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Maximum generalized Hasse-Witt invariants and an anabelian formula for topological types of pointed stable curves in positive characteristic

By

Yu YANG

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Maximum Generalized Hasse-Witt Invariants and An Anabelian Formula for Topological Types of Pointed Stable Curves in Positive Characteristic

Yu Yang

Abstract

In the present paper, we study the generalized Hasse-Witt invariants of cyclic coverings of curves in positive characteristic. Let $X^\bullet = (X, D_X)$ be a pointed stable curve of topological type (g_X, n_X) over an algebraically closed field of characteristic $p > 0$. We prove that, if $X^\bullet$ is component-generic, then the first generalized Hasse-Witt invariant of each prime-to-p cyclic admissible coverings of $X^\bullet$ attains the maximum under certain assumptions. This result generalizes a result of S. Nakajima concerning the ordinariness of prime-to-p cyclic étale coverings of smooth projective generic curves to the case of (possibly ramified) admissible coverings of (possibly singular) pointed stable curves. Moreover, without any assumptions, we prove that there exists a prime-to-p cyclic admissible covering of $X^\bullet$ such that the first generalized Hasse-Witt invariant of the cyclic admissible covering attains the maximum. This result can be regarded as an analogue of a result of M. Raynaud concerning the new-ordinariness of prime-to-p cyclic étale coverings of arbitrary smooth projective curves in the case of generalized Hasse-Witt invariants of prime-to-p cyclic admissible coverings of arbitrary pointed stable curves. As an application, we obtain a group-theoretical formula for (g_X, n_X). This formula generalizes a result of A. Tamagawa concerning a group-theoretical formula for topological types of smooth pointed stable curves to the case of arbitrary pointed stable curves.

Keywords: pointed stable curve, admissible covering, admissible fundamental group, generalized Hasse-Witt invariant, Raynaud-Tamagawa theta divisor, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14G32.

Contents

1 Introduction

In the present paper, we study the generalized Hasse-Witt invariants of coverings of curves in positive characteristic. Let

$$X^\bullet = (X, D)$$

be a pointed stable curve of topological type or type, for short, (g_X, n_X) over an algebraically closed field k, where X denotes the underlying curve, D_X denotes the set of marked points, g_X denotes the genus of X, and n_X denotes the cardinality $\#D_X$ of D_X. Moreover, by choosing a suitable base point of $X^\bullet$, we have the admissible fundamental group

$$\Pi_{X^\bullet}$$

of $X^\bullet$ (cf. Definition 2.2). In particular, if $X^\bullet$ is smooth over k, then $\Pi_{X^\bullet}$ is naturally (outer) isomorphic to the tame fundamental group $\pi_1^{\mathrm{t}}(X \setminus D_X)$.

Suppose that the characteristic char(k) of k is 0. Then the structure of $\Pi_{X^\bullet}$ is well-known, which is isomorphic to the profinite completion of the following free group (cf. [V, Théorème 2.2 (c)])

$$\langle a_1, \ldots, a_{g_X}, b_1, \ldots, b_{g_X}, c_1, \ldots, c_{n_X} \mid \prod_{i=1}^{g_X} [a_i, b_i] \prod_{j=1}^{n_X} c_j = 1 \rangle.$$

In particular, $\Pi_{X^\bullet}$ is a free profinite group with $2g_X + n_X - 1$ generators if $n_X > 0$. Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Suppose that $n_{X_i} > 0$, $i \in \{1, 2\}$. Then we see that $\Pi_{X_1^\bullet} \cong \Pi_{X_2^\bullet}$ if and only if $2g_{X_1} + n_{X_1} - 1 = 2g_{X_2} + n_{X_2} - 1$. Thus, we see that (g_X, n_X) cannot be determined group-theoretically from the isomorphism class of $\Pi_{X^\bullet}$.

On the other hand, when char$(k) = p > 0$, the situation is quite different from that in characteristic 0, and the structure of $\Pi_{X^\bullet}$ is no longer known. In the remainder of the introduction, we assume that char$(k) = p > 0$. The admissible fundamental group $\Pi_{X^\bullet}$ is very mysterious. In fact, some developments of F. Pop-M. Saïdi, M. Raynaud, A. Tamagawa, and the author (cf. [PS], [R2], [T1], [T2], [T3], [Y1], [Y2]) showed evidence for very strong *anabelian* phenomena for curves over *algebraically closed fields of characteristic* $p > 0$. In this situation, the Galois group of the base field is trivial, and the étale (or tame) fundamental group coincides with the geometric fundamental group, thus in a total absence of a Galois action of the base field. This kind of anabelian phenomena go beyond Grothendieck's anabelian geometry, and shows that the admissible (or tame) fundamental group of a smooth pointed stable curve over an algebraically closed field must encode"*moduli*" of the curve. This is the reason that we do not have an explicit description of the admissible (or tame) fundamental group of any pointed stable curve in positive characteristic. Note that since all the admissible coverings (cf. Definition 2.2) in positive characteristic can be lifted to characteristic 0 (cf. [V, Théorème 2.2 (c)]), we obtain that $\Pi_{X^\bullet}$ is topologically finitely generated. Then the isomorphism class of $\Pi_{X^\bullet}$ is determined by the set of finite quotients of $\Pi_{X^\bullet}$ (cf. [FJ, Proposition 16.10.6]).

Furthermore, the theory developed in [T2] and [Y2] implies that the isomorphism class of $X^\bullet$ as a scheme can possibly be determined by not only the isomorphism class of $\Pi_{X^\bullet}$ as a profinite group but also the isomorphism class of the maximal pro-solvable quotient of $\Pi_{X^\bullet}$. Then we may ask the following question:

Which finite solvable group can appear as a quotient of $\Pi_{X^\bullet}$*?*

Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open normal subgroup and $X_H^\bullet = (X_H, D_{X_H})$ the pointed stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H. We have an important invariant associated to $X_H^\bullet$ (or H) called *p-rank* (or *Hasse-Witt invariant*, see Definition 2.3). Roughly speaking, $\sigma_{X_H^\bullet}$ controls the finite quotients of $\Pi_{X^\bullet}$ which are extensions of the group $\Pi_{X^\bullet}/H$ by p-groups. Since the structures of maximal prime-to-p quotients of admissible fundamental groups have been known, in order to solve the question mentioned above, we need compute the p-rank $\sigma_{X_H^\bullet}$ when $\Pi_{X^\bullet}/H$ is abelian. If $\Pi_{X^\bullet}/H$ is a p-group, then $\sigma_{X_H^\bullet}$ can be computed by applying the Deuring-Shafarevich formula (cf. [C]). If $\Pi_{X^\bullet}/H$ is not a p-group, the situation of $\sigma_{X_H^\bullet}$ is very complicated, and the Deuring-Shafarevich formula implies that, to compute $\sigma_{X_H^\bullet}$, we only need to assume that $\Pi_{X^\bullet}/H$ is a prime-to-p group.

First, let us consider the case of generic curves. Suppose that $n_X = 0$, and that $X^\bullet$ is *smooth* over k. If $X^\bullet$ is a curve corresponding to a geometric generic point of moduli space (i.e., a geometric generic curve), S. Nakajima (cf. [N]) proved that, if $\Pi_{X^\bullet}/H$ is a cyclic group with a prime-to-p order, then $\sigma(X_H^\bullet)$ attains the maximum g_{X_H} (i.e., $X_H^\bullet$ is *ordinary*). Moreover, B. Zhang (cf. [Z]) extended Nakajima's result to the case where $\Pi_{X^\bullet}/H$ is an arbitrary abelian group. Recently, E. Ozman and R. Pries (cf. [OP]) generalized Nakajima's result to the case where $\Pi_{X^\bullet}/H$ is a cyclic group with a prime order distinct from p, and where $X^\bullet$ is a curve corresponding to a geometric point of p-rank stratas of moduli space. Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p. In other words, the results of Nakajima, Zhang, and Ozman-Pries say that, for *each* Galois étale covering of $X^\bullet$ with Galois group $\mathbb{Z}/m\mathbb{Z}$, the *generalized Hasse-Witt invariants* (cf. [N]) associated to non-trivial characters of $\mathbb{Z}/m\mathbb{Z}$ attain the maximum $g_X - 1$ except for the eigenspaces associated with eigenvalue 1. The first main result of the present paper is as follows (see Theorem 3.11 for more precisely):

Theorem 1.1. *Let $X^\bullet$ be a component-generic pointed stable curve (cf. Section 2.1 for the definition). Then the "first" generalized Hasse-Witt invariant (cf. Section 2.2 for the definition) of each prime-to-p cyclic admissible covering of $X^\bullet$ attains the maximum under certain assumptions.*

If $n_X = 0$ and $X^\bullet$ is smooth over k, then Theorem 1.1 is equivalent to [N, Proposition 4]. Thus, Theorem 1.1 generalizes [N, Proposition 4] to the case of (possibly ramified) admissible coverings of (possibly singular) pointed stable curves. Moreover, by applying this result, we generalize [N, Theorem 2] to the case of tame coverings (cf. Corollary 3.13).

Next, let us consider the general case. If $X^\bullet$ is not geometric generic, $\sigma_{X^\bullet_H}$ cannot be computed explicitly in general. Suppose that $X^\bullet$ is *smooth* over k, and that $n_X = 0$. Raynaud (cf. [R1]) developed his theory of theta divisors and proved that, if $\ell >> 0$ is a prime number distinct from p, then there *exists* a Galois étale coverings of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the generalized Hasse-Witt invariants associated to non-trivial characters of $\mathbb{Z}/\ell\mathbb{Z}$ attain the maximum $g_X - 1$ except for the eigenspaces associated with eigenvalue 1 (i.e., the étale covering is *new-ordinary*). Moreover, as a consequence, Raynaud obtained that $\Pi_{X^\bullet}$ is not a prime-to-p profinite group. This is the first deep result concerning the global structures of étale fundamental groups of projective curves over algebraically closed fields of characteristic $p > 0$.

Suppose that $X^\bullet$ is *smooth* over k, and that $n_X \geq 0$. The computations of generalized Hasse-invariants of admissible coverings of $X^\bullet$ (i.e., tame coverings of $X \setminus D_X$) are much more difficult than the case where $n_X = 0$. In the remainder of the introduction, let t be an arbitrary positive natural number and $n \stackrel{\text{def}}{=} p^t - 1$. Tamagawa observed that Raynaud's theory of theta divisors can be generalized to the case of tame coverings, and established a tamely ramified version of the theory of Raynaud's theta divisors. By applying the theory of theta divisors, under certain assumptions, Tamagawa proved that, if $n >> 0$ and $n_X > 1$, then the "first" generalized Hasse-Witt invariants of *almost* all of the Galois admissible coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ are equal to g_X. Furthermore, he introduced a kind of group-theoretical invariant $\mathrm{Avr}_p(\Pi_{X^\bullet})$ associated to $\Pi_{X^\bullet}$ (i.e., depends only on the isomorphism class of $\Pi_{X^\bullet}$) called the limit of p-averages (cf. Remark 5.4.1), and proved a highly non-trivial result as follows (cf. [T1, Theorem 0.5]):

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X, & \text{if } n_X \geq 2. \end{cases}$$

By applying the formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$, the following *group-theoretical* formula for (g_X, n_X) was essentially obtained by Tamagawa. In particular, we obtain that g_X and n_X are group-theoretical invariants associated to $\Pi_{X^\bullet}$. This result is the main goal of the theory developed in [T2] (cf. [T2, Theorem 0.1]).

Theorem 1.2. *Let Π be an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups. Suppose that $X^\bullet$ is smooth over k. Then we have (see Section 5 for the definitions of group-theoretical invariants b^1_Π, b^2_Π, and c_Π associated to Π)*

$$g_X = \mathrm{Avr}_p(\Pi) + c_\Pi,\ n_X = b^1_\Pi - 2\mathrm{Avr}_p(\Pi) - 2c_\Pi - b^2_\Pi + 1.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π.

Remark 1.2.1. Before Tamagawa proved Theorem 1.2, he also obtained an étale fundamental group version formula for (g_X, n_X) in a completely different way (by using wildly ramified coverings) which is much simpler than the case of tame fundamental groups (cf. [T1, §1]). Note that, for any smooth pointed stable curve over an algebraically closed field of positive characteristic, since the tame fundamental group can be recovered group-theoretically from the étale fundamental group (cf. [T1, Corollary 1.10]), the tame fundamental group version is stronger than the étale fundamental group version.

Remark 1.2.2. The formulas for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ and (g_X, n_X) are key results in the theory of tame anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (cf. [T2], [Y2]). On the other hand, if $W^\bullet$ is a smooth pointed stable curve of type (g_W, n_W) over an *arithmetic* field (e.g. number field, p-adic field, finite field), then a group-theoretical formula for (g_W, n_W) can be deduced immediately by computing "weight" (e.g. by applying the weight monodromy conjecture or p-adic Hodge theory).

Let us return to the case where $X^\bullet$ is an *arbitrary* pointed stable curve over k. Here our main question of the present paper is the following:

> *Does there exist a group-theoretical formula for (g_X, n_X) when $X^\bullet$ is an arbitrary pointed stable curve over k?*

We want to mention that the approach to finding a group-theoretical formula for (g_X, n_X) by applying $\mathrm{Avr}_p(\Pi_{X^\bullet})$ explained above *cannot be generalized* to the case where $X^\bullet$ is an arbitrary pointed stable curve. The reason is that the formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ is very complicated in general when $X^\bullet$ is not smooth over k, and $\mathrm{Avr}_p(\Pi_{X^\bullet})$ depends not only on the type (g_X, n_X) but also on *the structure of the dual semi-graph* of $X^\bullet$ (cf. [Y3, Theorem 1.3 and Theorem 1.4]).

In the present paper, we solve the problem mentioned above by considering the maximum generalized Hasse-Witt invariants. Let $\overline{\mathbb{F}}_p$ be an arbitrary algebraic closure of $\mathbb{F}_p$, Π an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups, and

$$\gamma_\Pi^{\max} \overset{\text{def}}{=} \max\{\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})) \mid \chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}_p^\times) \text{ such that } \chi \neq 1\},$$

where $\Pi_\chi \subseteq \Pi$ denotes the kernel of χ. Since the prime number p is a group-theoretical invariant associated to Π (cf. Lemma 5.2 (ii)), we see that $\gamma_\Pi^{\max}$ is also a group-theoretical invariant associated to Π. Then the main theorem of the present paper is as follows (see also Theorem 5.4):

Theorem 1.3. *Let $X^\bullet$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$ and Π an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups. Then we have*

$$g_X = b_\Pi^1 - \gamma_\Pi^{\max} - 1, \; n_X = 2\gamma_\Pi^{\max} - b_\Pi^1 - b_\Pi^2 + 3.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π.

Note that $\gamma_\Pi^{\max}$ is equal to the maximum of generalized Hasse-Witt invariants $\gamma_{X^\bullet}^{\max}$ of prime-to-p cyclic admissible coverings (cf. Definition 3.2). Then the main theorem follows from the following key observation (see Theorem 4.5 for more precisely):

Theorem 1.4. *We maintain the notation introduced above. Then there exist a natural number $m \in \mathbb{N}$ prime to p and a Galois admissible covering of $X^\bullet$ over k with Galois group $\mathbb{Z}/m\mathbb{Z}$ such that the "first" generalized Hasse-Witt invariant of the Galois admissible covering attains the maximum*

$$\gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \textit{if } n_X = 0, \\ g_X + n_X - 2, & \textit{if } n_X \neq 0. \end{cases}$$

Theorem 1.4 can be regarded as an analogue of [R1, Théorème 4.3.1] in the case of generalized Hasse-Witt invariants of prime-to-p cyclic admissible coverings of pointed stable curves.

The present paper is organized as follows. In Section 2, we recall some definitions and properties of admissible coverings, admissible fundamental groups, generalized Hasse-Witt invariants, and Raynaud-Tamagawa theta divisors. In Section 3, we study the maximum generalized Hasse-Witt invariants when $X^\bullet$ is a component-generic pointed stable curve. In Section 4, we study the maximum generalized Hasse-Witt invariants when $X^\bullet$ is an arbitrary pointed stable curve, and prove Theorem 1.4. In Section 5, by applying Theorem 1.4, we prove Theorem 1.3.

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2 Preliminaries

2.1 Admissible coverings and admissible fundamental groups

In this subsection, we recall some definitions and results which will be used in the present paper.

Definition 2.1. Let $\mathbb{G} \overset{\text{def}}{=} (v(\mathbb{G}), e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G}), \{\zeta_e^{\mathbb{G}}\}_{e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})})$ be a semi-graph (cf. [M3, Section 1]). Here, $v(\mathbb{G})$, $e^{\text{op}}(\mathbb{G})$, $e^{\text{cl}}(\mathbb{G})$, and $\{\zeta_e^{\mathbb{G}}\}_{e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})}$ denote the set of vertices of $\mathbb{G}$, the set of closed edges of $\mathbb{G}$, the set of open edges of $\mathbb{G}$, and the set of coincidence maps of $\mathbb{G}$, respectively. Note that, for each $e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})$, $e \overset{\text{def}}{=} \{b_e^1, b_e^2\}$ is a set of cardinality 2. Then e is a closed edge if $\zeta_e^{\mathbb{G}}(e) \subseteq v(\mathbb{G})$, and e is an open edge if $\zeta_e^{\mathbb{G}}(e) = \{\zeta_e^{\mathbb{G}}(e) \cap v(\mathbb{G}), \{v(\mathbb{G})\}\}$. We denote by $e^{\text{lp}}(\mathbb{G}) \subseteq e^{\text{cl}}(\mathbb{G})$ the subset of closed edges such that $\#\zeta_e^{\mathbb{G}}(e) = 1$ for each $e \in e^{\text{lp}}(\mathbb{G})$ (i.e., a closed edge which abuts to a unique vertex of $\mathbb{G}$). For each $e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})$, we denote by $v^{\mathbb{G}}(e) \subseteq v(\mathbb{G})$ the set of vertices of $\mathbb{G}$ to which e abuts. For each $v \in v(\mathbb{G})$, we denote by $e^{\mathbb{G}}(v) \subseteq e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})$ the set of edges of $\mathbb{G}$ to which v is abutted. Moreover, we shall say that $\mathbb{G}$ is a tree if the Betti number $\dim_{\mathbb{Q}}(H^1(\mathbb{G}, \mathbb{Q}))$ of $\mathbb{G}$ is equal to 0.

In the present paper, let

$$X^\bullet = (X, D)$$

be a pointed stable curve over an algebraically closed field k of characteristic $p > 0$, where X denotes the underlying curve, D_X denotes the set of marked points, g_X denotes the genus of X, and n_X denotes the cardinality $\#D_X$ of D_X. We shall say that (g_X, n_X) is the topological type (or type for short) of $X^\bullet$. Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$ and $r_X \overset{\text{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^\bullet}$. Let $v \in v(\Gamma_{X^\bullet})$ and $e \in e^{\text{op}}(\Gamma_{X^\bullet}) \cup e^{\text{cl}}(\Gamma_{X^\bullet})$. We write X_v for the irreducible component of X corresponding to v, write x_e for the node of X corresponding to e if $e \in e^{\text{cl}}(\Gamma_{X^\bullet})$, and write x_e for the marked point of X corresponding to e if $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$. Moreover, write X_v for the normalization of $\widetilde{X}_v$ and $\mathrm{norm}_v : \widetilde{X}_v \to X_v$ for the normalization morphism. We define a smooth pointed stable curve of type (g_v, n_v) over k to be

$$\widetilde{X}_v^\bullet = (\widetilde{X}_v, D_{\widetilde{X}_v} \stackrel{\mathrm{def}}{=} \mathrm{nom}_v^{-1}((X_v \cap X^{\mathrm{sing}}) \cup (D_X \cap X_v))).$$

We shall say that $X^\bullet$ is a *component-generic* pointed stable curve over k if $\widetilde{X}_v^\bullet$, $v \in v(\Gamma_{X^\bullet})$, is a geometric generic pointed stable curve of type (g_v, n_v) over k (i.e., a curve corresponding to a geometric generic point of the moduli space).

Definition 2.2. Let $Y^\bullet = (Y, D_Y)$ be a pointed stable curve over k, $f^\bullet : Y^\bullet \to X^\bullet$ a morphism of pointed stable curves over k, and $f : Y \to X$ the morphism of underlying curves induced by $f^\bullet$.

We shall say $f^\bullet$ a *Galois admissible covering* over k (or Galois admissible covering for short) if the following conditions are satisfied: (i) There exists a finite group $G \subseteq \mathrm{Aut}_k(Y^\bullet)$ such that $Y^\bullet/G = X^\bullet$, and $f^\bullet$ is equal to the quotient morphism $Y^\bullet \to Y^\bullet/G$. (ii) For each $y \in Y^{\mathrm{sm}} \setminus D_Y$, f is étale at y, where $(-)^{\mathrm{sm}}$ denotes the smooth locus of $(-)$. (iii) For any $y \in Y^{\mathrm{sing}}$, the image $f(y)$ is contained in X^{sing}, where $(-)^{\mathrm{sing}}$ denotes the set of singular points of $(-)$. (iv) For each $y \in Y^{\mathrm{sing}}$, the local morphism between two nodes induced by f may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[u,v]]/uv & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[s,t]]/st \\ u & \mapsto & s^n \\ v & \mapsto & t^n, \end{array}$$

where $(n, \mathrm{char}(k)) = 1$ if $\mathrm{char}(k) > 0$. Moreover, if we write $D_y \subseteq G$ for the decomposition group of y and $\#D_y$ for the cardinality of D_y, then $\tau(s) = \zeta_{\#D_y} s$ and $\tau(t) = \zeta_{\#D_y}^{-1} t$ for each $\tau \in D_y$, where $\zeta_{\#D_y}$ is a primitive $\#D_y$-th root of unit, and $\#(-)$ denotes the cardinality of $(-)$. (v) The local morphism between two marked points induced by f may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[a]] & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[b]] \\ a & \mapsto & b^m, \end{array}$$

where $(m, \mathrm{char}(k)) = 1$ if $\mathrm{char}(k) > 0$ (i.e., a tamely ramified extension).

Moreover, we shall say $f^\bullet$ an *admissible covering* if there exists a morphism of pointed stable curves $(f^\bullet)' : (Y^\bullet)' \to Y^\bullet$ over k such that the composite morphism $f^\bullet \circ (f^\bullet)' : (Y^\bullet)' \to X^\bullet$ is a Galois admissible covering over k. One can check easily that the definition of admissible covering coincides with the definition of [M1, §3.9 Definition] when the base scheme is k. We shall say an admissible covering $f^\bullet$ *étale* if f is an étale morphism.

Let $Z^\bullet$ be a disjoint union of finitely many pointed stable curves over k. We shall say a morphism $f^\bullet_{Z^\bullet} : Z^\bullet \to X^\bullet$ over k *multi-admissible covering* if the restriction of $f^\bullet_{Z^\bullet}$ to each connected component of $Z^\bullet$ is admissible. For any category $\mathscr{C}$, we write $\mathrm{Ob}(\mathscr{C})$ for the class of objects of $\mathscr{C}$, and write $\mathrm{Hom}(\mathscr{C})$ for the class of morphisms of $\mathscr{C}$. We denote by

$$\mathrm{Cov}^{\mathrm{adm}}(X^\bullet) \stackrel{\mathrm{def}}{=} (\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)), \mathrm{Hom}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)))$$

the category which consists of the following data: (i) $\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$ consists of an empty object and all the pairs $(Z^\bullet, f^\bullet_{Z^\bullet} : Z^\bullet \to X^\bullet)$, where $Z^\bullet$ is a disjoint union of finitely many pointed stable curves over k, and $f^\bullet_{Z^\bullet}$ is a multi-admissible covering over k; (ii) for any $(Z^\bullet, f^\bullet_{Z^\bullet})$, $(Y^\bullet, f^\bullet_{Y^\bullet}) \in \mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$, we define

$$\mathrm{Hom}((Z^\bullet, f^\bullet_{Z^\bullet}), (Y^\bullet, f^\bullet_{Y^\bullet})) \overset{\mathrm{def}}{=} \{g^\bullet \in \mathrm{Hom}_k(Z^\bullet, Y^\bullet) \mid f^\bullet_{Y^\bullet} \circ g^\bullet = f^\bullet_{Z^\bullet}\},$$

where $\mathrm{Hom}_k(Z^\bullet, Y^\bullet)$ denotes the set of k-morphisms of pointed stable curves. By applying [M1, §3.11 Proposition] and the theory of Kummer log étale coverings, we may see that $\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)$ is a Galois category. Thus, by choosing a base point $x \in X^{\mathrm{sm}} \setminus D_X$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ which is called the *admissible fundamental group* of $X^\bullet$. For simplicity of notation, we omit the base point and denote the admissible fundamental group by $\Pi_{X^\bullet}$. Write $\Pi^{\text{ét}}_{X^\bullet}$ for the étale fundamental group of the underlying curve X of $X^\bullet$ and $\Pi^{\mathrm{top}}_{X^\bullet}$ for the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$. Note that we have the following natural continuous surjective homomorphisms (for suitable choices of base points)

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{top}}_{X^\bullet}.$$

For each $v \in v(\Gamma_{X^\bullet})$, we denote by

$$\Pi_{\widetilde{X}^\bullet_v}$$

the admissible fundamental group of $\widetilde{X}^\bullet_v$. Then we have a natural (outer) injective homomorphism $\Pi_{\widetilde{X}^\bullet_v} \hookrightarrow \Pi_{X^\bullet}$.

For more details on the theory of admissible coverings and admissible fundamental groups for pointed stable curves, see [M1], [M2].

Remark 2.2.1. Let $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ be the moduli stack of pointed stable curves of type (g_X, n_X) over $\mathrm{Spec}\,\mathbb{Z}$ and $\mathcal{M}_{g_X,n_X,\mathbb{Z}}$ the open substack of $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ parametrizing smooth pointed stable curves. Write $\overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \setminus \mathcal{M}_{g_X,n_X,\mathbb{Z}} \subset \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ relative to $\mathrm{Spec}\,\mathbb{Z}$.

The pointed stable curve $X^\bullet$ over k induces a morphism $\mathrm{Spec}\,k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$. Write $s^{\log}_X$ for the log scheme whose underlying scheme is $\mathrm{Spec}\,k$, and whose log structure is the pulling-back log structure induced by the morphism $\mathrm{Spec}\,k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$. We obtain a natural morphism $s^{\log}_X \to \overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}$ induced by the morphism $\mathrm{Spec}\,k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ and a stable log curve $X^{\log} \overset{\mathrm{def}}{=} s^{\log}_X \times_{\overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}} \overline{\mathcal{M}}^{\log}_{g_X,n_X+1,\mathbb{Z}}$ over $s^{\log}_X$ whose underlying scheme is X. Let $Y^{\log} \to X^{\log}$ be an arbitrary Kummer log étale covering. One can prove that there exists a Kummer log étale covering $t^{\log}_X \to s^{\log}_X$ such that $Y^{\log} \times_{s^{\log}_X} t^{\log}_X \to X^{\log} \times_{s^{\log}_X} t^{\log}_X$ is a log admissible covering (cf. [M1, §3.5 Definition]) over $t^{\log}_X$. Then the admissible fundamental group of $X^\bullet$ does not depend on the log structure of $X^{\log}$, and [M1, §3.11 Proposition] implies that the admissible fundamental group $\Pi_{X^\bullet}$ of $X^\bullet$ is naturally isomorphic to the geometric log étale fundamental group of $X^{\log}$ (i.e., $\ker(\pi_1(X^{\log}) \to \pi_1(s^{\log}_X))$).

Remark 2.2.2. Suppose that $X^\bullet$ is smooth over k. By the definition of admissible fundamental groups, the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X \setminus D_X$.

In the remainder of the present paper, we suppose that the characteristic of k is $p > 0$.

Definition 2.3. We define the *p-rank* (or *Hasse-Witt invariant*) of $X^\bullet$ to be

$$\sigma_{X^\bullet} \overset{\text{def}}{=} \dim_{\mathbb{F}_p}(H^1_{\text{ét}}(X, \mathbb{F}_p)) = \dim_{\mathbb{F}_p}(\Pi^{\text{ab}}_{X^\bullet} \otimes \mathbb{F}_p),$$

where $(-)^{\text{ab}}$ denotes the abelianization of $(-)$.

Remark 2.3.1. The definition of p-rank implies that $\sigma_{X^\bullet} = \sigma_X$. Moreover, it is easy to see that

$$\sigma_{X^\bullet} = \sum_{v \in v(\Gamma_{X^\bullet})} \sigma_{\widetilde{X}^\bullet_v} + r_X = \sigma_X = \sum_{v \in v(\Gamma_{X^\bullet})} \sigma_{\widetilde{X}_v} + r_X.$$

2.2 Generalized Hasse-Witt invariants of cyclic admissible coverings

In this subsection, we recall some notation concerning generalized Hasse-Witt invariants of cyclic admissible coverings.

We maintain the notation introduced in Section 2.1, and let $X^\bullet = (X, D_X)$ be a pointed stable curve of type (g_X, n_X) over k, and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. Let n be an arbitrary positive natural number prime to p and $\mu_n \subseteq k^\times$ the group of nth roots of unity. Fix a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the map $\zeta^i \mapsto i$. Let $\alpha \in \text{Hom}(\Pi^{\text{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$. We denote by $X^\bullet_\alpha = (X_\alpha, D_{X_\alpha})$ the Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to α. Write F_{X_α} for the absolute Frobenius morphism on X_α. Then there exists a decomposition (cf. [S, Section 9])

$$H^1(X_\alpha, \mathcal{O}_X) = H^1(X_\alpha, \mathcal{O}_X)^{\text{st}} \oplus H^1(X_\alpha, \mathcal{O}_X)^{\text{ni}},$$

where F_{X_α} is a bijection on $H^1(X_\alpha, \mathcal{O}_X)^{\text{st}}$ and is nilpotent on $H^1(X_\alpha, \mathcal{O}_X)^{\text{ni}}$. Moreover, we have

$$H^1(X_\alpha, \mathcal{O}_X)^{\text{st}} = H^1(X_\alpha, \mathcal{O}_X)^{F_{X_\alpha}} \otimes_{\mathbb{F}_p} k,$$

where $(-)^{F_{X_\alpha}}$ denotes the subspace of $(-)$ on which F_{X_α} acts trivially. Then Artin-Schreier theory implies that we may identify

$$H_\alpha \overset{\text{def}}{=} H^1_{\text{ét}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$$

with the largest subspace of $H^1(X_\alpha, \mathcal{O}_X)$ on which F_{X_α} is a bijection.

The finite dimensional k-vector spaces H_α is a finitely generated $k[\mu_n]$-module induced by the natural action of μ_n on X_α. We have the following canonical decomposition

$$H_\alpha = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} H_{\alpha,i},$$

where $\zeta \in \mu_n$ acts on $H_{\alpha,i}$ as the ζ^i-multiplication. We define

$$\gamma_{\alpha,i} \overset{\text{def}}{=} \dim_k(H_{\alpha,i}),\ i \in \mathbb{Z}/n\mathbb{Z}.$$

These invariants are called *generalized Hasse-Witt invariants* (cf. [N]). Moreover, we shall say that $\gamma_{\alpha,1}$ is the *first* generalized Hasse-Witt invariant of the Galois multi-admissible covering $X_\alpha^\bullet \to X^\bullet$. Note that the decomposition above implies that

$$\dim_k(H_\alpha) = \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \gamma_{\alpha,i}.$$

In particular, if X_α is connected, then $\dim_k(H_\alpha) = \sigma_{X_\alpha}$.

We write $\mathbb{Z}[D_X]$ for the group of divisors whose supports are contained in D_X. Note that $\mathbb{Z}[D_X]$ is a free $\mathbb{Z}$-module with basis D_X. We define

$$c'_n : \mathbb{Z}/n\mathbb{Z}[D_X] \overset{\text{def}}{=} \mathbb{Z}[D_X] \otimes \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z},\ D \bmod n \mapsto \deg(D) \bmod n.$$

Then $\ker(c'_n)$ can be regarded as a subset of $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$, where $(\mathbb{Z}/n\mathbb{Z})^\sim$ denotes the set $\{0, 1, \dots, n-1\}$, and $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$ denotes the subset of $\mathbb{Z}[D_X]$ consisting of the elements whose coefficients are contained in $(\mathbb{Z}/n\mathbb{Z})^\sim$. We denote by $\mathbb{Z}/n\mathbb{Z}[D_X]^0$ the kernel of c'_n and by $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ the subset of $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$ corresponding to $\mathbb{Z}/n\mathbb{Z}[D_X]^0$ under the natural bijection $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X] \overset{\sim}{\to} \mathbb{Z}/n\mathbb{Z}[D_X]$. Note that, for each $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$, we have $n|\deg(D)$. Then

$$\deg(D) = s(D)n$$

for some integer $s(D)$ such that

$$0 \leq s(D) \leq \begin{cases} 0, & \text{if } n_X \leq 1, \\ n_X - 1, & \text{if } n_X \geq 2. \end{cases}$$

Let $X^{\bullet,*} = (X^*, D_{X^*}) \to X^\bullet$ be a universal admissible covering corresponding to $\Pi_{X^\bullet}$. For each $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{op}}(\Gamma_{X^\bullet})$, write x_e for the marked point corresponding to e, and let x_{e^*} be a point of the inverse image of x_e in D_{X^*}. Write $I_{e^*} \subseteq \Pi_{X^\bullet}$ for the inertia subgroup of x_{e^*}. Note that I_{e^*} is isomorphic to $\widehat{\mathbb{Z}}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Suppose that x_e is contained in X_v. Then we have an injection

$$\phi_{e^*} : I_{e^*} \hookrightarrow \Pi_{X^\bullet}^{\mathrm{ab}}$$

induced by the composition of (outer) injective homomorphisms $I_{e^*} \hookrightarrow \Pi_{\widetilde{X}_v^\bullet} \hookrightarrow \Pi_{X^\bullet}$, where $\Pi_{\widetilde{X}_v^\bullet}$ denotes the admissible fundamental group of $\widetilde{X}_v^\bullet$. Since the image of ϕ_{e^*} depends only on e, we may write I_e for the image $\phi_{e^*}(I_{e^*})$. Moreover, the specialization theorem of the maximal prime-to-p quotients of admissible fundamental groups of pointed stable curves (cf. [V, Théorème 2.2 (c)]) implies that, there exists a generator $[s_e]$ of I_e for each $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ such that the following holds

$$\sum_{e\in e^{\mathrm{op}}(\Gamma_{X^\bullet})} [s_e] = 0$$

in $\Pi_{X^\bullet}^{\mathrm{ab}}$.

Definition 2.4. We maintain the notation introduced above.

(i) We put

$$D_\alpha \overset{\text{def}}{=} \sum_{e \in e^{\text{op}}(\Gamma_{X^\bullet})} \alpha([s_e])x_e,\ \alpha \in \text{Hom}(\Pi^{\text{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}).$$

Note that we have $D_\alpha \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$. On the other hand, for each $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$, we denote by

$$\text{Rev}^{\text{adm}}_D(X^\bullet)$$

the subset of $\text{Hom}(\Pi^{\text{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$ such that $D_\alpha = D$ for each $\alpha \in \text{Rev}^{\text{adm}}_D(X^\bullet)$. Moreover, we put

$$\gamma_{(\alpha,D)} \overset{\text{def}}{=} \gamma_{\alpha,1}.$$

(ii) Let $t \in \mathbb{N}$ be an arbitrary positive natural number and $n \overset{\text{def}}{=} p^t - 1$. Let

$$u = \sum_{j=0}^{t-1} u_j p^j,\ u \in \{0, \dots, n\},$$

be the p-adic expansion with $u_j \in \{0, \dots, p-1\}$. We identify $\{0, \dots, t-1\}$ with $\mathbb{Z}/t\mathbb{Z}$ naturally, and put

$$u^{(i)} \overset{\text{def}}{=} \sum_{j=0}^{t-1} u_{i+j} p^j,\ i \in \{0, \dots, t-1\}.$$

Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$. We put

$$D^{(i)} \overset{\text{def}}{=} \sum_{x \in D_X} (\text{ord}_x(D))^{(i)} x,\ i \in \{0, 1, \dots, t-1\},$$

which is an effective divisor on X.

In the remainder of the present paper, we may assume that

$$n \overset{\text{def}}{=} p^t - 1$$

for some positive natural number $t \in \mathbb{N}$.

2.3 Raynaud-Tamagawa theta divisors

In this subsection, we recall some notation and results concerning theta divisors defined by Raynaud and Tamagawa (see also [T2, Section 2]).

We maintain the notation introduced in Section 2.2. Moreover, in the present subsection, we suppose that $X^\bullet$ is *smooth* over k. The generalized Hasse-Witt invariants can be also described in terms of line bundles and divisors. We denote by $\text{Pic}(X)$ the Picard group of X. Consider the following complex of abelian groups:

$$\mathbb{Z}[D_X] \overset{a_n}{\to} \text{Pic}(X) \oplus \mathbb{Z}[D_X] \overset{b_n}{\to} \text{Pic}(X),$$

where $a_n(D) = ([\mathcal{O}_X(-D)], nD)$, $b_n(([\mathcal{L}], D)) = [\mathcal{L}^n \otimes \mathcal{O}_X(-D)]$. We denote by

$$\mathscr{P}_{X^\bullet,n} \overset{\mathrm{def}}{=} \ker(b_n)/\mathrm{Im}(a_n)$$

the homology group of the complex. Moreover, we have the following exact sequence

$$0 \to \mathrm{Pic}(X)[n] \overset{a'_n}{\to} \mathscr{P}_{X^\bullet,n} \overset{b'_n}{\to} \mathbb{Z}/n\mathbb{Z}[D_X] \overset{c'_n}{\to} \mathbb{Z}/n\mathbb{Z},$$

where $[n]$ means the n-torsion subgroup, and

$$a'_n([\mathcal{L}]) = ([\mathcal{L}], 0) \bmod \mathrm{Im}(a_n),$$

$$b'_n(([\mathcal{L}], D)) \bmod \mathrm{Im}(a_n)) = D \bmod n,$$

$$c'_n(D \bmod n) = \deg(D) \bmod n.$$

We shall define

$$\widetilde{\mathscr{P}}_{X^\bullet,n}$$

to be the inverse image of $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^\sim[D_X] \subseteq \mathbb{Z}[D_X]$ under the projection $\ker(b_n) \to \mathbb{Z}[D_X]$. It is easy to see that $\mathscr{P}_{X^\bullet,n}$ and $\widetilde{\mathscr{P}}_{X^\bullet,n}$ are free $\mathbb{Z}/n\mathbb{Z}$-groups with rank $2g_X + n_X - 1$ if $n_X \neq 0$ and with rank $2g_X$ if $n_X = 0$. Moreover, [T2, Proposition 3.5] implies that

$$\widetilde{\mathscr{P}}_{X^\bullet,n} \cong \mathscr{P}_{X^\bullet,n} \cong \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}).$$

Then every element of $\widetilde{\mathscr{P}}_{X^\bullet,n}$ induces a Galois multi-admissible covering of $X^\bullet$ over k with Galois group $\mathbb{Z}/n\mathbb{Z}$.

Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$. We fix an isomorphism $\mathcal{L}^n \cong \mathcal{O}_X(-D)$. Note that D is an effective divisor on X. We have the following composition of morphisms of line bundles

$$\mathcal{L} \overset{p^t}{\to} \mathcal{L}^{\otimes p^t} = \mathcal{L}^{\otimes n} \otimes \mathcal{L} \overset{\sim}{\to} \mathcal{O}_X(-D) \otimes \mathcal{L} \hookrightarrow \mathcal{L}.$$

The composite morphism induces a morphism $\phi_{([\mathcal{L}],D)} : H^1(X, \mathcal{L}) \to H^1(X, \mathcal{L})$. We denote by

$$\gamma_{([\mathcal{L}],D)} \overset{\mathrm{def}}{=} \dim_k(\bigcap_{r \geq 1} \mathrm{Im}(\phi^r_{([\mathcal{L}],D)})).$$

Write $\alpha_{\mathcal{L}} \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$ for the element corresponding to $([\mathcal{L}], D)$ and F_X for the absolute Frobenius morphism on X. Then [S, Section 9] implies that $\gamma_{\alpha_{\mathcal{L}},1}$ is equal to the dimension over k of the largest subspace of $H^1(X, \mathcal{L})$ on which F_X is a bijection. Moreover, we have

$$\gamma_{\alpha_{\mathcal{L}},1} = \dim_k(H^1(X, \mathcal{L})^{F_X} \otimes_{\mathbb{F}_p} k),$$

where $(-)^{F_X}$ denotes the subspace of $(-)$ on which F_X acts trivially. It is easy to check that

$$H^1(X, \mathcal{L})^{F_X} \otimes_{\mathbb{F}_p} k = \bigcap_{r \geq 1} \mathrm{Im}(\phi^r_{([\mathcal{L}],D)}).$$

Then we obtain that $\gamma_{([\mathcal{L}],D)} = \gamma_{\alpha_{\mathcal{L}},1}$. Moreover, we observer that $D_{\alpha_{\mathcal{L}}} = D$. Then we obtain that

$$\gamma_{([\mathcal{L}],D)} = \gamma_{\alpha_{\mathcal{L}},1} = \gamma_{(\alpha_{\mathcal{L}},D)}.$$

Lemma 2.5. *We maintain the notation introduced above. Suppose that $X^\bullet$ is smooth k. Then we have*

$$\gamma_{(\alpha_{\mathcal{L}},D)} \leq \dim_k(H^1(X,\mathcal{L})) = \begin{cases} g_X, & \textit{if } ([\mathcal{L}],D) = ([\mathcal{O}_X],0), \\ g_X - 1, & \textit{if } s(D) = 0, \\ g_X + s(D) - 1, & \textit{if } s(D) \geq 1. \end{cases}$$

Proof. The first inequality follows from the definition of generalized Hasse-Witt invariants. The Riemann-Roch theorem implies that

$$\dim_k(H^1(X,\mathcal{L})) = g_X - 1 - \deg(\mathcal{L}) + \dim_k(H^0(X,\mathcal{L}))$$

$$= g_X - 1 + \frac{1}{n}\deg(D) + \dim_k(H^0(X,\mathcal{L})) = g_X - 1 + s(D) + \dim_k(H^0(X,\mathcal{L})).$$

This completes the proof of the lemma. $\square$

Next, let us explain the Raynaud-Tamagawa theta divisors. Let F_k be the absolute Frobenius morphism on $\mathrm{Spec}\, k$ and $F_{X/k}$ the relative Frobenius morphism $X \to X_1 \overset{\mathrm{def}}{=} X \times_{k,F_k} k$ over k. We define

$$X_t \overset{\mathrm{def}}{=} X \times_{k,F_k^t} k,$$

and define a morphism

$$F_{X/k}^t : X \to X_t$$

over k to be $F_{X/k}^t \overset{\mathrm{def}}{=} F_{X_{t-1}/k} \circ \cdots \circ F_{X_1/k} \circ F_{X/k}$.

Let $([\mathcal{L}],D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$ and $\mathcal{L}_t$ the pull-back of $\mathcal{L}$ by the natural morphism $X_t \to X$. Note that $\mathcal{L}$ and $\mathcal{L}_t$ are line bundles of degree $-s(D)$. We put

$$\mathcal{B}_D^t \overset{\mathrm{def}}{=} (F_{X/k}^t)_*(\mathcal{O}_X(D))/\mathcal{O}_{X_t}, \ \mathcal{E}_D \overset{\mathrm{def}}{=} \mathcal{B}_D^t \otimes \mathcal{L}_t.$$

Write $\mathrm{rk}(\mathcal{E}_D)$ for the rank of $\mathcal{E}_D$. Then we have

$$\chi(\mathcal{E}_D) = \deg(\det(\mathcal{E}_D)) - (g_X - 1)\mathrm{rk}(\mathcal{E}_D).$$

Moreover, $\chi(\mathcal{E}_D) = 0$ (cf. [T2, Lemma 2.3 (ii)]). In [R1], Raynaud investigated the following property of the vector bundle $\mathcal{E}_D$ on X.

Condition 2.6. We shall say that $\mathcal{E}_D$ satisfies $(\star)$ if there exists a line bundle $\mathcal{L}_t'$ of degree 0 on X_t such that

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}_t')), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}_t'))\}.$$

Let J_{X_t} be the Jacobian variety of X_t, and $\mathcal{L}_t$ a universal line bundle on $X_t \times J_{X_t}$. Let $\mathrm{pr}_{X_t} : X_t \times J_{X_t} \to X_t$ and $\mathrm{pr}_{J_{X_t}} : X_t \times J_{X_t} \to J_{X_t}$ be the natural projections. We denote by $\mathcal{F}$ the coherent $\mathcal{O}_{X_t}$-module $\mathrm{pr}_{X_t}^*(\mathcal{E}_D) \otimes \mathcal{L}_t$, and by

$$\chi_{\mathcal{F}} \overset{\mathrm{def}}{=} \dim_k(H^0(X_t \times_k k(y), \mathcal{F} \otimes k(y))) - \dim_k(H^1(X_t \times_k k(y), \mathcal{F} \otimes k(y)))$$

for each $y \in J_{X_t}$, where $k(y)$ denotes the residue field of y. Note that since $\mathrm{pr}_{J_{X_t}}$ is flat, $\chi_{\mathcal{F}}$ is independent of $y \in J_{X_t}$. Write $(-\chi_{\mathcal{F}})^+$ for $\max\{0, -\chi_{\mathcal{F}}\}$. We denote by

$$\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$$

the closed subscheme of J_{X_t} defined by the $(-\chi_{\mathcal{F}})^+$-th Fitting ideal

$$\mathrm{Fitt}_{(-\chi_{\mathcal{F}})^+}(R^1(\mathrm{pr}_{J_{X_t}})_*(\mathrm{pr}_{X_t}^*(\mathcal{E}_D) \otimes \mathcal{L}_t)).$$

The definition of $\Theta_{\mathcal{E}_D}$ is independent of the choice of $\mathcal{L}_t$. Moreover, for each line bundle $\mathcal{L}''$ of degree 0 on X_t, we have that $[\mathcal{L}''] \notin \Theta_{\mathcal{E}_D}$ if and only if

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}'')), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}''))\},$$

where $[\mathcal{L}'']$ denotes the point of J_{X_t} corresponding to $\mathcal{L}''$ (cf. [T2, Proposition 2.2 (i) (ii)]).

Suppose that $\mathcal{E}_D$ satisfies $(\star)$. [R1, Proposition 1.8.1] implies that $\Theta_{\mathcal{E}_D}$ is algebraically equivalent to $\mathrm{rk}(\mathcal{E}_D)\Theta$, where Θ is the classical theta divisor (i.e., the image of $X_t^{g_X-1}$ in J_{X_t}). Then we have the following definition.

Definition 2.7. We shall say $\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$ the *Raynaud-Tamagawa theta divisor* associated to $\mathcal{E}_D$ if $\mathcal{E}_D$ satisfies $(\star)$.

Remark 2.7.1. The definition of $\mathcal{E}_D$ implies that the following natural exact sequence holds

$$0 \to \mathcal{L}_t \to (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \to \mathcal{E}_D \to 0.$$

Let $\mathcal{I}$ be a line bundle of degree 0 on X. Write $\mathcal{I}_t$ for the pull-back of $\mathcal{I}$ by the natural morphism $X_t \to X$. we obtain the following exact sequence

$$\ldots \to H^0(X_t, \mathcal{E}_D \otimes \mathcal{I}_t) \to H^1(X_t, \mathcal{L}_t \otimes \mathcal{I}_t) \overset{\phi_{\mathcal{L}_t \otimes \mathcal{I}_t}}{\to} H^1(X_t, (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \otimes \mathcal{I}_t)$$
$$\to H^1(X_t, \mathcal{E}_D \otimes \mathcal{I}_t) \to \ldots.$$

Note that we have that

$$H^1(X_t, \mathcal{L}_t \otimes \mathcal{I}_t) \cong H^1(X, \mathcal{L} \otimes \mathcal{I}),$$

and that

$$H^1(X_t, (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \otimes \mathcal{I}_t) \cong H^1(X, \mathcal{O}_X(D) \otimes (F^t_{X/k})^*(\mathcal{L}_t \otimes \mathcal{I}_t))$$
$$\cong H^1(X, \mathcal{O}_X(D) \otimes (\mathcal{L} \otimes \mathcal{I})^{\otimes p^t}) \cong H^1(X, \mathcal{L} \otimes \mathcal{I}).$$

Moreover, it is easy to see that the homomorphism

$$H^1(X, \mathcal{L} \otimes \mathcal{I}) \to H^1(X, \mathcal{L} \otimes \mathcal{I})$$

induced by $\phi_{\mathcal{L}_t \otimes \mathcal{I}_t}$ coincides with $\phi_{([\mathcal{L} \otimes \mathcal{I}], D)}$ if $[\mathcal{I}] \in \mathrm{Pic}(X)[n]$. Thus, we obtain that if

$$\gamma_{(([\mathcal{L} \otimes \mathcal{I}], D))} = \dim_k(H^1(X, \mathcal{L} \otimes \mathcal{I}))$$

for some line bundle $[\mathcal{I}] \in \mathrm{Pic}(X)[n]$, then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists (i.e., $[\mathcal{I}_t] \notin \Theta_{\mathcal{E}_D}$).

Let N be an arbitrary non-negative integer. We put

$$C(N) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } N = 0, \\ 3^{N-1}N!, & \text{if } N \neq 0. \end{cases}$$

Then we have the following proposition.

Proposition 2.8. *We maintain the notation introduced above. Suppose that the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists, and that*

$$n = p^t - 1 > C(g_X) + 1.$$

Then there exists a line bundle $\mathcal{I}$ of degree 0 on X such that $[\mathcal{I}] \neq [\mathcal{O}_X]$, that $[\mathcal{I}^{\otimes n}] = [\mathcal{O}_X]$, and that $\gamma_{(([\mathcal{L}\otimes\mathcal{I}],D))} = \dim_k(H^1(X, \mathcal{L} \otimes \mathcal{I}))$ (i.e., $[\mathcal{I}_t] \notin \Theta_{\mathcal{E}_D}$).

Proof. By applying similar arguments to the arguments given in the proof of [T2, Corollary 3.10], the proposition follows immediately from Remark 2.7.1. □

The following fundamental theorem of theta divisors was proved by Raynaud and Tamagawa.

Theorem 2.9. *Suppose that $s(D) \in \{0, 1\}$. Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists (i.e., $\mathcal{E}_D$ satisfies $(\star)$).*

Remark 2.9.1. Theorem 2.9 was proved by Raynaud if $s(D) = 0$ (cf. [R1, Théorème 4.1.1]), and by Tamagawa if $s(D) \leq 1$ (cf. [T2, Theorem 2.5]).

We may ask whether or not the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ exists in general when $X^\bullet$ is smooth over k.

> *Suppose that $s(D) \geq 2$. Does the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ exist?*

Note that since the existence of $\Theta_{\mathcal{E}_D}$ implies that $\mathcal{E}_D$ is a semi-stable bundle, we obtain that $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, 1, \ldots, t-1\}$ (cf. [T2, Lemma 2.15]). Then we may consider the following problem:

> *Suppose that $X^\bullet$ is smooth over k, that $s(D) \geq 2$, and that $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, 1, \ldots, t-1\}$. Does the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ exist?*

In fact, the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ does not exist in general. Here, we have an example as follows. Suppose that $p = 3$. Let $X = \mathbb{P}^1_k$, $D_X = \{0, 1, \infty, \lambda\}$, where $w \notin \{0, 1\}$, and

$$D = \sum_{x \in D_X} \frac{p-1}{2} x.$$

Then we have $s(D) = 2$. Let $([\mathcal{L}], D)$ be an arbitrary element of $\widetilde{\mathscr{P}}_{X^\bullet,n}$. We see immediately that $\mathcal{E}_D$ satisfies $(\star)$ if and only if the elliptic curve defined by the equation

$$y^2 = x(x-1)(x-\lambda)$$

is ordinary. Thus, we cannot expect that $\Theta_{\mathcal{E}_D}$ exists in general. On the other hand, we have the following open problem posed by Tamagawa (cf. [T2, Question 2.20]).

Problem 2.10. *Suppose that $X^\bullet$ is a geometric generic pointed stable curve of type (g_X, n_X) over k. Let $([\mathcal{L}], D)$ be an arbitrary element of $\widetilde{\mathscr{P}}_{X^\bullet,n}$. Moreover, suppose that $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, 1, \ldots, t-1\}$. Does the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exist?*

In the next section, we solve Problem 2.10 under the assumption $s(D) = n_X - 1$ (cf. Corollary 3.10 below).

3 Maximum generalized Hasse-Witt invariants of cyclic admissible coverings of component-generic pointed stable curves

In this section, we discuss the maximum generalized Hasse-Witt invariants of cyclic admissible coverings of a component-generic pointed stable curve. We maintain the notation introduced in Section 2.2.

Lemma 3.1. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ and $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$. Write*

$$f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$$

for the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. For each $v \in v(\Gamma_{X^\bullet})$, $f^\bullet$ induces a Galois multi-admissible covering

$$\widetilde{f}^\bullet_v : \widetilde{Y}^\bullet_v \to \widetilde{X}^\bullet_v$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\widetilde{\alpha}_v$ for an element of $\mathrm{Hom}(\Pi^{\mathrm{ab}}_{\widetilde{X}^\bullet_v}, \mathbb{Z}/n\mathbb{Z})$ *induced by $\widetilde{f}^\bullet_v$. Then we have*

$$\gamma_{(\alpha,D)} = \max\{\gamma_{(\alpha',D)} \mid \alpha' \in \mathrm{Rev}_D(X^\bullet),\ \alpha' \neq 0\}$$

$$= \begin{cases} g_X - 1, & \textit{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \textit{if } \mathrm{Supp}(D) \neq \emptyset \end{cases}$$

if and only if

$$\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})} = \begin{cases} g_v, & \textit{if } \widetilde{\alpha}_v = 0, \\ g_v - 1, & \textit{if } \widetilde{\alpha}_v \neq 0,\ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) = \emptyset, \\ g_v + s(D_{\widetilde{\alpha}_v}) - 1, & \textit{if } \widetilde{\alpha}_v \neq 0,\ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) \neq \emptyset, \end{cases}$$

where $\mathrm{Supp}(-)$ *denotes the support of* $(-)$*.*

Proof. We will prove the lemma by induction on the cardinality $\#v(\Gamma_{X^\bullet})$ of $v(\Gamma_{X^\bullet})$. Suppose that $\#v(\Gamma_{X^\bullet}) = 1$ (i.e., $X^\bullet$ is irreducible, and $\widetilde{X}_v$ is the normalization of X). Then we have that $D_{\widetilde{\alpha}_v}|_{\mathrm{norm}_v^{-1}(D)} = \mathrm{norm}_v^*(D)$ and

$$g_v = g_X - \#X^{\mathrm{sing}}.$$

Moreover, since $\widetilde{X}_v^\bullet$ is smooth over k, we write $([\mathcal{L}_{\widetilde{\alpha}_v}], D_{\widetilde{\alpha}_v}) \in \widetilde{\mathscr{P}}_{\widetilde{X}_v^\bullet, n}$ for the pair induced by $\widetilde{\alpha}_v$. First, we suppose that $\#\mathrm{Supp}(D) \leq 1$. Then the structures of maximal prime-to-p quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]) implies that f is étale over $\mathrm{Supp}(D)$ if $\#\mathrm{Supp}(D) \leq 1$. Write $\mathscr{N}_X^{\mathrm{ra}} \subseteq X^{\mathrm{sing}}$ for the subset of nodes over which f is ramified and $\mathscr{N}_X^{\mathrm{et}} \subseteq X^{\mathrm{sing}}$ for the subset of nodes over which f is étale. Then we have $s(D_{\widetilde{\alpha}_v}) = s(D) + \#\mathscr{N}_X^{\mathrm{ra}}$.

On the other hand, we see immediately that

$$\dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\widetilde{\alpha}_v})) = \begin{cases} g_v, & \text{if } \widetilde{\alpha}_v = 0, \\ g_v - 1, & \text{if } \widetilde{\alpha}_v \neq 0,\ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) = \emptyset, \\ g_v + s(D_{\widetilde{\alpha}_v}) - 1, & \text{if } \widetilde{\alpha}_v \neq 0,\ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) \neq \emptyset, \end{cases}$$

Write $\Gamma_{Y^\bullet}$ for the dual semi-graph of $Y^\bullet$. The natural $k[\mu_n]$-submodule

$$H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k \subseteq H^1_{\mathrm{\acute{e}t}}(Y, \mathbb{F}_p) \otimes k$$

admits the following canonical decomposition

$$H^1(\Gamma_{\widetilde{Y}_v^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{\widetilde{Y}_v^\bullet}}(j).$$

Moreover, we observer that $\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \#\mathscr{N}_X^{\mathrm{et}}$. Then we obtain that

$$\dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\widetilde{\alpha}_v})) + \dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \begin{cases} g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset. \end{cases}$$

Since $\gamma_{(\alpha, D)} = \gamma_{([\mathcal{L}_{\widetilde{\alpha}_v}], D_{\widetilde{\alpha}_v})} + \dim_k(M_{\Gamma_{Y^\bullet}}(1))$ and $\#X^{\mathrm{sing}} = \#\mathscr{N}_X^{\mathrm{ra}} + \#\mathscr{N}_X^{\mathrm{et}}$, we have that

$$\gamma_{(\alpha, D)} = \begin{cases} g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset \end{cases}$$

if and only if $\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})} = \gamma_{([\mathcal{L}_{\widetilde{\alpha}_v}], D_{\widetilde{\alpha}_v})} = \dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\widetilde{\alpha}_v}))$. This completes the proof of the lemma when $\#v(\Gamma_{X^\bullet}) = 1$.

Suppose $m \overset{\mathrm{def}}{=} \#v(\Gamma_{X^\bullet}) \geq 2$. Let $v_0 \in v(\Gamma_{X^\bullet})$ be a vertex such that $\Gamma_{X^\bullet} \setminus \{v_0, e^{\Gamma_{X^\bullet}}(v_0)\}$ is connected (note that it is easy to see that such v_0 exists). Write X_1 for the topological closure of $X \setminus X_{v_0}$ in X and X_2 for X_{v_0}. Note that X_1 is connected. We define a pointed stable curve

$$X_i^\bullet = (X_i, D_{X_i} \overset{\mathrm{def}}{=} (X_i \cap D_X) \cup (X_1 \cap X_2)),\ i \in \{1, 2\},$$

over k. Then $f^\bullet$ induces a Galois multi-admissible covering

$$f_i^\bullet : Y_i^\bullet \to X_i^\bullet,\ i \in \{1, 2\},$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, we denote by

$$\alpha_i \in \mathrm{Hom}(\Pi_{X_i^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z}),\ i \in \{1, 2\},$$

an element induced by $f_i^\bullet$, where $\Pi_{X_i^\bullet}$ denotes the admissible fundamental group of $X_i^\bullet$. Write $\mathscr{N}_{X_1 \cap X_2}^{\mathrm{ra}} \subseteq X_1 \cap X_2$ for the subset of nodes over which f is ramified and $\mathscr{N}_{X_1 \cap X_2}^{\mathrm{et}} \subseteq$

$X_1 \cap X_2$ for the subset of nodes over which f is étale. Note that $\#\mathscr{N}^{\mathrm{ra}}_{X_1\cap X_2} + \#\mathscr{N}^{\mathrm{et}}_{X_1\cap X_2} = \#(X_1 \cap X_2)$. By induction, we obtain that, for each $i \in \{1, 2\}$,

$$\gamma_{(\alpha_i,D_{\alpha_i})} \leq \begin{cases} g_{X_i}, & \text{if } \alpha_i = 0, \\ g_{X_i} - 1, & \text{if } \alpha_i \neq 0,\ \mathrm{Supp}(D_{\alpha_i}) = \emptyset, \\ g_{X_i} + s(D_{\alpha_i}) - 1, & \text{if } \alpha_i \neq 0,\ \mathrm{Supp}(D_{\alpha_i}) \neq \emptyset, \end{cases}$$

where g_{X_i} denotes the genus of X_i. Note that the definition of admissible coverings implies that $s(D_{\alpha_1}) + s(D_{\alpha_2}) = s(D) + \#\mathscr{N}^{\mathrm{ra}}_{X_1\cap X_2}$.

On the other hand, the natural $k[\mu_n]$-modules $H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k$ and $H^1(\Gamma_{Y_i^\bullet}, \mathbb{F}_p) \otimes k$, $i \in \{1, 2\}$, admit the following canonical decomposition

$$H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j\in\mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y^\bullet}}(j)$$

and

$$H^1(\Gamma_{Y_i^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j\in\mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y_i^\bullet}}(j),\ i \in \{1, 2\},$$

respectively, where $\Gamma_{Y_i^\bullet}$ denotes the dual semi-graph of $Y_i^\bullet$. We put

$$\dim_k(M_{\Gamma_{X_1\cap X_2}}(1)) \overset{\mathrm{def}}{=} \dim_k(M_{\Gamma_{Y^\bullet}}(1)) - \dim_k(M_{\Gamma_{Y_1^\bullet}}) - \dim_k(M_{\Gamma_{Y_2^\bullet}}).$$

Then we see immediately that $\dim_k(M_{\Gamma_{X_1\cap X_2}}(1)) = \#\mathscr{N}^{\mathrm{et}}_{X_1\cap X_2}$, and that

$$\gamma_{(\alpha,D)} = \gamma_{(\alpha_1,D_{\alpha_1})} + \gamma_{(\alpha_2,D_{\alpha_2})} + \dim_k(M_{\Gamma_{X_1\cap X_2}}(1))$$

$$\leq \begin{cases} g_{X_1} + g_{X_2} + \#(X_1 \cap X_2) - 2 = g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_{X_1} + s(D_{\alpha_1}) + g_{X_2} + s(D_{\alpha_2}) + \#\mathscr{N}^{\mathrm{et}}_{X_1\cap X_2} - 2 = g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset. \end{cases}$$

Thus, we have that

$$\gamma_{(\alpha,D)} = \begin{cases} g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset \end{cases}$$

if and only if, for each $i \in \{1, 2\}$,

$$\gamma_{(\alpha_i,D_{\alpha_i})} = \begin{cases} g_{X_i}, & \text{if } \alpha_i = 0, \\ g_{X_i} - 1, & \text{if } \alpha_i \neq 0,\ \mathrm{Supp}(D_{\alpha_i}) = \emptyset, \\ g_{X_i} + s(D_{\alpha_i}) - 1, & \text{if } \alpha_i \neq 0,\ \mathrm{Supp}(D_{\alpha_i}) \neq \emptyset. \end{cases}$$

By induction, the lemma follows from the lemma when $\#v(\Gamma_{X^\bullet}) = m-1$ and $\#v(\Gamma_{X^\bullet}) = 1$. This completes the proof of the lemma. □

Definition 3.2. We put

$$\gamma^{\max}_{X^\bullet} \overset{\mathrm{def}}{=} \max_{t\in\mathbb{N}}\{\gamma_{(\alpha,D_\alpha)} \mid \alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}) \text{ and } \alpha \neq 0\}$$

$$= \max_{m\in\mathbb{N} \text{ s.t. } (m,p)=1}\{\gamma_{(\alpha,D_\alpha)} \mid \alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/m\mathbb{Z}) \text{ and } \alpha \neq 0\}.$$

We shall say $\gamma^{\max}_{X^\bullet}$ *the maximum of generalized Hasse-Witt invariant of prime-to-p cyclic admissible coverings* of $X^\bullet$.

Remark 3.2.1. Note that Lemma 3.1 implies that

$$\gamma^{\max}_{X^\bullet} \leq \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Lemma 3.3. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ and*

$$\mathrm{ord}_x(D) = \sum_{j=0}^{t-1} d_{x,j}p^j, \ x \in D_X$$

the p-adic expansion. Suppose that $s(D) = n_X - 1$ (i.e., $\deg(D) = (n_X - 1)n$). Then $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, 1, \ldots, t-1\}$ if and only if

$$\sum_{x \in D_X} d_{x,j} = (n_X - 1)(p-1), \ j \in \{0, \ldots, t-1\}.$$

Proof. The "if" part of the lemma is trivial. We only prove the "only if" part of the lemma. Let $\mathrm{ord}_x(D) \overset{\text{def}}{=} d_x$, $x \in D_X$. Since $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, \ldots, t-1\}$ and $n|\deg(D^{(i)})$, we have

$$\deg(D^{(i)}) = \sum_{x \in D_X} (\mathrm{ord}_x(D))^{(i)} = \sum_{x \in D_X} d_x^{(i)} = (n_X - 1)n,$$

where $d_x^{(i)} \overset{\text{def}}{=} (\mathrm{ord}_x(D))^{(i)}$. Moreover, for each $i \in \{0, \ldots, t-1\}$, we have

$$d_x^{(i+1)} = d_{x,i}p^{t-1} + \frac{d_x^{(i)} - d_{x,i}}{p} = \frac{1}{p}d_x^{(i)} + \frac{p^t - 1}{p}d_{x,i} = \frac{1}{p}d_x^{(i)} + \frac{n}{p}d_{x,i}.$$

Thus, we obtain that

$$\begin{aligned} (n_X - 1)n = \sum_{x \in D_x} d_x^{(i+1)} &= \frac{1}{p}\sum_{x \in D_X} d_x^{(i)} + \frac{n}{p}\sum_{x \in D_x} d_{x,i} \\ &= \frac{1}{p}(n_X - 1)n + \frac{n}{p}\sum_{x \in D_x} d_{x,i}. \end{aligned}$$

This means that

$$\sum_{x \in D_X} d_{x,i} = (n_X - 1)(p-1), \ i \in \{0, \ldots, t-1\}.$$

We complete the proof of the lemma. $\square$

Remark 3.3.1. Note that there exists $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ such that $s(D) = (n_X - 1)n$ if and only if $n > n_X - 1$. Lemma 3.3 implies that, if $n > n_X - 1$, then there exists $\alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$ such that $\alpha \neq 0$, that $s(D_\alpha) = n_X - 1$, and that

$$\deg(D_\alpha^{(i)}) \geq \deg(D_\alpha), \ i \in \{0, 1, \ldots, t-1\}.$$

Lemma 3.4. *We maintain the notation introduced in Lemma 3.3. We put* $D_X \stackrel{\mathrm{def}}{=} \{x_1, \dots, x_{n_X}\}$ *and put*

$$a_{l,l+1} \stackrel{\mathrm{def}}{=} [\sum_{r=l+1}^{n_X} d_{x_r}],\ b_{l,l+1} \stackrel{\mathrm{def}}{=} [\sum_{r=1}^{l} d_{x_r}],\ l \in \{2, \dots, n_X - 2\},$$

where $[(-)]$ *denotes the integer which is equal to the image of* $(-)$ *in* $\mathbb{Z}/n\mathbb{Z}$ *when we identify* $\{0, \dots, n-1\}$ *with* $\mathbb{Z}/n\mathbb{Z}$ *naturally. Then, for each* $i \in \{0, \dots, t-1\}$, *we have*

$$d^{(i)}_{x_1} + d^{(i)}_{x_2} + a^{(i)}_{2,3} = 2n,$$

$$b^{(i)}_{n_X-2,n_X-1} + d^{(i)}_{x_{n_X-1}} + d^{(i)}_{x_{n_X}} = 2n,$$

$$b^{(i)}_{l,l+1} + d^{(i)}_{x_{l+1}} + a^{(i)}_{l+1,l+2} = 2n,\ l \in \{2, \dots, n_X - 2\},$$

and

$$a^{(i)}_{l,l+1} + b^{(i)}_{l,l+1} = n,\ l \in \{2, \dots, n_X - 2\}.$$

Proof. Let $l \in \{2, \dots, n_X - 2\}$ and $i \in \{0, \dots, t-1\}$. The fourth equality follows immediately from the definitions of $a_{l,l+1}$ and $b_{l,l+1}$. Let us treat the third equality.

Let

$$a^{(i)}_{l,l+1} \stackrel{\mathrm{def}}{=} \sum_{j=0}^{t-1} a^{(i)}_{l,l+1,j} p^j,\ b^{(i)}_{l,l+1} \stackrel{\mathrm{def}}{=} \sum_{j=0}^{t-1} b^{(i)}_{l,l+1,j} p^j,\ \text{and } d^{(i)}_x \stackrel{\mathrm{def}}{=} \sum_{j=0}^{t-1} d^{(i)}_{x,j} p^j,\ x \in D_X,$$

be the p-adic expansions of $a^{(i)}_{l,l+1}$, $b^{(i)}_{l,l+1}$, and $d^{(i)}_x$, $x \in D_X$, respectively. Lemma 3.3 implies immediately that

$$a^{(i)}_{l,l+1,j} = [\sum_{r=l+1}^{n_X} d^{(i)}_{x_r,j}],\ b^{(i)}_{l,l+1,j} = [\sum_{r=1}^{l} d^{(i)}_{x_r,j}],\ j \in \{0, \dots, t-1\}.$$

Moreover, we have

$$b^{(i)}_{l,l+1,j} + d^{(i)}_{x_{l+1},j} + a^{(i)}_{l+1,l+2,j} = 2(p-1),\ j \in \{0, \dots, t-1\}.$$

This means that

$$b^{(i)}_{l,l+1} + d^{(i)}_{x_{l+1}} + a^{(i)}_{l+1,l+2} = 2n.$$

By applying similar arguments to the arguments given in the proof above, the first and the second equality hold. This completes the proof of the lemma. □

Lemma 3.5. *Let* $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ *and* $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ *such that* $\alpha \neq 0$, *that* $s(D) = n_X - 1$ *if* $n_X \neq 0$, *and that*

$$\deg(D^{(i)}) \geq \deg(D),\ i \in \{0, 1, \dots, t-1\}.$$

Moreover, suppose that $X^\bullet = (X, D_X \stackrel{\mathrm{def}}{=} \{x_1, \dots, x_{n_X}\})$ *is a component-generic pointed stable curve over* k, *and that* $X^\bullet$ *is smooth over* k, *and that* $(g_X, n_X) = (0, 3)$. *Then the Raynaud-Tamagawa theta divisor* $\Theta_{\mathcal{E}_D}$ *associated to* $\mathcal{E}_D$ *exists. Moreover, we have*

$$\gamma_{([\mathcal{L}],D)} = \dim_k(H^1(X, \mathcal{L}))$$

for each pair $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$.

Proof. This follows immediately from [B, Corollary 6.8]. $\square$

Remark 3.5.1. Note that, if $n_X = 3$, then we have $s(D) \in \{0, 1, 2\}$.

Let R be a discrete valuation ring with algebraically closed residue field k_R, K_R the quotient field of R, and $\overline{K}_R$ an algebraic closure of K_R. Suppose that $k \subseteq K_R$. Let

$$\mathcal{X}^\bullet = (\mathcal{X}, D_\mathcal{X} \stackrel{\text{def}}{=} \{e_1, \dots, e_{n_X}\})$$

be a pointed stable curve of type (g_X, n_X) over R. We shall write $\mathcal{X}^\bullet_\eta = (\mathcal{X}_\eta, D_{\mathcal{X}_\eta} \stackrel{\text{def}}{=} \{e_{\eta,1}, \dots, e_{\eta,n_X}\})$, $\mathcal{X}^\bullet_{\overline{\eta}} = (\mathcal{X}_{\overline{\eta}}, D_{\mathcal{X}_{\overline{\eta}}} \stackrel{\text{def}}{=} \{e_{\overline{\eta},1}, \dots, e_{\overline{\eta},n_X}\})$, $\mathcal{X}^\bullet_s = (\mathcal{X}_s, D_{\mathcal{X}_s} \stackrel{\text{def}}{=} \{e_{s,1}, \dots, e_{s,n_X}\})$ for the generic fiber $\mathcal{X}^\bullet \times_R K_R$ of $\mathcal{X}^\bullet$, the geometric generic fiber $\mathcal{X}^\bullet \times_R \overline{K}_R$ of $\mathcal{X}^\bullet$, and the special fiber $\mathcal{X}^\bullet \times_R k_R$ of $\mathcal{X}^\bullet$, respectively. Write $\Pi_{\mathcal{X}^\bullet_{\overline{\eta}}}$ and $\Pi_{\mathcal{X}^\bullet_s}$ for the admissible fundamental groups of $\mathcal{X}^\bullet_{\overline{\eta}}$ and $\mathcal{X}^\bullet_s$, respectively. Since the admissible fundamental groups do not depend on the base fields, $\Pi_{\mathcal{X}^\bullet_{\overline{\eta}}}$ is naturally isomorphic to $\Pi_{X^\bullet}$. Moreover, we shall say that $X^\bullet$ *admits a (DEG)* if the following conditions hold, where "(DEG)" means "degeneration":

(i) The geometric generic fiber $\mathcal{X}^\bullet_{\overline{\eta}}$ of $\mathcal{X}^\bullet$ is $\overline{K}_R$-isomorphic to $X^\bullet \times_k \overline{K}_R$. Then without loss of generality, we may identify $e_{\overline{\eta},r}$, $r \in \{1, \dots, n_X\}$, with $x_r \times_k \overline{K}_R$ via this isomorphism.

(ii) $\mathcal{X}^\bullet_s$ is a component-generic pointed stable curve over k_R.

(iii) If $n_X \leq 1$ and $\#X^{\text{sing}} = 0$ (i.e., $X^\bullet$ is smooth over k_R), we have $\mathcal{X}^\bullet \to \operatorname{Spec} R$ is isotrivial (i.e., the image of the natural morphism $\operatorname{Spec} R \to \overline{\mathcal{M}}_{g_X,n_X,k_R} \to \overline{M}_{g_X,n_X,k_R}$ determined by $\mathcal{X}^\bullet \to \operatorname{Spec} R$ is a point, where $\overline{\mathcal{M}}_{g_X,n_X,k_R} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \times_{\mathbb{F}_p} k_R$ and $\overline{M}_{g_X,n_X,k_R}$ denotes the coarse moduli space of $\overline{\mathcal{M}}_{g_X,n_X,k_R}$).

(iv) If $(g_X, n_X) = (1, 1)$ and $\#X^{\text{sing}} = 1$, we have that $\mathcal{X}^\bullet \to \operatorname{Spec} R$ is isotrivial.

(v) If $n_X \leq 1$ and $\#X^{\text{sing}} \geq 2$ hold, we have

$$\mathcal{X}_s = (\bigcup_{T \in \mathscr{T}} T) \cup C$$

such that the following conditions hold: (a) $\mathscr{T}$ is a set of singular pointed stable curves of type $(1, 0)$ over k_R such that $\#\mathscr{T} = \#X^{\text{sing}}$. (b) C is an empty set when $n_X = 0$, $\#X^{\text{sing}} = 2$, and $g_X = \#\mathscr{T}$; otherwise, C is a geometric generic pointed stable curve of type $(g_X - \#\mathscr{T}, 0)$ over k_R. (c) If C is empty, we have $\mathscr{T} \stackrel{\text{def}}{=} \{T_1, T_2\}$ such that $\#(T_1 \cap T_2) = 1$. (d) If C is not empty, we have that $T' \cap T'' \neq \emptyset$ if and only if $T' = T''$ for each $T', T'' \in \mathscr{T}$, that $\#(T \cap C) = 1$ for each $T \in \mathscr{T}$, and that $D_{\mathcal{X}_s} \subseteq C$.

(vi) If $n_X = 2$, we have

$$\mathcal{X}_s = (\bigcup_{T \in \mathscr{T}} T) \cup C \cup P$$

such that the following conditions hold: (a) $\mathscr{T}$ is a set of singular pointed stable curves of type $(1, 0)$ over k_R such that, for each $T, T' \in \mathscr{T}$, $T \cap T' \neq \emptyset$ if and only if $T = T'$, and that $\#\mathscr{T} = \#X^{\text{sing}}$. (b) C is either an empty set when $g_X - \#\mathscr{T} = 0$ or a geometric generic pointed stable curve of type $(g_X - \#\mathscr{T}, 0)$ over k_R when $g_X - \#\mathscr{T} \geq 1$. (c) P is k_R-isomorphic to $\mathbb{P}^1_{k_R}$. (d) If C is empty, we have $\#(P \cap T) = 1$ for each $T \in \mathscr{T}$. (e) If

C is not empty, we have that $\#(C \cap T) = 1$, that $\#(C \cap P) = 1$, and that $P \cap T = \emptyset$ for each $T \in \mathscr{T}$. (f) $D_{\mathcal{X}_s} \subseteq P$.

(vii) If $n_X \geq 3$, we have

$$\mathcal{X}_s = (\bigcup_{T \in \mathscr{T}} T) \cup C_1 \cup (\bigcup_{v=2}^{n_X - 1} P_v)$$

such that the following conditions hold: (a) $\mathscr{T}$ is a set of singular pointed stable curves of type $(1, 0)$ over k_R such that, for each $T, T' \in \mathscr{T}$, $T \cap T' \neq \emptyset$ if and only if $T = T'$, and that $\#\mathscr{T} = \#X^{\mathrm{sing}}$. (b) C_1 is either an empty set when $g_X - \#\mathscr{T} = 0$ or a geometric generic pointed stable curve of type $(g_X - \#\mathscr{T}, 0)$ over k_R when $g_X - \#\mathscr{T} \geq 1$. (c) P_v, $v \in \{2, \ldots, n_X - 1\}$, is k_R-isomorphic to $\mathbb{P}^1_{k_R}$. (d) If C_1 is empty, we have $\#(P_2 \cap T) = 1$ and $P_v \cap T = \emptyset$ for each $T \in \mathscr{T}$ and each $v \in \{3, \ldots, n_X - 1\}$. (e) If C_1 is not empty, we have $\#(C_1 \cap T) = 1$, $\#(C_1 \cap P_2) = 1$, and $P_v \cap T = \emptyset$ for each $T \in \mathscr{T}$ and each $v \in \{2, \ldots, n_X - 1\}$. (f) For each $v \in \{2, \ldots, n_X - 2\}$, $\#(P_v \cap P_{v+1}) = 1$ and $P_v \cap P_{v'} = \emptyset$ when $v' \notin \{v - 1, v, v + 1\}$. (g) If $n_X = 3$, we have $D_{\mathcal{X}_s} \cap P_2 = \{e_{s,1}, e_{s,2}, e_{s,3}\}$. (i) If $n_X = 4$, we have $D_{\mathcal{X}_s} \cap P_2 = \{e_{s,1}, e_{s,2}\}$ and $D_{\mathcal{X}_s} \cap P_3 = \{e_{s,3}, e_{s,4}\}$. (j) If $n_X \geq 5$, we have $D_{\mathcal{X}_s} \cap P_2 = \{e_{s,1}, e_{s,2}\}$, $D_{\mathcal{X}_s} \cap P_{n_X - 1} = \{e_{s,n_X - 1}, e_{s,n_X}\}$, and $D_{\mathcal{X}_s} \cap P_v = \{e_{s,v}\}$, $v \in \{3, \ldots, n_X - 2\}$.

Proposition 3.6. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ and $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$, that $s(D) = n_X - 1$ if $n_X \neq 0$, and that*

$$\deg(D^{(i)}) \geq \deg(D),\ i \in \{0, 1, \ldots, t - 1\}.$$

Moreover, suppose that $X^\bullet = (X, D_X \overset{\mathrm{def}}{=} \{x_1, \ldots, x_{n_X}\})$ is a component-generic pointed stable curve over k, and that $X^\bullet$ is irreducible. Then we have that $\gamma_{(\alpha,D)}$ attains the maximum

$$\gamma^{\max}_{X^\bullet} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. Let $f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$ be the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. We note that, to verify the proposition, we only need to prove the proposition in the case where $Y^\bullet$ is *connected*. Then we may assume that $Y^\bullet$ is connected.

Since $X^\bullet$ is a component-generic pointed stable curve, $X^\bullet$ admits a (DEG). Furthermore, we write $Q_{\overline{\eta}}$ (resp. Q_s) for the effective divisor on $\mathcal{X}_{\overline{\eta}}$ (resp. $\mathcal{X}_s$) induced by D and $\alpha_{\overline{\eta}} \in \mathrm{Rev}^{\mathrm{adm}}_{Q_{\overline{\eta}}}(\mathcal{X}^\bullet_{\overline{\eta}})$ for the element induced by α. Then we have

$$\gamma_{(\alpha,D)} = \gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})}.$$

Suppose that $X^\bullet$ satisfies (DEG)-(iii). If $n_X \leq 1$ and $g_X = 1$, then the proposition is trivial. If $n_X \leq 1$ and $g_X \geq 2$, then the proposition follows immediately from [N, Proposition 4] (or [Z, Théorème 3.1]).

Suppose that $X^\bullet$ satisfies (DEG)-(iv). Then we see immediately that $Y^\bullet$ is a pointed stable curve of type $(1, n)$ such that one of the following conditions holds: (1) $\#Y^{\mathrm{sing}} = 1$

and the normalization of Y is a rational curve; (2) $\#Y^{\mathrm{sing}} = n$ and the normalization of each irreducible component of Y is a rational curve. Thus, we obtain $\gamma_{(\alpha,D)} = 1$.

Suppose that $X^\bullet$ satisfies (DEG)-(vii). Moreover, we suppose that $C_1 \neq \emptyset$, and that $n_X \geq 5$. For each $v \in \{2, \ldots, n_X - 2\}$, we write

$$y_{v,v+1} \text{ and } z_{v,v+1}$$

for the inverse image of $P_v \cap P_{v+1}$ of the natural closed immersion $P_v \hookrightarrow \mathcal{X}_s$ and the inverse image of $P_v \cap P_{v+1}$ of the natural closed immersion $P_{v+1} \hookrightarrow \mathcal{X}_s$, respectively. We define

$$P_2^\bullet = (P_2, D_{P_2} \overset{\mathrm{def}}{=} \{e_{s,1}, e_{s,2}, y_{2,3}\} \cup (C_1 \cap P_2)),$$

$$P_{n_X-1}^\bullet = (P_{n_X-1}, D_{P_{n_X-1}} \overset{\mathrm{def}}{=} \{z_{n_X-2,n_X-1}, e_{s,n_X-1}, e_{s,n_X}\}),$$

and

$$P_v^\bullet = (P_v, D_{P_v} \overset{\mathrm{def}}{=} \{z_{v-1,v}, e_{s,v}, y_{v,v+1}\}),\ v \in \{3, \ldots, n_X - 2\},$$

to be smooth pointed stable curves of types $(0,4)$, $(0,3)$, and $(0,3)$ over k_R, respectively. Moreover, we define

$$C_1^\bullet = (C_1, D_{C_1} \overset{\mathrm{def}}{=} (C_1 \cap P_2) \cup (\bigcup_{T \in \mathscr{T}} T) \cap C_1)$$

and

$$T^\bullet = (T, D_T \overset{\mathrm{def}}{=} \{T \cap C_1\}),\ T \in \mathscr{T},$$

to be smooth pointed stable curves of types $(g_X, 1 + \#\mathscr{T})$ and $(1,1)$ over k_R, respectively. Note that $\sigma_{C_1^\bullet} = \sigma_{C_1} = g_X$. Let

$$f_{\overline{\eta}}^\bullet \overset{\mathrm{def}}{=} f^\bullet \times_k \overline{K}_R : \mathcal{Y}_{\overline{\eta}}^\bullet = (\mathcal{Y}_{\overline{\eta}}, D_{\mathcal{Y}_{\overline{\eta}}}) \overset{\mathrm{def}}{=} Y^\bullet \times_k \overline{K}_R \to \mathcal{X}_{\overline{\eta}}^\bullet$$

be the Galois admissible covering over $\overline{K}_R$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by $f^\bullet$, and $\Pi_{\mathcal{Y}_{\overline{\eta}}^\bullet} \subseteq \Pi_{\mathcal{X}_{\overline{\eta}}^\bullet}$ the admissible fundamental group of $\mathcal{Y}_{\overline{\eta}}^\bullet$. By the specialization theorem of maximal prime-to-p quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]), we have

$$sp_R^{p'} : \Pi_{\mathcal{X}_{\overline{\eta}}^\bullet}^{p'} \xrightarrow{\sim} \Pi_{\mathcal{X}_s^\bullet}^{p'},$$

where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Then we obtain a normal open subgroup $\Pi_{\mathcal{Y}_s^\bullet}^{p'} \overset{\mathrm{def}}{=} sp_R^{p'}(\Pi_{\mathcal{Y}_{\overline{\eta}}^\bullet}^{p'}) \subseteq \Pi_{\mathcal{X}_s^\bullet}^{p'}$. Write $\Pi_{\mathcal{Y}_s^\bullet} \subseteq \Pi_{\mathcal{X}_s^\bullet}$ for the inverse image of $\Pi_{\mathcal{Y}_s^\bullet}^{p'}$ of the natural surjection $\Pi_{\mathcal{X}_s^\bullet} \twoheadrightarrow \Pi_{\mathcal{X}_s^\bullet}^{p'}$. Then $\Pi_{\mathcal{Y}_s^\bullet}$ determines a Galois admissible covering

$$f_s^\bullet : \mathcal{Y}_s^\bullet = (\mathcal{Y}_s, D_{\mathcal{Y}_s}) \to \mathcal{X}_s^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_s \in \mathrm{Rev}_{Q_s}^{\mathrm{adm}}(\mathcal{X}_s^\bullet)$ for an element induced by $f_s^\bullet$.

The structure of the maximal prime-to-p quotients of admissible fundamental groups implies that f_s is étale over $(\bigcup_{T \in \mathscr{T}} T) \cap C_1$. Then we obtain that f_s is étale over $C_1 \cap P_2$. Thus, f_s is étale over D_{C_1}. Let $Y_v \overset{\mathrm{def}}{=} f_s^{-1}(P_v)$, $v \in \{2, \ldots, n_X - 1\}$. We put

$$Y_v^\bullet \overset{\mathrm{def}}{=} (Y_v, D_{Y_v} \overset{\mathrm{def}}{=} f_s^{-1}(D_{P_v})),\ v \in \{2, \ldots, n_X - 1\}.$$

Then $f_s^\bullet$ induces a Galois multi-admissible covering

$$f_v^\bullet : Y_v^\bullet \to P_v^\bullet,\ v \in \{2, \dots, n_X - 1\},$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. We maintain the notation introduced in Lemma 3.4 and define the following effective divisors

$$Q_2 \stackrel{\text{def}}{=} d_{x_1} e_{s,1} + d_{x_2} e_{s,2} + a_{2,3} y_{2,3},$$

$$Q_{n_X-1} \stackrel{\text{def}}{=} b_{n_X-2,n_X-1} z_{n_X-2,n_X-1} + d_{x_{n_X-1}} e_{s,n_X-1} + d_{x_{n_X}} e_{s,n_X},$$

and

$$Q_v \stackrel{\text{def}}{=} b_{v-1,v} z_{v-1,v} + d_{x_v} e_{s,v} + a_{v,v+1} y_{v,v+1},\ v \in \{3, \dots, n_X - 2\},$$

on P_2, P_{n_X-1}, and P_v, $v \in \{3, \dots, n_X - 2\}$, respectively. Since f_s is étale over $C_1 \cap P_2$, we see immediately that $f_v^\bullet$, $v \in \{2, \dots, n_X-1\}$, induces a pair $([\mathcal{L}_v], Q_v) \in \widetilde{\mathscr{P}}_{P_v^\bullet, n}$. Moreover, the $k_R[\mu_n]$-module $H^1_{\text{ét}}(Y_v, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Y_v, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{Y_v}(j),$$

where $\zeta \in \mu_n$ acts on $M_{Y_v}(j)$ as the ζ^j-multiplication. Lemma 3.4 implies that $\deg(Q_v^{(i)}) = \deg(Q_v) = 2n$, $i \in \{0, \dots, t-1\}$. Then Lemma 3.5 implies that

$$\gamma_{([\mathcal{L}_v], Q_v)} = \dim_{k_R}(M_{Y_v}(1)) = \dim_{k_R}(H^1(P_v, \mathcal{L}_v)) = 1.$$

Let $Z_1 \stackrel{\text{def}}{=} f_s^{-1}(C_1)$ and $\pi_0(Z_1)$ the set of connected components of Z_1. Then $f_s^\bullet$ induces a Galois étale covering (not necessarily connected)

$$f_{C_1}^\bullet : Z_1^\bullet = (Z_1, D_{Z_1} \stackrel{\text{def}}{=} f_s^{-1}(D_{C_1})) \to C_1^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, $f_{C_1}^\bullet$ induces an element $\alpha_{C_1} \in \text{Rev}_0^{\text{adm}}(C_1^\bullet)$. Suppose that $\#\pi_0(Z_1) \neq n$. Then we have $\alpha_{C_1} \neq 0$. The $k_R[\mu_n]$-module $H^1_{\text{ét}}(Z_1, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Z_1, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{Z_1}(j),$$

where $\zeta \in \mu_n$ acts on $M_{Z_1}(j)$ as the ζ^j-multiplication. [N, Proposition 4] (or [Z, Théorème 3.1]) implies that

$$\gamma_{(\alpha_{C_1}, 0)} = \dim_{k_R}(M_{Z_1}(1)) = g_X - \#\mathscr{T} - 1 = g_{C_1} - 1,$$

where g_{C_1} denotes the genus of C_1. Suppose that $\#\pi_0(Z_1) = n$. Then we have $\alpha_{C_1} = 0$. Since C_1 is ordinary, we obtain immediately that

$$\gamma_{(\alpha_{C_1}, 0)} = \sigma(C_1) = g_X - \#\mathscr{T} = g_{C_1}.$$

Let $V_T \stackrel{\text{def}}{=} f_s^{-1}(T)$, $T \in \mathscr{T}$, and $\mathrm{norm}_T : \widetilde{T} \to T$ the normalization morphism. Then $f_s^\bullet$ induces a Galois multi-admissible covering

$$f_T^\bullet : V_T^\bullet = (V_T, D_{V_T} \stackrel{\text{def}}{=} f_s^{-1}(D_T)) \to T^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Since f_s is étale over D_T, we have that the normalization of each irreducible component of V_T is a rational curve over k_R. We put

$$\widetilde{T}^\bullet \stackrel{\text{def}}{=} (\widetilde{T}, D_{\widetilde{T}} \stackrel{\text{def}}{=} \mathrm{norm}_T^{-1}(D_T)).$$

Then $f_T^\bullet$ induces a Galois multi-admissible covering

$$f_{\widetilde{T}}^\bullet : V_{\widetilde{T}}^\bullet = (V_{\widetilde{T}}, D_{V_{\widetilde{T}}}) \to \widetilde{T}^\bullet$$

over k_R, where $V_{\widetilde{T}}$ is the normalization of V_T. Write $\alpha_{\widetilde{T}} \in \mathrm{Rev}_0(\widetilde{T}^\bullet)$ for an element induced by $f_{\widetilde{T}}^\bullet$. Then we obtain that $\gamma_{(\alpha_{\widetilde{T}},0)} = 1$ if V_T is not connected, and that $\gamma_{(\alpha_{\widetilde{T}},0)} = 0$ if V_T is connected. Thus, Lemma 3.1 implies that

$$\gamma_{(\alpha_s,Q_s)} = g_X + n_X - 2.$$

On the other hand, the $k_R[\mu_n]$-modules $H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$ and $H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R$ admit the following canonical decompositions

$$H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_{\overline{\eta}}}(j)$$

and

$$H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_s}(j),$$

respectively. Moreover, we have an injection as $k_R[\mu_n]$-modules

$$H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R \hookrightarrow H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$$

induced by the specialization map $\Pi_{\mathcal{Y}_{\overline{\eta}}^\bullet} \twoheadrightarrow \Pi_{\mathcal{Y}_s^\bullet}$. Thus, we have

$$g_X + n_X - 2 \leq \gamma_{(\alpha_s,Q_s)} = \dim_{k_R}(M_{\mathcal{Y}_s}(1)) \leq \gamma_{(\alpha_{\overline{\eta}},Q_{\overline{\eta}})} = \dim_{k_R}(M_{\mathcal{Y}_{\overline{\eta}}}(1)).$$

We write $\widetilde{\mathcal{X}}_{\overline{\eta}}$ for the normalization of $\mathcal{X}_{\overline{\eta}}$ and $\mathrm{norm} : \widetilde{\mathcal{X}}_{\overline{\eta}} \to \mathcal{X}_{\overline{\eta}}$ for the normalization morphism. We define

$$\widetilde{\mathcal{X}}_{\overline{\eta}}^\bullet = (\widetilde{\mathcal{X}}_{\overline{\eta}}, D_{\widetilde{\mathcal{X}}_{\overline{\eta}}} \stackrel{\text{def}}{=} \mathrm{norm}^{-1}(D_{\mathcal{X}_{\overline{\eta}}}))$$

to be a pointed stable curve of type $(g_{\widetilde{X}}, n_X)$ over $\overline{K}_R$, where $g_{\widetilde{X}} = g_X - \#\mathcal{X}_{\overline{\eta}}^{\text{sing}}$. Let $\widetilde{Q}_{\overline{\eta}} \stackrel{\text{def}}{=} \mathrm{norm}^*(Q_{\overline{\eta}})$ and $\widetilde{\alpha}_{\overline{\eta}} \in \mathrm{Hom}(\Pi^{\text{ab}}_{\widetilde{\mathcal{X}}_{\overline{\eta}}^\bullet}, \mathbb{Z}/n\mathbb{Z})$ the element induced by α via the natural (outer) injection $\Pi_{\widetilde{\mathcal{X}}_{\overline{\eta}}^\bullet} \hookrightarrow \Pi_{\mathcal{X}_{\overline{\eta}}^\bullet}$. Note that $\widetilde{\alpha}_{\overline{\eta}} \in \mathrm{Rev}^{\text{adm}}_{\widetilde{Q}_{\overline{\eta}}}(\widetilde{X}_{\overline{\eta}}^\bullet)$. Then we have

$$\gamma_{(\widetilde{\alpha}_{\overline{\eta}},\widetilde{Q}_{\overline{\eta}})} \leq g_{\widetilde{X}} + n_X - 2.$$

Since $\gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = \gamma_{(\widetilde{\alpha}_{\overline{\eta}}, \widetilde{Q}_{\overline{\eta}})} + \#\mathcal{X}^{\mathrm{sing}}$, we obtain that $\gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} \leq g_X + n_X - 2$. Then we obtain that

$$\gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = g_X + n_X - 2.$$

This completes the proof of the proposition when $X^\bullet$ satisfies (DEG)-(vii), $C_1 \neq \emptyset$, and $n_X \geq 5$. By applying similar arguments to the arguments given in the proof above, one can prove the proposition immediately when $X^\bullet$ satisfies (DEG)-(vii) and either $C_1 = \emptyset$ or $n_X \leq 4$ holds.

Moreover, similar arguments to the arguments given in the proof above imply the proposition holds when $X^\bullet$ satisfies either (DEG)-(v) or (DEG)-(vi). We complete the proof of the proposition. □

Definition 3.7. Let $W^\bullet = (W, D_W)$ be a pointed stable curve of type (g_W, n_W) over k and $\Gamma_{W^\bullet}$ the dual semi-graph of $W^\bullet$.

(i) Let $\Gamma \stackrel{\mathrm{def}}{=} (v(\Gamma), e^{\mathrm{op}}(\Gamma) \cup e^{\mathrm{cl}}(\Gamma), \{\zeta_e^\Gamma\}_{e \in e^{\mathrm{op}}(\Gamma) \cup e^{\mathrm{cl}}(\Gamma)})$ be a *connected* semi-graph such that the following conditions hold: (a) $\Gamma \setminus e^{\mathrm{lp}}(\Gamma)$ is a tree. (b) $v(\Gamma) \subseteq v(\Gamma_{W^\bullet})$. (c) $e^{\mathrm{cl}}(\Gamma) \subseteq e^{\mathrm{cl}}(\Gamma_{W^\bullet})$ and $e^{\mathrm{lp}}(\Gamma) \subseteq e^{\mathrm{lp}}(\Gamma_{X^\bullet})$. Moreover, we have $\zeta_e^\Gamma(e) = \zeta_e^{\Gamma_{W^\bullet}}(e)$ if $e \in e^{\mathrm{cl}}(\Gamma)$. (d) $e^{\mathrm{op}}(\Gamma_{W^\bullet}) \subseteq e^{\mathrm{op}}(\Gamma)$. Moreover, we have $\zeta_e^\Gamma(e) \cap v(\Gamma) = \zeta_e^{\Gamma_{W^\bullet}}(e) \cap v(\Gamma_{W^\bullet})$ if $e \in e^{\mathrm{op}}(\Gamma_{W^\bullet})$. (e) Let $e \in e^{\mathrm{op}}(\Gamma) \setminus e^{\mathrm{op}}(\Gamma_{W^\bullet})$. Then e satisfies one of the following conditions:

(1) There exists $e' \in e^{\mathrm{cl}}(\Gamma_{W^\bullet}) \setminus e^{\mathrm{lp}}(\Gamma_{W^\bullet})$ such that $v^{\Gamma_{W^\bullet}}(e') \cap v(\Gamma) \neq \emptyset$, $v^{\Gamma_{W^\bullet}}(e') \cap (v(\Gamma_{W^\bullet}) \setminus v(\Gamma)) \neq \emptyset$, and $v^\Gamma(e) \cap v(\Gamma) = v^{\Gamma_{W^\bullet}}(e') \cap v(\Gamma)$. Moreover, we have $\zeta_e^\Gamma(e) = \{v^\Gamma(e) \cap v(\Gamma), \{v(\Gamma)\}\}$.

(2) There exists $e' \in e^{\mathrm{cl}}(\Gamma_{W^\bullet}) \setminus e^{\mathrm{lp}}(\Gamma_{W^\bullet})$ such that $v^{\Gamma_{W^\bullet}}(e') \subseteq v(\Gamma)$ and $v^\Gamma(e) \cap v^{\Gamma_{W^\bullet}}(e') \neq \emptyset$. Moreover, we have $\zeta_e^\Gamma(e) = \{v^\Gamma(e) \cap v^{\Gamma_{W^\bullet}}(e'), \{v(\Gamma)\}\}$.

By the definition of Γ, we obtain a natural morphism of semi-graphs

$$\phi_\Gamma : \Gamma \to \Gamma_{W^\bullet}$$

defined as follows: $v \mapsto v, v \in v(\Gamma)$, $e \mapsto e, e \in e^{\mathrm{cl}}(\Gamma) \cup e^{\mathrm{op}}(\Gamma_{W^\bullet})$, $e \mapsto e', e \in e^{\mathrm{op}}(\Gamma) \setminus e^{\mathrm{op}}(\Gamma_{W^\bullet})$. We shall say that Γ is a *quasi-tree associated* D_W if the map of $e^\Gamma(v) \to e^{\Gamma_{W^\bullet}}(v)$, $v \in v(\Gamma)$, induced by ϕ_Γ is a surjection.

Let $E_j \subseteq e^{\mathrm{cl}}(\Gamma_{W^\bullet})$, $j \in \{1, 2\}$, be the subset of closed edges whose elements are the images of open edges of Γ satisfied condition (e)-(j) above. We define a semi-graph Γ^{im} as follows: (a) $v(\Gamma^{\mathrm{im}}) \stackrel{\mathrm{def}}{=} v(\Gamma)$; (b) $e^{\mathrm{cl}}(\Gamma^{\mathrm{im}}) \stackrel{\mathrm{def}}{=} \phi_\Gamma(e^{\mathrm{cl}}(\Gamma)) \cup E_2$; (c) $e^{\mathrm{op}}(\Gamma^{\mathrm{im}}) \stackrel{\mathrm{def}}{=} E_1$; (d) $\zeta_e^{\Gamma^{\mathrm{im}}}(e) \stackrel{\mathrm{def}}{=} \zeta_e^\Gamma(\phi_\Gamma^{-1}(e)) \cap v(\Gamma)$ for each $e \in e^{\mathrm{cl}}(\Gamma^{\mathrm{im}})$; (e) $\zeta_e^{\Gamma^{\mathrm{im}}}(e) \stackrel{\mathrm{def}}{=} \{\zeta_e^\Gamma(\phi_\Gamma^{-1}(e)) \cap v(\Gamma), \{v(\Gamma^{\mathrm{im}})\}\}$ for each $e \in e^{\mathrm{op}}(\Gamma^{\mathrm{im}})$. We shall say that Γ^{im} is the *image of the morphism* ϕ_Γ. Then we obtain a sub-semi-stable curve $W_{\Gamma^{\mathrm{im}}}$ of W whose irreducible components are the irreducible components corresponding to $v(\Gamma)$, and whose set of nodes are the set of nodes corresponding to $\phi_\Gamma(e^{\mathrm{cl}}(\Gamma)) \cup E_2$. Moreover, we obtain a set of marked points $D_{W_{\Gamma^{\mathrm{im}}}})$ of $W_{\Gamma^{\mathrm{im}}}$ whose elements are the closed points of the inverse images of E_1 via the natural closed immersion $W_{\Gamma^{\mathrm{im}}} \hookrightarrow W$. Then we define a pointed stable curve associated to Γ^{im} over k to be

$$W^\bullet_{\Gamma^{\mathrm{im}}} = (W_{\Gamma^{\mathrm{im}}}, D_{W_{\Gamma^{\mathrm{im}}}}).$$

We see immediately that the dual semi-graph of $W^\bullet_{\Gamma^{\mathrm{im}}}$ is isomorphic to Γ^{im}. Let $\mathrm{norm}_\Gamma : W_\Gamma \to W_{\Gamma^{\mathrm{im}}}$ be the morphism which is an isomorphism over $W_{\Gamma^{\mathrm{im}}} \setminus \{w_e\}_{e \in E_2}$ and is a normalization over $\{w_e\}_{e\in E_2}$, where w_e denotes the closed point of W corresponding to e. We define a pointed stable curve over k associated to Γ to be

$$W^\bullet_\Gamma = (X_\Gamma, D_{X_\Gamma} \stackrel{\text{def}}{=} \text{norm}^{-1}_\Gamma(D_{W_{\Gamma^{\text{im}}}} \cup \{w_e\}_{e\in E_2})).$$

Then we see immediately that the dual semi-graph of $W^\bullet_\Gamma$ is isomorphic to Γ.

(ii) We shall say that

$$\Gamma_{D_W}$$

is a *minimal quasi-tree associated to* D_W if the following conditions hold: (a) Γ_{D_W} is a quasi-tree associated to D_W. (b) Let Γ be an arbitrary quasi-tree associated to D_W such that $\Gamma \subseteq \Gamma_{D_W}$; then $\Gamma = \Gamma_{D_W}$. Note that the definition of Γ_{D_W} implies that $\Gamma_{D_W} = \emptyset$ if $n_W = 0$.

Lemma 3.8. *Let $W^\bullet = (W, D_W)$ be a pointed stable curve of type (g_W, n_W) over k. Suppose that $n_W \neq 0$. Then the set of minimal quasi-trees associated to D_W is not empty.*

Proof. The lemma follows immediately from the definition of minimal quasi-trees associated to D_W. □

Proposition 3.9. *(i) Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ and $\alpha \in \text{Rev}^{\text{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$, that $s(D) = n_X - 1$ if $n_X \neq 0$, and that*

$$\deg(D^{(i)}) \geq \deg(D),\ i \in \{0, 1, \dots, t-1\}.$$

Moreover, suppose that $X^\bullet = (X, D_X)$ is a component-generic pointed stable curve over k, and that either $\Gamma_{X^\bullet} \setminus e^{\text{lp}}(\Gamma_{X^\bullet})$ is a tree when $n_X = 0$ or $\Gamma_{X^\bullet}$ is a minimal quasi-tree associated to D_X when $n_X \neq 0$. Then we have that $\gamma_{(\alpha,D)}$ attains the maximum

$$\gamma^{\max}_{X^\bullet} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

(ii) Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ such that $s(D) = n_X - 1$ if $n_X \neq 0$, and that

$$\deg(D^{(i)}) \geq \deg(D),\ i \in \{0, 1, \dots, t-1\}.$$

Moreover, suppose that $X^\bullet = (X, D_X)$ is a component-generic pointed stable curve over k. Then there exists an element $\beta \in \text{Rev}^{\text{adm}}_D(X^\bullet)$ such that $\beta \neq 0$, and that the generalized Hasse-Witt invariant $\gamma_{(\beta,D)}$ attains the maximum

$$\gamma^{\max}_{X^\bullet} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. (i) Let $f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$ be a Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. To verify (i), we only need to prove (i) in the case where $Y^\bullet$ is *connected*. Then we may assume that $Y^\bullet$ is connected.

Suppose that $n_X = 0$. We see immediately that f is étale over $X^{\text{sing}} \setminus \{x_e\}_{e\in e^{\text{lp}}(\Gamma_{X^\bullet})}$. Then (i) follows from Lemma 3.1 and Proposition 3.6.

Suppose that $n_X > 0$. Let $v \in v(\Gamma_{X^\bullet})$ and $\pi_0(\overline{X \setminus X_v})$ the set of connected components of $\overline{X \setminus X_v}$, where $\overline{X \setminus X_v}$ denotes the topological closure of $X \setminus X_v$ in X. We put

$$D_v \overset{\text{def}}{=} (D_X \cap X_v) \cup (\bigcup_{C \in \pi_0(\overline{X \setminus X_v})} (C \cap X_v)).$$

Note that since we assume that $\Gamma_{X^\bullet}$ is a quasi-tree associated to D_X, we have $\#(C \cap X_v) = 1$ for each $C \in \pi_0(\overline{X \setminus X_v})$. Let $x_C \overset{\text{def}}{=} C \cap X_v$, $C \in \pi_0(\overline{X \setminus X_v})$, be the unique closed point and $Q_v \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_v]^0$ an effective divisor on X_v defined as follows:

$$\mathrm{ord}_x(Q_v) \overset{\text{def}}{=} \mathrm{ord}_x(D),\ x \in D_X \cap X_v,$$

$$\mathrm{ord}_{x_C}(Q_v) \overset{\text{def}}{=} [\sum_{c \in D_X \cap C} \mathrm{ord}_c(D)],\ C \in \pi_0(\overline{X \setminus X_v}),$$

where $[(-)]$ denotes the integer which is equal to the image of $(-)$ in $\mathbb{Z}/n\mathbb{Z}$ when we identify $\{0, \dots, n-1\}$ with $\mathbb{Z}/n\mathbb{Z}$ naturally. By applying similar arguments to the arguments given in the proof of Lemma 3.4, we see immediately that

$$\deg(Q_v) = (\#D_v - 1)n \text{ and } \deg(Q_v^{(i)}) \geq \deg(Q_v),\ i \in \{0, \dots, t-1\}.$$

On the other hand, let

$$X_v^\bullet = (X_v, D_{X_v} \overset{\text{def}}{=} D_v),\ v \in v(\Gamma_{X^\bullet}),$$

be a pointed stable curve of type (g_{X_v}, n_{X_v}) over k. Then $f^\bullet$ induces a Galois multi-admissible covering

$$f_v^\bullet : Y_v^\bullet \to X_v^\bullet,\ v \in v(\Gamma_{X^\bullet}),$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_v \in \mathrm{Rev}_{Q_v}^{\mathrm{adm}}(X_v^\bullet)$ for the element induced by $f_v^\bullet$. If $\alpha_v = 0$, since $X^\bullet$ is component-generic, we have $\gamma_{(\alpha_v, Q_v)} = g_{X_v}$. Then Proposition 3.6 implies that

$$\gamma_{(\alpha_v, Q_v)} = \begin{cases} g_{X_v}, & \text{if } \alpha_v = 0, \\ g_{X_v} - 1, & \text{if } \alpha_v \neq 0,\ \mathrm{Supp}(Q_v) = \emptyset, \\ g_{X_v} + s(Q_v) - 2, & \text{if } \alpha_v \neq 0,\ \mathrm{Supp}(Q_v) \neq \emptyset. \end{cases}$$

Thus, Lemma 3.1 implies that

$$\gamma_{(\alpha, D)} = \gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

This completes the proof of (i).

(ii) Suppose that $n_X \leq 1$. Then $D = 0$. Let $\beta \in \mathrm{Rev}_0^{\mathrm{adm}}(X^\bullet)$ such that $\beta \neq 0$ and the Galois multi-admissible covering induced by β is étale. By applying [N, Proposition 4] (or [Z, Théoréme 3.1]), we have

$$\gamma_{(\beta, 0)} = \gamma_{X^\bullet}^{\max} = g_X - 1.$$

Then we may assume that $n_X \geq 2$.

Let $\Gamma \stackrel{\mathrm{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X, Γ^{im} the image of the natural morphism $\phi_\Gamma : \Gamma \to \Gamma_{X^\bullet}$, and

$$X^\bullet_\Gamma = (X_\Gamma, D_{X_\Gamma}),\ X^\bullet_{\Gamma^{\mathrm{im}}} = (X_{\Gamma^{\mathrm{im}}}, D_{X_{\Gamma^{\mathrm{im}}}})$$

the pointed stable curves over k associated to Γ, Γ^{im}, respectively. Note that D is also an effective divisor on $X_{\Gamma^{\mathrm{im}}}$.

Write D_Γ for $\mathrm{norm}^*_\Gamma(D)$ (cf. Definition 3.7 for the definition of norm_Γ). Let $\alpha_\Gamma \in \mathrm{Rev}^{\mathrm{adm}}_{D_\Gamma}(X^\bullet_\Gamma)$ be an arbitrary element such that $\alpha_\Gamma \neq 0$. Then (i) implies that $\gamma_{(\alpha_\Gamma, D_\Gamma)} = \gamma^{\max}_{X^\bullet_\Gamma} = g_{X_\Gamma} + n_X - 2$, where g_{X_Γ} denotes the genus of X_Γ. We denote by

$$g^\bullet_\Gamma : Z^\bullet_\Gamma \to X^\bullet_\Gamma$$

the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_Γ. By gluing $Z^\bullet_\Gamma$ along $g_\Gamma^{-1}(D_{X_\Gamma} \setminus \mathrm{norm}_\Gamma^{-1}(D_{X_{\Gamma^{\mathrm{im}}}}))$ that is compatible with the gluing of $X^\bullet_\Gamma$ that gives rise to $X^\bullet_{\Gamma^{\mathrm{im}}}$, we obtain a pointed stable curve $Z^\bullet_{\Gamma^{\mathrm{im}}}$ over k. Moreover, $g^\bullet_\Gamma$ induces a Galois multi-admissible covering

$$g^\bullet_{\Gamma^{\mathrm{im}}} : Z^\bullet_{\Gamma^{\mathrm{im}}} \to X^\bullet_{\Gamma^{\mathrm{im}}}$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\Pi_{X^\bullet_{\Gamma^{\mathrm{im}}}}$ for the admissible fundamental group of $X^\bullet_{\Gamma^{\mathrm{im}}}$ and $\alpha_{\Gamma^{\mathrm{im}}}$ for an element of $\mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet_{\Gamma^{\mathrm{im}}}}, \mathbb{Z}/n\mathbb{Z})$ induced by $g^\bullet_{\Gamma^{\mathrm{im}}}$. We put $D_{\Gamma^{\mathrm{im}}} \stackrel{\mathrm{def}}{=} D_{\alpha_{\Gamma^{\mathrm{im}}}}$. Then Lemma 3.1 implies that

$$\gamma_{(\alpha_{\Gamma^{\mathrm{im}}}, D_{\Gamma^{\mathrm{im}}})} = \gamma^{\max}_{X^\bullet_{\Gamma^{\mathrm{im}}}} = g_{X_{\Gamma^{\mathrm{im}}}} + n_X - 2,$$

where $g_{X_{\Gamma^{\mathrm{im}}}}$ denotes the genus of $X_{\Gamma^{\mathrm{im}}}$.

On the other hand, we write $\pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$ for the set of connected components of $\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}$, where $\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}$ denotes the topological closure of $X \setminus X_{\Gamma^{\mathrm{im}}}$ in X. We define the following pointed stable curve

$$C^\bullet = (C, D_C \stackrel{\mathrm{def}}{=} C \cap X_{\Gamma^{\mathrm{im}}}),\ C \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}),$$

over k. Note that since $X^\bullet$ is component-generic, we have that $C^\bullet$ is also component-generic. Then $\sigma_{C^\bullet}$ is equal to the genus of $C^\bullet$.

Let $C \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$. We put

$$Z^\bullet_C \stackrel{\mathrm{def}}{=} \coprod_{i \in \mathbb{Z}/n\mathbb{Z}} C^\bullet_i,$$

where $C^\bullet_i$ is a copy of $C^\bullet$. Then we obtain a Galois multi-admissible covering

$$g^\bullet_C : Z^\bullet_C \to C^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$, where the restriction morphism $g^\bullet_C|_{C_i}$ is an identity, and the Galois action is $j(C_i) = C_{i+j}$ for each $i, j \in \mathbb{Z}/n\mathbb{Z}$. By gluing $Z^\bullet_{\Gamma^{\mathrm{im}}}$ and

$\{Z^\bullet_C\}_{C\in\pi_0(\overline{X\setminus X_{\Gamma^{\rm im}}})}$ along $g_\Gamma^{-1}(X_{\Gamma^{\rm im}}\cap(\bigcup_{C\in\pi_0(\overline{X\setminus X_{\Gamma^{\rm im}}})}C))$ and $\{g_C^{-1}(X_{\Gamma^{\rm im}}\cap C)\}_{C\in\pi_0(\overline{X\setminus X_{\Gamma^{\rm im}}})}$ that is compatible with the gluing of $\{X^\bullet_{\Gamma^{\rm im}}\}\cup\{C^\bullet\}_{C\in\pi_0(\overline{X\setminus X_{\Gamma^{\rm im}}})}$ that gives rise to $X^\bullet$, we obtain a Galois multi-admissible covering

$$g: Z^\bullet\to X^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, we write $\beta\in\mathrm{Rev}^{\rm adm}_D(X^\bullet)$ for an element induced by $g^\bullet$. By applying Lemma 3.1, we see immediately that

$$\gamma_{(\beta,D)}=\gamma^{\rm max}_{X^\bullet}=g_X+n_X-2.$$

Then we complete the proof of (ii). $\square$

Remark 3.9.1. Proposition 3.9 (i) does not hold in general. For example, we suppose that $p>>0$, that $n_X=0$, and that there exist $v_1,v_2\in v(\Gamma_{X^\bullet})$ such that $\#(X_{v_1}\cap X_{v_2})\geq 3$. Then one can construct a Galois admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that (i) of the theorem does not hold.

Corollary 3.10. *Let $D\in(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ such that $s(D)=n_X-1$ if $n_X\neq 0$, and that*

$$\deg(D^{(i)})\geq\deg(D),\ i\in\{0,1,\dots,t-1\}.$$

Moreover, suppose that $X^\bullet=(X,D_X\overset{\rm def}{=}\{x_1,\dots,x_{n_X}\})$ is a component-generic pointed stable curve over k, and that $X^\bullet$ is smooth over k. Then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists.

Proof. Since $X^\bullet$ is smooth over k, the corollary follows immediately from Proposition 3.9 and Remark 2.7.1. $\square$

The main result of the present section is as follows.

Theorem 3.11. *Let $m\in\mathbb{N}$ be an arbitrary positive natural number prime to p and $D\in(\mathbb{Z}/m\mathbb{Z})^\sim[D_X]^0$. Let $t\in\mathbb{N}$ be a positive natural number such that $p^t=1$ in $(\mathbb{Z}/m\mathbb{Z})^\times$. Write D' for the divisor $m'D\in(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order m, where $n\overset{\rm def}{=}p^t-1$ and $m'\overset{\rm def}{=}n/m$. Suppose that $X^\bullet=(X,D_X)$ is a component-generic pointed stable curve over k.*

(i) Let $\alpha\in\mathrm{Rev}^{\rm adm}_D(X^\bullet)$ be an arbitrary element such that $\alpha\neq 0$. Suppose that either $\Gamma_{X^\bullet}\setminus e^{\rm lp}(\Gamma_{X^\bullet})$ is a tree when $n_X=0$ or $\Gamma_{X^\bullet}$ is a minimal quasi-tree associated to D_X when $n_X\neq 0$. Then we have that $\gamma_{(\alpha,D)}$ attains the maximum

$$\gamma^{\rm max}_{X^\bullet}=\begin{cases}g_X-1, & \textit{if } n_X=0,\\ g_X+n_X-2, & \textit{if } n_X\neq 0\end{cases}$$

if and only if

$$s(D)=\begin{cases}0, & \textit{if } n_X=0,\\ n_X-1, & \textit{if } n_X\neq 0\end{cases}$$

and $\deg((D')^{(i)})\geq\deg(D'),\ i\in\{0,1,\dots,t-1\}$.

(ii) There exists an element $\beta \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet)$ such that $\beta \neq 0$, and that the generalized Hasse-Witt invariant $\gamma_{(\beta,D)}$ attains the maximum

$$\gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0 \end{cases}$$

if and only if

$$s(D) = \begin{cases} 0, & \text{if } n_X = 0, \\ n_X - 1, & \text{if } n_X \neq 0, \end{cases}$$

and $\deg((D')^{(i)}) \geq \deg(D'),\ i \in \{0, 1, \ldots, t-1\}$.

Proof. (i) Write $\alpha' \in \mathrm{Rev}_{D'}^{\mathrm{adm}}(X^\bullet)$ for the element induced by α. Then we see immediately that $\gamma_{(\alpha,D)} = \gamma_{(\alpha',D')}$. The "only if" part of (i) follows immediately from Lemma 3.1 and [T2, Lemma 2.15]. Moreover, the "if" part of (i) follows immediately from Proposition 3.9 (i).

(ii) Write $\beta' \in \mathrm{Rev}_{D'}^{\mathrm{adm}}(X^\bullet)$ for the element induced by β. Then we see immediately that $\gamma_{(\alpha,D)} = \gamma_{(\beta',D')}$. The "only if" part of (ii) follows immediately from Lemma 3.1 and [T2, Lemma 2.15]. Moreover, the proof of Proposition 3.9 (ii) implies that the "if" part of (ii) holds. □

Definition 3.12. Let $W^\bullet = (W, D_W)$ be a pointed stable curve of type (g_W, n_W) over an algebraically closed field of characteristic $p > 0$. Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p. We shall say that $W^\bullet$ is (m, n_W)-*ordinary* if, for each $Q \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_W]^0$, the following conditions hold: (i) $Q = 0$ if $n_W = 0$, and $\deg(Q) = (n_W - 1)m$ if $n_W \neq 0$. (ii) There exists a positive natural number $d \in \mathbb{N}$ such that $p^d = 1$ in $(\mathbb{Z}/m\mathbb{Z})^\times$. (iii) Write Q' for the divisor $m'Q \in (\mathbb{Z}/(p^d-1)\mathbb{Z})^{\sim}[D_W]^0$ when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/(p^d-1)\mathbb{Z}$ of order m, where $m' \stackrel{\mathrm{def}}{=} (p^d-1)/m$. (iv) $\deg((Q')^{(i)}) \geq \deg(Q'),\ i \in \{0, 1, \ldots, d-1\}$. (v) For each $\omega \in \mathrm{Rev}_Q^{\mathrm{adm}}(W^\bullet)$ such that $\omega \neq 0$, we have that $\gamma_{(\omega,Q)}$ attains the maximum

$$\gamma_{W^\bullet}^{\max} = \begin{cases} g_W - 1, & \text{if } n_W = 0, \\ g_W + n_W - 2, & \text{if } n_W \neq 0. \end{cases}$$

Note that, if $n_W = 0$, then the definition of (m, n_W)-ordinary coincides with the definition of m-ordinary defined by Nakajima (cf. [N, §4]).

Corollary 3.13. *Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$ and $\overline{M}_{g_X,n_X}$ (resp. M_{g_X,n_X}) the coarse moduli space of the moduli stack $\overline{\mathcal{M}}_{g_X,n_X} \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ (resp. $\mathcal{M}_{g_X,n_X} \stackrel{\mathrm{def}}{=} \mathcal{M}_{g_X,n_X,\mathbb{Z}} \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$).*

(i) Let m be a positive natural number prime to p. We denote by

$$U_{(m,n_X)} \subseteq \overline{M}_{g_X,n_X}$$

the subset of $\overline{M}_{g_X,n_X}$ consisting of all points which correspond to (m, n_X)-ordinary pointed stable curves. Then

$$U_{(m,n_X)} \cap M_{g_X,n_X}$$

is a non-empty open subset.

(ii) Suppose that $n_X \leq 1$. We have that

$$M^{\mathrm{cl}}_{g_X,n_X} \cap \Big(\bigcap_{m\in\mathbb{N} \text{ s.t. } (m,p)=1} (U_{(m,n_X)} \cap M_{g_X,n_X}) \Big) = \emptyset,$$

where $M^{\mathrm{cl}}_{g_X,n_X}$ denotes the set of closed points of M_{g_X,n_X}.

(iii) Let $q \in \overline{M}_{g_X,n_X}$ be an arbitrary point. We denote by Π_q the admissible fundamental group of a pointed stable curve corresponding to a geometric point over q. Note that the isomorphism class of Π_q as a profinite group does not depend on the choices of geometric points over q. Suppose that $n_X \leq 1$. Let $U \subseteq M_{g_X,n_X}$ be an arbitrary non-empty open subset. Then there exist closed points $q_1, q_2 \in M^{\mathrm{cl}}_{g_X,n_X}$ such that $\Pi_{q_1} \not\cong \Pi_{q_2}$.

Proof. (i) By applying similar arguments to the arguments given in the proof of [N, Theorem 2], (i) follows immediately from Theorem 3.11.

(ii) We maintain the notation introduced in Section 2.2. Suppose that $k = \overline{\mathbb{F}}_p$, and that $X^\bullet$ is a smooth pointed stable curve of type (g_X, n_X) over k. To verify (ii), we only need to prove that, if $n_X \leq 1$, $X^\bullet$ is not (m, n_X)-ordinary for some positive natural number $m \in \mathbb{N}$ prime to p.

Since $n_X \leq 1$, we have that $(\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0 = \{0\}$. We denote by Θ_X the Raynaud-Tamagawa divisor associated to $\mathcal{B}_0$ and by Θ' an arbitrary irreducible component of Θ_X. Write J^1_X for the pull-back of the Jacobian J_X of X by the Frobenius F_k. If $X^\bullet$ is (m, n_X)-ordinary for every positive natural number m prime to p, then we have

$$J^1_X\{p'\} \cap \Theta_X(k) \subseteq \{0_{J^1_X}\},$$

where $0_{J^1_X}$ denotes the zero point of J^1_X, and $J^1_X\{p'\}$ denotes the set of prime-to-p torsion points of $J^1_X(k)$. Since $\dim(\Theta') > 0$, we have

$$\Theta'\{p'\} \overset{\text{def}}{=} J^1_X\{p'\} \cap \Theta'(k) \subseteq J^1_X\{p'\} \cap \Theta_X(k)$$

is not dense in Θ'.

On the other hand, since Θ' is defined over $k = \overline{\mathbb{F}}_p$, by applying a result of Anderson-Indik (cf. [T3, §5]), we have that Θ' is a subvariety of a translate of a proper sub-abelian varietry of J^1_X. But this contradict to a result of Raynaud (cf. [R2, Proposition 1.2.1]) which says that there exists an irreducible component Θ' of Θ_X such that Θ' is not contained in a translate of a proper sub-abelian variety of J^1_X. This completes the proof of (ii).

(iii) Suppose that (iii) does not hold. Then there exists a closed point $q' \in U^{\mathrm{cl}}$ such that $\Pi_{q^{\mathrm{gen}}} \cong \Pi_{q'}$. Let q be an arbitrary closed point of U and q^{gen} the generic point of M_{g_X,n_X}. Then there exist a discrete valuation ring R and a morphism $c_R : \operatorname{Spec} R \to M_{g_X,n_X}$ such that $c_R(\eta_R) = q^{\mathrm{gen}}$ and $c_R(s_R) = q$, where η_R is the generic point of $\operatorname{Spec} R$ and s_R is the closed point of $\operatorname{Spec} R$. By replacing R by a finite extension of R, we have a smooth pointed stable curve

$$\mathcal{X}^\bullet$$

of type (g_X, n_X) over Spec R determined by c_R. Moreover, we obtain a specialization map

$$sp_R : \Pi_{q^{\mathrm{gen}}} \to \Pi_q.$$

On the other hand, (ii) implies that there exist a positive integer m prime to p and a connected Galois étale covering $\mathcal{Y} \to \mathcal{X}$ over R with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that the geometric generic fiber of $\mathcal{Y}$ is ordinary and the geometric special fiber of $\mathcal{Y}$ is not ordinary. This means that sp_R is not an isomorphism. This is a contradiction. We complete the proof of (iii). $\square$

Remark 3.13.1. If $n_X \leq 1$, then Corollary 3.13 (i) was proved by Nakajima (cf. Theorem 2). Then Corollary 3.13 (i) is a generalized version of [N, Theorem 2] to the case of admissible coverings of smooth pointed stable curves. Moreover, Nakajima asked whether or not

$$\bigcap_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} (U_{(m,n_X)} \cap M_{g_X,n_X})$$

is a non-empty open subset of M_{g_X,n_X} (cf. [N, §4 Remark]). Then Corollary 3.13 (ii) gives an answer of Nakajima's question. Furthermore, we may ask the following question:

Does

$$M^{\mathrm{cl}}_{g_X,n_X} \cap (\bigcap_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} (U_{(m,n_X)} \cap M_{g_X,n_X})) = \emptyset$$

hold for each non-negative integer n_X?

Remark 3.13.2. Corollary 3.13 (iii) gives an answer of a question of D. Harbater (cf. [H, 4.2]) which was first solved by F. Pop and M. Saïdi (cf. [PS, Corollary]). In [PS], Pop and Saïdi proved a result which says that the specialization map of geometric étale fundamental groups of smooth projective curves in positive characteristic is not an isomorphism under certain assumptions. Then together with a result of C-L. Chai-F. Oort, and a result of J-P. Serre, they obtained Corollary 3.13 (iii).

4 Maximum generalized Hasse-Witt invariants of cyclic admissible coverings of pointed stable curves

In the present section, we discuss the maximum generalized Hasse-Witt invariants of cyclic admissible coverings of an arbitrary pointed stable curve. Let us return to the case where $X^\bullet$ is an arbitrary pointed stable curve over k, and we maintain the notation introduced in Section 2.2. First, by applying Theorem 2.9, we have the following lemma (cf. [T2, Corollary 2.6 and Lemma 2.12 (ii)]).

Lemma 4.1. *(i) Let $Q \in \mathbb{Z}[D_X]$ be an effective divisor on X of degree $\deg(Q) = s(Q)n$, $\mathcal{L}_Q$ a line bundle on X such that $\mathcal{L}_Q^{\otimes n} \cong \mathcal{O}_X(-Q)$, and $\mathcal{L}_{Q,t}$ the pull-back of $\mathcal{L}_Q$ by the natural morphism $X_t \to X$. Suppose that $X^\bullet$ is smooth over k, and that*

$$\#\{x \in X \mid \mathrm{ord}_x(Q) = n\} \geq s(Q) - 1.$$

Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}^t_Q \otimes \mathcal{L}_{Q,t}$ exists.

(ii) Let t_i, $i \in \{1,2\}$, be an arbitrary positive natural number and $n_i \stackrel{\text{def}}{=} p^{t_i} - 1$. Let $Q_i \in \mathbb{Z}[D_X]$ be an effective divisor on X of degree $\deg(Q_i) = s(Q_i)n_i$, $\mathcal{L}_{Q_i}$ a line bundle on X such that $\mathcal{L}_{Q_i}^{\otimes n_i} \cong \mathcal{O}_X(-Q_i)$, and $\mathcal{L}_{Q_i,t_i}$ the pull-back of $\mathcal{L}_{Q_i}$ by the natural morphism $X_{t_i} \to X$. Suppose that $s \stackrel{\text{def}}{=} s(Q_1) = s(Q_2)$. Let $t \stackrel{\text{def}}{=} t_1 + t_2$, $n \stackrel{\text{def}}{=} n_1 + p^{t_1}n_2$,

$$Q \stackrel{\text{def}}{=} Q_1 + p^{t_1}Q_2 \in \mathbb{Z}[D_X]$$

an effective divisor on X of degree $\deg(Q) = sn$, $\mathcal{L}_Q$ a line bundle on X such that $\mathcal{L}_Q^{\otimes n} \cong \mathcal{O}_X(-Q)$, and $\mathcal{L}_{Q,t}$ the pull-back of $\mathcal{L}_Q$ by the natural morphism $X_t \to X$. Suppose that $X^\bullet$ is smooth over k. Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}^t_Q \otimes \mathcal{L}_{Q,t}$ exists if and only if the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}^{t_i}_{Q_i} \otimes \mathcal{L}_{Q_i,t_i}$ exists for each $i \in \{1,2\}$.

Lemma 4.1 implies the following proposition.

Proposition 4.2. *Suppose that $X^\bullet$ is irreducible. Then there exist a positive natural number $n \stackrel{\text{def}}{=} p^t - 1 \in \mathbb{N}$, an effective divisor $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$, and an element $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$, and that the generalized Hasse-Witt invariant $\gamma_{(\alpha,D)}$ attains the maximum*

$$\gamma^{\max}_{X^\bullet} = \begin{cases} g_X - 1, & \textit{if } n_X = 0, \\ g_X + n_X - 2, & \textit{if } n_X \neq 0. \end{cases}$$

Proof. We write $\widetilde{X}$ for the normalization of X and $\mathrm{norm} : \widetilde{X} \to X$ for the normalization morphism. We define

$$\widetilde{X}^\bullet = (\widetilde{X}, D_{\widetilde{X}} \stackrel{\text{def}}{=} \mathrm{norm}^{-1}(D_X \cup X^{\mathrm{sing}}))$$

to be a pointed stable curve of type $(g_{\widetilde{X}}, n_X)$ over k. Note that $g_{\widetilde{X}} = g_X - \#X^{\mathrm{sing}}$. Moreover, we put $\widetilde{D}_X \stackrel{\text{def}}{=} \mathrm{norm}^{-1}(D_X)$. By applying Lemma 3.1, to verify the proposition, it is sufficient to prove that there exist a positive natural number $n \stackrel{\text{def}}{=} p^t - 1 \in \mathbb{N}$, an effective (Weil) divisor $\widetilde{D} \in (\mathbb{Z}/n\mathbb{Z})^\sim[\widetilde{D}_X]^0$, and an element $\widetilde{\alpha} \in \mathrm{Rev}^{\mathrm{adm}}_{\widetilde{D}}(\widetilde{X}^\bullet)$ such that $\widetilde{\alpha} \neq 0$, and that the generalized Hasse-Witt invariant $\gamma_{(\widetilde{\alpha},\widetilde{D})}$ attains the maximum

$$\gamma^{\max}_{\widetilde{X}^\bullet} = \begin{cases} g_{\widetilde{X}} - 1, & \text{if } n_X = 0, \\ g_{\widetilde{X}} + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Suppose that $n_X \leq 2$. Then $s(\widetilde{D}) \leq 1$ for each $\widetilde{D} \in (\mathbb{Z}/n\mathbb{Z})^\sim[\widetilde{D}_X]^0$. Thus, the proposition follows immediately from Proposition 2.8 and Theorem 2.9.

Suppose that $n_X \geq 3$. Let $\widetilde{D}_X \stackrel{\text{def}}{=} \{x_1, \dots, x_{n_X}\}$, $n_i \stackrel{\text{def}}{=} p^{t_i} - 1$ for each $i \in \{1, \dots, n_X - 1\}$ such that $n_i > \max\{C(g_X) + 1, \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{op}}(\Gamma_{X^\bullet}))\}$, and $0 < a_{i,1}, a_{i,2} < n_i$ for each $i \in \{1, \dots, n_X - 1\}$ such that $a_{i,1} + a_{i,2} = n_i$. We put

$$D_i \stackrel{\text{def}}{=} a_{i,1}x_i + a_{i,2}x_{i+1} + \sum_{x \in \widetilde{D}_X \setminus \{x_i, x_{i+1}\}} n_i x, \ i \in \{1, \dots, n_X - 1\},$$

which is an effective divisor on $\widetilde{X}$ with degree $\deg(D_i) = (n_X - 1)n_i$. Moreover, we put

$$\widetilde{D} \stackrel{\text{def}}{=} \sum_{i=1}^{n_X-1} p^{\sum_{j=0}^{i-1} t_j} D_i$$

and

$$n \stackrel{\text{def}}{=} p^{\sum_{i=0}^{n_X-1} t_j} - 1 = \sum_{i=1}^{n_X-1} p^{\sum_{j=0}^{i-1} t_j}(p^{t_i} - 1),$$

where $t_0 \stackrel{\text{def}}{=} 0$. We see immediately that $\deg(\widetilde{D}) = (n_X - 1)n$, and that $\widetilde{D} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$. Let $\mathcal{L}_{\widetilde{D}}$ a line bundle on $\widetilde{X}$ such that $\mathcal{L}_{\widetilde{D}}^{\otimes n} \cong \mathcal{O}_X(-\widetilde{D})$, and $\mathcal{L}_{\widetilde{D},t}$ the pull-back of $\mathcal{L}_{\widetilde{D}}$ by the natural morphism $\widetilde{X}_t \to \widetilde{X}$. Then Lemma 4.1 implies the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}^t_{\widetilde{D}} \otimes \mathcal{L}_{\widetilde{D},t}$ exists. Moreover, Proposition 2.8 implies that there exists a line bundle $\widetilde{\mathcal{I}}$ of degree 0 on $\widetilde{X}$ such that $[\widetilde{\mathcal{I}}] \neq [\mathcal{O}_{\widetilde{X}}]$, that $[\widetilde{\mathcal{I}}^{\otimes n}] = [\mathcal{O}_{\widetilde{X}}]$, and that

$$\gamma_{([\mathcal{L}_{\widetilde{D}} \otimes \widetilde{\mathcal{I}}], \widetilde{D})} = \begin{cases} g_{\widetilde{X}} - 1, & \text{if } n_X = 0, \\ g_{\widetilde{X}} + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Let $\widetilde{\alpha} \in \mathrm{Rev}^{\mathrm{adm}}_{\widetilde{D}}(\widetilde{X}^\bullet)$ be the element corresponding to the pair $([\mathcal{L}_{\widetilde{D}} \otimes \widetilde{\mathcal{I}}], \widetilde{D}) \in \widetilde{\mathscr{P}}_{\widetilde{X}^\bullet,n}$. This completes the proof of the proposition. $\square$

Remark 4.2.1. We maintain the notation introduced in the proof of Proposition 4.2. By choosing a suitable $a_{i,2}$ and $a_{i,2}$ for each $i \in \{1, \dots, n_X - 1\}$, we may obtain that the Galois multi-admissible covering induced by α is connected.

In the remainder of this section, we will generalizes Proposition 4.2 to the case where $X^\bullet$ is an arbitrary pointed stable curve over k.

Definition 4.3. Let $\mathbb{G}$ be a connected semi-graph and $v \in v(\mathbb{G})$ an arbitrary vertex. Moreover, we suppose that $\mathbb{G}$ is a *tree*. For each $v' \in v(\mathbb{G})$, there exists a path $p_{v,v'}$ connecting v and v' in $\mathbb{G}$. We define

$$\mathrm{leng}(p_{v,v'}) \stackrel{\text{def}}{=} \#\{p_{v,v'} \cap v(\mathbb{G})\} - 1$$

to be the *length* of the path $p_{v,v'}$. Moreover, since $\mathbb{G}$ is a tree, there exists a unique path connecting v and v' whose length is equal to $\min\{\mathrm{leng}(p_{v,v'})\}_{p_{v,v'}}$. We shall write

$$p(\mathbb{G}, v, v')$$

for this unique path connecting v and v' in $\mathbb{G}$, and say that $p(\mathbb{G}, v, v')$ is the *minimal path* connecting v and v' in $\mathbb{G}$.

Lemma 4.4. *Let $\Gamma \stackrel{\text{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X,*

$$X^\bullet_\Gamma = (X_\Gamma, D_{X_\Gamma})$$

the pointed stable curve of type $(g_{X_\Gamma}, n_{X_\Gamma})$ associated to Γ, and $\Pi_{X^\bullet_\Gamma}$ the admissible fundamental group of $X^\bullet_\Gamma$. Suppose that $n_X \geq 2$. Then there exist a positive natural number $n \stackrel{\text{def}}{=} p^t - 1 \in \mathbb{N}$, an effectiv divisor $D_\Gamma \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$, and an element $\alpha_\Gamma \in \mathrm{Rev}^{\mathrm{adm}}_{D_\Gamma}(X^\bullet_\Gamma)$ such that $\alpha_\Gamma \neq 0$, and that the generalized Hasse-Witt invariant

$$\gamma_{(\alpha_\Gamma, D_\Gamma)} = g_{X_\Gamma} + n_X - 2.$$

Proof. Since Γ is a minimal quasi-tree associated to D_X, we obtain that $\Gamma' \overset{\text{def}}{=} \Gamma \setminus e^{\text{lp}}(\Gamma)$ is a tree. Then we have $v(\Gamma) = v(\Gamma')$. Note that $D_X \subseteq D_{X_\Gamma}$. Let $v \in v(\Gamma)$ be an arbitrary vertex and $n_0 = p^{t_0} - 1 \in \mathbb{N}$ a positive natural number such that

$$n_0 > \max\{C(g_X) + 1, \#(e^{\text{cl}}(\Gamma_{X^\bullet}) \cup e^{\text{op}}(\Gamma_{X^\bullet}))\}.$$

We put

$$D'_v \overset{\text{def}}{=} D_X \cap X_v,\ m_v \overset{\text{def}}{=} \#D'_v,\ \text{and}\ D'_v \overset{\text{def}}{=} \{x_{v,1}, \dots, x_{v,m_v}\} \text{ if } m_v \neq 0.$$

Moreover, we put

$$D_v \overset{\text{def}}{=} D'_v \cup (X_v \cap \overline{X_\Gamma \setminus X_v}),$$

where $\overline{X_\Gamma \setminus X_v}$ denotes the topological closure of $X_\Gamma \setminus X_v$ in X_Γ. Note that since $n_X > 0$, we have $\#D_v > 0$. Let $w \in v(\Gamma)$ be an arbitrary vertex distinct from v. Since Γ' is a tree, there exists a unique node

$$x^-_{v,w}$$

such that the closed edge of Γ' corresponding to $x^-_{v,w}$ is contained in the minimal path $p(\Gamma', v, w)$ connecting v and w in Γ'. On the other hand, we define a set of nodes to be

$$\text{Node}^+_{v,w} \overset{\text{def}}{=} \{X_w \cap X_{w'},\ w' \in v(\Gamma) \mid \text{leng}(p(\Gamma', v, w')) = \text{leng}(p(\Gamma', v, w)) + 1\}.$$

Note that $\text{Node}^+_{v,w}$ may possibly be an empty set, and that $D_w = \{x^-_{v,w}\} \cup \text{Node}^+_{v,w} \cup D'_w$.

First, we define two sets of effective divisors

$$\text{Div}^{\text{irr-st}}_v,\ \text{Div}^{\text{st}}_v$$

associated to v as follows, where "st" means that "standard", and "irr" means that "irreducible components". Let $i \in \{1, \dots, m_v - 1\}$ and $0 < a_{v,i,1}, a_{v,i,2} < n_0$ such that $a_{v,i,1} + a_{v,i,2} = n_0$. Suppose that $m_v \leq 1$. Then we put

$$\text{Div}^{\text{st}}_v \overset{\text{def}}{=} \emptyset,\ \text{Div}^{\text{st}}_v \overset{\text{def}}{=} \emptyset.$$

Suppose that $m_v \geq 2$. We define

$$Q_{v,v,i} \overset{\text{def}}{=} a_{v,i,1}x_{v,i} + a_{v,i,2}x_{v,i+1} + \sum_{x' \in D'_v \setminus \{x_{v,i}, x_{v,i+1}\}} n_0 x' + \sum_{x \in D_v \setminus D'_v} n_0 x$$

to be an effective divisor on X_v whose support is D_v, and whose degree is equal to $(\#D_v - 1)n_0$. We define

$$Q_{v,w,i} \overset{\text{def}}{=} \sum_{x \in D_w \setminus \{x^-_{v,w}\}} n_0 x,\ w \in v(\Gamma) \setminus \{v\},$$

to be an effective divisor on X_w whose support is $D_w \setminus \{x^-_{v,w}\}$, and whose degree is equal to $(\#D_w - 1)n_0$. Moreover, we define

$$Q^v_i \overset{\text{def}}{=} a_{v,i,1}x_{v,i} + a_{v,i,2}x_{v,i+1} + \sum_{x \in D_X \setminus \{x_{v,i}, x_{v,i+1}\}} n_0 x,$$

to be an effective divisor on X_Γ whose support is D_X, and whose degree is $(n_X - 1)n_0$. Then we put

$$\mathrm{Div}_{v,i}^{\text{irr-st}} \stackrel{\text{def}}{=} \bigcup_{u \in v(\Gamma)} \{Q_{v,u,i}\}, \ \mathrm{Div}_v^{\text{irr-st}} \stackrel{\text{def}}{=} \bigcup_{i=1}^{m_v - 1} \mathrm{Div}_{v,i}^{\text{irr-st}}, \ \mathrm{Div}_v^{\text{st}} \stackrel{\text{def}}{=} \bigcup_{i=1}^{m_v-1} \{Q_i^v\}.$$

Next, we define two sets of effective divisors

$$\mathrm{Div}_v^{\text{irr-md}}, \ \mathrm{Div}_v^{\text{md}}$$

associated to v as follows, where "md" means that "modification". Let $z \in D_X \setminus D'_v$ and $0 < b_{v,z,1}, b_{v,z,2} < n_0$ such that $b_{v,z,1} + b_{v,z,2} = n_0$. Suppose that $m_v = 0$. Then we put

$$\mathrm{Div}_v^{\text{irr-md}} \stackrel{\text{def}}{=} \emptyset, \ \mathrm{Div}_v^{\text{md}} \stackrel{\text{def}}{=} \emptyset.$$

Suppose that $m_v \neq 0$. Let w_z be the vertex such that the irreducible component X_{w_z} corresponding to w_z contains z (i.e., $z \in D'_{w_z}$), $p(\Gamma', v, w_z)$ the minimal path connecting v and w_z in Γ', and $w \in v(\Gamma)$ an arbitrary vertex distinct from w_z such that $w \subseteq p(\Gamma', v, w_z)$. Since Γ' is a tree, we have that $\#(\mathrm{Node}_{v,w}^+ \cap p(\Gamma', v, w_z)) = 1$. We put

$$x_{v,w}^+ \stackrel{\text{def}}{=} \mathrm{Node}_{v,w}^+ \cap p(\Gamma', v, w_z).$$

We define

$$Q_{v,v,z} \stackrel{\text{def}}{=} b_{v,z,1} x_{v,m_v} + b_{v,z,2} x_{v,v}^+ + \sum_{x \in D_v \setminus \{x_{v,m_v}, x_{v,v}^+\}} n_0 x$$

and

$$Q_{v,w_z,z} \stackrel{\text{def}}{=} b_{v,z,1} x_{v,w_z}^- + b_{v,z,2} z + \sum_{x \in D_{w_z} \setminus \{x_{v,w_z}, z\}} n_0 x$$

to be effective divisors on X_v and X_{w_z} whose supports are D_v and D_{w_z}, and whose degrees are equal to $(\#D_v - 1)n_0$ and $(\#D_{w_z} - 1)n_0$, respectively. Let $w \in v(\Gamma) \setminus \{v, w_z\}$ be an arbitrary vertex such that $w \subseteq p(\Gamma', v, w_z)$. Then we define

$$Q_{v,w,z} \stackrel{\text{def}}{=} b_{v,z,1} x_{v,w}^- + b_{v,z,2} x_{v,w}^+ + \sum_{x \in D_w \setminus \{x_{v,w}^-, x_{v,w}^+\}} n_0 x$$

to be an effective divisor on X_w whose support is D_w, and whose degree is equal to $(\#D_w - 1)n_0$. Let $w' \in v(\Gamma)$ be an arbitrary vertex such that $w' \not\subseteq p(\Gamma', v, w_z)$. Then we define

$$Q_{v,w',z} \stackrel{\text{def}}{=} \sum_{x \in D_{w'} \setminus \{x_{v,w'}^-\}} n_0 x$$

to be an effective divisor on $X_{w'}$ whose support is $D_{w'} \setminus \{x_{v,w'}^-\}$, and whose degree is equal to $(\#D_{w'} - 1)n_0$. Moreover, we define

$$Q_z^v \stackrel{\text{def}}{=} b_{v,z,1} x_{v,m_v} + b_{v,z,2} z + \sum_{x \in D_X \setminus \{x_{v,m_v}, z\}} n_0 x$$

to be an effective divisor on X_Γ whose support is D_X, and whose degree is equal to $(n_X-1)n_0$. Then we put

$$\mathrm{Div}_{v,z}^{\text{irr-md}} \overset{\text{def}}{=} \bigcup_{u\in v(\Gamma)} \{Q_{v,u,z}\},\ \mathrm{Div}_{v}^{\text{irr-md}} \overset{\text{def}}{=} \bigcup_{z\in D_X\setminus D'_v} \mathrm{Div}_{v,z}^{\text{irr-md}},\ \mathrm{Div}_{v}^{\text{md}} \overset{\text{def}}{=} \bigcup_{z\in D_X\setminus D'_v} \{Q_z^v\}.$$

We put

$$\mathrm{Div}_X^{\text{irr}} \overset{\text{def}}{=} \bigcup_{v\in v(\Gamma)} (\mathrm{Div}_v^{\text{irr-st}} \cup \mathrm{Div}_v^{\text{irr-md}})$$

and

$$\mathrm{Div}_X \overset{\text{def}}{=} \bigcup_{v\in v(\Gamma)} (\mathrm{Div}_v^{\text{st}} \cup \mathrm{Div}_v^{\text{md}}).$$

We denote by $\mathrm{Div}_X^{\text{irr}}(X_u)$, $u\in v(\Gamma)$, the subset of $\mathrm{Div}_X^{\text{irr}}$ whose elements are effective divisors on X_u. Note that $d \overset{\text{def}}{=} \#\mathrm{Div}_X^{\text{irr}}(X_{u_1}) = \#\mathrm{Div}_X^{\text{irr}}(X_{u_2}) = \#\mathrm{Div}_X$ for each $u_1, u_2\in v(\Gamma)$. Moreover, let

$$\mathrm{Div}_X^{\text{irr}}(X_u) \overset{\text{def}}{=} \{P_{u,1},\dots,P_{u,d}\},\ u\in v(\Gamma),$$

be an order of $\mathrm{Div}_X^{\text{irr}}(X_u)$ such that, for each $u_1,u_2\in v(\Gamma)$ and each $j\in\{1,\dots,d\}$, one of the following conditions is satisfied: (i) if $P_{u_1,j}\in \mathrm{Div}_{v,i}^{\text{irr-st}}$ for some $v\in v(\Gamma)$ and some $i\in\{1,\dots,m_v-1\}$, then $P_{u_2,j}\in\mathrm{Div}_{v,i}^{\text{irr-st}}$; (ii) if $P_{u_1,j}\in\mathrm{Div}_{v,z}^{\text{irr-md}}$ for some $v\in v(\Gamma)$ and some $z\in D_X\setminus D'_v$, then $P_{u_2,j}\in\mathrm{Div}_{v,z}^{\text{irr-md}}$. Then, by the construction of Div_X, the order of $\mathrm{Div}_X^{\text{irr}}(X_u)$, $u\in v(\Gamma)$, induces an order of Div_X. We may put

$$\mathrm{Div}_X \overset{\text{def}}{=} \{P_1,\dots,P_d\}.$$

Let $t\overset{\text{def}}{=} dt_0$ and $n\overset{\text{def}}{=} \sum_{j=1}^d p^{(j-1)t_0}(p^{t_0}-1) = p^t-1$. We define

$$P_u \overset{\text{def}}{=} \sum_{j=1}^d p^{(j-1)t_0} P_{u,j} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_u]^0,\ u\in v(\Gamma),$$

and

$$P_\Gamma \overset{\text{def}}{=} \sum_{j=1}^d p^{(j-1)t_0} P_j \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$$

to be effective divisors on X_u and X_Γ, respectively. We see immediately that the support of P_u, $u\in v(\Gamma)$, is D_u, that the support of P_Γ is D_X, that $\deg(P_u)=(\#D_u-1)n$, and that $\deg(P_\Gamma)=(n_X-1)n$.

Let $u\in v(\Gamma)$ be an arbitrary vertex and $\widetilde{P}_u \overset{\text{def}}{=} \mathrm{norm}_v^*(P_u)$ an effective divisor on $\widetilde{X}_u$. By applying similar arguments to the arguments given in the proof of Proposition 4.2, there exists $\widetilde{\alpha}_u\in \mathrm{Rev}_{\widetilde{P}_u}^{\text{adm}}(\widetilde{X}_u^\bullet)$ such that

$$\gamma_{(\widetilde{\alpha}_u,\widetilde{P}_u)} = g_u + \#D_u - 2.$$

We define

$$X_u^\bullet = (X_u, D_{X_u} \overset{\text{def}}{=} D_u)$$

to be a pointed stable curve over k. Then Lemma 3.1 implies that the element $\alpha_u \in \mathrm{Rev}^{\mathrm{adm}}_{P_u}(X^\bullet_u)$ induced by $\widetilde{\alpha}_u$ such that $\gamma_{(\alpha_u, P_u)}$ attains the maximum

$$\gamma^{\max}_{X^\bullet_u} = g_{X_u} + \#D_u - 2,$$

where g_{X_u} denotes the genus of X_u. Write

$$f^\bullet_u : Y^\bullet_u \to X^\bullet_u$$

for the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_u. By gluing $\{Y^\bullet_u\}_{u \in v(\Gamma)}$ along $\{f_u^{-1}(D_u \setminus D'_u)\}_{u \in v(\Gamma)}$ that is compatible with the gluing of $\{X^\bullet_u\}_{u \in v(\Gamma)}$ that gives rise to $X^\bullet_\Gamma$, we obtain a Galois multi-admissible covering

$$f^\bullet_\Gamma : Y^\bullet_\Gamma \to X^\bullet_\Gamma$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Note that the construction of $f^\bullet_\Gamma$ implies that $f^\bullet_\Gamma$ is étale over $D_{X_\Gamma} \setminus D_X$. We denote by $\alpha_\Gamma \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet_\Gamma}, \mathbb{Z}/n\mathbb{Z})$ an element induced by $f^\bullet_\Gamma$. We put $D_\Gamma \overset{\mathrm{def}}{=} P_\Gamma$. By the construction of D_Γ, we see immediately that

$$\alpha_\Gamma \in \mathrm{Rev}^{\mathrm{adm}}_{D_\Gamma}(X^\bullet_\Gamma).$$

Moreover, Lemma 3.1 implies that

$$\gamma_{(\alpha_\Gamma, D_\Gamma)} = g_{X_\Gamma} + n_X - 2.$$

We complete the proof of the lemma. $\square$

Next, we prove the main result of the present section.

Theorem 4.5. *There exist a positive natural number $n \overset{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$, an effective divisor $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$, and an element $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$, and that the generalized Hasse-Witt invariant $\gamma_{(\alpha, D)}$ attains the maximum*

$$\gamma^{\max}_{X^\bullet} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. Let $t \in \mathbb{N}$ be an arbitrary positive natural number and $n \overset{\mathrm{def}}{=} p^t - 1$ such that

$$n > \max\{C(g_X) + 1, \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{op}}(\Gamma_{X^\bullet}))\}.$$

First, we suppose that $n_X \leq 1$. We denote by $v(\Gamma_{X^\bullet})^{>0} \subseteq v(\Gamma_{X^\bullet})$ the set of vertices such that $g_v > 0$ for each $v \in v(\Gamma_{X^\bullet})^{>0}$. Suppose that $v(\Gamma_{X^\bullet})^{>0} = \emptyset$. Then $n_X \leq 1$ implies that $\Gamma_{X^\bullet}$ is not a tree. This means that $\Pi^{\mathrm{top}}_{X^\bullet}$ is not trivial. Let $\alpha' : \Pi^{\mathrm{top,ab}}_{X^\bullet} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ be a surjection and

$$\alpha : \Pi^{\mathrm{ab}}_{X^\bullet} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$$

the composite morphism $\Pi^{\mathrm{ab}}_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{top,ab}}_{X^\bullet} \overset{\alpha'}{\twoheadrightarrow} \mathbb{Z}/n\mathbb{Z}$. Then the theorem follows immediately from Lemma 3.1.

Suppose that $v(\Gamma_{X^\bullet})^{>0} \neq \emptyset$. Let $v \in v(\Gamma_{X^\bullet})^{>0}$. Then Proposition 2.8 and Theorem 2.9 imply that there exists an element $\widetilde{\alpha}_v \in \mathrm{Rev}_0^{\mathrm{adm}}(\widetilde{X}_v^\bullet)$ such that $\widetilde{\alpha}_v : \Pi^{\mathrm{ab}}_{\widetilde{X}_v^\bullet} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ is a surjective, and that

$$\gamma_{(\widetilde{\alpha}_v, 0)} = g_v - 1.$$

Write $\widetilde{f}_v^\bullet : \widetilde{Y}_v^\bullet \to \widetilde{X}_v^\bullet$ for the connected Galois étale covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by $\widetilde{\alpha}_v$. Let

$$\pi_0(\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v})$$

be the set of connected components of $\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v}$ and $C \in \pi_0(\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v})$, where $\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v}$ denotes the topological closure of $X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v$ in X. We define

$$C^\bullet = (C, D_C \overset{\mathrm{def}}{=} (C \cap (\bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v)) \cup (D_X \cap C))$$

to be a pointed stable curve over k. Note that the normalization of each irreducible component of C is a rational curve over k. Let

$$Y_C^\bullet \overset{\mathrm{def}}{=} \coprod_{i \in \mathbb{Z}/n\mathbb{Z}} C_i^\bullet,$$

where $C_i^\bullet$ is a copy of $C^\bullet$. Then we obtain a Galois multi-admissible covering

$$f_C^\bullet : Y_C^\bullet \to C^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$, where the restriction morphism $f_C^\bullet|_{C_i}$ is an identity, and the Galois action is $j(C_i) = C_{i+j}$ for each $i, j \in \mathbb{Z}/n\mathbb{Z}$. By gluing

$$\{\widetilde{Y}_v^\bullet\}_{v \in v(\Gamma_{X^\bullet})^{>0}} \text{ and } \{Y_C^\bullet\}_{C \in \pi_0(\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v})}$$

along $\{D_{\widetilde{X}_v}\}_{v \in v(\Gamma_{X^\bullet})^{>0}}$ and $\{D_C\}_{C \in \pi_0(\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v})}$ that is compatible with the gluing of $\{\widetilde{X}_v^\bullet\}_{v \in v(\Gamma_{X^\bullet})^{>0}} \cup \{C^\bullet\}_{C \in \pi_0(\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v})}$ that gives rise to $X^\bullet$, we obtain a Galois (étale) admissible covering

$$f^\bullet : Y^\bullet \to X^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then theorem follows immediately from Lemma 3.1.

Next, we suppose that $n_X \geq 2$. Let $\Gamma \overset{\mathrm{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X, Γ^{im} the image of the natural morphism $\phi_\Gamma : \Gamma \to \Gamma_{X^\bullet}$, and

$$X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma}),\ X_{\Gamma^{\mathrm{im}}}^\bullet = (X_{\Gamma^{\mathrm{im}}}, D_{X_{\Gamma^{\mathrm{im}}}})$$

the pointed stable curves over k associated to Γ, Γ^{im}, respectively.

Lemma 4.4 implies that there exist a natural number $n \overset{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$, an effective divisor $D \overset{\mathrm{def}}{=} D_\Gamma \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ on X_Γ whose degree is $(n_X - 1)n$, and an element $\alpha_\Gamma \in \mathrm{Rev}_D^{\mathrm{adm}}(X_\Gamma^\bullet)$ such that

$$\gamma_{(\alpha_\Gamma, D)} = g_{X_\Gamma} + n_X - 2,$$

where g_{X_Γ} denotes the genus of X_Γ. We denote by

$$f^\bullet_\Gamma : Z^\bullet_\Gamma \to X^\bullet_\Gamma$$

the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_Γ. Note that $f^\bullet_\Gamma$ is étale over $D_{X_\Gamma} \setminus D_X$. By gluing $Z^\bullet_\Gamma$ along $f_\Gamma^{-1}(D_{X_\Gamma} \setminus (D_X \cup \{x_e\}_{e\in\phi_\Gamma^{-1}(e^{\mathrm{op}}(\Gamma^{\mathrm{im}}))}))$ that is compatible with the gluing of $X^\bullet_\Gamma$ that gives rise to $X^\bullet_{\Gamma^{\mathrm{im}}}$, we obtain a pointed stable curve $Z^\bullet_{\Gamma^{\mathrm{im}}}$ over k. Moreover, $f^\bullet_\Gamma$ induces a Galois multi-admissible covering

$$f^\bullet_{\Gamma^{\mathrm{im}}} : Z^\bullet_{\Gamma^{\mathrm{im}}} \to X^\bullet_{\Gamma^{\mathrm{im}}}$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\Pi_{X^\bullet_{\Gamma^{\mathrm{im}}}}$ for the admissible fundamental group of $X^\bullet_{\Gamma^{\mathrm{im}}}$ and $\alpha_{\Gamma^{\mathrm{im}}}$ for an element of $\mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet_{\Gamma^{\mathrm{im}}}}, \mathbb{Z}/n\mathbb{Z})$ induced by $f^\bullet_{\Gamma^{\mathrm{im}}}$. Note that we have $D_{\alpha_{\Gamma^{\mathrm{im}}}} = D$. Then Lemma 3.1 implies that $\gamma_{(\alpha_{\Gamma^{\mathrm{im}}}, D)} = g_{X_{\Gamma^{\mathrm{im}}}} + n_X - 2$, where $g_{X_{\Gamma^{\mathrm{im}}}}$ denotes the genus of $X_{\Gamma^{\mathrm{im}}}$.

On the other hand, we write $\pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$ for the set of connected components of $\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}$. We define the following pointed stable curve

$$E^\bullet = (E, D_E \overset{\mathrm{def}}{=} E \cap X_{\Gamma^{\mathrm{im}}}),\ E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}),$$

over k. We denote by $\pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})^{>0}$ the set of curves of $\pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$ such that the genus of curves are > 0.

Let $E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})^{>0}$. Similar arguments to the arguments given in the proof of the case where $n_X \leq 1$ and $v(\Gamma_{X^\bullet}) \neq \emptyset$ above imply that there exists a Galois étale covering

$$f^\bullet_E : Z^\bullet_E = (Z_E, D_{Z_E}) \to E^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that

$$\gamma_{(\alpha_E, 0)} = g_E - 1,$$

where g_E denotes the genus of E, and $\alpha_E \in \mathrm{Rev}^{\mathrm{adm}}_0(E^\bullet)$ is an element induced by $f^\bullet_E$.

Let $E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}) \setminus \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})^{>0}$. We put

$$Z^\bullet_E \overset{\mathrm{def}}{=} \coprod_{i\in\mathbb{Z}/n\mathbb{Z}} E^\bullet_i,$$

where $E^\bullet_i$ is a copy of $E^\bullet$. Then we obtain a Galois multi-admissible covering

$$f^\bullet_E : Z^\bullet_E \to E^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$, where the restriction morphism $f^\bullet_E|_{E_i}$ is an identity, and the Galois action is $j(E_i) = E_{i+j}$ for each $i, j \in \mathbb{Z}/n\mathbb{Z}$.

We may glue $Z^\bullet_{\Gamma^{\mathrm{im}}}$ and $\{Z^\bullet_E\}_{E\in\pi_0(\overline{X\setminus X_{\Gamma^{\mathrm{im}}}})}$ along $f_\Gamma^{-1}(X_{\Gamma^{\mathrm{im}}} \cap (\bigcup_{E\in\pi_0(\overline{X\setminus X_{\Gamma^{\mathrm{im}}}})} E))$ and $\{f_E^{-1}(X_{\Gamma^{\mathrm{im}}} \cap E)\}_{E\in\pi_0(\overline{X\setminus X_{\Gamma^{\mathrm{im}}}})}$ that is compatible with the gluing of $\{X^\bullet_{\Gamma^{\mathrm{im}}}\} \cup \{E^\bullet\}_{E\in\pi_0(\overline{X\setminus X_{\Gamma^{\mathrm{im}}}})}$ that gives rise to $X^\bullet$, then we obtain a Galois multi-admissible covering

$$f^\bullet : Z^\bullet \to X^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, we write

$$\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$$

for an element induced by $f^\bullet$. We see immediately that $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet)$. By applying Lemma 3.1, we obtain that

$$\gamma_{(\alpha,D)} = g_X + n_X - 2.$$

This completes the proof of the theorem.

□

5 A group-theoretical formula for topological types of pointed stable curves

In this section, by using Theorem 4.5, we prove a group-theoretical formula for the topological type of an arbitrary pointed stable curve over an algebraically closed field of characteristic $p > 0$.

Definition 5.1. (i) Let Δ be an arbitrary profinite group and $m, N \in \mathbb{N}$ positive natural numbers. We define the closed normal subgroup

$$D_N(\Delta)$$

of Δ to be the topological closure of $[\Delta, \Delta]\Delta^N$, where $[\Delta, \Delta]$ denotes the commutator subgroup of Δ. Moreover, we define the closed normal subgroup

$$D_N^{(m)}(\Delta)$$

of Δ inductively by $D_N^{(1)}(\Delta) \overset{\mathrm{def}}{=} D_N(\Delta)$ and $D_N^{(i+1)}(\Delta) \overset{\mathrm{def}}{=} D_N^{(i)}(\Delta)$, $i \in \{1, \dots, m-1\}$. Note that $\#(\Delta/D_N^{(m)}(\Delta)) \leq \infty$ when Δ is topologically finitely generated.

(ii) Let ℓ be a prime number and $r, m \in \mathbb{N}$ natural numbers. We denote by

$$F_{r,m}^{\ell}$$

the finite group $\widehat{F}_r/D_\ell^{(m)}(\widehat{F}_r)$, where $\widehat{F}_r$ denotes the free profinite group of rank r.

Let $X^\bullet = (X, D_X)$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$ and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. In this section, let

$$\Pi$$

be *an abstract profinite group* which is isomorphic to $\Pi_{X^\bullet}$ as profinite groups. Moreover, we denote by $\pi_A(\Pi)$ be the set of finite quotients of Π. We put

$$b_\Pi^1 \overset{\mathrm{def}}{=} \max\{r \mid \text{there exists a prime number } \ell \text{ such that } (\mathbb{Z}/\ell\mathbb{Z})^{\oplus r} \in \pi_A(\Pi)\}$$

and

$$b_\Pi^2 \overset{\mathrm{def}}{=} \begin{cases} 0, & F_{b_\Pi^1,2}^{\ell} \in \pi_A(\Pi) \text{ for some prime number } \ell, \\ 1, & \text{otherwise.} \end{cases}$$

Note that b_Π^i, $i \in \{1, 2\}$, is a group-theoretical invariant associated to Π (i.e., depends only on the isomorphism class of Π). First, we have the following lemma.

Lemma 5.2. *(i) We have that*

$$b^2_\Pi = \begin{cases} 1, & \text{if } n_X = 0, \\ 0, & \text{if } n_X \neq 0. \end{cases}$$

and

$$b^1_\Pi = 2g_X + n_X - 1 + b^2_\Pi.$$

(ii) There exists a unique prime number p_Π such that $(\mathbb{Z}/p_\Pi\mathbb{Z})^{\oplus b^1_\Pi} \notin \pi_A(\Pi)$. In particular, we have $p = p_\Pi$.

Proof. (i) Let $r_\Pi \stackrel{\text{def}}{=} \dim_{\mathbb{F}_\ell}(\Pi^{\text{ab}} \otimes \mathbb{F}_\ell)$, where ℓ an arbitrary prime number $\mathfrak{Primes} \setminus \{p\}$, and $\mathfrak{Primes}$ denotes the set of prime numbers. Then the structures of maximal prime-to-p quotients of admissible fundamental groups imply that

$$\Pi^{\text{ab}} \cong \mathbb{Z}_p^{\sigma_{X^\bullet}} \times \prod_{\ell \in \mathfrak{Primes}\setminus\{p\}} \mathbb{Z}_\ell^{r_\Pi}.$$

Since $X^\bullet$ is a pointed stable curve, we have that

$$\sigma_{X^\bullet} < r_\Pi.$$

This implies that $b^1_\Pi = r_\Pi$. Moreover, we have

$$b^1_\Pi = \begin{cases} 2g_X, & \text{if } n_X = 0, \\ 2g_X + n_X - 1, & \text{if } n_X \neq 0. \end{cases}$$

Suppose that $n_X > 0$. Let $\ell_1 \in \mathfrak{Primes} \setminus \{p\}$. The specialization theorem of maximal pro-ℓ_2 quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]) implies that the maximal pro-ℓ_1 quotient Π^{ℓ_1} of Π is a free pro-ℓ_1 profinite group of rank b^1_Π. Then we obtain immediately that

$$F^{\ell_1}_{b^1_\Pi, 2} \in \pi_A(\Pi).$$

Thus, we obtain that $b^2_\Pi = 0$ if $n_X > 0$.

Conversely, we assume that $F^{\ell_2}_{b^1_\Pi, 2} \in \pi_A(\Pi)$ for some prime number ℓ_2. Then we have $\ell_2 \neq p$. Note that we have the following natural exact sequence

$$1 \to (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus \ell_2^{b^1_\Pi}(b^1_\Pi - 1)+1} \to F^{\ell_2}_{b^1_\Pi, 2} \to (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus b^1_\Pi} \to 1.$$

Let $\phi : \Pi \twoheadrightarrow F^{\ell_2}_{b^1_\Pi, 2}$ be a surjection. We denote by $X^\bullet_{\ell_2}$ the pointed stable curve over k corresponding to the kernel of the natural surjection

$$\Pi_{X^\bullet} \xrightarrow{\sim} \Pi \stackrel{\phi}{\twoheadrightarrow} F^{\ell_2}_{b^1_\Pi, 2} \twoheadrightarrow (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus b^1_\Pi}$$

and by $\Pi_{\ell_2} \subseteq \Pi$ the kernel of the surjection $\Pi \stackrel{\phi}{\twoheadrightarrow} F^{\ell_2}_{b^1_\Pi, 2} \twoheadrightarrow (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus b^1_\Pi}$. Then we have

$$(\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus \ell_2^{b^1_\Pi}(b^1_\Pi - 1)+1} \in \pi_A(\Pi_{\ell_2}).$$

Thus, $b^1_{\Pi_{\ell_2}} \geq \ell_2^{b^1_\Pi}(b^1_\Pi - 1) + 1$. If $n_X = 0$, the Riemann-Hurwitz formula implies that

$$g_{X_{\ell_2}} = \ell_2^{b^1_\Pi}(g_X - 1) + 1,$$

where $g_{X_{\ell_2}}$ denotes the genus of $X^\bullet_{\ell_2}$. Then we have

$$b^1_{\Pi_{\ell_2}} = 2(\ell_2^{b^1_\Pi}(g_X - 1) + 1) = \ell_2^{b^1_\Pi}(b^1_\Pi - 2) + 2.$$

On the other hand,

$$\ell_2^{b^1_\Pi}(b^1_\Pi - 2) + 2 < \ell_2^{b^1_\Pi}(b^1_\Pi - 1) + 1.$$

This contradicts to the fact that $b^1_{\Pi_{\ell_2}} \geq \ell_2^{b^1_\Pi}(b^1_\Pi - 1) + 1$. Then we obtain that $n_X > 0$ if $b^2_\Pi = 0$. Moreover, we see immediately that

$$b^1_\Pi = 2g_X + n_X - 1 + b^2_\Pi.$$

(ii) This follows immediately from the structure of Π^{ab}. We complete the proof of the lemma. □

Let $\overline{\mathbb{F}}_{p_\Pi}$ be an arbitrary algebraic closure of $\mathbb{F}_{p_\Pi}$. Let $\chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}^\times_{p_\Pi})$. We denote by $\Pi_\chi \subseteq \Pi$ the kernel of χ. Moreover, we put

$$\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z})[\chi] \overset{\mathrm{def}}{=} \{\pi \in \mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z}) \otimes_{\mathbb{F}_{p_\Pi}} \overline{\mathbb{F}}_{p_\Pi} \mid \tau(\pi) = \chi(\tau)\pi \\ \text{for all } \pi \in \Pi\},$$

and put $\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z})) \overset{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_{p_\Pi}}(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z})[\chi])$. We define a group-theoretical invariant associated to Π as follows:

$$\gamma^{\max}_\Pi \overset{\mathrm{def}}{=} \max\{\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z})) \mid \chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}^\times_{p_\Pi}) \text{ and } \chi \neq 1\}.$$

Then we have the following lemma.

Lemma 5.3. *Let $\gamma^{\max}_{X^\bullet}$ be the maximum of generalized Hasse-Witt invariant of prime-to-p cyclic admissible coverings of $X^\bullet$ defined in Section 3. Then we have $\gamma^{\max}_\Pi = \gamma^{\max}_{X^\bullet}$. In particular, we have*

$$\gamma^{\max}_\Pi = g_X + n_X - 2 + b^2_\Pi.$$

Proof. The first part of lemma follows immediately from the definitions of generalized Hasse-Witt invariants and $\gamma^{\max}_{X^\bullet}$. The "in particular" part of the lemma follows immediately from Theorem 4.5 and Lemma 5.2 (i). □

The main theorem of the present paper is as follows.

Theorem 5.4. *Let $X^\bullet$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$, and Π an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups. Then we have that*

$$g_X = b^1_\Pi - \gamma^{\max}_\Pi - 1, \ n_X = 2\gamma^{\max}_\Pi - b^1_\Pi - b^2_\Pi + 3.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π.

Proof. The theorem follows immediately from Lemma 5.2 and Lemma 5.3. $\square$

Remark 5.4.1. We maintain the notation introduced above. Moreover, suppose that $X^\bullet$ is *smooth* over k. In this remark, we discuss a formula for (g_X, n_X) which was essentially obtained by Tamagawa. Let $n \stackrel{\text{def}}{=} p^t - 1$ and K_n the kernel of the natural surjection $\Pi \twoheadrightarrow \Pi^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z}$. In [T2], Tamagawa defined the limit of p-averages associated to Π to be

$$\mathrm{Avr}_p(\Pi) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K_n^{\text{ab}} \otimes \mathbb{F}_p)}{\#(\Pi^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}.$$

Note that since $p = p_\Pi$ (cf. Lemma 5.2 (ii)), we have that $\mathrm{Avr}_p(\Pi)$ is a group-theoretical invariant associated to Π. Then the main theorem of [T2] (i.e., Tamagawa's p-average theorem, see [T2, Theorem 0.5]) says that

$$\mathrm{Avr}_p(\Pi) = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X, & \text{if } n_X \geq 2. \end{cases}$$

Let $\ell' \in \mathfrak{Primes} \setminus \{p_\Pi = p\}$ be an arbitrary prime number distinct from p_Π. Write $\mathrm{Nom}_{\ell'}(\Pi)$ for the set of normal subgroups of Π such that $\#(\Pi/\Pi(\ell')) = \ell'$ for each $\Pi(\ell') \in \mathrm{Nom}_{\ell'}(\Pi)$. Suppose that $b^2_\Pi = 0$ (i.e., $n_X \neq 0$). By applying Riemann-Hurwitz formula, we see immediately that

$$\mathrm{Avr}_p(\Pi(\ell)) - 1 = \ell(\mathrm{Avr}_p(\Pi))$$

holds for each $\ell \in \mathfrak{Primes} \setminus \{p_\Pi\}$ and each $\Pi(\ell) \in \mathrm{Nom}_\ell(\Pi)$ if and only if $n_X = 1$. We define a group-theoretical invariant associated to Π as follows:

$$c_\Pi \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } b^2_\Pi = 1, \\ 1, & \text{if } b^2_\Pi = 0,\ \mathrm{Avr}_p(\Pi(\ell)) - 1 = \ell(\mathrm{Avr}_p(\Pi)),\ \ell \in \mathfrak{Primes} \setminus \{p_\Pi\},\ \Pi(\ell) \in \mathrm{Nom}_\ell(\Pi), \\ 0, & \text{otherwise.} \end{cases}$$

Then the p-average theorem above implies immediately the following formula

$$g_X = \mathrm{Avr}_p(\Pi) + c_\Pi,\ n_X = b^1_\Pi - 2\mathrm{Avr}_p(\Pi) - 2c_\Pi - b^2_\Pi + 1.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π (cf. [T2, Theorem 0.1]). This result is the main goal of the theory developed in [T2], which plays a key role in the theory of tame anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (cf. [T2], [Y2]).

On the other hand, the approach to finding a group-theoretical formula for (g_X, n_X) by applying the limit of p-averages associated to Π explained above *cannot be generalized* to the case where $X^\bullet$ is an arbitrary (possibly singular) pointed stable curve. The reason is as follows. In [Y3], the author generalized Tamagawa's p-average theorem to the case of pointed stable curves (cf. [Y3, Theorem 1.3 and Theorem 1.4]). The generalized formula concerning the limit of p-averages associated to Π is very complicated in general when $X^\bullet$ is not smooth over k, and $\mathrm{Avr}_p(\Pi)$ depends not only on the topological type (g_X, n_X) but also on the structure of dual semi-graph $\Gamma_{X^\bullet}$.

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Yu Yang

Address: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail: yuyang@kurims.kyoto-u.ac.jp