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Abstract

In this article, we give equations about homology classes given by the self-intersection of singularity sets of fold maps. As an application, we give an obstruction to the existence of fold maps up to cobordism.

1 Introduction

Let X be a closed n–dimensional manifold. For a smooth map $f : X \to \mathbb{R}^p$, we denote by $\Sigma^k(f) \subset X$ the Thom-Boardman singularity set (See Definition 2.1 for the details). It is known that, for a generic f, the closure $\overline{\Sigma}^k(f)$ of $\Sigma^k(f)$ gives a homology class with $\mathbb{Z}/2$ coefficients and its Poincaré dual $[\overline{\Sigma}^k(f)]_{P.D}$ is written by a polynomial of Stiefel-Whitney classes, called Thom polynomial ([12], [5], [7], [6]). Sakuma in [11], Ohmoto, Saeki and Sakuma in [4] studied the self-intersection of $\overline{\Sigma}^k(f)$. In particular Ohmoto, Saeki and Sakuma gave equations among the cohomology class given by the self-intersection of $\overline{\Sigma}^k(f)$ and the characteristic classes of X.

Let $f : X \to \mathbb{R}^p$ be a fold map and let $\widetilde{f} : X \to \mathbb{R}^{p+k-1}$ be a generic map which is a lift of f. In this article, we show the following equation (Proposition 4.1).

$$[\overline{\Sigma}^{n-p+1}(f)]^k = [\overline{\Sigma}^{n-p+1}(\widetilde{f})] \in H_{n-k(n-p+1)}(X; \mathbb{Z}/2).$$

Since $[\overline{\Sigma}^{n-p+1}(f)]$ and $[\overline{\Sigma}^{n-p+1}(\widetilde{f})]$ are invariant under cobordism, the above equation also holds for any generic f and generic $\widetilde{f}$ under the following each condition:

(1) There is a closed manifold X' cobrdant to X and there is a fold map $f' : X \to \mathbb{R}^p$. In addition, n, p, k satisfy $n - k(n-p+1) = 0$ (Theorem 1).

(2) There is a fold map $f_0 : X \to \mathbb{R}^p$ (Theorem 2).

In this article, we prove these equations by using arguments about homology intersections.

We give two applications of Theorem 1 and Theorem 2.

As the first application, we give an obstruction to admitting a fold map from the manifold into $\mathbb{R}^p$ up to cobordism. The existence problem of fold map was

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deeply studied by Ando [1], Saeki [10], Sadykov and Saeki [8], Sadykov, Saeki and Sakuma [9].

As the second application, we give some formulas among Stiefel-Whitney classes of X by using Ronga's formula about Thom polynomials [7], when X admit a fold map up to cobordism.

The organization of this paper is as follows. In Section 2 we introduce some notations and definitions. In Section 3 we give two equations among the homology classes determined by singularity sets and then we give proof of these equations in Section 4. In Section 5 we give two applications of our equations.

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2 Preliminaries

2.1 Singularity sets

Let X be a smooth compact n–dimensional manifold and let $f : X \to \mathbb{R}^p$ be a smooth map. Let $j^1 f : X \to Hom(TX, T\mathbb{R}^p)$ be the 1-jet extension of f, namely $j^1 f(x) \in Hom(T_x M, T_{f(x)}\mathbb{R}^p)$ is the derivative of f. The typical fiber of the vector bundle $Hom(TX, T\mathbb{R}^p)$ can be identified with the space $Hom(\mathbb{R}^n, \mathbb{R}^p)$ of all $n \times p$ matrices. Let Σ^k be the Thom-Boardman singularity set ([12], [2], [3]), namely

$$\Sigma^k = \{f \in Hom(\mathbb{R}^n, \mathbb{R}^p) \mid \dim \ker(f) = k\} \subset Hom(\mathbb{R}^n, \mathbb{R}^p).$$

We denote by $\Sigma^k(M, \mathbb{R}^p)$ the associated Σ^k-bundle of $Hom(TX, T\mathbb{R}^p)$. A smooth map $f : M \to \mathbb{R}^p$ is said to be *generic* if $j^1 f$ is transverse to $\Sigma^k(M, \mathbb{R}^p)$ for all k. Thanks to Thom's transversality theorem, for generic map f, $f^{-1}(\Sigma^k(M, \mathbb{R}^p))$ is a smooth (may not be closed) submanifold of X of codimension $k(n - p + k)$, for $n - k \geq p$.

definition 2.1. *Let*

$$\Sigma^k(f) = \Sigma^k(f : X \to \mathbb{R}^p) = (j^1 f)^{-1}(\Sigma^k(X, \mathbb{R}^p)),$$

and we denote by $\overline{\Sigma}^k(f)$ *its closure in* X.

In particular when $k \leq n$, $\overline{\Sigma}^{n-p+1}(f)$ is the set of singular points of f.

definition 2.2. *A smooth map $f : X \to \mathbb{R}^p$ is said to be a fold map if f is locally written by*

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_{p-1}, \sum_{i=p}^{n} \pm x_i^2)$$

near any singular point.

We remark that the closure $\overline{\Sigma}^k(f)$ may not a manifold. It is known that, however, $\overline{\Sigma}^k(f)$ gives a homology class in $H_*(X; \mathbb{Z}/2)$ independent from the choice of f (For example, See [7]).

If f is a fold map, then $\overline{\Sigma}^{n-p+1}(f) = \Sigma^{n-p+1}(f)$ is a submanifold of X.

2.2 A self-intersection of a homology class

Let X be a smooth closed n–dimensional manifold. Let k, l be positive integers. For a homology class $\alpha \in H_k(X; \mathbb{Z}/2)$, the *$l$-th self-intersection* $\alpha^l \in H_{n-l(n-k)}(X; \mathbb{Z}/2)$ *of* α is defined to be

$$\alpha^l = ((\alpha_{P.D.})^l)_{P.D.} \in H_{n-l(n-k)}(X; \mathbb{Z}/2).$$

Here $\alpha_{P.D.} \in H^{n-k}(X; \mathbb{Z}/2)$ is the Poincaré dual of α.

Let $A \subset X$ be a submanifold of X of dimension k. We denote by $[A] \in H_k(X; \mathbb{Z}/2)$ the image of the fundamental homology class of A via the homomorphism $H_k(A; \mathbb{Z}/2) \to H_n(M; \mathbb{Z}/2)$ induced from the inclusion map $A \hookrightarrow X$. When α is given as $\alpha = [A]$, the l-th self-intersection $\alpha^l = [A]^l$ can be described as the intersection of l copies of A:

Let $v_2, \ldots, v_l$ be sections of the normal bundle $\nu(A)$ of A in X or sections of $TX|_A$. Let v_1 be the zero section of $\nu(A)$. Let $A_{v_i} = v_i(A)$ be the manifold given by pushing A along v_i, for $i = 1, \ldots, l$. For generic $v_2, \ldots, v_l$, the manifolds $A_{v_1}(= A)$, $A_{v_2}, \ldots, A_{v_{l-1}}$ intersect transversally. Thus the intersection $\bigcap_{i=1}^{l} A_{v_i}$ is a manifold representing the homology class $[A]^l$:

$$[\bigcap_{i=1}^{l} A_{v_i}] = [A]^l.$$

Remark 2.3. *When l is an even number, we can define l-th self-intersection as a homology class with integer coefficient $\mathbb{Z}$ as follows: For $x \in \bigcap_i A_{v_i}$, take a basis $B_1 = \{b_1, \ldots, b_{n-k}\}$ of $\nu_x(A)$. Let $B_i = \{b_1^i, \ldots, b_{n-k}^i\}$ be the basis of $\nu_x(A'_{v_i})$ given by pushing $b_1, \ldots, b_{n-k}$ along v_i, for $i = 2, \ldots, l$. Then a collection of vectors $B_1 \sqcup \ldots \sqcup B_l$ gives a basis of $\nu_x(\bigcap_i A_{v_i})$ and then it gives an orientation of $\nu_x(\bigcap_i A_{v_i})$. We orient $T_x(\bigcap_i A_{v_i})$ as $T_x(\bigcap_i A_{v_i}) \oplus \nu_x(\bigcap_i A_{v_i}) = T_x X$. When l is even, this orientation is independent of the choice of B_1:*

Lemma 2.4. *The orientation of $\bigcap_i A_{v_i}$ determined by B_1 is independent of the choice of the basis B_1.*

Proof. Let B'_1 be an alternative choice of B_1. Let $U \in M_{n-k}(\mathbb{R})$ be the transformation matrix U from B_1 to B'_1. Then the transformation matrix from the basis of $\nu_x(\bigcap_i A_{v_i})$ given by B_1 to that of B'_1 is represented by the matrix $U^{\oplus l} \in M_{l(n-k)}(\mathbb{R})$. Since l is even, $\det(U^{\oplus l}) = (\det U)^l = 1$. This implies that the orientation of $T_x(\bigcap_i A_{v_i})$ determined by B'_1 coincides with that of B_1. $\square$

3 Equations among singularity sets.

We first define a notion of *"to admitting a fold map up to cobordism"*.

definition 3.1. *A smooth closed n–dimensional manifold X admits a fold map into $\mathbb{R}^p$ up to cobordism if there exist a smooth n–dimensional manifold X' cobordant to X and there exists a fold map $f' : X' \to \mathbb{R}^p$ on X'.*

Theorem 1. *Let X be a closed oriented n–dimensional manifold admitting a fold map into $\mathbb{R}^p$ up to cobordism. Let $k \geq 2$ be an integer satisfying*

$$k(n-p+1) = n.$$

Then for any generic map $f : X \to \mathbb{R}^p$ and for any generic map $\widetilde{f} : X \to \mathbb{R}^{p+k-1}$,

$$[\overline{\Sigma}^{n-p+1}(f)]^k = [\overline{\Sigma}^{n-p+1}(\widetilde{f})] \in H_0(X; \mathbb{Z}/2).$$

If X itself admits a fold map (not up to cobordism), the following more precisely equation holds:

Theorem 2. *Let X be a closed oriented n–dimensional manifold that there is a fold map from X to $\mathbb{R}^p$. Let k be a positive integer. Then for any generic map $f : X \to \mathbb{R}^p$ and for any generic map $\widetilde{f} : X \to \mathbb{R}^{p+k-1}$,*

$$[\overline{\Sigma}^{n-p+1}(f)]^k = [\overline{\Sigma}^{n-p+1}(\widetilde{f})] \in H_{n-k(n-p+1)}(X; \mathbb{Z}/2).$$

Example 3.2. *(1) Since Theorem 1.1 of [10], any closed connected oriented 4–manifold X admits a fold map into $\mathbb{R}^3$ up to cobordism. Thus Theorem 1 implies that for any generic maps $f : X \to \mathbb{R}^3$ and $\widetilde{f} : X \to \mathbb{R}^4$,*

$$[\overline{\Sigma}^2(f)]^2 = [\overline{\Sigma}^2(\widetilde{f})] \in H_0(X; \mathbb{Z}/2).$$

We remark that, Ohmoto, Saeki and Sakuma proved in [4] that

$$[\overline{\Sigma}^2(f)]^2 = 3\mathrm{Sign}X \in \mathbb{Z}$$

by using kinds of classifying spaces of singularities.

(2) Let X be a connected closed 8–dimensional manifold admitting a fold map into $\mathbb{R}^7$ up to cobordism. Then, for any generic maps $f : X \to \mathbb{R}^7$ and $\widetilde{f} : X \to \mathbb{R}^{10}$,

$$[\overline{\Sigma}^2(f)]^4 = [\overline{\Sigma}^2(\widetilde{f})] \in H_0(X; \mathbb{Z}/2).$$

(3) *Let X be a connected closed 10–dimensional manifold admitting a fold map into $\mathbb{R}^8$. Then, for any generic maps $f : X \to \mathbb{R}^8$ and $\widetilde{f} : X \to \mathbb{R}^{10}$,*

$$[\overline{\Sigma}^3(f)]^3 = [\overline{\Sigma}^3(\widetilde{f})] \in H_1(X; \mathbb{Z}/2).$$

Theorem 1, 2 will be proved in Section 4.

4 Proofs

In this section we prove Theorem 1, 2. Then we give an equation with $\mathbb{Z}$ coefficient.

4.1 An equation for special f and $\widetilde{f}$

Let $f : X \to \mathbb{R}^p$ be a fold map and let $\widetilde{f} : X \to \mathbb{R}^{p+k-1}$ be a generic map that is a lift of f, namely

$$\pi_{\mathbb{R}^k} \circ \widetilde{f} = f,$$

where $\pi_{\mathbb{R}^k} : \mathbb{R}^{p+k-1} \to \mathbb{R}^p$ is the projection defined as

$$\pi_{\mathbb{R}^k}(x_1, \dots, x_{p+k-1}) = (x_1, \dots, x_p).$$

For $i = 1, \dots, p+k-1$, we denote by

$$\pi_i : \mathbb{R}^{p+k-1} \to \mathbb{R}$$

the i-th projection defined as

$$\pi_i(x_1, \dots, x_{p+k-1}) = x_i.$$

Let $f_1, \dots, f_p : X \to \mathbb{R}$ and $h_1, \dots, h_{k-1} : X \to \mathbb{R}$ be smooth functions given by

(1) $f_i = \pi_i \circ \widetilde{f}(= \pi_i \circ f)$, for $i = 1, \dots, p$,

(2) $h_i = \pi_{p+i} \circ \widetilde{f} = h_i$, for $i = 1, \dots, k-1$,

Namely,

$$\widetilde{f} = (f_1, \dots, f_p, h_1, \dots, h_{k-1}) : X \to \mathbb{R}^{p+k-1},$$
$$f = (f_1, \dots, f_p) : X \to \mathbb{R}^p.$$

We first prove an equation for the above f and $\widetilde{f}$.

Proposition 4.1. *For the above f and $\widetilde{f}$, we have*

$$[\overline{\Sigma}^{n-p+1}(f)]^k = [\overline{\Sigma}^{n-p+1}(\widetilde{f})] \in H_{n-k(n-p+1)}(X; \mathbb{Z}/2).$$

Proof. Take a Riemannian metric on X. For a smooth function $\varphi : X \to \mathbb{R}$, we denote by $\{\partial_x \varphi \in T_x X\}_{x \in X}$ the gradient vector field of φ with respect to the chosen metric.

Since f is a fold map,

$$\overline{\Sigma}^{n-p+1}(f) = \Sigma^{n-p+1}(f)$$

and, for any $x \in \Sigma^{n-p+1}(f)$, the vectors $\partial_x f_1, \ldots, \partial_x f_p \in T_x X$ span a $(p-1)$–dimensional linear subspace of $T_x X$. We denote by

$$J_x(f) = \langle \partial_x f_1, \ldots, \partial_x f_p \rangle \subset T_x X$$

this linear subspace. The projection $J(f) = \sqcup_x J_x X \to \Sigma^{n-p+1}(f)$ is a $\mathbb{R}^{p-1}$–bundle over $\Sigma^{n-p+1}(f)$. On the other hand, $T\Sigma^{n-p+1}(f)$ is also a $\mathbb{R}^{p-1}$-bundle over $\Sigma^{n-p+1}(f)$. The following lemma says that $J(f)$ coincides with $T\Sigma^{n-p+1}(f)$ for an appropriate metric.

Lemma 4.2. *There exists a Riemannian metric on X such that*

$$J_x(f) = T_x \Sigma^{n-p+1}(f)$$

for any $x \in \Sigma^{n-p+1}(f)$.

Proof. Let $x \in \Sigma^{n-p+1}(f)$. Let $U_x \subset X$ be a neighborhood of x such that $f|_{U_x}$ is locally written as follows:

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, \pm x_{k+1}^2 \pm \ldots \pm x_n^2).$$

Here $x_1, \ldots, x_n$ is a local coordinate of U_x with $x = 0$. Let g^{U_x} be a Riemanian metric on U_x given by pulling back the standard metric of $\mathbb{R}^n$ via the local coordinate system. Under the metric g^{U_x},

$$\Sigma^{n-p+1}(f) = \{(x_1, \ldots, x_{p-1}, 0, \ldots, 0)\} \subset U_x,$$

$$T_y \Sigma^{n-p+1}(f) = \{(v_1, \ldots, v_{p-1}, 0, \ldots, 0) \in \mathbb{R}^n\} \subset \mathbb{R}^n = T_y U_x,$$

$$J_y(f) = \langle e_1, \ldots, e_{p-1} \rangle \subset \mathbb{R}^n.$$

Here $e_1 \ldots, e_n \in \mathbb{R}^n$ is a standard basis of $\mathbb{R}^n$. Therefore, $J_y(f) = T_y S(f)$ holds for any $y \in U_x$. Finally, by using a partition of the unity, we obtain a required metric on X. $\square$

We take and fix a metric on X so that $T\Sigma^{n-p+1}(f) = J(f)$ given in Lemma 4.2. By the definition of $\Sigma^{n-p+1}(\widetilde{f})$, for any $z \in \overline{\Sigma}^{n-p+1}(\widetilde{f})$,

$$\dim \langle \partial_z f_1, \ldots, \partial_z f_p, \partial_z h_1, \ldots, \partial_z h_{k-1} \rangle \leq p - 1.$$

Since $\langle \partial_z f_1, \ldots, \partial_z f_p \rangle \subset \langle \partial_z f_1, \ldots, \partial_z f_p, \partial_z h_1, \ldots, \partial_z h_{k-1} \rangle$,

$$(p - 1 \leq) \dim \langle \partial_z f_1, \ldots, \partial_z f_p \rangle \leq \dim \langle \partial_z f_1, \ldots, \partial_z f_p, \partial_z h_1, \ldots, \partial_z h_{k-1} \rangle.$$

Thus for any $z \in \Sigma^{n-p+1}(\widetilde{f})$,

$$\langle \partial_z f_1, \ldots, \partial_z f_p \rangle = \langle \partial_z f_1, \ldots, \partial_z f_p, \partial_z h_1, \ldots, \partial_z h_{k-1} \rangle.$$

Therefore $x \in \Sigma^{n-p+1}(f)$ belongs to $\Sigma^{n-p+1}(\widetilde{f})$ if and only if

$$\partial_x h_1, \ldots, \partial_x h_{k-1} \in J_x(f) = T_x \Sigma^{n-p+1}(f).$$

We denote by $T_x\Sigma^{n-p+1}(f)^{\perp} \subset T_x X$ the orthogonal complement of $T_x\Sigma^{n-p+1}(f)$ with respect to the metric. Let

$$v_x^i = \pi_{\Sigma^{n-p+1}(f)^{\perp}}(\partial_x h_i) \in T_x\Sigma^{n-p+1}(f)^{\perp}$$

for $i = 1, \ldots, k-1$. Here $\pi_{\Sigma^{n-p+1}(f)^{\perp}} : T_x X \to T_x\Sigma^{n-p+1}(f)^{\perp}$ is the orthogonal projection. Since $\partial_x h_i \in T_x\Sigma^{n-p+1}(f)$ if and only if $v_x^i = 0$, we have

$$\Sigma^{n-p+1}(\widetilde{f}) = \{x \in \Sigma^{n-p+1}(f) | v_x^1 = \ldots = v_x^{k-1} = 0\}.$$

Each v_i is a vector field on $\Sigma^{n-p+1}(f)$ perpendicular to $T\Sigma^{n-p+1}(f)$. So we can push $\Sigma^{n-p+1}(f)$ by using each v_i:

$$\Sigma^{n-p+1}(f)_{v_i}.$$

Therefore, for generic $h_1, \ldots, h_{k-1}$,

$$\begin{aligned}
[\Sigma^{n-p+1}(f)]^k &= [\Sigma^{n-p+1}(f) \cap \Sigma^{n-p+1}(f)_{v_1} \cap \ldots \cap \Sigma^{n-p+1}(f)_{v_{k-1}}] \\
&= [\{x \in \Sigma^{n-p+1}(f) | v_x^1 = \ldots = v_x^{k-1} = 0\}] \\
&= [\Sigma^{n-p+1}(\widetilde{f})].
\end{aligned}$$

This completes the proof. □

4.2 Proof of Theorem 1 and 2

It is known that the Poincaré dual of the homology classes $[\overline{\Sigma}^{n-p+1}(f)], [\overline{\Sigma}^{n-p+1}(\widetilde{f})]$ are independent under cobordism of a maps (See [5],[7] for the details). Since $\mathbb{R}^p$ and $\mathbb{R}^{p+k-1}$ are contractible, a map $f : X \to \mathbb{R}^p$ cobordant to a map $f' : X' \to \mathbb{R}^p$ if and only if X cobordant to X'. Therefore, the equations in Theorem 1, 2 are reduced to Proposition 4.1.

4.3 An equation with integer coefficients

Let X be a closed oriented n–dimensional manifold. Let k be a positive even number. Let $f : X \to \mathbb{R}^p$ be a fold map and $\widetilde{f} : X \to \mathbb{R}^{p+k-1}$ be a generic map that is a lift of f:

$$\pi_{\mathbb{R}^k} \circ \widetilde{f} = f.$$

Since k is even, k-th self-intersection of $\overline{\Sigma}^{n-p+1}(f)$ is oriented (See Remark 2.3). We can apply the proof of Proposition 4.1 with orientations. Therefore we get the following Proposition (an equation with integer coefficients):

Proposition 4.3.

$$[\overline{\Sigma}^{n-p+1}(f : X \to \mathbb{R}^p)]^k_{P.D.} = [\overline{\Sigma}^{n-p+1}(\widetilde{f} : X \to \mathbb{R}^{p+k-1})]_{P.D.} \in H^{k(n-p+1)}(X; \mathbb{Z}).$$

Here the orientation of $[\overline{\Sigma}^{n-p+1}(\widetilde{f})]$ is given as the k-the self-intersection of $[\overline{\Sigma}^{n-p+1}(f)]$.

Example 4.4. *(1) Let X be a closed oriented 4-dimensional manifold. For any fold map $f : X \to \mathbb{R}^3$ and its generic lift $\widetilde{f} : X \to \mathbb{R}^4$,*

$$[\overline{\Sigma}^2(f)]^2 = [\overline{\Sigma}^2(\widetilde{f})] \in H_0(X; \mathbb{Z}).$$

(2) Let X be a closed 12–dimensional manifold. Then, for any fold map $f : X \to \mathbb{R}^{11}$ and its generic lift $\widetilde{f} : X \to \mathbb{R}^{14}$,

$$[\overline{\Sigma}^2(f)]^4 = [\overline{\Sigma}^2(\widetilde{f})] \in H_4(X; \mathbb{Z}).$$

5 Applications

5.1 An obstruction to admitting a fold map up to cobordism

In this subsection, we define an obstruction to admitting a fold map up to cobordism, by using Theorem1. Let X be a closed n–dimensional manifold. Let $\varepsilon : H_0(X; \mathbb{Z}/2) \to \mathbb{Z}/2$ be the homomorphism defined to be

$$\varepsilon([*]) = 1$$

for any point $* \in X$. Let p, k be positive integers satisfying $k(n - p + 1) = n$. Let $f : X \to \mathbb{R}^p$ and $\widetilde{f} : X \to \mathbb{R}^{p+k-1}$ be generic maps. The number

$$\varepsilon([\overline{\Sigma}^{n-p+1}(f)]^k - [\overline{\Sigma}^{n-p+1}(\widetilde{f})]) \in \mathbb{Z}/2$$

is independent from f and $\widetilde{f}$. We denote

$$d_{n,p,k}(X) = \varepsilon([\overline{\Sigma}^{n-p+1}(f)]^k - [\overline{\Sigma}^{n-p+1}(\widetilde{f})]).$$

d_n is a cobordism invariant, thus we have a homomorphism

$$d_n : \Omega^n \to \mathbb{Z}/2.$$

Here Ω^n is the cobordism group of n-dimensional manifolds. Theorem 1 implies that d_n gives an obstruction to admitting a fold map into $\mathbb{R}^p$ up to cobordism:

Proposition 5.1. *If X admit a fold map into $\mathbb{R}^p$ up to cobordism, then*

$$d_{n,p,k}(X) = 0.$$

Example 5.2. *If a closed 8–manifold X admits a fold map into $\mathbb{R}^7$ up to cobordism, then $d_{8,7,4} = 0$.*

Remark 5.3. *We can also define*

$$\overline{d}_{n,p,k} = [\overline{\Sigma}^{n-p+1}(f)]^k - [\overline{\Sigma}^{n-p+1}(\widetilde{f})] \in H_{n-k(n-p+1)}(X;\mathbb{Z}/2)$$

for other n, p, k may not satisfying $n = k(n-p+1)$. $\overline{d}_{n,p,k}$ is an obstruction to admitting a fold map (not up to cobordism).

5.2 Some formulas about Stiefel–Whitney classes

Ronga in [7] gave the following formulas:

(Ronga1) $[\overline{\Sigma}^{n-p+1}(f : X^n \to \mathbb{R}^p)]_{P.D.} = w_{n-p+1}(X)$,

(Ronga2) $[\overline{\Sigma}^{n-p+1}(\widetilde{f} : X^n \to \mathbb{R}^{p+k-1})]_{P.D.} = \det \begin{pmatrix} w_{n-p+1}(X) & \cdots & w_{n-p-k+2}(X) \\ \vdots & & \vdots \\ w_{n-p+k}(X) & \cdots & w_{n-p+1}(X) \end{pmatrix}$.

Here $w_i(X) \in H^i(X;\mathbb{Z}/2)$ is the i-th Stiefel–Whitney class of X. Thank to Ronga's formula, we have

$$d_{n,p,k}(X) = w_{n+p-1}(X)^k - \det \begin{pmatrix} w_{n-p+1}(X) & \cdots & w_{n-p-k+2}(X) \\ \vdots & & \vdots \\ w_{n-p+k}(X) & \cdots & w_{n-p+1}(X) \end{pmatrix}.$$

Example 5.4. *In this example, w_i means $w_i(X)$.*

(1) Let X be a closed 8–manifold. $d_{8,7,4}(X) = w_2^4 - \det \begin{pmatrix} w_2 & w_1 & 0 & 0 \\ w_3 & w_2 & w_1 & 0 \\ w_4 & w_3 & w_2 & w_1 \\ w_5 & w_4 & w_3 & w_2 \end{pmatrix} =$ $w_1(w_2^2 w_3 + w_2 w_4 + w_1 w_5)$. *If X admit a fold map into $\mathbb{R}^7$ up to cobordism, then $w_1(w_2^2 w_3 + w_2 w_4 + w_1 w_5) = 0$.*

(2) Let X be a 16–dimensional closed manifold.

$$\overline{d}_{16,12,3}(X) = w_5^3 - \det \begin{pmatrix} w_5 & w_4 & w_3 \\ w_6 & w_5 & w_4 \\ w_7 & w_6 & w_5 \end{pmatrix} = w_5^3 + w_4^2 w_7 + w_4 \det \begin{pmatrix} w_6 & w_5 \\ w_7 & w_6 \end{pmatrix}.$$

If X admit a fold map f into $\mathbb{R}^{12}$, $\overline{\Sigma}^6(f) = \emptyset$. Thus $[\overline{\Sigma}^6(f)]_{P.D.} =$ $\det \begin{pmatrix} w_6 & w_5 \\ w_7 & w_6 \end{pmatrix} = 0$. *Therefore if X admits a fold map into $\mathbb{R}^{12}$ then*

$$w_5^3 + w_4^2 w_7 = 0.$$

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