RIMS-1899

Combinatorial Belyi Cuspidalization and Arithmetic Subquotients of the Grothendieck-Teichmüller Group

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March 2019

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March 26, 2019

Abstract

In this paper, we develop a certain combinatorial version of the theory of Belyi cuspidalization developed in [AbsTopII]. We also give applications of these techniques to certain natural closed subgroups of the Grothendieck-Teichmüller group associated to the field of p-adic numbers and the maximal abelian extension of the field of rational numbers.

Contents

Introduction

In [AbsTopII], §3 [cf. [AbsTopII], Corollary 3.7], the theory of Belyi cuspidalization was developed and applied to reconstruct the decomposition groups of the closed points of a hyperbolic orbicurve of strictly Belyi type over a mixed characteristic local field [cf. [AbsTopII], Definition 3.5; [AbsTopII], Remark 3.7.2].

In the present paper, we develop a certain combinatorial version of the theory of Belyi cuspidalization developed in [AbsTopII], §3. To begin, let us recall the Grothendieck-Teichmüller group GT, which may be regarded as a closed subgroup of the outer automorphism group of the étale fundamental group Π_X [cf. Notations and Conventions] of $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1], where $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ denotes the projective line over the field of algebraic numbers $\overline{\mathbb{Q}}$ [cf. Notations and Conventions], minus the three points "0", "1", "∞". Recall, further, that the natural outer action of $G_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Π_X determines natural inclusions

$$G_{\mathbb{Q}} \ \subseteq \ \mathrm{GT} \ \subseteq \ \mathrm{Out}(\Pi_X),$$

and that Π_X is topologically finitely generated and slim [cf., e.g., [MT], Remark 1.2.2; [MT], Proposition 1.4]. By pulling-back the exact sequence of profinite groups

$$1 \longrightarrow \Pi_X \ \ (\stackrel{\sim}{\to} \mathrm{Inn}(\Pi_X)) \longrightarrow \mathrm{Aut}(\Pi_X) \longrightarrow \mathrm{Out}(\Pi_X) \longrightarrow 1$$

via the natural inclusion $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$, we obtain an exact sequence of profinite groups

$$1 \longrightarrow \Pi_X \longrightarrow \Pi_X \stackrel{\mathrm{out}}{\rtimes} \mathrm{GT} \longrightarrow \mathrm{GT} \longrightarrow 1$$

[cf. Notations and Conventions].

We shall develop a combinatorial version for $\Pi_X \stackrel{\mathrm{out}}{\rtimes} \mathrm{GT}$ — i.e., which we regard as a sort of group-theoretic version of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$, where "$\mathbb{Q}$" is replaced by "GT"— of the theory of Belyi cuspidalization. We shall refer to this combinatorial version of the theory of Belyi cuspidalization as the theory of *combinatorial Belyi cuspidalization*. We construct combinatorial Belyi cuspidalizations and, in particular, the "GT analogue" of the set (equipped with a natural action of GT) of decomposition groups of $\Pi_X \stackrel{\mathrm{out}}{\rtimes} \mathrm{GT}$, by applying the technique of *tripod synchronization* developed in [CbTpII], together with the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [Tama1], Theorem 0.4; [LocAn], Theorem A].

Let $U \to X$ be a connected finite étale covering of X, $U \hookrightarrow X$ an open immersion. Then the morphisms $U \to X$, $U \hookrightarrow X$ determine, respectively, the vertical and horizontal arrows in a diagram of outer homomorphisms of profinite groups as follows:

$$\begin{array}{ccc} \Pi_U & \longrightarrow & \Pi_X \\ \big\downarrow & & \\ \Pi_X. & & \end{array}$$

We shall refer to any pair consisting of

- a diagram obtained in this way;
- an open subgroup of Π_X, which, by a slight of abuse of notation, we denote by $\Pi_U \subseteq \Pi_X$, that belongs to the Π_X-conjugacy class of open subgroups that arises as the image of the vertical arrow of the diagram

as a *Belyi diagram*.

Let $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ be a pair consisting of

- an abstract topological group Π;
- a closed subgroup G of $\mathrm{Out}(\Pi)$.

If there exists an isomorphism of such pairs

$$(\Pi, G \subseteq \mathrm{Out}(\Pi)) \xrightarrow{\sim} (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X))$$

[i.e., if there exist isomorphisms $\Pi \xrightarrow{\sim} \Pi_X$ and $G \xrightarrow{\sim} \mathrm{GT}$ of topological groups compatible with the inclusions $G \subseteq \mathrm{Out}(\Pi)$ and $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$], then we shall refer to the pair $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ as a *tripodal pair*.

Let $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ be a tripodal pair; $J \subseteq G$ a closed subgroup of G; Π^* an open subgroup of Π. Then one verifies easily [cf. Lemma 1.2] that, for any sufficiently small normal open subgroup $M \subseteq J$, there exist an outer action of M on Π^* and an open injection $\Pi^* \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} J$ such that

(a) the outer action of M preserves and induces the identity automorphism on the set of the conjugacy classes of cuspidal inertia subgroups of Π^* [cf. Theorem A, (i)];

(b) the injection $\Pi^* \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi \overset{\mathrm{out}}{\rtimes} J$ is compatible with the inclusions between respective subgroups $\Pi^* \subseteq \Pi$ and quotients $M \subseteq J$.

Then our first main result is the following [cf. Theorem 1.3]:

Theorem A (Combinatorial Belyi cuspidalization for a tripod). *Fix a Belyi diagram*

$$\begin{array}{ccc} \Pi_U & \longrightarrow & \Pi_X \\ \big\downarrow & & \\ \Pi_X & & \end{array}$$

that arises from a connected finite étale covering $U \to X$ and an open immersion $U \hookrightarrow X$ [as in the above discussion]. Then:

(i) Let $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ *be a* **tripodal pair**. *Fix an isomorphism of pairs* $\alpha : (\Pi, G \subseteq \mathrm{Out}(\Pi)) \xrightarrow{\sim} (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X))$. *Then the set of subgroups of* Π *determined, via* α*, by the cuspidal inertia subgroups of* Π_X*, may be* **reconstructed**, *in a* **purely group-theoretic** *way, from the pair* $(\Pi, G \subseteq \mathrm{Out}(\Pi))$. *We shall refer to the subgroups of* Π *constructed in this way as the* **cuspidal inertia subgroups** *of* Π. *In particular, for each open subgroup* $\Pi^* \subseteq \Pi$ *of* Π*, the pair* $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ *determines a set* $I(\Pi^*)$ *(respectively,* $\mathrm{Cusp}(\Pi^*)$*) of cuspidal inertia subgroups of* Π^* *(respectively, cusps of* Π^**), namely, the set of intersections of* Π^* *with cuspidal inertia subgroups of* Π *(respectively, the conjugacy classes of cuspidal inertia subgroups of* Π^**).*

(ii) Let $N \subseteq \mathrm{GT}$ *be a normal open subgroup. Suppose that we are given an outer action of* N *on* Π_U *and an open injection* $\Pi_U \overset{\mathrm{out}}{\rtimes} N \hookrightarrow \Pi_X \overset{\mathrm{out}}{\rtimes} \mathrm{GT}$ *such that the above conditions (a), (b) in the case of "*$\Pi^* \subseteq \Pi$*", "*$M \subseteq J$*" hold for* $\Pi_U \subseteq \Pi_X$*,* $N \subseteq \mathrm{GT}$. *Then the original* **outer action** *of* $N \subseteq \mathrm{GT}$ *on* Π_X **coincides** *with the outer action of* N *on* Π_X *induced [cf. condition (a)] by the outer action of* N *on* Π_U *and the outer surjection* $\Pi_U \twoheadrightarrow \Pi_X$ *[i.e., the horizontal arrow in the above Belyi diagram].*

(iii) Let

$$C(\Pi) = (\Pi, G \subseteq \mathrm{Out}(\Pi), \Pi^*, \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi), \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi^*))$$

be a 5*-tuple consisting of the following data:*

- *a topological group* Π*;*
- *a closed subgroup* $G \subseteq \mathrm{Out}(\Pi)$ *such that the pair* $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ *is a tripodal pair;*
- *an open subgroup* $\Pi^* \subseteq \Pi$ *of* Π *of genus* 0*, where we observe that the genus of an open subgroup of* Π *may be defined by using the cuspidal inertia subgroups of the open subgroup [cf. (i)];*
- *a subset* $\{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi)$ *[cf. (i)] of cardinality* 3 *[equipped with labels "*0*", "*1*", "*∞*"] of the set* $\mathrm{Cusp}(\Pi)$*;*
- *a subset* $\{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi^*)$ *[cf. (i)] of cardinality* 3 *[equipped with labels "*0*", "*1*", "*∞*"] of the set* $\mathrm{Cusp}(\Pi^*)$.

Suppose that the collection of data $C(\Pi)$ *is isomorphic to the collection of data*

$$\begin{aligned} C(\Pi_X) &= (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X), \Pi_U, \\ &\qquad \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi_X), \{0, 1, \infty\} \subseteq \mathrm{Cusp}(\Pi_U)) \end{aligned}$$

determined, in a natural way, by the given Belyi diagram. [Here, we observe that the horizontal arrow in the given Belyi diagram determines, in *a natural way, data* $\{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi_U)$*.] Fix an isomorphism of collections of data* $C(\Pi) \overset{\sim}{\to} C(\Pi_X)$*. Thus, the outer surjection* $\Pi_U \twoheadrightarrow \Pi_X$ *[i.e., the horizontal arrow in the given Belyi diagram], together with the isomorphism* $C(\Pi) \overset{\sim}{\to} C(\Pi_X)$*, determine an outer surjection* $\Pi^* \twoheadrightarrow \Pi$*. Let* $N \subseteq G$ *be a normal open subgroup such that the conditions (a), (b) considered above in the case of "*$M \subseteq J$*" hold for* $N \subseteq G$*. Then the* **outer surjection** $\Pi^* \twoheadrightarrow \Pi$ *may be* **reconstructed***, in a* **purely group-theoretic** *way, from the collection of data* $C(\Pi)$ *as the outer surjection induced by the unique* Π*-outer surjection* $\Pi^* \overset{\text{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\text{out}}{\rtimes} N$ *[i.e., surjection well-defined up to composition with inner automorphisms arising from elements of* Π*] that lies over the identity morphism of* N *such that*

- *the kernel of this* Π*-outer surjection* $\Pi^* \overset{\text{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\text{out}}{\rtimes} N$ *is topologically generated by the cuspidal inertia subgroups of* Π^* *which are not associated to* $0, 1, \infty \in \mathrm{Cusp}(\Pi^*)$*;*
- *the conjugacy class of cuspidal inertia subgroups of* Π^* *associated to* 0 *(respectively,* 1*,* ∞*)* $\in \mathrm{Cusp}(\Pi^*)$ *maps to the conjugacy class of cuspidal inertia subgroups of* Π *associated to* 0 *(respectively,* 1*,* ∞*)* $\in \mathrm{Cusp}(\Pi)$.

Next, let us consider the situation discussed in Theorem A, (ii). Let J be a closed subgroup of GT. Thus, for each normal open subgroup M of J such that $M \subseteq N \cap J$, we have a diagram

$$\begin{array}{ccc} \Pi_U \overset{\text{out}}{\rtimes} M & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} M \\ \big\downarrow & & \\ \Pi_X \overset{\text{out}}{\rtimes} M & & \end{array}$$

of Π_X*-outer homomorphisms* [i.e., homomorphisms well-defined up to composition with inner automorphisms arising from elements of Π_X] of profinite groups. We shall refer to a diagram obtained in this way as an *arithmetic Belyi diagram.*

Fix an arithmetic Belyi diagram $\mathbb{B}^{\rtimes}$ as above. Write

$$\mathbb{D}(\mathbb{B}^{\rtimes}, M, J)$$

for the set of the images via the natural composite Π_X-outer homomorphism $\Pi_U \overset{\text{out}}{\rtimes} M \twoheadrightarrow \Pi_X \overset{\text{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\text{out}}{\rtimes} J$ of the normalizers in $\Pi_U \overset{\text{out}}{\rtimes} M$ of cuspidal inertia subgroups of Π_U;

$$\mathbb{D}(\mathbb{B}^{\rtimes}, J)$$

for the quotient set $\left(\sqcup_{M \subseteq J} \mathbb{D}(\mathbb{B}^{\rtimes}, M, J)\right)/\sim$, where M ranges over all sufficiently small normal open subgroups of J, and we write $\mathbb{D}(\mathbb{B}^{\rtimes}, M, J) \ni G_M \sim G_{M^\dagger} \in \mathbb{D}(\mathbb{B}^{\rtimes}, M^\dagger, J)$ if $G_M \cap G_{M^\dagger}$ is open in both G_M and $G_{M^\dagger}$.

Write

$$\mathbb{D}(J)$$

for the quotient set $\left(\sqcup_{\mathbb{B}^{\rtimes}} \mathbb{D}(\mathbb{B}^{\rtimes}, J)\right)/\sim$, where $\mathbb{B}^{\rtimes}$ ranges over all arithmetic Belyi diagrams, and we write $\mathbb{D}({}^{\dagger}\mathbb{B}^{\rtimes}, J) \ni G_{{}^{\dagger}\mathbb{B}^{\rtimes}} \sim G_{{}^{\ddagger}\mathbb{B}^{\rtimes}} \in \mathbb{D}({}^{\ddagger}\mathbb{B}^{\rtimes}, J)$ if $G_{M^{\dagger}} \cap G_{M^{\ddagger}}$ is open in both $G_{M^{\dagger}}$ and $G_{M^{\ddagger}}$ for some representative $G_{M^{\dagger}}$ (respectively, $G_{M^{\ddagger}}$) of $G_{{}^{\dagger}\mathbb{B}^{\rtimes}}$ (respectively, $G_{{}^{\ddagger}\mathbb{B}^{\rtimes}}$). We shall refer to $\mathbb{D}(J)$ as the set of *decomposition subgroup-germs of* $\Pi_X \overset{\text{out}}{\rtimes} J$. One verifies immediately that the natural conjugation action of $\Pi_X \overset{\text{out}}{\rtimes} J$ on itself induces a natural action of $\Pi_X \overset{\text{out}}{\rtimes} J$ on $\mathbb{D}(J)$ [cf. Corollary 1.6].

Write

$$D(J)$$

for the quotient set $\mathbb{D}(J)/\Pi_X$. Thus, $D(J)$ admits a natural action by J. Here, we recall that, by the ["usual"] theory of Belyi cuspidalization developed in [AbsTopII], §3, we have a *natural bijection*

$$D(G_{\mathbb{Q}}) \overset{\sim}{\leftarrow} \overline{\mathbb{Q}}$$

[cf. Corollary 1.7].

Next, let J_1 and J_2 be closed subgroups of GT. If $J_1 \subseteq J_2 \subseteq \mathrm{GT}$, then one verifies immediately from the definition of $D(J)$ that the inclusion $J_1 \subseteq J_2$ induces, by considering the intersection of subgroups of $\Pi_X \overset{\text{out}}{\rtimes} J_2$ with $\Pi_X \overset{\text{out}}{\rtimes} J_1$, a natural surjection $D(J_2) \twoheadrightarrow D(J_1)$ that is equivariant with respect to the natural actions of J_1 ($\subseteq J_2$) on the domain and codomain [cf. Corollary 1.6]. Thus, we obtain the following commutative diagram

$$\begin{array}{ccccc} \mathrm{GT} & \supseteq & G_{\mathbb{Q}} & & \\ \curvearrowright & & \curvearrowright & & \\ D(\mathrm{GT}) & \twoheadrightarrow & D(G_{\mathbb{Q}}) & \overset{\sim}{\leftarrow} & \overline{\mathbb{Q}} \end{array}$$

[cf. Corollary 1.7]. In particular,

> if one could prove that the surjection $D(\mathrm{GT}) \twoheadrightarrow D(G_{\mathbb{Q}})$ is a *bijection*, then it would follow that GT *naturally acts on the set* $\overline{\mathbb{Q}}$.

In fact, at the time of writing of the present paper, the author does not know

> *whether or not the surjection* $D(\mathrm{GT}) \twoheadrightarrow D(G_{\mathbb{Q}})$ *is a bijection*,

or indeed, more generally,

> *whether or not* GT *admits a natural action on the set* $\overline{\mathbb{Q}}$.

On the other hand, we obtain the following result concerning the p-adic analogue of this sort of issue [cf. Corollary 2.4]:

Corollary B (**Natural surjection from $\mathrm{GT}_p^{\mathrm{tp}}$ to $G_{\mathbb{Q}_p}$**)**.** *Let p be a prime number; $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ [cf. Notations and Conventions]. Write $\mathrm{GT}_p^{\mathrm{tp}}$ for the p-adic version of the Grothendieck-Teichmüller group defined in Definition 2.1 [cf. also Remark 2.1.2]. Then there exists a surjection $\mathrm{GT}_p^{\mathrm{tp}} \twoheadrightarrow G_{\mathbb{Q}_p} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ whose restriction to $G_{\mathbb{Q}_p}$ is the identity automorphism.*

The key point of the proof of the above corollary is the following theorem [cf. Theorem 2.2]:

Theorem C (**Determination of moduli of certain types of p-adic hyperbolic curves by data arising from geometric tempered fundamental groups**)**.** *We maintain the notation of Corollary B. Write $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, 1, \infty\}$, where $\mathbb{C}_p$ denotes the p-adic completion of $\overline{\mathbb{Q}}_p$. Let $Y \to X$ be a connected finite étale covering of X; y, y' elements of $Y(\mathbb{C}_p)$. Write Y_y (respectively, $Y_{y'}$) for $Y \setminus \{y\}$ (respectively, $Y \setminus \{y'\}$); Π_Y^{tp} (respectively, $\Pi_{Y_y}^{\mathrm{tp}}$, $\Pi_{Y_{y'}}^{\mathrm{tp}}$) for the tempered fundamental group of Y (respectively, Y_y, $Y_{y'}$). Suppose that there exists an isomorphism $\Pi_{Y_y}^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{Y_{y'}}^{\mathrm{tp}}$ that fits into a commutative diagram*

$$\begin{array}{ccc} \Pi_{Y_y}^{\mathrm{tp}} & \xrightarrow{\sim} & \Pi_{Y_{y'}}^{\mathrm{tp}} \\ \downarrow & & \downarrow \\ \Pi_Y^{\mathrm{tp}} & = & \Pi_Y^{\mathrm{tp}}, \end{array}$$

where the vertical arrows are the surjections [determined up to composition with an inner automorphism] induced by the natural open immersions of hyperbolic curves. Then $y = y'$.

Finally, we consider yet another interesting closed subgroup of GT which acts on the set of algebraic numbers $\overline{\mathbb{Q}}$. Write $\mathbb{Q}^{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ for the maximal abelian extension of $\mathbb{Q}$. Since $G_{\mathbb{Q}^{\mathrm{ab}}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{\mathrm{ab}})$ is a normal subgroup of $G_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the commensurator $C_{\mathrm{GT}}(G_{\mathbb{Q}^{\mathrm{ab}}})$ of $G_{\mathbb{Q}^{\mathrm{ab}}}$ in GT [cf. Notations and Conventions] contains $G_{\mathbb{Q}}$ as a closed subgroup. As an application of the theory of combinatorial Belyi cuspidalization developed in §1, we also obtain the following [cf. Corollary 3.4]:

Corollary D (**Natural surjection from the commensurator of the absolute Galois group of the maximal abelian extension of $\mathbb{Q}$ to $G_{\mathbb{Q}}$**)**.** *There exists a surjection $C_{\mathrm{GT}}(G_{\mathbb{Q}^{\mathrm{ab}}}) \twoheadrightarrow G_{\mathbb{Q}}$ whose restriction to $G_{\mathbb{Q}}$ is the identity automorphism.*

The key point of the proof of the above corollary is the injectivity portion of the section conjecture for hyperbolic curves over maximal cyclotomic extensions of number fields [cf. Corollary 3.2].

This paper is organized as follows. In §1, we develop the theory of combinatorial Belyi cuspidalization. In §2, we first show that the moduli of a hyperbolic curve over $\overline{\mathbb{Q}}_p$ of genus 0 with 4 points removed are completely determined by the geometric tempered fundamental group of the curve, regarded as an extension of the geometric tempered fundamental group of the tripod [cf. Notations and Conventions] over $\overline{\mathbb{Q}}_p$ [cf. Theorem C]. This result, together with the theory of combinatorial Belyi cuspidalization developed in §1, implies that there exists a surjection $\mathrm{GT}_p^{\mathrm{tp}} \twoheadrightarrow G_{\mathbb{Q}_p}$ whose restriction to $G_{\mathbb{Q}_p}$ is the identity automorphism [cf. Corollary B]. In §3, we observe that the injectivity portion of the section conjecture for hyperbolic curves over maximal cyclotomic extensions of number fields holds [by a well-known argument!] and prove that there exists a surjection $C_{\mathrm{GT}}(G_{\mathbb{Q}^{\mathrm{ab}}}) \twoheadrightarrow G_{\mathbb{Q}}$ whose restriction to $G_{\mathbb{Q}}$ is the identity automorphism.

Notations and Conventions

In this paper, we follow the notations and conventions of [CbTpI].

Numbers: The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. The notation $\mathbb{C}$ will be used to denote the field of complex numbers. The notation $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ will be used to denote the set or field of algebraic numbers $\in \mathbb{C}$. We shall refer to a finite extension field of $\mathbb{Q}$ as a *number field*. If p is a prime number, then the notation $\mathbb{Q}_p$ will be used to denote the p-adic completion of $\mathbb{Q}$.

Topological groups: Let G be a topological group and $H \subseteq G$ a closed subgroup of G. Then we shall denote by $Z_G(H)$ (respectively, $N_G(H)$, $C_G(H)$) the *centralizer* (respectively, *normalizer*, *commensurator*) of $H \subseteq G$, i.e.,

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}$$
$$\text{(respectively, } N_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$
$$C_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid H \cap g \cdot H \cdot g^{-1} \text{ is of finite index in } H \text{ and } g \cdot H \cdot g^{-1}\}).$$

We shall say that G is *slim* if $Z_G(U) = \{1\}$ for any open subgroup U of G.

Let G be a topological group. Then we shall write $\mathrm{Aut}(G)$ for the group of automorphisms of the topological group G, $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G, and $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$. We shall refer to an element of $\mathrm{Out}(G)$ as an *outomorphism* of G. Now suppose that G is *center-free* [i.e., $Z_G(G) = \{1\}$]. Then we have a natural exact sequence of groups

$$1 \longrightarrow G \ \ (\stackrel{\sim}{\to} \mathrm{Inn}(G)) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1.$$

If J is a group, and $\rho : J \to \mathrm{Out}(G)$ is a homomorphism, then we shall denote by

$$G \stackrel{\mathrm{out}}{\rtimes} J$$

the group obtained by pulling back the above exact sequence of groups via ρ. Thus, we have a *natural exact sequence* of groups

$$1 \longrightarrow G \longrightarrow G \overset{\text{out}}{\rtimes} J \longrightarrow J \longrightarrow 1.$$

Suppose further that G is *profinite* and *topologically finitely generated*. Then one verifies immediately that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ with respect to which the above exact sequence relating $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ determines an exact sequence of *profinite groups*. In particular, one verifies easily that if, moreover, J is *profinite*, and $\rho : J \to \mathrm{Out}(G)$ is *continuous*, then the above exact sequence involving $G \overset{\text{out}}{\rtimes} J$ determines an exact sequence of *profinite groups*.

Curves: A smooth hyperbolic curve of genus 0 over a field k with precisely 3 cusps [i.e., points at infinity], all of which are defined over k, will be referred to as a "*tripod*".

Fundamental groups: For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint.

1 Combinatorial Belyi cuspidalization

In this section, we develop the theory of combinatorial Belyi cuspidalization. First, we introduce the notion of a Belyi diagram as follows.

Definition 1.1.

(i) Write X for $\mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$, where $\mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$ denotes the projective line over the field of algebraic numbers $\overline{\mathbb{Q}}$ [cf. Notations and Conventions], minus the three points "0", "1", "∞". Let $U \to X$ be a connected finite étale covering of X, $U \hookrightarrow X$ an open immersion. Then the morphisms $U \to X$, $U \hookrightarrow X$ determine, respectively, the vertical and horizontal arrows in a diagram of outer homomorphisms of profinite groups as follows:

$$\begin{array}{ccc} \Pi_U & \longrightarrow & \Pi_X \\ \big\downarrow & & \\ \Pi_X. & & \end{array}$$

We shall refer to any pair consisting of

- a diagram obtained in this way;
- an open subgroup of Π_X, which, by a slight abuse of notation, we denote by $\Pi_U \subseteq \Pi_X$, that belongs to the Π_X-conjugacy class of open subgroups that arises as the image of the vertical arrow of the diagram

as a *Belyi diagram.*

(ii) Fix a Belyi diagram

$$\begin{array}{ccc}\Pi_U & \longrightarrow & \Pi_X \\ \downarrow & & \\ \Pi_X & & \end{array}$$

[cf. (i)]. Recall the Grothendieck-Teichmüller group GT, which may be regarded as a closed subgroup of the outer automorphism group of the étale fundamental group Π_X [cf. Notations and Conventions] of $X = \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1]. Let $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ be a pair consisting of

- an abstract topological group Π;
- a closed subgroup G of $\mathrm{Out}(\Pi)$.

If there exists an isomorphism of such pairs

$$(\Pi, G \subseteq \mathrm{Out}(\Pi)) \xrightarrow{\sim} (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X))$$

[i.e., if there exist isomorphisms $\Pi \xrightarrow{\sim} \Pi_X$ and $G \xrightarrow{\sim} \mathrm{GT}$ of topological groups compatible with the inclusions $G \subseteq \mathrm{Out}(\Pi)$ and $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$], then we shall refer to the pair $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ as a *tripodal pair.*

Lemma 1.2. *Let $J \subseteq$ GT be a closed subgroup of* GT*. Fix a Belyi diagram*

$$\begin{array}{ccc}\Pi_U & \longrightarrow & \Pi_X \\ \downarrow & & \\ \Pi_X. & & \end{array}$$

Write $\phi_U : \mathrm{Aut}(\Pi_U) \twoheadrightarrow \mathrm{Out}(\Pi_U)$, $\phi_X : \mathrm{Aut}(\Pi_X) \twoheadrightarrow \mathrm{Out}(\Pi_X)$ for the natural surjections. Then, for any sufficiently small normal open subgroup $M \subseteq J$, there exist an outer action of M on Π_U and an open injection $\Pi_U \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\mathrm{out}}{\rtimes} J$ such that

(a) the outer action of M preserves and induces the identity automorphism on the set of the conjugacy classes of cuspidal inertia subgroups of Π_U;

(b) the injection $\Pi_U \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\mathrm{out}}{\rtimes} J$ is compatible with the inclusions $\Pi_U \subseteq \Pi_X$ and $M \subseteq J$.

Proof. First, we recall that Π_X is slim [cf., e.g., [MT], Proposition 1.4]. Write

$$\mathrm{Aut}^{\Pi_U}(\Pi_X) \subseteq \mathrm{Aut}(\Pi_X)$$

for the subgroup of $\mathrm{Aut}(\Pi_X)$ consisting of elements that induce automorphisms of Π_U that fix each of the conjugacy classes of cuspidal inertia subgroups of Π_U;

$$\mathrm{Inn}^{\Pi_U}(\Pi_X) \subseteq \mathrm{Aut}^{\Pi_U}(\Pi_X)$$

for the image of Π_U by the natural isomorphism $\Pi_X \xrightarrow{\sim} \mathrm{Inn}(\Pi_X)$. It follows immediately from the slimness of Π_X [cf., e.g., [MT], Proposition 1.4] that the natural homomorphism $\mathrm{Aut}^{\Pi_U}(\Pi_X) \to \mathrm{Aut}(\Pi_U)$ is injective. This injectivity implies that $\mathrm{Ker}(\mathrm{Aut}^{\Pi_U}(\Pi_X) \to \mathrm{Out}(\Pi_U)) \subseteq \mathrm{Inn}^{\Pi_U}(\Pi_X)$.

Since Π_U is a finite index subgroup of Π_X, and the cardinality of the conjugacy classes of cuspidal inertia subgroups of Π_U is finite, there exists a normal open subgroup M_{Aut} of $\phi_X^{-1}(J) \subseteq \mathrm{Aut}(\Pi_X)$ satisfying the following conditions:

(i) $M_{\mathrm{Aut}} \cap \mathrm{Inn}(\Pi_X) \subseteq \mathrm{Inn}^{\Pi_U}(\Pi_X)$;

(ii) $M_{\mathrm{Aut}} \subseteq \mathrm{Aut}^{\Pi_U}(\Pi_X)$.

Write $M_U \subseteq \mathrm{Out}(\Pi_U)$ (respectively, $M \subseteq \mathrm{Out}(\Pi_X)$) for the image of the composite $M_{\mathrm{Aut}} \hookrightarrow \mathrm{Aut}(\Pi_U) \overset{\phi_U}{\twoheadrightarrow} \mathrm{Out}(\Pi_U)$ (respectively, by the composite $M_{\mathrm{Aut}} \hookrightarrow \mathrm{Aut}(\Pi_X) \overset{\phi_X}{\twoheadrightarrow} \mathrm{Out}(\Pi_X)$). Now it follows from condition (ii), together with the discussion of the preceding paragraph, that we obtain a surjection $M_U \twoheadrightarrow M$. Finally, it follows immediately from condition (i) that this surjection is bijective. This completes the proof of Lemma 1.2. $\square$

Theorem 1.3 (**Combinatorial Belyi cuspidalization for a tripod**). *Fix a Belyi diagram*

$$\begin{tikzcd} \Pi_U \arrow[r] \arrow[d] & \Pi_X \\ \Pi_X & \end{tikzcd}$$

that arises from a connected finite étale covering $U \to X$ and an open immersion $U \hookrightarrow X$ [cf. Definition 1.1, (i)]. Then:

(i) Let $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ be a **tripodal pair**. *Fix an isomorphism of pairs $\alpha : (\Pi, G \subseteq \mathrm{Out}(\Pi)) \xrightarrow{\sim} (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X))$. Then the set of subgroups of Π determined, via α, by the cuspidal inertia subgroups of Π_X, may be* **reconstructed**, *in a* **purely group-theoretic** *way, from the pair $(\Pi, G \subseteq \mathrm{Out}(\Pi))$. We shall refer to the subgroups of Π constructed in this way as the* **cuspidal inertia subgroups** *of Π. In particular, for each open subgroup $\Pi^* \subseteq \Pi$ of Π, the pair $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ determines a set $I(\Pi^*)$ (respectively, $\mathrm{Cusp}(\Pi^*)$) of cuspidal inertia subgroups of Π^* (respectively, cusps of Π^*), namely, the set of intersections of Π^* with cuspidal inertia subgroups of Π (respectively, the conjugacy classes of cuspidal inertia subgroups of Π^*).*

(ii) *Let $N \subseteq \mathrm{GT}$ a normal open subgroup. Suppose that we are given an outer action of N on Π_U and an open injection $\Pi_U \overset{\mathrm{out}}{\rtimes} N \hookrightarrow \Pi_X \overset{\mathrm{out}}{\rtimes} \mathrm{GT}$ such that the conditions (a), (b) in Lemma 1.2 in the case of "$M \subseteq J$" hold for $N \subseteq \mathrm{GT}$. Then the original* **outer action** *of $N \subseteq \mathrm{GT}$ on Π_X* **coincides** *with the outer action of N on Π_X induced [cf. condition (a)] by the outer action of N on Π_U and the outer surjection $\Pi_U \twoheadrightarrow \Pi_X$ [i.e., the horizontal arrow in the above Belyi diagram].*

(iii) *Let*

$$C(\Pi) = (\Pi, G \subseteq \mathrm{Out}(\Pi), \Pi^*, \{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi), \{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi^*))$$

be a 5-tuple consisting of the following data:

- *a topological group Π;*
- *a closed subgroup $G \subseteq \mathrm{Out}(\Pi)$ such that the pair $(\Pi, G \subseteq \mathrm{Out}(\Pi))$ is a tripodal pair;*
- *an open subgroup $\Pi^* \subseteq \Pi$ of Π of genus 0, where we observe that the genus of an open subgroup of Π may be defined by using the cuspidal inertia subgroups of the open subgroup [cf. (i)];*
- *a subset $\{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi)$ [cf. (i)] of cardinality 3 [equipped with labels "0", "1", "∞"] of the set $\mathrm{Cusp}(\Pi)$;*
- *a subset $\{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi^*)$ [cf. (i)] of cardinality 3 [equipped with labels "0", "1", "∞"] of the set $\mathrm{Cusp}(\Pi^*)$.*

Suppose that the collection of data $C(\Pi)$ is isomorphic to the collection of data

$$\begin{aligned} C(\Pi_X) \;=\; & (\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X), \Pi_U, \\ & \{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi_X), \{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi_U)) \end{aligned}$$

determined, in a natural way, by the given Belyi diagram. [Here, we observe that the horizontal arrow in the given Belyi diagram determines, in a natural way, data $\{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi_U)$.] Fix an isomorphism of collections of data $C(\Pi) \overset{\sim}{\to} C(\Pi_X)$. Thus, the outer surjection $\Pi_U \twoheadrightarrow \Pi_X$ [i.e., the horizontal arrow in the given Belyi diagram], together with the isomorphism $C(\Pi) \overset{\sim}{\to} C(\Pi_X)$, determine an outer surjection $\Pi^* \twoheadrightarrow \Pi$. Let $N \subseteq G$ be a normal open subgroup such that similar conditions to the conditions (a), (b) considered in Lemma 1.2 in the case of "$M \subseteq J$" hold for $N \subseteq G$. Then the **outer surjection** *$\Pi^* \twoheadrightarrow \Pi$ may be* **reconstructed***, in a* **purely group-theoretic** *way, from the collection of data $C(\Pi)$ as the outer surjection induced by the unique Π-outer surjection $\Pi^* \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\mathrm{out}}{\rtimes} N$ [i.e., surjection well-defined up to composition with inner automorphisms arising from elements of Π] that lies over the identity morphism of N such that*

- *the kernel of this* Π*-outer surjection* $\Pi^* \overset{\text{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\text{out}}{\rtimes} N$ *is topologically generated by the cuspidal inertia subgroups of* Π^* *which are not associated to* $0, 1, \infty \in \mathrm{Cusp}(\Pi^*)$*;*
- *the conjugacy class of cuspidal inertia subgroups of* Π^* *associated to* 0 *(respectively,* 1*,* ∞*)* $\in \mathrm{Cusp}(\Pi^*)$ *maps to the conjugacy class of cuspidal inertia subgroups of* Π *associated to* 0 *(respectively,* 1*,* ∞*)* $\in \mathrm{Cusp}(\Pi)$.

Proof. First, we verify assertion (i). Since the outer action of GT on Π_X determined by the inclusion $\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$ is l-cyclotomically full [cf. [CmbGC], Definition 2.3, (ii)], assertion (i) follows immediately from [CmbGC], Corollary 2.7, (i), and its proof.

Next, we verify assertion (ii). First, we observe that:

> Claim 1.3.A: It suffices to prove assertion (ii) for a sufficiently small normal open subgroup $N^\dagger \subseteq N$.

Indeed, let $\sigma \in N$. Write

- $\rho' : N \to \mathrm{Out}(\Pi_X)$ for the original outer action;
- $\rho'' : N \to \mathrm{Out}(\Pi_X)$ for the outer action of N on Π_X induced [cf. condition (a)] by the outer action of N on Π_U and the outer surjection $\Pi_U \twoheadrightarrow \Pi_X$.

Suppose that $\rho'|_{N^\dagger} = \rho''|_{N^\dagger}$. Write $\rho \overset{\text{def}}{=} \rho'|_{N^\dagger}$; $\sigma' \overset{\text{def}}{=} \rho'(\sigma)$; $\sigma'' \overset{\text{def}}{=} \rho''(\sigma)$. Our goal is to prove that $\sigma' = \sigma''$. Since $N^\dagger$ is a normal subgroup in N, for each $\tau \in N^\dagger$, $\sigma'\rho(\tau)(\sigma')^{-1} = \rho'(\sigma\tau\sigma^{-1}) = \rho''(\sigma\tau\sigma^{-1}) = \sigma''\rho(\tau)(\sigma'')^{-1}$. Thus, $(\sigma'')^{-1}\sigma' \in Z_{\mathrm{Out}(\Pi_X)}(\rho(N))$. By the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [Tama1], Theorem 0.4], $(\sigma'')^{-1}\sigma'$ is induced by a geometric automorphism of X. Since the condition (a) in Lemma 1.2 in the case of "$M \subseteq J$" holds for $N \subseteq \mathrm{GT}$, $(\sigma'')^{-1}\sigma'$ preserves and fixes each conjugacy class of cuspidal inertia subgroups of Π_X. Thus, we conclude that $\sigma' = \sigma''$. This completes the proof of Claim 1.3.A.

Write

- Π_{X_3} for the étale fundamental group of the third configuration space X_3 of X [cf. [MT], Definition 2.1, (i)];
- $\mathrm{pr}_i : \Pi_{X_3} \twoheadrightarrow \Pi_X$ $(i = 1, 2, 3)$ for choices of surjections that induce the natural outer surjections determined by the natural scheme-theoretic projections;
- $U^{\times 3} \overset{\text{def}}{=} U \times U \times U$, $X^{\times 3} \overset{\text{def}}{=} X \times X \times X$, ${\Pi_U}^{\times 3} \overset{\text{def}}{=} \Pi_U \times \Pi_U \times \Pi_U$, ${\Pi_X}^{\times 3} \overset{\text{def}}{=} \Pi_X \times \Pi_X \times \Pi_X$;
- $V_3 \overset{\text{def}}{=} X_3 \times_{X^{\times 3}} U^{\times 3}$, where the fiber product is with respect to the open immersion $X_3 \hookrightarrow X^{\times 3}$ that arises from the definition of the configuration space X_3 and the finite étale covering $U^{\times 3} \to X^{\times 3}$ determined by the given connected finite étale covering $U \to X$.

Next, we make the following *observations*:

- the projection $V_3 \to U^{\times 3}$ is an open immersion that factors as the composite of an open immersion $V_3 \hookrightarrow U_3$ and the open immersion $U_3 \hookrightarrow U^{\times 3}$ that arises from the definition of the configuration space U_3;

- by choosing a suitable basepoint of V_3, we may regard Π_{V_3} as the open subgroup $\Pi_{V_3} \subseteq \Pi_{X_3}$ given by forming the inverse image of the open subgroup $\Pi_U^{\times 3} \subseteq \Pi_X^{\times 3}$ (determined by the open subgroup $\Pi_U \subseteq \Pi_X$) via the surjection $\Pi_{X_3} \twoheadrightarrow \Pi_X^{\times 3}$ determined by $\mathrm{pr}_i : \Pi_{X_3} \twoheadrightarrow \Pi_X$ $(i = 1, 2, 3)$;

- the open immersion $V_3 \hookrightarrow U_3$ induces a natural outer surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$;

- the open immersion $U_3 \hookrightarrow X_3$ determined by the open immersion $U \hookrightarrow X$ induces a natural outer surjection $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$;

- we have natural inclusions $N \subseteq \mathrm{GT} \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3}) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_X)$ [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1; [CmbCsp], Theorem 4.1, (i); [CmbCsp], Corollary 4.2, (i), (ii)].

For each $\sigma \in N \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})$, let $\tilde{\sigma}_3 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_{X_3})$ be a lifting of the image $\sigma_3 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})$ of σ such that the automorphisms of Π_X induced by $\tilde{\sigma}_3$ via the surjections $\mathrm{pr}_i : \Pi_{X_3} \twoheadrightarrow \Pi_X$ $(i = 1, 2, 3)$ coincide and stabilize the subgroup $\Pi_U \subseteq \Pi_X$ [cf. our hypotheses on N]. Thus, it follows from the various *observations* made above concerning the open subgroup $\Pi_{V_3} \subseteq \Pi_{X_3}$ that $\tilde{\sigma}_3$ induces an automorphism $\tilde{\sigma}_{V_3}$ of Π_{V_3}.

Next, we verify the following assertion:

> Claim 1.3.B: There exists a normal open subgroup $N^\dagger$ of GT such that $N^\dagger \subseteq N$, and, moreover, the following condition holds:
>
> > For each element $\sigma \in N^\dagger$, $\tilde{\sigma}_{V_3} \in \mathrm{Aut}(\Pi_{V_3})$ preserves the kernel of the outer surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ (respectively, $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$) induced by the open immersion $V_3 \hookrightarrow U_3$ (respectively, the composite of open immersions $V_3 \hookrightarrow U_3 \hookrightarrow X_3$).
>
> In particular, $\tilde{\sigma}_{V_3} \in \mathrm{Aut}(\Pi_{V_3})$ induces outer automorphisms of Π_{U_3} and Π_{X_3} compatible with the outer surjections $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ and $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$, respectively.

Write

- I_{X_3} for the set of inertia subgroups $\subseteq \Pi_{X_3}$ associated to the irreducible divisors contained in the complement of the interior of the third log configuration space of X [cf. [MT], Definition 2.1, (i)];

- $I_{V_3} \stackrel{\mathrm{def}}{=} \{I \cap \Pi_{V_3}\ (\subseteq \Pi_{X_3}) \mid I \in I_{X_3}\}$;

- I_{U_3} for the set of images of elements of I_{V_3} by the outer surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$;

- $|I_{X_3}|$ (respectively, $|I_{V_3}|$) for the set of Π_{X_3}- (respectively, Π_{V_3}-)conjugacy classes of elements of I_{X_3} (respectively, I_{V_3}).

Next, we make the following *observations*:

- $\tilde{\sigma}_3$ acts on I_{X_3} and induces the identity automorphism of $|I_{X_3}|$ [cf. condition (a) in Lemma 1.2; [CmbCsp], Proposition 1.3, (vi)];

- for each $\sigma \in N$, the action of $\tilde{\sigma}_3$ on I_{X_3} induces a natural action of $\tilde{\sigma}_{V_3}$ on I_{V_3}, and hence on $|I_{V_3}|$;

- since, for each $\sigma \in N$, $\tilde{\sigma}_3$ is completely determined [cf. condition (a) in Lemma 1.2; the fact that U is of genus 0; the definition of $\tilde{\sigma}_3$] up to composition with an inner automorphism of Π_{X_3} arising from Π_{V_3}, we conclude that the natural action of $\tilde{\sigma}_3$ on I_{V_3} determines a natural action of N on $|I_{V_3}|$;

- $|I_{X_3}|$ and $|I_{V_3}|$ are finite sets.

Thus, it follows immediately from the above *observations* that, if we take $N^\dagger$ to be a sufficiently small normal open subgroup of GT, then $\tilde{\sigma}_{V_3}$ induces the identity automorphism of $|I_{V_3}|$ for each $\sigma \in N^\dagger$. Since the kernel of the outer surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ (respectively, $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$) is topologically normally generated by a certain collection of elements of I_{V_3} (respectively, I_{U_3}), we obtain the desired conclusion. This completes the proof of Claim 1.3.B.

By applying Claim 1.3.A and Claim 1.3.B, we may assume [by replacing N by a suitable normal open subgroup of GT] that, for each element $\sigma \in N$, $\tilde{\sigma}_{V_3} \in \mathrm{Aut}(\Pi_{V_3})$ induces outer automorphisms $\sigma_{V_3} \in \mathrm{Out}(\Pi_{V_3})$, $\sigma_{U_3} \in \mathrm{Out}(\Pi_{U_3})$, and $\sigma_{X_3} \in \mathrm{Out}(\Pi_{X_3})$ compatible with the outer surjections $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ and $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$, respectively. Our goal is to prove that

$$\sigma_3 = \sigma_{X_3} \in \mathrm{Out}(\Pi_{X_3}).$$

Note that $\sigma_{X_3} \in \mathrm{Out}^{\mathrm{F}}(\Pi_{X_3})$ by construction. Since $\mathrm{Out}^{\mathrm{F}}(\Pi_{X_3}) = \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})$ [cf. [CbTpII], Theorem A, (ii)], $\sigma_{X_3} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})$.

In the following discussion, we *fix a surjection* $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ (respectively, $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$) that induces the outer surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ (respectively, $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$) of Claim 1.3.B.

Next, write C for the set of central tripods in Π_{X_3} [cf, [CbTpII], Definition 3.7, (ii)]; C_V for the set of central tripods Π^{ctpd} of Π_{X_3} that satisfy the following condition:

> $\Pi^{\mathrm{ctpd}} \subseteq \Pi_{V_3}$; the image of Π^{ctpd} ($\subseteq \Pi_{V_3}$) by the surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$ is a central tripod of Π_{U_3}.

Then:

Claim 1.3.C: The natural action of Π_{V_3} by conjugation on C_V is *transitive*; moreover,

$$C \supseteq C_V = \{\Pi^{\mathrm{ctpd}} \in C \mid \Pi^{\mathrm{ctpd}} \cap \mathrm{Ker}(\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}) = \{1\}\} \neq \emptyset.$$

Write $\Delta \subseteq X^{\times 3}$ (respectively, $\Delta_U \subseteq U^{\times 3}$) for the image of X (respectively, U) under the diagonal embedding $X \hookrightarrow X^{\times 3}$ (respectively, $U \hookrightarrow U^{\times 3}$). Note that it follows immediately from the definition of the subgroup $\Pi_{V_3} \subseteq \Pi_{X_3}$ [cf. also [CbTpII], Definitions 3.3, (ii); 3.7, (ii)] that every $\Pi^{\mathrm{ctpd}} \in C$ is contained in Π_{V_3}, and that any two subgroups $\in C$ are Π_{X_3}-conjugate. Moreover, one verifies immediately that the Π_{V_3}-conjugacy classes of subgroups $\in C$ are in natural bijective correspondence with the irreducible [or, equivalently, connected] components of the inverse image of Δ by the finite étale covering $U^{\times 3} \to X^{\times 3}$. Thus, by considering the Π_{V_3}-conjugacy class of subgroups $\in C$ corresponding to Δ_U, we obtain that $C_V \neq \emptyset$. On the other hand, by considering the scheme-theoretic geometry of tripods that give rise to Π_{V_3}-conjugacy classes of subgroups $\in C$ that do *not* correspond to Δ_U, we conclude that such subgroups $\in C$ have *nontrivial intersection* with the kernel of the surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$. This completes the proof of Claim 1.3.C.

Let $\Pi^{\mathrm{ctpd}} \in C_V$. Write Π_U^{ctpd} for the image of Π^{ctpd} by the surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3}$; Π_X^{ctpd} for the image of Π_U^{ctpd} by the surjection $\Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$. Thus, Π_U^{ctpd} is a central tripod of Π_{U_3}, and Π_X^{ctpd} is a central tripod of Π_{X_3} [hence Π_{X_3}-conjugate to Π^{ctpd}].

By the theory of tripod synchronization [cf. [CbTpII], Theorem C, (ii), (iii)] and the injectivity of $\mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3}) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_X)$ [cf. [CmbCsp], Theorem 4.1, (i)], we obtain injective tripod homomorphisms

$$T : \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})^{\mathrm{cusp}} \to \mathrm{Out}(\Pi^{\mathrm{ctpd}}), \quad T_X : \mathrm{Out}^{\mathrm{FC}}(\Pi_{X_3})^{\mathrm{cusp}} \to \mathrm{Out}(\Pi_X^{\mathrm{ctpd}})$$

[cf. [CmbCsp], Definition 1.1, (v)], which are related to one another via composition with the isomorphism $\zeta : \mathrm{Out}(\Pi^{\mathrm{ctpd}}) \xrightarrow{\sim} \mathrm{Out}(\Pi_X^{\mathrm{ctpd}})$ induced by the geometric outer isomorphism $\Pi^{\mathrm{ctpd}} \xrightarrow{\sim} \Pi_X^{\mathrm{ctpd}}$ [cf. [CbTpII], Definition 3.4, (ii)] determined by the composite surjection $\Pi_{V_3} \twoheadrightarrow \Pi_{U_3} \twoheadrightarrow \Pi_{X_3}$. Since $\tilde{\sigma}_{V_3}$ preserves the Π_{V_3}-conjugacy class of $\Pi^{\mathrm{ctpd}} \subseteq \Pi_{V_3}$ [cf. Claims 1.3.B, 1.3.C; [CbTpII], Theorem C, (ii)], we conclude that $\zeta(T(\sigma_3)) = T_X(\sigma_{X_3})$. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). The existence of a Π-outer surjection $\Pi^* \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\mathrm{out}}{\rtimes} N$ as in the statement of assertion (iii) follows immediately from assertion (ii) and the various definitions involved. Since $G_{\mathbb{Q}} \subseteq \mathrm{GT} \xrightarrow{\sim} G$, the uniqueness of a Π-outer surjection $\Pi^* \overset{\mathrm{out}}{\rtimes} N \twoheadrightarrow \Pi \overset{\mathrm{out}}{\rtimes} N$ as in the statement of assertion (iii) follows immediately from the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [Tama1], Theorem 0.4], applied to the case of $\mathbb{P}^1_{\mathbb{Q}} \backslash \{0, 1, \infty\}$. This completes the proof of assertion (iii), hence also the proof of Theorem 1.3. $\square$

Definition 1.4. Let $J \subseteq \mathrm{GT}$ be a closed subgroup of GT. In the situation of Theorem 1.3, (ii), for each normal open subgroup M of J satisfying $M \subseteq N \cap J$, we obtain a diagram

$$\begin{array}{ccc} \Pi_U \overset{\mathrm{out}}{\rtimes} M & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} M \\ \downarrow & & \\ \Pi_X \overset{\mathrm{out}}{\rtimes} M & & \end{array}$$

of Π_X*-outer homomorphisms* [i.e., homomorphisms well-defined up to composition with inner automorphisms arising from elements of Π_X] of profinite groups. We shall refer to a diagram obtained in this way as an *arithmetic Belyi diagram.*

Definition 1.5.

(i) Fix an arithmetic Belyi diagram $\mathbb{B}^{\rtimes}$ as in Definition 1.4. Write

$$\mathbb{D}(\mathbb{B}^{\rtimes}, M, J)$$

for the set of the images via the natural composite Π_X-outer homomorphism $\Pi_U \overset{\mathrm{out}}{\rtimes} M \twoheadrightarrow \Pi_X \overset{\mathrm{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\mathrm{out}}{\rtimes} J$ of the normalizers in $\Pi_U \overset{\mathrm{out}}{\rtimes} M$ of cuspidal inertia subgroups of Π_U;

$$\mathbb{D}(\mathbb{B}^{\rtimes}, J)$$

for the quotient set $\big(\sqcup_{M \subseteq J} \mathbb{D}(\mathbb{B}^{\rtimes}, M, J)\big)/\sim$, where M ranges over all sufficiently small normal open subgroups of J, and we write $\mathbb{D}(\mathbb{B}^{\rtimes}, M, J) \ni G_M \sim G_{M^\dagger} \in \mathbb{D}(\mathbb{B}^{\rtimes}, M^\dagger, J)$ if $G_M \cap G_{M^\dagger}$ is open in both G_M and $G_{M^\dagger}$.

(ii) Write

$$\mathbb{D}(J)$$

for the quotient set $\big(\sqcup_{\mathbb{B}^{\rtimes}} \mathbb{D}(\mathbb{B}^{\rtimes}, J)\big)/\sim$, where $\mathbb{B}^{\rtimes}$ ranges over all arithmetic Belyi diagrams, and we write $\mathbb{D}({}^{\dagger}\mathbb{B}^{\rtimes}, J) \ni G_{{}^{\dagger}\mathbb{B}^{\rtimes}} \sim G_{{}^{\ddagger}\mathbb{B}^{\rtimes}} \in \mathbb{D}({}^{\ddagger}\mathbb{B}^{\rtimes}, J)$ if $G_{M^\dagger} \cap G_{M^\ddagger}$ is open in both $G_{M^\dagger}$ and $G_{M^\ddagger}$ for some representative $G_{M^\dagger}$ (respectively, $G_{M^\ddagger}$) of $G_{{}^{\dagger}\mathbb{B}^{\rtimes}}$ (respectively, $G_{{}^{\ddagger}\mathbb{B}^{\rtimes}}$). We shall refer to $\mathbb{D}(J)$ as the set of *decomposition subgroup-germs of* $\Pi_X \overset{\mathrm{out}}{\rtimes} J$.

(iii) We shall refer to the technique of constructing decomposition subgroup-germs of $\Pi_X \overset{\mathrm{out}}{\rtimes} J$ as in (ii) as *combinatorial Belyi cuspidalization.*

Corollary 1.6. *In the situation of Definition 1.5:*

(i) The natural conjugation action of $\Pi_X \overset{\text{out}}{\rtimes} J$ *on itself induces a natural action of* $\Pi_X \overset{\text{out}}{\rtimes} J$ *on* $\mathbb{D}(J)$*.*

(ii) Write

$$D(J)$$

for the quotient set $\mathbb{D}(J)/\Pi_X$*. Then* $D(J)$ *admits a natural action by* J*.*

(iii) Let J_1 *and* J_2 *be closed subgroups of* GT*. If* $J_1 \subseteq J_2 \subseteq \mathrm{GT}$*, then the inclusion* $J_1 \subseteq J_2$ *induces, by considering the intersection of subgroups of* $\Pi_X \overset{\text{out}}{\rtimes} J_2$ *with* $\Pi_X \overset{\text{out}}{\rtimes} J_1$*, a natural surjection*

$$D(J_2) \twoheadrightarrow D(J_1)$$

that is equivariant with respect to the natural actions of J_1 $(\subseteq J_2)$ *on the domain and codomain.*

Proof. First, we verify assertion (i). Let $\sigma \in \Pi_X \overset{\text{out}}{\rtimes} J$ $(\subseteq \mathrm{Aut}(\Pi_X))$. Fix an arithmetic Belyi diagram $\mathbb{B}^{\rtimes}$

$$\begin{array}{ccc} \Pi_U \overset{\text{out}}{\rtimes} M & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} M \\ \downarrow & & \\ \Pi_X \overset{\text{out}}{\rtimes} M. & & \end{array}$$

Next, we *observe* that σ, the inclusion $\Pi_U \subseteq \Pi_X$, and the outer action of M on Π_U determine

- an open subgroup $\Pi_{U^\sigma} \overset{\text{def}}{=} \sigma(\Pi_U)\sigma^{-1} \subseteq \Pi_X$ that belongs to the Π_X-conjugacy class of open subgroups that arises as the image of the outer injection $\Pi_{U^\sigma} \hookrightarrow \Pi_X$ determined by some connected finite étale covering $U^\sigma \to X$;
- an isomorphism $\Pi_U \xrightarrow{\sim} \Pi_{U^\sigma}$ [induced by conjugating by σ] that induces a bijection of the set of cuspidal inertia subgroups;
- an outer action [induced by conjugating by σ] of M on Π_{U^σ};
- a collection of data [induced by conjugating by σ]

$$\begin{aligned} C(\Pi_X)^\sigma \overset{\text{def}}{=} &(\Pi_X, \mathrm{GT} \subseteq \mathrm{Out}(\Pi_X), \Pi_{U^\sigma}, \\ &\{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi_X), \{0,1,\infty\} \subseteq \mathrm{Cusp}(\Pi_{U^\sigma})) \end{aligned}$$

  [cf. Theorem 1.3, (i), (iii)];

- an isomorphism $C(\Pi_X) \xrightarrow{\sim} C(\Pi_X)^\sigma$ [induced by conjugating by σ].

Since M is a normal subgroup of J, by conjugating by σ, we obtain an automorphism $\sigma_M : \Pi_X \overset{\text{out}}{\rtimes} M \xrightarrow{\sim} \Pi_X \overset{\text{out}}{\rtimes} M$ and an isomorphism $\sigma_M|_{\Pi_U} : \Pi_U \overset{\text{out}}{\rtimes} M \xrightarrow{\sim} \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} M$ compatible with the natural inclusions $\Pi_U \overset{\text{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\text{out}}{\rtimes} M$ and $\Pi_{U^\sigma} \overset{\text{out}}{\rtimes} M \hookrightarrow \Pi_X \overset{\text{out}}{\rtimes} M$. Thus, it follows immediately from the above *observations*, together with Theorem 1.3, (ii), (iii), that we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccc}
\Pi_X \overset{\text{out}}{\rtimes} M & \longleftarrow & \Pi_U \overset{\text{out}}{\rtimes} M & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} M \\
{\scriptstyle \sigma_M}\downarrow\wr & & {\scriptstyle \sigma_M|_{\Pi_U}}\downarrow\wr & & {\scriptstyle \sigma_M}\downarrow\wr \\
\Pi_X \overset{\text{out}}{\rtimes} M & \longleftarrow & \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} M & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} M,
\end{array}$$

where the upper horizontal arrows "$\leftarrow$", "$\rightarrow$" are, respectively, the vertical and horizontal arrows of $\mathbb{B}^{\rtimes}$; the arrow $\Pi_X \overset{\text{out}}{\rtimes} M \leftarrow \Pi_{U^\sigma} \overset{\text{out}}{\rtimes} M$ is the natural inclusion discussed above; the arrow $\Pi_{U^\sigma} \overset{\text{out}}{\rtimes} M \to \Pi_X \overset{\text{out}}{\rtimes} M$ is the Π_X-outer surjection induced [cf. Theorem 1.3, (ii), (iii)] by the outer surjection $\Pi_{U^\sigma} \to \Pi_X$ determined by the open immersion $U^\sigma \hookrightarrow X$ that maps the cusp 0 (respectively, 1, ∞) of U^σ to the cusp 0 (respectively, 1, ∞) of X. Thus, by the above *observations* and the definition of $\mathbb{D}(J)$, we conclude that the natural conjugation action of $\Pi_X \overset{\text{out}}{\rtimes} J$ on itself induces a natural action of $\Pi_X \overset{\text{out}}{\rtimes} J$ on $\mathbb{D}(J)$. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i). Assertion (iii) follows immediately from the various definitions involved. This completes the proof of Corollary 1.6. $\square$

Corollary 1.7. *In the notation of Corollary 1.6, there exist a natural surjection $D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}}$ and a natural bijection $D(G_{\mathbb{Q}}) \xrightarrow{\sim} \overline{\mathbb{Q}}$.*

Proof. The usual theory of Belyi cuspidalization [cf. [AbsTopIII], Theorem 1.9, (a)] yields a natural bijection $D(G_{\mathbb{Q}}) \xrightarrow{\sim} \overline{\mathbb{Q}}$. Next, by applying the natural inclusion $G_{\mathbb{Q}} \subseteq \mathrm{GT}$ [cf. the discussion at the beginning of the Introduction], we obtain a natural surjection $D(\mathrm{GT}) \twoheadrightarrow D(G_{\mathbb{Q}})$ [cf. Corollary 1.6, (iii)]. Thus, by considering the composite $D(\mathrm{GT}) \twoheadrightarrow D(G_{\mathbb{Q}}) \xrightarrow{\sim} \overline{\mathbb{Q}}$, we obtain a natural surjection $D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}}$. This completes the proof of Corollary 1.7. $\square$

Remark 1.7.1. The author does not know, at the time of writing, whether or not the *surjection*

$$D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}}$$

in Corollary 1.7 is *bijective*.

2 Construction of an action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the field $\overline{\mathbb{Q}}$

In this section, we construct [cf. Corollary 2.4] a certain natural action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the field $\overline{\mathbb{Q}}$, where $\mathrm{GT}_p^{\mathrm{tp}}$ denotes [cf. Definition 2.1] a certain subgroup of GT that contains the p-adic version of the Grothendieck-Teichmüller group GT_p defined by Y. André [cf. [André], Definition 8.6.3] by using the theory of tempered fundamental groups [cf. [André], §4, for the definition and basic properties of tempered fundamental groups]. First, we define $\mathrm{GT}_p^{\mathrm{tp}}$.

Definition 2.1. Let p be a prime number, $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ [cf. Notations and Conventions]. Write

- $X \overset{\mathrm{def}}{=} \mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, 1, \infty\}$, where $\mathbb{C}_p$ denotes the p-adic completion of $\overline{\mathbb{Q}}_p$;
- Π_X^{tp} for the tempered fundamental group of X, relative to a suitable choice of basepoint.

We shall denote by $\mathrm{GT}_p^{\mathrm{tp}}$ the intersection of GT and $\mathrm{Out}(\Pi_X^{\mathrm{tp}})$ in $\mathrm{Out}(\Pi_X)$ [cf. Remark 2.1.1].

Remark 2.1.1. Observe that [for suitable choices of basepoints] Π_X may be regarded as the profinite completion of Π_X^{tp}, and Π_X^{tp} may be regarded as a subgroup of Π_X [cf. [André], §4.5]. Then the operation of passing to the profinite completion induces a natural homomorphism

$$\mathrm{Out}(\Pi_X^{\mathrm{tp}}) \to \mathrm{Out}(\Pi_X).$$

It follows immediately from the normal terminality of Π_X^{tp} in Π_X, i.e., $N_{\Pi_X}(\Pi_X^{\mathrm{tp}}) = \Pi_X^{\mathrm{tp}}$ [cf. [André], Corollary 6.2.2; [SemiAn], Lemma 6.1, (ii)], that this natural homomorphism is *injective*. Thus, we shall use this natural injection to regard $\mathrm{Out}(\Pi_X^{\mathrm{tp}})$ as a subgroup of $\mathrm{Out}(\Pi_X)$.

Remark 2.1.2. Various p-adic versions of the Grothendieck-Teichmüller group appear in the literature. It follows immediately from [André], Definition 8.6.3; [CbTpIII], Theorem B, (ii); [CbTpIII], Theorem D, (i); [CbTpIII], Theorem E; [CbTpIII], Proposition 3.6, (i), (ii); [CbTpIII], Definition 3.7, (i); [CbTpIII], Remark 3.13.1, (i); [CbTpIII], Remark 3.19.2; [CbTpIII], Remark 3.20.1, that

$$\begin{array}{ccccccccc} G_{\mathbb{Q}_p} & \subseteq & \mathrm{GT}^{\mathrm{M}} & \subseteq & \mathrm{GT}^{\mathrm{G}} & \subseteq & \mathrm{GT} \cap \mathrm{Out}^{\mathrm{G}}(\Pi_X) & = & \mathrm{GT}_p^{\mathrm{tp}} \\ \| & & \| & & & & \| & & \| \\ G_{\mathbb{Q}_p} & \subseteq & \mathrm{GT}^{\mathrm{M}} & \subseteq & \mathrm{GT}_p & \subseteq & \mathrm{GT} \cap \mathrm{Out}^{\mathrm{G}}(\Pi_X) & = & \mathrm{GT}_p^{\mathrm{tp}}. \end{array}$$

Remark 2.1.3. It follows immediately from the fact that the subgroup "$\mathrm{Out}^{\mathrm{G}}(\Pi_1) \subseteq \mathrm{Out}(\Pi_1)$" [cf. [CbTpIII], Proposition 3.6, (i), (ii); [CbTpIII], Definition 3.7, (i); [CbTpIII], Remark 3.13.1, (i)] is *closed* [cf. [CbTpIII], Theorem 3.17, (iv)] that $\mathrm{GT}_p^{\mathrm{tp}}$ is a *closed* subgroup of GT.

Next, we construct a natural action of $\mathrm{GT}_p^{\mathrm{tp}}$ on the set $\overline{\mathbb{Q}}$. The following theorem plays a central role in this construction. We prove this theorem by applying various "resolution of nonsingularities" results [cf. [Tama2], Theorem 0.2, (v); [Lpg], Theorem 2.7], as well as the reconstruction theorem of the dual semi-graph from the tempered fundamental group of a pointed stable curve [cf. [SemiAn], Corollary 3.11].

Theorem 2.2. *In the notation of Definition 2.1, let $\phi : Y \to X$ be a connected finite étale covering of X; y, y' elements of $Y(\mathbb{C}_p)$. Write Y_y (respectively, $Y_{y'}$) for $Y\backslash\{y\}$ (respectively, $Y\backslash\{y'\}$); Π_Y^{tp} (respectively, $\Pi_{Y_y}^{\mathrm{tp}}$, $\Pi_{Y_{y'}}^{\mathrm{tp}}$) for the tempered fundamental group of Y (respectively, Y_y, $Y_{y'}$), relative to a suitable choice of basepoint. Suppose that there exists an isomorphism $\Pi_{Y_y}^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{Y_{y'}}^{\mathrm{tp}}$ that fits into a commutative diagram*

$$\begin{array}{ccc} \Pi_{Y_y}^{\mathrm{tp}} & \xrightarrow{\sim} & \Pi_{Y_{y'}}^{\mathrm{tp}} \\ \downarrow & & \downarrow \\ \Pi_Y^{\mathrm{tp}} & = & \Pi_Y^{\mathrm{tp}}, \end{array}$$

where the vertical arrows are the surjections [determined up to composition with an inner automorphism] induced by the natural open immersions $Y_y \hookrightarrow Y$, $Y_{y'} \hookrightarrow Y$ of hyperbolic curves. Then $y = y'$.

Proof. Suppose that $y \neq y'$. Write

- $\mathcal{O}_{\mathbb{C}_p}$ for the ring of integers of $\mathbb{C}_p$;
- Y^{cpt} for the smooth compactification of Y (over $\mathbb{C}_p$);
- S for $Y^{\mathrm{cpt}} \setminus Y$;
- $\mathcal{Y}_{y,y'}$ for the stable model over $\mathcal{O}_{\mathbb{C}_p}$ of the pointed stable curve $(Y^{\mathrm{cpt}}, S \cup \{y, y'\})$;
- $\mathcal{Y}$ for the semi-stable model over $\mathcal{O}_{\mathbb{C}_p}$ of the pointed stable curve (Y^{cpt}, S) obtained by forgetting the data of the horizontal divisors of $\mathcal{Y}_{y,y'}$ determined by y, y';
- $\overline{y}$ (respectively, $\overline{y}'$) for the closed point of $\mathcal{Y}$ determined by y (respectively, y').

Let

- $\tilde{\mathcal{Y}}$ be a proper normal model of Y^{cpt} over $\mathcal{O}_{\mathbb{C}_p}$ that dominates $\mathcal{Y}$, and whose special fiber contains an irreducible component $\tilde{y}$ (respectively, $\tilde{y}'$) that maps to $\overline{y}$ (respectively, $\overline{y}'$) in $\mathcal{Y}$;

- $\hat{y}$ (respectively, $\hat{y}'$) the valuation of the function field of $\mathcal{Y}$ determined by $\tilde{y}$ (respectively, $\tilde{y}'$).

Then, by applying [Lpg], Theorem 2.7 [cf. also the discussion at the beginning of [Lpg], §1; the discussion immediately preceding [Lpg], Definition 2.1; the discussion immediately preceding [Lpg], Corollary 2.9], to Y, we conclude that there exists a finite étale Galois covering

$$\phi : Z \to Y$$

such that, if we write

- $Y^{\mathrm{an}}_{(2)}$ for the set of type 2 points of the Berkovich space Y^{an} associated to Y [so that, by a slight abuse of notation, we may regard $\hat{y}$, $\hat{y}'$ as points of $Y^{\mathrm{an}}_{(2)}$];

- $V(\mathcal{Y})$ for the set of type 2 points of Y^{an} corresponding to the irreducible components of the special fiber of $\mathcal{Y}$;

- Z^{cpt} for the smooth compactification of Z (over $\mathbb{C}_p$);

- $\mathcal{Z}$ for the stable model of the pointed stable curve $(Z^{\mathrm{cpt}}, \phi^{-1}(S))$;

- $V(\mathcal{Z})$ for the set of type 2 points of the Berkovich space Z^{an} associated to Z corresponding to the irreducible components of the special fiber of $\mathcal{Z}$;

- $\mathrm{Im}(V(\mathcal{Z})) \subseteq Y^{\mathrm{an}}_{(2)}$ for the image of $V(\mathcal{Z})$ by the natural map $Z^{\mathrm{an}} \to Y^{\mathrm{an}}$ induced by ϕ,

then

$$\{\hat{y}, \hat{y}'\} \cup V(\mathcal{Y}) \ \subseteq \ \mathrm{Im}(V(\mathcal{Z})) \ \subseteq \ Y^{\mathrm{an}}_{(2)}.$$

Since $\mathcal{Y}$ is normal, it follows immediately, via a well-known argument [involving the closure in $\mathcal{Z} \times_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{Y}$ of the graph of ϕ], from Zariski's Main Theorem, together with the first inclusion of the above display, that ϕ determines a morphism $f : \mathcal{Z} \to \mathcal{Y}$ such that

- the morphism f induces ϕ on the generic fiber;

- the image in the special fiber of $\mathcal{Y}$ of the vertical components of the special fiber of $\mathcal{Z}$ [i.e., the irreducible components of this special fiber that map to a point in the special fiber of $\mathcal{Y}$] contains $\overline{y}$ and $\overline{y}'$.

Fix a vertical component v in the special fiber of $\mathcal{Z}$ such that $f(v) = \overline{y}$. Write $\tilde{\mathcal{Y}}$ for the normalization of $\mathcal{Y}$ in the function field of Z; $\tilde{f} : \mathcal{Z} \to \tilde{\mathcal{Y}}$ for the morphism induced by the universal property of the normalization morphism $h : \tilde{\mathcal{Y}} \to \mathcal{Y}$. Since h is finite, $\tilde{f}(v)$ is a closed point of $\tilde{\mathcal{Y}}$. By Zariski's Main Theorem, $\tilde{f}^{-1}(\tilde{f}(v))$ is connected. In particular, every irreducible component of $\tilde{f}^{-1}(\tilde{f}(v))$ is of dimension 1. Let $z \in Z(\mathbb{C}_p)$ be such that

- $f(z) = y$;

- $\overline{z} \in \tilde{f}^{-1}(\tilde{f}(v))$, where $\overline{z}$ denotes the closed point of $\mathcal{Z}$ determined by z.

Observe that the set C_z of irreducible components of the special fiber of $\mathcal{Z}$ that contain $\overline{z}$ is nonempty and of cardinality ≤ 2. Write $C_z \overset{\mathrm{def}}{=} \{v_z, w_z\}$, where we note that it may or may not be the case that $v_z = w_z$. Without loss of generality, we may assume that $\overline{z} \in v_z \subseteq \tilde{f}^{-1}(\tilde{f}(v))$.

By [SemiAn], Corollary 3.11, any isomorphism of tempered fundamental groups preserves cuspidal inertia subgroups. Thus, the given commutative diagram of tempered fundamental groups

$$\begin{array}{ccc} \Pi^{\mathrm{tp}}_{Y_y} & \xrightarrow{\sim} & \Pi^{\mathrm{tp}}_{Y_{y'}} \\ \downarrow & & \downarrow \\ \Pi^{\mathrm{tp}}_{Y} & = & \Pi^{\mathrm{tp}}_{Y}, \end{array}$$

implies the existence of a $\mathbb{C}_p$-valued point z' of Z such that $\phi(z') = y'$, together with a commutative diagram of tempered fundamental groups

$$\begin{array}{ccc} \Pi^{\mathrm{tp}}_{Z_z} & \xrightarrow{\sim} & \Pi^{\mathrm{tp}}_{Z_{z'}} \\ \downarrow & & \downarrow \\ \Pi^{\mathrm{tp}}_{Z} & = & \Pi^{\mathrm{tp}}_{Z}, \end{array}$$

where $Z_z \overset{\mathrm{def}}{=} Z \setminus \{z\}$; $Z_{z'} \overset{\mathrm{def}}{=} Z \setminus \{z'\}$; Π^{tp}_{Z} (respectively, $\Pi^{\mathrm{tp}}_{Z_z}$, $\Pi^{\mathrm{tp}}_{Z_{z'}}$) denotes the tempered fundamental group of Z (respectively, Z_z, $Z_{z'}$), relative to a suitable choice of basepoint; the vertical arrows are the surjections [determined up to composition with an inner automorphism] induced by the natural open immersions $Z_z \hookrightarrow Z$ and $Z_{z'} \hookrightarrow Z$ of hyperbolic curves.

Write

- $\overline{z}'$ for the closed point of $\mathcal{Z}$ determined by z';
- $\mathcal{Z}_z$ for the stable model of the pointed stable curve $(Z^{\mathrm{cpt}}, \phi^{-1}(S) \cup \{z\})$;
- $\mathcal{Z}_{z'}$ for the stable model of the pointed stable curve $(Z^{\mathrm{cpt}}, \phi^{-1}(S) \cup \{z'\})$;
- v_z^* (respectively, w_z^*) for the unique irreducible component of the special fiber of $\mathcal{Z}_z$ that maps surjectively [via the natural morphism $\mathcal{Z}_z \to \mathcal{Z}$] onto v_z (respectively, w_z);
- Γ for the dual semi-graph of the special fiber of $\mathcal{Z}$;
- Γ_z for the dual semi-graph of the special fiber of $\mathcal{Z}_z$;
- $\Gamma_{z'}$ for the dual semi-graph of the special fiber of $\mathcal{Z}_{z'}$.

Since, by [SemiAn], Corollary 3.11 [and its proof], the isomorphism $\Pi^{\mathrm{tp}}_{Z_z} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{Z_{z'}}$ induces an isomorphism between the dual semi-graphs of special fibers of the respective stable models, the preceding commutative diagram of tempered fundamental groups induces a commutative diagram of *"generalized morphisms"* of dual semi-graphs

$$\begin{array}{ccc} \Gamma_z & \xrightarrow{\sim} & \Gamma_{z'} \\ \downarrow & & \downarrow \\ \Gamma & = & \Gamma, \end{array}$$

where the term "generalized morphism" refers to a *functor* between the respective *categories* "Cat$(-)$" associated to the semi-graphs in the domain and codomain [cf. the discussion immediately preceding [SemiAn], Definition 2.11].

Write

- $v^*_{z'}$ (respectively, $w^*_{z'}$) for the irreducible component of the special fiber of $\mathcal{Z}_{z'}$ corresponding to v^*_z (respectively, w^*_z) via the isomorphism $\Gamma_z \xrightarrow{\sim} \Gamma_{z'}$;
- $v_{z'}$ (respectively, $w_{z'}$) for the irreducible component of the special fiber of $\mathcal{Z}$ obtained by mapping $v^*_{z'}$ (respectively, $w^*_{z'}$) via the generalized morphism $\Gamma_{z'} \to \Gamma$.

Then the commutativity of the above diagram of generalized morphisms of dual semi-graphs implies that $\{v_z, w_z\} = \{v_{z'}, w_{z'}\}$. On the other hand, it follows from the definitions of the various objects involved that $\overline{z} \in v_z \cap w_z = v_{z'} \cap w_{z'} \ni \overline{z}'$. Thus, [if, by a slight abuse of notation, we regard closed points as closed subschemes, then] we conclude that

$$\tilde{f}(\overline{z}') \subseteq \tilde{f}(v_{z'} \cap w_{z'}) = \tilde{f}(v_z \cap w_z) \subseteq \tilde{f}(v_z) = \tilde{f}(v),$$

hence that

$$\overline{y}' = f(\overline{z}') = h(\tilde{f}(\overline{z}')) = h(\tilde{f}(v)) = f(v) = \overline{y}.$$

However, this contradicts our assumption that $\overline{y} \neq \overline{y}'$. This completes the proof of Theorem 2.2. □

Our goal in this section is to prove the following corollaries of Theorem 2.2.

Corollary 2.3. $\mathrm{GT}_p^{\mathrm{tp}}$ *acts naturally on the set of algebraic numbers* $\overline{\mathbb{Q}}$.

Proof. Write $X \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$, where we think of "$\overline{\mathbb{Q}}$" as the subfield of $\mathbb{C}_p$ consisting of the elements algebraic over $\mathbb{Q}$. [Thus, we have a *natural embedding* $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.] In the following discussion, we shall identify $X(\overline{\mathbb{Q}})$ with $\overline{\mathbb{Q}} \setminus \{0, 1\}$. We take the "natural action" in the statement of Corollary 2.3 on $\{0, 1\} \subseteq \overline{\mathbb{Q}}$ to be the trivial action. Let $x \in X(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \setminus \{0, 1\}$; $\sigma \in \mathrm{GT}_p^{\mathrm{tp}}$; $\mathbb{B}$ a Belyi diagram

$$\begin{array}{ccc} \Pi_U & \longrightarrow & \Pi_X \\ \downarrow & & \\ \Pi_X & & \end{array}$$

such that $x \notin U(\overline{\mathbb{Q}})$, where we identify U with the image scheme of the open immersion $U \hookrightarrow X$. Thus, we obtain an element $x_{\mathbb{B}} \in D(\mathrm{GT})$ [cf. Definitions 1.4, 1.5; Corollary 1.6, (ii)]. Write $(x_{\mathbb{B}})^{\sigma} \in \overline{\mathbb{Q}}$ for the image of the composite

$$D(\mathrm{GT}) \xrightarrow{\sim} D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}},$$

where the first arrow denotes the bijection induced by σ [cf. Corollary 1.6, (ii), in the case where $J = \mathrm{GT}$]; the second arrow denotes the surjection in Corollary 1.7. Thus, to complete the proof of Corollary 2.3, it suffices to show that $(x_{\mathbb{B}})^{\sigma} = (x_{\mathbb{B}^{\dagger}})^{\sigma} \in \overline{\mathbb{Q}}$ for any Belyi diagram $\mathbb{B}^{\dagger}$

$$\begin{tikzcd} \Pi_{U'} \arrow[r] \arrow[d] & \Pi_X \\ \Pi_X & \end{tikzcd}$$

such that $x \notin U'(\overline{\mathbb{Q}})$, where we identify U' with the image scheme of the open immersion $U' \hookrightarrow X$. Write

- $X_x \overset{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, x, \infty\}$;
- $X_{(x_{\mathbb{B}})^{\sigma}} \overset{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, (x_{\mathbb{B}})^{\sigma}, \infty\}$;
- $X_{(x_{\mathbb{B}^{\dagger}})^{\sigma}} \overset{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, (x_{\mathbb{B}^{\dagger}})^{\sigma}, \infty\}$.

By recalling the [right-hand square in the final display of the] proof of Corollary 1.6, (i), in the case where $J = \mathrm{GT}$, we obtain a commutative diagram of outer homomorphisms

$$\begin{tikzcd} \Pi_{X_{(x_{\mathbb{B}})^{\sigma}}} \arrow[d] & \Pi_{X_x} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d] & \Pi_{X_{(x_{\mathbb{B}^{\dagger}})^{\sigma}}} \arrow[d] \\ \Pi_X & \Pi_X \arrow[l, "\sim", "\sigma"'] \arrow[r, "\sim", "\sigma"'] & \Pi_X, \end{tikzcd}$$

where the vertical arrows are the outer surjections induced by the natural open immersions $X_x \hookrightarrow X$, $X_{(x_{\mathbb{B}})^{\sigma}} \hookrightarrow X$, $X_{(x_{\mathbb{B}^{\dagger}})^{\sigma}} \hookrightarrow X$ of hyperbolic curves; the horizontal arrows are outer isomorphisms of topological groups. Since $\sigma \in \mathrm{GT}_p^{\mathrm{tp}}$, by recalling the [construction of the diagram in the final display of the] proof of Corollary 1.6, (i), in the case where $J = \mathrm{GT}$, we conclude that the above commutative diagram is induced by the following tempered version of the above commutative diagram

$$\begin{tikzcd} \Pi^{\mathrm{tp}}_{X_{(x_{\mathbb{B}})^{\sigma}}} \arrow[d] & \Pi^{\mathrm{tp}}_{X_x} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d] & \Pi^{\mathrm{tp}}_{X_{(x_{\mathbb{B}^{\dagger}})^{\sigma}}} \arrow[d] \\ \Pi^{\mathrm{tp}}_X & \Pi^{\mathrm{tp}}_X \arrow[l, "\sim", "\sigma"'] \arrow[r, "\sim", "\sigma"'] & \Pi^{\mathrm{tp}}_X, \end{tikzcd}$$

where Π^{tp}_X (respectively, $\Pi^{\mathrm{tp}}_{X_{(x_{\mathbb{B}})^\sigma}}$, $\Pi^{\mathrm{tp}}_{X_{(x_{\mathbb{B}^\dagger})^\sigma}}$) denotes the tempered fundamental group of the base extension of X_x (respectively, $X_{(x_{\mathbb{B}})^\sigma}$, $X_{(x_{\mathbb{B}^\dagger})^\sigma}$) by the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$; the vertical arrows are the outer surjections induced by the natural open immersions $X_x \hookrightarrow X$, $X_{(x_{\mathbb{B}})^\sigma} \hookrightarrow X$, $X_{(x_{\mathbb{B}^\dagger})^\sigma} \hookrightarrow X$ of hyperbolic curves; the horizontal arrows are outer isomorphisms of topological groups. Note, moreover, that it follows from the surjectivity of the vertical arrows in the diagram of the preceding display that the inner automorphism indeterminacies in this diagram may be eliminated in a consistent fashion. Thus, by applying Theorem 2.2 [in the case where "ϕ" is taken to be the identity morphism], we conclude that $(x_{\mathbb{B}})^\sigma = (x_{\mathbb{B}^\dagger})^\sigma \in \overline{\mathbb{Q}}$. This completes the proof of Corollary 2.3. $\square$

Corollary 2.4. *There exists a surjection* $\mathrm{GT}^{\mathrm{tp}}_p \twoheadrightarrow G_{\mathbb{Q}_p}$ *whose restriction to* $G_{\mathbb{Q}_p}$ *[cf. Remark 2.1.2] is the identity automorphism.*

Proof. We continue to use the notation $X = \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ of the proof of Corollary 2.3. Write $Y \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}}$. [Thus, $X \subseteq Y$ is an open subscheme of Y.] It suffices to show that the action of $\mathrm{GT}^{\mathrm{tp}}_p$ on the set $\overline{\mathbb{Q}}$ ($\subseteq \overline{\mathbb{Q}} \cup \{\infty\} = Y(\overline{\mathbb{Q}})$) [cf. Corollary 2.3] is compatible with the field structure of $\overline{\mathbb{Q}}$ and the p-adic topology of $\overline{\mathbb{Q}}$ induced by the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Fix $\sigma \in \mathrm{GT}^{\mathrm{tp}}_p \subseteq \mathrm{GT}$.

First, we verify the compatibility with the field structure of $\overline{\mathbb{Q}}$. We begin by verifying the following assertion:

> Claim 2.4.A: The action of $\mathrm{GT}^{\mathrm{tp}}_p$ on the set $Y(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \cup \{\infty\}$ induced by the action of $\mathrm{GT}^{\mathrm{tp}}_p$ on the set $\overline{\mathbb{Q}}$ commutes with the natural action of $\mathrm{Aut}_{\overline{\mathbb{Q}}}(X)$ [i.e., the group of scheme-theoretic automorphisms of X over $\overline{\mathbb{Q}}$] on the set $Y(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \cup \{\infty\}$.

Recall that every element of $\mathrm{GT}^{\mathrm{tp}}_p$ commutes with the outomorphisms of Π_X induced by elements of $\mathrm{Aut}_{\overline{\mathbb{Q}}}(X)$ [cf. [CmbCsp], Definition 1.11, (i); [CmbCsp], Remark 1.11.1]. Thus, Claim 2.4.A follows immediately from the definition of the action of $\mathrm{GT}^{\mathrm{tp}}_p$ on $\overline{\mathbb{Q}}$ in the proof of Corollary 2.3 via the action discussed in the proof of Corollary 1.6, (i), (ii) [cf., especially, the right-hand vertical isomorphism in the final display of the proof of Corollary 1.6, (i)].

Next, we verify the following assertion:

> Claim 2.4.B: Suppose that
>
> (∗) the action of $\mathrm{GT}^{\mathrm{tp}}_p$ on the set $\overline{\mathbb{Q}}^\times \stackrel{\mathrm{def}}{=} \overline{\mathbb{Q}} \backslash \{0\}$ is compatible with the multiplicative group structure of $\overline{\mathbb{Q}}^\times$.
>
> Then the action of $\mathrm{GT}^{\mathrm{tp}}_p$ on the set $\overline{\mathbb{Q}}$ is compatible with the field structure of $\overline{\mathbb{Q}}$.

Indeed, suppose that $(*)$ holds. Since $-1 \in \overline{\mathbb{Q}}$ may be characterized as the unique element $x \in \overline{\mathbb{Q}} \setminus \{1\}$ such that $x^2 = 1$, we conclude that σ preserves $-1 \in \overline{\mathbb{Q}}$. Let $a, b \in \overline{\mathbb{Q}}^{\times}$. Then $a + b = a \cdot (1 - ((-1) \cdot a^{-1} \cdot b))$. Since the action of σ commutes with the action of the automorphism of X over $\overline{\mathbb{Q}}$ given [relative to the standard coordinate "t" on $Y = \mathbb{P}^1_{\overline{\mathbb{Q}}}$] by $t \mapsto 1 - t$ [cf. Claim 2.4.A], we obtain the desired conclusion. This completes the proof of Claim 2.4.B.

Thus, by Claim 2.4.B, it suffices to show that $(*)$ holds. Let $x, y \in \overline{\mathbb{Q}}^{\times} \setminus \{1\}$; $\mathbb{B}^{\rtimes}$ an arithmetic Belyi diagram [in the case where N is a normal open subgroup of $J = \mathrm{GT}$]

$$\begin{array}{ccc} \Pi_U \overset{\text{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} N \\ \big\downarrow & & \\ \Pi_X \overset{\text{out}}{\rtimes} N & & \end{array}$$

such that $x^{-1}, y \notin U(\overline{\mathbb{Q}})$, where we identify U with the image scheme of the open immersion $U \hookrightarrow X$. Write

$$U_x \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, x, \infty\} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, x, \infty\}$$

for the image scheme of the composite of the open immersion $U \hookrightarrow X$ with the isomorphism $X \overset{\sim}{\to} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, x, \infty\}$ induced by multiplication by x. Thus, we obtain an arithmetic Belyi diagram $\mathbb{B}^{\rtimes}_x$

$$\begin{array}{ccc} \Pi_{U_x} \overset{\text{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\text{out}}{\rtimes} N \\ \big\downarrow & & \\ \Pi_X \overset{\text{out}}{\rtimes} N, & & \end{array}$$

where the horizontal arrow $\Pi_{U_x} \overset{\text{out}}{\rtimes} N \to \Pi_X \overset{\text{out}}{\rtimes} N$ denotes the Π_X-outer homomorphism induced by the composite of inclusions

$$U_x \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, x, \infty\} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\} = X;$$

the vertical arrow $\Pi_{U_x} \overset{\text{out}}{\rtimes} N \to \Pi_X \overset{\text{out}}{\rtimes} N$ denotes the composite of the vertical arrow

$$\Pi_U \overset{\text{out}}{\rtimes} N \to \Pi_X \overset{\text{out}}{\rtimes} N$$

in the arithmetic Belyi diagram $\mathbb{B}^{\rtimes}$ with an isomorphism

$$\mu_{x^{-1}} : \Pi_{U_x} \overset{\text{out}}{\rtimes} N \overset{\sim}{\to} \Pi_U \overset{\text{out}}{\rtimes} N$$

over N induced by the natural scheme-theoretic isomorphism $U_x \overset{\sim}{\to} U$.

Next, by recalling the right-hand square in the final display of the proof of Corollary 1.6, (i), in the case where $N = M \subseteq J = \mathrm{GT}$, we obtain commutative diagrams of outer homomorphisms of profinite groups

$$\begin{array}{ccc} \Pi_U \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\ {\scriptstyle\sigma}\downarrow{\scriptstyle\wr} & & {\scriptstyle\sigma}\downarrow{\scriptstyle\wr} \\ \Pi_{U^\sigma} \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N, \end{array}$$

$$\begin{array}{ccc} \Pi_{U_x} \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N \\ {\scriptstyle\sigma}\downarrow{\scriptstyle\wr} & & {\scriptstyle\sigma}\downarrow{\scriptstyle\wr} \\ \Pi_{(U_x)^\sigma} \overset{\mathrm{out}}{\rtimes} N & \longrightarrow & \Pi_X \overset{\mathrm{out}}{\rtimes} N. \end{array}$$

Write

$$(U_x)^\sigma_{(x^\sigma)^{-1}} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, (x^\sigma)^{-1}, \infty\} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, (x^\sigma)^{-1}, \infty\}$$

for the image scheme of the composite of the open immersion $(U_x)^\sigma \hookrightarrow X$ [cf. the proof of Corollary 1.6, (i)] with the isomorphism $X \xrightarrow{\sim} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, (x^\sigma)^{-1}, \infty\}$ induced by multiplication by $(x^\sigma)^{-1}$. Note that there exists a natural $\Pi_{(U_x)^\sigma}$-outer isomorphism

$$\mu_{x^\sigma} : \Pi_{(U_x)^\sigma_{(x^\sigma)^{-1}}} \overset{\mathrm{out}}{\rtimes} N \xrightarrow{\sim} \Pi_{(U_x)^\sigma} \overset{\mathrm{out}}{\rtimes} N$$

over N induced by the natural scheme-theoretic isomorphism $(U_x)^\sigma_{(x^\sigma)^{-1}} \xrightarrow{\sim} (U_x)^\sigma$.

Thus, by taking the *composite of the* $\Pi_{(-)}$*-outer isomorphisms*

- $\mu_{x^\sigma} : \Pi_{(U_x)^\sigma_{(x^\sigma)^{-1}}} \overset{\mathrm{out}}{\rtimes} N \xrightarrow{\sim} \Pi_{(U_x)^\sigma} \overset{\mathrm{out}}{\rtimes} N$,

- the inverse of $\Pi_{U_x} \overset{\mathrm{out}}{\rtimes} N \xrightarrow{\sim} \Pi_{(U_x)^\sigma} \overset{\mathrm{out}}{\rtimes} N$ [cf. the second of the above two commutative diagrams],

- $\mu_{x^{-1}} : \Pi_{U_x} \overset{\mathrm{out}}{\rtimes} N \xrightarrow{\sim} \Pi_U \overset{\mathrm{out}}{\rtimes} N$, and

- $\Pi_U \overset{\mathrm{out}}{\rtimes} N \xrightarrow{\sim} \Pi_{U^\sigma} \overset{\mathrm{out}}{\rtimes} N$ [cf. the first of the above two commutative diagrams],

we obtain a Π_{U^σ}-outer isomorphism

$$\Pi_{(U_x)^\sigma_{(x^\sigma)^{-1}}} \overset{\mathrm{out}}{\rtimes} N \xrightarrow{\sim} \Pi_{U^\sigma} \overset{\mathrm{out}}{\rtimes} N$$

over N. Note that the conjugacy class of cuspidal inertia subgroups of $\Pi_{(U_x)^\sigma_{(x^\sigma)^{-1}}}$ associated to

$$0 \text{ (respectively, } 1,\ (x^\sigma)^{-1},\ (x^\sigma)^{-1}(xy)^\sigma,\ \infty)$$

maps, via the above *composite of* $\Pi_{(-)}$*-outer isomorphisms*, to the conjugacy classes of cuspidal inertia subgroups of $\Pi_{(-)}$ given as follows:

$$\begin{aligned} &\rightsquigarrow \quad 0 \text{ (respectively, } x^\sigma,\ 1,\ (xy)^\sigma,\ \infty) \\ &\rightsquigarrow \quad 0 \text{ (respectively, } x,\ 1,\ xy,\ \infty) \\ &\rightsquigarrow \quad 0 \text{ (respectively, } 1,\ x^{-1},\ y,\ \infty) \\ &\rightsquigarrow \quad 0 \text{ (respectively, } 1,\ (x^{-1})^\sigma,\ y^\sigma,\ \infty). \end{aligned}$$

Thus, by restricting to $G_{\mathbb{Q}} \subseteq \mathrm{GT} = J$ [cf. Corollary 1.7], we conclude that

$$(x^\sigma)^{-1}(xy)^\sigma = y^\sigma \ (\Leftrightarrow (xy)^\sigma = x^\sigma y^\sigma).$$

This completes the proof of $(*)$ and hence of the compatibility of the action of σ with the field structure of $\overline{\mathbb{Q}}$.

Next, we verify the compatibility with the p-adic topology of $\overline{\mathbb{Q}}$. Write

- X_x (respectively, X_{x^σ}) for $\mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, 1, x, \infty\}$ (respectively, $\mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, 1, x^\sigma, \infty\}$);
- $\Pi^{\mathrm{tp}}_{X_x}$ (respectively, $\Pi^{\mathrm{tp}}_{X_{x^\sigma}}$) for the tempered fundamental group of X_x (respectively, X_{x^σ}), relative to a suitable choice of basepoint;
- Γ_x (respectively, Γ_{x^σ}) for the dual semi-graph of the special fiber of the stable model of X_x (respectively, X_{x^σ});
- $V_x(y)$ (respectively, $V_{x^\sigma}(y)$) for the vertex of Γ_x (respectively, Γ_{x^σ}) to which the open edge determined by a cusp y of X_x (respectively, X_{x^σ}) abuts;
- $v_p : \overline{\mathbb{Q}}^\times \to \mathbb{Q}$ for the p-adic valuation normalized so that $v_p(p) = 1$.

Recall [cf. the upper horizontal isomorphisms in the final display of the proof of Corollary 2.3] that there exists an isomorphism of topological groups

$$\Pi^{\mathrm{tp}}_{X_x} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{X_{x^\sigma}}$$

such that the conjugacy class of cuspidal inertia subgroups associated to 0 (respectively, 1, x, ∞) maps to the conjugacy class of cuspidal inertia subgroups associated to 0 (respectively, 1, x^σ, ∞). Thus, by applying [SemiAn], Corollary 3.11, we conclude that the isomorphism of topological groups of the above display induces an isomorphism of semi-graphs $\Gamma_x \xrightarrow{\sim} \Gamma_{x^\sigma}$, and hence that

$$\begin{aligned} v_p(x) > 0 &\Leftrightarrow V_x(x) = V_x(0) \neq V_x(1) \\ &\Leftrightarrow V_{x^\sigma}(x^\sigma) = V_{x^\sigma}(0) \neq V_{x^\sigma}(1) \\ &\Leftrightarrow v_p(x^\sigma) > 0. \end{aligned}$$

This completes the proof of the compatibility of the action of σ with the p-adic topology of $\overline{\mathbb{Q}}$ and hence of Corollary 2.4. $\square$

3 Construction of an action of $C_{\mathrm{GT}}(G_{\mathbb{Q}^{\mathrm{ab}}})$ on the field $\overline{\mathbb{Q}}$

Write $\mathbb{Q}^{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ [cf. Notations and Conventions] for the maximal abelian extension field of $\mathbb{Q}$, i.e., the subfield generated by the roots of unity $\in \overline{\mathbb{Q}}$. In this section, we construct [cf. Corollary 3.4] a natural action of $C_{\mathrm{GT}}(G_{\mathbb{Q}^{\mathrm{ab}}})$ [cf. Notations and Conventions] on the field of algebraic numbers. This construction is obtained as a consequence of the injectivity portion of the Section Conjecture for abelian varieties over the finite extensions of $\mathbb{Q}^{\mathrm{ab}}$ [cf. Theorem 3.1].

Theorem 3.1. *Let $K \subseteq \overline{\mathbb{Q}}$ be a number field, i.e., a finite extension of $\mathbb{Q}$; A an abelian variety over K. Write $K^{\mathrm{cycl}} = K \cdot \mathbb{Q}^{\mathrm{ab}}$; $A(K^{\mathrm{cycl}})$ for the group of K^{cycl}-valued points of A; $A_{K^{\mathrm{cycl}}}$ (respectively, $A_{\overline{\mathbb{Q}}}$) for the base extension of A to K^{cycl} (respectively, $\overline{\mathbb{Q}}$). Then the natural map*

$$A(K^{\mathrm{cycl}}) \to H^1(G_{K^{\mathrm{cycl}}}, \Pi_{A_{\overline{\mathbb{Q}}}})$$

— i.e., obtained by taking the difference between the two sections [each of which is well-defined up to composition with an inner automorphism induced by an element of $\Pi_{A_{\overline{\mathbb{Q}}}}$] of $\Pi_{A_{K^{\mathrm{cycl}}}} \twoheadrightarrow G_{K^{\mathrm{cycl}}}$ induced by an element of $A(K^{\mathrm{cycl}})$ and the origin — is injective.

Proof. By considering the Kummer exact sequence for $A(K^{\mathrm{cycl}})$, we obtain natural maps

$$A(K^{\mathrm{cycl}}) \to \varprojlim_{n} A(K^{\mathrm{cycl}})/A[n](K^{\mathrm{cycl}}) \hookrightarrow H^1(G_{K^{\mathrm{cycl}}}, \Pi_{A_{\overline{\mathbb{Q}}}}),$$

where $A[n](K^{\mathrm{cycl}})$ denotes the group of n-torsion points of $A(K^{\mathrm{cycl}})$; the first map is the natural homomorphism; the second map is injective; the inverse limit is indexed by the positive integers, regarded multiplicatively. By a well-known general nonsense argument [cf., e.g., the proof of [Cusp], Proposition 2.2, (i)], it follows that the composite map of the above display coincides with the natural map in the statement of Theorem 3.1. Thus, it suffices to show that $A(K^{\mathrm{cycl}})$ has no divisible elements. But this follows immediately from [KLR], Appendix, Theorem 1, and [Moon], Proposition 7. This completes the proof of Theorem 3.1. $\square$

Corollary 3.2. *Let $K \subseteq \overline{\mathbb{Q}}$ be a number field, i.e., a finite extension of $\mathbb{Q}$; Y a hyperbolic curve over K. Write $K^{\mathrm{cycl}} = K \cdot \mathbb{Q}^{\mathrm{ab}}$; $Y(K^{\mathrm{cycl}})$ for the set of K^{cycl}-valued points of Y; $Y_{\overline{\mathbb{Q}}}$ for the base extension of Y to $\overline{\mathbb{Q}}$. Suppose that $Y(K^{\mathrm{cycl}}) \neq \emptyset$. Fix a K^{cycl}-valued point $y \in Y(K^{\mathrm{cycl}})$. Then the natural map*

$$Y(K^{\mathrm{cycl}}) \to H^1(G_{K^{\mathrm{cycl}}}, \Pi_{Y_{\overline{\mathbb{Q}}}})$$

— *i.e., obtained by taking the difference between the two sections [each of which is well-defined up to composition with an inner automorphism induced by an element of* $\Pi_{Y_{\overline{\mathbb{Q}}}}$*] of* $\Pi_{Y_{K^{\mathrm{cycl}}}} \twoheadrightarrow G_{K^{\mathrm{cycl}}}$ *induced by an element of* $Y(K^{\mathrm{cycl}})$ *and* $y \in Y(K^{\mathrm{cycl}})$ — *is injective.*

Proof. One verifies immediately that, by replacing Y by a suitable finite étale covering of Y, we may assume without loss of generality Y is of genus ≥ 1. Then the desired injectivity follows immediately from Theorem 3.1 by considering the Albanese embedding of Y. $\square$

Corollary 3.3. $C_{\mathrm{GT}}(G_{\mathbb{Q}^{\mathrm{ab}}})$ *acts naturally on the set of algebraic numbers* $\overline{\mathbb{Q}}$.

Proof. In the following discussion, we shall identify $X(\overline{\mathbb{Q}})$ with $\overline{\mathbb{Q}} \setminus \{0,1\}$. We take the "natural action" in the statement of Corollary 3.3 on $\{0,1\} \subseteq \overline{\mathbb{Q}}$ to be the trivial action. Let $x \in X(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \setminus \{0,1\}$; $\sigma \in C_{\mathrm{GT}}(G_{\mathbb{Q}^{\mathrm{ab}}})$; $\mathbb{B}$ a Belyi diagram

$$\begin{array}{ccc} \Pi_U & \longrightarrow & \Pi_X \\ \big\downarrow & & \\ \Pi_X & & \end{array}$$

such that $x \notin U(\overline{\mathbb{Q}})$, where we identify U with the image scheme of the open immersion $U \hookrightarrow X$. Thus, we obtain an element $x_{\mathbb{B}} \in D(\mathrm{GT})$ [cf. Definitions 1.4, 1.5; Corollary 1.6, (ii)]. Write $(x_{\mathbb{B}})^{\sigma} \in \overline{\mathbb{Q}}$ for the image of the composite

$$D(\mathrm{GT}) \xrightarrow{\sim} D(\mathrm{GT}) \twoheadrightarrow \overline{\mathbb{Q}},$$

where the first arrow denotes the bijection induced by σ [cf. Corollary 1.6, (ii), in the case where $J = \mathrm{GT}$]; the second arrow denotes the surjection in Corollary 1.7. Thus, to complete the proof of Corollary 3.3, it suffices to show that $(x_{\mathbb{B}})^{\sigma} = (x_{\mathbb{B}^{\dagger}})^{\sigma} \in \overline{\mathbb{Q}}$ for any Belyi diagram $\mathbb{B}^{\dagger}$

$$\begin{array}{ccc} \Pi_{U'} & \longrightarrow & \Pi_X \\ \big\downarrow & & \\ \Pi_X & & \end{array}$$

such that $x \notin U'(\overline{\mathbb{Q}})$, where we identify U' with the image scheme of the open immersion $U' \hookrightarrow X$. For any finite extension $L \subseteq \overline{\mathbb{Q}}$ of $\mathbb{Q}^{\mathrm{ab}}$, write $G_L \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/L) \subseteq G_{\mathbb{Q}^{\mathrm{ab}}}$. Since $\sigma \in C_{\mathrm{GT}}(G_{\mathbb{Q}^{\mathrm{ab}}})$, there exists a finite extension $K \subseteq \overline{\mathbb{Q}}$ of $\mathbb{Q}^{\mathrm{ab}}(x)$ such that we have inclusions

$$\sigma G_K \sigma^{-1} \subseteq G_{\mathbb{Q}^{\mathrm{ab}}((x_{\mathbb{B}})^{\sigma})} \cap G_{\mathbb{Q}^{\mathrm{ab}}((x_{\mathbb{B}^{\dagger}})^{\sigma})} \subseteq G_{\mathbb{Q}^{\mathrm{ab}}}$$

of open subgroups of $G_{\mathbb{Q}^{\mathrm{ab}}}$. Fix such a finite extension K.

Write

- $K^\sigma \subseteq \overline{\mathbb{Q}}$ for the finite extension of $\mathbb{Q}^{\mathrm{ab}}$ such that $G_{K^\sigma} = \sigma G_K \sigma^{-1} \subseteq G_{\mathbb{Q}^{\mathrm{ab}}}$;
- $X_x \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, x, \infty\}$;
- $X_{(x_{\mathbb{B}})^\sigma} \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, (x_{\mathbb{B}})^\sigma, \infty\}$;
- $X_{(x_{\mathbb{B}^\dagger})^\sigma} \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, (x_{\mathbb{B}^\dagger})^\sigma, \infty\}$.

Thus, it follows from our choice of K that $x \in K$ and $(x_{\mathbb{B}})^\sigma, (x_{\mathbb{B}^\dagger})^\sigma \in K^\sigma$.

By recalling the [right-hand square in the final display of the] proof of Corollary 1.6, (i), in the case where $J = \mathrm{GT}$, and possibly replacing K by a finite extension of K if necessary, we obtain a commutative diagram of outer homomorphisms

$$\begin{tikzcd}
\Pi_{X_{(x_{\mathbb{B}})^\sigma}} \overset{\mathrm{out}}{\rtimes} G_{K^\sigma} \arrow[d] & \Pi_{X_x} \overset{\mathrm{out}}{\rtimes} G_K \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d] & \Pi_{X_{(x_{\mathbb{B}^\dagger})^\sigma}} \overset{\mathrm{out}}{\rtimes} G_{K^\sigma} \arrow[d] \\
\Pi_X \overset{\mathrm{out}}{\rtimes} G_{K^\sigma} & \Pi_X \overset{\mathrm{out}}{\rtimes} G_K \arrow[l, "\sim", "\sigma"'] \arrow[r, "\sim", "\sigma"'] & \Pi_X \overset{\mathrm{out}}{\rtimes} G_{K^\sigma},
\end{tikzcd}$$

where the vertical arrows are the Π_X-outer surjections induced by the natural open immersions $X_x \hookrightarrow X$, $X_{(x_{\mathbb{B}})^\sigma} \hookrightarrow X$, $X_{(x_{\mathbb{B}^\dagger})^\sigma} \hookrightarrow X$ of hyperbolic curves; the horizontal arrows are outer isomorphisms of topological groups.

Thus, by applying Corollary 3.2 in the case where $Y = \mathbb{P}^1_{K^\sigma} \backslash \{0, 1, \infty\}$, we conclude that $(x_{\mathbb{B}})^\sigma = (x_{\mathbb{B}^\dagger})^\sigma \in \overline{\mathbb{Q}}$. This completes the proof Corollary 3.3. $\square$

Corollary 3.4. *There exists a surjection $C_{\mathrm{GT}}(G_{\mathbb{Q}^{\mathrm{ab}}}) \twoheadrightarrow G_{\mathbb{Q}}$ whose restriction to $G_{\mathbb{Q}}$ is the identity automorphism.*

Proof. It suffices to show that the natural action of $C_{\mathrm{GT}}(G_{\mathbb{Q}^{\mathrm{ab}}})$ on the set $\overline{\mathbb{Q}}$ [cf. Corollary 3.3] is compatible with the field structure of $\overline{\mathbb{Q}}$. This compatibility with the field structure follows from a similar argument to the argument given in the proof of Corollary 2.4. This completes the proof of Corollary 3.4. $\square$

Acknowledgements

The author would like to thank Professor Yuichiro Hoshi and Professor Shinichi Mochizuki for suggesting the topic [especially, Theorem 1.3], and for many helpful discussions, as well as for their warm encouragement. Moreover, the author deeply appreciates Professor Shinichi Mochizuki for taking an enormous amount of time to fix his master's thesis and the present paper. The author also deeply appreciates Professor Yuichiro Hoshi for answering his questions concerning general algebraic geometry and anabelian geometry in detail. Finally, the author also would like to thank Yu Yang and Arata Minamide for many stimulating discussions concerning various topics in anabelian geometry.

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