RIMS-1902

Integrable Connections I: Two Fundamental Correspondences

By

Yuichiro HOSHI

July 2019

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Integrable Connections I: Two Fundamental Correspondences

Yuichiro Hoshi

July 2019

Abstract. — The purpose of the present paper is to establish a certain abstract theory related to the notion of an integrable connection. More concretely, to establish certain two bijections related to the notion of an integrable connection is the main purpose of the present paper.

Contents

Introduction

The purpose of the present paper is to establish a certain abstract theory related to the notion of an *integrable connection*. More concretely, to establish certain *two bijections* [cf. the respective discussions following Theorem A and Theorem B below] related to the notion of an *integrable connection* is the main purpose of the present paper.

In the present Introduction, let us fix a scheme S. Write Sch_S for the category of schemes over S and morphisms of schemes over S [cf. Definition 1.6, (i)]. Let us also fix a scheme X which is *smooth* and *separated* over S and a category $\mathcal{F} \to \mathrm{Sch}_S$ fibered in groupoids over Sch_S. Write $X^{(2)} \stackrel{\mathrm{def}}{=} X \times_S X$ for the fiber product of two copies of X over S [cf. Definition 2.1, (i)] and ${}^X\mathrm{pr}_i^{(2)} \colon X^{(2)} \to X$ for the projection onto the i-th factor [where $i \in \{1, 2\}$] [cf. Definition 2.1, (iii)].

2010 Mathematics Subject Classification. — 14D15.
Key words and phrases. — connection, curvature, integrable connection, stratification, stratification structure.

One main object of the present paper is a *stratification structure* [cf. Definition 2.5]. We shall refer to a collection of data

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big)$$

consisting of

- a scheme ${}^{\mathfrak{S}}P^n$ over S for each nonnegative integer n,
- a morphism ${}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1}$ over S for each nonnegative integer n,
- a morphism ${}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)}$ over S for each nonnegative integer n, and
- a morphism ${}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2}$ over S for each nonnegative integers n_1, n_2

— where we write ${}^{\mathfrak{S}}P^{n_1,n_2}$ for the fiber product of the composite ${}^{\mathfrak{S}}P^{n_1} \overset{{}^{\mathfrak{S}}\sigma^{n_1}}{\to} X^{(2)} \overset{{}^{X}\mathrm{pr}_2^{(2)}}{\to} X$ and the composite ${}^{\mathfrak{S}}P^{n_2} \overset{{}^{\mathfrak{S}}\sigma^{n_2}}{\to} X^{(2)} \overset{{}^{X}\mathrm{pr}_1^{(2)}}{\to} X$ [cf. Definition 2.4, (i)]

that satisfies certain conditions as a *stratification structure* on X/S.

An example of a stratification structure on X/S is given as follows [cf. Remark 2.5.1]: Let Y be a *separated* scheme over S and $X \to Y$ an *affine* morphism over S. Write $P \overset{\mathrm{def}}{=} X \times_Y X$; q_1, $q_2 \colon P \to X$ for the projections onto the first, second factors, respectively; $q_{1,3} \colon X \times_Y X \times_Y X \to P$ for the projection onto the first and third factors. Let

$$P^0 \subseteq P^1 \subseteq \ldots \subseteq P^n \subseteq P^{n+1} \subseteq \ldots \subseteq P$$

be a sequence of closed subschemes of P. Suppose that the following two conditions are satisfied:

- The natural closed immersion $P^0 \hookrightarrow P$ determines an *isomorphism* of P^0 with the diagonal closed subscheme $X \subseteq P$ of P.

- For each nonnegative integers n_1, n_2, if we write $P^{n_1,n_2} \overset{\mathrm{def}}{=} P^{n_1} \times_X P^{n_2}$ for the fiber product of the composite $P^{n_1} \hookrightarrow P \overset{q_2}{\to} X$ and the composite $P^{n_2} \hookrightarrow P \overset{q_1}{\to} X$, then the composite

$$P^{n_1,n_2} \hookrightarrow X \times_Y X \times_Y X \xrightarrow{q_{1,3}} P$$

factors through the closed subscheme $P^{n_1+n_2} \subseteq P$ of P. Write

$$\delta^{n_1,n_2} \colon P^{n_1,n_2} \longrightarrow P^{n_1+n_2}$$

for the resulting morphism.

For each nonnegative integer n, write $\iota^n \colon P^n \hookrightarrow P^{n+1}$ for the natural closed immersion and $\sigma^n \colon P^n \to X^{(2)}$ for the composite of the natural closed immersion $P^n \hookrightarrow P$ and the closed immersion $P \hookrightarrow X^{(2)}$ induced by the morphism $Y \to S$. Then the collection of data

$$\big((P^n)_{n\geq 0},\ (\iota^n \colon P^n \hookrightarrow P^{n+1})_{n\geq 0},\ (\sigma^n \colon P^n \to X^{(2)})_{n\geq 0},\ (\delta^{n_1,n_2} \colon P^{n_1,n_2} \to P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big)$$

gives an example of a stratification structure on X/S.

Let us fix a stratification structure on X/S

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\,n_2\geq 0}\big).$$

Note that one condition imposed on a stratification structure is that the morphism ${}^{\mathfrak{S}}\sigma^0 \colon {}^{\mathfrak{S}}P^0 \to X^{(2)}$ determines an *isomorphism* of ${}^{\mathfrak{S}}P^0$ with the diagonal closed subscheme $X \subseteq X^{(2)}$ of $X^{(2)}$ [cf. condition (2) of Definition 2.2]. In the remainder of the present Introduction, let us identify ${}^{\mathfrak{S}}P^0$ with X by means of the isomorphism determined by the morphism ${}^{\mathfrak{S}}\sigma^0$. For each nonnegative integer n, write ${}^{\mathfrak{S}}\iota^0_n \colon X = {}^{\mathfrak{S}}P^0 \to {}^{\mathfrak{S}}P^n$ for the morphism over S obtained by forming the composite of the morphisms ${}^{\mathfrak{S}}\iota^0, \ldots, {}^{\mathfrak{S}}\iota^{n-1}$ [cf. Definition 2.3, (i)]; ${}^{\mathfrak{S}}\mathrm{pr}^n_i \colon {}^{\mathfrak{S}}P^n \to X$ for the morphism over S obtained by forming the composite of the morphism ${}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)}$ and the morphism ${}^{X}\mathrm{pr}^{(2)}_i \colon X^{(2)} \to X$ [where $i \in \{1,2\}$] [cf. Definition 2.3, (ii)]. Note, moreover, that the composite $X = {}^{\mathfrak{S}}P^0 \stackrel{{}^{\mathfrak{S}}\iota^0_n}{\to} {}^{\mathfrak{S}}P^n \stackrel{{}^{\mathfrak{S}}\mathrm{pr}^n_i}{\to} X$ coincides with the *identity automorphism* of X [cf. Remark 2.3.1].

Other main objects of the present paper are *connections* and *stratifications*. Let us fix an object ξ of $\mathcal{F}$ over X. Then we shall refer to an isomorphism

$$({}^{\mathfrak{S}}\mathrm{pr}^n_2)^*\xi \xrightarrow{\sim} ({}^{\mathfrak{S}}\mathrm{pr}^n_1)^*\xi$$

in $\mathcal{F}$ over the identity automorphism of ${}^{\mathfrak{S}}P^n$ whose pull-back by ${}^{\mathfrak{S}}\iota^0_n$ is the identity automorphism of $\xi = ({}^{\mathfrak{S}}\iota^0_n)^*({}^{\mathfrak{S}}\mathrm{pr}^n_2)^*\xi = ({}^{\mathfrak{S}}\iota^0_n)^*({}^{\mathfrak{S}}\mathrm{pr}^n_1)^*\xi$ and which satisfies a certain "cocycle condition" as an n-$\mathfrak{S}$-*connection* on ξ [cf. Definition 4.4]; moreover, we shall refer to a 1-$\mathfrak{S}$-connection as an $\mathfrak{S}$-*connection* [cf. Definition 4.1, (iii); Remark 4.1.1]. Write

$${}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi)$$

for the set of n-$\mathfrak{S}$-*connections* on ξ [cf. Definition 4.4]. Then one may prove that the pull-back of an $(n+1)$-$\mathfrak{S}$-connection by ${}^{\mathfrak{S}}\iota^n$ is an n-$\mathfrak{S}$-connection, which thus implies that one obtains a map ${}^{\mathfrak{S}}\mathrm{Cnn}^{n+1}(\xi) \to {}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi)$ [cf. Lemma 4.5]. Write

$${}^{\mathfrak{S}}\mathrm{Strt}(\xi) \stackrel{\mathrm{def}}{=} \varprojlim_{n\geq 0} {}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi)$$

and refer to an element of the set ${}^{\mathfrak{S}}\mathrm{Strt}(\xi)$ as an $\mathfrak{S}$-*stratification* on ξ [cf. Definition 4.6]. Now let us observe that, in the situation of the above example of a stratification structure on X/S, if there exists a nonnegative integer n_0 such that the closed immersion $P_{n_0} \hookrightarrow P$ is an *isomorphism*, then one verifies easily that the notion of a *stratification on* ξ [i.e., with respect to the above stratification structure] is the *same* as the notion of a *descent datum on* ξ *with respect to the morphism* $X \to Y$ [cf. Remark 4.6.2].

In the present paper, we give the definition of the notion of the $\mathfrak{S}$-*curvature* of an $\mathfrak{S}$-connection [cf. Definition 4.7, (i)]. The $\mathfrak{S}$-curvature of an $\mathfrak{S}$-connection on ξ is defined to be an automorphism of the pull-back of ξ to a certain scheme over S. Moreover, we shall say that an $\mathfrak{S}$-connection is $\mathfrak{S}$-*integrable* if the $\mathfrak{S}$-curvature of the $\mathfrak{S}$-connection is the identity automorphism [cf. Definition 4.7, (ii)]. Write

$${}^{\mathfrak{S}}\mathrm{IntCnn}(\xi) \subseteq {}^{\mathfrak{S}}\mathrm{Cnn}^1(\xi)$$

for the set of $\mathfrak{S}$-integrable $\mathfrak{S}$-connections on ξ [cf. Definition 4.7, (ii)]. Note that one may prove that if $n \geq 2$, then the map ${}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \to {}^{\mathfrak{S}}\mathrm{Cnn}^1(\xi)$ discussed above *factors* through the subset ${}^{\mathfrak{S}}\mathrm{IntCnn}(\xi) \subseteq {}^{\mathfrak{S}}\mathrm{Cnn}^1(\xi)$ [cf. Lemma 4.8].

We are now ready to state the first main result of the present paper. Roughly speaking, the first main result of the present paper asserts that if the stratification structure $\mathfrak{S}$ and the category $\mathcal{F}$ fibered in groupoids over Sch_S have certain "good" properties, then we have a *natural bijection* between the set of $\mathfrak{S}$-*stratifications* and the set of $\mathfrak{S}$-*integrable* $\mathfrak{S}$-*connections*. The first main result of the present paper is as follows [cf. Theorem 5.6].

THEOREM A. — *Let S be a scheme, X a scheme which is* **smooth** *and* **separated** *over S,*

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big)$$

a stratification structure on X/S [cf. Definition 2.5],

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S, and ξ an object of $\mathcal{F}$ over X. Suppose that the following two conditions are satisfied:

(1) *The stratification structure $\mathfrak{S}$ is* **integrable** [*cf. Definition* 3.2] (*respectively,* **strictly integrable** [*cf. Definition* 3.3]).

(2) *The category $\mathcal{F}$ fibered in groupoids over Sch_S is* **integrable** [*cf. Definition* 1.7] (*respectively,* **weakly integrable** [*cf. Definition* 1.8]).

Then, for each integer $n \geq 2$, the natural maps

$${}^{\mathfrak{S}}\mathrm{Strt}(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{IntCnn}(\xi)$$

[*cf. Definition* 4.4*; Definition* 4.6*; Definition* 4.7, (ii)*; Lemma* 4.8] *are* **bijective**.

In the situation of Theorem A, we shall refer to the bijection obtained by Theorem A

$${}^{\mathfrak{S}}\mathrm{Strt}(\xi) \xrightarrow{\sim} {}^{\mathfrak{S}}\mathrm{IntCnn}(\xi)$$

as the *first fundamental correspondence* [cf. Definition 5.7]. In a sequel to the present paper, we will observe that this *first fundamental correspondence* may be regarded as a *generalization* of a well-known bijection related to *integrable connections*.

Next, to explain the second main result of the present paper, let us introduce the notion of a *morphism of stratification structures* [cf. Definition 2.8]. Let us fix another stratification structure on X/S

$$\mathfrak{T} = \big(({}^{\mathfrak{T}}P^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\iota^n \colon {}^{\mathfrak{T}}P^n \to {}^{\mathfrak{T}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{T}}\sigma^n \colon {}^{\mathfrak{T}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{T}}\delta^{n_1,n_2} \colon {}^{\mathfrak{T}}P^{n_1,n_2} \to {}^{\mathfrak{T}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big).$$

Then we shall refer to a collection of data

$$\Phi = (\Phi^n)_{n\geq 0}$$

consisting of a morphism $\Phi^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{T}}P^n$ over S for each nonnegative integer n that satisfies certain conditions as a *morphism of stratification structures*.

Let us fix a morphism of stratification structures

$$\Phi = (\Phi^n)_{n\geq 0} \colon \mathfrak{S} \longrightarrow \mathfrak{T}.$$

Then one may prove that the pull-back of an n-$\mathfrak{T}$-connection by Φ^n is an n-$\mathfrak{S}$-connection, which thus implies that one obtains a map ${}^{\mathfrak{T}}\mathrm{Cnn}^n(\xi) \to {}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi)$, hence also a map ${}^{\mathfrak{T}}\mathrm{Strt}(\xi) \to {}^{\mathfrak{S}}\mathrm{Strt}(\xi)$ [cf. Lemma 4.12, (i)].

In the present paper, for a nonnegative integer n, we give the definition of the notion of the (Φ, n)-*curvature* of an $\mathfrak{S}$-stratification [cf. Definition 4.13, (i)]. The (Φ, n)-curvature of an $\mathfrak{S}$-stratification on ξ is defined to be an automorphism of the pull-back of ξ to a certain scheme over S. We shall say that an $\mathfrak{S}$-stratification is (Φ, n)-*dormant* if the (Φ, n)-curvature of the $\mathfrak{S}$-stratification is the identity automorphism [cf. Definition 4.7, (ii)]. Moreover, we shall say that an $\mathfrak{S}$-connection is (Φ, n)-*dormant* if the $\mathfrak{S}$-connection extends to a (Φ, n)-dormant $\mathfrak{S}$-stratification [cf. Definition 4.7, (iii)]. Write

$$ {}^{\Phi,n}\mathrm{DrmStrt}(\xi) \subseteq {}^{\mathfrak{S}}\mathrm{Strt}(\xi), \qquad {}^{\Phi,n}\mathrm{DrmCnn}(\xi) \subseteq {}^{\mathfrak{S}}\mathrm{IntCnn}(\xi)$$

for the sets of (Φ, n)-dormant $\mathfrak{S}$-stratifications, (Φ, n)-dormant $\mathfrak{S}$-connections on ξ, respectively [cf. Definition 4.13, (ii), (iii)]. Note that one may prove that the map ${}^{\mathfrak{T}}\mathrm{Strt}(\xi) \to {}^{\mathfrak{S}}\mathrm{Strt}(\xi)$ discussed above *factors* through the subset ${}^{\Phi,n}\mathrm{DrmStrt}(\xi) \subseteq {}^{\mathfrak{S}}\mathrm{Strt}(\xi)$ [cf. Lemma 4.14].

We are now ready to state the second main result of the present paper. Roughly speaking, the second main result of the present paper asserts that if the morphism Φ and the category $\mathcal{F}$ fibered in groupoids over Sch_S have certain "good" properties, then we have a *natural bijection* between the set of $\mathfrak{T}$-*stratifications* and the set of (Φ, n)-*dormant* $\mathfrak{S}$-*connections*. The second main result of the present paper is as follows [cf. Theorem 6.6].

THEOREM B. — *Let S be a scheme; X a scheme which is* **smooth** *and* **separated** *over S;*

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big),$$

$$\mathfrak{T} = \big(({}^{\mathfrak{T}}P^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\iota^n \colon {}^{\mathfrak{T}}P^n \to {}^{\mathfrak{T}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{T}}\sigma^n \colon {}^{\mathfrak{T}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{T}}\delta^{n_1,n_2} \colon {}^{\mathfrak{T}}P^{n_1,n_2} \to {}^{\mathfrak{T}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big)$$

stratification structures on X/S [*cf. Definition* 2.5]*;*

$$\Phi = (\Phi^n)_{n\geq 0} \colon \mathfrak{S} \longrightarrow \mathfrak{T}$$

a morphism of stratification structures [*cf. Definition* 2.8]*;*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S; ξ an object of $\mathcal{F}$ over X; r a nonnegative integer. Suppose that the following two conditions are satisfied:

(1) *The morphism* Φ *is* $\boldsymbol{r}$**-integrable** [*cf. Definition* 3.8] (*respectively,* **strictly** $\boldsymbol{r}$**-integrable** [*cf. Definition* 3.9]).

(2) *The category* $\mathcal{F}$ *fibered in groupoids over* Sch_S *is* **integrable** [*cf. Definition* 1.7] (*respectively,* **weakly integrable** [*cf. Definition* 1.8]).

Then the natural maps

$$^{\mathfrak{T}}\mathrm{Strt}(\xi) \xrightarrow{\Phi^*} {}^{\Phi,r}\mathrm{DrmStrt}(\xi) \longrightarrow {}^{\Phi,r}\mathrm{DrmCnn}(\xi)$$

[*cf. Definition* 4.6*; Definition* 4.13, (ii), (iii)*; Lemma* 4.14] *are* **bijective**.

In the situation of Theorem B, we shall refer to the bijection obtained by Theorem B

$$^{\mathfrak{T}}\mathrm{Strt}(\xi) \xrightarrow{\sim} {}^{\Phi,r}\mathrm{DrmCnn}(\xi)$$

as the *second fundamental correspondence* [cf. Definition 6.7]. In a sequel to the present paper, we will observe that this *second fundamental correspondence* may be regarded as a *generalization* of a well-known bijection related to *dormant connections*, i.e., *integrable connections of p-curvature zero.*

ACKNOWLEDGMENTS

This research was supported by JSPS KAKENHI Grant Number 18K03239 and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

1. INTEGRABLE CATEGORIES FIBERED IN GROUPOIDS

In the present §1, we introduce certain properties of a category fibered in groupoids over categories of schemes [cf. Definition 1.7 and Definition 1.8 below].

LEMMA 1.1. — *Let* $f\colon X \to Y$ *be a morphism of schemes. Suppose that* f *is a* **homeomorphism**. *Then* f *is* **affine**.

PROOF. — To verify Lemma 1.1, we may assume without loss of generality, by replacing Y by an affine open subscheme of Y, that Y is *affine*. Let $y \in Y$ be a point of Y and $U \subseteq X$ an affine open subscheme of X such that $y \in f(U)$. Then since $f(U) \subseteq Y$ is an open neighborhood of $y \in Y$, there exists a section $s \in \Gamma(Y, \mathcal{O}_Y)$ such that $y \in D(s) \subseteq f(U)$ — where we write "$D(-)$" for the maximal [necessarily *affine*] open subscheme on which "$(-)$" is invertible. Then since f is a *homeomorphism*, and $D(s)$ is *contained* in $f(U)$, one verifies easily that the [necessarily *affine*] open subscheme $D(f^{-1}s) \subseteq U$ of the affine scheme U determined by the section $f^{-1}s \in \Gamma(U, \mathcal{O}_X)$ satisfies the equality $D(f^{-1}s) = f^{-1}(D(s))$. Thus, one obtains an *affine* open neighborhood $D(s) \subseteq Y$ of $y \in Y$ whose pull-back by f is *affine*, as desired. This completes the proof of Lemma 1.1. $\square$

DEFINITION 1.2. — Let X, Y be schemes; $f \colon X \to Y$ a morphism of schemes.

(i) We shall say that a closed subscheme of the scheme X is *square-nilpotent* if the square of the quasi-coherent ideal of $\mathcal{O}_X$ that defines the closed subscheme is zero.

(ii) Suppose that f is a closed immersion. Then we shall say that the closed immersion f is *square-nilpotent* if the closed subscheme of Y determined by f is square-nilpotent.

(iii) We shall say that the morphism f is *quasi-nil-retraction-like* if f induces an isomorphism $X_{\mathrm{red}} \xrightarrow{\sim} Y_{\mathrm{red}}$, where we write "$(-)_{\mathrm{red}}$" for the closed subscheme of "$(-)$" defined by the quasi-coherent ideal of "$\mathcal{O}_{(-)}$" consisting of nilpotent local sections.

(iv) We shall say that the morphism f is *nil-retraction-like* if f is quasi-nil-retraction-like, and, moreover, the homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ determined by f is injective.

REMARK 1.2.1. — One verifies easily from Lemma 1.1, together with the various definitions involved, that we have the following implications:

$$
\begin{array}{ccc}
 & & \text{nil-retraction-like morphism} \\
 & & \Downarrow \\
\text{square-nilpotent closed immersion} & \Longrightarrow & \text{quasi-nil-retraction-like morphism} \\
 & & \Downarrow \\
 & & \text{universal homeomorphism} \\
 & & \Downarrow \\
 & & \text{affine morphism.}
\end{array}
$$

DEFINITION 1.3. — Let X, Y, $\underline{Y}$ be schemes; $\underline{Y} \hookrightarrow Y$ a square-nilpotent closed immersion; $f \colon X \to Y$ a morphism of schemes. Write $\underline{X} \stackrel{\mathrm{def}}{=} X \times_Y \underline{Y} \hookrightarrow X$ for the [necessarily square-nilpotent] closed immersion obtained by forming the base-change of the closed immersion $\underline{Y} \hookrightarrow Y$ by f; $\mathcal{I}_X$, $\mathcal{I}_Y$ for the conormal sheaves of the square-nilpotent closed immersions $\underline{X} \hookrightarrow X$, $\underline{Y} \hookrightarrow Y$, respectively [i.e., the quasi-coherent ideals of $\mathcal{O}_X$, $\mathcal{O}_Y$ that define the closed subschemes of X, Y determined by the closed immersions $\underline{X} \hookrightarrow X$, $\underline{Y} \hookrightarrow Y$, respectively]. Then we shall say that the morphism f is *conormally strict* with respect to $\underline{Y} \hookrightarrow Y$ if

(1) the morphism f is quasi-nil-retraction-like [hence also affine — cf. Remark 1.2.1], and

(2) the homomorphism of $\mathcal{O}_Y$-modules

$$\mathcal{I}_Y \longrightarrow f_*\mathcal{I}_X$$

induced by the homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ determined by f is injective,

and, moreover, there exist schemes $\overline{X}$, $\overline{Y}$ and closed immersions $i_X \colon \overline{X} \hookrightarrow X$, $i_Y \colon \overline{Y} \hookrightarrow Y$ such that

(3) the natural homomorphisms

$$\mathcal{I}_X \longrightarrow (i_X)_* i_X^* \mathcal{I}_X, \qquad \mathcal{I}_Y \longrightarrow (i_Y)_* i_Y^* \mathcal{I}_Y$$

are isomorphisms [or, alternatively, the quasi-coherent ideals $\mathcal{I}_X$, $\mathcal{I}_Y$ of $\mathcal{O}_X$, $\mathcal{O}_Y$ are annihilated by the quasi-coherent ideals that define the closed subschemes of X, Y determined by the closed immersions $i_X \colon \overline{X} \hookrightarrow X$, $i_Y \colon \overline{Y} \hookrightarrow Y$, respectively], and

(4) the morphism $f \colon X \to Y$ induces [relative to i_X, i_Y] an isomorphism of schemes

$$\overline{f} \colon \overline{X} \xrightarrow{\sim} \overline{Y}.$$

REMARK 1.3.1. — In the situation of Definition 1.3, it is immediate from Remark 1.2.1 that if f is *nil-retraction-like*, then conditions (1), (2) are always satisfied.

LEMMA 1.4. — *In the situation of Definition* 1.3*, suppose that the morphism f is* **conormally strict** *with respect to $\underline{Y} \hookrightarrow Y$, i.e., satisfies the four conditions of Definition* 1.3*. Then the homomorphism of $\mathcal{O}_{\overline{Y}}$-module*

$$i_Y^* \mathcal{I}_Y \longrightarrow \overline{f}_* i_X^* \mathcal{I}_X$$

determined by $\overline{f}$ is an **isomorphism***.*

PROOF. — Since f is *affine* [cf. condition (1) of Definition 1.3], we may assume without loss of generality, by replacing Y by an affine open subscheme of Y, that both X and Y are *affine*. Write $B \stackrel{\text{def}}{=} \Gamma(X, \mathcal{O}_X)$, $A \stackrel{\text{def}}{=} \Gamma(Y, \mathcal{O}_Y)$, and $I_B \subseteq B$ (respectively, $J_B \subseteq B$; $I_A \subseteq A$; $J_A \subseteq A$) for the ideal that defines the closed subscheme determined by the closed immersion $\underline{X} \hookrightarrow X$ (respectively, $\overline{X} \hookrightarrow X$; $\underline{Y} \hookrightarrow Y$; $\overline{Y} \hookrightarrow Y$). Now observe that, to complete the verification of Lemma 1.4, it suffices to verify that the homomorphism I_A ($\stackrel{\sim}{\to} I_A \otimes_A (A/J_A)$ — cf. condition (3) of Definition 1.3) $\to I_B = I_A B$ ($\stackrel{\sim}{\to} (I_A B) \otimes_B (B/J_B)$ — cf. condition (3) of Definition 1.3) induced by the homomorphism $A \to B$ determined by f is an *isomorphism*. On the other hand, it follows from condition (2) of Definition 1.3 that the homomorphism $I_A \to I_A B$ is *injective*. Moreover, since the homomorphism $A \to B$ induces an *isomorphism* $A/J_A \stackrel{\sim}{\to} B/J_B$ [cf. condition (4) of Definition 1.3], the homomorphism $I_A \otimes_A (A/J_A) \to (I_A B) \otimes_B (B/J_B)$ is *surjective*. This completes the proof of Lemma 1.4. $\square$

DEFINITION 1.5. — Let

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"'] & Y \arrow[d, "c"] \\ Z \arrow[r, "d"'] & W \end{tikzcd}$$

be a commutative diagram of schemes. Then we shall say that the diagram is *strictly cocartesian* if the following two conditions are satisfied:

(1) The morphisms a, b, c, and d are nil-retraction-like [hence also affine — cf. Remark 1.2.1].

(2) The affine open subschemes $U_W \subseteq W$ of W that satisfy the following condition form an open basis of W: If one regards the rings $\Gamma(c^{-1}(U_W), \mathcal{O}_Y)$, $\Gamma(d^{-1}(U_W), \mathcal{O}_Z)$, $\Gamma(U_W, \mathcal{O}_W)$ as subrings of $\Gamma(a^{-1}(c^{-1}(U_W)), \mathcal{O}_X) = \Gamma(b^{-1}(d^{-1}(U_W)), \mathcal{O}_X)$ by the relevant injective [cf. (1)] homomorphisms, then the equality

$$\Gamma(U_W, \mathcal{O}_W) = \Gamma(c^{-1}(U_W), \mathcal{O}_Y) \cap \Gamma(d^{-1}(U_W), \mathcal{O}_Z)$$

in $\Gamma(a^{-1}(c^{-1}(U_W)), \mathcal{O}_X) = \Gamma(b^{-1}(d^{-1}(U_W)), \mathcal{O}_X)$ holds.

REMARK 1.5.1. — One verifies immediately that a *strictly cocartesian* diagram of schemes is *cocartesian* in the category of schemes.

DEFINITION 1.6. — Let S be a scheme.

(i) We shall write

$$\mathrm{Sch}_S$$

for the category of schemes over S and morphisms of schemes over S.

(ii) Let

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

be a category fibered in groupoids over Sch_S and X a scheme over S. Then we shall write

$$\mathcal{F}|_X$$

for the groupoid of objects of $\mathcal{F}$ over X and isomorphisms in $\mathcal{F}$ over the identity automorphism of X.

(iii) Let $f\colon X \to Y$ be a morphism in Sch_S; ξ_1, ξ_2 objects of $\mathcal{F}$ over Y; $\phi\colon f^*\xi_1 \xrightarrow{\sim} f^*\xi_2$ an isomorphism in $\mathcal{F}|_X$. Then we shall write

$$\mathrm{Lift}_f(\xi_1, \xi_2; \phi) \subseteq \mathrm{Isom}_{\mathcal{F}|_Y}(\xi_1, \xi_2)$$

for the set of isomorphisms $\psi\colon \xi_1 \xrightarrow{\sim} \xi_2$ in $\mathcal{F}|_Y$ such that the equality $f^*\psi = \phi$ holds.

DEFINITION 1.7. — Let S be a scheme and

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S. Then we shall say that the category $\mathcal{F}$ fibered in groupoids over Sch_S is *integrable* if the following three conditions are satisfied:

(1) Let

$$\begin{tikzcd} \underline{X} \arrow[r, "f"] \arrow[d, hook, "j_X"'] & \underline{Y} \arrow[d, hook, "j_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

be a cartesian diagram in Sch_S such that the morphism j_Y is a square-nilpotent closed immersion, and, moreover, the morphism f is conormally strict with respect to the square-nilpotent closed immersion j_Y; ξ_1, ξ_2 objects of $\mathcal{F}$ over Y; $\phi\colon j_Y^*\xi_1 \xrightarrow{\sim} j_Y^*\xi_2$ an isomorphism in $\mathcal{F}|_{\underline{Y}}$. Then the map induced by f

$$\mathrm{Lift}_{j_Y}(\xi_1, \xi_2; \phi) \longrightarrow \mathrm{Lift}_{j_X}(f^*\xi_1, f^*\xi_2; \underline{f}^*\phi)$$

is bijective.

(2) Let

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"'] & Y \arrow[d, "c"] \\ Z \arrow[r, "d"'] & W \end{tikzcd}$$

be a commutative diagram in Sch_S which is strictly cocartesian; V a scheme over S;

$$V \xrightarrow{e} X, \qquad \begin{tikzcd} W \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & V \end{tikzcd}$$

quasi-nil-retraction-like morphisms over S such that

$$f_1 \circ c \circ a \circ e = (f_1 \circ d \circ b \circ e =)\, f_2 \circ c \circ a \circ e = (f_2 \circ d \circ b \circ e =)\, \mathrm{id}_V;$$

ξ an object of $\mathcal{F}$ over V; $\phi\colon c^*f_1^*\xi \xrightarrow{\sim} c^*f_2^*\xi$ an isomorphism in $\mathcal{F}|_Y$ such that the pull-back $e^*a^*\phi$ is the identity automorphism of $\xi = e^*a^*c^*f_1^*\xi = e^*a^*c^*f_2^*\xi$. Then the map induced by d

$$\mathrm{Lift}_c(f_1^*\xi, f_2^*\xi; \phi) \longrightarrow \mathrm{Lift}_b(d^*f_1^*\xi, d^*f_2^*\xi; a^*\phi)$$

is bijective.

(3) Let $f\colon X \to Y$ be a nil-retraction-like morphism over S and ξ an object of $\mathcal{F}$ over Y. Then the homomorphism of groups induced by f

$$\mathrm{Aut}_{\mathcal{F}|_Y}(\xi) \longrightarrow \mathrm{Aut}_{\mathcal{F}|_X}(f^*\xi)$$

is injective.

REMARK 1.7.1. — In sequels to the present paper, we give some examples of *integrable* categories fibered in groupoids over Sch_S.

DEFINITION 1.8. — Let S be a scheme and

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S. Then we shall say that the category $\mathcal{F}$ fibered in groupoids over Sch_S is *weakly integrable* if the following three conditions are satisfied:

(1) Let

$$\begin{tikzcd} \underline{X} \arrow[r, "\underline{f}"] \arrow[d, hook, "j_X"'] & \underline{Y} \arrow[d, hook, "j_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

be a cartesian diagram in Sch_S such that the morphism j_Y is a square-nilpotent closed immersion, and, moreover, the morphism f is conormally strict with respect to the square-nilpotent closed immersion j_Y; V a scheme over S;

$$V \xrightarrow{g} \underline{X}, \qquad \begin{tikzcd} Y \arrow[r, bend left, "h_1"] \arrow[r, bend right, "h_2"'] & V \end{tikzcd}$$

quasi-nil-retraction-like morphisms over S such that

$$h_1 \circ j_Y \circ \underline{f} \circ g = (h_1 \circ f \circ j_X \circ g =)\, h_2 \circ j_Y \circ \underline{f} \circ g = (h_2 \circ f \circ j_X \circ g =)\, \mathrm{id}_V,$$

and, moreover, the morphisms

$$h_1 \circ j_Y \circ \underline{f}, \qquad h_2 \circ j_Y \circ \underline{f}, \qquad h_1 \circ j_Y, \qquad h_2 \circ j_Y,$$

$$h_1 \circ f, \qquad h_2 \circ f, \qquad h_1, \qquad h_2$$

are finite and flat; ξ an object of $\mathcal{F}$ over V; $\phi\colon j_Y^* h_1^* \xi \xrightarrow{\sim} j_Y^* h_2^* \xi$ an isomorphism in $\mathcal{F}|_{\underline{Y}}$ such that the pull-back $g^* \underline{f}^* \phi$ is the identity automorphism of $\xi = g^* \underline{f}^* j_Y^* h_1^* \xi = g^* \underline{f}^* j_Y^* h_2^* \xi$. Then the map induced by f

$$\mathrm{Lift}_{j_Y}(h_1^*\xi, h_2^*\xi; \phi) \longrightarrow \mathrm{Lift}_{j_X}(f^* h_1^* \xi, f^* h_2^* \xi; \underline{f}^* \phi)$$

is bijective.

(2) Let

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"'] & Y \arrow[d, "c"] \\ Z \arrow[r, "d"'] & W \end{tikzcd}$$

be a commutative diagram in Sch_S which is strictly cocartesian; V a scheme over S;

$$V \xrightarrow{e} X, \qquad \begin{tikzcd} W \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & V \end{tikzcd}$$

quasi-nil-retraction-like morphisms over S such that

$$f_1 \circ c \circ a \circ e = (f_1 \circ d \circ b \circ e =)\, f_2 \circ c \circ a \circ e = (f_2 \circ d \circ b \circ e =)\, \mathrm{id}_V,$$

and, moreover, the morphisms

$$f_1 \circ c \circ a, \qquad f_2 \circ c \circ a, \qquad f_1 \circ c, \qquad f_2 \circ c,$$

$$f_1 \circ d, \qquad f_2 \circ d, \qquad f_1, \qquad f_2$$

are finite and flat; ξ an object of $\mathcal{F}$ over V; $\phi\colon c^*f_1^*\xi \xrightarrow{\sim} c^*f_2^*\xi$ an isomorphism in $\mathcal{F}|_Y$ such that the pull-back $e^*a^*\phi$ is the identity automorphism of $\xi = e^*a^*c^*f_1^*\xi = e^*a^*c^*f_2^*\xi$. Then the map induced by d

$$\mathrm{Lift}_c(f_1^*\xi, f_2^*\xi; \phi) \longrightarrow \mathrm{Lift}_b(d^*f_1^*\xi, d^*f_2^*\xi; a^*\phi)$$

is bijective.

(3) Condition (3) of Definition 1.7 is satisfied.

REMARK 1.8.1. — One verifies easily that an *integrable* category fibered in groupoids over Sch_S is *weakly integrable*.

2. STRATIFICATION STRUCTURES

In the present §2, we introduce and discuss the notion of a *stratification structure* on a smooth scheme [cf. Definition 2.5 below]. In the present §2, let S be a scheme and X a scheme which is *smooth* and *separated* over S.

DEFINITION 2.1. — Let n be a positive integer.

(i) We shall write

$$X^{(n)} \overset{\mathrm{def}}{=} \overbrace{X \times_S \cdots \times_S X}^{n}$$

for the fiber product of n copies of X over S.

(ii) Let $I \subseteq \{1, \ldots, n\}$ be a nonempty subset of $\{1, \ldots, n\}$. Then we shall write

$${}^X\mathrm{pr}_I^{(n)}\colon X^{(n)} \longrightarrow X^{(\sharp I)}$$

for the projection onto the factors labeled by the elements of I, i.e., the morphism given by "$(x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_{\sharp I}})$", where $I = \{i_1, \ldots, i_{\sharp I}\}$ and $i_1 < \cdots < i_{\sharp I}$.

(iii) Let i be an element of $\{1, \ldots, n\}$. Then we shall write

$${}^X\mathrm{pr}_i^{(n)} \overset{\mathrm{def}}{=} {}^X\mathrm{pr}_{\{i\}}^{(n)}\colon X^{(n)} \longrightarrow X^{(1)} = X.$$

DEFINITION 2.2. — We shall refer to a collection of data

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n\colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},\ ({}^{\mathfrak{S}}\sigma^n\colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0}\big)$$

consisting of

- a scheme ${}^{\mathfrak{S}}P^n$ over S for each nonnegative integer n,
- an affine morphism ${}^{\mathfrak{S}}\iota^n\colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1}$ over S for each nonnegative integer n, and
- an affine morphism ${}^{\mathfrak{S}}\sigma^n\colon {}^{\mathfrak{S}}P^n \to X^{(2)}$ over S for each nonnegative integer n

as a *pre-stratification structure* on X/S if the following two conditions are satisfied:

(1) Let n be a nonnegative integer. Then the diagram of schemes over S

$$\begin{tikzcd} {}^{\mathfrak{S}}P^n \arrow[rr, "{}^{\mathfrak{S}}\iota^n"] \arrow[dr, "{}^{\mathfrak{S}}\sigma^n"'] & & {}^{\mathfrak{S}}P^{n+1} \arrow[dl, "{}^{\mathfrak{S}}\sigma^{n+1}"] \\ & X^{(2)} & \end{tikzcd}$$

is commutative.

(2) The morphism ${}^{\mathfrak{S}}\sigma^0 \colon {}^{\mathfrak{S}}P^0 \to X^{(2)}$ determines an isomorphism of ${}^{\mathfrak{S}}P^0$ with the diagonal closed subscheme $X \subseteq X^{(2)}$ of $X^{(2)}$.

Let us identify ${}^{\mathfrak{S}}P^0$ with X by means of the isomorphism determined by the morphism ${}^{\mathfrak{S}}\sigma^0$:

$$ {}^{\mathfrak{S}}P^0 = X. $$

DEFINITION 2.3. — Let

$$ \mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\sigma^n)_{n\geq 0}\big) $$

be a pre-stratification structure on X/S and n a nonnegative integer.

(i) Let m be a nonnegative integer such that $m < n$. Then we shall write

$$ {}^{\mathfrak{S}}\iota^m_n \colon {}^{\mathfrak{S}}P^m \longrightarrow {}^{\mathfrak{S}}P^n $$

for the morphism over S obtained by forming the composite of the morphisms ${}^{\mathfrak{S}}\iota^m, \dots, {}^{\mathfrak{S}}\iota^{n-1}$. Moreover, we shall write

$$ {}^{\mathfrak{S}}\iota^n_n \stackrel{\mathrm{def}}{=} \mathrm{id}_{{}^{\mathfrak{S}}P^n}. $$

(ii) Let i be an element of $\{1,2\}$. Then we shall write

$$ {}^{\mathfrak{S}}\mathrm{pr}^n_i \colon {}^{\mathfrak{S}}P^n \longrightarrow X $$

for the morphism over S obtained by forming the composite of the morphism ${}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)}$ and the morphism ${}^{X}\mathrm{pr}^{(2)}_i \colon X^{(2)} \to X$.

REMARK 2.3.1. — In the situation of Definition 2.3, it follows from conditions (1), (2) of Definition 2.2 that, for each nonnegative integer n and $i \in \{1,2\}$, the diagram of schemes over S

$$\begin{tikzcd} X = {}^{\mathfrak{S}}P^0 \arrow[rr, "{}^{\mathfrak{S}}\iota^0_n"] \arrow[dr, equal] & & {}^{\mathfrak{S}}P^n \arrow[dl, "{}^{\mathfrak{S}}\mathrm{pr}^n_i"] \\ & X & \end{tikzcd}$$

is *commutative*. In particular, the morphism ${}^{\mathfrak{S}}\iota^0_n \colon X = {}^{\mathfrak{S}}P^0 \to {}^{\mathfrak{S}}P^n$ is a *closed immersion*.

DEFINITION 2.4. — Let

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\sigma^n)_{n\geq 0} \big)$$

be a pre-stratification structure on X/S; r a positive integer; $n_1, \ldots, n_r$ nonnegative integers.

(i) We shall write

$${}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \stackrel{\text{def}}{=} {}^{\mathfrak{S}}P^{n_1} \times_X \cdots \times_X {}^{\mathfrak{S}}P^{n_r}$$

— where, for each $i \in \{1, \ldots, r-1\}$, the morphism ${}^{\mathfrak{S}}P^{n_i} \to X$ (respectively, ${}^{\mathfrak{S}}P^{n_{i+1}} \to X$) implicit in the i-th "$\times_X$" is the morphism ${}^{\mathfrak{S}}\mathrm{pr}_2^{n_i}$ (respectively, ${}^{\mathfrak{S}}\mathrm{pr}_1^{n_{i+1}}$).

(ii) Let $m_1, \ldots, m_r$ be nonnegative integers such that $m_i \leq n_i$ for each $i \in \{1, \ldots, r\}$. Then we shall write

$${}^{\mathfrak{S}}\iota^{m_1,\ldots,m_r}_{n_1,\ldots,n_r} \colon {}^{\mathfrak{S}}P^{m_1,\ldots,m_r} \longrightarrow {}^{\mathfrak{S}}P^{n_1,\ldots,n_r}$$

for the morphism over S determined [cf. condition (1) of Definition 2.2] by the morphisms ${}^{\mathfrak{S}}\iota^{m_1}_{n_1}, \ldots, {}^{\mathfrak{S}}\iota^{m_r}_{n_r}$.

(iii) We shall write

$${}^{\mathfrak{S}}\sigma^{n_1,\ldots,n_r} \colon {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \longrightarrow X^{(r+1)}$$

for the morphism over S determined [cf. condition (1) of Definition 2.2] by the morphisms ${}^{\mathfrak{S}}\sigma^{n_1}, \ldots, {}^{\mathfrak{S}}\sigma^{n_r}$.

(iv) Let i be an element of $\{1, \ldots, r+1\}$. Then we shall write

$${}^{\mathfrak{S}}\mathrm{pr}_i^{n_1,\ldots,n_r} \colon {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \longrightarrow X$$

for the morphism over S obtained by forming the composite of the morphism ${}^{\mathfrak{S}}\sigma^{n_1,\ldots,n_r} \colon {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \to X^{(r+1)}$ and the morphism ${}^{X}\mathrm{pr}_i^{(r+1)} \colon X^{(r+1)} \to X$.

(v) Let i be an element of $\{1, \ldots, r\}$. Then we shall write

$${}^{\mathfrak{S}}\mathrm{pr}^{n_1,\ldots,n_r}_{\{i,i+1\}} \colon {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \longrightarrow {}^{\mathfrak{S}}P^{n_i}$$

for the projection onto the i-th factor.

(vi) We shall write

$${}^{\mathfrak{S}}P^{\times r}, \qquad {}^{\mathfrak{S}}\sigma^{\times r} \colon {}^{\mathfrak{S}}P^{\times r} \longrightarrow X^{(r+1)},$$

$${}^{\mathfrak{S}}\mathrm{pr}_i^{\times r} \colon {}^{\mathfrak{S}}P^{\times r} \longrightarrow X, \qquad {}^{\mathfrak{S}}\mathrm{pr}^{\times r}_{\{i,i+1\}} \colon {}^{\mathfrak{S}}P^{\times r} \longrightarrow {}^{\mathfrak{S}}P^1$$

for the "${}^{\mathfrak{S}}P^{n_1,\ldots,n_r}$", "${}^{\mathfrak{S}}\sigma^{n_1,\ldots,n_r}$", "${}^{\mathfrak{S}}\mathrm{pr}_i^{n_1,\ldots,n_r}$", "${}^{\mathfrak{S}}\mathrm{pr}^{n_1,\ldots,n_r}_{\{i,i+1\}}$" in the case where we take the "$(n_1, \ldots, n_r)$" to be $(1, \ldots, 1)$.

REMARK 2.4.1. — In the situation of Definition 2.4:

(i) It follows from condition (2) of Definition 2.2 that the morphism ${}^{\mathfrak{S}}\sigma^{0,\ldots,0} \colon X = {}^{\mathfrak{S}}P^{0,\ldots,0} \to X^{(r+1)}$ determines an *isomorphism* of $X = {}^{\mathfrak{S}}P^{0,\ldots,0}$ with the diagonal closed subscheme $X \subseteq X^{(r+1)}$ of $X^{(r+1)}$.

(ii) It follows from Remark 2.3.1 that the diagram of schemes over S

$$\begin{tikzcd}
X = {}^{\mathfrak{S}}P^{0,\ldots,0} \arrow[rr, "{}^{\mathfrak{S}}\iota^{0,\ldots,0}_{n_1,\ldots,n_r}"] \arrow[dr, equal] & & {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \arrow[dl, "{}^{\mathfrak{S}}\mathrm{pr}^{n_1,\ldots,n_r}_i"] \\
& X &
\end{tikzcd}$$

is *commutative*, and the morphism ${}^{\mathfrak{S}}\iota^{0,\ldots,0}_{n_1,\ldots,n_r} \colon X = {}^{\mathfrak{S}}P^{0,\ldots,0} \to {}^{\mathfrak{S}}P^{n_1,\ldots,n_r}$ is a *closed immersion.*

DEFINITION 2.5. — We shall refer to a collection of data

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big)$$

consisting of

- a pre-stratification structure on X/S

$$\big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},\ ({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0}\big)$$

and

- an affine morphism ${}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2}$ over S for each nonnegative integers n_1, n_2

as a *stratification structure* on X/S if the following four conditions are satisfied:

(1) Let n_1, n_2 be nonnegative integers. Then the diagram of schemes over S

$$\begin{tikzcd}
{}^{\mathfrak{S}}P^{n_1,n_2} \arrow[r, "{}^{\mathfrak{S}}\delta^{n_1,n_2}"] \arrow[d, "{}^{\mathfrak{S}}\sigma^{n_1,n_2}"'] & {}^{\mathfrak{S}}P^{n_1+n_2} \arrow[d, "{}^{\mathfrak{S}}\sigma^{n_1+n_2}"] \\
X^{(3)} \arrow[r, "{}^{X}\mathrm{pr}^{(3)}_{\{1,3\}}"'] & X^{(2)}
\end{tikzcd}$$

is commutative.

(2) Let n_1, n_2, n_3 be nonnegative integers. Then the diagram of schemes over S

$$\begin{tikzcd}
{}^{\mathfrak{S}}P^{n_1,n_2,n_3} \arrow[r, "({}^{\mathfrak{S}}\delta^{n_1,n_2},\, \mathrm{id}_{{}^{\mathfrak{S}}P^{n_3}})"] \arrow[d, "(\mathrm{id}_{{}^{\mathfrak{S}}P^{n_1}},\, {}^{\mathfrak{S}}\delta^{n_2,n_3})"'] & {}^{\mathfrak{S}}P^{n_1+n_2,n_3} \arrow[d, "{}^{\mathfrak{S}}\delta^{n_1+n_2,n_3}"] \\
{}^{\mathfrak{S}}P^{n_1,n_2+n_3} \arrow[r, "{}^{\mathfrak{S}}\delta^{n_1,n_2+n_3}"'] & {}^{\mathfrak{S}}P^{n_1+n_2+n_3}
\end{tikzcd}$$

[cf. (1)] is commutative.

(3) Let n be a nonnegative integer. Then the morphisms ${}^{\mathfrak{S}}\delta^{0,n}$, ${}^{\mathfrak{S}}\delta^{n,0}$ coincide — relative to the identification of ${}^{\mathfrak{S}}P^0$ with X [cf. condition (2) of Definition 2.2] — with the identity automorphisms of ${}^{\mathfrak{S}}P^{0,n}$, ${}^{\mathfrak{S}}P^{n,0}$, respectively.

(4) Let n_1, n_2 be nonnegative integers. Then the diagrams of schemes over S

$$\begin{tikzcd} {}^{\mathfrak{S}}P^{n_1,n_2} \arrow[r, "{}^{\mathfrak{S}}\delta^{n_1,n_2}"] \arrow[d, "{}^{\mathfrak{S}}\iota^{n_1,n_2}_{n_1+1,n_2}"'] & {}^{\mathfrak{S}}P^{n_1+n_2} \arrow[d, "{}^{\mathfrak{S}}\iota^{n_1+n_2}"] \\ {}^{\mathfrak{S}}P^{n_1+1,n_2} \arrow[r, "{}^{\mathfrak{S}}\delta^{n_1+1,n_2}"'] & {}^{\mathfrak{S}}P^{n_1+n_2+1} \end{tikzcd} \qquad \begin{tikzcd} {}^{\mathfrak{S}}P^{n_1,n_2} \arrow[r, "{}^{\mathfrak{S}}\delta^{n_1,n_2}"] \arrow[d, "{}^{\mathfrak{S}}\iota^{n_1,n_2}_{n_1,n_2+1}"'] & {}^{\mathfrak{S}}P^{n_1+n_2} \arrow[d, "{}^{\mathfrak{S}}\iota^{n_1+n_2}"] \\ {}^{\mathfrak{S}}P^{n_1,n_2+1} \arrow[r, "{}^{\mathfrak{S}}\delta^{n_1,n_2+1}"'] & {}^{\mathfrak{S}}P^{n_1+n_2+1} \end{tikzcd}$$

are commutative.

REMARK 2.5.1. — An example of a stratification structure on X/S is given as follows: Let Y be a *separated* scheme over S and $X \to Y$ an *affine* morphism over S. Write $P \stackrel{\text{def}}{=} X \times_Y X$; q_1, $q_2\colon P \to X$ for the projections onto the first, second factors, respectively; $q_{1,3}\colon X \times_Y X \times_Y X \to P$ for the projection onto the first and third factors. Let

$$P^0 \subseteq P^1 \subseteq \ldots \subseteq P^n \subseteq P^{n+1} \subseteq \ldots \subseteq P$$

be a sequence of closed subschemes of P. Suppose that the following two conditions are satisfied:

• The natural closed immersion $P^0 \hookrightarrow P$ determines an *isomorphism* of P^0 with the diagonal closed subscheme $X \subseteq P$ of P.

• For each nonnegative integers n_1, n_2, if we write $P^{n_1,n_2} \stackrel{\text{def}}{=} P^{n_1} \times_X P^{n_2}$ $(\subseteq X \times_Y X \times_Y X)$ for the fiber product of the composite $P^{n_1} \hookrightarrow P \stackrel{q_2}{\to} X$ and the composite $P^{n_2} \hookrightarrow P \stackrel{q_1}{\to} X$, then the composite

$$P^{n_1,n_2} \hookrightarrow X \times_Y X \times_Y X \xrightarrow{q_{1,3}} P$$

factors through the closed subscheme $P^{n_1+n_2} \subseteq P$ of P. Write

$$\delta^{n_1,n_2}\colon P^{n_1,n_2} \longrightarrow P^{n_1+n_2}$$

for the resulting morphism.

For each nonnegative integer n, write $\iota^n\colon P^n \hookrightarrow P^{n+1}$ for the natural closed immersion and $\sigma^n\colon P^n \to X^{(2)}$ for the composite of the natural closed immersion $P^n \hookrightarrow P$ and the closed immersion $P \hookrightarrow X^{(2)}$ induced by the morphism $Y \to S$. Then one verifies easily that the collection of data

$$\big((P^n)_{n\geq 0},\ (\iota^n\colon P^n \hookrightarrow P^{n+1})_{n\geq 0},\ (\sigma^n\colon P^n \to X^{(2)})_{n\geq 0},\ (\delta^{n_1,n_2}\colon P^{n_1,n_2} \to P^{n_1+n_2})_{n_1,\,n_2\geq 0}\big)$$

gives an example of a stratification structure on X/S.

DEFINITION 2.6. — Let

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\sigma^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2})_{n_1,\,n_2\geq 0}\big)$$

be a stratification structure on X/S; r a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n \stackrel{\text{def}}{=} \sum_{i=1}^r n_i$. Then it follows from conditions (1), (2) of Definition 2.5 that the various ${}^{\mathfrak{S}}\delta^{m_1,m_2}$'s determine a commutative diagram of schemes over S

$$\begin{tikzcd}
{}^{\mathfrak{S}}P^{\times n} \arrow[rr] \arrow[dr] & & {}^{\mathfrak{S}}P^{n_1,\dots,n_r} \arrow[dl] \\
& {}^{\mathfrak{S}}P^{n}. &
\end{tikzcd}$$

We shall write

$${}^{\mathfrak{S}}\delta^{\times n}_{n_1,\dots,n_r} \colon {}^{\mathfrak{S}}P^{\times n} \longrightarrow {}^{\mathfrak{S}}P^{n_1,\dots,n_r}, \qquad {}^{\mathfrak{S}}\delta^{\times n} \colon {}^{\mathfrak{S}}P^{\times n} \longrightarrow {}^{\mathfrak{S}}P^{n},$$

$${}^{\mathfrak{S}}\delta^{n_1,\dots,n_r} \colon {}^{\mathfrak{S}}P^{n_1,\dots,n_r} \longrightarrow {}^{\mathfrak{S}}P^{n}$$

for the top horizontal, left-hand lower, right-hand lower arrows of this diagram, respectively.

REMARK 2.6.1. — In the situation of Definition 2.6, it follows from condition (1) of Definition 2.5 that the diagrams of schemes over S

$$\begin{tikzcd}
{}^{\mathfrak{S}}P^{n_1,\dots,n_r} \arrow[rr, "{}^{\mathfrak{S}}\delta^{n_1,\dots,n_r}"] \arrow[dr, "{}^{\mathfrak{S}}\mathrm{pr}_1^{n_1,\dots,n_r}"'] & & {}^{\mathfrak{S}}P^{n} \arrow[dl, "{}^{\mathfrak{S}}\mathrm{pr}_1^{n}"] \\
& X, &
\end{tikzcd}
\qquad
\begin{tikzcd}
{}^{\mathfrak{S}}P^{n_1,\dots,n_r} \arrow[rr, "{}^{\mathfrak{S}}\delta^{n_1,\dots,n_r}"] \arrow[dr, "{}^{\mathfrak{S}}\mathrm{pr}_{r+1}^{n_1,\dots,n_r}"'] & & {}^{\mathfrak{S}}P^{n} \arrow[dl, "{}^{\mathfrak{S}}\mathrm{pr}_2^{n}"] \\
& X, &
\end{tikzcd}$$

$$\begin{tikzcd}
{}^{\mathfrak{S}}P^{\times n} \arrow[rr, "{}^{\mathfrak{S}}\delta^{\times n}_{n_1,\dots,n_r}"] \arrow[dr, "{}^{\mathfrak{S}}\mathrm{pr}_1^{\times n}"'] & & {}^{\mathfrak{S}}P^{n_1,\dots,n_r} \arrow[dl, "{}^{\mathfrak{S}}\mathrm{pr}_1^{n_1,\dots,n_r}"] \\
& X, &
\end{tikzcd}
\qquad
\begin{tikzcd}
{}^{\mathfrak{S}}P^{\times n} \arrow[rr, "{}^{\mathfrak{S}}\delta^{\times n}_{n_1,\dots,n_r}"] \arrow[dr, "{}^{\mathfrak{S}}\mathrm{pr}_{n+1}^{\times n}"'] & & {}^{\mathfrak{S}}P^{n_1,\dots,n_r} \arrow[dl, "{}^{\mathfrak{S}}\mathrm{pr}_{r+1}^{n_1,\dots,n_r}"] \\
& X &
\end{tikzcd}$$

are *commutative.*

DEFINITION 2.7. — Let

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^{n})_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^{n})_{n\geq 0},\ ({}^{\mathfrak{S}}\sigma^{n})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2})_{n_1,\,n_2\geq 0}\big)$$

be a stratification structure on X/S; r a positive integer; $n_1,\dots,n_r$ nonnegative integers. Write $n \overset{\mathrm{def}}{=} \sum_{i=1}^{r} n_i$. Suppose that $n \geq 1$.

(i) We shall write

$${}^{\mathfrak{S}}\underline{P}^{n_1,\dots,n_r} \overset{\mathrm{def}}{=} {}^{\mathfrak{S}}P^{n_1,\dots,n_r} \times_{{}^{\mathfrak{S}}P^{n}} {}^{\mathfrak{S}}P^{n-1}$$

for the fiber product of the morphism ${}^{\mathfrak{S}}\delta^{n_1,\dots,n_r} \colon {}^{\mathfrak{S}}P^{n_1,\dots,n_r} \to {}^{\mathfrak{S}}P^{n}$ and the morphism ${}^{\mathfrak{S}}\iota^{n-1} \colon {}^{\mathfrak{S}}P^{n-1} \to {}^{\mathfrak{S}}P^{n}$.

(ii) We shall write

$${}^{\mathfrak{S}}\underline{\iota}^{n_1,\dots,n_r} \colon {}^{\mathfrak{S}}\underline{P}^{n_1,\dots,n_r} \longrightarrow {}^{\mathfrak{S}}P^{n_1,\dots,n_r}, \qquad {}^{\mathfrak{S}}\underline{\delta}^{n_1,\dots,n_r} \colon {}^{\mathfrak{S}}\underline{P}^{n_1,\dots,n_r} \longrightarrow {}^{\mathfrak{S}}P^{n-1}$$

for the first, second projections, respectively, Thus, we have a cartesian diagram of schemes over S

$$\begin{tikzcd} {}^{\mathfrak{S}}\underline{P}^{n_1,\dots,n_r} \arrow[r, "{}^{\mathfrak{S}}\underline{\delta}^{n_1,\dots,n_r}"] \arrow[d, "{}^{\mathfrak{S}}\underline{\iota}^{n_1,\dots,n_r}"'] & {}^{\mathfrak{S}}P^{n-1} \arrow[d, "{}^{\mathfrak{S}}\iota^{n-1}"] \\ {}^{\mathfrak{S}}P^{n_1,\dots,n_r} \arrow[r, "{}^{\mathfrak{S}}\delta^{n_1,\dots,n_r}"'] & {}^{\mathfrak{S}}P^{n}. \end{tikzcd}$$

(iii) Let i be an element of $\{1,\dots,r+1\}$. Then we shall write

$${}^{\mathfrak{S}}\underline{\mathrm{pr}}_i^{n_1,\dots,n_r} \colon {}^{\mathfrak{S}}\underline{P}^{n_1,\dots,n_r} \longrightarrow X$$

for the morphism over S obtained by forming the composite of the morphism ${}^{\mathfrak{S}}\underline{\iota}^{n_1,\dots,n_r} \colon {}^{\mathfrak{S}}\underline{P}^{n_1,\dots,n_r} \to {}^{\mathfrak{S}}P^{n_1,\dots,n_r}$ and the morphism ${}^{\mathfrak{S}}\mathrm{pr}_i^{n_1,\dots,n_r} \colon {}^{\mathfrak{S}}P^{n_1,\dots,n_r} \to X$.

(iv) It follows from conditions (3), (4) of Definition 2.5 that the diagram of schemes over S

$$\begin{tikzcd} X = {}^{\mathfrak{S}}P^0 = {}^{\mathfrak{S}}P^{0,\dots,0} \arrow[r, "{}^{\mathfrak{S}}\iota^0_{n-1}"] \arrow[d, "{}^{\mathfrak{S}}\iota^{0,\dots,0}_{n_1,\dots,n_r}"'] & {}^{\mathfrak{S}}P^{n-1} \arrow[d, "{}^{\mathfrak{S}}\iota^{n-1}"] \\ {}^{\mathfrak{S}}P^{n_1,\dots,n_r} \arrow[r, "{}^{\mathfrak{S}}\delta^{n_1,\dots,n_r}"'] & {}^{\mathfrak{S}}P^{n} \end{tikzcd}$$

is commutative. We shall write

$${}^{\mathfrak{S}}\underline{\iota}^0_{n_1,\dots,n_r} \colon X \longrightarrow {}^{\mathfrak{S}}\underline{P}^{n_1,\dots,n_r}$$

for the morphism over S determined by this commutative diagram and the cartesian diagram of (ii).

(v) We shall write

$${}^{\mathfrak{S}}\underline{P}^{\times r}, \qquad {}^{\mathfrak{S}}\underline{\iota}^{\times r} \colon {}^{\mathfrak{S}}\underline{P}^{\times r} \longrightarrow {}^{\mathfrak{S}}P^{\times r}, \qquad {}^{\mathfrak{S}}\underline{\delta}^{\times r} \colon {}^{\mathfrak{S}}\underline{P}^{\times r} \longrightarrow {}^{\mathfrak{S}}P^{r-1},$$

$${}^{\mathfrak{S}}\underline{\mathrm{pr}}_i^{\times r} \colon {}^{\mathfrak{S}}\underline{P}^{\times r} \longrightarrow X, \qquad {}^{\mathfrak{S}}\underline{\iota}^0_{\times r} \colon X \longrightarrow {}^{\mathfrak{S}}\underline{P}^{\times r}$$

for "${}^{\mathfrak{S}}\underline{P}^{n_1,\dots,n_r}$", "${}^{\mathfrak{S}}\underline{\iota}^{n_1,\dots,n_r}$", "${}^{\mathfrak{S}}\underline{\delta}^{n_1,\dots,n_r}$", "${}^{\mathfrak{S}}\underline{\mathrm{pr}}_i^{n_1,\dots,n_r}$", "${}^{\mathfrak{S}}\underline{\iota}^0_{n_1,\dots,n_r}$" in the case where we take the "$(n_1,\dots,n_r)$" to be $(1,\dots,1)$.

REMARK 2.7.1. — In the situation of Definition 2.7:

(i) It follows from Remark 2.4.1, (ii), that the diagram of schemes over S

$$\begin{tikzcd} X \arrow[r, "{}^{\mathfrak{S}}\underline{\iota}^0_{n_1,\dots,n_r}"] \arrow[dr, equal] & {}^{\mathfrak{S}}\underline{P}^{n_1,\dots,n_r} \arrow[d, "{}^{\mathfrak{S}}\underline{\mathrm{pr}}_i^{n_1,\dots,n_r}"] \\ & X \end{tikzcd}$$

is *commutative*, and the morphism ${}^{\mathfrak{S}}\underline{\iota}^0_{n_1,\dots,n_r} \colon X \to {}^{\mathfrak{S}}\underline{P}^{n_1,\dots,n_r}$ is a *closed immersion.*

(ii) It follows from condition (1) of Definition 2.2 and Remark 2.6.1 that the diagrams of schemes over S

$$\begin{tikzcd} {}^{\mathfrak{S}}\underline{P}^{n_1,\ldots,n_r} \arrow[rr, "{}^{\mathfrak{S}}\underline{\delta}^{n_1,\ldots,n_r}"] \arrow[dr, "{}^{\mathfrak{S}}\underline{\mathrm{pr}}_1^{n_1,\ldots,n_r}"'] & & {}^{\mathfrak{S}}P^{n-1} \arrow[dl, "{}^{\mathfrak{S}}\mathrm{pr}_1^{n-1}"] \\ & X, & \end{tikzcd} \qquad \begin{tikzcd} {}^{\mathfrak{S}}\underline{P}^{n_1,\ldots,n_r} \arrow[rr, "{}^{\mathfrak{S}}\underline{\delta}^{n_1,\ldots,n_r}"] \arrow[dr, "{}^{\mathfrak{S}}\underline{\mathrm{pr}}_{r+1}^{n_1,\ldots,n_r}"'] & & {}^{\mathfrak{S}}P^{n-1} \arrow[dl, "{}^{\mathfrak{S}}\mathrm{pr}_2^{n-1}"] \\ & X & \end{tikzcd}$$

are *commutative.*

DEFINITION 2.8. — Let

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\sigma^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2})_{n_1,\,n_2\geq 0}\big),$$

$$\mathfrak{T} = \big(({}^{\mathfrak{T}}P^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\iota^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\sigma^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\delta^{n_1,n_2})_{n_1,\,n_2\geq 0}\big)$$

be stratification structures on X/S. Then we shall define a *morphism of stratification structures*

$$\mathfrak{S} \longrightarrow \mathfrak{T}$$

to be a collection of data

$$\Phi = (\Phi^n)_{n\geq 0}$$

consisting of an affine morphism $\Phi^n\colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{T}}P^n$ over S for each nonnegative integer n that satisfies the following three conditions:

(1) Let n be a nonnegative integer. Then the diagram of schemes over S

$$\begin{tikzcd} {}^{\mathfrak{S}}P^n \arrow[r, "\Phi^n"] \arrow[d, "{}^{\mathfrak{S}}\iota^n"'] & {}^{\mathfrak{T}}P^n \arrow[d, "{}^{\mathfrak{T}}\iota^n"] \\ {}^{\mathfrak{S}}P^{n+1} \arrow[r, "\Phi^{n+1}"'] & {}^{\mathfrak{T}}P^{n+1} \end{tikzcd}$$

is commutative.

(2) Let n be a nonnegative integer. Then the diagram of schemes over S

$$\begin{tikzcd} {}^{\mathfrak{S}}P^n \arrow[rr, "\Phi^n"] \arrow[dr, "{}^{\mathfrak{S}}\sigma^n"'] & & {}^{\mathfrak{T}}P^n \arrow[dl, "{}^{\mathfrak{T}}\sigma^n"] \\ & X^{(2)} & \end{tikzcd}$$

is commutative.

(3) Let n_1, n_2 be nonnegative integers. Then the diagram of schemes over S

$$\begin{tikzcd} {}^{\mathfrak{S}}P^{n_1,n_2} \arrow[r] \arrow[d, "{}^{\mathfrak{S}}\delta^{n_1,n_2}"'] & {}^{\mathfrak{T}}P^{n_1,n_2} \arrow[d, "{}^{\mathfrak{T}}\delta^{n_1,n_2}"] \\ {}^{\mathfrak{S}}P^{n_1+n_2} \arrow[r, "\Phi^{n_1+n_2}"'] & {}^{\mathfrak{T}}P^{n_1+n_2} \end{tikzcd}$$

— where the upper horizontal arrow is the morphism over S determined [cf. (2)] by the morphisms Φ^{n_1}, Φ^{n_2} — is commutative.

REMARK 2.8.1. — In the situation of Definition 2.8, it follows from condition (2) of Definition 2.2 and condition (2) of Definition 2.8 that the morphism $\Phi^0 \colon X = {}^{\mathfrak{S}}P^0 \to {}^{\mathfrak{T}}P^0 = X$ coincides with the *identity automorphism* of X.

DEFINITION 2.9. — Let

$$\mathfrak{S} = \left(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\sigma^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2})_{n_1,\,n_2\geq 0}\right),$$

$$\mathfrak{T} = \left(({}^{\mathfrak{T}}P^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\iota^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\sigma^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\delta^{n_1,n_2})_{n_1,\,n_2\geq 0}\right)$$

be stratification structures on X/S;

$$\Phi = (\Phi^n)_{n\geq 0} \colon \mathfrak{S} \longrightarrow \mathfrak{T}$$

a morphism of stratification structures; r a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Then we shall write

$$\Phi^{n_1,\ldots,n_r} \colon {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \longrightarrow {}^{\mathfrak{T}}P^{n_1,\ldots,n_r}$$

for the morphism over S determined [cf. condition (2) of Definition 2.8] by the morphisms $\Phi^{n_1}, \ldots, \Phi^{n_r}$. [So the upper horizontal arrow of the diagram of Definition 2.8, (3), is the morphism Φ^{n_1,n_2}.]

DEFINITION 2.10. — Let

$$\mathfrak{S} = \left(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\sigma^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2})_{n_1,\,n_2\geq 0}\right),$$

$$\mathfrak{T} = \left(({}^{\mathfrak{T}}P^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\iota^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\sigma^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\delta^{n_1,n_2})_{n_1,\,n_2\geq 0}\right)$$

be stratification structures on X/S;

$$\Phi = (\Phi^n)_{n\geq 0} \colon \mathfrak{S} \longrightarrow \mathfrak{T}$$

a morphism of stratification structures; $m \leq n$ nonnegative integers.

(i) We shall write

$${}^{\Phi}\underline{P}^{n|m} \stackrel{\mathrm{def}}{=} {}^{\mathfrak{S}}P^n \times_{{}^{\mathfrak{T}}P^n} {}^{\mathfrak{T}}P^m$$

for the fiber product of the morphism $\Phi^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{T}}P^n$ and the morphism ${}^{\mathfrak{T}}\iota^m_n \colon {}^{\mathfrak{T}}P^m \to {}^{\mathfrak{T}}P^n$. [So ${}^{\Phi}\underline{P}^{n|n} = {}^{\mathfrak{S}}P^n$.]

(ii) We shall write

$${}^{\Phi}\underline{\iota}^{n|m} \colon {}^{\Phi}\underline{P}^{n|m} \longrightarrow {}^{\mathfrak{S}}P^n, \qquad \underline{\Phi}^{n|m} \colon {}^{\Phi}\underline{P}^{n|m} \longrightarrow {}^{\mathfrak{T}}P^m$$

for the first, second projections, respectively, Thus, we have a cartesian diagram of schemes over S

$$
\begin{tikzcd}
{}^{\Phi}\underline{P}^{n|m} \arrow[r, "\underline{\Phi}^{n|m}"] \arrow[d, "{}^{\Phi}\underline{\iota}^{n|m}"'] & {}^{\mathfrak{T}}P^{m} \arrow[d, "{}^{\mathfrak{T}}\iota_n^m"] \\
{}^{\mathfrak{S}}P^{n} \arrow[r, "\Phi^{n}"'] & {}^{\mathfrak{T}}P^{n}.
\end{tikzcd}
$$

(iii) Let $m' \leq m$ be a nonnegative integer. Then we shall write

$$
{}^{\Phi}\underline{\iota}_{n|m}^{n|m'} \colon {}^{\Phi}\underline{P}^{n|m'} \longrightarrow {}^{\Phi}\underline{P}^{n|m}
$$

for the morphism over S determined by the morphism ${}^{\mathfrak{T}}\iota_m^{m'} \colon {}^{\mathfrak{T}}P^{m'} \to {}^{\mathfrak{T}}P^{m}$.

(iv) Let i be an element of $\{1, 2\}$. Then we shall write

$$
{}^{\Phi}\underline{\mathrm{pr}}_i^{n|m} \colon {}^{\Phi}\underline{P}^{n|m} \longrightarrow X
$$

for the morphism over S obtained by forming the composite of the morphism ${}^{\Phi}\underline{\iota}^{n|m} \colon {}^{\Phi}\underline{P}^{n|m} \to {}^{\mathfrak{S}}P^{n}$ and the morphism ${}^{\mathfrak{S}}\mathrm{pr}_i^n \colon {}^{\mathfrak{S}}P^n \to X$.

(v) It follows from condition (1) of Definition 2.8 and Remark 2.8.1 that the diagram of schemes over S

$$
\begin{tikzcd}
X = {}^{\mathfrak{S}}P^{0} = {}^{\mathfrak{T}}P^{0} \arrow[r, "{}^{\mathfrak{T}}\iota_m^0"] \arrow[d, "{}^{\mathfrak{S}}\iota_n^0"'] & {}^{\mathfrak{T}}P^{m} \arrow[d, "{}^{\mathfrak{T}}\iota_n^m"] \\
{}^{\mathfrak{S}}P^{n} \arrow[r, "\Phi^{n}"'] & {}^{\mathfrak{T}}P^{n}
\end{tikzcd}
$$

is commutative. We shall write

$$
{}^{\Phi}\underline{\iota}_{n|m}^{0} \colon X \longrightarrow {}^{\Phi}\underline{P}^{n|m}
$$

for the morphism over S determined by this commutative diagram and the cartesian diagram of (ii).

REMARK 2.10.1. — In the situation of Definition 2.10:

(i) It follows from Remark 2.3.1 that the diagram of schemes over S

$$
\begin{tikzcd}
X \arrow[rr, "{}^{\Phi}\underline{\iota}_{n|m}^{0}"] \arrow[dr, equal] & & {}^{\Phi}\underline{P}^{n|m} \arrow[dl, "{}^{\Phi}\underline{\mathrm{pr}}_i^{n|m}"] \\
& X &
\end{tikzcd}
$$

is *commutative*, and the morphism ${}^{\Phi}\underline{\iota}_{n|m}^{0} \colon X \to {}^{\Phi}\underline{P}^{n|m}$ is a *closed immersion*.

(ii) It follows from condition (1) of Definition 2.2 and condition (2) of Definition 2.8 that the diagram of schemes over S

$$\begin{tikzcd} {}^{\Phi}\underline{P}^{n|m} \arrow[rr, "\underline{\Phi}^{n|m}"] \arrow[dr, "{}^{\Phi}\underline{\mathrm{pr}}_i^{n|m}"'] & & {}^{\mathfrak{S}}P^m \arrow[dl, "{}^{\mathfrak{S}}\mathrm{pr}_i^m"] \\ & X & \end{tikzcd}$$

is *commutative.*

(iii) It follows from (ii) and Remark 2.3.1 that the equalities

$$\underline{\Phi}^{n|0} = {}^{\Phi}\underline{\mathrm{pr}}_1^{n|0} = {}^{\Phi}\underline{\mathrm{pr}}_2^{n|0} \colon {}^{\Phi}\underline{P}^{n|0} \longrightarrow {}^{\mathfrak{S}}P^0 = X$$

hold.

3. Integrable Stratification Structures

In the present §3, we introduce the notion of an *integrable stratification structure* on a smooth scheme [cf. Definition 3.2 below] and the notion of an *integrable morphism* of stratification structures [cf. Definition 3.8 below]. In the present §3, let S be a scheme, X a scheme which is *smooth* and *separated* over S, and

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\,n_2\geq 0}\big)$$

a stratification structure on X/S.

Definition 3.1.

(i) We shall say that the stratification structure $\mathfrak{S}$ is *ι-quasi-nil-retraction-like* if, for each nonnegative integer n, the morphism ${}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1}$ is quasi-nil-retraction-like.

(ii) We shall say that the stratification structure $\mathfrak{S}$ is pr-*finite flat* if, for each nonnegative integer n and $i \in \{1, 2\}$, the morphism ${}^{\mathfrak{S}}\mathrm{pr}_i^n \colon {}^{\mathfrak{S}}P^n \to X$ is finite and flat.

(iii) We shall say that the stratification structure $\mathfrak{S}$ is *δ-nil-retraction-like* if the following condition is satisfied: Let r be a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n \stackrel{\mathrm{def}}{=} \sum_{i=1}^r n_i$. Then the morphisms in the commutative diagram of Definition 2.6

$$\begin{tikzcd} {}^{\mathfrak{S}}P^{\times n} \arrow[rr, "{}^{\mathfrak{S}}\delta^{\times n}_{n_1,\ldots,n_r}"] \arrow[dr, "{}^{\mathfrak{S}}\delta^{\times n}"'] & & {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \arrow[dl, "{}^{\mathfrak{S}}\delta^{n_1,\ldots,n_r}"] \\ & {}^{\mathfrak{S}}P^n & \end{tikzcd}$$

are nil-retraction-like.

(iv) Let r be a positive integer. Then we shall say that the stratification structure $\mathfrak{S}$ is *δ-strictly cocartesian of level $\geq r$* if, for each integer $n \geq r$, the commutative diagram of schemes over S

$$\begin{tikzcd} {}^{\mathfrak{S}}P^{\times n} \arrow[r, "{}^{\mathfrak{S}}\delta^{\times n}_{1,n-1}"] \arrow[d, "{}^{\mathfrak{S}}\delta^{\times n}_{n-1,1}"'] & {}^{\mathfrak{S}}P^{1,n-1} \arrow[d, "{}^{\mathfrak{S}}\delta^{1,n-1}"] \\ {}^{\mathfrak{S}}P^{n-1,1} \arrow[r, "{}^{\mathfrak{S}}\delta^{n-1,1}"'] & {}^{\mathfrak{S}}P^{n} \end{tikzcd}$$

is strictly cocartesian.

REMARK 3.1.1. — Suppose that the stratification structure $\mathfrak{S}$ is *ι-quasi-nil-retraction-like.*

(i) In the situation of Definition 2.3, (ii), it follows from Remark 2.3.1 that the morphism ${}^{\mathfrak{S}}\mathrm{pr}^n_i \colon {}^{\mathfrak{S}}P^n \to X$ is *quasi-nil-retraction-like.*

(ii) In the situation of Definition 2.4, (ii), it follows from (i) that the morphism ${}^{\mathfrak{S}}\iota^{m_1,\ldots,m_2}_{n_1,\ldots,n_2} \colon {}^{\mathfrak{S}}P^{m_1,\ldots,m_2} \to {}^{\mathfrak{S}}P^{n_1,\ldots,n_2}$ is *quasi-nil-retraction-like.*

(iii) In the situation of Definition 2.6, it follows from (ii) and conditions (3), (4) of Definition 2.5 that the morphisms in the commutative diagram

$$\begin{tikzcd} {}^{\mathfrak{S}}P^{\times n} \arrow[rr, "{}^{\mathfrak{S}}\delta^{\times n}_{n_1,\ldots,n_r}"] \arrow[dr, "{}^{\mathfrak{S}}\delta^{\times n}"'] & & {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \arrow[dl, "{}^{\mathfrak{S}}\delta^{n_1,\ldots,n_r}"] \\ & {}^{\mathfrak{S}}P^{n} & \end{tikzcd}$$

are *quasi-nil-retraction-like.*

REMARK 3.1.2. — Suppose that the stratification structure $\mathfrak{S}$ is pr-*finite flat.* Then, in the situation of Definition 2.4, (iv), one verifies immediately that the morphism ${}^{\mathfrak{S}}\mathrm{pr}^{n_1,\ldots,n_r}_i \colon {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \to X$ is *finite* and *flat.*

DEFINITION 3.2. — We shall say that the stratification structure $\mathfrak{S}$ is *integrable* if the following four conditions are satisfied:

(1) The stratification structure $\mathfrak{S}$ is ι-quasi-nil-retraction-like and δ-nil-retraction-like.

(2) The stratification structure $\mathfrak{S}$ is δ-strictly cocartesian of level ≥ 3.

(3) The morphism ${}^{\mathfrak{S}}\iota^1 \colon {}^{\mathfrak{S}}P^1 \to {}^{\mathfrak{S}}P^2$ is a square-nilpotent closed immersion.

(4) The morphism ${}^{\mathfrak{S}}\delta^{\times 2} \colon {}^{\mathfrak{S}}P^{\times 2} \to {}^{\mathfrak{S}}P^2$ is conormally strict with respect to the square-nilpotent closed immersion ${}^{\mathfrak{S}}\iota^1 \colon {}^{\mathfrak{S}}P^1 \hookrightarrow {}^{\mathfrak{S}}P^2$ [cf. (3)].

REMARK 3.2.1. — Suppose that the stratification structure $\mathfrak{S}$ is *integrable.*

(i) It follows from condition (3) of Definition 3.2 that the morphism ${}^{\mathfrak{S}}\underline{\iota}^{\times 2}\colon {}^{\mathfrak{S}}\underline{P}^{\times 2} \to {}^{\mathfrak{S}}P^{\times 2}$ is a *square-nilpotent* closed immersion, hence also [cf. Remark 1.2.1] *quasi-nil-retraction-like.*

(ii) It follows from Remark 3.1.1, (ii), and condition (1) of Definition 3.2 that the morphism ${}^{\mathfrak{S}}\iota^{0,0}_{1,1}\colon X = {}^{\mathfrak{S}}P^{0,0} \to {}^{\mathfrak{S}}P^{\times 2}$ is *quasi-nil-retraction-like.* Thus, it follows from (i) that the morphism ${}^{\mathfrak{S}}\underline{\iota}^{0}_{\times 2}\colon X \to {}^{\mathfrak{S}}\underline{P}^{\times 2}$ is *quasi-nil-retraction-like.*

REMARK 3.2.2. — Let us recall from Definition 2.7, (ii), that the diagram

$$\begin{tikzcd} {}^{\mathfrak{S}}\underline{P}^{\times 2} \arrow[r, "{}^{\mathfrak{S}}\underline{\delta}^{\times 2}"] \arrow[d, "{}^{\mathfrak{S}}\underline{\iota}^{\times 2}"'] & {}^{\mathfrak{S}}P^{1} \arrow[d, "{}^{\mathfrak{S}}\iota^{1}"] \\ {}^{\mathfrak{S}}P^{\times 2} \arrow[r, "{}^{\mathfrak{S}}\delta^{\times 2}"'] & {}^{\mathfrak{S}}P^{2} \end{tikzcd}$$

is *cartesian.* Now suppose that the stratification structure $\mathfrak{S}$ is *integrable.* Then it follows from conditions (3), (4) of Definition 3.2 that the right-hand vertical arrow of this diagram is a *square-nilpotent* closed immersion, and the lower horizontal arrow of this diagram is *conormally strict* with respect to the right-hand vertical arrow.

REMARK 3.2.3. — Suppose that the stratification structure $\mathfrak{S}$ is *integrable.* Let $n \geq 3$ be an integer. Let us recall the diagram of schemes over S

$$\begin{tikzcd} X = {}^{\mathfrak{S}}P^{0,\ldots,0} \arrow[r, "{}^{\mathfrak{S}}\iota^{0,\ldots,0}_{1,\ldots,1}"] & {}^{\mathfrak{S}}P^{\times n} \arrow[r, "{}^{\mathfrak{S}}\delta^{\times n}_{1,n-1}"] \arrow[d, "{}^{\mathfrak{S}}\delta^{\times n}_{n-1,1}"'] & {}^{\mathfrak{S}}P^{1,n-1} \arrow[d, "{}^{\mathfrak{S}}\delta^{1,n-1}"] & \\ & {}^{\mathfrak{S}}P^{n-1,1} \arrow[r, "{}^{\mathfrak{S}}\delta^{n-1,1}"'] & {}^{\mathfrak{S}}P^{n} \arrow[r, bend left, "{}^{\mathfrak{S}}\mathrm{pr}^{n}_{1}"] \arrow[r, bend right, "{}^{\mathfrak{S}}\mathrm{pr}^{n}_{2}"'] & X. \end{tikzcd}$$

(i) It follows from condition (2) of Definition 3.2 that the diagram

$$\begin{tikzcd} {}^{\mathfrak{S}}P^{\times n} \arrow[r, "{}^{\mathfrak{S}}\delta^{\times n}_{1,n-1}"] \arrow[d, "{}^{\mathfrak{S}}\delta^{\times n}_{n-1,1}"'] & {}^{\mathfrak{S}}P^{1,n-1} \arrow[d, "{}^{\mathfrak{S}}\delta^{1,n-1}"] \\ {}^{\mathfrak{S}}P^{n-1,1} \arrow[r, "{}^{\mathfrak{S}}\delta^{n-1,1}"'] & {}^{\mathfrak{S}}P^{n} \end{tikzcd}$$

is [commutative and] *strictly cocartesian.*

(ii) It follows from Remark 3.1.1, (ii), and condition (1) of Definition 3.2 that the morphism

$$X = {}^{\mathfrak{S}}P^{0,\ldots,0} \xrightarrow{{}^{\mathfrak{S}}\iota^{0,\ldots,0}_{1,\ldots,1}} {}^{\mathfrak{S}}P^{\times n}$$

is *quasi-nil-retraction-like.*

(iii) It follows from Remark 3.1.1, (i), and condition (1) of Definition 3.2 that the morphisms

$$\begin{tikzcd} {}^{\mathfrak{S}}P^n \arrow[r, shift left, "{}^{\mathfrak{S}}\mathrm{pr}_1^n"] \arrow[r, shift right, "{}^{\mathfrak{S}}\mathrm{pr}_2^n"'] & X \end{tikzcd}$$

are *quasi-nil-retraction-like.*

(iv) It follows from Remark 2.4.1, (ii), and Remark 2.6.1 that the equalities

$${}^{\mathfrak{S}}\mathrm{pr}_1^n \circ {}^{\mathfrak{S}}\delta^{1,n-1} \circ {}^{\mathfrak{S}}\delta^{\times n}_{1,n-1} \circ {}^{\mathfrak{S}}\iota^{0,\ldots,0}_{1,\ldots,1} = {}^{\mathfrak{S}}\mathrm{pr}_2^n \circ {}^{\mathfrak{S}}\delta^{1,n-1} \circ {}^{\mathfrak{S}}\delta^{\times n}_{1,n-1} \circ {}^{\mathfrak{S}}\iota^{0,\ldots,0}_{1,\ldots,1} = \mathrm{id}_X$$

hold.

DEFINITION 3.3. — We shall say that the stratification structure $\mathfrak{S}$ is *strictly integrable* if the following three conditions are satisfied:

(1) The stratification structure $\mathfrak{S}$ is integrable.

(2) The stratification structure $\mathfrak{S}$ is pr-finite flat.

(3) Let i be an element of $\{1, 3\}$. Then the morphism ${}^{\mathfrak{S}}\underline{\mathrm{pr}}_i^{\times 2} \colon {}^{\mathfrak{S}}\underline{P}^{\times 2} \to X$ is finite and flat.

REMARK 3.3.1. — Suppose that the stratification structure $\mathfrak{S}$ is *strictly integrable.* Let us recall the diagram of schemes over S

$$\begin{tikzcd} X \arrow[r, "{}^{\mathfrak{S}}\underline{\iota}^0_{\times 2}"] & {}^{\mathfrak{S}}\underline{P}^{\times 2} \arrow[r, "{}^{\mathfrak{S}}\underline{\delta}^{\times 2}"] \arrow[d, "{}^{\mathfrak{S}}\underline{\iota}^{\times 2}"'] & {}^{\mathfrak{S}}P^1 \arrow[d, "{}^{\mathfrak{S}}\iota^1"] & \\ & {}^{\mathfrak{S}}P^{\times 2} \arrow[r, "{}^{\mathfrak{S}}\delta^{\times 2}"'] & {}^{\mathfrak{S}}P^2 \arrow[r, shift left, "{}^{\mathfrak{S}}\mathrm{pr}_1^2"] \arrow[r, shift right, "{}^{\mathfrak{S}}\mathrm{pr}_2^2"'] & X. \end{tikzcd}$$

(i) It follows from Remark 3.2.1, (ii), and condition (1) of Definition 3.3 that the morphism

$${}^{\mathfrak{S}}\underline{\iota}^0_{\times 2} \colon X \longrightarrow {}^{\mathfrak{S}}\underline{P}^{\times 2}$$

is *quasi-nil-retraction-like.*

(ii) It follows from Remark 3.1.1, (i), condition (1) of Definition 3.2, and condition (1) of Definition 3.3 that the morphisms

$$\begin{tikzcd} {}^{\mathfrak{S}}P^2 \arrow[r, shift left, "{}^{\mathfrak{S}}\mathrm{pr}_1^2"] \arrow[r, shift right, "{}^{\mathfrak{S}}\mathrm{pr}_2^2"'] & X \end{tikzcd}$$

are *quasi-nil-retraction-like.*

(iii) It follows from Remark 2.4.1, (ii), and Remark 2.6.1 that the equalities

$${}^{\mathfrak{S}}\mathrm{pr}_1^2 \circ {}^{\mathfrak{S}}\iota^1 \circ {}^{\mathfrak{S}}\underline{\delta}^{\times 2} \circ {}^{\mathfrak{S}}\underline{\iota}^0_{\times 2} = {}^{\mathfrak{S}}\mathrm{pr}_2^2 \circ {}^{\mathfrak{S}}\iota^1 \circ {}^{\mathfrak{S}}\underline{\delta}^{\times 2} \circ {}^{\mathfrak{S}}\underline{\iota}^0_{\times 2} = \mathrm{id}_X$$

hold.

(iv) It follows from Remark 3.1.2 and conditions (2), (3) of Definition 3.3 [cf. also condition (1) of Definition 2.2; Remark 2.6.1; Remark 2.7.1, (ii)] that the morphisms

$$
\begin{aligned}
{}^{\mathfrak{S}}\underline{\mathrm{pr}}_1^{\times 2} &= {}^{\mathfrak{S}}\mathrm{pr}_1^2 \circ {}^{\mathfrak{S}}\iota^1 \circ {}^{\mathfrak{S}}\underline{\delta}^{\times 2}, & {}^{\mathfrak{S}}\underline{\mathrm{pr}}_3^{\times 2} &= {}^{\mathfrak{S}}\mathrm{pr}_2^2 \circ {}^{\mathfrak{S}}\iota^1 \circ {}^{\mathfrak{S}}\underline{\delta}^{\times 2}, \\
{}^{\mathfrak{S}}\mathrm{pr}_1^1 &= {}^{\mathfrak{S}}\mathrm{pr}_1^2 \circ {}^{\mathfrak{S}}\iota^1, & {}^{\mathfrak{S}}\mathrm{pr}_2^1 &= {}^{\mathfrak{S}}\mathrm{pr}_2^2 \circ {}^{\mathfrak{S}}\iota^1, \\
{}^{\mathfrak{S}}\mathrm{pr}_1^{\times 2} &= {}^{\mathfrak{S}}\mathrm{pr}_1^2 \circ {}^{\mathfrak{S}}\delta^{\times 2}, & {}^{\mathfrak{S}}\mathrm{pr}_3^{\times 2} &= {}^{\mathfrak{S}}\mathrm{pr}_2^2 \circ {}^{\mathfrak{S}}\delta^{\times 2}, \\
{}^{\mathfrak{S}}\mathrm{pr}_1^2&, & {}^{\mathfrak{S}}\mathrm{pr}_2^2&
\end{aligned}
$$

are *finite* and *flat*.

REMARK 3.3.2. — Suppose that the stratification structure $\mathfrak{S}$ is *strictly integrable*. Let $n \geq 3$ be an integer. Let us recall the diagram of schemes over S

$$
\begin{tikzcd}
X = {}^{\mathfrak{S}}P^{0,\ldots,0} \arrow[r, "{}^{\mathfrak{S}}\iota^{0,\ldots,0}_{1,\ldots,1}"] & {}^{\mathfrak{S}}P^{\times n} \arrow[r, "{}^{\mathfrak{S}}\delta^{\times n}_{1,n-1}"] \arrow[d, "{}^{\mathfrak{S}}\delta^{\times n}_{n-1,1}"'] & {}^{\mathfrak{S}}P^{1,n-1} \arrow[d, "{}^{\mathfrak{S}}\delta^{1,n-1}"] & \\
 & {}^{\mathfrak{S}}P^{n-1,1} \arrow[r, "{}^{\mathfrak{S}}\delta^{n-1,1}"'] & {}^{\mathfrak{S}}P^{n} \arrow[r, shift left, "{}^{\mathfrak{S}}\mathrm{pr}_1^n"] \arrow[r, shift right, "{}^{\mathfrak{S}}\mathrm{pr}_2^n"'] & X.
\end{tikzcd}
$$

Then it follows from Remark 3.1.2 and condition (2) of Definition 3.3 [cf. also Remark 2.6.1] that the morphisms

$$
\begin{aligned}
{}^{\mathfrak{S}}\mathrm{pr}_1^{\times n} &= {}^{\mathfrak{S}}\mathrm{pr}_1^n \circ {}^{\mathfrak{S}}\delta^{1,n-1} \circ {}^{\mathfrak{S}}\delta^{\times n}_{1,n-1}, & {}^{\mathfrak{S}}\mathrm{pr}_{n+1}^{\times n} &= {}^{\mathfrak{S}}\mathrm{pr}_2^n \circ {}^{\mathfrak{S}}\delta^{1,n-1} \circ {}^{\mathfrak{S}}\delta^{\times n}_{1,n-1}, \\
{}^{\mathfrak{S}}\mathrm{pr}_1^{1,n-1} &= {}^{\mathfrak{S}}\mathrm{pr}_1^n \circ {}^{\mathfrak{S}}\delta^{1,n-1}, & {}^{\mathfrak{S}}\mathrm{pr}_3^{1,n-1} &= {}^{\mathfrak{S}}\mathrm{pr}_2^n \circ {}^{\mathfrak{S}}\delta^{1,n-1}, \\
{}^{\mathfrak{S}}\mathrm{pr}_1^{n-1,1} &= {}^{\mathfrak{S}}\mathrm{pr}_1^n \circ {}^{\mathfrak{S}}\delta^{n-1,1}, & {}^{\mathfrak{S}}\mathrm{pr}_3^{n-1,1} &= {}^{\mathfrak{S}}\mathrm{pr}_2^n \circ {}^{\mathfrak{S}}\delta^{n-1,1}, \\
{}^{\mathfrak{S}}\mathrm{pr}_1^n&, & {}^{\mathfrak{S}}\mathrm{pr}_2^n&
\end{aligned}
$$

are *finite* and *flat*.

REMARK 3.3.3. — In sequels to the present paper, we give some examples of *strictly integrable stratification structures*.

DEFINITION 3.4.

(i) We shall write

$$T^1 \subseteq X^{(3)}$$

for the closed subscheme of $X^{(3)}$ defined by the quasi-coherent ideal of $\mathcal{O}_{X^{(3)}}$ obtained by forming the square of the quasi-coherent ideal that defines the diagonal closed subscheme $X \subseteq X^{(3)}$ of $X^{(3)}$.

(ii) Let i be an element of $\{1,2,3\}$. Then we shall write

$$^{X}\mathrm{pr}_i^T : T^1 \longrightarrow X$$

for the morphism over S obtained by forming the composite of the natural closed immersion $T^1 \hookrightarrow X^{(3)}$ and the morphism $^{X}\mathrm{pr}_i^{(3)} : X^{(3)} \to X$.

LEMMA 3.5. — *Suppose that the morphism* $^{\mathfrak{S}}\sigma^1 \colon {}^{\mathfrak{S}}P^1 \to X^{(2)}$ *determines an* **isomorphism** *of* $^{\mathfrak{S}}P^1$ *with the closed subscheme of* $X^{(2)}$ *defined by the quasi-coherent ideal of* $\mathcal{O}_{X^{(2)}}$ *obtained by forming the* **square** *of the quasi-coherent ideal that defines the diagonal closed subscheme* $X \subseteq X^{(2)}$ *of* $X^{(2)}$ [*which thus implies that the morphisms*

$$^{\mathfrak{S}}\sigma^{1,0} \colon {}^{\mathfrak{S}}P^1 = {}^{\mathfrak{S}}P^{1,0} \longrightarrow X^{(3)}, \qquad {}^{\mathfrak{S}}\sigma^{0,1} \colon {}^{\mathfrak{S}}P^1 = {}^{\mathfrak{S}}P^{0,1} \longrightarrow X^{(3)},$$

$$^{\mathfrak{S}}\sigma^{\times 2} \colon {}^{\mathfrak{S}}P^{\times 2} \longrightarrow X^{(3)}$$

are **closed immersions**]. *Then the following hold:*

(i) *The closed immersion* $^{\mathfrak{S}}\sigma^{0,0} \colon X = {}^{\mathfrak{S}}P^{0,0} \hookrightarrow X^{(3)}$ [*cf. Remark* 2.4.1, (i)] *factors as the composite of a* **square-nilpotent** *closed immersion* $X \hookrightarrow T^1$ *whose* **conormal sheaf** *is isomorphic to* $\Omega^1_{X/S} \oplus \Omega^1_{X/S}$ *and the natural closed immersion* $T^1 \hookrightarrow X^{(3)}$.

Write

$$^{\mathfrak{S}}\iota^0_T \colon X \hookrightarrow T^1$$

for the resulting square-nilpotent closed immersion [*whose conormal sheaf is isomorphic to* $\Omega^1_{X/S} \oplus \Omega^1_{X/S}$].

(ii) *The closed immersion* $^{\mathfrak{S}}\sigma^{1,0} \colon {}^{\mathfrak{S}}P^1 = {}^{\mathfrak{S}}P^{1,0} \hookrightarrow X^{(3)}$ *factors as the composite of a* **square-nilpotent** *closed immersion* $^{\mathfrak{S}}P^1 \hookrightarrow T^1$ *whose* **conormal sheaf** *is isomorphic to* $^{\mathfrak{S}}\iota^0_*\Omega^1_{X/S}$ *and the natural closed immersion* $T^1 \hookrightarrow X^{(3)}$.

Write

$$^{\mathfrak{S}}\iota^{1,0}_T \colon {}^{\mathfrak{S}}P^1 \hookrightarrow T^1$$

for the resulting square-nilpotent closed immersion [*whose conormal sheaf is isomorphic to* $^{\mathfrak{S}}\iota^0_*\Omega^1_{X/S}$].

(iii) *The closed immersion* $^{\mathfrak{S}}\sigma^{0,1} \colon {}^{\mathfrak{S}}P^1 = {}^{\mathfrak{S}}P^{0,1} \hookrightarrow X^{(3)}$ *factors as the composite of a* **square-nilpotent** *closed immersion* $^{\mathfrak{S}}P^1 \hookrightarrow T^1$ *whose* **conormal sheaf** *is isomorphic to* $^{\mathfrak{S}}\iota^0_*\Omega^1_{X/S}$ *and the natural closed immersion* $T^1 \hookrightarrow X^{(3)}$.

Write

$$^{\mathfrak{S}}\iota^{0,1}_T \colon {}^{\mathfrak{S}}P^1 \hookrightarrow T^1$$

for the resulting square-nilpotent closed immersion [*whose conormal sheaf is isomorphic to* $^{\mathfrak{S}}\iota^0_*\Omega^1_{X/S}$].

(iv) *The natural closed immersion* $T^1 \hookrightarrow X^{(3)}$ *factors as the composite of a* **square-nilpotent** *closed immersion* $T^1 \hookrightarrow {}^{\mathfrak{S}}P^{\times 2}$ *whose* **conormal sheaf** *is isomorphic to* $({}^{\mathfrak{S}}\iota^0_T)_*(\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^1_{X/S})$ *and the closed immersion* $^{\mathfrak{S}}\sigma^{\times 2} \colon {}^{\mathfrak{S}}P^{\times 2} \hookrightarrow X^{(3)}$.

Write

$$^{\mathfrak{S}}\iota^T_{\times 2} \colon T^1 \hookrightarrow {}^{\mathfrak{S}}P^{\times 2}$$

for the resulting square-nilpotent closed immersion [*whose conormal sheaf is isomorphic to* $({}^{\mathfrak{S}}\iota^0_T)_*(\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^1_{X/S})$].

Proof. — These assertions follow immediately from elementary algebraic geometry. $\square$

Definition 3.6. — We shall say that the stratification structure $\mathfrak{S}$ is *of standard type* if the following three conditions are satisfied:

(1) The morphism ${}^{\mathfrak{S}}\sigma^1 \colon {}^{\mathfrak{S}}P^1 \to X^{(2)}$ determines an isomorphism of ${}^{\mathfrak{S}}P^1$ with the closed subscheme of $X^{(2)}$ defined by the quasi-coherent ideal of $\mathcal{O}_{X^{(2)}}$ obtained by forming the square of the quasi-coherent ideal that defines the diagonal closed subscheme $X \subseteq X^{(2)}$ of $X^{(2)}$.

(2) The closed immersion ${}^{\mathfrak{S}}\iota^T_{\times 2} \colon T^1 \hookrightarrow {}^{\mathfrak{S}}P^{\times 2}$ of Lemma 3.5, (iv) [cf. (1)], factors as the composite of a square-nilpotent closed immersion $T^1 \hookrightarrow {}^{\mathfrak{S}}\underline{P}^{\times 2}$ and the closed immersion ${}^{\mathfrak{S}}\underline{\iota}^{\times 2} \colon {}^{\mathfrak{S}}\underline{P}^{\times 2} \hookrightarrow {}^{\mathfrak{S}}P^{\times 2}$ [cf. (1)].

(3) If we write $\mathcal{N}$ for the conormal sheaf of the square-nilpotent closed immersion $T^1 \hookrightarrow {}^{\mathfrak{S}}\underline{P}^{\times 2}$ of (2), then the surjective homomorphism $({}^{\mathfrak{S}}\iota^0_T)_*(\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) \twoheadrightarrow \mathcal{N}$ of $\mathcal{O}_{T^1}$-modules determined [cf. Lemma 3.5, (iv)] by the closed immersion ${}^{\mathfrak{S}}\underline{\iota}^{\times 2} \colon {}^{\mathfrak{S}}\underline{P}^{\times 2} \hookrightarrow {}^{\mathfrak{S}}P^{\times 2}$ induces an isomorphism $({}^{\mathfrak{S}}\iota^0_T)_*\Omega^2_{X/S} \xrightarrow{\sim} \mathcal{N}$ of $\mathcal{O}_{T^1}$-modules.

In this situation, we shall write

$$ {}^{\mathfrak{S}}\underline{\iota}^T_{\times 2} \colon T^1 \hookrightarrow {}^{\mathfrak{S}}\underline{P}^{\times 2} $$

for the square-nilpotent closed immersion [whose conormal sheaf is isomorphic to $({}^{\mathfrak{S}}\iota^0_T)_*\Omega^2_{X/S}$ — cf. (3)] of (2) and

$$ {}^{\mathfrak{S}}\underline{\delta}^T_1 \colon T^1 \longrightarrow {}^{\mathfrak{S}}P^1 $$

for the morphism over S obtained by forming the composite of the morphism ${}^{\mathfrak{S}}\underline{\iota}^T_{\times 2} \colon T^1 \to {}^{\mathfrak{S}}\underline{P}^{\times 2}$ and the morphism ${}^{\mathfrak{S}}\underline{\delta}^{\times 2}_1 \colon {}^{\mathfrak{S}}\underline{P}^{\times 2} \to {}^{\mathfrak{S}}P^1$.

Remark 3.6.1. — Suppose that the stratification structure $\mathfrak{S}$ is *of standard type.* Let us recall the diagram of schemes over S

$$\begin{tikzcd}
X \arrow[r, "{}^{\mathfrak{S}}\iota^0"] \arrow[d, "{}^{\mathfrak{S}}\iota^0"'] & {}^{\mathfrak{S}}P^1 \arrow[d, "{}^{\mathfrak{S}}\iota^{0,1}_T"] & \\
{}^{\mathfrak{S}}P^1 \arrow[r, "{}^{\mathfrak{S}}\iota^{1,0}_T"'] & T^1 \arrow[r, bend left, "{}^{X}\mathrm{pr}^T_1"] \arrow[r, bend right, "{}^{X}\mathrm{pr}^T_3"'] & X.
\end{tikzcd}$$

(i) It follows from Lemma 3.5, (iii), and condition (1) of Definition 3.6, together with elementary algebraic geometry, that the diagram

$$\begin{tikzcd} X \arrow[r, "{}^{\mathfrak{S}}\iota^0"] \arrow[d, "{}^{\mathfrak{S}}\iota^0"'] & {}^{\mathfrak{S}}P^1 \arrow[d, "{}^{\mathfrak{S}}\iota_T^{0,1}"] \\ {}^{\mathfrak{S}}P^1 \arrow[r, "{}^{\mathfrak{S}}\iota_T^{1,0}"'] & T^1 \end{tikzcd}$$

is *cartesian*, and the right-hand vertical arrow of this diagram is a *square-nilpotent* closed immersion. Moreover, again by condition (1) of Definition 3.6, together with elementary algebraic geometry, the lower horizontal arrow of this diagram is *conormally strict* with respect to the right-hand vertical arrow.

(ii) It follows from elementary algebraic geometry that the morphisms

$$\begin{tikzcd} T^1 \arrow[r, shift left, "{}^{X}\mathrm{pr}_1^T"] \arrow[r, shift right, "{}^{X}\mathrm{pr}_3^T"'] & X \end{tikzcd}$$

are *quasi-nil-retraction-like*.

(iii) It follows from Remark 2.3.1 and condition (1) of Definition 3.6 that the equalities

$${}^{X}\mathrm{pr}_1^T \circ {}^{\mathfrak{S}}\iota_T^{0,1} \circ {}^{\mathfrak{S}}\iota^0 = {}^{X}\mathrm{pr}_3^T \circ {}^{\mathfrak{S}}\iota_T^{0,1} \circ {}^{\mathfrak{S}}\iota^0 = \mathrm{id}_X$$

hold.

(iv) It follows from condition (1) of Definition 3.6, together with elementary algebraic geometry [cf. also Remark 2.3.1], that the morphisms

$$\mathrm{id}_X = {}^{X}\mathrm{pr}_1^T \circ {}^{\mathfrak{S}}\iota_T^{0,1} \circ {}^{\mathfrak{S}}\iota^0, \qquad \mathrm{id}_X = {}^{X}\mathrm{pr}_3^T \circ {}^{\mathfrak{S}}\iota_T^{0,1} \circ {}^{\mathfrak{S}}\iota^0,$$

$${}^{\mathfrak{S}}\mathrm{pr}_1^1 = {}^{X}\mathrm{pr}_1^T \circ {}^{\mathfrak{S}}\iota_T^{0,1}, \qquad {}^{\mathfrak{S}}\mathrm{pr}_2^1 = {}^{X}\mathrm{pr}_3^T \circ {}^{\mathfrak{S}}\iota_T^{0,1},$$

$${}^{\mathfrak{S}}\mathrm{pr}_1^1 = {}^{X}\mathrm{pr}_1^T \circ {}^{\mathfrak{S}}\iota_T^{1,0}, \qquad {}^{\mathfrak{S}}\mathrm{pr}_2^1 = {}^{X}\mathrm{pr}_3^T \circ {}^{\mathfrak{S}}\iota_T^{1,0},$$

$${}^{X}\mathrm{pr}_1^T, \qquad {}^{X}\mathrm{pr}_3^T$$

are *finite* and *flat*.

REMARK 3.6.2. — Suppose that the stratification structure $\mathfrak{S}$ is *of standard type*. Then let us recall that the conormal sheaf of the square-nilpotent closed immersion ${}^{\mathfrak{S}}\underline{\iota}_{\times 2}^T \colon T^1 \hookrightarrow {}^{\mathfrak{S}}\underline{P}^{\times 2}$ of Definition 3.6 is *isomorphic* to the $\mathcal{O}_{T^1}$-module $({}^{\mathfrak{S}}\iota_T^0)_*\Omega^2_{X/S}$. Thus, since [one verifies easily that] the morphism ${}^{X}\mathrm{pr}_i^T \colon T^1 \to X$ is *finite* and *flat* for each $i \in \{1, 2, 3\}$, and the $\mathcal{O}_X$-module $\Omega^2_{X/S}$ is *coherent* and *locally free*, one verifies immediately that the stratification structure $\mathfrak{S}$ satisfies condition (3) of Definition 3.3.

REMARK 3.6.3. — In sequels to the present paper, we give some examples of *stratification structures of standard type*.

In the remainder of the present §3, let

$$\mathfrak{T} = \big(({}^{\mathfrak{T}}P^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\iota^n\colon {}^{\mathfrak{T}}P^n \to {}^{\mathfrak{T}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{T}}\sigma^n\colon {}^{\mathfrak{T}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{T}}\delta^{n_1,n_2}\colon {}^{\mathfrak{T}}P^{n_1,n_2} \to {}^{\mathfrak{T}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big)$$

be a stratification structure on X/S and

$$\Phi = (\Phi^n)_{n\geq 0}\colon \mathfrak{S} \longrightarrow \mathfrak{T}$$

a morphism of stratification structures.

DEFINITION 3.7.

(i) We shall say that the morphism Φ is *nil-retraction-like* if, for each positive integer r and nonnegative integers $n_1, \ldots, n_r$, the morphism $\Phi^{n_1,\ldots,n_r}\colon {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \to {}^{\mathfrak{T}}P^{n_1,\ldots,n_r}$ is nil-retraction-like.

(ii) Let r be a nonnegative integer. Then we shall say that the morphism Φ is r-$\underline{\mathrm{pr}}$-*finite flat* if, for each nonnegative integer $n \leq r$ and $i \in \{1, 2\}$, the morphism ${}^{\Phi}\underline{\mathrm{pr}}_i^{r|n}\colon {}^{\Phi}\underline{P}^{r|n} \to X$ is finite and flat.

REMARK 3.7.1. — Suppose that the stratification structures $\mathfrak{S}$ and $\mathfrak{T}$ are *ι-quasi-nil-retraction-like*. Then, in the situation of Definition 2.9, it follows from conditions (1), (3) of Definition 2.8 and Remark 2.8.1 that the morphism $\Phi^{n_1,\ldots,n_r}\colon {}^{\mathfrak{S}}P^{n_1,\ldots,n_r} \to {}^{\mathfrak{T}}P^{n_1,\ldots,n_r}$ is *quasi-nil-retraction-like*.

DEFINITION 3.8. — Let r be a nonnegative integer. Then we shall say that the morphism Φ is r-*integrable* if the following five conditions are satisfied:

(1) The morphism Φ is nil-retraction-like.

(2) The stratification structure $\mathfrak{S}$ is ι-quasi-nil-retraction-like and δ-nil-retraction-like.

(3) The stratification structure $\mathfrak{T}$ is ι-quasi-nil-retraction-like, δ-nil-retraction-like, and δ-strictly cocartesian of level $\geq r + 1$.

(4) Let $n \leq r$ be a positive integer. Then the morphism ${}^{\mathfrak{T}}\iota^{n-1}\colon {}^{\mathfrak{T}}P^{n-1} \to {}^{\mathfrak{T}}P^n$ is a square-nilpotent closed immersion.

(5) Let $n \leq r$ be a positive integer. Then the morphism $\underline{\Phi}^{r|n}\colon {}^{\Phi}\underline{P}^{r|n} \to {}^{\mathfrak{T}}P^n$ is conormally strict with respect to the square-nilpotent closed immersion ${}^{\mathfrak{T}}\iota^{n-1}\colon {}^{\mathfrak{T}}P^{n-1} \hookrightarrow {}^{\mathfrak{T}}P^n$ [cf. (4)].

REMARK 3.8.1. — Let r be a nonnegative integer. Suppose that the morphism Φ is *r-integrable*

(i) It follows from Remark 1.2.1 and condition (4) of Definition 3.8 that, for each nonnegative integers $m \leq n \leq r$, the morphism ${}^{\mathfrak{T}}\iota_n^m\colon {}^{\mathfrak{T}}P^m \to {}^{\mathfrak{T}}P^n$ is *quasi-nil-retraction-like*.

(ii) In the situation of Definition 2.10, (v), it follows from (i) and conditions (1), (3) of Definition 3.8 that the morphism ${}^{\Phi}\underline{\iota}^{0}_{n|m} \colon X \to {}^{\Phi}\underline{P}^{n|m}$ is *quasi-nil-retraction-like.*

REMARK 3.8.2. — Let $n \leq r$ be positive integers. Then one verifies easily from the various definitions involved that the diagram of schemes over S

$$\begin{tikzcd}
{}^{\Phi}\underline{P}^{r|n-1} \arrow[r, "\underline{\Phi}^{r|n-1}"] \arrow[d, "{}^{\Phi}\underline{\iota}^{r|n-1}_{r|n}"'] & {}^{\mathfrak{T}}P^{n-1} \arrow[d, "{}^{\mathfrak{T}}\iota^{n-1}"] \\
{}^{\Phi}\underline{P}^{r|n} \arrow[r, "\underline{\Phi}^{r|n}"'] & {}^{\mathfrak{T}}P^{n}
\end{tikzcd}$$

is *cartesian.* Now suppose that the morphism Φ is r*-integrable.* Then it follows from conditions (4), (5) of Definition 3.8 that the right-hand vertical arrow of this diagram is a *square-nilpotent* closed immersion, and the lower horizontal arrow of this diagram is *conormally strict* with respect to the right-hand vertical arrow.

REMARK 3.8.3. — Let r be a nonnegative integer. Suppose that the morphism Φ is r*-integrable.* Let $n \geq r+1$ be an integer. Let us recall the diagram of schemes over S

$$\begin{tikzcd}
X = {}^{\mathfrak{T}}P^{0,\dots,0} \arrow[r, "{}^{\mathfrak{T}}\iota^{0,\dots,0}_{1,\dots,1}"] & {}^{\mathfrak{T}}P^{\times n} \arrow[r, "{}^{\mathfrak{T}}\delta^{\times n}_{1,n-1}"] \arrow[d, "{}^{\mathfrak{T}}\delta^{\times n}_{n-1,1}"'] & {}^{\mathfrak{T}}P^{1,n-1} \arrow[d, "{}^{\mathfrak{T}}\delta^{1,n-1}"] & \\
& {}^{\mathfrak{T}}P^{n-1,1} \arrow[r, "{}^{\mathfrak{T}}\delta^{n-1,1}"'] & {}^{\mathfrak{T}}P^{n} \arrow[r, shift left, "{}^{\mathfrak{T}}\mathrm{pr}^{n}_{1}"] \arrow[r, shift right, "{}^{\mathfrak{T}}\mathrm{pr}^{n}_{2}"'] & X.
\end{tikzcd}$$

(i) It follows from condition (3) of Definition 3.8 that the diagram

$$\begin{tikzcd}
{}^{\mathfrak{T}}P^{\times n} \arrow[r, "{}^{\mathfrak{T}}\delta^{\times n}_{1,n-1}"] \arrow[d, "{}^{\mathfrak{T}}\delta^{\times n}_{n-1,1}"'] & {}^{\mathfrak{T}}P^{1,n-1} \arrow[d, "{}^{\mathfrak{T}}\delta^{1,n-1}"] \\
{}^{\mathfrak{T}}P^{n-1,1} \arrow[r, "{}^{\mathfrak{T}}\delta^{n-1,1}"'] & {}^{\mathfrak{T}}P^{n}
\end{tikzcd}$$

is [commutative and] *strictly cocartesian.*

(ii) It follows from Remark 3.1.1, (ii), and condition (3) of Definition 3.8 that the morphism

$$X = {}^{\mathfrak{T}}P^{0,\dots,0} \xrightarrow{{}^{\mathfrak{T}}\iota^{0,\dots,0}_{1,\dots,1}} {}^{\mathfrak{T}}P^{\times n}$$

is *quasi-nil-retraction-like.*

(iii) It follows from Remark 3.1.1, (i), and condition (3) of Definition 3.8 that the morphisms

$$\begin{tikzcd}
{}^{\mathfrak{T}}P^{n} \arrow[r, shift left, "{}^{\mathfrak{T}}\mathrm{pr}^{n}_{1}"] \arrow[r, shift right, "{}^{\mathfrak{T}}\mathrm{pr}^{n}_{2}"'] & X
\end{tikzcd}$$

are *quasi-nil-retraction-like*.

(iv) It follows from Remark 2.4.1, (ii), and Remark 2.6.1 that the equalities

$$ {}^{\mathfrak{T}}\mathrm{pr}_1^n \circ {}^{\mathfrak{T}}\delta^{1,n-1} \circ {}^{\mathfrak{T}}\delta_{1,n-1}^{\times n} \circ {}^{\mathfrak{T}}\iota_{1,\ldots,1}^{0,\ldots,0} = {}^{\mathfrak{T}}\mathrm{pr}_2^n \circ {}^{\mathfrak{T}}\delta^{1,n-1} \circ {}^{\mathfrak{T}}\delta_{1,n-1}^{\times n} \circ {}^{\mathfrak{T}}\iota_{1,\ldots,1}^{0,\ldots,0} = \mathrm{id}_X $$

hold.

DEFINITION 3.9. — Let r be a nonnegative integer. Then we shall say that the morphism Φ is *strictly r-integrable* if the following three conditions are satisfied:

(1) The morphism Φ is r-integrable.

(2) The morphism Φ is r-$\underline{\mathrm{pr}}$-finite flat.

(3) The stratification structure $\mathfrak{T}$ is pr-finite flat.

REMARK 3.9.1. — Let $n \leq r$ be positive integers. Suppose that the morphism Φ is *strictly r-integrable*. Let us recall the diagram of schemes over S

$$ \begin{tikzcd} X \arrow[r, "{}^{\Phi}\underline{\iota}_{r|n-1}^{0}"] & {}^{\Phi}\underline{P}^{r|n-1} \arrow[r, "\underline{\Phi}^{r|n-1}"] \arrow[d, "{}^{\Phi}\underline{\iota}_{r|n}^{r|n-1}"'] & {}^{\mathfrak{T}}P^{n-1} \arrow[d, "{}^{\mathfrak{T}}\iota^{n-1}"] & \\ & {}^{\Phi}\underline{P}^{r|n} \arrow[r, "\underline{\Phi}^{r|n}"'] & {}^{\mathfrak{T}}P^{n} \arrow[r, shift left, "{}^{\mathfrak{T}}\mathrm{pr}_1^n"] \arrow[r, shift right, "{}^{\mathfrak{T}}\mathrm{pr}_2^n"'] & X. \end{tikzcd} $$

(i) It follows from Remark 3.8.1, (ii), and condition (1) of Definition 3.9 that the morphism

$$ {}^{\Phi}\underline{\iota}_{r|n-1}^{0} \colon X \longrightarrow {}^{\Phi}\underline{P}^{r|n-1} $$

is *quasi-nil-retraction-like*.

(ii) It follows from Remark 3.1.1, (i), condition (3) of Definition 3.8, and condition (1) of Definition 3.9 that the morphisms

$$ \begin{tikzcd} {}^{\mathfrak{T}}P^{n} \arrow[r, shift left, "{}^{\mathfrak{T}}\mathrm{pr}_1^n"] \arrow[r, shift right, "{}^{\mathfrak{T}}\mathrm{pr}_2^n"'] & X \end{tikzcd} $$

are *quasi-nil-retraction-like*.

(iii) It follows from condition (1) of Definition 2.2 and Remark 2.10.1, (ii), that the equalities

$$ {}^{\mathfrak{T}}\mathrm{pr}_1^n \circ {}^{\mathfrak{T}}\iota^{n-1} \circ \underline{\Phi}^{r|n-1} \circ {}^{\Phi}\underline{\iota}_{r|n-1}^{0} = {}^{\mathfrak{T}}\mathrm{pr}_2^n \circ {}^{\mathfrak{T}}\iota^{n-1} \circ \underline{\Phi}^{r|n-1} \circ {}^{\Phi}\underline{\iota}_{r|n-1}^{0} = \mathrm{id}_X $$

hold.

(iv) It follows from conditions (2), (3) of Definition 3.9 [cf. also condition (1) of Definition 2.2; Remark 2.10.1, (ii)] that the morphisms

$$ {}^{\Phi}\underline{\mathrm{pr}}_1^{r|n-1} = {}^{\mathfrak{T}}\mathrm{pr}_1^n \circ {}^{\mathfrak{T}}\iota^{n-1} \circ \underline{\Phi}^{r|n-1}, \qquad {}^{\Phi}\underline{\mathrm{pr}}_2^{r|n-1} = {}^{\mathfrak{T}}\mathrm{pr}_2^n \circ {}^{\mathfrak{T}}\iota^{n-1} \circ \underline{\Phi}^{r|n-1}, $$

$$ {}^{\mathfrak{T}}\mathrm{pr}_1^{n-1} = {}^{\mathfrak{T}}\mathrm{pr}_1^{n} \circ {}^{\mathfrak{T}}\iota^{n-1}, \qquad {}^{\mathfrak{T}}\mathrm{pr}_2^{n-1} = {}^{\mathfrak{T}}\mathrm{pr}_2^{n} \circ {}^{\mathfrak{T}}\iota^{n-1}, $$

$$ {}^{\Phi}\underline{\mathrm{pr}}_1^{r|n} = {}^{\mathfrak{T}}\mathrm{pr}_1^{n} \circ \underline{\Phi}^{r|n}, \qquad {}^{\Phi}\underline{\mathrm{pr}}_2^{r|n} = {}^{\mathfrak{T}}\mathrm{pr}_2^{n} \circ \underline{\Phi}^{r|n}, $$

$$ {}^{\mathfrak{T}}\mathrm{pr}_1^{n}, \qquad {}^{\mathfrak{T}}\mathrm{pr}_2^{n} $$

are *finite* and *flat.*

REMARK 3.9.2. — Let r be a nonnegative integer. Suppose that the morphism Φ is *strictly r-integrable.* Let $n \geq r+1$ be an integer. Let us recall the diagram of schemes over S

$$
\begin{tikzcd}
X = {}^{\mathfrak{T}}P^{0,\ldots,0} \arrow[r, "{}^{\mathfrak{T}}\iota^{0,\ldots,0}_{1,\ldots,1}"] & {}^{\mathfrak{T}}P^{\times n} \arrow[r, "{}^{\mathfrak{T}}\delta^{\times n}_{1,n-1}"] \arrow[d, "{}^{\mathfrak{T}}\delta^{\times n}_{n-1,1}"'] & {}^{\mathfrak{T}}P^{1,n-1} \arrow[d, "{}^{\mathfrak{T}}\delta^{1,n-1}"] & \\
& {}^{\mathfrak{T}}P^{n-1,1} \arrow[r, "{}^{\mathfrak{T}}\delta^{n-1,1}"'] & {}^{\mathfrak{T}}P^{n} \arrow[r, shift left, "{}^{\mathfrak{T}}\mathrm{pr}_1^n"] \arrow[r, shift right, "{}^{\mathfrak{T}}\mathrm{pr}_2^n"'] & X.
\end{tikzcd}
$$

Then it follows from Remark 3.1.2 and condition (3) of Definition 3.9 [cf. also Remark 2.6.1] that the morphisms

$$ {}^{\mathfrak{T}}\mathrm{pr}_1^{\times n} = {}^{\mathfrak{T}}\mathrm{pr}_1^{n} \circ {}^{\mathfrak{T}}\delta^{1,n-1} \circ {}^{\mathfrak{T}}\delta^{\times n}_{1,n-1}, \qquad {}^{\mathfrak{T}}\mathrm{pr}_{n+1}^{\times n} = {}^{\mathfrak{T}}\mathrm{pr}_2^{n} \circ {}^{\mathfrak{T}}\delta^{1,n-1} \circ {}^{\mathfrak{T}}\delta^{\times n}_{1,n-1}, $$

$$ {}^{\mathfrak{T}}\mathrm{pr}_1^{1,n-1} = {}^{\mathfrak{T}}\mathrm{pr}_1^{n} \circ {}^{\mathfrak{T}}\delta^{1,n-1}, \qquad {}^{\mathfrak{T}}\mathrm{pr}_3^{1,n-1} = {}^{\mathfrak{T}}\mathrm{pr}_2^{n} \circ {}^{\mathfrak{T}}\delta^{1,n-1}, $$

$$ {}^{\mathfrak{T}}\mathrm{pr}_1^{n-1,1} = {}^{\mathfrak{T}}\mathrm{pr}_1^{n} \circ {}^{\mathfrak{T}}\delta^{n-1,1}, \qquad {}^{\mathfrak{T}}\mathrm{pr}_3^{n-1,1} = {}^{\mathfrak{T}}\mathrm{pr}_2^{n} \circ {}^{\mathfrak{T}}\delta^{n-1,1}, $$

$$ {}^{\mathfrak{T}}\mathrm{pr}_1^{n}, \qquad {}^{\mathfrak{T}}\mathrm{pr}_2^{n} $$

are *finite* and *flat.*

REMARK 3.9.3. — In sequels to the present paper, we give some examples of *strictly r-integrable morphisms of stratification structures.*

4. STRATIFICATIONS AND INTEGRABLE CONNECTIONS

In the present §4, we introduce and discuss the notions of a *connection* [cf. Definition 4.4 below], a *stratification* [cf. Definition 4.6 below], and an *integrable connection* [cf. Definition 4.7, (ii), below]. In the present §4, let S be a scheme, X a scheme which is *smooth* and *separated* over S,

$$ \mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0}, $$

$$ ({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\, n_2\geq 0} \big) $$

a stratification structure on X/S,

$$ \mathcal{F} \longrightarrow \mathrm{Sch}_S $$

a category fibered in groupoids over Sch_S, and

$$\xi$$

an object of $\mathcal{F}$ over X.

DEFINITION 4.1. — Let n be a nonnegative integer.

(i) We shall write

$$^{\mathfrak{S}}\mathrm{qCnn}^n(\xi) \stackrel{\mathrm{def}}{=} \mathrm{Lift}_{^{\mathfrak{S}}\iota_n^0}\big((^{\mathfrak{S}}\mathrm{pr}_2^n)^*\xi, (^{\mathfrak{S}}\mathrm{pr}_1^n)^*\xi; \mathrm{id}_\xi\big)$$

and refer to an element of the set $^{\mathfrak{S}}\mathrm{qCnn}^n(\xi)$ as a *quasi-n-$\mathfrak{S}$-connection* on ξ.

(ii) Let ∇ be a quasi-n-$\mathfrak{S}$-connection on ξ and $m \leq n$ a nonnegative integer. Then one verifies easily that the isomorphism

$$(^{\mathfrak{S}}\iota_n^m)^*\nabla \colon (^{\mathfrak{S}}\mathrm{pr}_2^m)^*\xi \xrightarrow{\sim} (^{\mathfrak{S}}\mathrm{pr}_1^m)^*\xi$$

[cf. condition (1) of Definition 2.2] is a quasi-m-$\mathfrak{S}$-connection on ξ. We shall write

$$\nabla|_m \stackrel{\mathrm{def}}{=} (^{\mathfrak{S}}\iota_n^m)^*\nabla$$

for this quasi-m-$\mathfrak{S}$-connection on ξ. Thus, we have a map of sets

$$^{\mathfrak{S}}\mathrm{qCnn}^n(\xi) \longrightarrow {^{\mathfrak{S}}\mathrm{qCnn}^m(\xi)}$$

given by mapping $\nabla \in {^{\mathfrak{S}}\mathrm{qCnn}^n(\xi)}$ to $\nabla|_m \in {^{\mathfrak{S}}\mathrm{qCnn}^m(\xi)}$.

(iii) We shall refer to a quasi-1-$\mathfrak{S}$-connection on ξ as an $\mathfrak{S}$-*connection* on ξ [cf. Remark 4.1.1 below].

REMARK 4.1.1. — One verifies easily that the notion of an $\mathfrak{S}$-*connection* is the *same* as the notion of a 1-$\mathfrak{S}$-*connection* in the sense of Definition 4.4 below.

DEFINITION 4.2. — Let n be a nonnegative integer; ξ_1, ξ_2 objects of $\mathcal{F}$ over X; ∇_1, ∇_2 quasi-n-$\mathfrak{S}$-connections on ξ_1, ξ_2, respectively; $\phi\colon \xi_1 \xrightarrow{\sim} \xi_2$ an isomorphism in $\mathcal{F}|_X$. Then we shall say that the isomorphism ϕ is $\mathfrak{S}$-*horizontal* [with respect to ∇_1, ∇_2] if the diagram in $\mathcal{F}|_{^{\mathfrak{S}}P^1}$

$$\begin{tikzcd}
(^{\mathfrak{S}}\mathrm{pr}_2^n)^*\xi_1 \arrow[r, "(^{\mathfrak{S}}\mathrm{pr}_2^n)^*\phi", "\sim"'] \arrow[d, "\nabla_1"', "\wr"] & (^{\mathfrak{S}}\mathrm{pr}_2^n)^*\xi_2 \arrow[d, "\nabla_2", "\wr"'] \\
(^{\mathfrak{S}}\mathrm{pr}_1^n)^*\xi_1 \arrow[r, "\sim", "(^{\mathfrak{S}}\mathrm{pr}_1^n)^*\phi"'] & (^{\mathfrak{S}}\mathrm{pr}_1^n)^*\xi_2
\end{tikzcd}$$

is commutative.

DEFINITION 4.3. — Let $r \geq 2$ be an integer; $n_1, \ldots, n_r$ positive integers. Write $n \stackrel{\text{def}}{=} \sum_{i=1}^{r} n_i$ [so $1 \leq n_i \leq n-1$ for each $i \in \{1, \ldots, r\}$]. Let ∇ be a quasi-$(n-1)$-$\mathfrak{S}$-connection on ξ. Then we shall write

$$\nabla^{n_1,\ldots,n_r} \colon ({}^{\mathfrak{S}}\mathrm{pr}_{r+1}^{n_1,\ldots,n_r})^*\xi \xrightarrow{\sim} ({}^{\mathfrak{S}}\mathrm{pr}_1^{n_1,\ldots,n_r})^*\xi$$

for the isomorphism in $\mathcal{F}|_{{}^{\mathfrak{S}}P^{n_1,\ldots,n_r}}$ obtained by forming the composite

$$({}^{\mathfrak{S}}\mathrm{pr}_{r+1}^{n_1,\ldots,n_r})^*\xi = ({}^{\mathfrak{S}}\mathrm{pr}_{\{r,r+1\}}^{n_1,\ldots,n_r})^*({}^{\mathfrak{S}}\mathrm{pr}_2^{n_r})^*\xi \xrightarrow[\sim]{({}^{\mathfrak{S}}\mathrm{pr}_{\{r,r+1\}}^{n_1,\ldots,n_r})^*(\nabla|_{n_r})} ({}^{\mathfrak{S}}\mathrm{pr}_{\{r,r+1\}}^{n_1,\ldots,n_r})^*({}^{\mathfrak{S}}\mathrm{pr}_1^{n_r})^*\xi$$

$$= ({}^{\mathfrak{S}}\mathrm{pr}_r^{n_1,\ldots,n_r})^*\xi = ({}^{\mathfrak{S}}\mathrm{pr}_{\{r-1,r\}}^{n_1,\ldots,n_r})^*({}^{\mathfrak{S}}\mathrm{pr}_2^{n_{r-1}})^*\xi \xrightarrow[\sim]{({}^{\mathfrak{S}}\mathrm{pr}_{\{r-1,r\}}^{n_1,\ldots,n_r})^*(\nabla|_{n_{r-1}})} ({}^{\mathfrak{S}}\mathrm{pr}_{\{r-1,r\}}^{n_1,\ldots,n_r})^*({}^{\mathfrak{S}}\mathrm{pr}_1^{n_{r-1}})^*\xi$$

$$\cdots$$

$$= ({}^{\mathfrak{S}}\mathrm{pr}_2^{n_1,\ldots,n_r})^*\xi = ({}^{\mathfrak{S}}\mathrm{pr}_{\{1,2\}}^{n_1,\ldots,n_r})^*({}^{\mathfrak{S}}\mathrm{pr}_2^{n_1})^*\xi \xrightarrow[\sim]{({}^{\mathfrak{S}}\mathrm{pr}_{\{1,2\}}^{n_1,\ldots,n_r})^*(\nabla|_{n_1})} ({}^{\mathfrak{S}}\mathrm{pr}_{\{1,2\}}^{n_1,\ldots,n_r})^*({}^{\mathfrak{S}}\mathrm{pr}_1^{n_1})^*\xi$$

$$= ({}^{\mathfrak{S}}\mathrm{pr}_1^{n_1,\ldots,n_r})^*\xi.$$

Moreover, we shall write

$$\nabla^{\times r} \colon ({}^{\mathfrak{S}}\mathrm{pr}_{r+1}^{\times r})^*\xi \xrightarrow{\sim} ({}^{\mathfrak{S}}\mathrm{pr}_1^{\times r})^*\xi$$

for "$\nabla^{n_1,\ldots,n_r}$" in the case where we take the "$(n_1, \ldots, n_r)$" to be $(1, \ldots, 1)$.

DEFINITION 4.4. — Let n be a nonnegative integer and ∇ a quasi-n-$\mathfrak{S}$-connection on ξ. Then we shall say that ∇ is an n-$\mathfrak{S}$-*connection* if, for each positive integers n_1, n_2 such that $n_1 + n_2 = n$, the "cocycle condition"

$$(\nabla|_{n-1})^{n_1,n_2} = ({}^{\mathfrak{S}}\delta^{n_1,n_2})^*\nabla$$

[cf. Remark 2.6.1] is satisfied. We shall write

$${}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \subseteq {}^{\mathfrak{S}}\mathrm{qCnn}^n(\xi)$$

for the set of n-$\mathfrak{S}$-connections on ξ.

LEMMA 4.5. — *The map ${}^{\mathfrak{S}}\mathrm{qCnn}^n(\xi) \to {}^{\mathfrak{S}}\mathrm{qCnn}^m(\xi)$ of Definition* 4.1, (ii), **restricts** *to a map of subsets*

$${}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Cnn}^m(\xi).$$

PROOF. — Let ∇ be an n-$\mathfrak{S}$-connection on ξ; m_1, m_2 positive integers such that $m_1 + m_2 = m$. Then since ∇ is an n-$\mathfrak{S}$-*connection*, the "cocycle condition"

$$(\nabla|_{n-1})^{m_1+n-m,m_2} = ({}^{\mathfrak{S}}\delta^{m_1+n-m,m_2})^*\nabla$$

is satisfied. Thus, by pulling back this equality by the morphism ${}^{\mathfrak{S}}\iota^{m_1,m_2}_{m_1+n-m,m_2} \colon {}^{\mathfrak{S}}P^{m_1,m_2} \to {}^{\mathfrak{S}}P^{m_1+n-m,m_2}$, we conclude from condition (1) of Definition 2.2 and condition (4) of Definition 2.5 that the "cocycle condition"

$$(\nabla|_{m-1})^{m_1,m_2} = ({}^{\mathfrak{S}}\delta^{m_1,m_2})^*(\nabla|_m)$$

is satisfied, as desired. This completes the proof of Lemma 4.5. □

DEFINITION 4.6. — We shall write

$${}^{\mathfrak{S}}\mathrm{Strt}(\xi) \stackrel{\mathrm{def}}{=} \varprojlim_{n\geq 0} {}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi)$$

[cf. Lemma 4.5] and refer to an element of the set ${}^{\mathfrak{S}}\mathrm{Strt}(\xi)$ as an $\mathfrak{S}$-*stratification* on ξ.

REMARK 4.6.1. — A typical example of an $\mathfrak{S}$-*stratification* is as follows: Suppose that there exists an object of $\mathcal{F}$ *over* S such that ξ is given by the pull-back of the object by $X \to S$. Then, for each nonnegative integer n, since [one verifies easily that] the two composites

$${}^{\mathfrak{S}}P^n \xrightarrow{{}^{\mathfrak{S}}\mathrm{pr}^n_1} X \longrightarrow S, \qquad {}^{\mathfrak{S}}P^n \xrightarrow{{}^{\mathfrak{S}}\mathrm{pr}^n_2} X \longrightarrow S$$

coincide, we have a natural identification

$$\nabla_n \colon ({}^{\mathfrak{S}}\mathrm{pr}^n_2)^*\xi = ({}^{\mathfrak{S}}\mathrm{pr}^n_1)^*\xi.$$

Now one verifies immediately that these ∇_n's form an $\mathfrak{S}$-*stratification* on ξ.

REMARK 4.6.2. — Suppose that we are in the situation of Remark 2.5.1. Thus, we have a *stratification structure* on X/S

$$\big((P^n)_{n\geq 0},\ (\iota^n \colon P^n \hookrightarrow P^{n+1})_{n\geq 0},\ (\sigma^n \colon P^n \to X^{(2)})_{n\geq 0},\ (\delta^{n_1,n_2} \colon P^{n_1,n_2} \to P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big).$$

Suppose, moreover, that there exists a nonnegative integer n_0 such that the closed immersion $P_{n_0} \hookrightarrow P$ is an *isomorphism*. In this situation, one verifies easily that the notion of a *stratification on* ξ [i.e., with respect to the above stratification structure] is the *same* as the notion of a *descent datum on* ξ *with respect to the morphism* $X \to Y$ [in the usual sense] — i.e., an isomorphism in $\mathcal{F}|_P$

$$q_2^*\xi \xrightarrow{\sim} q_1^*\xi$$

that satisfies a certain cocycle condition.

DEFINITION 4.7.

(i) Let ∇ be an $\mathfrak{S}$-connection on ξ. Then we shall refer to the automorphism of $({}^{\mathfrak{S}}\underline{\mathrm{pr}}_1^{\times 2})^*\xi$ in $\mathcal{F}|_{{}^{\mathfrak{S}}\underline{P}^{\times 2}}$ obtained by forming the composite

$$({}^{\mathfrak{S}}\underline{\iota}^{\times 2})^*(\nabla^{\times 2}) \circ ({}^{\mathfrak{S}}\underline{\delta}^{\times 2})^*(\nabla^{-1})$$

— where we write ∇^{-1} for the inverse of the isomorphism ∇ — as the $\mathfrak{S}$-*curvature* of ∇.

(ii) We shall say that an $\mathfrak{S}$-connection on ξ is $\mathfrak{S}$-*integrable* if the $\mathfrak{S}$-curvature of the $\mathfrak{S}$-connection is the identity automorphism of $({}^{\mathfrak{S}}\underline{\mathrm{pr}}_1^{\times 2})^*\xi$. We shall write

$$ {}^{\mathfrak{S}}\mathrm{IntCnn}(\xi) \subseteq {}^{\mathfrak{S}}\mathrm{Cnn}^1(\xi) \left(= {}^{\mathfrak{S}}\mathrm{qCnn}^1(\xi)\right) $$

for the set of $\mathfrak{S}$-integrable $\mathfrak{S}$-connections on ξ.

LEMMA 4.8. — *If $n > m = 1$, then the map ${}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \to {}^{\mathfrak{S}}\mathrm{Cnn}^m(\xi) = {}^{\mathfrak{S}}\mathrm{Cnn}^1(\xi)$ of Lemma* 4.5 **restricts** *to a map of subsets*

$$ {}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{IntCnn}(\xi). $$

PROOF. — Let us first observe that it follows from Lemma 4.5 that we may assume without loss of generality that $n = 2$. Let ∇ be a 2-$\mathfrak{S}$-connection on ξ. Thus, the "cocycle condition"

$$ (\nabla|_1)^{\times 2} = ({}^{\mathfrak{S}}\delta^{\times 2})^*\nabla $$

is satisfied. In particular, by pulling back this equality by the morphism ${}^{\mathfrak{S}}\underline{\iota}^{\times 2}\colon {}^{\mathfrak{S}}\underline{P}^{\times 2} \to {}^{\mathfrak{S}}P^{\times 2}$, we obtain an equality

$$ ({}^{\mathfrak{S}}\underline{\iota}^{\times 2})^*\left((\nabla|_1)^{\times 2}\right) = ({}^{\mathfrak{S}}\underline{\delta}^{\times 2})^*(\nabla|_1). $$

Thus, the $\mathfrak{S}$-*curvature* of $\nabla|_1$ is the *identity automorphism*, as desired. This completes the proof of Lemma 4.8. $\square$

LEMMA 4.9. — *Suppose that the following two conditions are satisfied:*

(1) *The stratification structure $\mathfrak{S}$ is* **of standard type**.

(2) *The category $\mathcal{F}$ fibered in groupoids over* Sch_S *satisfies condition* (1) *of Definition* 1.8.

Then the pull-back by the closed immersion ${}^{\mathfrak{S}}\underline{\iota}^T_{\times 2}\colon T^1 \hookrightarrow {}^{\mathfrak{S}}\underline{P}^{\times 2}$ of Definition 3.6 *of the* **$\mathfrak{S}$-curvature** *of an $\mathfrak{S}$-connection on ξ is the* **identity automorphism** *of $({}^X\mathrm{pr}_1^T)^*\xi$. In particular, the* **$\mathfrak{S}$-curvature** *of an $\mathfrak{S}$-connection on ξ is an element of*

$$ \mathrm{Lift}_{{}^{\mathfrak{S}}\underline{\iota}^T_{\times 2}}\left(({}^{\mathfrak{S}}\underline{\mathrm{pr}}_1^{\times 2})^*\xi, ({}^{\mathfrak{S}}\underline{\mathrm{pr}}_1^{\times 2})^*\xi; \mathrm{id}_{({}^X\mathrm{pr}_1^T)^*\xi}\right) \subseteq \mathrm{Aut}_{\mathcal{F}|_{{}^{\mathfrak{S}}\underline{P}^{\times 2}}}\left(({}^{\mathfrak{S}}\underline{\mathrm{pr}}_1^{\times 2})^*\xi\right). $$

PROOF. — Let ∇ be an $\mathfrak{S}$-connection on ξ. Write α for the pull-back of the $\mathfrak{S}$-curvature of ∇ by ${}^{\mathfrak{S}}\underline{\iota}^T_{\times 2}$, i.e.,

$$ \alpha = ({}^{\mathfrak{S}}\underline{\iota}^T_{\times 2})^*({}^{\mathfrak{S}}\underline{\iota}^{\times 2})^*(\nabla^{\times 2}) \circ ({}^{\mathfrak{S}}\underline{\iota}^T_{\times 2})^*({}^{\mathfrak{S}}\underline{\delta}^{\times 2})^*(\nabla^{-1}) = ({}^{\mathfrak{S}}\iota^T_{\times 2})^*(\nabla^{\times 2}) \circ ({}^{\mathfrak{S}}\delta^T_1)^*(\nabla^{-1}). $$

Now let us observe that it follows from our assumption that $\mathfrak{S}$ is *of standard type*, and $\mathcal{F}$ satisfies condition (1) of Definition 1.8 [cf. also Remark 3.6.1] that the morphism ${}^{\mathfrak{S}}\iota_T^{1,0}$ determines a *bijection*

$$ \mathrm{Lift}_{{}^{\mathfrak{S}}\iota_T^{0,1}}\left(({}^X\mathrm{pr}_1^T)^*\xi, ({}^X\mathrm{pr}_1^T)^*\xi; \mathrm{id}_{({}^{\mathfrak{S}}\mathrm{pr}_1^1)^*\xi}\right) \xrightarrow{\sim} \mathrm{Lift}_{{}^{\mathfrak{S}}\iota^0}\left(({}^{\mathfrak{S}}\mathrm{pr}_1^1)^*\xi, ({}^{\mathfrak{S}}\mathrm{pr}_1^1)^*\xi; \mathrm{id}_\xi\right). $$

Next, let us observe that since [one verifies easily that]

$$^{\mathfrak{S}}\iota^{T}_{\times 2} \circ {}^{\mathfrak{S}}\iota^{0,1}_{T} = {}^{\mathfrak{S}}\iota^{0,1}_{1,1}, \qquad {}^{\mathfrak{S}}\delta^{T}_{1} \circ {}^{\mathfrak{S}}\iota^{0,1}_{T} = \mathrm{id}_{{}^{\mathfrak{S}}P^{1}},$$

the pull-back $({}^{\mathfrak{S}}\iota^{0,1}_{T})^{*}\alpha$ is the *identity automorphism* of $({}^{\mathfrak{S}}\mathrm{pr}^{1}_{1})^{*}\xi$, which thus implies that α is an element of the domain of the above bijection. Next, let us observe that since [one verifies easily that]

$$^{\mathfrak{S}}\iota^{T}_{\times 2} \circ {}^{\mathfrak{S}}\iota^{1,0}_{T} = {}^{\mathfrak{S}}\iota^{1,0}_{1,1}, \qquad {}^{\mathfrak{S}}\delta^{T}_{1} \circ {}^{\mathfrak{S}}\iota^{1,0}_{T} = \mathrm{id}_{{}^{\mathfrak{S}}P^{1}},$$

the pull-back $({}^{\mathfrak{S}}\iota^{1,0}_{T})^{*}\alpha$ is the *identity automorphism* of $({}^{\mathfrak{S}}\mathrm{pr}^{1}_{1})^{*}\xi$, which thus implies that the image by the above bijection of α is the *identity automorphism* of $({}^{\mathfrak{S}}\mathrm{pr}^{1}_{1})^{*}\xi$. Thus, we conclude that α is the *identity automorphism*, as desired. This completes the proof of Lemma 4.9. □

PROPOSITION 4.10. — *Suppose that the following three conditions are satisfied:*

(1) *The stratification structure* $\mathfrak{S}$ *is* **of standard type**.

(2) *The category* $\mathcal{F}$ *fibered in groupoids over* Sch_S *is* **weakly integrable**.

(3) *The scheme* X *is* **of relative dimension ≤ 1** *over* S.

Then every $\mathfrak{S}$*-connection on* ξ *is* **$\mathfrak{S}$-integrable**:

$${}^{\mathfrak{S}}\mathrm{qCnn}^{1}(\xi) = {}^{\mathfrak{S}}\mathrm{Cnn}^{1}(\xi) = {}^{\mathfrak{S}}\mathrm{IntCnn}(\xi).$$

PROOF. — Since $\mathfrak{S}$ is *of standard type*, and X is *of relative dimension* ≤ 1 over S, it follows from condition (3) of Definition 3.6 that the closed immersion ${}^{\mathfrak{S}}\underline{\iota}^{T}_{\times 2} \colon T^{1} \hookrightarrow {}^{\mathfrak{S}}\underline{P}^{\times 2}$ of Definition 3.6 is an *isomorphism*. Thus, Proposition 4.10 follows from Lemma 4.9. This completes the proof of Proposition 4.10. □

In the remainder of the present §4, let

$$\mathfrak{T} = \big(({}^{\mathfrak{T}}P^{n})_{n\geq 0},\ ({}^{\mathfrak{T}}\iota^{n} \colon {}^{\mathfrak{T}}P^{n} \to {}^{\mathfrak{T}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{T}}\sigma^{n} \colon {}^{\mathfrak{T}}P^{n} \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{T}}\delta^{n_1,n_2} \colon {}^{\mathfrak{T}}P^{n_1,n_2} \to {}^{\mathfrak{T}}P^{n_1+n_2})_{n_1,\,n_2\geq 0}\big)$$

be a stratification structure on X/S and

$$\Phi = (\Phi^{n})_{n\geq 0} \colon \mathfrak{S} \longrightarrow \mathfrak{T}$$

a morphism of stratification structures.

DEFINITION 4.11. — Let n be a nonnegative integer and ∇ a quasi-n-$\mathfrak{T}$-connection on ξ. Then one verifies easily from condition (1) of Definition 2.8 and Remark 2.8.1 that the isomorphism

$$(\Phi^{n})^{*}\nabla \colon ({}^{\mathfrak{S}}\mathrm{pr}^{n}_{2})^{*}\xi \xrightarrow{\sim} ({}^{\mathfrak{S}}\mathrm{pr}^{n}_{1})^{*}\xi$$

[cf. condition (2) of Definition 2.8] is a quasi-n-$\mathfrak{S}$-connection on ξ. We shall write

$$\Phi^{*}\nabla \stackrel{\mathrm{def}}{=} (\Phi^{n})^{*}\nabla$$

for this quasi-n-$\mathfrak{S}$-connection on ξ. Thus, we have a map of sets

$$\Phi^* \colon {}^{\mathfrak{T}}\mathrm{qCnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{qCnn}^n(\xi)$$

given by mapping $\nabla \in {}^{\mathfrak{T}}\mathrm{qCnn}^n(\xi)$ to $\Phi^*\nabla \in {}^{\mathfrak{S}}\mathrm{qCnn}^n(\xi)$.

LEMMA 4.12. — *Let n be a nonnegative integer. Then the following hold:*

(i) *The map $\Phi^* \colon {}^{\mathfrak{T}}\mathrm{qCnn}^n(\xi) \to {}^{\mathfrak{S}}\mathrm{qCnn}^n(\xi)$ of Definition* 4.11 **restricts** *to a map of subsets*

$$\Phi^* \colon {}^{\mathfrak{T}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi).$$

In particular, we have a map of sets

$$\Phi^* \colon {}^{\mathfrak{T}}\mathrm{Strt}(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Strt}(\xi)$$

[*cf. condition* (1) *of Definition* 2.8].

(ii) *Suppose that the category $\mathcal{F}$ fibered in groupoids over* Sch_S *satisfies condition* (3) *of Definition* 1.7*, and that the morphism Φ is* **nil-retraction-like**. *Then the diagram of sets*

$$\begin{tikzcd}
{}^{\mathfrak{T}}\mathrm{Cnn}^n(\xi) \arrow[r, "\Phi^*"] \arrow[d, hook] & {}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \arrow[d, hook] \\
{}^{\mathfrak{T}}\mathrm{qCnn}^n(\xi) \arrow[r, "\Phi^*"'] & {}^{\mathfrak{S}}\mathrm{qCnn}^n(\xi)
\end{tikzcd}$$

[*cf.* (i)] *is* **cartesian**.

PROOF. — First, we verify assertion (i). Let ∇ be an n-$\mathfrak{T}$-connection on ξ; n_1, n_2 positive integers such that $n_1 + n_2 = n$. Then since ∇ is an n-$\mathfrak{T}$-*connection*, the "cocycle condition"

$$(\nabla|_{n-1})^{n_1,n_2} = ({}^{\mathfrak{T}}\delta^{n_1,n_2})^*\nabla$$

is satisfied. Thus, by pulling back this equality by the morphism $\Phi^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{T}}P^{n_1,n_2}$, we conclude from conditions (1), (2), and (3) of Definition 2.8 that the "cocycle condition"

$$\big((\Phi^*\nabla)|_{n-1}\big)^{n_1,n_2} = ({}^{\mathfrak{S}}\delta^{n_1,n_2})^*(\Phi^*\nabla)$$

is satisfied, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let ∇ be a quasi-n-$\mathfrak{T}$-connection on ξ such that the quasi-n-$\mathfrak{S}$-connection $\Phi^*\nabla$ is an n-$\mathfrak{S}$-*connection*; n_1, n_2 positive integers such that $n_1 + n_2 = n$. Then since $\mathcal{F}$ satisfies condition (3) of Definition 1.7, and the morphism Φ is *nil-retraction-like*, to verify the "cocycle condition"

$$(\nabla|_{n-1})^{n_1,n_2} = ({}^{\mathfrak{T}}\delta^{n_1,n_2})^*\nabla$$

for ∇, it suffices to verify the pull-back of this equality by the morphism $\Phi^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{T}}P^{n_1,n_2}$, i.e., the equality

$$((\Phi^*\nabla)|_{n-1})^{n_1,n_2} = ({}^{\mathfrak{S}}\delta^{n_1,n_2})^*(\Phi^*\nabla)$$

[cf. conditions (1), (2), and (3) of Definition 2.8]. On the other hand, since the quasi-n-$\mathfrak{S}$-connection $\Phi^*\nabla$ is an n-$\mathfrak{S}$-*connection*, this equality is satisfied, as desired. This completes the proof of assertion (ii), hence also of Lemma 4.12. $\square$

DEFINITION 4.13. — Let $n \leq r$ be nonnegative integers.

(i) Let ∇ be a quasi-r-$\mathfrak{S}$-connection on ξ. Then we shall refer to the automorphism of $(\underline{\Phi}^{n|0})^*\xi$ in $\mathcal{F}|_{{}^{\Phi}\underline{P}^{n|0}}$

$$(\underline{\Phi}^{n|0})^*\xi = ({}^{\Phi}\underline{\mathrm{pr}}_2^{n|0})^*\xi \xrightarrow[\sim]{({}^{\Phi}\underline{\iota}^{n|0})^*(\nabla|_n)} ({}^{\Phi}\underline{\mathrm{pr}}_1^{n|0})^*\xi = (\underline{\Phi}^{n|0})^*\xi$$

[cf. Remark 2.10.1, (iii)] as the (Φ, n)-*curvature* of ∇. Moreover, we shall refer to the (Φ, n)-curvature of the quasi-n-$\mathfrak{S}$-connection on ξ determined by an $\mathfrak{S}$-stratification on ξ as the (Φ, n)-*curvature* of the $\mathfrak{S}$-stratification.

(ii) We shall say that a quasi-r-$\mathfrak{S}$-connection on ξ is (Φ, n)-*dormant* if the (Φ, n)-curvature of the quasi-r-$\mathfrak{S}$-connection is the identity automorphism of $(\underline{\Phi}^{n|0})^*\xi$. Moreover, we shall say that an $\mathfrak{S}$-stratification on ξ is (Φ, n)-*dormant* if the quasi-n-$\mathfrak{S}$-connection determined by the $\mathfrak{S}$-stratification is (Φ, n)-dormant. We shall write

$${}^{\Phi,n}\mathrm{DrmStrt}(\xi) \subseteq {}^{\mathfrak{S}}\mathrm{Strt}(\xi)$$

for the set of (Φ, n)-dormant $\mathfrak{S}$-stratifications on ξ.

(iii) We shall say that an $\mathfrak{S}$-connection on ξ is (Φ, n)-*dormant* if the $\mathfrak{S}$-connection is contained in the image of the composite

$${}^{\Phi,n}\mathrm{DrmStrt}(\xi) \hookrightarrow {}^{\mathfrak{S}}\mathrm{Strt}(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Cnn}(\xi).$$

[In particular, every (Φ, n)-dormant $\mathfrak{S}$-connection is $\mathfrak{S}$-integrable — cf. Lemma 4.8.] We shall write

$${}^{\Phi,n}\mathrm{DrmCnn}(\xi) \subseteq {}^{\mathfrak{S}}\mathrm{IntCnn}(\xi)$$

for the set of (Φ, n)-dormant $\mathfrak{S}$-connections on ξ. Thus, we have a natural surjective map

$${}^{\Phi,n}\mathrm{DrmStrt}(\xi) \twoheadrightarrow {}^{\Phi,n}\mathrm{DrmCnn}(\xi).$$

LEMMA 4.14. — *Let n be a nonnegative integer. Then the map $\Phi^* \colon {}^{\mathfrak{T}}\mathrm{Strt}(\xi) \to {}^{\mathfrak{S}}\mathrm{Strt}(\xi)$ of Lemma* 4.12, (i), **factors** *through the subset ${}^{\Phi,n}\mathrm{DrmStrt}(\xi) \subseteq {}^{\mathfrak{S}}\mathrm{Strt}(\xi)$. In particular, we obtain a map of sets*

$$\Phi^* \colon {}^{\mathfrak{T}}\mathrm{Strt}(\xi) \longrightarrow {}^{\Phi,n}\mathrm{DrmStrt}(\xi).$$

PROOF. — This assertion follows immediately from the various definitions involved [cf. the cartesian diagram of Definition 2.10, (ii)]. $\square$

5. The First Fundamental Correspondence

In the present §5, we prove the first main result of the present paper [cf. Theorem 5.6 below]. In the present §5, let S be a scheme, X a scheme which is *smooth* and *separated* over S,

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big)$$

a stratification structure on X/S,

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S, and

$$\xi$$

an object of $\mathcal{F}$ over X. Suppose that

- the stratification structure $\mathfrak{S}$ is *integrable* (respectively, *strictly integrable*), and that
- the category $\mathcal{F}$ fibered in groupoids over Sch_S is *integrable* (respectively, *weakly integrable*).

LEMMA 5.1. — *Let $r \geq 2$ be an integer; $n_1, \ldots, n_r$ positive integers. Write $n \overset{\mathrm{def}}{=} \sum_{i=1}^r n_i$ [so $n \geq 2$]. Let ∇ be an $(n-1)$-$\mathfrak{S}$-connection on ξ. Then the equality*

$$\nabla^{\times n} = ({}^{\mathfrak{S}}\delta^{\times n}_{n_1,\ldots,n_r})^*\nabla^{n_1,\ldots,n_r}$$

holds.

PROOF. — This assertion follows immediately from Lemma 4.5, together with the "cocycle conditions" of Definition 4.4 for $\nabla|_1, \ldots, \nabla|_{n-1}$. □

LEMMA 5.2. — *Let $m \leq n$ be nonnegative integers. Then the map*

$${}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Cnn}^m(\xi)$$

[cf. Lemma 4.5] is **injective**.

PROOF. — Let us first observe that it is immediate that we may assume without loss of generality that $m = n-1$. Let ∇, ∇' be n-$\mathfrak{S}$-connections on ξ such that $\nabla|_{n-1} = \nabla'|_{n-1}$. Now since $\mathcal{F}$ satisfies condition (3) of Definition 1.7, and the morphism ${}^{\mathfrak{S}}\delta^{1,n-1} \colon {}^{\mathfrak{S}}P^{1,n-1} \to {}^{\mathfrak{S}}P^n$ is *nil-retraction-like* [cf. condition (1) of Definition 3.2; condition (1) of Definition 3.3], to verify Lemma 5.2, it suffices to verify the equality

$$({}^{\mathfrak{S}}\delta^{1,n-1})^*\nabla = ({}^{\mathfrak{S}}\delta^{1,n-1})^*\nabla'.$$

On the other hand, since both ∇ and ∇' are n-$\mathfrak{S}$-*connections*, it follows from the "cocycle condition" of Definition 4.4 that $({}^{\mathfrak{S}}\delta^{1,n-1})^*\nabla$, $({}^{\mathfrak{S}}\delta^{1,n-1})^*\nabla'$ are *completely determined* by $\nabla|_{n-1}$, $\nabla'|_{n-1}$, respectively. In particular, we conclude that $({}^{\mathfrak{S}}\delta^{1,n-1})^*\nabla = ({}^{\mathfrak{S}}\delta^{1,n-1})^*\nabla'$, as desired. This completes the proof of Lemma 5.2. □

LEMMA 5.3. — *Let $n \geq 2$ be an integer and ∇ a quasi-n-$\mathfrak{S}$-connection on ξ. Then it holds that ∇ is an* **n-$\mathfrak{S}$-connection** *on ξ if and only if the following two conditions are satisfied:*

(1) *The quasi-$(n-1)$-$\mathfrak{S}$-connection $\nabla|_{n-1}$ is an* **$(n-1)$-$\mathfrak{S}$-connection**.

(2) *The equality*

$$(\nabla|_{n-1})^{\times n} = ({}^{\mathfrak{S}}\delta^{\times n})^* \nabla$$

holds.

PROOF. — First, we verify the *necessity*. Suppose that ∇ is an n-$\mathfrak{S}$-*connection* on ξ. Then it follows from Lemma 4.5 that condition (1) is satisfied. Next, to verify condition (2), let us observe that it follows from Lemma 5.1, together with condition (1), that

$$(\nabla|_{n-1})^{\times n} = ({}^{\mathfrak{S}}\delta^{\times n}_{1,n-1})^* \big((\nabla|_{n-1})^{1,n-1}\big).$$

Thus, since ∇ is an n-$\mathfrak{S}$-*connection*, which thus implies that the equality

$$(\nabla|_{n-1})^{1,n-1} = ({}^{\mathfrak{S}}\delta^{1,n-1})^* \nabla$$

holds, condition (2) is satisfied, as desired. This completes the proof of the *necessity*.

Next, we verify the *sufficiency*. Suppose that ∇ satisfies the two conditions in the statement of Lemma 5.3. Let n_1, n_2 be positive integers such that $n_1+n_2 = n$. Then since $\mathcal{F}$ satisfies condition (3) of Definition 1.7, and the morphism ${}^{\mathfrak{S}}\delta^{\times n}_{n_1,n_2} \colon {}^{\mathfrak{S}}P^{\times n} \to {}^{\mathfrak{S}}P^{n_1,n_2}$ is *nil-retraction-like* [cf. condition (1) of Definition 3.2; condition (1) of Definition 3.3], to verify the "cocycle condition" $(\nabla|_{n-1})^{n_1,n_2} = ({}^{\mathfrak{S}}\delta^{n_1,n_2})^*\nabla$, it suffices to verify the equality

$$({}^{\mathfrak{S}}\delta^{\times n}_{n_1,n_2})^* \big((\nabla|_{n-1})^{n_1,n_2}\big) = ({}^{\mathfrak{S}}\delta^{\times n})^* \nabla,$$

or, alternatively [cf. condition (2)], the equality

$$({}^{\mathfrak{S}}\delta^{\times n}_{n_1,n_2})^* \big((\nabla|_{n-1})^{n_1,n_2}\big) = (\nabla|_{n-1})^{\times n}.$$

On the other hand, since $\nabla|_{n-1}$ is an $(n-1)$-$\mathfrak{S}$-*connection* [cf. condition (1)], this equality follows from Lemma 5.1. This completes the proof of the *sufficiency*, hence also of Lemma 5.3. $\square$

LEMMA 5.4. — *The map*

$${}^{\mathfrak{S}}\mathrm{Cnn}^2(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{IntCnn}(\xi)$$

[*cf. Lemma* 4.8] *is* **surjective**.

PROOF. — Let us recall that we have assumed that

- the stratification structure $\mathfrak{S}$ is *integrable* (respectively, *strictly integrable*), and that
- the category $\mathcal{F}$ fibered in groupoids over Sch_S is *integrable* (respectively, *weakly integrable*).

Let ∇ be an $\mathfrak{S}$-*integrable* $\mathfrak{S}$-connection on ξ. Then it follows from condition (1) of Definition 1.7 (respectively, condition (1) of Definition 1.8), together with Remark 3.2.2 and Remark 3.3.1, that the morphism ${}^{\mathfrak{S}}\delta^{\times 2}$ determines a *bijection*

$$\mathrm{Lift}_{\mathfrak{S}_{\iota^1}}\big(({}^{\mathfrak{S}}\mathrm{pr}_2^2)^*\xi, ({}^{\mathfrak{S}}\mathrm{pr}_1^2)^*\xi; \nabla\big) \xrightarrow{\sim} \mathrm{Lift}_{\mathfrak{S}_{\underline{\iota}^{\times 2}}}\big(({}^{\mathfrak{S}}\mathrm{pr}_3^{\times 2})^*\xi, ({}^{\mathfrak{S}}\mathrm{pr}_1^{\times 2})^*\xi; ({}^{\mathfrak{S}}\underline{\delta}^{\times 2})^*\nabla\big).$$

Now since ∇ is $\mathfrak{S}$-*integrable*, which thus implies that the equality

$$({}^{\mathfrak{S}}\underline{\iota}^{\times 2})^*(\nabla^{\times 2}) = ({}^{\mathfrak{S}}\underline{\delta}^{\times 2})^*\nabla$$

holds, the isomorphism $\nabla^{\times 2}$ is an element of the codomain of this bijection. Thus, we obtain an element $\widetilde{\nabla}$ of the domain of this bijection whose image *coincides* with $\nabla^{\times 2}$, i.e.,

$$({}^{\mathfrak{S}}\delta^{\times 2})^*\widetilde{\nabla} = \nabla^{\times 2}.$$

On the other hand, since ∇ is an $\mathfrak{S}$-*connection*, it follows from Lemma 5.3, together with this equality, that $\widetilde{\nabla}$ is a 2-$\mathfrak{S}$-*connection* on ξ. This completes the proof of Lemma 5.4. $\square$

LEMMA 5.5. — *Let $n \geq 3$ be an integer. Then the map*

$${}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Cnn}^{n-1}(\xi)$$

[*cf. Lemma* 4.5] *is* **surjective**.

PROOF. — Let us recall that we have assumed that

- the stratification structure $\mathfrak{S}$ is *integrable* (respectively, *strictly integrable*), and that
- the category $\mathcal{F}$ fibered in groupoids over Sch_S is *integrable* (respectively, *weakly integrable*).

Let ∇ be an $(n-1)$-$\mathfrak{S}$-connection on ξ. Then it follows from Lemma 5.1 that

$$({}^{\mathfrak{S}}\delta^{\times n}_{1,n-1})^*(\nabla^{1,n-1}) = \nabla^{\times n} = ({}^{\mathfrak{S}}\delta^{\times n}_{n-1,1})^*(\nabla^{n-1,1}).$$

Next, let us observe that it follows from condition (2) of Definition 1.7 (respectively, condition (2) of Definition 1.8), together with Remark 3.2.3 and Remark 3.3.2, that the morphism ${}^{\mathfrak{S}}\delta^{n-1,1}$ determines a *bijection*

$$\mathrm{Lift}_{\mathfrak{S}_{\delta^{1,n-1}}}\big(({}^{\mathfrak{S}}\mathrm{pr}_2^n)^*\xi, ({}^{\mathfrak{S}}\mathrm{pr}_1^n)^*\xi; \nabla^{1,n-1}\big)$$

$$\xrightarrow{\sim} \mathrm{Lift}_{\mathfrak{S}_{\delta^{\times n}_{n-1,1}}}\big(({}^{\mathfrak{S}}\mathrm{pr}_3^{n-1,1})^*\xi, ({}^{\mathfrak{S}}\mathrm{pr}_1^{n-1,1})^*\xi; ({}^{\mathfrak{S}}\delta^{\times n}_{1,n-1})^*\nabla^{1,n-1}\big).$$

Next, let us observe that it follows from the two equalities of the first display of the present proof of Lemma 5.5 that $\nabla^{n-1,1}$ is an element of the codomain of this bijection. In particular, we obtain an element $\widetilde{\nabla}$ of the domain of this bijection whose image *coincides* with $\nabla^{n-1,1}$, i.e.,

$$({}^{\mathfrak{S}}\delta^{n-1,1})^*\widetilde{\nabla} = \nabla^{n-1,1}.$$

On the other hand, this equality, together with the second equality of the first display in the present proof of Lemma 5.5, implies the equalities

$$({}^{\mathfrak{S}}\delta^{\times n})^*\widetilde{\nabla} = ({}^{\mathfrak{S}}\delta^{\times n}_{n-1,1})^*({}^{\mathfrak{S}}\delta^{n-1,1})^*\widetilde{\nabla} = ({}^{\mathfrak{S}}\delta^{\times n}_{n-1,1})^*\nabla^{n-1,1} = \nabla^{\times n}.$$

Thus, it follows from Lemma 5.3 that $\widetilde{\nabla}$ is an *n-connection* on ξ. This completes the proof of Lemma 5.5. □

The first main result of the present paper is as follows.

THEOREM 5.6. — *Let S be a scheme, X a scheme which is* **smooth** *and* **separated** *over S,*

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big)$$

a stratification structure on X/S [cf. Definition 2.5],

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S, and ξ an object of $\mathcal{F}$ over X. Suppose that the following two conditions are satisfied:

(1) *The stratification structure $\mathfrak{S}$ is* **integrable** [*cf. Definition* 3.2] (*respectively,* **strictly integrable** [*cf. Definition* 3.3]).

(2) *The category $\mathcal{F}$ fibered in groupoids over Sch_S is* **integrable** [*cf. Definition* 1.7] (*respectively,* **weakly integrable** [*cf. Definition* 1.8]).

Then, for each integer $n \geq 2$, the natural maps

$${}^{\mathfrak{S}}\mathrm{Strt}(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{IntCnn}(\xi)$$

[*cf. Definition* 4.4*; Definition* 4.6*; Definition* 4.7, (ii)*; Lemma* 4.8] *are* **bijective**.

PROOF. — The injectivity of the two maps under consideration follows from Lemma 5.2. The surjectivity of the first map under consideration follows from Lemma 5.5. The surjectivity of the second map under consideration follows from Lemma 5.4 and Lemma 5.5. This completes the proof of Theorem 5.6. □

DEFINITION 5.7. — In the situation of Theorem 5.6, we shall refer to the bijection obtained by Theorem 5.6

$${}^{\mathfrak{S}}\mathrm{Strt}(\xi) \xrightarrow{\sim} {}^{\mathfrak{S}}\mathrm{IntCnn}(\xi)$$

as the *first fundamental correspondence*.

6. THE SECOND FUNDAMENTAL CORRESPONDENCE

In the present §6, we prove the second main result of the present paper [cf. Theorem 6.6 below]. In the present §6, let S be a scheme; X a scheme which is *smooth* and *separated* over S;

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big),$$

$$\mathfrak{T} = \big(({}^{\mathfrak{T}}P^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\iota^n \colon {}^{\mathfrak{T}}P^n \to {}^{\mathfrak{T}}P^{n+1})_{n\geq 0},$$
$$({}^{\mathfrak{T}}\sigma^n \colon {}^{\mathfrak{T}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{T}}\delta^{n_1,n_2} \colon {}^{\mathfrak{T}}P^{n_1,n_2} \to {}^{\mathfrak{T}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big)$$

stratification structures on X/S;

$$\Phi = (\Phi^n)_{n\geq 0} \colon \mathfrak{S} \longrightarrow \mathfrak{T}$$

a morphism of stratification structures;

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S;

$$\xi$$

an object of $\mathcal{F}$ over X; r a nonnegative integer. Suppose that

- the morphism Φ is *r-integrable* (respectively, *strictly r-integrable*), and that
- the category $\mathcal{F}$ fibered in groupoids over Sch_S is *integrable* (respectively, *weakly integrable*).

LEMMA 6.1. — *Let $m \leq n$ be nonnegative integers. Then the maps*

$${}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Cnn}^m(\xi), \qquad {}^{\mathfrak{T}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{T}}\mathrm{Cnn}^m(\xi)$$

[*cf. Lemma* 4.5] *are* **injective**.

PROOF. — This assertion follows immediately from a similar argument to the argument applied in the proof of Lemma 5.2, together with conditions (2), (3) of Definition 3.8 and condition (1) of Definition 3.9. ☐

LEMMA 6.2. — *Let n be a nonnegative integer. Then the map*

$$\Phi^* \colon {}^{\mathfrak{T}}\mathrm{qCnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{qCnn}^n(\xi)$$

[*cf. Definition* 4.11]*, hence also the maps*

$$\Phi^* \colon {}^{\mathfrak{T}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Cnn}^n(\xi), \qquad \Phi^* \colon {}^{\mathfrak{T}}\mathrm{Strt}(\xi) \longrightarrow {}^{\mathfrak{S}}\mathrm{Strt}(\xi),$$

[*Lemma* 4.12*,* (i)]*, is* **injective**.

PROOF. — This assertion follows from condition (3) of Definition 1.7 and condition (1) of Definition 3.8 and condition (1) of Definition 3.9. ☐

LEMMA 6.3. — *Let $n \geq r+1$ be an integer. Then the map*

$${}^{\mathfrak{T}}\mathrm{Cnn}^n(\xi) \longrightarrow {}^{\mathfrak{T}}\mathrm{Cnn}^{n-1}(\xi)$$

[*cf. Lemma* 4.5] *is* **surjective**.

PROOF. — This assertion follows immediately from a similar argument to the argument applied in the proof of Lemma 5.5, together with condition (3) of Definition 3.8 and condition (1) of Definition 3.9 [cf. also Remark 3.8.3 and Remark 3.9.2]. ☐

LEMMA 6.4. — *Let* ∇ *be a* $(\mathbf{\Phi}, \boldsymbol{r})$**-dormant** r-$\mathfrak{S}$-*connection on* ξ. *Then, for each positive integer* $n \leq r$, *there exists a quasi-*n*-*$\mathfrak{T}$*-connection* $\widetilde{\nabla}_n$ *on* ξ *such that*

$$(\underline{\Phi}^{r|n})^* \widetilde{\nabla}_n = ({}^{\Phi}\underline{\iota}^{r|n}_{r|r})^* \nabla.$$

[*Note that* ${}^{\Phi}\underline{P}^{r|r} = {}^{\mathfrak{S}}P^r$ — *cf. Definition* 2.10, (i).]

PROOF. — Let us recall that we have assumed that

- the morphism Φ is r-*integrable* (respectively, *strictly* r-*integrable*), and that
- the category $\mathcal{F}$ fibered in groupoids over Sch_S is *integrable* (respectively, *weakly integrable*).

Let us verify Lemma 6.4 by induction on n. Suppose that $n = 1$. Then it follows from condition (1) of Definition 1.7 (respectively, condition (1) of Definition 1.8), together with Remark 3.8.2 and Remark 3.9.1, that the morphism $\underline{\Phi}^{r|1}$ determines a *bijection*

$$\mathrm{Lift}_{{}^{\mathfrak{T}}\iota^0}\big(({}^{\mathfrak{T}}\mathrm{pr}^1_2)^*\xi, ({}^{\mathfrak{T}}\mathrm{pr}^1_1)^*\xi; \mathrm{id}_\xi\big) \xrightarrow{\sim} \mathrm{Lift}_{{}^{\Phi}\underline{\iota}^{r|0}_{r|1}}\big(({}^{\Phi}\underline{\mathrm{pr}}^{r|1}_2)^*\xi, ({}^{\Phi}\underline{\mathrm{pr}}^{r|1}_1)^*\xi; \mathrm{id}_{(\underline{\Phi}^{r|0})^*\xi}\big).$$

Then since ∇ is (Φ, r)-*dormant*, it follows from the definition of the (Φ, r)-*curvature* that the pull-back $({}^{\Phi}\underline{\iota}^{r|1}_{r|r})^*\nabla$ is an element of the codomain of this bijection. In particular, we obtain an element $\widetilde{\nabla}_1$ of the domain of this bijection whose image *coincides* with $({}^{\Phi}\underline{\iota}^{r|1}_{r|r})^*\nabla$, i.e.,

$$(\underline{\Phi}^{r|1})^* \widetilde{\nabla}_1 = ({}^{\Phi}\underline{\iota}^{r|1}_{r|r})^* \nabla,$$

as desired.

Next, suppose that $n \geq 2$, and that there exists a quasi-$(n-1)$-$\mathfrak{T}$-connection $\widetilde{\nabla}_{n-1}$ on ξ such that

$$(\underline{\Phi}^{r|n-1})^* \widetilde{\nabla}_{n-1} = ({}^{\Phi}\underline{\iota}^{r|n-1}_{r|r})^* \nabla.$$

Then it follows from condition (1) of Definition 1.7 (respectively, condition (1) of Definition 1.8), together with Remark 3.8.2 and Remark 3.9.1, that the morphism $\underline{\Phi}^{r|n}$ determines a *bijection*

$$\mathrm{Lift}_{{}^{\mathfrak{T}}\iota^{n-1}}\big(({}^{\mathfrak{T}}\mathrm{pr}^n_2)^*\xi, ({}^{\mathfrak{T}}\mathrm{pr}^n_1)^*\xi; \widetilde{\nabla}_{n-1}\big)$$

$$\xrightarrow{\sim} \mathrm{Lift}_{{}^{\Phi}\underline{\iota}^{r|n-1}_{r|n}}\big(({}^{\Phi}\underline{\mathrm{pr}}^{r|n}_2)^*\xi, ({}^{\Phi}\underline{\mathrm{pr}}^{r|n}_1)^*\xi; (\underline{\Phi}^{r|n-1})^*\widetilde{\nabla}_{n-1}\big).$$

Next, let us observe that since $(\underline{\Phi}^{r|n-1})^* \widetilde{\nabla}_{n-1} = ({}^{\Phi}\underline{\iota}^{r|n-1}_{r|r})^* \nabla$, it is immediate that the pull-back $({}^{\Phi}\underline{\iota}^{r|n}_{r|r})^*\nabla$ is an element of the codomain of this bijection. In particular, we obtain an element $\widetilde{\nabla}_n$ of the domain of this bijection whose image *coincides* with $({}^{\Phi}\underline{\iota}^{r|n}_{r|r})^*\nabla$, i.e.,

$$(\underline{\Phi}^{r|n})^* \widetilde{\nabla}_n = ({}^{\Phi}\underline{\iota}^{r|n}_{r|r})^* \nabla,$$

as desired. This completes the proof of Lemma 6.4. $\square$

LEMMA 6.5. — *Let $n \geq r$ be an integer and ∇ a* **(Φ, r)-dormant** *n-$\mathfrak{S}$-connection on ξ. Then there exists an n-$\mathfrak{T}$-connection $\widetilde{\nabla}$ on ξ such that $\Phi^*\widetilde{\nabla} = \nabla$.*

PROOF. — Let us first observe that it follows from Lemma 6.4 that there exists a quasi-r-$\mathfrak{T}$-connection $\widetilde{\nabla}_r$ such that

$$\Phi^*\widetilde{\nabla}_r = \nabla|_r.$$

Thus, since $\nabla|_r$ is an r-$\mathfrak{S}$-*connection* on ξ [cf. Lemma 4.5], it follows from Lemma 4.12, (ii), together with condition (1) of Definition 3.8 and condition (1) of Definition 3.9, that $\widetilde{\nabla}_r$ is an r-$\mathfrak{T}$-*connection* on ξ.

Next, let us observe that it follows from Lemma 6.3 that there exists an n-$\mathfrak{T}$-*connection* $\widetilde{\nabla}$ on ξ such that

$$\widetilde{\nabla}|_r = \widetilde{\nabla}_r.$$

Thus, it follows immediately from Lemma 6.1 that the equality

$$\Phi^*\widetilde{\nabla} = \nabla$$

holds. This completes the proof of Lemma 6.5. $\square$

The second main result of the present paper is as follows.

THEOREM 6.6. — *Let S be a scheme; X a scheme which is* **smooth** *and* **separated** *over S;*

$$\mathfrak{S} = \big(({}^{\mathfrak{S}}P^n)_{n\geq 0},\ ({}^{\mathfrak{S}}\iota^n \colon {}^{\mathfrak{S}}P^n \to {}^{\mathfrak{S}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{S}}\sigma^n \colon {}^{\mathfrak{S}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{S}}\delta^{n_1,n_2} \colon {}^{\mathfrak{S}}P^{n_1,n_2} \to {}^{\mathfrak{S}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big),$$

$$\mathfrak{T} = \big(({}^{\mathfrak{T}}P^n)_{n\geq 0},\ ({}^{\mathfrak{T}}\iota^n \colon {}^{\mathfrak{T}}P^n \to {}^{\mathfrak{T}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathfrak{T}}\sigma^n \colon {}^{\mathfrak{T}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathfrak{T}}\delta^{n_1,n_2} \colon {}^{\mathfrak{T}}P^{n_1,n_2} \to {}^{\mathfrak{T}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\big)$$

stratification structures on X/S [cf. Definition 2.5];

$$\Phi = (\Phi^n)_{n\geq 0} \colon \mathfrak{S} \longrightarrow \mathfrak{T}$$

a morphism of stratification structures [cf. Definition 2.8];

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S; ξ an object of $\mathcal{F}$ over X; r a nonnegative integer. Suppose that the following two conditions are satisfied:

(1) *The morphism Φ is* **r-integrable** *[cf. Definition 3.8] (respectively,* **strictly r-integrable** *[cf. Definition 3.9]).*

(2) *The category $\mathcal{F}$ fibered in groupoids over Sch_S is* **integrable** *[cf. Definition 1.7] (respectively,* **weakly integrable** *[cf. Definition 1.8]).*

Then the natural maps

$${}^{\mathfrak{T}}\mathrm{Strt}(\xi) \xrightarrow{\Phi^*} {}^{\Phi,r}\mathrm{DrmStrt}(\xi) \longrightarrow {}^{\Phi,r}\mathrm{DrmCnn}(\xi)$$

[*cf. Definition* 4.6*; Definition* 4.13*,* (ii)*,* (iii)*; Lemma* 4.14] *are* **bijective***.*

PROOF. — The injectivity of the first map under consideration follows from Lemma 6.2. The surjectivity of the first map under consideration follows from Lemma 6.2 and Lemma 6.5. The bijectivity of the second map under consideration follows from Lemma 6.1. This completes the proof of Theorem 6.6. □

DEFINITION 6.7. — In the situation of Theorem 6.6, we shall refer to the bijection obtained by Theorem 6.6

$$^{\mathfrak{T}}\mathrm{Strt}(\xi) \xrightarrow{\sim} {}^{\Phi,r}\mathrm{DrmCnn}(\xi)$$

as the *second fundamental correspondence*.

(Yuichiro Hoshi) RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN
Email address: yuichiro@kurims.kyoto-u.ac.jp