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FINITENESS OF ISOMORPHISM CLASSES OF HYPERBOLIC POLYCURVES WITH PRESCRIBED FUNDAMENTAL GROUPS

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Abstract. In the present paper, we show that there are at most finitely many isomorphism classes of hyperbolic polycurves (i.e., successive extensions of families of hyperbolic curves) over certain types of fields (for example, finitely generated extension fields over $\mathbb{Q}$) whose (geometrically pro-p) étale fundamental group is isomorphic to a prescribed profinite group.

Introduction

Let p be a prime number, k a field of characteristic zero, and X a variety over k. Write G_k for the absolute Galois group of k, Π_X for the étale fundamental group of X, $\Delta_{X/k}$ for the kernel of the natural (outer) surjection $\Pi_X \twoheadrightarrow G_k$ induced by the structure morphism $X \to \operatorname{Spec} k$, $\Delta^p_{X/k}$ for the maximal pro-p quotient of $\Delta_{X/k}$, and $\Pi^p_{X/k} := \Pi_X / \ker(\Delta_{X/k} \twoheadrightarrow \Delta^p_{X/k})$ (which we call geometrically pro-p étale fundamental group). A. Grothendieck proposed that, for certain types of k, if X is an "anabelian variety" over k, then the isomorphism class of X may be completely determined by $\Pi_X \twoheadrightarrow G_k$ (cf. [1],[2]), which we often call "Grothendieck conjecture". Although we do not have any general definition of the notion of an "anabelian variety", a successive extension of families of hyperbolic curves, i.e., a hyperbolic polycurve (see Definition 2.2(ii)), has been regarded as a typical example of an anabelian variety. In [3], the Grothendieck conjecture for hyperbolic polycurves of dimension ≤ 4 was proved. Moreover, in [12], we show that the pro-p version of the Grothendieck conjecture (where we consider $\Pi^p_{X/k} \twoheadrightarrow G_k$ instead of $\Pi_X \twoheadrightarrow G_k$) for hyperbolic polycurves of dimension ≤ 4 satisfying condition $(*)_p$ (see Definition 2.4) holds.

On the other hand, the (pro-p) Grothendieck conjecture for hyperbolic polycurves of dimension ≥ 5 is still open. In the present paper, we give a partial result on the Grothendieck conjecture for hyperbolic polycurves. That is to say, we show that the isomorphism class of a hyperbolic polycurve is determined by the étale fundamental group (equipped with the surjective homomorphism to the absolute Galois group of the base field) up to finitely many possibilities. More precisely, we show the following, among other things.

Theorem (Corollary 2.8)**.** *Let p be a prime number, k a generalized sub-p-adic field (see Definition 2.5), G a profinite group, and $G \twoheadrightarrow G_k$ a surjective homomorphism. Then there are at most finitely many k-isomorphism classes of hyperbolic polycurves over k (resp. hyperbolic polycurves over k satisfying condition $(*)_p$) whose étale fundamental group (resp. geometrically pro-p étale fundamental group) is isomorphic to G over G_k.*

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1. Finiteness of SESG-filtrations

In the present §1, we introduce the notion of an SESG-filtration and discuss the finiteness of SESG-filtrations for a given profinite group. Main arguments of the present §1 are essentially due to [4], which treats the discrete case. Let us fix a real number $q > 1$. Write $\mathfrak{Primes}$ for the set of all prime numbers.

First, we review some properties of cohomology groups of profinite groups.

Definition 1.1. Let G be a profinite group.

(i) A *G-module* A is a discrete abelian group A together with a continuous action of G on A.

(ii) Let A be a G-module and n a nonnegative integer. Then we shall write

$$H^n(G, A)$$

for the n-th cohomology group of G with coefficients in A.

(iii) If G is topologically finitely generated, then we shall write $r(G)$ for the minimum number of (topological) generators of G.

Definition 1.2 (cf. [13] Definition 1.3)**.** Let G be a profinite group.

(i) Let A be a G-module. For each nonnegative integer i, we shall write

$$h^i(G, A) := \log_q(\sharp H^i(G, A)).$$

(ii) Let A be a G-module. Suppose that $h^i(G, A) < \infty$ for any nonnegative integer i, and that $h^i(G, A) = 0$ for all but finitely many nonnegative integers i. Then we shall write

$$\chi(G, A) := \sum_{i=0}^{\infty}(-1)^i h^i(G, A).$$

In this case, we shall say that "$\chi(G, A)$ is defined".

(iii) Let $\Sigma \subset \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$. Suppose that there exists a (unique) constant $b \in \mathbb{R}$ such that, for any finite Σ-torsion G-module A (i.e., for any $a \in A$, there exists a positive integer n such that $na = 0$ and that every prime factor of n is contained in Σ), it holds that $\chi(G, A)$ is defined, and $\chi(G, A) = b \log_q(\sharp A)$. Then we shall write

$$\chi_\Sigma(G) := b.$$

In this case, we shall say that "$\chi_\Sigma(G)$ is defined".

Remark 1.2.1.

(i) It is clear by definition that if $\chi_\Sigma(G)$ is defined, then $\chi_\Sigma(G)$ does not depend on q and $\chi_\Sigma(G) \in \mathbb{Z}$. Moreover, if $\chi_\Sigma(G)$ is defined, then, for any nonempty subset $\Sigma' \subset \Sigma$ of Σ, $\chi_{\Sigma'}(G)$ is also defined and it holds that $\chi_{\Sigma'}(G) = \chi_\Sigma(G)$.

(ii) If G is a pro-p group such that $\chi(G, \mathbb{F}_p)$ is defined, then it is well-known that $\chi_{\{p\}}(G)$ is defined. The value $\chi_{\{p\}}(G)$ is often called the Euler-Poincaré characteristic of G (cf. e.g., [14] §4.1).

Lemma 1.3 ([13] Lemma 1.4)**.** *Let $\Sigma \subset \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$. Then the following hold:*

(i) *Let G be a profinite group and U an open subgroup of G. Suppose that $\chi_\Sigma(G)$ is defined. Then $\chi_\Sigma(U)$ is also defined, and it holds that $\chi_\Sigma(U) = [G:U]\chi_\Sigma(G)$.*

(ii) *Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be a short exact sequence of profinite groups. Suppose that $\chi_\Sigma(G_3)$ is defined. Then for any finite Σ-torsion G_2-module A, if $\chi(G_1, A)$ is defined, then $\chi(G_2, A)$ is also defined, and it holds that $\chi(G_2, A) = \chi(G_1, A) \cdot \chi_\Sigma(G_3)$. In particular, if $\chi_\Sigma(G_1)$ is defined, then $\chi_\Sigma(G_2)$ is also defined, and it holds that $\chi_\Sigma(G_2) = \chi_\Sigma(G_1) \cdot \chi_\Sigma(G_3)$.*

Definition 1.4. Let G be a group and $\Sigma \subset \mathfrak{Primes}$ a subset of $\mathfrak{Primes}$. Then we shall write

$$G^\Sigma$$

for the pro-Σ completion of G. Note that if G is a topologically finitely generated profinite group, then, since every homomorphism from G to any finite group is continuous (cf. [10] Theorem 1.1), G^Σ is the maximal pro-Σ quotient of G. Let p be a prime number. Then we shall write simply

$$G^p$$

for the pro-p group $G^{\{p\}}$. Moreover, we shall write simply

$$G^\wedge$$

for the profinite group $G^{\mathfrak{Primes}}$.

Definition 1.5.

(i) Let (g, r) be a pair of nonnegative integers. Then we shall write

$$\Pi_{g,r} := \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_r \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_r = 1 \rangle.$$

Note that if $r > 0$, then $\Pi_{g,r}$ is a free group of rank $2g + r - 1$.

(ii) Let $\Sigma \subset \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$ and (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$. Then we shall refer to a profinite group isomorphic to $\Pi^\Sigma_{g,r}$ as a *(pro-Σ) surface group* (cf. [8] Definition 1.2).

Remark 1.5.1.

(i) Let X be a curve of type (g, r) over an algebraically closed field of characteristic zero and $\Sigma \subset \mathfrak{Primes}$ a subset of $\mathfrak{Primes}$. Then it holds that the maximal pro-Σ quotient of the étale fundamental group $\pi_1(X)$ is isomorphic to $\Pi^\Sigma_{g,r}$ (cf. e.g., [15] Proposition (1.1)(i)).

(ii) Any open subgroup of a pro-Σ surface group is a pro-Σ surface group.

Proposition 1.6. *Let $\Sigma \subset \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$, $p \in \Sigma$, and (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$. Write $\varepsilon := \begin{cases} 0 & (r > 0) \\ 1 & (r = 0). \end{cases}$ Then the following hold:*

(i) *It holds that* $\mathrm{cd}_p(\Pi^\Sigma_{g,r}) = 1 + \varepsilon$.

(ii) $\chi_\Sigma(\Pi^\Sigma_{g,r})$ *is defined, and it holds that* $\chi_\Sigma(\Pi^\Sigma_{g,r}) = 2 - 2g - r$.

(iii) *It holds that* $\dim_{\mathbb{F}_p} H^0(\Pi^\Sigma_{g,r}, \mathbb{F}_p) = 1$, $\dim_{\mathbb{F}_p} H^1(\Pi^\Sigma_{g,r}, \mathbb{F}_p) = 2g + r + \varepsilon - 1$, $\dim_{\mathbb{F}_p} H^2(\Pi^\Sigma_{g,r}, \mathbb{F}_p) = \varepsilon$.

Proof. Assertion (i) is [13] Proposition 2.7(i), and assertion (ii) is [13] Proposition 2.7(iv). Assertion (iii) follows from assertions (i), (ii), together with [13] Proposition 2.7(ii), (iii). □

Corollary 1.7. *Let $\Sigma \subset \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$, $p \in \Sigma$, G a pro-Σ surface group, and $U \subset G$ an open subgroup of G. Then it holds that $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H^1(U, \mathbb{F}_p)$.*

Proof. This follows from Lemma 1.3(i), Remark 1.5.1(ii), Proposition 1.6(ii), (iii). □

Next, we consider the finiteness of normal closed subgroups of a given profinite group such that the quotient group relative to the subgroup is isomorphic to a surface group. Note that Lemma 1.9 (resp. Theorem 1.10) below is a pro-Σ analogue of [4] Lemma 2.1 (resp. [4] Theorem 2.3).

Proposition 1.8 (cf. e.g., [8] Theorem 1.5)**.** *Let G be a surface group. Then G is elastic, i.e., for any open subgroup $U \subset G$ of G and topologically finitely generated nontrivial normal closed subgroup $N \subset U$ of U, N is open in G.*

Lemma 1.9. *Let $\Sigma \subset \mathfrak{Primes}$ be a nonempty subset of $\mathfrak{Primes}$, $p \in \Sigma$, H a pro-Σ surface group, G a profinite group, and $N_1, N_2 \subset G$ normal closed subgroups of G such that $G/N_1 \cong G/N_2 \cong H$. Suppose that N_1 is topologically finitely generated and $\dim_{\mathbb{F}_p} H^1(N_1, \mathbb{F}_p) < \dim_{\mathbb{F}_p} H^1(H, \mathbb{F}_p) (< \infty)$. Then it holds that $N_1 = N_2$.*

Proof. Write $p_2 : G \twoheadrightarrow G/N_2$ for the natural surjection. Then the surjection $N_1 \twoheadrightarrow p_2(N_1)$ induces an injection $H^1(p_2(N_1), \mathbb{F}_p) \hookrightarrow H^1(N_1, \mathbb{F}_p)$. In particular, it holds that $\dim_{\mathbb{F}_p} H^1(p_2(N_1), \mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H^1(N_1, \mathbb{F}_p) < \dim_{\mathbb{F}_p} H^1(H, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^1(G/N_2, \mathbb{F}_p)$. Thus, it follows from Corollary 1.7 that $p_2(N_1) \subset G/N_2$ is not open in G/N_2. On the other hand, $p_2(N_1) \subset G/N_2$ is a topologically finitely generated normal closed subgroup of G/N_2. Thus, it follows from Proposition 1.8 that $p_2(N_1)$ is trivial, i.e., $N_1 \subset N_2$. Now, since (we have assumed that) $G/N_1 \cong G/N_2 \cong H$, it follows from the fact that H is (topologically finitely generated, hence) hopfian (cf. [11] Proposition 2.5.2), that $N_1 = N_2$. This completes the proof of Lemma 1.9. □

Theorem 1.10. *Let p be a prime number, G a profinite group, and $\mathcal{N}$ a class of profinite groups which is closed under isomorphism. Suppose that the following hold:*

- *For each $N \in \mathcal{N}$, N is topologically finitely generated.*
- *There exists a real number M such that, for each $N \in \mathcal{N}$, it holds that $\dim_{\mathbb{F}_p} H^1(N, \mathbb{F}_p) \leq M$.*

Then G has at most finitely many normal closed subgroups $N \subset G$ such that $N \in \mathcal{N}$ and that G/N is a pro-Σ surface group, where $\Sigma = \Sigma(N)$ is a set of prime numbers such that $p \in \Sigma$.

Proof. Write S for the set of all normal closed subgroups $N \subset G$ satisfying the condition appearing in the statement of Theorem 1.10. We may assume that $S \neq \emptyset$. Then G is topologically finitely generated. Let us write $\varphi : G \twoheadrightarrow G^p$ for the natural surjective homomorphism.

First, we show that the set $\varphi(S) = \{\varphi(N) \subset G^p \mid N \in S\}$ is finite. Since the operation of taking the maximal pro-p quotient is right exact, for each $N \in S$,

$G^p/\varphi(N) \cong (G/N)^p$ is a pro-p surface group. For each isomorphism class of pro-p surface group C, let us write $\varphi(S)_C := \{\varphi(N) \subset G^p | N \in S,\ G^p/\varphi(N) \in C\}$. Then, since $r(G^p/\varphi(N)) \leq r(G) < \infty$, there are finitely many isomorphism classes of pro-p surface groups C such that $\varphi(S)_C \neq \emptyset$. Thus, to show the set $\varphi(S)$ $(= \bigcup_C \varphi(S)_C)$ is finite, it suffices to show that for each C, the set $\varphi(S)_C$ is finite.

Let C be an isomorphism class of pro-p surface group such that $\varphi(S)_C \neq \emptyset$ and $H \in C$ a pro-p surface group. Moreover, let us fix a positive real number M such that, for each $N \in \mathcal{N}$, it holds that $\dim_{\mathbb{F}_p} H^1(N, \mathbb{F}_p) \leq M$. Now, since it is well-known (and follows easily from Proposition 1.6(i)) that H is infinite, there exists an open subgroup $V \subset H$ of H such that $[H : V] > \frac{M-1}{|\chi_{\{p\}}(H)|}$. Then it follows from Lemma 1.3(i), Remark 1.5.1(ii), Proposition 1.6(ii), (iii) that $M < \dim_{\mathbb{F}_p} H^1(V, \mathbb{F}_p)$. Let us observe that, for each $N \in S$, $\dim_{\mathbb{F}_p} H^1(\varphi(N), \mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H^1(N, \mathbb{F}_p) \leq M < \dim_{\mathbb{F}_p} H^1(V, \mathbb{F}_p)$. Now, since G^p is topologically finitely generated, there exist finitely many open subgroups U of G^p such that $[G^p : U] = [H : V]$. Write T for the set of all open subgroups $U \subset G^p$ of G^p such that $[G^p : U] = [H : V]$, and $m := \sharp T (< \infty)$.

Now let us suppose that $\sharp\varphi(S)_C > m$. Let $N_1, \ldots, N_{m+1} \in S$ be elements of S such that $\varphi(N_1), \ldots, \varphi(N_{m+1})$ are distinct elements of $\varphi(S)_C$. For each integer i such that $1 \leq i \leq m+1$, let us choose an isomorphism $G^p/\varphi(N_i) \xrightarrow{\sim} H$ and write U_i for the inverse image of $V \subset H$ by the composite $G^p \twoheadrightarrow G^p/\varphi(N_i) \xrightarrow{\sim} H$. Then it is immediate that $U_i \in T$, $\varphi(N_i) \subset U_i$, and $U_i/\varphi(N_i) \cong V$. Since $\sharp T = m$, there exist two integers h, i such that $1 \leq h < i \leq m+1$ and that $U_h = U_i$. Then it follows from Lemma 1.9 that $\varphi(N_h) = \varphi(N_i)$, which contradicts the choice of N_h and N_i. Thus, it holds that $\sharp\varphi(S)_C \leq m$. This completes the proof of the finiteness of $\varphi(S)_C$, hence also that of $\varphi(S)$.

To conclude the proof of Theorem 1.10, it suffices to show that the surjective map $S \twoheadrightarrow \varphi(S)$, $N \mapsto \varphi(N)$ is bijective. Let $N_1, N_2 \in S$ be such that $\varphi(N_1) = \varphi(N_2)$. Write $p_2 : G \twoheadrightarrow G/N_2$ for the natural surjection. Then, since $p_2(N_1)$ is contained in the kernel of the natural surjection $G/N_2 \twoheadrightarrow (G/N_2)^p$, $p_2(N_1)$ is not open in G/N_2. Moreover, $p_2(N_1) \subset G/N_2$ is a topologically finitely generated normal closed subgroup of G/N_2. Thus, it follows from Proposition 1.8 that $p_2(N_1)$ is trivial, i.e., $N_1 \subset N_2$. Similarly, we have $N_1 \supset N_2$, which implies that $N_1 = N_2$. This completes the proof of Theorem 1.10. □

Remark 1.10.1. The proof of Theorem 1.10 implies that we can write down an upper bound of the number of normal closed subgroups $N \subset G$ satisfying the condition appearing in the statement of Theorem 1.10 only by using p, M, and $r(G)$. (Note that we can write down an upper bound of the number of open subgroups of G^p of given index and the number of isomorphism classes of pro-p surface groups C such that $\varphi(S)_C \neq \emptyset$ only by using $r(G^p) (\leq r(G))$. Moreover, the possible values for the index of V in H depend on p, M, and $r(H) (\leq r(G))$.)

In the remainder of the present §1, we consider profinite groups obtained by forming successive extensions of surface groups.

Definition 1.11 (cf. [13] Definition 2.6)**.** Let n be a positive integer. A *successive extension of surface groups* is data $(G, (G_j)_{0 \leq j \leq n}, (\Sigma_j)_{1 \leq j \leq n})$ consisting of

- a profinite group G;
- a sequence of profinite groups $(G_j)_{0 \leq j \leq n}$;

- a sequence of nonempty sets of prime numbers $(\Sigma_j)_{1\leq j\leq n}$

such that

- $G_0 = G$, $G_n = \{1\}$;
- for any integer j such that $1 \leq j \leq n$, G_j is a normal closed subgroup of G_{j-1}, and, moreover, G_{j-1}/G_j is a pro-Σ_j surface group.

We shall refer to n as the *dimension* of $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$.

Definition 1.12. Let G be a profinite group and $(G_j)_{0\leq j\leq n}$ a sequence of subgroups of G. Then we shall say that $(G_j)_{0\leq j\leq n}$ is an *SESG-filtration* (of dimension n) on G if there exists a sequence of nonempty sets of prime numbers $(\Sigma_j)_{1\leq j\leq n}$ such that $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ is a successive extension of surface groups. We shall say that a profinite group G is of *SESG-type* (of dimension n) if G has an SESG-filtration (of dimension n).

Lemma 1.13 (cf. [13] Proposition 2.13)**.** *Let $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ be a successive extension of surface groups and $\Sigma \subset \mathfrak{Primes}$ a nonempty subset of $\mathfrak{Primes}$. Then the following conditions are equivalent:*

1. $\Sigma \subset \bigcap_{j=1}^n \Sigma_j$.
2. *$\chi_\Sigma(G)$ is defined.*

In particular, $\bigcap_{j=1}^n \Sigma_j$ can be reconstructed from the profinite group G.

Theorem 1.14 (cf. [13] Theorem 2.15)**.** *Let $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ be a successive extension of surface groups and m a nonnegative integer. Write $\Sigma := \bigcap_{j=1}^n \Sigma_j$. Suppose that $\Sigma \neq \emptyset$. Then the following conditions are equivalent:*

1. *$m = n$.*
2. *For any positive real number M, there exists an open subgroup $V \subset G$ of G such that, for any open subgroup $U \subset V$ of V, any nonzero finite Σ-torsion U-module A, and any nonnegative integer i such that $i \neq m$, it holds that $h^m(U, A) > Mh^i(U, A)$.*

In particular, n can be reconstructed from the profinite group G.

Lemma 1.15. *Let p be a prime number, n a positive integer, and $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ a successive extension of surface groups of dimension n such that $p \in \bigcap_{j=1}^n \Sigma_j$. Then the following hold:*

(i) *G is topologically finitely generated and $\chi_{\{p\}}(G)$ is defined.*
(ii) $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) \leq r(G) \leq |\chi_{\{p\}}(G)| + 3n - 1$.

Proof. It is clear by definition that G is topologically finitely generated. Moreover, it follows from Lemma 1.13 that $\chi_{\{p\}}(G)$ is defined. We verify assertion (ii). Let us observe that, since there exists a surjection from the free profinite group of rank $r(G)$ to G, it holds that $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) \leq r(G)$. Thus, it suffices to show that $r(G) \leq |\chi_{\{p\}}(G)| + 3n - 1$. We verify this inequality by induction on n. If $n = 1$, then this follows from Proposition 1.6(ii). Now suppose that $n \geq 2$, and that the induction hypothesis is in force. Then it follows from Lemma 1.3(ii), together with Lemma 1.13 that $\chi_{\{p\}}(G) = \chi_{\{p\}}(G_1) \cdot \chi_{\{p\}}(G/G_1)$. Moreover, it follows from the induction hypothesis that $r(G_1) \leq |\chi_{\{p\}}(G_1)| + 3(n-1) - 1$ and $r(G/G_1) \leq |\chi_{\{p\}}(G/G_1)| + 3 - 1$. Thus, since $|\chi_{\{p\}}(G_1)| + |\chi_{\{p\}}(G/G_1)| \leq |\chi_{\{p\}}(G_1)| \cdot |\chi_{\{p\}}(G/G_1)| + 1$ (observe that it follows from Lemma 1.3(ii) and

Proposition 1.6(ii) that $|\chi_{\{p\}}(G_1)|, |\chi_{\{p\}}(G/G_1)| \geq 1$), we obtain that $r(G) \leq r(G_1) + r(G/G_1) \leq |\chi_{\{p\}}(G)| + 3n - 1$. This completes the proof of assertion (ii), hence also of Lemma 1.15. $\square$

Theorem 1.16. *Let G be a profinite group. Write Σ for the set consisting of all prime numbers p such that $\chi_{\{p\}}(G)$ is defined. Suppose that Σ is nonempty. Then G has at most finitely many SESG-filtrations.*

Proof. Let us fix a prime number $p \in \Sigma$. We may assume that G has an SESG-filtration of dimension n. First, let us observe that Lemma 1.13 and Theorem 1.14 imply that, for any successive extension of surface groups $(H, (H_j)_{0\leq j\leq n'}, (\Sigma_j)_{1\leq j\leq n'})$, if there exists an isomorphism $\alpha : H \xrightarrow{\sim} G$, then it holds that $\bigcap_{j=1}^{n} \Sigma_j = \Sigma$, and $n = n'$. We verify Theorem 1.16 by induction on n. If $n = 1$, then Theorem 1.16 is immediate. Now suppose that $n \geq 2$, and that the induction hypothesis is in force. Write S for the set of normal closed subgroups $N \subset G$ of G such that N is of SESG-type of dimension $n-1$ and that G/N is a surface group. Note that, it follows from the observation above that, if we write Σ' (resp. Σ'') for the set consisting of all prime numbers p such that $\chi_{\{p\}}(N)$ (resp. $\chi_{\{p\}}(G/N)$) is defined, then $\Sigma' \cap \Sigma'' = \Sigma$. Thus, it follows from the induction hypothesis that, for each $N \in S$, N has finitely many SESG-filtrations. Moreover, it follows from Lemma 1.3(ii), Proposition 1.6(ii), Lemma 1.15(ii) that $\dim_{\mathbb{F}_p} H^1(N, \mathbb{F}_p) \leq |\chi_{\{p\}}(N)| + 3(n-1) - 1 \leq |\chi_{\{p\}}(G)| + 3n - 4$. Thus, by applying Theorem 1.10, where we take "$\mathcal{N}$" to be the class of profinite groups N of SESG-type of dimension $n-1$ such that $\dim_{\mathbb{F}_p} H^1(N, \mathbb{F}_p) \leq |\chi_{\{p\}}(G)| + 3n - 4$, we obtain that $\sharp S < \infty$. This completes the proof of Theorem 1.16. $\square$

Remark 1.16.1. It follows from the proof of Theorem 1.16, together with Lemma 1.3(ii), Lemma 1.15(ii), and Remark 1.10.1, that we can write down an upper bound of the number of SESG-filtrations of G only by using the smallest prime number p in Σ, $\chi_\Sigma(G)$, and n (note that these numbers can be reconstructed group-theoretically from G).

Remark 1.16.2. In light of Lemma 1.15(ii), it follows from the proof of Lemma 1.9, Theorem 1.10 that, to verify Theorem 1.16, we can replace all "$\dim_{\mathbb{F}_p} H^1(-, \mathbb{F}_p)$"s appearing in Lemma 1.9, Theorem 1.10 with "$r(-)$"s, "$r((-)^{\mathrm{ab}})$"s, and so on. Indeed, we have used only the following properties of $f(-) := \dim_{\mathbb{F}_p} H^1(-, \mathbb{F}_p)$:

- For any G_1, G_2, if there exists a surjective homomorphism $G_1 \twoheadrightarrow G_2$, then it holds that $f(G_1) \geq f(G_2)$.
- For any pro-Σ surface group H and open subgroup $U \subset H$ of H, if $p \in \Sigma$, then $f(U) \geq f(H)$.
- For any pro-Σ surface group H and real number $M \in \mathbb{R}$, if $p \in \Sigma$, then there exists an open subgroup $V \subset H$ of H such that $f(V) > M$.
- For any successive extension of surface groups $(G, (G_j)_{0\leq j\leq n}, (\Sigma_j)_{1\leq j\leq n})$ of dimension n such that $\bigcap_{j=1}^{n} \Sigma_j \neq \emptyset$, there exists a real number M determined by $\chi_{\{p\}}(G)$ ($p \in \bigcap_{j=1}^{n} \Sigma_j$) and n such that $f(G) \leq M$ (cf. Lemma 1.15(ii)).

However, these properties also hold when $f(-) = r(-)$ or $f(-) = r((-)^{\mathrm{ab}})$.

Remark 1.16.3. Let us fix a prime number p. Let G be a profinite group. Write $\mathcal{C}_p$ for the class of profinite group isomorphic to G_k (cf. Definition 2.1(i)) or G_k^p, where

k is a finite extension field of $\mathbb{Q}_p$. Then $G \in \mathcal{C}_p$ satisfies some properties similar to the properties of pro-p surface groups. For example:

- Any open subgroup of G is in $\mathcal{C}_p$.
- $\mathrm{cd}_p(G) \leq 2$ (cf. [9] Theorem (7.1.8)(i), Proposition (7.5.8)).
- $\chi_{\{p\}}(G)$ is defined, and it holds that $\chi_{\{p\}}(G) < 0$ (cf. [9] Theorem (7.3.1), Proposition (7.5.8)).
- For any finite p-primary G-module A, it holds that $\sharp H^i(G, A) \leq \sharp A$ ($i = 0, 2$) (cf. [9] Theorem (7.2.6), Proposition (7.5.8)).
- G is topologically finitely generated and elastic (cf. [9] Theorem (7.4.1), [7] Theorem 1.7(ii)).

Moreover, it follows from local class field theory that, for any positive integer m, there are at most finitely many isomorphism classes of profinite groups C such that if $G \in C$, then $G \in \mathcal{C}_p$ and $r(G) \leq m$. Thus, by the argument of Theorem 1.17, we can show that there are at most finitely many (finite) filtrations of a given profinite group such that each subquotient is in $\mathcal{C}_p$ or a pro-Σ surface group, where Σ is a set of prime numbers such that $p \in \Sigma$.

2. Finiteness of Isomorphism Classes of Hyperbolic Polycurves

In the present §2, we discuss the finiteness of hyperbolic polycurves whose étale fundamental group is isomorphic to a given profinite group.

Definition 2.1. Let p be a prime number, k a field, X, S connected noetherian schemes, and $X \to S$ a morphism of schemes.

(i) We shall write
$$G_k$$
for the absolute Galois group of k (for some choice of a separable closure of k).

(ii) We shall write
$$\Pi_X$$
for the étale fundamental group of X (for some choice of basepoint).

(iii) We shall write
$$\Delta_{X/S} \subset \Pi_X$$
for the kernel of the outer homomorphism $\Pi_X \to \Pi_S$ induced by $X \to S$. If $S = \operatorname{Spec} k$, then by abuse of notation we sometimes write
$$\Delta_{X/k}$$
instead of $\Delta_{X/S}$.

(iv) We shall write
$$\Pi^p_{X/S}$$
for the quotient of Π_X by the kernel of the natural surjection from $\Delta_{X/S}$ to its maximal pro-p quotient (which is a characteristic subgroup of $\Delta_{X/S}$). If $S = \operatorname{Spec} k$, then by abuse of notation we sometimes write
$$\Pi^p_{X/k}$$
instead of $\Pi^p_{X/S}$. We shall refer to $\Pi^p_{X/k}$ as the *geometrically pro-p étale fundamental group* of X (over k).

Definition 2.2 (cf. [3] Definition 2.1)**.** Let S be a scheme and X a scheme over S.

(i) We shall say that X is a *hyperbolic curve* (*of type* (g, r)) over S if there exist
   - a pair of nonnegative integers (g, r);
   - a scheme X^{cpt} which is smooth, proper, geometrically connected, and of relative dimension one over S;
   - a (possibly empty) closed subscheme $D \subset X^{\mathrm{cpt}}$ of X^{cpt} which is finite and étale over S

   such that
   - $2g - 2 + r > 0$;
   - any geometric fiber of $X^{\mathrm{cpt}} \to S$ is (a necessarily smooth proper curve) of genus g;
   - the finite étale morphism $D \hookrightarrow X^{\mathrm{cpt}} \to S$ is of degree r;
   - X is isomorphic to $X^{\mathrm{cpt}} \setminus D$ over S.

(ii) We shall say that X is a *hyperbolic polycurve (of relative dimension n)* over S if there exist a positive integer n and a (not necessarily unique) factorization of the structure morphism $X \to S$

$$X = X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to S = X_0$$

such that, for each integer j such that $1 \le j \le n$, $X_j \to X_{j-1}$ is a hyperbolic curve. We shall refer to the above morphism $X \to X_{n-1}$ as a *parametrizing morphism* for X and refer to the above factorization of $X \to S$ as a *sequence of parametrizing morphisms*.

Definition 2.3. Let S be a scheme.

(i) A *parametrized hyperbolic polycurve (of relative dimension n)* is a pair $\mathfrak{X} = (X, X = X_n \to X_{n-1} \to \cdots \to X_1 \to S = X_0)$ consisting of a hyperbolic polycurve X (of relative dimension n) over S and a sequence of parametrizing morphisms $X = X_n \to X_{n-1} \to \cdots \to X_1 \to S = X_0$ of X/S. We shall refer to X (over S) as an *underlying hyperbolic polycurve* of $\mathfrak{X}$.

(ii) Let $\mathfrak{X}$ be a parametrized hyperbolic polycurve over S whose underlying hyperbolic polycurve is X. Then we shall write $\Delta_{\mathfrak{X}/S} := \Delta_{X/S}$, $\Pi_{\mathfrak{X}} := \Pi_X$, $\Pi^p_{\mathfrak{X}/S} := \Pi^p_{X/S}$. If $S = \operatorname{Spec} k$, then by abuse of notation we sometimes write $\Delta_{\mathfrak{X}/k}$ (resp. $\Pi^p_{\mathfrak{X}/k}$) instead of $\Delta_{\mathfrak{X}/S}$ (resp. $\Pi^p_{\mathfrak{X}/S}$). By abuse of terminology, we shall refer to $\Pi_{\mathfrak{X}}$ (resp. $\Pi^p_{\mathfrak{X}/k}$) as the étale fundamental group of $\mathfrak{X}$ (resp. geometrically pro-p étale fundamental group of $\mathfrak{X}$ over k).

(iii) Let T be a scheme and $\mathfrak{X} = (X, X = X_n \to X_{n-1} \to \cdots \to X_1 \to S = X_0)$, $\mathfrak{Y} = (Y, Y = Y_n \to Y_{n-1} \to \cdots \to Y_1 \to T = Y_0)$ parametrized hyperbolic polycurves of relative dimension n. An *isomorphism* from $\mathfrak{Y}$ to $\mathfrak{X}$ is defined to be a collection of isomorphisms of schemes $\{Y_j \xrightarrow{\sim} X_j\}_{0 \le j \le n}$ such that, for each integer j such that $1 \le j \le n$, the diagram

$$\begin{array}{ccc} Y_j & \xrightarrow{\sim} & X_j \\ \downarrow & & \downarrow \\ Y_{j-1} & \xrightarrow{\sim} & X_{j-1} \end{array}$$

commutes. If $S = T$, then an S-isomorphism $\mathfrak{Y} \xrightarrow{\sim} \mathfrak{X}$ is defined to be an isomorphism $\{Y_j \xrightarrow{\sim} X_j\}_{0 \leq j \leq n}$ such that $T = Y_0 \xrightarrow{\sim} X_0 = S$ is the identity morphism (i.e., each $Y_j \xrightarrow{\sim} X_j$ is an S-isomorphism).

Remark 2.3.1. Let k be a field of characteristic zero, S a connected noetherian separated normal scheme over k, and X a hyperbolic curve over S. Then it follows from Remark 1.5.1(i), together with [3] Proposition 2.4(ii), that $\Delta_{X/S}$ is a (pro-$\mathfrak{Primes}$) surface group.

Remark 2.3.2 (cf. [12] Remark 2.8). Let k be a field of characteristic zero, S a connected noetherian separated normal scheme over k, and $(X, X = X_n \to X_{n-1} \to \cdots \to X_1 \to S = X_0)$ a parametrized hyperbolic polycurve of relative dimension n over S. Then, for any triplet of integers (i, j, l) such that $0 \leq i < j < l \leq n$, we obtain a natural exact sequence of profinite groups

$$1 \to \Delta_{X_l/X_j} \to \Delta_{X_l/X_i} \to \Delta_{X_j/X_i} \to 1$$

(for any choice of basepoints).

Definition 2.4 (cf. [12] Definition 2.10)**.** Let p be a prime number, n a positive integer, k a field of characteristic zero, and S a connected noetherian separated normal scheme over k.

(i) Let X be a hyperbolic polycurve of relative dimension n over S and $X = X_n \to X_{n-1} \to \cdots \to X_1 \to S = X_0$ a sequence of parametrizing morphisms. Then we shall say that the sequence $X = X_n \to X_{n-1} \to \cdots \to X_1 \to S = X_0$ satisfies condition $(*)_p$ if for any triplet of integers (i, j, l) such that $0 \leq i < j < l \leq n$, the sequence of profinite groups

$$1 \to \Delta^p_{X_l/X_j} \to \Delta^p_{X_l/X_i} \to \Delta^p_{X_j/X_i} \to 1$$

is exact. We shall say that X/S satisfies condition $(*)_p$ if there exists a sequence of parametrizing morphisms of X/S satisfying condition $(*)_p$.

(ii) Let $\mathfrak{X} = (X, X = X_n \to X_{n-1} \to \cdots \to X_1 \to S = X_0)$ be a parametrized hyperbolic polycurve of relative dimension n over S. Then we shall say that $\mathfrak{X}/S$ satisfies condition $(*)_p$ if the sequence $X = X_n \to X_{n-1} \to \cdots \to X_1 \to S = X_0$ satisfies condition $(*)_p$.

Remark 2.4.1 (cf. [13] Remark 2.5.3). If X/S satisfies condition $(*)_p$, then $\Delta_{X/S}$ admits various group-theoretic properties (cf. e.g., [12] Proposition 2.16(iii), [13] Corollary 2.8). However, it is unknown whether the validity of condition $(*)_p$ for X/S only depends on the profinite group $\Delta_{X/S}$ or not.

Remark 2.4.2. Let n be a positive integer, k a field of characteristic zero, S a connected noetherian separated normal scheme over k, and $\mathfrak{X} = (X, X = X_n \to X_{n-1} \to \cdots \to X_1 \to S = X_0)$ a parametrized hyperbolic polycurve of relative dimension n over S. Then the data $(\Delta_{X/S}, (\Delta_{X/X_j})_{0 \leq j \leq n}, (\mathfrak{Primes})_{1 \leq j \leq n})$ is a successive extension of surface groups of dimension n. If, moreover, $\mathfrak{X}/S$ satisfies condition $(*)_p$ (where p is a prime number), then $(\Delta^p_{X/S}, (\Delta^p_{X/X_j})_{0 \leq j \leq n}, (\{p\})_{1 \leq j \leq n})$ is also a successive extension of surface groups of dimension n.

Definition 2.5 (cf. [6] Definition 4.11)**.** Let p be a prime number and k a field. Then we shall say that k is *generalized sub-p-adic* if k is isomorphic to a subfield of a finitely generated extension of the quotient field of $W(\overline{\mathbb{F}_p})$ (the ring of Witt vectors with coefficients in $\overline{\mathbb{F}_p}$).

Theorem 2.6. *Let p be a prime number, n a positive integer, k a generalized sub-p-adic field, and $\mathfrak{X} = (X, X = X_n \to X_{n-1} \to \cdots \to X_1 \to \operatorname{Spec} k = X_0)$, $\mathfrak{Y} = (Y, Y = Y_n \to Y_{n-1} \to \cdots \to Y_1 \to \operatorname{Spec} k = Y_0)$ parametrized hyperbolic polycurves of dimension n over k. Then the following hold:*

(i) *Let $\varphi : \Pi_{\mathfrak{Y}} \xrightarrow{\sim} \Pi_{\mathfrak{X}}$ be an isomorphism from $\Pi_{\mathfrak{Y}}$ to $\Pi_{\mathfrak{X}}$ over G_k. Suppose that for each integer j such that $0 \leq j \leq n$, it holds that $\varphi(\Delta_{Y/Y_j}) = \Delta_{X/X_j}$. Then φ arises from a unique isomorphism $\mathfrak{Y} \xrightarrow{\sim} \mathfrak{X}$ over k.*

(ii) *Let $\psi : \Pi^p_{\mathfrak{Y}/k} \xrightarrow{\sim} \Pi^p_{\mathfrak{X}/k}$ be an isomorphism from $\Pi^p_{\mathfrak{Y}/k}$ to $\Pi^p_{\mathfrak{X}/k}$ over G_k. Suppose that both of $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy condition $(*)_p$, and that for each integer j such that $0 \leq j \leq n$, it holds that $\psi(\Delta^p_{Y/Y_j}) = \Delta^p_{X/X_j}$. Then ψ arises from a unique isomorphism $\mathfrak{Y} \xrightarrow{\sim} \mathfrak{X}$ over k.*

Proof. Assertion (i) follows from [3] Proposition 3.2(i) and the proof of [3] Lemma 4.2(iii). Assertion (ii) follows from [12] Proposition 3.2(i) and the proof of [12] Lemma 4.3(ii). □

Remark 2.6.1. In [3] §4 and [12] §4 (especially [3] Lemma 4.2(iii) and [12] Lemma 4.3(ii)), we sometimes assumed that the base field is sub-p-adic (cf. [5] Definition 15.4(i)). That is because we have used [5] Theorem A (i.e., a "Hom-version" of the Grothendieck conjecture for hyperbolic curves over a sub-p-adic field). However, in these two sections, by using [6] Theorem 4.12 (i.e., an "Isom-version" of the Grothendieck conjecture for hyperbolic curves over a generalized sub-p-adic field) instead of [5] Theorem A, we can replace "sub-p-adic" with "generalized sub-p-adic".

Theorem 2.7. *Let p be a prime number, k a generalized sub-p-adic field, G a profinite group, and $G \twoheadrightarrow G_k$ a surjective homomorphism. Then there are at most finitely many (possibly none) k-isomorphism classes of parametrized hyperbolic polycurves over k (resp. parametrized hyperbolic polycurves over k satisfying condition $(*)_p$) whose étale fundamental group (resp. geometrically pro-p étale fundamental group) is isomorphic to G over G_k.*

Proof. Write $\Delta := \ker(G \twoheadrightarrow G_k)$ and S for the set of k-isomorphism classes of parametrized hyperbolic polycurves over k (resp. parametrized hyperbolic polycurves over k satisfying condition $(*)_p$) whose étale fundamental group (resp. geometrically pro-p étale fundamental group) is isomorphic to G over G_k. We may assume that $S \neq \emptyset$. Then it follows from Theorem 1.16 that Δ has finitely many SESG-filtrations. Write m for the number of SESG-filtrations of Δ. Suppose that $\sharp S \geq m+1$. Let $C^{(1)}, \ldots, C^{(m+1)} \in S$ be distinct elements of S. For each integer i such that $1 \leq i \leq m+1$, let us fix a parametrized hyperbolic polycurve $\mathfrak{X}^{(i)}$ over k whose k-isomorphism class is $C^{(i)}$, and an isomorphism $\alpha^{(i)} : \Pi_{\mathfrak{X}^{(i)}} \xrightarrow{\sim} G$ (resp. $\alpha^{(i)} : \Pi^p_{\mathfrak{X}^{(i)}/k} \xrightarrow{\sim} G$) over G_k. Let us write $(\Delta^{(i)}_j)_{0 \leq j \leq n}$ for the SESG-filtration of $\Delta^{(i)} := \Delta_{\mathfrak{X}^{(i)}/k}$ (resp. $\Delta^{(i)} := \Delta^p_{\mathfrak{X}^{(i)}/k}$) determined by $\mathfrak{X}/k$ as in Remark 2.4.2. Note that it follows from Theorem 1.14 that the dimension n does not depend on i. Now since $(\alpha^{(i)}(\Delta^{(i)}_j))_{0 \leq j \leq n}$ is an SESG-filtration of Δ, it follows from our choice of m that there exist two integers h, i such that $1 \leq h < i \leq m+1$ and that $\alpha^{(h)}(\Delta^{(h)}_j) = \alpha^{(i)}(\Delta^{(i)}_j)$ for each integer j such that $0 \leq j \leq n$. Then it follows from Theorem 2.6 that the isomorphism $(\alpha^{(i)})^{-1} \circ \alpha^{(h)}$ arises from a k-isomorphism $\mathfrak{X}^{(h)} \xrightarrow{\sim} \mathfrak{X}^{(i)}$ over k. However, since (we have assumed that) the k-isomorphism class of $\mathfrak{X}^{(h)}$ (i.e., $C^{(h)}$) and that of $\mathfrak{X}^{(i)}$ (i.e., $C^{(i)}$) are distinct from each other, we obtain a contradiction. Thus, we conclude that $\sharp S \leq m < \infty$. This completes the proof of Theorem 2.7. $\square$

Remark 2.7.1. It follows from the proof of Theorem 2.7, together with Remark 1.16.1, that, if the set of k-isomorphism classes of parametrized hyperbolic polycurves over k (resp. parametrized hyperbolic polycurves over k satisfying condition $(*)_p$) whose étale fundamental group (resp. geometrically pro-p étale fundamental group) is isomorphic to G over G_k is nonempty, then we can write down an upper bound of the cardinality of the set only by using $\chi_{\mathfrak{Primes}}(\ker(G \twoheadrightarrow G_k))$ and the dimension of $\ker(G \twoheadrightarrow G_k)$ (resp. p, $\chi_{\{p\}}(\ker(G \twoheadrightarrow G_k))$, and the dimension of $\ker(G \twoheadrightarrow G_k)$). Note that these numbers can be reconstructed group-theoretically from $\ker(G \twoheadrightarrow G_k)$.

The following corollary immediately follows from Theorem 2.7.

Corollary 2.8. *Let p be a prime number, k a generalized sub-p-adic field, G a profinite group, and $G \twoheadrightarrow G_k$ a surjective homomorphism. Then there are at most finitely many k-isomorphism classes of hyperbolic polycurves over k (resp. hyperbolic polycurves over k satisfying condition $(*)_p$) whose étale fundamental group (resp. geometrically pro-p étale fundamental group) is isomorphic to G over G_k.*

Moreover, by using an argument similar to Theorem 2.7, we can show the following theorem. Theorem 2.9 states that, roughly speaking, any hyperbolic polycurve (over a field of characteristic zero) has at most finitely many sequences of parametrizing morphisms (up to isomorphism).

Theorem 2.9. *Let k be a field of characteristic zero and X a hyperbolic polycurve over k. Then there are at most finitely many k-isomorphism classes of parametrized hyperbolic polycurves over k whose underlying hyperbolic polycurve is X/k.*

Proof. Since k is a direct limit of finitely generated subextensions of $k/\mathbb{Q}$, there exists a finitely generated subextension k_0 of $k/\mathbb{Q}$ such that X/k has a model X_{k_0}/k_0. For a subextension l of k/k_0, let us write P_l for the set of isomorphism class of parametrized hyperbolic polycurves over l whose underlying hyperbolic polycurve is $X_{k_0} \times_{k_0} l/l$. Note that it follows from [3] Proposition 2.4(ii) that $X \cong (X_{k_0} \times_{k_0} l) \times_l k \to X_{k_0} \times_{k_0} l$ determines an isomorphism $\Delta_{X/k} \xrightarrow{\sim} \Delta_{X_{k_0} \times_{k_0} l/l}$. Thus, if we write F for the set of SESG-filtrations of $\Delta_{X/k}$, there exists a natural map $P_l \to F$ (cf. Remark 2.4.2). Moreover, if l is finitely generated over $\mathbb{Q}$, then (since l is generalized sub-p-adic for any prime number p), it follows from Theorem 2.6(i) that the natural map $P_l \to F$ is injective. It is clear that this map is compatible with finitely generated subextensions $l' \supset l$ of k/k_0. Moreover, any parametrized hyperbolic polycurve over k whose underlying hyperbolic polycurve is X/k has a model for some finitely generated subextension of k/k_0. Thus, the natural map $P_k \to F$ is injective. Since it follows from Theorem 1.16 that F, hence also P_k, is a finite set. This completes the proof of Theorem 2.9. $\square$

Remark 2.9.1. The statement of Theorem 2.9 is purely algebro-geometric. However, the author does not know at the time of writing whether we can prove Theorem 2.9 only by using a purely algebro-geometric method (i.e., without using anabelian geometry) or not.

When the base field k is finitely generated over $\mathbb{Q}$, then we can also prove the "absolute version" of Theorem 2.7, i.e., we can consider only the (geometrically pro-p) étale fundamental group instead of the (geometrically pro-p) étale fundamental group equipped with the surjective homomorphism to the absolute Galois group of the base field. (However, since the automorphism group of k is not finite in general, we cannot prove the finiteness of k-isomorphism classes of parametrized hyperbolic polycurves.)

Proposition 2.10 ([3] Proposition 3.19)**.** *Let k_X, k_Y be finitely generated extension fields of $\mathbb{Q}$ and $\overline{k}_X, \overline{k}_Y$ algebraic closures of k_X, k_Y, respectively. Then the following hold:*

(i) *Let $H \subset G_{k_X}$ be a closed subgroup of G_{k_X}. Suppose that H is topologically finitely generated and normal in an open subgroup of G_{k_X}. Then H is trivial.*

(ii) *Write $\mathrm{Isom}(\overline{k}_X/k_X, \overline{k}_Y/k_Y)$ for the set of isomorphisms $\overline{k}_X \xrightarrow{\sim} \overline{k}_Y$ that determine isomorphisms $k_X \xrightarrow{\sim} k_Y$. Then the natural map $\mathrm{Isom}(\overline{k}_X/k_X, \overline{k}_Y/k_Y) \to \mathrm{Isom}(G_{k_Y}, G_{k_X})$ is bijective.*

Theorem 2.11. *Let p be a prime number and G a profinite group. Then the following hold:*

(i) *Suppose that there exist a finitely generated extension field k of $\mathbb{Q}$ and a hyperbolic polycurve X over k (resp. a hyperbolic polycurve X over k satisfying condition $(*)_p$) such that G is isomorphic to Π_X (resp. $\Pi^p_{X/k}$). Then there exists a unique maximal normal closed subgroup H of G which is topologically finitely generated. Moreover, for any isomorphism α from Π_X (resp. $\Pi^p_{X/k}$) to G, it holds that $H = \alpha(\Delta_{X/k})$ (resp. $H = \alpha(\Delta^p_{X/k})$) and α induces an isomorphism $G_k \xrightarrow{\sim} G/H$.*

(ii) *In the notation of (i), k is completely determined by G up to isomorphism. Moreover, for any finitely generated extension field k' of $\mathbb{Q}$, the map $\mathrm{Isom}(\overline{k'}/k', \overline{k}/k) \to \mathrm{Isom}(G_{k'}, G/H)$ determined by the isomorphism $G_k \xrightarrow{\sim} G/H$ appearing in (i) (for any fixed isomorphism α) is bijective.*

(iii) *There are at most finitely many isomorphism classes of parametrized hyperbolic polycurves over finitely generated extension fields of $\mathbb{Q}$ (resp. parametrized hyperbolic polycurves over finitely generated extension fields of $\mathbb{Q}$ satisfying condition $(*)_p$) whose étale fundamental group (resp. geometrically pro-p étale fundamental group) is isomorphic to G.*

Proof. Assertion (ii) follows from assertion (i), together with Proposition 2.10(ii). Assertion (iii) follows from assertion (ii), together with Theorem 2.7. We verify assertion (i). Let us write $\beta : \Pi_X \twoheadrightarrow G_k$ (resp. $\beta : \Pi^p_{X/k} \twoheadrightarrow G_k$). Let $H \subset G$ be a topologically finitely generated normal closed subgroup of G. Then it follows from Proposition 2.10(i) that the image of $H \subset G$ by the composite of α^{-1} and β is trivial. Thus, $H \subset \alpha(\ker \beta)$. Since $\ker \beta = \Delta_{X/k}$ (resp. $(\ker \beta = \Delta^p_{X/k})$), hence also $\alpha(\ker \beta)$, is of SESG-type, $\alpha(\ker \beta)$ is topologically finitely generated. Thus, $\alpha(\ker \beta)$ is the unique maximal normal closed subgroup of G which is topologically finitely generated. Moreover, since $G_k \cong \Pi_X/\Delta_{X/k}$ (resp. $G_k \cong \Pi^p_{X/k}/\Delta^p_{X/k}$), α induces an isomorphism $G_k \xrightarrow{\sim} G/\alpha(\ker \beta)$. This completes the proof of assertion (i), hence also of Theorem 2.11. □

Remark 2.11.1. If we consider the case where the base field k is a number field, then, since the automorphism group of a number field is finite, we can prove the finiteness of k-isomorphism class of parametrized hyperbolic polycurves over k whose étale fundamental group is isomorphic to a given profinite group.

Finally, as another application of Theorem 1.16, we give an alternative proof of [3] Theorem 4.4 and [12] Theorem 4.6 (see also Remark 2.6.1).

Theorem 2.12 ([3] Theorem 4.4, [12] Theorem 4.6)**.** *Let p be a prime number, k a generalized sub-p-adic field, X, Y hyperbolic polycurves over k. Write* $\mathrm{Isom}_{G_k}(\Pi_Y, \Pi_X)$ *(resp.* $\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})$*) for the set of isomorphisms of* Π_Y *(resp.* $\Pi^p_{Y/k}$*) with* Π_X *(resp.* $\Pi^p_{X/k}$*) over G_k. Then the set*

$$\mathrm{Isom}_{G_k}(\Pi_Y, \Pi_X)/\operatorname{Inn}(\Delta_{X/k})$$

is finite. Moreover, if at least one of $X/k, Y/k$ satisfies condition $(*)_p$, then the set

$$\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})/\operatorname{Inn}(\Delta^p_{X/k})$$

is finite.

Proof. We may assume that $\mathrm{Isom}_{G_k}(\Pi_Y, \Pi_X) \neq \emptyset$ (resp. $\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k}) \neq \emptyset$). Then any element of $\mathrm{Isom}_{G_k}(\Pi_Y, \Pi_X)$ (resp. $\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})$) determines a bijection between $\mathrm{Isom}_{G_k}(\Pi_Y, \Pi_X)/\operatorname{Inn}(\Delta_{X/k})$ (resp. $\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})/\operatorname{Inn}(\Delta^p_{X/k})$) and $\mathrm{Aut}_{G_k}(\Pi_X)/\operatorname{Inn}(\Delta_{X/k})$ (resp. $\mathrm{Aut}_{G_k}(\Pi^p_{X/k})/\operatorname{Inn}(\Delta^p_{X/k})$). Thus, to verify Theorem 2.12, we may assume without loss of generality that $X = Y$ (resp. $X = Y$, and that X/k satisfies condition $(*)_p$). For convenience, let us write $\Delta := \Delta_{X/k}, \Pi := \Pi_X$ (resp. $\Delta := \Delta^p_{X/k}, \Pi := \Pi^p_{X/k}$). Let us fix an SESG-filtration $(\Delta_j)_{0\leq j\leq \dim X}$ of Δ determined by a sequence of parametrizing morphisms of X/k (resp. a sequence of parametrizing morphisms of X/k satisfying condition $(*)_p$) as in Remark 2.4.2.

Now it follows from Theorem 1.16 that Δ has finitely many SESG-filtrations, and it follows from [3] Proposition 4.5 (note that the proof of [3] Proposition 4.5 does not use [3] Theorem 4.4) that $\mathrm{Aut}_k(X)$ is finite. Write S for the set of SESG-filtrations of Δ. Then $\mathrm{Aut}_{G_k}(\Pi)$ acts naturally on S. Write $A \subset \mathrm{Aut}_{G_k}(\Pi)$ for the stabilizer subgroup of $\mathrm{Aut}_{G_k}(\Pi)$ with respect to $(\Delta_j)_{0\leq j\leq \dim X} \in S$. Then it is immediate that $[\mathrm{Aut}_{G_k}(\Pi) : A] \leq \sharp S$. Moreover, it follows from Theorem 2.6 that the image of $A \subset \mathrm{Aut}_{G_k}(\Pi)$ by the natural surjection $\mathrm{Aut}_{G_k}(\Pi) \twoheadrightarrow \mathrm{Aut}_{G_k}(\Pi)/\operatorname{Inn}(\Delta)$ is contained in the image of the natural map $\mathrm{Aut}_k(X) \to \mathrm{Aut}_{G_k}(\Pi)/\operatorname{Inn}(\Delta)$, which implies that $\sharp(\mathrm{Aut}_{G_k}(\Pi)/\operatorname{Inn}(\Delta)) \leq \sharp S \cdot \sharp \mathrm{Aut}_k(X) < \infty$. This completes the proof of Theorem 2.12. $\square$

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