i=1}^{d}(t-a_i) \;\in\; k[t]^{\leq g_A-1}\setminus\{0\}$$

— *where $a_1,\ldots,a_d \in k$ — of $k[t]^{\leq g_A-1}$ that satisfies three conditions* (1), (2), (3) *of* (i), (ii).

PROOF. — First, we verify assertion (i). Let us first observe that it follows from Lemma 3.1, (iii), and [1], Proposition 4.1, that a $k^\times$-multiple of the square of $\omega_{f(t)}$ gives rise [i.e., in the sense of the discussion preceding [1], Proposition 4.1] to a *nilpotent indigenous bundle* on X. Next, let us observe that it follows from Lemma 2.4 and [1], Proposition 3.2, that the zero divisor of the square Hasse invariant of this nilpotent indigenous bundle *coincides* with $2D$. This completes the proof of assertion (i).

Assertions (ii), (iii) follow from a similar argument to the argument applied in the proof of assertion (i), together with Lemma 3.1, (ii), and [1], Theorem B. This completes the proof of Theorem 3.2. ☐

LEMMA 3.3. — *Suppose that $g_A \geq 2$. Let $\mathcal{L}$ be an invertible sheaf on X_A of order two. Write $\{A_1, A_2\}$ for the even decomposition of A* [*cf. Definition* 1.3, (iii)] *corresponding, via the bijection of Lemma* 2.6, (ii), *to the isomorphism class of $\mathcal{L}$. Then the following hold:*

(i) *For each $1 \leq i \leq 2$, the restriction of the isomorphism of Lemma* 2.7, (i),

$$\Gamma(X_{A_i},\omega_{A_i}) \;\hookrightarrow\; \Gamma(X_{A_1},\omega_{A_1})\oplus\Gamma(X_{A_2},\omega_{A_2}) \;\xrightarrow{\sim}\; \Gamma(X_A,\mathcal{L}\otimes_{\mathcal{O}_{X_A}}\omega_A)$$

is **compatible**, *up to $k^\times$-multiple, with the* [*usual*] *Cartier operator on $\Gamma(X_{A_i},\omega_{A_i})$ and the Cartier operator on $\Gamma(X_A,\mathcal{L}\otimes_{\mathcal{O}_{X_A}}\omega_A)$ associated to the square-trivialized invertible* *sheaf* [*cf.* [1], *Definition* A.3] *consisting of* $\mathcal{L}$ *and an arbitrary trivialization of* $\mathcal{L}^{\otimes 2}$ [*cf.* [1], *Definition* A.4].

(ii) *It holds that the invertible sheaf* $\mathcal{L}$ *is* **parabolically ordinary** [*cf.* [1], *Definition* A.7] *if and only if both the elements* D_{A_1}, $D_{A_2} \in k$ *are* **nonzero**.

(iii) *Suppose that* $g_{A_1} \geq 1$. *Let* ω *be a nonzero global section of* $\mathcal{L} \otimes_{\mathcal{O}_{X_A}} \omega_A$. *Suppose, moreover, that the image of* ω, *via the inverse of the isomorphism of Lemma* 2.7, (i), *is contained in the subspace*

$$\Gamma(X_{A_1}, \omega_{A_1}) \;\subseteq\; \Gamma(X_{A_1}, \omega_{A_1}) \oplus \Gamma(X_{A_2}, \omega_{A_2}).$$

Write $f(t) \in k[t]^{\leq g_{A_1}-1}$ *for the element of* $k[t]^{\leq g_{A_1}-1}$ *such that* ω *corresponds, via the isomorphism of Lemma* 2.7, (i), *to* $(\omega_{f(t)}, 0)$. *Then it holds that the global section* $\omega \in \Gamma(X_A, \mathcal{L} \otimes_{\mathcal{O}_{X_A}} \omega_A)$ *is a* **Cartier eigenform** *associated to* $\mathcal{L}$ *if and only if*

$$C_{A_1}(f(t)) \;\in\; k^{\times} \cdot f(t)^3.$$

PROOF. — First, we verify assertion (i). Let us first observe that it is immediate that, to verify assertion (i), it suffices to verify that the homomorphisms

$$\Gamma(X_{A_i}, \omega_{A_i}) \;\longrightarrow\; \Gamma(X_{A_1,A_2}, \omega_{A_1,A_2}), \quad \Gamma(X_A, \mathcal{L} \otimes_{\mathcal{O}_{X_A}} \omega_A) \;\longrightarrow\; \Gamma(X_{A_1,A_2}, \omega_{A_1,A_2})$$

of Lemma 2.7, (i), are *compatible*, up to $k^{\times}$-multiple, with the relevant Cartier operators, respectively. On the other hand, one verifies immediately — by considering the restrictions of these homomorphisms to suitable open subschemes of X_{A_i}, X_A, respectively — from the local description of the Cartier operator given in, for instance, [3], §2.1 — especially, the equality (2.1.13) in [3], §2.1 — that this desired compatibility holds. This completes the proof of assertion (i).

Assertion (ii) follows from assertion (i), together with Lemma 3.1, (ii). Assertion (iii) follows from assertion (i), together with Lemma 3.1, (i). This completes the proof of Lemma 3.3. □

THEOREM 3.4. — *In the notational conventions introduced at the beginning of the present* §3, *suppose that* $g_A \geq 2$ [*cf. Definition* 1.3, (ii)]. *Let* $\mathcal{L}$ *be an invertible sheaf on the projective hyperbolic curve* X_A [*cf. Definition* 2.1, *Remark* 2.1.1] **of order two**. *Write* $\{A_1, A_2\}$ *for the* **even decomposition** *of* A [*cf. Definition* 1.3, (iii)] *corresponding, via the bijection of Lemma* 2.6, (ii), *to the isomorphism class of* $\mathcal{L}$. *Suppose, moreover, that* $g_{A_1} \geq 1$. *Let* D *be an effective divisor on* X_A. *Then the following hold:*

(i) *Suppose that there exists a nonzero element*

$$f(t) \;=\; \prod_{i=1}^{d}(t - a_i) \;\in\; k[t]^{\leq g_{A_1}-1} \setminus \{0\}$$

— *where* $a_1, \ldots, a_d \in k$ — *of* $k[t]^{\leq g_{A_1}-1}$ *that satisfies the following two conditions:*

(1) *The divisor* D *is given by the* **sum** *of*

- *the pull-back, via* $\xi_A \colon X_A \to \mathbb{P}^1_k$ [*cf. Definition* 2.1], *of the divisor on* $\mathbb{P}^1_k$

$$(g_{A_1} - 1 - d)[\infty] + \sum_{i=1}^{d}[a_i]$$

— *where we write "$[-]$" for the prime divisor on* $\mathbb{P}^1_k$ *determined by the closed point of* $\mathbb{P}^1_k$ *corresponding to "$(-)$"* —

and

- *the* **reduced** *effective divisor on* X_A *whose support is given by the pull-back, via* $\xi_A\colon X_A \to \mathbb{P}^1_k$*, of the closed subset* $A_2 \subseteq \mathbb{P}^1_k$*.*

(2) *It holds that*

$$C_{A_1}(f(t)) \ \in \ k^\times \cdot f(t)^3$$

[*cf. Definition* 1.3*,* (ii)]*.*

Then the divisor $2D$ *coincides with the zero divisor of the* **square Hasse invariant** [*cf.* [4]*, Chapter* II*, Proposition* 2.6*,* (1)] *of a* **nilpotent** [*cf.* [4]*, Chapter* II*, Definition* 2.4] **indigenous bundle** [*cf.* [4]*, Chapter* I*, Definition* 2.2]*.*

(ii) *Suppose that there exists a nonzero element*

$$f(t) \ = \ \prod_{i=1}^{d}(t-a_i) \ \in \ k[t]^{\leq g_{A_1}-1} \setminus \{0\}$$

— *where* $a_1, \ldots, a_d \in k$ — *of* $k[t]^{\leq g_{A_1}-1}$ *that satisfies two conditions* (1)*,* (2) *of* (i) *and, moreover, the following condition:*

(3) *The* a_i*'s are* **distinct***,* $a_i \notin A$ *for every* $1 \leq i \leq d$*, and* $g_{A_1} - 2 \leq d \leq g_{A_1} - 1$*. Moreover, if* $\infty \in A$*, then* $d = g_{A_1} - 1$*.*

Then the divisor D *coincides with the* **supersingular divisor** [*cf.* [4]*, Chapter* II*, Proposition* 2.6*,* (3)] *of a* **nilpotent admissible** [*cf.* [4]*, Chapter* II*, Definition* 2.4] **indigenous bundle** *whose* **Hasse defect** [*cf.* [1]*, Definition* B.2] *is* **isomorphic** *to* $\mathcal{L}$ [*which thus implies that the divisor* D *is* **of CE-type** — *cf.* [1]*, Definition* 5.1*,* (iii)*;* [1]*, Theorem* B]*.*

(iii) *Suppose that there exists a nonzero element*

$$f(t) \ = \ \prod_{i=1}^{d}(t-a_i) \ \in \ k[t]^{\leq g_{A_1}-1} \setminus \{0\}$$

— *where* $a_1, \ldots, a_d \in k$ — *of* $k[t]^{\leq g_{A_1}-1}$ *that satisfies three conditions* (1)*,* (2)*,* (3) *of* (i)*,* (ii)*. Suppose, moreover, that*

$$D_{A_1} \cdot D_{A_2} \ \neq \ 0$$

[*cf. Definition* 1.3*,* (ii)]*. Then the divisor* D *coincides with the* **supersingular divisor** *of a* **nilpotent ordinary** [*cf.* [4]*, Chapter* II*, Definition* 3.1] **indigenous bundle** *whose* **Hasse defect** *is* **isomorphic** *to* $\mathcal{L}$ [*which thus implies that the divisor* D *is* **of CEO-type** — *cf.* [1]*, Definition* 5.1*,* (iii)*;* [1]*, Theorem* B]*.*

PROOF. — First, we verify assertion (i). Let us first observe that it follows from Lemma 3.3, (iii), and [1], Proposition 4.1, that a $k^\times$-multiple of the square of the global section of $\mathcal{L} \otimes_{\mathcal{O}_{X_A}} \omega_A$ determined by $\omega_{f(t)}$ [cf. Lemma 2.7, (i)] gives rise [i.e., in the sense of the discussion preceding [1], Proposition 4.1] to a *nilpotent indigenous bundle* on X. Next, let us observe that it follows from Lemma 2.7, (ii), and [1], Proposition 3.2, that the zero divisor of the square Hasse invariant of this nilpotent indigenous bundle *coincides* with $2D$. This completes the proof of assertion (i).

Assertions (ii), (iii) follow from a similar argument to the argument applied in the proof of assertion (i), together with Lemma 3.3, (ii), and [1], Theorem B. This completes the proof of Theorem 3.4. □

Corollary 3.5. — *In the notational conventions introduced at the beginning of the present* §3, *suppose that* $g_A \geq 2$ [*cf. Definition* 1.3, (ii)]. *Let* $A_1 \subseteq A$ *be a subset of* A **of cardinality 4**. *Write* D *for the* **reduced** *effective divisor on the projective hyperbolic curve* X_A [*cf. Definition* 2.1*, Remark* 2.1.1] *whose support is given by the pull-back, via* $\xi_A \colon X_A \to \mathbb{P}^1_k$ [*cf. Definition* 2.1], *of the closed subset* $A \setminus A_1 \subseteq \mathbb{P}^1_k$ *of* $\mathbb{P}^1_k$ [*i.e., of cardinality* $2g_A - 2$]. *Then the following hold:*

(i) *Suppose that* $D_{A_1} \neq 0$. *Then the effective divisor* D *on* X_A *is* **of CE-type** [*cf.* [1], *Definition* 5.1, (iii)].

(ii) *Suppose that* $D_{A_1} \cdot D_{A \setminus A_1} \neq 0$. *Then the effective divisor* D *on* X_A *is* **of CEO-type** [*cf.* [1], *Definition* 5.1, (iii)].

Proof. — Since A_1 is *of cardinality* 4, the vector space $k[t]^{\leq g_{A_1} - 1}$ is *of dimension* 1. Now observe that if $D_{A_1} \neq 0$, then the homomorphism C_{A_1} is *injective* [cf. Remark 1.1.1]. Thus, it follows immediately from Theorem 3.4, (ii), that the assumption that $D_{A_1} \neq 0$ implies that D is *of CE-type*. This completes the proof of assertion (i). Moreover, if $D_{A_1} \cdot D_{A \setminus A_1} \neq 0$, then it follows immediately from Theorem 3.4, (iii), that D is *of CEO-type*. This completes the proof of assertion (ii), hence also of Corollary 3.5. □

4. Hyperbolic Ordinariness of Hyperelliptic Curves of Lower Genus

In the present §4, by applying some results obtained in the preceding §3, let us prove the main result of the present paper.

Theorem 4.1. — *Let* k *be an algebraically closed field of characteristic three,* $2 \leq g \leq 3$ *an integer, and*

$$X$$

a **hyperelliptic** *projective* [*necessarily hyperbolic*] *curve of genus* g *over* k. *Suppose that* X *is* **parabolically ordinary** [*i.e., that the Jacobian variety of* X *is an* **ordinary** *abelian variety — cf. the discussion following* [4], *Chapter* II, *Definition* 3.3]. *Then the hyperbolic curve* X *has a* **nilpotent** [*cf.* [4], *Chapter* II, *Definition* 2.4] **ordinary** [*cf.* [4], *Chapter* II, *Definition* 3.1] **indigenous bundle** [*cf.* [4], *Chapter* I, *Definition* 2.2] *whose* **Hasse defect** [*cf.* [1], *Definition* B.2] *is* **trivial**.

Proof. — Theorem 4.1 in the case where $g = 2$ was already proved in [1], Theorem 6.1, (iii), (v). In the remainder of the proof of Theorem 4.1, suppose that $g = 3$.

Let us recall that since [we have assumed that] X is *hyperelliptic*, X admits a uniquely determined hyperelliptic involution, which determines a double covering

$$\xi_X \colon X \longrightarrow Q$$

— where we write Q for the [scheme-theoretic] quotient of X by the hyperelliptic involution of X. Now one verifies easily that, for each isomorphism $\phi \colon \mathbb{P}^1_k \xrightarrow{\sim} Q$ over k, if we write $A \subseteq \mathbb{P}^1_k$ for the closed subset of $\mathbb{P}^1_k$ obtained by forming the image, via ϕ^{-1}, of the branch locus of ξ_X, then we have a commutative diagram of schemes over k

$$\begin{array}{ccc} X_A & \xrightarrow{\sim} & X \\ {\scriptstyle \xi_A}\downarrow & & \downarrow{\scriptstyle \xi_X} \\ \mathbb{P}^1_k & \xrightarrow[\phi]{\sim} & Q \end{array}$$

[cf. Definition 2.1] — where the horizontal arrows are *isomorphisms*. Let us identify X with X_A by means of the upper horizontal arrow of this diagram.

Next, observe that since [we have assumed that] X is *parabolically ordinary*, one verifies immediately [cf., e.g., [7], Corollary, p.143] that the set of $k^\times$-orbits of nonzero elements of the vector space $k[t]^{\leq g-1} = k[t]^{\leq 2}$ [of dimension 3] that satisfies condition (2) of Theorem 3.2, (i), is *of cardinality* $\sharp \mathbb{P}^{3-1}_{\mathbb{F}_3}(\mathbb{F}_3) = 13$. Write $D_1, \ldots, D_{13}$ for the [13 distinct] effective divisors on X [necessarily of degree 4] determined by these $k^\times$-orbits, i.e., in the sense of condition (1) of Theorem 3.2, (i). Now let us observe that it follows from Lemma 3.1, (ii), and Theorem 3.2, (iii), that, to verify Theorem 4.1, it suffices to verify that there exists $1 \leq i \leq 13$ such that D_i is *reduced*. For each $1 \leq i \leq 13$, write d_i for the uniquely determined [necessarily positive] integer such that there exists a closed point of X at which the effective divisor D_i is *of multiplicity* d_i, and, moreover, for every closed point $x \in X$ of X, the effective divisor D_i is *of multiplicity* $\leq d_i$ at x. Thus, since D_i is *of degree* 4, it holds that $1 \leq d_i \leq 4$. Now it is immediate that, to verify Theorem 4.1, it suffices to verify that there exists $1 \leq i \leq 13$ such that $d_i = 1$.

Assume that $d_i = 2$ for some $1 \leq i \leq 13$. Now let us recall that it follows from Theorem 3.2, (i), that the effective divisor $2D_i$ coincides with the zero divisor of the *square Hasse invariant* of a *nilpotent indigenous bundle* on X. Thus, since $2d_i = 4 \in 3\mathbb{Z} + 1$, it follows from [1], Proposition 3.2, and [1], Lemma 3.5, that we obtain a *contradiction*.

Next, assume that $d_i = 3$ for some $1 \leq i \leq 13$. Then since D_i is *of degree* 4, there exists a [unique] closed point $x \in X$ of X such that the effective divisor D_i is *of multiplicity* 1 at x. Now observe that it follows from Lemma 2.4 that D_i, hence also x, is *fixed* by the hyperelliptic involution of X. In particular, again by Lemma 2.4, one verifies immediately that D_i is *of even multiplicity* at x. Thus, we obtain a *contradiction*.

Next, suppose that $d_i = 4$, which thus implies that the support of D_i consists of a *single closed point* $x \in X$ of X. Now observe that it follows from Lemma 2.4 that D_i, hence also x, is *fixed* by the hyperelliptic involution of X. In particular, the closed point x is *contained* in the ramification locus of ξ [which consists of 8 distinct closed points].

By these arguments, we conclude that at least *five* [i.e., $13 - 8$] "D_i" are *reduced*. This completes the proof of Theorem 4.1. □

THEOREM 4.2. — *Let k be an algebraically closed field of characteristic three, $g \geq 2$ an integer, and*

$$X$$

a **hyperelliptic** *projective [necessarily hyperbolic] curve of genus g over k. Then the following hold:*

(i) *The hyperbolic curve X has a* **nilpotent admissible** [*cf.* [4], *Chapter* II, *Definition* 2.4] **indigenous bundle** [*cf.* [4], *Chapter* I, *Definition* 2.2].

(ii) *Suppose that $g \leq 5$. Then the hyperbolic curve X has a* **nilpotent ordinary** [*cf.* [4], *Chapter* II, *Definition* 3.1] **indigenous bundle***.*

Proof. — Let us first recall that since [we have assumed that] X is *hyperelliptic*, X admits a uniquely determined hyperelliptic involution, which determines a double covering

$$\xi_X\colon\ X \longrightarrow Q$$

— where we write Q for the [scheme-theoretic] quotient of X by the hyperelliptic involution of X. Now one verifies easily that, for each isomorphism $\phi\colon \mathbb{P}^1_k \xrightarrow{\sim} Q$ over k, if we write $A \subseteq \mathbb{P}^1_k$ for the closed subset of $\mathbb{P}^1_k$ obtained by forming the image, via ϕ^{-1}, of the branch locus of ξ_X, then we have a commutative diagram of schemes over k

$$\begin{array}{ccc} X_A & \xrightarrow{\sim} & X \\ {\scriptstyle \xi_A}\big\downarrow & & \big\downarrow{\scriptstyle \xi_X} \\ \mathbb{P}^1_k & \xrightarrow[\phi]{\sim} & Q \end{array}$$

[cf. Definition 2.1] — where the horizontal arrows are *isomorphisms.*

Now we verify assertion (i). It follows immediately from Lemma 1.4, (i), that there exists a subset $A_1 \subseteq A$ *of cardinality* 4 such that $D_{A_1} \neq 0$. Thus, it follows from Corollary 3.5, (i), that an effective divisor on X_A, hence also X, is *of CE-type.* In particular, it follows from [1], Theorem B, that X has a *nilpotent admissible indigenous bundle.* This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that assertion (ii) in the case where $g = 2$ was already proved in [1], Theorem D. In the remainder of the proof of assertion (ii), suppose that $g = 3$ (respectively, $4 \leq g \leq 5$). Next, let us observe that, to verify assertion (ii), we may assume without loss of generality, by replacing ϕ by a suitable isomorphism $\mathbb{P}^1_k \xrightarrow{\sim} Q$ over k, that $0, \infty \notin A$ (respectively, $0, \infty \in A$). Then it follows from Lemma 1.6 (respectively, Lemma 1.7 and Lemma 1.8) that there exists a subset $A_1 \subseteq A$ *of cardinality* 4 such that $D_{A_1} \cdot D_{A \setminus A_1} \neq 0$. Thus, it follows from Corollary 3.5, (ii), that an effective divisor on X_A, hence also X, is *of CEO-type.* In particular, it follows from [1], Theorem B, that X has a *nilpotent ordinary indigenous bundle.* This completes the proof of assertion (ii), hence also of Theorem 4.2. ☐

Corollary 4.3. — *Every* **hyperelliptic** *projective hyperbolic curve* **of genus $\leq$ 5** *over a connected noetherian scheme of characteristic* 3 *is* **hyperbolically ordinary** [*cf.* [4], *Chapter* II, *Definition* 3.3].

Proof. — This assertion follows from Theorem 4.2, (ii), together with [4], Chapter II, Proposition 3.4. ☐

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