RIMS-1905

The reduced Dijkgraaf-Witten invariant of twist knots in the Bloch group of a finite field

By

Hiroaki KARUO

July 2019

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

The reduced Dijkgraaf–Witten invariant of twist knots in the Bloch group of a finite field

Hiroaki Karuo

Abstract

Let M be a closed oriented 3-manifold, and let G be a discrete group. We consider a representation $\rho\colon \pi_1(M) \to G$. For a 3-cocycle α, the Dijkgraaf–Witten invariant is given by $(\rho^*\alpha)[M]$, where $\rho^*: H^3(G) \to H^3(M)$ is the map indeced by ρ, and $[M]$ denotes the fundamental class of M. Noting that $(\rho^*\alpha)[M] = \alpha(\rho_*[M])$ where $\rho_*\colon H_3(M) \to H_3(G)$ is the map induced by ρ, we consider an equivalent invariant $\rho_*[M] \in H_3(G)$, and we also regard it as the Dijkgraaf–Witten invariant. In 2004, Neumann described the complex hyperbolic volume of M in terms of the image of the Dijkgraaf–Witten invariant for $G = \mathrm{SL}_2\mathbb{C}$ by the Bloch–Wigner map from $H_3(\mathrm{SL}_2\mathbb{C})$ to the Bloch group of $\mathbb{C}$.

In this paper, replacing $\mathbb{C}$ with a finite field $\mathbb{F}_p$, we calculate the reduced Dijkgraaf–Witten invariant of the complement of twist knots, where the reduced Dijkgraaf–Witten invariant is the image of the Dijkgraaf–Witten invariant for $\mathrm{SL}_2\mathbb{F}_p$ by the Bloch–Wigner map from $H_3(\mathbb{F}_p)$ to the Bloch group of $\mathbb{F}_p$.

1 Introduction

In 1990, Dijkgraaf and Witten [3] introduced a topological invariant for closed oriented 3-manifolds, called the Dijkgaaf–Witten invariant, from Chern–Simons theory; we abbreviate the invariant to the DW invariant. For a closed oriented 3-manifold M, a 3-cocycle α of a finite group G, and a representation $\rho\colon \pi_1(M) \to G$, the DW invariant is given by $(\rho^*\alpha)[M]$, where $[M]$ denotes the fundamental class of M, and $\rho^*\alpha$ is the pull back of α by ρ. Further, Wakui [18] reconstructed the DW invariant of M combinatorially from a 3-cocycle of G in terms of a triangulation of M. It is not easy to calculate the DW invariant by their way since a concrete presentation of a non-trivial 3-cocycle is often complicated in general. Noting that $(\rho^*\alpha)[M] = \alpha(\rho_*[M])$ where $\rho_*\colon H_3(M) \to H_3(G)$ is the map induced by ρ, we consider an equivalent invariant $\rho_*[M] \in H_3(G)$, and we also regard it as the DW invariant. For any algebraically closed field F, it is known that there exists a natural map from $H_3(\mathrm{SL}_2F;\mathbb{Z})$ to the Bloch group of F, called the Bloch–Wigner map, by collapsing a chain complex, which defines $H_3(\mathrm{SL}_2F;\mathbb{Z})$, by the action of SL_2F on $\mathbb{P}^1(F)$; see, for example, [4], [15], and [6]. Further, in 2004, Neumann [9] obtained the complex hyperbolic volume of M in terms of the image of $\rho_*[M]$ by a particular 3-cocycle of $\mathrm{PSL}_2\mathbb{C}$ to the extended Bloch group of $\mathbb{C}$, where the 3-cocycle is a map similar to the Bloch–Wigner map. Further, Hutchinson [6] gave a concrete construction of the Bloch–Wigner map, for any finite field F, from $H_3(\mathrm{SL}_2F;\mathbb{Z})$ to the Bloch group of F.

In the paper, we calculate the reduced DW invariant for the complement of n-twist knots ($n \geq 2$) and finite fields $\mathbb{F}_p$ ($p = 7,\ 11,$ and 13) (Theorems 3.1, 3.2, and 3.4), where the reduced DW invariant is the image of $\rho_*[M]$ by the Bloch–Wigner map from

$H_3(\mathrm{SL}_2\mathbb{F}_p)$ to the Bloch group of $\mathbb{F}_p$. Noting that the reduced DW invariant is naturally extended to an invariant of cusped 3-manifolds. An idea of the paper is to replace $\mathbb{C}$ of Neumann's construction [9] with finite fields. To show the main theorems, we prove that the reduced DW invariant can be calculated from moduli of ideal tetrahedra (Proposition 4.8). We expect that our invariant would be related to a "discrete version" of hyperbolic geometry. Although calculation of the DW invariant need many calculations to give non-trivial 3-cocycles explicitly even if the order of the group is lower, calculation of the reduced DW invariant avoids such complicated calculations and it is advantage that the reduced DW invariant is obtained more easily by only labelings of ideal vertices of an ideal triangulation by $\mathbb{P}^1(\mathbb{F}_p)(=\mathbb{F}_p \cup \{\infty\})$. We explain an outline of the proof of the theorems. When an ideal triangulation of a knot complement whose moduli does not collapse and $\mathbb{P}^1(\mathbb{F}_p)$-labeling of ideal vertices are given, we prove that the reduced DW invariant and the summation of moduli of ideal tetrahedra are equal (Proposition 4.8), and replace the ideal triangulation of the complement of the n-twist knot, which obtained by a method of Yokota [20], by an ideal triangulation whose moduli do not collapse, and calculate the reduced DW invariant by using moduli combinatorially.

The paper is organized as follows. In Section 2, we introduce the reduced DW invariant. In Section 3, as the main results, we give the concrete value of the reduced DW invariant of the complement of the n-twist knot ($n \geq 2$) and $\mathbb{F}_p$ ($p = 7, 11, 13$). In Section 4, we give combinatorial calculations of the reduced DW invariant from an ideal triangulation of the complement of the n-twist knot such that the modulus of any ideal tetrahedron does not collapse and a labeling for the ideal vertices of the ideal triangulation by $\mathbb{P}^1(\mathbb{F}_p)$. In Section 5, for each of $p = 7$, 11, and 13, we calculate the reduced DW invariants concretely by using the method which is given in Section 4, and give the proof of the theorems.

The author would like to thank Professor Tomotada Ohtsuki for encouraging him. The author would also like to thank Kevin Hutchinson, Takefumi Nosaka, and Michihisa Wakui for helpful comments.

2 Preliminaries

In this section, we review and define some definitions for theorems. In Section 2.1, we review the Bloch group of $\mathbb{F}_p$ and the third homology group of $\mathrm{SL}_2\mathbb{F}_p$. In Section 2.2, we review the way to construct an oriented closed 3-manifold from a knot complement called the Dehn filling and parabolic representations. In Section 2.3, we review the DW invariant and we introduce the reduced DW invariant.

2.1 The Bloch group of $\mathbb{F}_p$ and the third homology group of $\mathrm{SL}_2\mathbb{F}_p$

In this section we review the Bloch group of $\mathbb{F}_p$ and the third homology group of $\mathrm{SL}_2\mathbb{F}_p$. Let $\mathbb{F}_p$ be the finite field of order p, where p is an odd prime. The *pre-Bloch group* $\mathcal{P}(\mathbb{F}_p)$ of $\mathbb{F}_p$ is the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}(\mathbb{F}_p^\times \setminus \{1\})$ by all instances of the following relation.

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0 \quad (x, y \in \mathbb{F}_p^\times \setminus \{1\}) \tag{1}$$

The *Bloch group* $\mathcal{B}(\mathbb{F}_p)$ is the kernel of the map

$$\mathcal{P}(\mathbb{F}_p) \to \mathbb{F}_p^\times \underset{\mathbb{Z}}{\wedge} \mathbb{F}_p^\times \cong \mathbb{Z}/2\mathbb{Z} \qquad [z] \mapsto z \wedge (1-z).$$

Here, $\mathbb{F}_p^\times \underset{\mathbb{Z}}{\wedge} \mathbb{F}_p^\times$ means the second exterior power of $\mathbb{F}_p^\times$ over $\mathbb{Z}$.

Example 2.1 (Hutchinson [6])**.** It is known that

$$\mathcal{B}(\mathbb{F}_q) \cong \mathbb{Z}/\tfrac{q+1}{2}\mathbb{Z}$$

for any finite field $\mathbb{F}_q$ of odd characteristic by Hutchinson, where q is a power of an odd prime; see also [19].

Let $\check{\mathcal{P}}(\mathbb{F}_p)$ denote the quotient of pre-Bloch group $\mathcal{P}(\mathbb{F}_p)$ by all instances of the following relation.

$$[x] = \left[1 - \frac{1}{x}\right] = \left[\frac{1}{1-x}\right] = -\left[\frac{1}{x}\right] = -\left[\frac{x}{x-1}\right] = -[1-x] \quad (x \in \mathbb{F}_p \backslash \{0,1\}) \tag{2}$$

Let $\check{\mathcal{B}}(\mathbb{F}_p)$ denote the image of the Bloch group $\mathcal{B}(\mathbb{F}_p) \subset \mathcal{P}(\mathbb{F}_p)$ by the projective homomorphism $\mathcal{P}(\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p)$ induced by the relation (2).

Example 2.2. We give the proof of the following isomorphisms concretely in Appendix B; see also Ohtsuki [11].

$$\check{\mathcal{P}}(\mathbb{F}_7) \cong \mathbb{Z}/4\mathbb{Z},\ \check{\mathcal{B}}(\mathbb{F}_7) \cong \mathbb{Z}/2\mathbb{Z},$$
$$\check{\mathcal{P}}(\mathbb{F}_{11}) \cong \mathbb{Z}/2\mathbb{Z},\ \check{\mathcal{B}}(\mathbb{F}_{11}) = \{0\},$$
$$\check{\mathcal{P}}(\mathbb{F}_{13}) \cong \mathbb{Z}/7\mathbb{Z},\ \check{\mathcal{B}}(\mathbb{F}_{13}) \cong \mathbb{Z}/7\mathbb{Z}.$$

We review group homology; see, for example, [1] and [9] for details.

We review an algebraic definition of group homology. Let G be a discrete group. We consider the free module generated by $(n+1)$-tuple $(g_0, g_1, \ldots, g_n)$ of distinct $n+1$ elements of G. We consider a left action of G on this module given by

$$g(g_0, g_1, \ldots, g_n) = (gg_0, gg_1, \ldots, gg_n)\ (g \in G). \tag{3}$$

Let $C_n(G)$ denote the quotient module obtained from the above module by the action (3). We define a boundary map $\partial_n \colon C_n(G) \to C_{n-1}(G)$ by

$$\partial_n(g_0, g_1, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n).$$

Then $(C_*(G), \partial_*)$ forms a chain complex. It is known [1] that, for $k < |G|-1$, the homology group $H_k(C_*; \mathbb{Z})$ of $(C_*(G), \partial_*)$ and the homology $H_k(G; \mathbb{Z})$ of G are isomorphic. We note that, when $|G| \geq 4$, $H_3(G; \mathbb{Z}) \cong H_3(C_*; \mathbb{Z})$.

We review geometric definition of group homology. We regard the above $(g_0, g_1, \ldots, g_n)$ as an n-simplex, and consider a simplicial complex obtained by quotient of the action of G in (3). For degrees lower than $|G|$, this simplicial complex realizes simplexes of the degrees of the classifying space BG. Hence, for $k < |G| - 1$, since $H_k(BG; \mathbb{Z})$ and $H_k(C_*; \mathbb{Z})$ correspond and $H_k(BG; \mathbb{Z}) = H_k(G; \mathbb{Z})$ for any k, $H_k(C_*; \mathbb{Z})$ and $H_k(G; \mathbb{Z})$ correspond. Namely the geometric and algebraic definitions of group homology are consistent.

2.2 Dehn filling and parabolic representation

In this section, we review a method to construct closed orientable 3-manifolds from a knot complement, called Dehn filling, and parabolic representations.

We review Dehn fillings on a knot complement; for details, see Thurston [17]. Let K be a knot, and let $N(K)$ be an open tubular neighborhood of K in S^3. Let C be an essential simple closed curve on the boundary of $S^3 \setminus N(K)$. We can obtain a closed orientable 3-manifold by attaching a solid torus ($\cong D^2 \times S^1$) to $S^3 \setminus N(K)$ by a homomorphism from $\partial(S^3 \setminus N(K))$ to the boundary of the solid torus which takes C to the meridian of the boundary of the solid torus. We call this operation the *Dehn filling* of K along C. We take a meridian and a longitude of torus as a basis of $\pi_1(S^3 \setminus N(K))$. When C is presented by (a, b), we let $M_{a,b}(K)$ denote the closed 3-manifold obtained from $S^3 \setminus N(K)$ by Dehn filling along C.

When a parabolic representation $\rho\colon \pi_1(S^3 \setminus N(K)) \to \mathrm{SL}_2\mathbb{F}_{p^n}$ is given, we prove that there exists an (a, b) such that the following diagram commutes.

$$\begin{tikzcd}
\pi_1(S^3 \setminus N(K)) \arrow[r] \arrow[d] & \mathrm{SL}_2\mathbb{F}_p \\
\pi_1(M_{(a,b)}(K)) \arrow[ur] &
\end{tikzcd}$$

We consider an essential simple closed curve on $\partial(S^3 \setminus N(K))$ from an $\mathrm{SL}_2\mathbb{F}_{p^n}$ representation of the knot group. Let a representation $\rho\colon \pi_1(S^3 \setminus N(K)) \to \mathrm{SL}_2\mathbb{F}_p$ be given. Let ρ also denote the restrict of ρ to $\pi_1(\partial(S^3 \setminus N(K)))$. $\partial(S^3 \setminus N(K))$ is homeomorphic to a torus ($\cong S^1 \times S^1$). When we take meridian ($S^1 \times \{\text{a point}\}$) and longitude ($\{\text{a point}\} \times \mathrm{S}^1$) as a basis, we have $\pi_1(\partial(S^3 \setminus N(K))) \cong \mathbb{Z} \oplus \mathbb{Z}$. Moreover, since the above and that $\mathrm{SL}_2\mathbb{F}_p$ is a finite group, the kernel of $\pi_1(\partial(S^3 \setminus N(K))) \to \mathrm{SL}_2\mathbb{F}_p$ is non-trivial. Hence, one can take $(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \setminus \{(0, 0)\}$ from the kernel. In particular, for ρ, by Lemma D.1, one can take an (a, b) such that a and b are coprime; we note that there is ambiguity of the choice of (a, b). The above $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ is an essential simple closed curve on a torus, we call the curve the (a, b)-*curve* of ρ. We note that $\pi_1(M_{a,b}(K)) \to \mathrm{SL}_2\mathbb{F}_p$ is induced from ρ since the (a, b)-curve is in the kernel of ρ.

Let $\overline{\mathbb{F}_p}$ denote the algebraic closure of $\mathbb{F}_p$. For $A,\ B \in \mathrm{SL}_2\mathbb{F}_p$, A and B are *conjugate* if there exists $P \in \mathrm{SL}_2\overline{\mathbb{F}_p}$ such that

$$P^{-1}AP = B.$$

For a knot K, let $\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SL}_2\mathbb{F}_p$ is given. We call ρ is a *parabolic representation* if the image of each meridian by ρ is conjugate to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ ($* \in \mathbb{F}_p^{\times}$). Here, we note that all the image of meridians by ρ are conjugate.

2.3 Dijkgraaf–Witten invariant and reduced Dijkgraaf–Witten invariant

In this subsection, we review the Dijkgraaf–Witten invariant and define the reduced Dijkgraaf–Witten invariant of knot complements and finite fields.

We review the Dijkgraaf–Witten invariant; See [3], [18] for details. Let G be a discrete group and let BG be a classifying space of G. We note that $H_3(BG)$ and $H_3(G)$ can be identified as we mentioned in Section 2.1. Let M be an oriented closed 3-manifold. We consider a representation $\rho\colon \pi_1(M) \to G$. Then a map $\rho_*\colon M \to BG$ is induced from ρ. In addition, a homomorphism $\rho_*\colon H_3(M) \to H_3(BG) = H_3(G)$ is induced from ρ_*. For the fundamental class $[M] \in H_3(M)$ of M, we call $\rho_*([M]) \in H_3(G, \mathbb{Z})$ *the Dijkgraaf–Witten invariant* of (M, ρ) (we abbreviate the invariant to the DW invariant), and we denote it by $\mathrm{DW}(M, \rho)$.

Let an oriented closed 3-manifold M and a representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{F}_p$ be given. We consider the composite $H_3(\mathrm{SL}_2\mathbb{F}_p;\mathbb{Z}) \to \check{\mathcal{B}}(\mathbb{F}_p)$ of the map $H_3(\mathrm{SL}_2\mathbb{F}_p;\mathbb{Z}) \to \mathcal{B}(\mathbb{F}_p)$ given by Hutchinson [6] and the restriction $\mathcal{B}(\mathbb{F}_p) \to \check{\mathcal{B}}(\mathbb{F}_p)$ of the projection map $\mathcal{P}(\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p)$ to $\mathcal{B}(\mathbb{F}_p)$. We call the image of $\mathrm{DW}(M, \rho) \in H_3(\mathrm{SL}_2\mathbb{F}_p;\mathbb{Z})$ by the composite *the reduced Dijkgraaf–Witten invariant* of (M, ρ) (we abbreviate the invariant to the reduced DW invariant) and we denote it by $\widehat{\mathrm{DW}}(M, \rho) \in \check{\mathcal{B}}(\mathbb{F}_p)$.

For a knot K, we consider a parabolic representation $\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SL}_2\mathbb{F}_p$. Then, as we mentioned in Section 2.2, we can take an (a, b)-curve of ρ, and, for the closed 3-manifold $M_{a,b}(K)$ obtained by Dehn filling of the complement of K, a representation $\pi_1(M_{a,b}(K)) \to \mathrm{SL}_2\mathbb{F}_p$ is induced from ρ. We also denote the map by ρ. In general, $M_{a,b}(K)$ is not unique, depending on the choice of (a, b)-curve, but the reduced DW invariant $\widehat{\mathrm{DW}}(M_{a,b}(K), \rho)$ does not depend on the choice of (a, b)-curve by Lemma G.1 in Appendix G. In the following, we call $\widehat{\mathrm{DW}}(M_{a,b}(K), \rho)$ *the reduced DW invariant* of (K, ρ), and we denote it by $\widehat{\mathrm{DW}}(K, \rho)$.

In general, calculations of the DW invariant $\mathrm{DW}(M_{a,b}(K), \rho)$ is complicated since it needs to calculate the homology group of $\mathrm{SL}_2\mathbb{F}_p$. However, when an ideal triangulation of the complement of K and uncollapsed moduli of ideal tetrahedra are given, the reduced DW invariant $\widehat{\mathrm{DW}}(K, \rho)$ can be calculated combinatorially as the sum of moduli regarded as elements of $\check{\mathcal{P}}(\mathbb{F}_p)$.

We consider the value obtained by summing up $\widehat{\mathrm{DW}}(K, \rho)$ over all conjugacy classes of parabolic representations in $\mathbb{Z}[\check{\mathcal{B}}(\mathbb{F}_p)]$ and call it *the reduced DW invariant* of K and $\mathbb{F}_p$, and we denote it by $\widehat{\mathrm{DW}}(K, \mathbb{F}_p)$.

Remark 2.3. By the difinition of third homology group in Section 2.1, we need that $p \geq 4$. Hence, we are not able to define the reduced DW invariant for $\mathbb{F}_p$ ($p = 2$ and 3).

3 Main results

In this section, we give the calculations of the reduced DW invariant of n-twist knots ($n \geq$2) and $\mathbb{F}_p$ ($p = 7, 11, 13$).

The *n-twist knot* is the knot shown in Figure 1, and we denote it by $\mathcal{T}_n$.

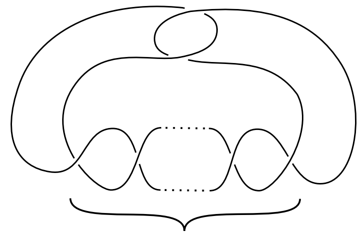

Figure 1: n-twist knot

Theorem 3.1 ($p=7$)**.** By the isomorphism $\check{\mathcal{B}}(\mathbb{F}_7) \cong \mathbb{Z}/2\mathbb{Z}$ in Lemma B.2, we identify $\check{\mathcal{B}}(\mathbb{F}_7)$ and $\mathbb{Z}/2\mathbb{Z}$, and we identify an additive group $\mathbb{Z}/2\mathbb{Z}$ and a multiplicative group $\langle\, t \,|\, t^2 = 1 \,\rangle$ naturally. Then the reduced DW invariant is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \mathbb{F}_7) = A_n + B_n \in \mathbb{Z}[\langle\, t \,|\, t^2 = 1 \,\rangle]$$

Here,

$$A_n = \begin{cases} t & \text{if } n \equiv 2, 3 \pmod 6, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_n = \begin{cases} t^{(n-1)/8} & \text{if } n \equiv 1 \pmod 8, \\ t^{(n+6)/8} & \text{if } n \equiv 2 \pmod 8, \\ t^{(n-5)/8} & \text{if } n \equiv 5 \pmod 8, \\ t^{(n+2)/8} & \text{if } n \equiv 6 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

We give a proof of Theorem 3.1 in Section 5.

Theorem 3.2 ($p=11$)**.** Since $\check{\mathcal{B}}(\mathbb{F}_{11}) = \{0\}$ in Lemma B.4, we have that $\mathbb{Z}[\check{\mathcal{B}}(\mathbb{F}_{11})] = \mathbb{Z}$. Then the reduced DW invariant is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \mathbb{F}_{11}) = A_n + B_n \in \mathbb{Z}$$

Here,

$$A_n = \begin{cases} 1 & \text{if } n \equiv 1, 3 \pmod 5, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_n = \begin{cases} 1 & \text{if } n \equiv 4, 5, 6, 7 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

We give a proof of the theorem in Section 5.

Remark 3.3. Since $\check{\mathcal{B}}(\mathbb{F}_{11}) = \{0\}$, the DW invariant of the theorem is equal to the number of the conjugacy classes of parabolic representations.

Theorem 3.4 ($p=13$)**.** By the isomorphism $\check{\mathcal{B}}(\mathbb{F}_{13}) \cong \mathbb{Z}/7\mathbb{Z}$ in Lemma B.6, we identify $\check{\mathcal{B}}(\mathbb{F}_{13})$ and $\mathbb{Z}/7\mathbb{Z}$, and we identify an additive group $\mathbb{Z}/7\mathbb{Z}$ and a multiplicative group $\langle\, t \,|\, t^7 = 1 \,\rangle$ naturally. Then the reduced DW invariant is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \mathbb{F}_{13}) = A_n + B_n \in \mathbb{Z}[\langle\, t \,|\, t^7 = 1 \,\rangle].$$

Here,

$$A_n = \begin{cases} t & \text{if } n \equiv 10 \pmod{13}, \\ t^4 & \text{if } n \equiv 2 \pmod{13}, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_n = \begin{cases} t^{(n-1)/7} & \text{if } n \equiv 1 \pmod 7 \\ t^{(2n+17)/7} & \text{if } n \equiv 2 \pmod 7, \\ t^{(2n-15)/7} & \text{if } n \equiv 4 \pmod 7, \\ t^{(n+2)/7} & \text{if } n \equiv 5 \pmod 7, \\ 0 & \text{otherwise.} \end{cases}$$

We give a proof of the theorem in Section 5.

Remark 3.5. The reduced DW invariant does not always consist of two different periodic terms. For example, in the case of $p = 17$, the reduced DW invariant consists of three different periodic terms.

4 Preliminaries for the proof of theorems

In this section, we prove some lemmas and propositions for the proof of theorems. In Section 4.1, we review a map found by Hutchinson [6] explicitly. In Section 4.2, we prove that the reduced DW invariant is calculated by an ideal triangulation of a knot complement and $\mathbb{P}^1(\mathbb{F}_p)$-labeling of ideal vertices. In Section 4.3, we prove that the conjugacy classes of parabolic representations of $\pi_1(S^3 \setminus \mathcal{T}_n)$ and zeros of a polynomial correspond. In Section 4.4, we review hyperbolicity equations for a knot diagram. In Section 4.5, we review hyperbolicity equations for an ideal triangulation, and review that this is equivalent to the hyperbolicity equations for the knot diagram. Moreover, we classify the conjugacy classes of parabolic representations into three types by representing the classes by sequences, and we give an ideal triangulation of the knot complement for each of types again, and we give a method to calculate the reduced DW invariant.

4.1 The third homologies of $\mathrm{SL}_2\mathbb{F}_p$ and $\mathrm{PGL}_2\mathbb{F}_p$ and pre-Bloch group

In this section, we review the map $H_3(\mathrm{SL}_2\mathbb{F}_p) \to \mathcal{P}(\mathbb{F}_p)$ which is given by using spectral sequences by Hutchinson [6]. This map is also obtained by extending a natural map $H_3(\mathrm{PGL}_2\mathbb{F}_p) \to \mathcal{P}(\mathbb{F}_p)$. In particular, we review a relation between these construction, and describe $H_3(\mathrm{SL}_2\mathbb{F}_p) \to \hat{\mathcal{P}}(\mathbb{F}_p)$ explicitly.

We set $\hat{G} = \mathrm{SL}_2\mathbb{F}_p$ and $X_m = \{m\text{-tuples of distinct } m \text{ points of } \mathbb{P}^1(\mathbb{F}_p)\}$. We define a homomorphism $\delta\colon \mathbb{Z}[X_{m+1}] \to \mathbb{Z}[X_m]$ by

$$\delta(x_0, x_1, \ldots, x_m) = \sum_{i=0}^{m} (-1)^i (x_0, \ldots, \hat{x_i}, \ldots x_m).$$

Then $(\mathbb{Z}[X_{*+1}], \delta)$ is a chain complex. We set $\hat{Y}_n = \{n\text{-tuples of distinct } n \text{ elements of } \hat{G}\}$. We define a homomorphism $\partial\colon \mathbb{Z}[\hat{Y}_{n+1}] \to \mathbb{Z}[\hat{Y}_n]$ by

$$\partial(g_0, g_1, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots g_n).$$

Then $(\mathbb{Z}[\hat{Y}_{*+1}], \partial)$ is a chain complex. We consider diagonal actions of $g \in \hat{G}$ on X_{m+1} and $\hat{Y}_{n+1}$ respectively. Then we define an action of $g \in \hat{G}$ on $\hat{Y}_{n+1} \times X_{m+1}$ by

$$g((g_0, g_1, \ldots, g_n), (z_0, z_1, \ldots, z_m)) = (g(g_0, g_1, \ldots, g_n), g(z_0, z_1, \ldots, z_m)).$$

We consider the $\hat{E}^0$-page of the spectral sequence of a double complex $\left(\mathbb{Z}\left[\dfrac{\hat{Y}_{n+1} \times X_{m+1}}{\hat{G}}\right], \partial, (-1)^n \delta\right)$ in Section 4.4 in [6]. Since $\hat{E}^0_{n,m} = \mathbb{Z}\left[\dfrac{\hat{Y}_{n+1} \times X_{m+1}}{\hat{G}}\right]$ and $d^0 = \partial$, $\hat{E}^0$-page is shown as follows.

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
\mathbb{Z}\left[\frac{\hat{Y}_1 \times X_4}{\hat{G}}\right] \longleftarrow \cdots \arrow[d, "\delta"] & \vdots \arrow[d] & \\
\mathbb{Z}\left[\frac{\hat{Y}_1 \times X_3}{\hat{G}}\right] \arrow[d, "\delta"] & \mathbb{Z}\left[\frac{\hat{Y}_2 \times X_3}{\hat{G}}\right] \arrow[l, "\partial"'] \arrow[d, "-\delta"] & \cdots \arrow[l] \\
\mathbb{Z}\left[\frac{\hat{Y}_1 \times X_2}{\hat{G}}\right] \arrow[d, "\delta"] & \mathbb{Z}\left[\frac{\hat{Y}_2 \times X_2}{\hat{G}}\right] \arrow[l, "\partial"'] \arrow[d, "-\delta"] & \cdots \arrow[l] \\
\mathbb{Z}\left[\frac{\hat{Y}_1 \times X_1}{\hat{G}}\right] & \mathbb{Z}\left[\frac{\hat{Y}_2 \times X_1}{\hat{G}}\right] \arrow[l, "\partial"'] & \cdots \arrow[l]
\end{tikzcd}$$

Next, we consider the $\hat{E}^1$-page. Since $\partial((g_0, g_1), (z_0, z_1, z_2, z_3)) = ((g_1), (z_0, z_1, z_2, z_3)) - ((g_0), (z_0, z_1, z_2, z_3))$ and $\hat{E}^1_{0,3} = \dfrac{\mathrm{kernel}(\hat{E}^0_{0,3} \to 0)}{\mathrm{image}(\hat{E}^0_{1,3} \to \hat{E}^0_{0,3})}$, we obtain that $\hat{E}^1_{0,3} = \mathbb{Z}\left[\dfrac{X_4}{\hat{G}}\right]$. We set $R_F = \mathbb{Z}[\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2] \cong \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \cong \mathbb{Z}[\{\pm 1\}]$ and $Z_k = \{k\text{-tuples of distinct } k \text{ points of } \mathbb{F}_p^\times \setminus \{1\}\}$. Then it is known that $\hat{E}^1_{0,3} \cong R_F[Z_1]$; see [6]. Further it follows that

$d^1 = (-1)^n\delta$. Then $\hat{E}^1$-page is shown as follows; see [6] for the details.

$$\begin{array}{c}
\vdots \\
\downarrow \\
\hat{E}^1_{0,4} = R_F[Z_2] \\
\downarrow d^1 \\
\hat{E}^1_{0,3} = R_F[Z_1] \\
\downarrow d^1 \\
\hat{E}^1_{0,2} = R_F \\
\downarrow d^1 \\
\mathbb{Z} \\
\downarrow d^1 \\
\mathbb{Z}
\end{array}$$

As in [6], we set $\mathcal{RP}(\mathbb{F}_p) = \hat{E}^1_{0,3}/\mathrm{image}\, d^1$. Since $d^1 = (-1)^n\delta\colon \hat{E}^1_{n,m} \to \hat{E}^1_{n,m-1}$ in the $\hat{E}^1$-page, it is follows that $\hat{E}^2_{0,3} = \dfrac{\mathrm{kernel}(\hat{E}^1_{0,3} \to \hat{E}^1_{0,2})}{\mathrm{image}(\hat{E}^1_{0,4} \to \hat{E}^1_{0,3})}$. For $k \geq 2$, since $d^k\colon \hat{E}^k_{n,m} \to \hat{E}^2_{n+k-1,m-k}$ in the $\hat{E}^k$-page and we consider a first quadrant spectral sequence, $\hat{E}^{k+1}_{0,3} = \dfrac{\mathrm{kernel}(\hat{E}^k_{0,3} \to \hat{E}^k_{k-1,3-k})}{\mathrm{image}(\hat{E}^k_{-k+1,k+3} \to \hat{E}^k_{0,3})} = \mathrm{kernel}(\hat{E}^k_{0,3} \to \hat{E}^k_{k-1,3-k})$. Moreover we consider a first quadrant spectral sequence, for $k \geq 4$, $d^k\colon \hat{E}^k_{0,3} \to \hat{E}^k_{k-1,3-k}$ is a zero map. Hence we obtain that

$$\hat{E}^2_{0,3} \supset \hat{E}^3_{0,3} \supset \hat{E}^4_{0,3} = \hat{E}^5_{0,3} = \cdots = \hat{E}^\infty_{0,3}.$$

On the other hand, there is a filtration of $H_3(\hat{G})$

$$H_3(\hat{G}) \supset \hat{F}_0 \supset \hat{F}_1 \supset \hat{F}_2 \supset \hat{F}_3 \supset \{0\}$$

and we obtain a quotient $H_3(\hat{G})/\hat{F}_0 \cong \hat{E}^\infty_{0,3}$ from the filtration. Then we obtain that a natural homomorphism $H_3(\hat{G}) \to H_3(\hat{G})/\hat{F}_0 \cong \hat{E}^\infty_{0,3}$. In particular, since $\hat{E}^\infty_{0,3} \subset \hat{E}^2_{0,3}$, we obtain that

$$H_3(\hat{G}) \to \hat{E}^2_{0,3} = \mathcal{RP}(\mathbb{F}_p). \tag{4}$$

We set $G = \mathrm{PGL}_2\mathbb{F}_p$. We consider a spectral sequence obtained by replacing $\hat{G}$ to G in the above spectral sequence. We set $Y_n = \{n\text{-tuples of distinct } n \text{ elements of } G\}$. We define a homomorphism $\partial\colon \mathbb{Z}[Y_{n+1}] \to \mathbb{Z}[Y_n]$ by

$$\partial(g_0, g_1, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots g_n).$$

Then $(\mathbb{Z}[Y_{*+1}], \partial)$ is a chain complex. We consider the E^1-page of the spectral sequence of a double complex $\left(\mathbb{Z}\left[\dfrac{Y_{n+1} \times X_{m+1}}{G}\right], \partial, (-1)^n\delta\right)$. Since $\partial((g_0, g_1), (z_0, z_1, z_2, z_3)) =$

$((g_1),(z_0,z_1,z_2,z_3)) - ((g_0),(z_0,z_1,z_2,z_3))$ and $E^1_{0,3} = \dfrac{\text{kernel}(E^0_{0,3} \to 0)}{\text{image}(E^0_{1,3} \to E^0_{0,3})}$, we obtain that $E^1_{0,3} = \mathbb{Z}\left[\dfrac{X_4}{G}\right]$. We consider an action of G on X_4. Then the action of G can send arbitrary distinct three points of $\mathbb{P}^1(\mathbb{F}_p)$ to arbitrary distinct three points. Such elements of G are unique. We let z denote the image of z_3 by an element of G which sends (z_0,z_1,z_2) to $(0,\infty,1)$, and consider a correspondence from (z_0,z_1,z_2,z_3) to z. By this correspondence, we identify $\dfrac{X_4}{G}$ and Z_1. We note that this correspondence is shown as $z = \dfrac{(z_3-z_0)(z_2-z_1)}{(z_3-z_1)(z_2-z_0)}$. Then we obtain that $E^1_{0,3} = \mathbb{Z}\left[\dfrac{X_4}{G}\right] \cong \mathbb{Z}[Z_1]$. Further it follows that $d^1 = (-1)^n\delta$. Then E^1-page is shown as follows.

$$
\begin{array}{c}
\vdots \\
\downarrow \\
\mathbb{Z}[Z_2] \\
\downarrow d^1 \\
\mathbb{Z}[Z_1] \\
\downarrow 0 \\
\mathbb{Z} \\
\downarrow \mathrm{id} \\
\mathbb{Z} \\
\downarrow 0 \\
\mathbb{Z}
\end{array}
$$

We set $\mathcal{P}(\mathbb{F}_p) = \mathbb{Z}[Z_1]/\text{Image}\, d^1$. By a projection $\mathcal{RP}(\mathbb{F}_p) \to \mathcal{RP}(\mathbb{F}_p)/\mathbb{F}_p^\times$ and an isomorphism $\mathcal{RP}(\mathbb{F}_p)/{\mathbb{F}_p}^\times \cong \mathcal{P}(\mathbb{F}_p)$ in Lemma 2.4 in [6], we obtain a homomorphism

$$\mathcal{RP}(\mathbb{F}_p) \to \mathcal{P}(\mathbb{F}_p). \tag{5}$$

Similarly as the case of $\hat{E}$, we obtain that

$$E^2_{0,3} \supset E^3_{0,3} \supset E^4_{0,3} = E^5_{0,3} = \cdots = E^\infty_{0,3}.$$

Since there is a filtration of $H_3(G)$

$$H_3(G) \supset F_0 \supset F_1 \supset F_2 \supset F_3 \supset \{0\},$$

we obtain the quotient $H_3(G)/F_0 \cong E^\infty_{0,3}$ from the filtration. Then we obtain a natural projective homomorphism $H_3(G) \to H_3(G)/F_0 \cong E^\infty_{0,3}$. In particular, since $E^\infty_{0,3} \subset E^2_{0,3}$, we obtain that

$$H_3(G) \to E^2_{0,3} = \mathcal{P}(\mathbb{F}_p). \tag{6}$$

We describe $H_3(\mathrm{SL}_2\mathbb{F}_p) \to \mathcal{P}(\mathbb{F}_p)$ by using $H_3(\mathrm{PGL}_2\mathbb{F}_p)$. By (4), (5), (6), and

$$H_3(\mathrm{SL}_2\mathbb{F}_p;\mathbb{Z}) \to H_3(\mathrm{PGL}_2\mathbb{F}_p;\mathbb{Z})$$

induced from

$$\mathrm{SL}_2\mathbb{F}_p \to \mathrm{GL}_2\mathbb{F}_\mathrm{p} \to \mathrm{PGL}_2\mathbb{F}_p,$$

We obtain the following diagram.

$$\begin{array}{ccccc} H_3(\mathrm{SL}_2\mathbb{F}_p;\mathbb{Z}) & \longrightarrow & \hat{E}^2_{0,3} & \longrightarrow & \mathcal{RP}(\mathbb{F}_p) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ H_3(\mathrm{PGL}_2\mathbb{F}_p;\mathbb{Z}) & \longrightarrow & E^2_{0,3} & \longrightarrow & \mathcal{P}(\mathbb{F}_p) \end{array} \tag{7}$$

Since the right two vertical maps in (7) are obtained by taking a quotient of the action of $\mathbb{F}_p^\times$, the right square of (7) is commutative. While a natural map $\mathrm{SL}_2\mathbb{F}_p \to \mathrm{PGL}_2\mathbb{F}_p$ induces a map from a double complex of $\mathbb{Z}\left[\dfrac{\hat{Y}_{n+1}\times X_{m+1}}{\hat{G}}\right]$ to a double complex of $\mathbb{Z}\left[\dfrac{Y_{n+1}\times X_{m+1}}{G}\right]$. Then it induces a map between spectral sequences of them. Moreover, $\mathrm{SL}_2\mathbb{F}_p \to \mathrm{PGL}_2\mathbb{F}_p$ also induces $H_3(\mathrm{SL}_2\mathbb{F}_p;\mathbb{Z}) \to H_3(\mathrm{PGL}_2\mathbb{F}_p;\mathbb{Z})$, and two horizontal maps on the left-hand side of (7) is obtained from a spectral sequence. Hence the left square of (7) is commutative. Therefore we represent $H_3(\mathrm{SL}_2\mathbb{F}_p;\mathbb{Z}) \to \mathcal{RP}(\mathbb{F}_p) \to \mathcal{P}(\mathbb{F}_p)$ by a composite

$$H_3(\mathrm{SL}_2\mathbb{F}_p;\mathbb{Z}) \to H_3(\mathrm{PGL}_2\mathbb{F}_p;\mathbb{Z}) \to \mathcal{P}(\mathbb{F}_p). \tag{8}$$

We construct a complex from $\{m\text{-tuples of } m \text{ elements of } \mathbb{P}^1(\mathbb{F}_p)\}$ which is an extension of X_m, and we describe (8) by using the complex. In the case of $\mathbb{C}$, Dupont–Zickert [5] mentioned that the images obtained maps are equal if we replace m-tuples of distinct m points by m-tuples of m points which allow duplicate points. We set $\hat{X}_m = \{m\text{-tuples of } m \text{ points of } \mathbb{P}^1(\mathbb{F}_p)\}$, and $C_m = \mathbb{Z}[\hat{X}_{m+1}]$. Further we define a homomorphism $\delta\colon C_m \to C_{m-1}$ by

$$\delta(z_0, z_1, \ldots, z_m) = \sum_{i=0}^{m} (-1)^i (z_0, \ldots, \hat{z}_i, \ldots z_m).$$

Then (C_*, δ) is a chain complex. In particular, the following lemma holds.

Lemma 4.1 (see [6])**.** $H_n(C_*, \mathbb{Z}) = 0$ for any $n \neq 0,\ p$.

Although we omit the proof, we can show the lemma similarly as the proof of Lemma 4.4 of [6].

We consider a diagonal action of G on $\hat{X}_{m+1}$, and set $\hat{C}_m = \mathbb{Z}\left[\dfrac{\hat{X}_{m+1}}{G}\right]$. Further we define $\hat{\delta}\colon \hat{C}_n \to \hat{C}_{n-1}$ by

$$\hat{\delta}(z_0, z_1, \ldots, z_n) = \sum_{i=0}^{n} (-1)^i (z_0, \ldots, \hat{z}_i, \ldots, z_n).$$

Then $(\hat{C}_*, \hat{\delta})$ is a chain complex. For $z \in \mathbb{P}^1(\mathbb{F}_p)$, we define a homomorphism

$$H_3(\mathrm{PGL}_2\mathbb{F}_p) \to H_3(\hat{C}_m), \quad (g_0, g_1, g_2, g_3) \mapsto (g_0 z, g_1 z, g_2 z, g_3 z). \tag{9}$$

For $z_0, z_1, z_2, z_3 \in \mathbb{P}^1(\mathbb{F}_p)$, we define

$$[z_0, z_1, z_2, z_3] = \begin{cases} \left[\dfrac{(z_3 - z_0)(z_2 - z_1)}{(z_3 - z_1)(z_2 - z_0)}\right] & \text{if } z_i \neq z_j \text{ for all distinct } i \text{ and } j, \\ 0 & \text{if } z_i = z_j \text{ for some distinct } i \text{ and } j. \end{cases} \tag{10}$$

Since $[z_0, z_1, z_2, z_3]$ is an element of $\mathbb{P}^1(\mathbb{F}_p)$ when z_0, z_1, z_2, z_3 are distinct, one can regard the element as an element of $\check{\mathcal{P}}(\mathbb{F}_p)$. Hence we obtain a homomorphism

$$\hat{C}_3 \to \mathcal{P}(\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p), \quad (z_0, z_1, z_2, z_3) \mapsto [z_0, z_1, z_2, z_3] \tag{11}$$

Lemma 4.2. The map (11) sends boundaries of $\hat{C}_3$ to zeros.

Proof. Since (10) does not change by the action of $\mathrm{PGL}_2\mathbb{F}_p$, the image of $(z_0, z_1, z_2, z_3, z_4) \in \hat{C}_4$ also does not change by the action. To prove the lemma, it is enough to prove that the image of $\hat{\delta}\colon \hat{C}_4 \to \hat{C}_3$ is in the kernel of (11). We consider $(z_0, z_1, z_2, z_3, z_4) \in \hat{C}$. We classify z_0, z_1, z_2, z_3, z_4 according to the number of duplicate elements.

We consider the case where all z_0, z_1, z_2, z_3, and z_4 are distinct. Then the image of $(z_0, z_1, z_2, z_3, z_4)$ by $\hat{\delta}$ is

$$\sum_i (-1)^i (z_0, \ldots, \hat{z}_i, \ldots, z_4), \tag{12}$$

this is mapped to zero by (1).

We consider the case where there is only one duplicate pair of z_0, z_1, z_2, z_3, and z_4. Then the image of $(z_0, z_1, z_2, z_3, z_4)$ by $\hat{\delta}$ is (12). Since three terms of that are 4-tuples which have duplicate elements, these terms are mapped to zeros, and it is enough to consider the remained two terms. These two terms are 4-tuples which correspond to the other by permutation. In particular, by concrete calculation, the signs of these are opposite if these correspond to the other by even permutation, and are same if these correspond to the other by odd permutation. By even permutation of a, b, c, and d, $z = [a, b, c, d]$ changes to z, $1 - \frac{1}{z}$, or $\frac{1}{1-z}$, by odd permutation, z changes to $\frac{1}{z}$, $\frac{z}{z-1}$, or $1 - z$. By those and (2), the summation of remained two terms is mapped to zero. Hence the image of $(z_0, z_1, z_2, z_3, z_4)$ by $\hat{\delta}$ is mapped to zero.

We consider the case where there is only one set of three or more duplicate elements of z_0, z_1, z_2, z_3, z_4. Then each term of the image of $(z_0, z_1, z_2, z_3, z_4)$ by $\hat{\delta}$ is a 4-tuple which has duplicate elements. Hence the image is mapped to zero.

We consider the case where there are two set which consist duplicate elements of z_0, z_1, z_2, z_3, z_4. Then each term of the image of $(z_0, z_1, z_2, z_3, z_4)$ by $\hat{\delta}$ is a 4-tuple which has duplicate elements. Hence the image is mapped to zero.

Hence the image of $\hat{\delta}$ is in the kernel of (11). Therefore the lemma holds. $\square$

Then the map

$$H_3(\hat{C}_*) \to \mathcal{P}(\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p) \tag{13}$$

obtained from (11) is well-defined by Lemma 4.2. As the composite of (9) and (13), we obtain

$$H_3(\mathrm{PGL}_2\mathbb{F}_p) \to H_3(\hat{C}_*) \to \mathcal{P}(\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p). \tag{14}$$

We prove that (14) does not depend on the choice of $z \in \mathbb{P}^1(\mathbb{F}_p)$.

Lemma 4.3. It follows that (14) does not depend on the choice of $z \in \mathbb{P}^1(\mathbb{F}_p)$.

Proof. We consider the map obtained from (14) by replacing $z \in \mathbb{P}^1(\mathbb{F}_p)$ with $z' \in \mathbb{P}^1(\mathbb{F}_p)$. Since there is $h \in \mathrm{PGL}_2\mathbb{F}_p$ such that $z' = hz$, the image of (g_0, g_1, g_2, g_3) by the map is as follows:

$$[g_0z', g_1z', g_2z', g_3z'] = [g_0hz, g_1hz, g_2hz, g_3hz]. \tag{15}$$

In particular, since (10) does not change by the action of $\mathrm{PGL}_2\mathbb{F}_p$, the right-hand side of (15) is equal to

$$[h^{-1}g_0hz, h^{-1}g_1hz, h^{-1}g_2hz, h^{-1}g_3hz].$$

By Proposition II.6.2 in [1], the conjugacy action of G on $H_3(\mathrm{PGL}_2\mathbb{F}_p)$ is trivial, and we obtain that

$$(h^{-1}g_0h, h^{-1}g_1h, h^{-1}g_2h, h^{-1}g_3h) = (g_0, g_1, g_2, g_3) \in H_3(\mathrm{PGL}_2\mathbb{F}_p).$$

Hence, by considering the image of the map, we obtain that

$$\begin{aligned}
[g_0z', g_1z', g_2z', g_3z'] &= [g_0hz, g_1hz, g_2hz, g_3hz] \\
&= [h^{-1}g_0hz, h^{-1}g_1hz, h^{-1}g_2hz, h^{-1}g_3hz] \\
&= [g_0z, g_1z, g_2z, g_3z] \in \mathcal{P}(\mathbb{F}_p).
\end{aligned}$$

Therefore (14) does not depend on the choice of z. $\square$

We construct a chain map from a complex $\mathbb{Z}\left[\dfrac{Y_*}{G}\right]$ to the total complex of a double complex $\mathbb{Z}\left[\dfrac{Y_{n+1} \times \hat{X}_{m+1}}{G}\right]$. For $z \in \mathbb{P}^1(\mathbb{F}_p)$, we define a map

$$Y_{n+m+1} \to Y_{n+1} \times \hat{X}_{m+1} \tag{16}$$

by

$$(g_0, g_1, \ldots, g_{n+m}) \mapsto ((g_0, g_1, \ldots, g_n), (g_nz, g_{n+1}z, \ldots, g_{n+m}z)).$$

Lemma 4.4. A map from a complex $\mathbb{Z}\left[\dfrac{Y_{k+1}}{G}\right]$ to the total complex of a double complex $\mathbb{Z}\left[\dfrac{Y_{n+1} \times \hat{X}_{m+1}}{G}\right]$ is obtained from (16) as follows:

$$\mathbb{Z}\left[\frac{Y_{n+1}}{G}\right] \to \bigoplus_i \mathbb{Z}\left[\frac{Y_i \times \hat{X}_{n+i-1}}{G}\right], \quad (g_0, g_1, \ldots, g_k) \mapsto ((g_0, g_1, \ldots, g_i), (g_iz, g_{i+1}z, \ldots, g_nz)). \tag{17}$$

Then (17) is a chain map.

Proof. It is enough to prove that (17) and boundary maps are commutative. By mapping $(g_0, g_1, \ldots, g_n) \in Y_{n+1}$ by (17), we have that

$$\sum_{i=0}^{n}((g_0, g_1, \ldots, g_i), (g_i z, g_{i+1} z, \ldots, g_n z)) \tag{18}$$

By mapping each $((g_0, g_1, \ldots, g_i), (g_i z, g_{i+1} z, \ldots, g_n z))$ by the differential of the total complex $D = \partial + (-1)^i \delta$, we obtain that

$$\sum_{j=0}^{i}(-1)^j((g_0, , \ldots, \hat{g}_j, \ldots, g_i), (g_i z, g_{i+1} z, \ldots, g_n z))$$
$$+(-1)^i \sum_{j=i}^{n}(-1)^{j-i}((g_0, g_1, \ldots, g_i), (g_i z, \ldots, \hat{g_j z}, \ldots, g_n z)).$$

In $((g_0, g_1, \ldots, g_{i+1}), (g_{i+1} z, g_{i+2} z, \ldots, g_n z))$ and the image of $((g_0, g_1, \ldots, g_i), (g_i z, g_{i+1} z, \ldots, g_n z))$ by D, the signs of $((g_0, g_1, \ldots, g_i), (g_{i+1} z, g_{i+2} z, \ldots, g_n z))$ are $(-1)^{i+1}$ and $(-1)^i$ respectively; these signs are opposite. Then these terms cancel in the image of (18) by D. Hence the image of (18) by D is

$$\sum_i \Big(\sum_{j=0}^{i-1}(-1)^j((g_0, \ldots, \hat{g}_j, \ldots, g_i), (g_i z, g_{i+1} z, \ldots, g_n z))$$
$$+(-1)^i \sum_{j=i+1}^{n}(-1)^{j-i}((g_0, g_1, \ldots, g_i), (g_i z, \ldots, \hat{g_j z}, \ldots, g_n z))\Big). \tag{19}$$

While the image of $(g_0, g_1, \ldots, g_n) \in Y_{n+1}$ by by $\partial\colon Y_{n+1} \to Y_n$ is

$$\sum_i (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n) \in Y_n. \tag{20}$$

the image of each $(g_0, \ldots, \hat{g}_i, \ldots, g_n)$ by (17) is

$$\begin{cases} \sum_j (-1)^j (g_0, \ldots, g_j, g_j z, \ldots, \hat{g_i z}, \ldots, g_n z) & \text{if } j < i, \\ \sum_j (-1)^j (g_0, \ldots, \hat{g}_i, \ldots, g_j, g_j z, \ldots, g_n z) & \text{if } j > i. \end{cases}$$

Hence the image of (20) by $Y_n \to \bigoplus_j \mathbb{Z}\left[\dfrac{Y_j \times X_{n-j}}{G}\right]$ is

$$\sum_i (-1)^i \Big(\sum_{j=i+1}^{n}((g_0, \ldots, \hat{g}_i, \ldots, g_j), (g_j z, g_{j+1} z, \ldots, g_n z))$$
$$+ \sum_{j=0}^{i-1}((g_0, g_1, \ldots, g_j), (g_j z, \ldots, \hat{g_i z}, \ldots, g_n z))\Big). \tag{21}$$

By summing up (19) in terms of i firstly, (19) and (21) are equal. Hence (17) and boundary maps are commutative. Therefore we obtain a map between chain complexes. $\square$

Lemma 4.5. The map (14) and the composite of $H_3(\mathrm{PGL}_2\mathbb{F}_p) \to \mathcal{P}(\mathbb{F}_p)$ in (7) and $\mathcal{P}(\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p)$ are equal.

Proof. We consider the spectral sequence of $\underset{n,m}{\oplus}\mathbb{Z}\left[\dfrac{Y_{n+1}\times \hat{X}_{m+1}}{G}\right]$, and see the E^1-page. Then we obtain a homomorphism $H_3(\mathrm{PGL}_2\mathbb{F}_p) \to \mathcal{P}(\mathbb{F}_p)$. This map is equal to (6) obtained from the spectral sequence of $\underset{n,m}{\oplus}\mathbb{Z}\left[\dfrac{Y_{n+1}\times X_{m+1}}{G}\right]$. Hence the map obtained from the spectral sequence is equal to (14). In particular, composites of the maps and $\mathcal{P}(\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p)$ are also equal. Therefore the lemma holds. □

4.2 Modulus and the reduced DW invariant

In this section, we review modulus of a labeled tetrahedron, and prove that the reduced DW invariant is obtained from an ideal triangulation of a knot complement and $\mathbb{P}^1(\mathbb{F}_p)$-labeling of ideal vertices (Proposition 4.6). It is known a fact which is similar to Proposition 4.6 in the case where a field is $\mathbb{C}$ and M is a hyperbolic 3-manifold by Neumann–Yang [10]. However they have used hyperbolicity of M in the proof. We give the proof without hyperbolicity of M.

We review modulus of a labeled tetrahedron following Thurston [17]. Although Thurston [17] mentioned the case of $\mathbb{C}$, the case where we replace $\mathbb{C}$ with $\mathbb{F}_p$ is also holds. For $v \in \mathbb{F}_p \cup \{\infty\} = \mathbb{P}^1(\mathbb{F}_p)$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2\mathbb{F}_p$, we define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \frac{av+b}{cv+d}.$$

Then $\mathrm{PGL}_2\mathbb{F}_p$ acts on $\mathbb{F}_p \cup \{\infty\}$ by linear fractional transformation.

We consider an oriented tetrahedron whose vertices are labeled by distinct elements of $\mathbb{F}_p \cup \{\infty\}$. We fix an orientation of an edge of the tetrahedron. For example, we consider a left tetrahedron shown in Figure 2. We set $z = \dfrac{(a-d)(b-c)}{(a-c)(b-d)} \in \mathbb{F}_p \setminus \{0,1\}$, and we call z *the modulus* of the tetrahedron. We can check by concrete calculation that the equivalent classes of labeled tetrahedra by the action of $\mathrm{PGL}_2\mathbb{F}_p$ are parametrized by z elementary. We note that the modulus of a tetrahedron depends on the choice of the edge, that is, by using the above z, the modulus is z, $\dfrac{z}{1-z}$, or $1-\dfrac{1}{z}$. Moreover, for the tetrahedron whose modulus is z, modulus of the tetrahedron with reversed orientation is $\dfrac{1}{z}$.

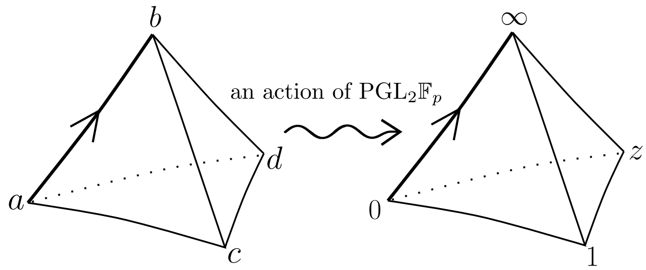

Figure 2

We consider a triangulation of a closed 3-manifold M. We let moduli($\in \mathbb{F}_p \backslash \{0,1\}$) be given for tetrahedra and these moduli satisfies hyperbolicity equations for the triangulation, where we define hyperbolicity equations for a triangulation in Section 4.5. Then the holonomy representation ($\pi_1(M) \to \mathrm{PGL}_2\mathbb{F}_p$) is constructed as follows. We consider a universal covering $\tilde{M}$ of M and fix a tetrahedron Δ of $\tilde{M}$. We give $\mathbb{P}^1(\mathbb{F}_p)$-labels of the vertices of Δ so that modulus defined by the labeling corresponds to the modulus of a tetrahedron which corresponds to the image of Δ by the projection $\tilde{M} \to M$. We consider $\gamma \in \pi_1(M)$. By using moduli of tetrahedra of $\tilde{M}$, we extend $\mathbb{P}^1(\mathbb{F}_p)$-labels of vertices of Δ to all vertices of $\tilde{M}$. One can uniquely define an element $g \in \mathrm{PGL}_2\mathbb{F}_p$ which sends $\mathbb{P}^1(\mathbb{F}_p)$-labels of the vertices of Δ to $\mathbb{P}^1(\mathbb{F}_p)$-labels of corresponding. In particular, g does not depend on the choice of Δ and a $\mathbb{P}^1(\mathbb{F}_p)$-label of the vertices of Δ. By corresponding γ to g, a map $\pi_1(M) \to \mathrm{PGL}_2\mathbb{F}_p$ is defined. We call the homomorphism $\pi_1(M) \to \mathrm{PGL}_2\mathbb{F}_p$ *the holonomy representation.*

Proposition 4.6. Let M be a closed 3-manifold. We consider a triangulation of M with moduli. We assume that the moduli ($\in \mathbb{F}_p \setminus \{0,1\}$) of the tetrahedra of a triangulation M satisfies hyperbolicity equations and that the holonomy representation ($\pi_1(M) \to \mathrm{PGL}_2\mathbb{F}_p$) lifts to an $\mathrm{SL}_2\mathbb{F}_p$ representation ρ. Then we obtain that

$$\widehat{\mathrm{DW}}(M,\rho) = \sum[\text{the modulus of a tetrahedron of } M] \in \check{\mathcal{P}}(\mathbb{F}_p),$$

where the summation on the right-hand side is over all tetrahedra of M.

Proof. We prove that the reduced DW invariant of M and a lift ρ of the holonomy representation to an $\mathrm{SL}_2\mathbb{F}_p$ representation is the summation of the moduli of the tetrahedra of a triangulation of M. Let M be a closed 3-manifold, and let a triangulation of M be given. We consider moduli of tetrahedra which satisfy hyperbolicity equations. In the following of this proof, we obtain a $\mathbb{P}^1(\mathbb{F}_p)$-labeling of vertices of the universal covering $\tilde{M}$, and construct a map $H_3(M) \to \check{\mathcal{P}}(\mathbb{F}_p)$ by using the labeling, and prove that the map sends to the summation of the moduli of the tetrahedra of M.

We set $\Gamma = \pi_1(M)$. As we mentioned just before Proposition 4.6, one can obtain Γ-equivariant $\mathbb{P}^1(\mathbb{F}_p)$-labeling of vertices of a universal covering $\tilde{M}$ of M, and obtain $\mathbb{P}^1(\mathbb{F}_p)$-labeling by the labeling. In particular, this $\mathbb{P}^1(\mathbb{F}_p)$-labeling is Γ-equivariant, that is, the labeling does not change by the action of Γ. One can lift this Γ-equivariant $\mathbb{P}^1(\mathbb{F}_p)$-labeling of the vertices of $\tilde{M}$ to a Γ-equivariant $\mathrm{PGL}_2\mathbb{F}_p$-labeling. For $z \in \mathbb{P}^1(\mathbb{F}_p)$, we define a $\mathrm{PGL}_2\mathbb{F}_p$-labeling $g_1, g_2, \ldots, g_n$ of representative vertices of equivalent classes of the vertices of $\tilde{M}$ for the action of Γ such that $g_1 z, g_2 z, \ldots, g_n z$ correspond to the original $\mathbb{P}^1(\mathbb{F}_p)$-labels, where n is the number of equivalent classes. By the action of Γ, we extend the $\mathrm{PGL}_2\mathbb{F}_p$-labels to $\tilde{M}$ similarly as the case of $\mathbb{P}^1(\mathbb{F}_p)$-labeling. Hence we obtain a Γ-equivariant $\mathrm{PGL}_2\mathbb{F}_p$-labeling of the vertices of $\tilde{M}$.

We construct a map $H_3(M) \to \check{\mathcal{P}}(\mathbb{F}_p)$. We note that $H_3(M) \cong \mathbb{Z}$ and this is generated by the fundamental class $[M]$. We consider Γ-equivalent $\mathrm{PGL}_2\mathbb{F}_p$-labeling of the vertices of $\tilde{M}$ which is given in the above, and fix a fundamental region of $\tilde{M}$ with respect to the action of Γ and a total order of the set of the vertices of M. We let vertices of each tetrahedron of a fundamental region $\tilde{M}$ are labeled by ${g_0}^{(i)}, {g_1}^{(i)}, {g_2}^{(i)}, {g_3}^{(i)} \in \mathrm{PGL}_2\mathbb{F}_p$ in ascending order, where the above tetrahedron correspond to a tetrahedron of M through the projection $\tilde{M} \to M$. We correspond these tetrahedra to a signed 4-tuple $\varepsilon_i({g_0}^{(i)}, {g_1}^{(i)}, {g_2}^{(i)}, {g_3}^{(i)})$, where

$$\varepsilon_i = \begin{cases} 1 & \text{if the orientation of } M \text{ and that of a tetrahedron obtained by the total order correspond,} \\ -1 & \text{otherwise.} \end{cases}$$

By extending this correspondence linearly, we obtain a map

$$H_3(M) \to H_3(\mathrm{PGL}_2\mathbb{F}_p), \quad [M] \mapsto \sum_i \varepsilon_i({g_0}^{(i)}, {g_1}^{(i)}, {g_2}^{(i)}, {g_3}^{(i)}). \tag{22}$$

We set $G = \mathrm{PGL}_2\mathbb{F}_p$. Since $H_*(\mathrm{PGL}_2\mathbb{F}_p)$ is obtained from a complex $\mathbb{Z}\left[\dfrac{Y_*}{G}\right]$, (22) does not depend on the choice of the $\mathrm{PGL}_2\mathbb{F}_p$-labeling of the vertices of $\tilde{M}$. As a composite of (22), (14), and $\mathcal{P}(\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p)$, we obtain $H_3(M) \to \check{\mathcal{P}}(\mathbb{F}_p)$.

We prove that $H_3(M) \to \check{\mathcal{P}}(\mathbb{F}_p)$ is a map which corresponds 3-cycle to the summation of the moduli of tetrahedra of M. The image of $[M]$ by $H_3(M) \to \check{\mathcal{P}}(\mathbb{F}_p)$ is

$$\sum_i \varepsilon_i[{g_0}^{(i)} z, {g_1}^{(i)} z, {g_2}^{(i)} z, {g_3}^{(i)} z]. \tag{23}$$

For tetrahedra of $\tilde{M}$ which corresponds to tetrahedra of M, we let the vertices of each tetrahedron be labeled by ${z_0}^{(i)}, {z_1}^{(i)}, {z_2}^{(i)}, {z_3}^{(i)} \in \mathcal{P}(\mathbb{F}_p)$ in ascending order, where the order is the total order of the vertices of M defined in the above. Since we defined Γ-equivariant $\mathrm{PGL}_2\mathbb{F}_p$-labeling as a lift of Γ-equivariant $\mathbb{P}^1(\mathbb{F}_p)$-labeling of the vertices of $\tilde{M}$, for $g \in \mathrm{PGL}_2\mathbb{F}_p$, we obtain that

$$\sum_i \varepsilon_i[{g_0}^{(i)} z, {g_1}^{(i)} z, {g_2}^{(i)} z, {g_3}^{(i)} z] = \sum_i \varepsilon_i[g z_0^{(i)}, g z_1^{(i)}, g z_2^{(i)}, g z_3^{(i)}]. \tag{24}$$

Moreover the right-hand side of (24) is equal to $\sum_i \varepsilon_i[z_0^{(i)}, z_1^{(i)}, z_2^{(i)}, z_3^{(i)}]$, where each $\varepsilon_i[z_0^{(i)}, z_1^{(i)}, z_2^{(i)}, z_3^{(i)}]$ corresponds to the modulus of the tetrahedron. Hence $H_3(M) \to \check{\mathcal{P}}(\mathbb{F}_p)$ is a map which corresponds a 3-cycle to the summation of moduli.

We verify that the resulting value is the reduced DW invariant of (M, ρ). We consider the following diagram:

$$\begin{tikzcd} & H_3(\mathrm{SL}_2\mathbb{F}_p) \arrow[d] \arrow[dr] & \\ H_3(M) \arrow[ur] \arrow[r] & H_3(\mathrm{PGL}_2\mathbb{F}_p) \arrow[r] & \check{\mathcal{P}}(\mathbb{F}_p) \end{tikzcd} \tag{25}$$

where $H_3(M) \to H_3(\mathrm{SL}_2\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p)$ appeared in the definition of the reduced DW invariant, and $H_3(M) \to H_3(\mathrm{PGL}_2\mathbb{F}_p) \to \check{\mathcal{P}}(\mathbb{F}_p)$ is the above $H_3(M) \to \check{\mathcal{P}}(\mathbb{F}_p)$. The left triangular diagram commutes since the holonomy representation lifts to an $\mathrm{SL}_2\mathbb{F}_p$ representation ρ, and the right triangular diagram commutes by (7). Hence, (25) commutes. Therefore, the resulting value is the reduced DW invariant of (M, ρ).

Therefore, Proposition 4.6 holds. □

By Proposition 4.6, it follows that the reduced DW invariant of (M, ρ) is obtained as the summation of moduli of tetrahedra of M in $\check{\mathcal{P}}(\mathbb{F}_p)$. In particular, the reduced DW invariant does not depend on the choice of edges and orientations of tetrahedra by the following lemma.

Lemma 4.7. The reduced DW invariant does not depend on the choice of edges and orientations of tetrahedra which are required in the definition of moduli.

Proof. A modulus is z, $1 - \dfrac{1}{z}$, or $\dfrac{1}{1-z}$ depending on the choice of an oriented edge of a tetrahedron. Moreover a modulus is z or $\frac{1}{z}$ depending on the choice of an orientation. Since

$$[z] = \left[1 - \frac{1}{z}\right] = \left[\frac{1}{1-z}\right] = -\left[\frac{1}{z}\right] = -\left[\frac{z}{z-1}\right] = -[1-z]$$

by (2), it follows that the summation of moduli in $\check{\mathcal{P}}(\mathbb{F}_p)$ does not depend on the choice of edges and orientations. Therefore the lemma holds. □

We give a proposition for knot complements which is similar to Propositon 4.6.

Proposition 4.8. Let K be a knot. We consider an ideal triangulation of $S^3 \setminus K$ with moduli. We assume that the moduli ($\in \mathbb{F}_p \setminus \{0, 1\}$) of ideal tetrahedra of $S^3 \setminus K$ satisfies hyperbolicity equations and that the holonomy representation ($\pi_1(S^3 \setminus K) \to \mathrm{PGL}_2\mathbb{F}_p$) lifts to an $\mathrm{SL}_2\mathbb{F}_p$ representation ρ. Then we obtain that

$$\widehat{\mathrm{DW}}(K, \rho) = \sum [\text{modulus of an ideal tetrahedron of } S^3 \setminus K] \in \check{\mathcal{P}}(\mathbb{F}_p),$$

where the summation on the right-hand side is over all ideal tetrahedra of $S^3 \setminus K$.

Proof. We obtain a closed oriented 3-manifold by Dehn filling, and we apply Proposition 4.6 to the closed 3-manifold. Let K be a knot. When we take an (a,b)-curve on $\partial(S^3 \setminus N(K))$, we can obtain a closed 3-manifold $M_{a,b}(K)$. Moreover, we take the (a, b)-curve in the kernel of a lift of the holonomy represention $\rho \colon \pi_1(S^3 \setminus K) \to \mathrm{SL}_2\mathbb{F}_p$, where the holonomy representation ($\pi_1(S^3 \setminus K) \to \mathrm{PGL}_2\mathbb{F}_p$) is obtained from moduli of ideal tetrahedra. Then we can also obtain a representation $\pi_1(M_{a,b}(K)) \to \mathrm{SL}_2\mathbb{F}_p$ from ρ. We also denote the representation by ρ. It is enough to verify two assumptions in Proposition 4.6; one is that the moduli ($\in \mathbb{F}_p\backslash\{0,1\}$) of the ideal tetrahedra of $S^3 \setminus K$ satisfies hyperbolicity equations, and the other is that the holonomy representation ($\pi_1(S^3 \setminus K) \to \mathrm{PGL}_2\mathbb{F}_p$) lifts to an $\mathrm{SL}_2\mathbb{F}_p$ representation ρ.

We verify that the former assumption is satisfied. By Dehn filling in Appendix G, the moduli of tetrahedra of $M_{a,b}(K)$ satisfy hyperbolicity equations easily. Hence, the former assumption is satisfied.

We verify that the latter assumption is satisfied. Since the holonomy representation ($\pi_1(S^3 \setminus K) \to \mathrm{PGL}_2\mathbb{F}_p$) lifts to an $\mathrm{SL}_2\mathbb{F}_p$ representation,

$$\mathrm{kernel}(\pi_1(S^3 \setminus K) \to \mathrm{SL}_2\mathbb{F}_p) \subset \mathrm{kernel}(\pi_1(S^3 \setminus K) \to \mathrm{PGL}_2\mathbb{F}_p).$$

Hence, we can obtain a representation $\pi_1(M_{a,b}(K)) \to \mathrm{PGL}_2\mathbb{F}_p$ for the above (a,b)-curve. Therefore the following quadrilateral diagram commutes.

$$\begin{tikzcd}
\pi_1(S^3 \setminus K)) \arrow[r] \arrow[d] & \mathrm{SL}_2\mathbb{F}_p \arrow[d] \\
\pi_1(M_{(a,b)}(K)) \arrow[r] \arrow[ur] & \mathrm{PGL}_2\mathbb{F}_p
\end{tikzcd}$$

Moreover, the upper left triangular diagram also commutes. Hence, the lower right triangular diagram commutes. Therefore, the latter assumption is satisfied.

Therefore, we can apply Proposition 4.6 for $M_{a,b}(K)$, and we obtain that

$$\begin{aligned}
\widehat{\mathrm{DW}}(K,\rho) &= \sum[\text{the modulus of a tetrahedron of } M_{a,b}(K)] \\
&= \sum[\text{the modulus of an ideal tetrahedron of } S^3 \setminus K] \in \check{\mathcal{P}}(\mathbb{F}_p),
\end{aligned}$$

where we obtain the last equality since Dehn filling does not change the value of the reduced DW invariant by Remark G.2. □

Remark 4.9. The invariance of the reduced DW invariant $\widehat{\mathrm{DW}}(K,\rho)$ follows similarly as Lemma 4.7.

4.3 Parabolic representations of the twist-knot group

In this section we prove the one-to-one correspondence between the conjugacy classes of parabolic representations $\pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p$ and zeros of a polynomial in a finite field.

Let $\mathcal{T}_n$ be the n-twist knot. We consider an $\mathrm{SL}_2\mathbb{F}_p$ representation of $\pi_1(S^3 \setminus \mathcal{T}_n)$. We describe $\mathcal{T}_n$ by a 1-tangle diagram as in Figure 3. The arrows in Figure 3 denote elements of $\pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p$; see, for example, [8]. Hence $\pi_1(S^3 \setminus \mathcal{T}_n)$ has a presentation

$$\begin{aligned}
\pi_1(S^3 \setminus \mathcal{T}_n) &\cong \left\langle X, Y, W, Z_1, \ldots, Z_n \;\middle|\; \begin{array}{l} W = Y^{-1}XY, \quad Z_1 = WY^{-1}W^{-1}, \quad Z_2 = Z_1^{-1}YZ_1, \\ Z_k = Z_{k-1}^{-1}Z_{k-2}Z_{k-1} \ (k = 3, 4, \ldots, n), \\ W = Z_n^{-1}Z_{n-1}^{-1}Z_n, \quad Z_n = X \end{array} \right\rangle \\
&\cong \begin{cases} \left\langle X, Y \;\middle|\; (Y^{-1}XYX^{-1})^{\frac{n-1}{2}}Y^{-1}XY^{-1}X^{-1}Y(XY^{-1}X^{-1}Y)^{\frac{n-1}{2}} = X \right\rangle & \text{if } n \text{ is odd,} \\ \left\langle X, Y \;\middle|\; (Y^{-1}XYX^{-1})^{\frac{n}{2}}Y(XY^{-1}X^{-1}Y)^{\frac{n}{2}} = X \right\rangle & \text{if } n \text{ is even.} \end{cases}
\end{aligned} \tag{26}$$

The second isomorphism is obtained by a presentation of Z_{n-1} by only $X^{\pm 1}$ and $Y^{\pm 1}$, obtained by substituting $W = Y^{-1}XY$ for Z_1 and substituting Z_{i-2}, Z_{i-1} for Z_i $(i = 3, 4, \ldots, n-1)$ repeatedly, and substituting $W = Y^{-1}XY$ for $W = Z_n^{-1}Z_{n-1}^{-1}Z_n = X^{-1}Z_{n-1}^{-1}X$.

We note that $\pi_1(S^3 \setminus \mathcal{T}_n)$ has a presentation with two generators which are conjugate to each other and one relator. Twist knots are special cases of two-bridge knots, and it is known that the knot group of a two-bridge knot has a presentation with two generators and one relator; see, for example, [2].

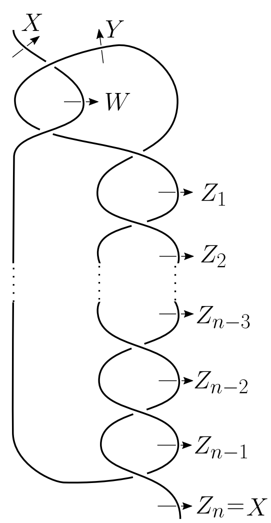

Figure 3

We defined parabolic representations of $\pi_1(S^3 \setminus \mathcal{T}_n)$ and conjugation of them in Section 2.2. To describe the set of conjugacy classes of parabolic representations $\rho\colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p$, we define a map

$$\phi\colon \{\text{parabolic representations } \rho\colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p\}/\text{conjugation} \longrightarrow \mathbb{F}_p \tag{27}$$

by $\phi([\rho]) = $trace $\rho(XY) - 2$. Here $[\rho]$ denotes the conjugacy class of ρ. We note that $\phi([\rho])$ depends on only the conjugacy class of ρ since trace is an invariant under conjugation. In the case of $\mathrm{SL}_2\mathbb{C}$, a similar lemma as the following lemma is shown by Riley [13].

Lemma 4.10. For a parabolic representation $\rho\colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p$ and generators X and Y of $\pi_1(S^3 \setminus \mathcal{T}_n)$ in Figure 3, there is a $P \in \mathrm{SL}_2\overline{\mathbb{F}_p}$ such that

$$P^{-1}\rho(X)P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ P^{-1}\rho(Y)P = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},$$

where $u \in \mathbb{F}_p^\times$. Moreover u is $\phi([\rho])$.

We give the proof of Lemma 4.10 in Appendix C.

We specify the image of (27). We define polynomials in u with integral coefficients by

$$F_{k+2}(u) = -uF_{k+1}(u) + F_k(u) \tag{28}$$

$$F_0(u) = 1,\ F_1(u) = 1 - u.$$

In particular, since $F_k(0) = 1$ for any k, we have $F_k(u) \neq 0$.

Proposition 4.11. The conjugacy classes of parabolic representations $\pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p$ and the solutions of $F_n(u) = 0$ over $\mathbb{F}_p$ are one-to-one correspondence by ϕ. Namely, the map induced by ϕ

$$\{\text{parabolic representations } \rho\colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p\}/\text{conjugation} \longrightarrow \{u \in \mathbb{F}_p \mid F_n(u) = 0\} \tag{29}$$

is bijective.

We note that, for 2-bridge knots, Riley [14] proved a similar result as Proposition 4.11.

Before we prove Proposition 4.11, we show Lemmas 4.12 and 4.13 below.

Lemma 4.12. We set $u = \alpha - \alpha^{-1}$. Then $F_k(u)$ $(k \geq 0)$ can be expressed by

$$F_k(u) = F_k(\alpha - \alpha^{-1}) = \frac{\alpha^{-k-1} + \alpha^{-k} - (-\alpha)^k - (-\alpha)^{k+1}}{\alpha + \alpha^{-1}}.$$

Proof. When $k = 0$, we have

$$F_0(\alpha - \alpha^{-1}) = 1 = \frac{\alpha^{-1} + \alpha^0 - (-\alpha)^0 - (-\alpha)^1}{\alpha + \alpha^{-1}}.$$

Hence the lemma holds.

When $k = 1$, we have

$$F_1(\alpha - \alpha^{-1}) = 1 - \left(\alpha - \alpha^{-1}\right) = \frac{\alpha^{-2} + \alpha^{-1} - (-\alpha) - (-\alpha)^2}{\alpha + \alpha^{-1}}.$$

Hence the lemma holds.

When $k \geq 2$ we prove the equation in the lemma by induction on k. By the assumption of the induction,

$$F_{k-1}(\alpha - \alpha^{-1}) = \frac{\alpha^{-k} + \alpha^{-k+1} - (-\alpha)^{k-1} - (-\alpha)^k}{\alpha + \alpha^{-1}},$$

$$F_{k-2}(\alpha - \alpha^{-1}) = \frac{\alpha^{-k+1} + \alpha^{-k+2} - (-\alpha)^{k-2} - (-\alpha)^{k-1}}{\alpha + \alpha^{-1}}.$$

Hence,

$$(\alpha+\alpha^{-1})\left\{-(\alpha - \alpha^{-1})F_{k-1}(\alpha - \alpha^{-1}) + F_{k-2}(\alpha - \alpha^{-1})\right\} = \alpha^{-k-1}+\alpha^{-k}-(-\alpha)^k-(-\alpha)^{k+1},$$

Therefore, by (28) we obtain

$$\begin{aligned}(\alpha + \alpha^{-1})F_k(\alpha - \alpha^{-1}) &= \left(\alpha + \alpha^{-1}\right)\left\{-\left(\alpha - \alpha^{-1}\right)F_{k-1}(\alpha - \alpha^{-1}) + F_{k-2}(\alpha - \alpha^{-1})\right\} \\ &= \alpha^{-k-1} + \alpha^{-k} - (-\alpha)^k - (-\alpha)^{k+1}.\end{aligned}$$

Then we obtain $F_k(\alpha - \alpha^{-1}) = \dfrac{\alpha^{-k-1} + \alpha^{-k} - (-\alpha)^k - (-\alpha)^{k+1}}{\alpha + \alpha^{-1}}$, which is the formula of the lemma in the case of k.

Therefore, we obtain the lemma for any k. $\square$

We set $\widehat{X},\ \widehat{Y},\ \widehat{W}$,and $\widehat{Z}_k$ $(k \geq 3)$ as follows:

$$\widehat{X} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \widehat{Y} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},$$

$$\widehat{W} = \widehat{Y}^{-1}\widehat{X}\widehat{Y},\ \widehat{Z}_1 = \widehat{W}\widehat{Y}^{-1}\widehat{W}^{-1},\ \widehat{Z}_2 = \widehat{Z}_1^{-1}\widehat{Y}\widehat{Z}_1,\ \widehat{Z}_k = \widehat{Z}_{k-1}^{-1}\widehat{Z}_{k-2}\widehat{Z}_{k-1}\ (k \geq 3).$$

Lemma 4.13. We set $u = \alpha - \alpha^{-1}$. Then $\widehat{Z}_k$ $(k \geq 1)$ is expressed as follows:

$$\widehat{Z}_k = F_k(u) \begin{pmatrix} \frac{(-\alpha)^{-k-1}-(-\alpha)^{-k+1}+\alpha^{k-1}-\alpha^{k+1}}{\alpha+\alpha^{-1}} & \frac{(-\alpha)^{-k}+(-\alpha)^{-k+1}-\alpha^{k-1}-\alpha^{k}}{\alpha+\alpha^{-1}} \\ (-1)^k u F_k(u) & \frac{-(-\alpha)^{-k-1}+(-\alpha)^{-k+1}-\alpha^{k-1}+\alpha^{k+1}}{\alpha+\alpha^{-1}} \end{pmatrix} + \widehat{X}. \tag{30}$$

Proof. When $k = 1$ or 2, we obtain

$$\widehat{Z}_1 = F_1(u) \begin{pmatrix} \frac{(-\alpha)^{-2}-\alpha^2}{\alpha+\alpha^{-1}} & \frac{(-\alpha)^{-1}-\alpha}{\alpha+\alpha^{-1}} \\ -uF_1(u) & \frac{-(-\alpha)^{-2}+\alpha^2}{\alpha+\alpha^{-1}} \end{pmatrix} + \widehat{X},$$

$$\widehat{Z}_2 = F_2(u) \begin{pmatrix} \frac{(-\alpha)^{-3}-(-\alpha)^{-1}+\alpha-\alpha^3}{\alpha+\alpha^{-1}} & \frac{(-\alpha)^{-2}+(-\alpha)^{-1}-\alpha-\alpha^2}{\alpha+\alpha^{-1}} \\ uF_2(u) & \frac{-(-\alpha)^{-3}+(-\alpha)^{-1}-\alpha+\alpha^3}{\alpha+\alpha^{-1}} \end{pmatrix} + \widehat{X}$$

by concrete calculation. Hence the lemma holds in these cases.

When $k \geq 3$, we prove the lemma by induction on k. Since $\widehat{Z}_j = \widehat{Z}_{j-1}^{-1}\widehat{Z}_{j-2}\widehat{Z}_{j-1}$ we obtain $\widehat{Z}_{j-1}\widehat{Z}_j = \widehat{Z}_{j-2}\widehat{Z}_{j-1}$ for each i. Then we set $P = \widehat{Z}_1\widehat{Z}_2$, then we obtain $\widehat{Z}_{k-1}\widehat{Z}_k = P$, that is, $\widehat{Z}_k = \widehat{Z}_{k-1}^{-1}P$. In the following, we assume that the lemma follows for $\widehat{Z}_{k-1}$, we prove the lemma for $\widehat{Z}_k$.

By the definition of P, we obtain

$$P = \begin{pmatrix} 1-u+u^2 & u \\ u^2 & 1+u \end{pmatrix} = \begin{pmatrix} \alpha^2+\alpha-1-\alpha^{-1}+\alpha^{-2} & -\alpha+\alpha^{-1} \\ \alpha^2-2+\alpha^{-2} & -\alpha+1+\alpha^{-1} \end{pmatrix}.$$

This matrix is diagonalized, by $Q = \begin{pmatrix} 1 & 1 \\ 1-\alpha^{-1} & \alpha+1 \end{pmatrix}$, as

$$Q^{-1}PQ = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^{-2} \end{pmatrix}. \tag{31}$$

We set

$$R_k = \begin{pmatrix} \frac{(-\alpha)^{-k-1}-(-\alpha)^{-k+1}+\alpha^{k-1}-\alpha^{k+1}}{\alpha+\alpha^{-1}} & \frac{(-\alpha)^{-k}-(-\alpha)^{-k+1}-\alpha^{k-1}-\alpha^{k}}{\alpha+\alpha^{-1}} \\ (-1)^k u F_k(u) & \frac{-(-\alpha)^{-k-1}+(-\alpha)^{-k+1}-\alpha^{k-1}+\alpha^{k+1}}{\alpha+\alpha^{-1}} \end{pmatrix}.$$

(What we prove is $\widehat{Z}_k = F_k(u)R_k + \widehat{X}$.) By concrete calculation, we obtain

$$Q^{-1}R_kQ = \begin{pmatrix} 0 & -\frac{\alpha+1}{\alpha^k} \\ (-\alpha)^{k-1}(\alpha-1) & 0 \end{pmatrix}. \tag{32}$$

Since the determinants of $\widehat{Z}_k$ and X are 1, the equality in the lemma is also rewritten as

$$\widehat{Z}_k^{-1} = F_k(u)\begin{pmatrix} \frac{-(-\alpha)^{-k-1}+(-\alpha)^{-k+1}-\alpha^{k-1}+\alpha^{k+1}}{\alpha+\alpha^{-1}} & -\frac{(-\alpha)^{-k}-(-\alpha)^{-k+1}-\alpha^{k-1}-\alpha^{k}}{\alpha+\alpha^{-1}} \\ -(-1)^k uF_k(u) & \frac{(-\alpha)^{-k-1}-(-\alpha)^{-k+1}+\alpha^{k-1}-\alpha^{k+1}}{\alpha+\alpha^{-1}} \end{pmatrix} + \widehat{X}^{-1}.$$

We set

$$\overline{R}_k = \begin{pmatrix} \frac{-(-\alpha)^{-k-1}+(-\alpha)^{-k+1}-\alpha^{k-1}+\alpha^{k+1}}{\alpha+\alpha^{-1}} & -\frac{(-\alpha)^{-k}-(-\alpha)^{-k+1}-\alpha^{k-1}-\alpha^{k}}{\alpha+\alpha^{-1}} \\ -(-1)^k uF_k(u) & \frac{(-\alpha)^{-k-1}-(-\alpha)^{-k+1}+\alpha^{k-1}-\alpha^{k+1}}{\alpha+\alpha^{-1}} \end{pmatrix}.$$

(By the assumption of the induction, $\widehat{Z}_{k-1}^{-1} = F_{k-1}(u)\overline{R}_{k-1} + \widehat{X}^{-1}$.) By concrete calculation, we obtain

$$Q^{-1}\overline{R}_k Q = \begin{pmatrix} 0 & \frac{\alpha+1}{\alpha^k} \\ (-\alpha)^{k-1}(1-\alpha) & 0 \end{pmatrix}. \tag{33}$$

What we prove is, by $\widehat{Z}_k = \widehat{Z}_{k-1}^{-1}P$ and $\widehat{Z}_{k-1}^{-1} = F_{k-1}(u)\overline{R}_{k-1} + \widehat{X}^{-1}$, to prove $\widehat{Z}_k = F_k(u)R_k + \widehat{X}$. To prove the equation, it is enough to prove

$$F_k(u)R_k + \widehat{X} = (F_{k-1}(u)\overline{R}_{k-1} + \widehat{X}^{-1})P$$

which is obtained by substituting $\widehat{Z}_k = F_kR_k + \widehat{X}$ and $\widehat{Z}_{k-1}^{-1} = F_{k-1}(u)\overline{R}_{k-1} + \widehat{X}^{-1}$ for $\widehat{Z}_k = \widehat{Z}_{k-1}^{-1}P$. Further, to prove the above equation, it is enough to prove

$$F_k(u)\cdot Q^{-1}R_kQ + Q^{-1}(\widehat{X} - \widehat{X}^{-1}P)Q = F_{k-1}(u)\cdot Q^{-1}\overline{R}_{k-1}Q\cdot Q^{-1}PQ$$

By concrete calculation, we obtain $Q^{-1}(\widehat{X} - \widehat{X}^{-1}P)Q = \begin{pmatrix} 0 & \frac{(a+1)^2}{a} \\ -\frac{(a-1)^2}{a} & 0 \end{pmatrix}$. Hence, by (31), (32), (33), it is enough to prove

$$F_k(u)\begin{pmatrix} 0 & -\frac{\alpha+1}{\alpha^k} \\ (-\alpha)^{k-1}(\alpha-1) & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{(a+1)^2}{a} \\ -\frac{(a-1)^2}{a} & 0 \end{pmatrix} = F_{k-1}(u)\begin{pmatrix} 0 & \frac{\alpha+1}{\alpha^{k+1}} \\ (-\alpha)^k(1-\alpha) & 0 \end{pmatrix}.$$

To prove the equation, it is enough to prove

$$-\frac{F_k(u)}{\alpha^k} + \frac{\alpha+1}{\alpha} = \frac{F_{k-1}(u)}{\alpha^{k+1}}, \qquad (-\alpha)^{k-1}F_k(u) - \frac{\alpha-1}{\alpha} = -(-\alpha)^kF_{k-1}(u).$$

By concrete calculation by using Lemma 4.12, we can confirm these two equations hold. Therefore, we obtain the lemma. □

Proof (Proof of Proposition 4.11)**.** We have the injectivity of ϕ, since a parabolic representation ρ such that $\phi(\rho) = u$ is conjugate to ρ' in Lemma 4.10 by the lemma. Hence, to prove the lemma, it is enough to prove that the image of ϕ is equal to the set on the right-hand side of (29), that is,

$$\mathrm{Image}\,\phi = \{u \in \mathbb{F}_p \mid F_n(u) = 0\}.$$

In other words, it is enough to prove that

$$\operatorname{Image}\phi \subset \{u \in \mathbb{F}_p \mid F_n(u) = 0\}, \tag{34}$$
$$\operatorname{Image}\phi \supset \{u \in \mathbb{F}_p \mid F_n(u) = 0\}. \tag{35}$$

In the following, we let $\widehat{Z}_j(c)$ be the matrix obtained from $\widehat{Z}_j$ by substituting $u = c$.

We prove (34). Let $c \in$ Image ϕ. Then, there exists a parabolic representation $\rho\colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p$ such that $c = \phi([\rho])$. Since ρ is a representation of $\pi_1(S^3 \setminus \mathcal{T}_n)$, we obtain $\widehat{Z}_n(c) = \widehat{X}$. It is for the equation that the first term of the equation obtained from (30) by substituting c for u needs to be a zero matrix. Moreover, since the $(2,1)$-entry offers $(-1)^k c(F_k(c))^2 = 0$, it needs to $c = 0$ or $F_n(c) = 0$. Since ρ is parabolic, $\phi([\rho]) \in \mathbb{F}_p^{\times}$, that is, $c \neq 0$. Hence $F_n(c) = 0$, that is, $c \in \{u \in \mathbb{F}_p \mid F_n(u) = 0\}$. Therefore we obtain (34).

We prove (35). Let $c \in \{u \in \mathbb{F}_p \mid F_n(u) = 0\}$. Since the first term of (30) obtained by substituting c for u is equal to a zero matrix, that is, we have $\widehat{Z}_n(c) = \widehat{X}$. Since $\widehat{Z}_{n-1}\widehat{Z}_n = P$, we have

$$\widehat{Z}_{n-1}(c)\widehat{Z}_n(c) = \begin{pmatrix} c^2 - c + 1 & c \\ c^2 & c + 1 \end{pmatrix},$$

and the matrix is equal to the matrix obtained from

$$\widehat{X}\widehat{W}^{-1}\widehat{X}^{-1} = \widehat{X}\widehat{Y}^{-1}\widehat{X}^{-1}\widehat{Y}\widehat{X}^{-1}$$

by substituting c for u. By these equations, if we set

$$\rho(X) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \rho(Y) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$

ρ satisfies the relation of $\pi_1(S^3 \setminus \mathcal{T}_n)$ and become an $\mathrm{SL}_2\mathbb{F}_p$ representation of $\pi_1(S^3 \setminus \mathcal{T}_n)$. In particular, the representation is parabolic by Lemma 4.10. Since $\phi([\rho]) = c$, we obtain $c \in \operatorname{Image}\phi$. Hence we obtain (35).

Therefore we obtain Proposition 4.11. □

4.4 Hyperbolicity equations for knot diagrams

In this section, we review hyperbolicity equations for a knot diagram. We review hyperbolicity equations for a knot diagram following Thurston [16] and Yokota [21]. Hyperbolicity equations for $\mathbb{C}$ of a knot complement are required equations to allow a hyperbolic structure on the complement. It is known that $\mathrm{SL}_2\mathbb{C}$ representations are obtained from solutions of the hyperbolicity equations and one of the representations gives the holonomy representation which allow the hyperbolic structure. In the case of $\mathbb{F}_p$ instead of $\mathbb{C}$, we obtain $\mathrm{SL}_2\mathbb{F}_p$ representations in the same way. In the following, we call the same equations in the case of $\mathbb{F}_p$ hyperbolicity equations, but we note that the solutions of the equations in $\mathbb{F}_p$ do not relate to a hyperbolic structure directly. In this section, we describe hyperbolic equations by using $F_k(u)$ defined in Section 4.3.

We review the hyperbolicity equations of a diagram of the n-twist knot. As in Figure 4, we describe the n-twist knot by a 1-tangle diagram and parametrize each semi-arc of the diagram as Figure 4. Here, we label semi-arcs adjacent to unbounded regions by 1. we label semi-arcs next to the terminal edges by 0 or ∞ as in Figure 4; we parameterize such a semi-arc by ∞ (resp. 0) if it is connected to the terminal semi-arc by an underpath (resp. an overpath). We parameterize the other semi-arcs located as in Figure 5 by solutions of the equations as follows:

$$(1-\frac{w}{y})(1-\frac{z'}{w}) = (1-\frac{w}{y'})(1-\frac{z}{w}).$$

In particular, in the case of the n-twist knot, we obtain the following equations by apply the above relations:

$$\begin{cases} x_2 = 1 - x_1 + x_1^2, \\ x_k = 1 - \dfrac{x_{k-1}}{x_{k-2}} + x_{k-1} \quad (k = 3, 4, \dots, n), \\ x_n = 0. \end{cases} \tag{36}$$

We call the equations *hyperbolicity equations* for the diagram in Figure 4.

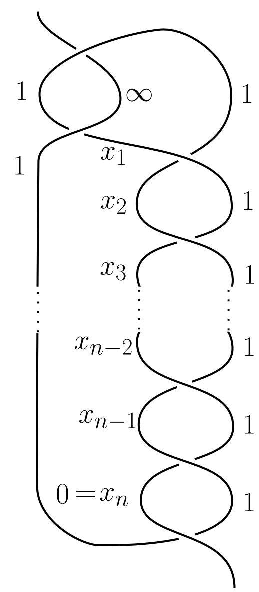

Figure 4

Let u be an indeterminate, and we put $x_1 = 1 - u$. We describe x_k as rational polynomial by the first and second equations in (36). Thus x_k is expressed concretely as follows.

Lemma 4.14. For $k \geq 2$, $x_k = \dfrac{F_k(u)}{F_{k-2}(u)}$. Here, $F_k(u)$ is a polynomial defined by (28).

Remark 4.15. In the case of $\mathbb{C}$, it is known that the holonomy representation $\pi_1(S^3 \setminus K) \to \mathrm{PGL}_2\mathbb{C}$ can be obtained from a solution of hyperbolicity equations for a diagram of a two-bridge knot; see Ohtsuki and Takata [12].

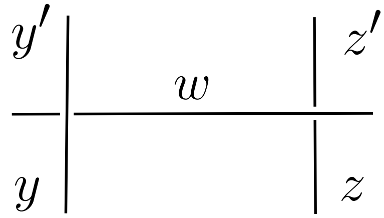

Figure 5

Proof. When $k = 2$ and 3, by concrete calculation, we obtain

$$x_2 = u^2 - u + 1 = \frac{F_2(u)}{F_0(u)},$$

$$x_3 = \frac{u^3 - u^2 + 2u - 1}{1 - u} = \frac{F_3(u)}{F_1(u)}.$$

Hence, we obtain the lemma in this case.

When $k \geq 4$, we prove the lemma by induction on k. By Lemma 4.12, we obtain

$$\begin{aligned}
F_{k-2}(u)F_{k+1}(u) - F_{k-1}(u)F_k(u) &= F_{k-2}(\alpha - \alpha^{-1})F_{k+1}(\alpha - \alpha^{-1}) - F_{k-1}(\alpha - \alpha^{-1})F_k(\alpha - \alpha^{-1}) \\
&= (-1)^{k-1}\frac{\alpha^4 + 2 + \alpha^{-4}}{(\alpha + \alpha^{-1})^2} \\
&= (-1)^{k-1}(\alpha - \alpha^{-1})^2 \\
&= (-1)^{k-1}u^2.
\end{aligned}$$

In the same way, we obtain

$$F_{k-1}(u)F_{k+2}(u) - F_k(u)F_{k+1}(u) = (-1)^k u^2.$$

Because of these, we obtain

$$(F_{k-1}(u)F_{k+2}(u) - F_k(u)F_{k+1}(u)) + (F_{k-2}(u)F_{k+1}(u) - F_{k-1}(u)F_k(u)) = (-1)^k u^2 + (-1)^{k-1}u^2 = 0.$$

Since we have the lemma for x_{k-1}, x_{k-2} by the assumption of the induction and $F_j(u) \neq 0\,(j \geq 0)$, we obtain

$$\frac{F_k(u)}{F_{k-2}(u)} = 1 - \left(\frac{F_{k-1}(u)}{F_{k-3}(u)}\right)\left(\frac{F_{k-4}(u)}{F_{k-2}(u)}\right) + \frac{F_{k-1}(u)}{F_{k-3}(u)} = 1 - \frac{x_{k-1}}{x_{k-2}} + x_{k-1} = x_k.$$

Hence the lemma holds for x_k.

Therefore we obtain the lemma for any $k \geq 2$. $\square$

4.5 Hyperbolicity equations for a triangulation

In this section, we review hyperbolicity equations for a triangulation of a closed 3-manifold and hyperbolicity equations for an ideal triangulation of a knot complement, and the latter equations equivalent to hyperbolicity equations of a diagram of the knot. Moreover, we describe the conjugacy classes of parabolic representations as sequences, and classify the sequences into three types, and regive an ideal triangulation of the knot complement for each case, and give an method to calculate reduced DW invariants.

We consider a triangulation of a closed 3-manifold M. Let orientations of edges and moduli of tetrahaedra be given. Then *hyperbolicity equations* for the triangulation of M is a condition that the summation of moduli of tetrahedra around each edge is 0 in $\check{\mathcal{P}}(\mathbb{F}_p)$.

For the complement of the n-twist knot, we review the method to give an ideal triangulation of the complement by Yokota [20]; see also [8]. We consider a 1-tangle diagram of the n-twist knot; see Figure 6. We assign a tetrahedron to each triangle as in Figure 6 and 7. These tetrahedra form a polyhedron by gluing each other appropriately. We construct an ideal triangulation of a knot complement by collapsing one of faces of the tetrahedron of a black triangle to a point. By collapsing glued tetrahedra linearly along this collapsing, the tetrahedron of each black triangle collapses to a segment, and the tetrahedron of each gray triangle collapses to a triangle: the tetrahedron of each white triangle is not affected. From the resulting polyhedron, we eliminate the point obtained by collapsing a face as above. Then we have an open 3-manifold, and this 3-manifold is homeomorphic to $S^3 \setminus K$. Thus we obtain an ideal triangulation of the complement of the n-twist knot by Yokota's method. We show an ideal triangulation obtained by applying Yokota's method to the complement of the n-twist knot as Figure 7.

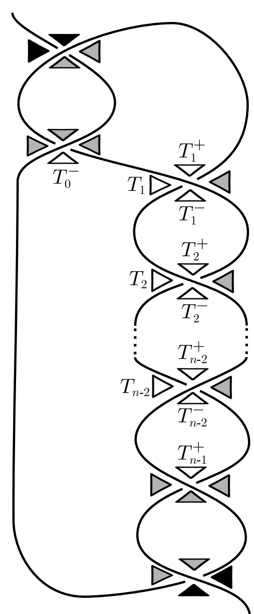

Figure 6: Correspondence between crossings of a diagram of the n-twist knot and ideal tetrahedra of an ideal triangulation

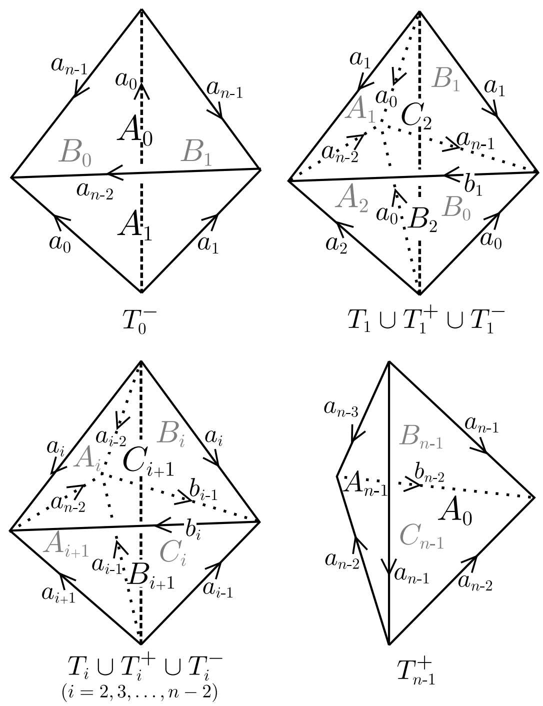

Figure 7: An ideal triangulation of the complement of the n-twist knot:
$T_k\,(1 \leq k \leq n-2)$, $T_k^+\,(1 \leq k \leq n-1)$, and $T_k^-\,(0 \leq k \leq n-2)$ denote ideal tetrahedra, and a_k, $(0 \leq k \leq n-1)$, $b_k\,(1 \leq k \leq n-2)$ denotes an edge of an ideal tetrahedron, and $A_k, B_k\,(0 \leq k \leq n-1)$, $C_k\,(2 \leq k \leq n-1)$ denotes an face of an ideal tetrahedron. Gray characters are on back faces, and faces with same symbols are glued to each other with the information of orientations.

We reduce the ideal triangulation of Figure 6. Triangles in Figure 8 denote ideal tetrahedra (which are denoted by white triangles) in Figure 6. We remove ideal tetrahedra which correspond to gray triangles in Figure 8 as follows. By Figure 7, for each region which makes digon in the twist part in Figure 8, two ideal tetrahedra which correspond to gray triangles are glued to each other sharing just two faces. By applying a $(0,2)$-Pachner move (Figure 17) to such two ideal tetrahedra, we remove these ideal tetrahedra. For all regions in the twist parts which make digons, we reduce the ideal triangulation of the complement of the n-twist knot by the above operation. Then we obtain an ideal triangulation as in Figure 9. In particular, $T_0, T_1, \ldots, T_{n-2}$ are glued in this order around an edge a_{n-1} like a fan as in Figure 10.

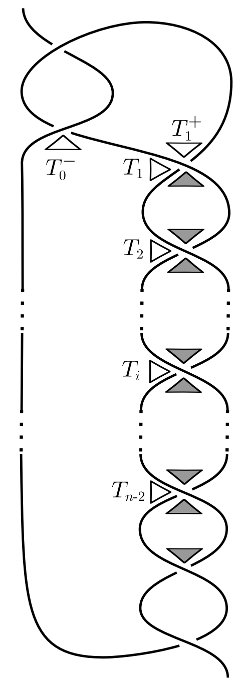

Figure 8: A reduction of the ideal triagulation in Figure 6

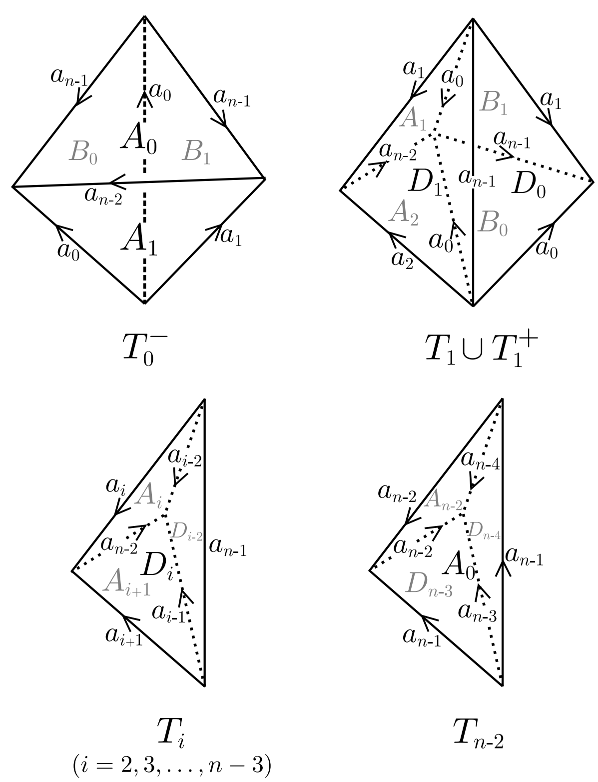

Figure 9: A reduced ideal triangulation of the complement of the n-twist knot

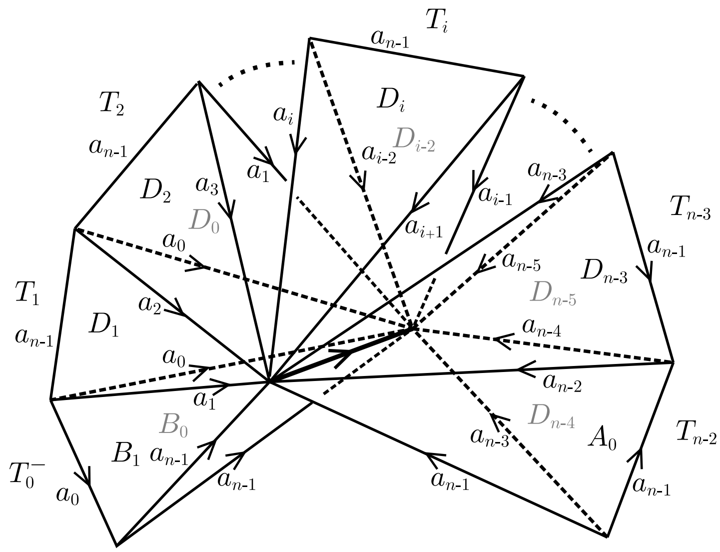

Figure 10

We give an orientation of each edge of the ideal triangulation of the complement of a knot. Around each edge, tetrahedra which have the edge are glued to each other. For any edge, if the product of moduli of ideal tetrahedra which are glued around the edge is 1, then the condition is called *gluing equations.*

Thurston [17] observed that, when an ideal triangulation of the complement of a knot K is given, the torus $\partial(S^3\backslash N(K))$ has a natural triangulation. We show such a triangulation in Figure 11 when K is $\mathcal{T}_5$. Let each edge in Figure 11 denote an edge which transverse $\partial N(K)$. Under the orientation of each edge of the ideal triangulation defined in Figure 9, we denote $\odot$ if the initial of the edge is in the knot side, and denote $\otimes$ if the terminal of the edge is in the knot side.

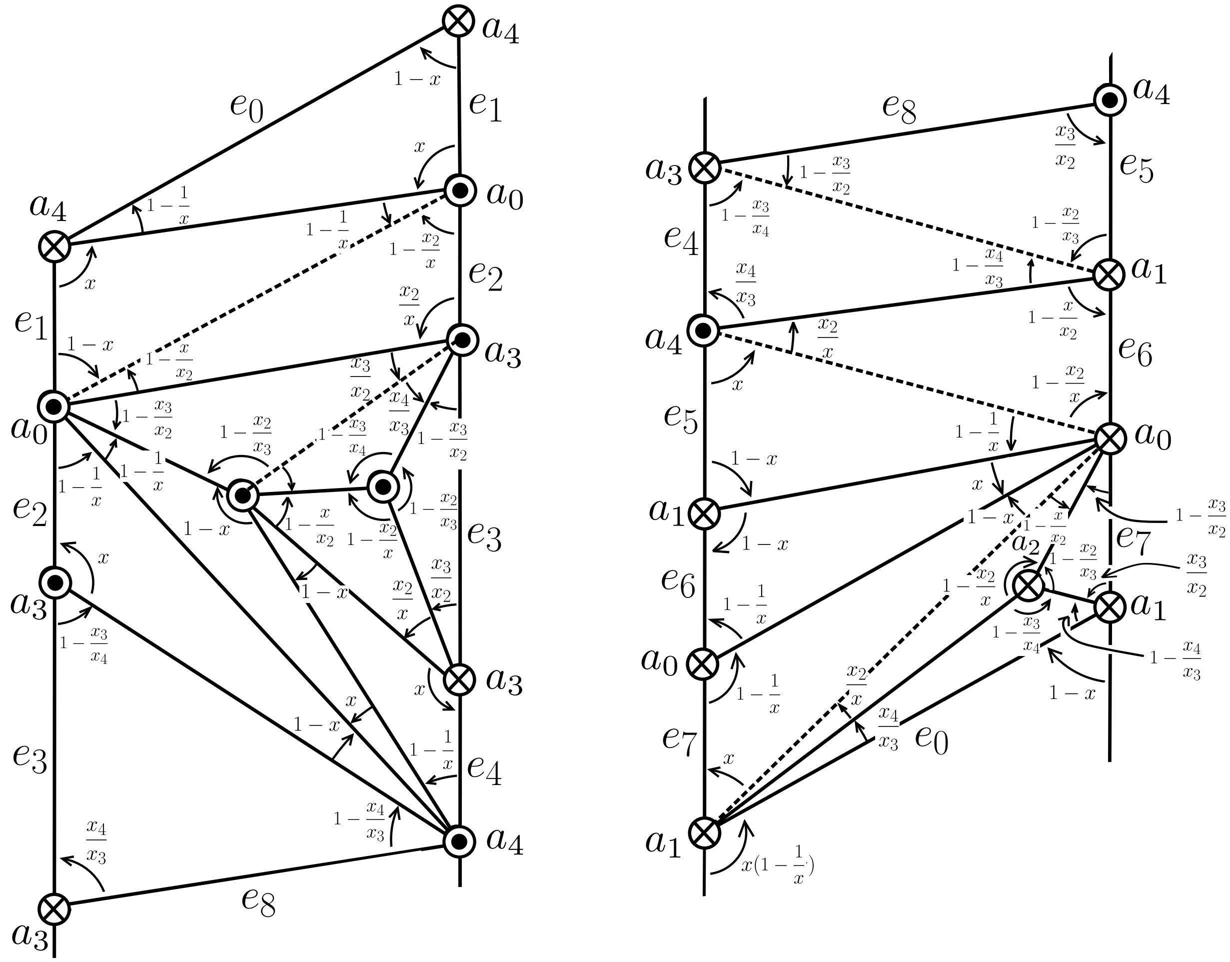

Figure 11: A triangulation of the torus $\partial(S^3 \setminus N(7_2))$
Let $e_0, e_1, \ldots, e_8$ denote edges, and edges with same symbols are glued to each other with the information of orientations.

We consider the condition the holonomy representation is to be parabolic. It is known that the condition is equivalent to that the product of moduli along a path connecting the same edge in Figure 11 is 1 by Thurston [16], where the product of moduli along a path means the product of moduli which are depicted at the corner consisting of two edges which the path transverse successively.

Example 4.16. A path connecting e_1 Figure 11 is depicted in Figure 12, and the product of moduli along the path is

$$x\,\frac{1}{x} = 1$$

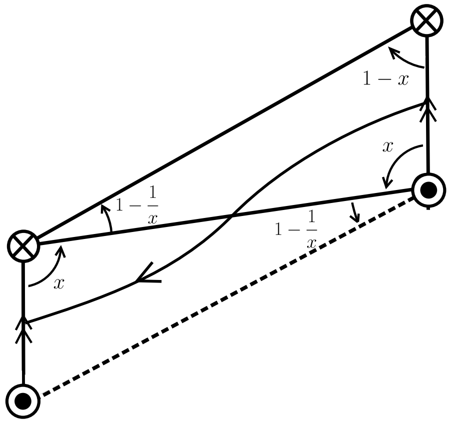

Figure 12: The product of moduli along a path

We consider the equations obtained from a gluing equations and the condition that the products of moduli along paths connecting a same edge are 1 as in Figure 12, and we call these equations *hyperbolicity equations* for the ideal triangulation of the complement of the n-twist knot in Figure 9. It is known that the hyperbolicity equations are equivalent to hyperbolicity equations for the diagram in Figure 4 by Thurston [16].

Example 4.17. We consider the 5-twist knot. The hyperbolicity equations for the diagram in Figure 8 are given as follows:

$$\begin{cases} x_2 = 1 - x_1 + x_1^2, \\ x_3 = 1 - \dfrac{x_2}{x_1} + x_2, \\ x_4 = 1 - \dfrac{x_3}{x_2} + x_3, \\ x_5 = 1 - \dfrac{x_4}{x_3} + x_4, \\ x_5 = 0. \end{cases} \tag{37}$$

On the other hand, by hyperbolicity equations for the ideal triangulation in Figure 9, we have

$$\begin{cases} (1-\frac{1}{x})^2(1-\frac{x_3}{x_2})^{-1}(1-\frac{x}{x_2})(1-x)^{-1}x(1-\frac{1}{x})(1-\frac{x_2}{x})^{-1}=1 & \text{around a vertex } a_0, \\ (1-x)^2(1-\frac{x}{x_2})^{-1}(1-\frac{x_4}{x_3})(1-\frac{x_2}{x_3})^{-1}=1 & \text{around a vertex } a_1, \\ (1-\frac{x_2}{x_3})(1-\frac{x_3}{x_4})(1-\frac{x_2}{x})^{-1}=1 & \text{around a vertex } a_2, \\ (1-\frac{x_3}{x_4})x(\frac{x_2}{x})(\frac{x_3}{x_2})(\frac{x_4}{x_3})(1-\frac{x_3}{x_2})^{-1}=1 & \text{around a vertex } a_3, \\ (1-\frac{1}{x})^{-1}x^{-1}(1-x)(1-\frac{x_4}{x_3})(\frac{x_3}{x_2})^{-1}x^{-1}(\frac{x_2}{x})^{-1}(\frac{x_4}{x_3})^{-1}=1 & \text{around a vertex } a_4 \\ (1-\frac{x_2}{x})^{-1}(1-\frac{1}{x})(1-x)=1 & \text{a path connecting an edge } e_6. \end{cases} \tag{38}$$

It is follows from concrete calculation that (38) is equivalent to (37).

For $F_n(u)$ defined in Section 4.3, we fix a solution $c \in \mathbb{F}_p$ of $F_n(u) = 0$ in $\mathbb{F}_p$. Then we define $c_k \in \mathbb{F}_p \cup \{\infty\}$ by

$$c_1 = F_1(c), \quad c_k = \begin{cases} \dfrac{F_k(c)}{F_{k-2}(c)} & \text{if } F_{k-2}(c) \neq 0 \quad (2 \leq k \leq n-1), \\ \infty & \text{if } F_{k-2}(c) = 0 \text{ and } F_k(c) \neq 0, \end{cases} \tag{39}$$

noting that we do not have the case where $F_{k-2}(c) = F_k(c) = 0$ by the following remark.

Remark 4.18. We note that there exists no $c \in \mathbb{F}_p$ such that $F_k(c) = F_{k-2}(c) = 0$.

Proof. if we assume there existed $c \in \mathbb{F}_p$ such that $F_k(c) = F_{k-2}(c) = 0$, it is follows that $cF_{k-1}(c) = 0$ by the recursion with respect to $F_k(u)$ (28) and $F_k(c) = F_{k-2}(c) = 0$. Then we have that $c = 0$ or $F_{k-1}(c) = 0$. In the case where $c = 0$, this contradicts that $F_k(0) = F_{k-2}(0) = 1$. Hence we have that $F_{k-1}(c) = 0$. By applying (28) repeatedly, it should follow that $F_0(c) = 0$, but this contradicts that $F_0(u) = 1$. Therefore there exists no $c \in \mathbb{F}_p$ such that $F_k(c) = F_{k-2}(c) = 0$. $\square$

Lemma 4.19. We consider an element $c \in \{u \in \mathbb{F}_p \mid F_n(u) = 0\}$ and $(c_1, c_2, \ldots, c_n)$ obtained from c by (39). Then, $c_n = 0$ since $F_n(c) = 0$,and $F_{n-2}(c) \neq 0$, and $(c_i, c_{i+1}, c_{i+2}) = (0, 1, \infty)$ when $c_i = 0$ $(1 \leq i \leq n-3)$. Moreover, $(c_1, c_2, \ldots, c_{n-1}, c_n)$ is either of the following form (i), (ii), and (iii):
(i) There exist no 0, 1, and ∞ in $(c_1, c_2, \ldots, c_{n-1})$, and $c_i \neq c_{i+1}$ $(1 \leq i \leq n-1)$,
(ii) There exist 0, 1, and ∞ in $(c_1, c_2, \ldots, c_{n-1})$ successively in order, and 3-tuples of $0, 1, \infty$ does not appear successively, and $c_1 \neq 0$, and $c_i \neq c_{i+1}$ $(1 \leq i \leq n-1)$.
(iii) $(c_1, c_2, c_3) = (0, 1, \infty)$, and 3-tuples $0, 1, \infty$ do not appear successively in $(c_1, c_2, \ldots, c_{n-1})$, and $c_i \neq c_{i+1}$ $(1 \leq i \leq n-1)$.

We give the proof in Appendix E.

In the following, we calculate the reduced DW invariant of the n-twist knot $\mathcal{T}_n$ and $\mathbb{F}_p$. For that, it is enough to calculate $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho)$ by the definition in Section 2.3 for each parabolic representation $\rho\colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p$. Let a parabolic representation $\rho\colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p$ be given. By the correspondence in Lemma 4.11, we can take $c = \phi([\rho]) \in \{u \in \mathbb{F}_p \mid F_n(u) = 0\}$ which corresponds to the conjugacy class $[\rho]$ of ρ. We consider $(c_1, c_2, \ldots, c_n)$ obtained from c by (39). By Lemma 4.19, this sequence is either of the form (i), (ii), and (iii). In either of the cases (i), (ii), and (iii), we calculate $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho)$.

The case where $(c_1, c_2, \ldots, c_n)$ is of the form (i) of Lemma 4.19 We consider the case of (i) in Lemma 4.19, that is, there exist no 0, 1, and ∞ in $(c_1, c_2, \ldots, c_{n-1})$, and $c_i \neq c_{i+1}$ $(1 \leq i \leq n-1)$. We give an ideal triangulation of the complement of the n-twist knot as in Figure 9, and a label of each ideal vertex by $\mathbb{P}^1(\mathbb{F}_p)$ as in Figure 9. Then all moduli of ideal vertices do not collapse. In particular, since the ideal triangulations in Figure 9 and Figure13 are same, the hyperbolicity equations for the ideal triangulation in Figure 13 is equivalent to (36). By Lemma 4.14 and the definition of c_k $(k = 1, 2, \ldots, n-1)$, $(c_1, c_2, \ldots, c_{n-2}, c_{n-1})$ satisfies the hyperbolicity equations for the ideal triangulation in Figure 13. Hence we obtain the holonomy representation $\rho'\colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{PGL}_2\mathbb{F}_p$.

We prove that we can take a lift of ρ' to an $\mathrm{SL}_2\mathbb{F}_p$ representation. By hyperbolicity equations, ρ' is a parabolic representation. Namely, all the images of meridians are conjugate with $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $(* \in \mathbb{F}_p^\times)$. Hence, $\rho'(X),\ \rho'(Y) \in \mathrm{PSL}_2\mathbb{F}_p$, where X and Y are the generators of the presentation (26) of $\pi_1(S^3 \backslash \mathcal{T}_n)$. Therefore ρ' is a $\mathrm{PSL}_2\mathbb{F}_p$ representation. We take a lift $\widetilde{\rho}$ of ρ' to an $\mathrm{SL}_2\mathbb{F}_p$ representation whose eigenvalues are 1's. For some P which is a word of X and Y, we can rewritten (26) as

$$\left\langle X, Y \;\middle|\; P X^{\varepsilon} P^{-1} = Y \right\rangle,$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Hence, by

$$\rho'(P)\rho'(X)^{\varepsilon}\rho'(P)^{-1} = \rho'(Y),$$

we can take a lift $\widetilde{\rho'}$ of ρ' to an $\mathrm{SL}_2\mathbb{F}_p$ representation such that

$$\widetilde{\rho'}(P)\widetilde{\rho'}(X)^{\varepsilon}\widetilde{\rho'}(P)^{-1} = \pm\widetilde{\rho'}(Y).$$

We note that the ambiguity of ± 1 on the right-hand side comes from taking a lift. Since $\widetilde{\rho'}$ is also parabolic, we have both eigenvalues of $\widetilde{\rho'}(X), \widetilde{\rho'}(Y)$ are 1's. Hence, we obtain that

$$\widetilde{\rho'}(P)\widetilde{\rho'}(X)^{\varepsilon}\widetilde{\rho'}(P)^{-1} = \widetilde{\rho'}(Y).$$

Therefore we can take a lift of ρ' to an $\mathrm{SL}_2\mathbb{F}_p$ representation $\widetilde{\rho'}$ canonically.

We prove that the parabolic representation ρ and the holonomy representation $\widetilde{\rho'}$ are conjugate. We set $c' = \operatorname{trace} \widetilde{\rho'}(XY) - 2$. Since we obtained the holonomy representation $(\pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{PGL}_2\mathbb{F}_p)$ by applying the construction of the holonomy representations by Ohtsuki and Takata [12] to the case of $\mathrm{PGL}_2\mathbb{F}_p$, X_0, X_0', Z_0, W_0 in Figure 14 are given as follows:

$$X_0 \sim \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ Z_0 \sim \pm\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix},\ X_0' \sim \pm\begin{pmatrix} 1 & c_1 - 1 \\ 0 & 1 \end{pmatrix},\ W_0 \sim \pm\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Here " $\sim$ " means the equality in $\mathrm{PGL}_2\mathbb{F}_p$. We note that there are ambiguities of ± 1 since we lifted a $\mathrm{PGL}_2\mathbb{F}_p$ representation to an $\mathrm{SL}_2\mathbb{F}_p$ representation. Hence it follows that

$$\widetilde{\rho'}(XY) \sim X_0 Z_0 X_0' \sim \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & c_1 - 1 \\ 0 & 1 \end{pmatrix} = \pm\begin{pmatrix} 0 & 1 \\ -1 & 3 - c_1 \end{pmatrix},$$

and

$$c' + 2 = \operatorname{trace} \widetilde{\rho'}(XY) = \operatorname{trace} X_0 Z_0 X_0' = \pm(3 - c_1) = \pm(c + 2) \tag{40}$$

where we obtain the last equality since $c_1 = F_1(c) = 1 - c$. Moreover it follows that

$$\widetilde{\rho'}(Y^{-1}X) \sim X_0' W_0 X_0 \sim \pm\begin{pmatrix} 1 & c_1 - 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \pm\begin{pmatrix} c_1 + 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence it follows that

$$2 - c' = \operatorname{trace} \widetilde{\rho}'(Y^{-1}X) = \operatorname{trace} X_0' W_0 X_0 = \pm(c_1 + 1) = \pm(2 - c). \tag{41}$$

By (40) and (41), we obtain the following system of equations

$$\begin{cases} c' + 2 = \pm(c + 2), \\ c' - 2 = \pm(c - 2). \end{cases} \tag{42}$$

If $c \neq c'$, then (42) is rewritten as

$$\begin{cases} c + c' = 4, \\ c + c' = -4, \end{cases}$$

and it contradicts. Hence we have that $c = c'$. Therefore, by Lemma 4.11, ρ and ρ' are conjugate.

By Proposition 4.8, $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho)$ is equal to the summation of the moduli of the ideal tetrahedra of an ideal triangulation of $S^3 \backslash \mathcal{T}_n$ in $\check{\mathcal{P}}(\mathbb{F}_p)$. By this and the ideal triangulation in Figure 13, the reduced DW invariant is given as follows:

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho) &= (\text{modulus of } T_0^-) + (\text{modulus of } T_1^+) + \sum_{\mathrm{i=1}}^{\mathrm{n-2}} (\text{modulus of } T_i) \in \check{\mathcal{P}}(\mathbb{F}_\mathrm{p}) \\ &= 2[c_1] + \sum_{i=1}^{n-2} \left[\frac{c_{i+1}}{c_i} \right] \in \check{\mathcal{P}}(\mathbb{F}_p). \end{aligned} \tag{43}$$

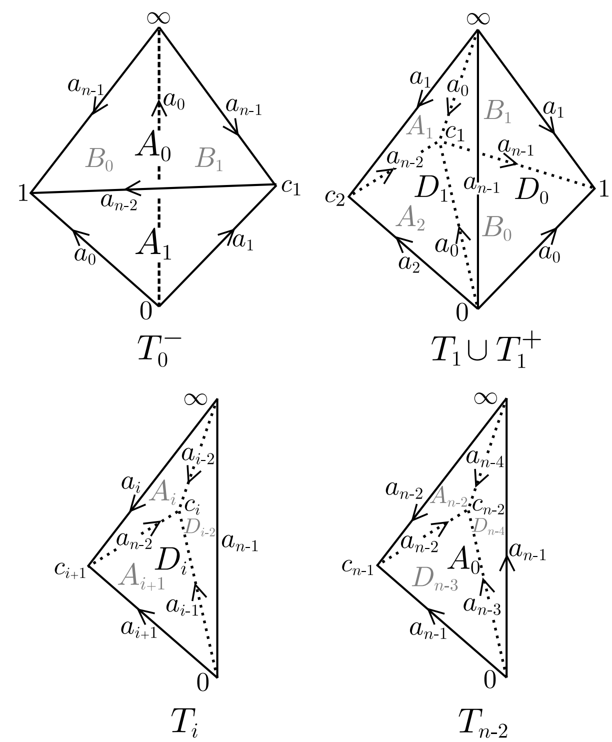

Figure 13

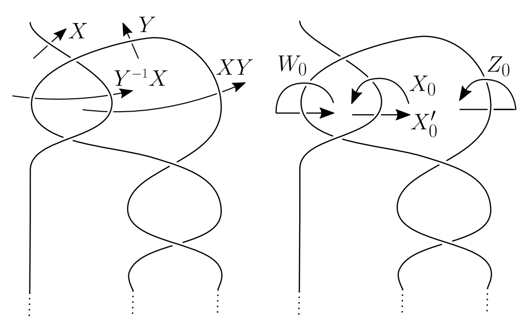

Figure 14

The case where $(c_1, c_2, \dots, c_n)$ **is of the form (ii) of Lemma 4.19** We consider the case of (ii) in Lemma 4.19, that is, there exist 0, 1, and ∞ in $(c_1, c_2, \dots, c_{n-1})$ successively in order, and 3-tuples $0, 1, \infty$ does not appear successively, and $c_1 \neq 0$, and $c_i \neq c_{i+1}$ $(1 \leq i \leq n-1)$. When $(c_j, c_{j+1}, c_{j+2}) = (0, 1, \infty)$, we give an ideal triangulation of the complement of the n-twist knot and a $\mathbb{P}^1(\mathbb{F}_p)$-labeling of ideal vertices as in Figure 15. In the case where there are more zeros, we can give a labeling similarly. Then all ideal tetrahedra do not collapse.

We obtain the hyperbolicity equations and the equation for the condition that the holonomy representation is parabolic. The system of such equations can be rewritten as the following system of equations,

$$\begin{cases} 1-\frac{c_{j-1}}{c_{j-2}}+c_{j-1}=0, \\ c_{j+4}-c_{j+3}-1=0, \\ c_{i+2}=1-\frac{c_{j+1}}{c_i}+c_{i+1}, \quad (i=1,2,\ldots,j\hat{-}2,\ j\hat{-}1,\ \hat{j},\ j\hat{+}1,\ j\hat{+}2,\ldots,n-3), \\ 1-\frac{c_{n-1}}{c_{n-2}}+c_{n-1}=0. \end{cases} \tag{44}$$

Since c_i is defined by $F_k(c)$, (44) satisfies by the recursive formula of $F_k(c)$. Hence we obtain the holonomy representation. Similarly as (i), ρ can lift to an $\mathrm{SL}_2\mathbb{F}_p$-representation. Moreover it follows that the lift of ρ is conjugate to a parabolic representation which correspond with $c \in \{u \in \mathbb{F}_p \mid F_n(u)=0\}$.

Lemma 4.20. When $(c_k, c_{k+1}, c_{k+2}) = (0, 1, \infty)$, we obtain that $c_{k-1}c_{k+3} = 1$.

Proof. What we should prove is that $\frac{F_{k-1}(c)}{F_{k-3}(c)}\frac{F_{k+3}(c)}{F_{k+1}(c)} = 1$. We obtain that $F_{k+1}(c) = F_{k-1}(c)$ by $c_{k+1} = 1$, and that

$$\frac{F_{k-1}(c)}{F_{k-3}(c)}\frac{F_{k+3}(c)}{F_{k+1}(c)} = \frac{F_{k+3}(c)}{F_{k-3}(c)}.$$

Hence it is enough to prove that $F_{k+3}(c) = F_{k-3}(c)$. Since $c_k = 0$, we have $F_k(c) = 0$. Therefore, by the recursive formula of $F_k(u)$, we obtain that $-cF_{k-1}(c) + F_{k-2}(c) = F_k(c) = 0$, that is, $F_{k-2}(c) = cF_{k-1}(c)$ and also obtain that $F_{k+2}(c)+cF_{k+1}(c) = F_k(c) = 0$, that is, $F_{k+2}(c) = -cF_{k+1}(c)$. By substituting these for

$$F_{k+3}(c) = -cF_{k+2}(c) + F_{k+1}(c),\ F_{k-3}(c) = F_{k-1}(c) + cF_{k-2}(c)$$

and using $F_{k+1}(c) = F_{k-1}(c)$, we obtain that

$$F_{k+3}(c) = (c^2+1)F_{k+1}(c) = (c^2+1)F_{k-1}(c) + cF_{k-2}(c) = F_{k-3}(c).$$

Therefore the lemma holds. □

By Proposition 4.8, $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho)$ is equal to the summation of the moduli of the ideal tetrahedra of an ideal triangulation of $S^3\backslash\mathcal{T}_n$ in $\check{\mathcal{P}}(\mathbb{F}_p)$. By this and the ideal triangulation in Figure 15, the reduced DW invariant is given as follows:

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_n,\rho) &= [\text{modulus of } T_0^-] + [\text{modulus of } T_1^+] + \sum_{i=j-1}^{j+2}[\text{modulus of } S_i] \\ &+ \sum_{i=1}^{j-2}[\text{modulus of } T_i] + \sum_{i=j+3}^{n-2}[\text{modulus of } T_i] \in \check{\mathcal{P}}(\mathbb{F}_p). \end{aligned}$$

Moreover, by Remark 4.21 below, the reduced DW invariant is given as follows:

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_n,\rho) &= [\text{modulus of } T_0^-] + [\text{modulus of } T_1^+] \\ &+ \sum_{i=1}^{j-2}[\text{modulus of } T_i] + \sum_{i=j+3}^{n-2}[\text{modulus of } T_i] \\ &= 2[c_1] + \sum_{i=1}^{j-2}\left[\frac{c_{i+1}}{c_i}\right] + \sum_{i=j+3}^{n-2}\left[\frac{c_{i+1}}{c_i}\right] \in \check{\mathcal{P}}(\mathbb{F}_p). \end{aligned} \tag{45}$$

In the case that there are more zeros $0, 1, \infty$ in $(c_1, c_2, \dots, c_{n-2}, c_{n-1})$, $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho)$ can be calculated similarly.

Remark 4.21. In Figure 15, it follows that

$$[\text{modulus of } S_k] = -[\text{modulus of } S_{k+1}] \quad (k = j-1, j+1)$$

by (2). Hence the summation of moduli of S_{j-1}, S_j, S_{j+1}, and S_{j+2} is 0 in $\check{\mathcal{P}}(\mathbb{F}_p)$.

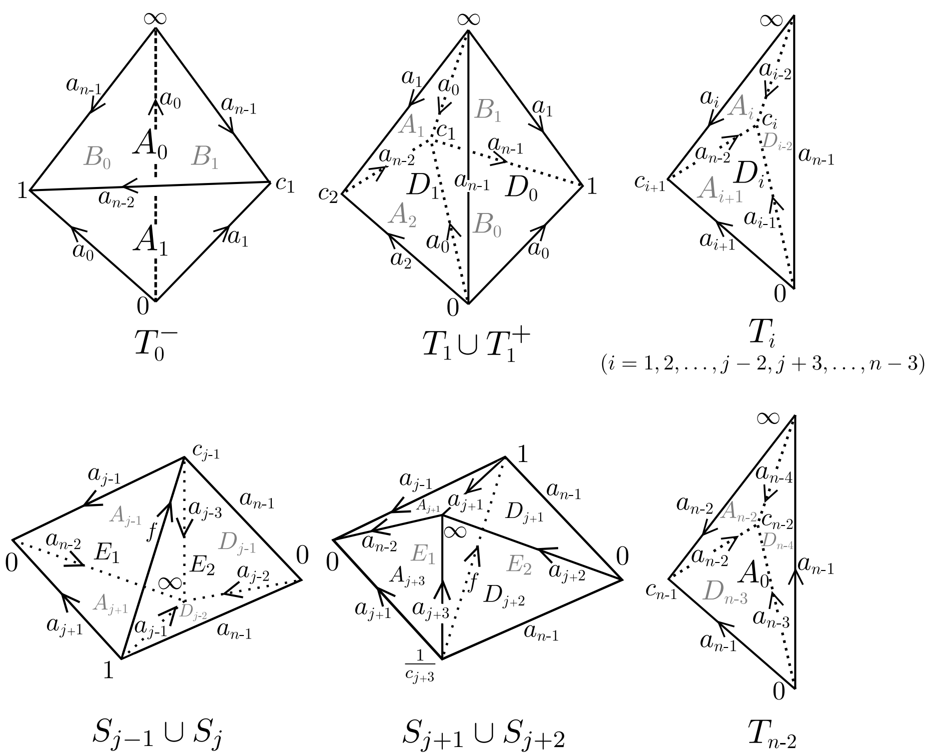

Figure 15

The case where $(c_1, c_2, \dots, c_n)$ **is of the form (iii) of Lemma 4.19** We consider the case of (iii) in Lemma 4.19, that is, it follows that $(c_1, c_2, c_3) = (0, 1, \infty)$, and 3-tuples $0, 1, \infty$ does not appear successively in $(c_1, c_2, \dots, c_{n-1})$, and $c_i \neq c_{i+1}$ $(1 \leq i \leq n-1)$. we give an ideal triangulation of the complement of the n-twist knot and a $\mathbb{P}^1(\mathbb{F}_p)$-labeling of ideal vertices as in Figure 16. Then all ideal tetrahedra do not collapse.

We obtain the hyperbolicity equations and the equation for the condition that the holonomy representation is parabolic. The system of such equations can be rewritten as the following system of equations,

$$\begin{cases} c_{i+2} = 1 - \frac{c_{j+1}}{c_i} + c_{i+1} \quad (i = 4, 5, \dots, n-3), \\ 1 - \frac{c_{n-1}}{c_{n-2}} + c_{n-1} = 0. \end{cases} \tag{46}$$

Noting that c_i is defined by $F_k(c)$, we obtain (46) from the recursive formula of $F_k(c)$. Hence we obtain the holonomy representation. Similarly as the case (i), ρ can lift to an

$\mathrm{SL}_2\mathbb{F}_p$-representation. Moreover, by considering ideal triangulations concretely, it follows that the lift of ρ is conjugate to a parabolic representation which correspond with $c \in \{u \in \mathbb{F}_p \mid F_n(u) = 0\}$.

By Proposition 4.8, $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho)$ is equal to the summation of the moduli of the ideal tetrahedra of an ideal triangulation of $S^3 \setminus \mathcal{T}_n$ in $\check{\mathcal{P}}(\mathbb{F}_p)$. By this and the ideal triangulation in Figure 16, the reduced DW invariant is given as follows:

$$\widehat{\mathrm{DW}}(T_n, \rho) = \sum_{i=1}^{6}[\text{modulus of } S_i'] + \sum_{i=4}^{n-2}[\text{modulus of } T_i] \in \check{\mathcal{P}}(\mathbb{F}_p).$$

We note that the above is the case where moduli of T_i $(i = 4, 5, \ldots, n-2)$ are not collapse (the case can result in (i)); if a modulus collapses, we result in (ii). Moreover by Remark 4.22, the reduced DW invariant is rewritten as follows:

$$\begin{aligned} \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho) &= \sum_{i=4}^{n-2}[\text{modulus of } T_i] \\ &= \sum_{i=4}^{n-2}\left[\frac{c_{i+1}}{c_i}\right]. \end{aligned} \tag{47}$$

Remark 4.22. In Figure 16, $S_1' \cup S_2' \cup S_3' \cup S_4' \cup S_5' \cup S_6'$ are symmetric with respect to the central horizontal face. By using (2), the summation of moduli of $S_1', S_2', S_3', S_4', S_5', S_6'$ in $\check{\mathcal{P}}(\mathbb{F}_p)$ is zero.

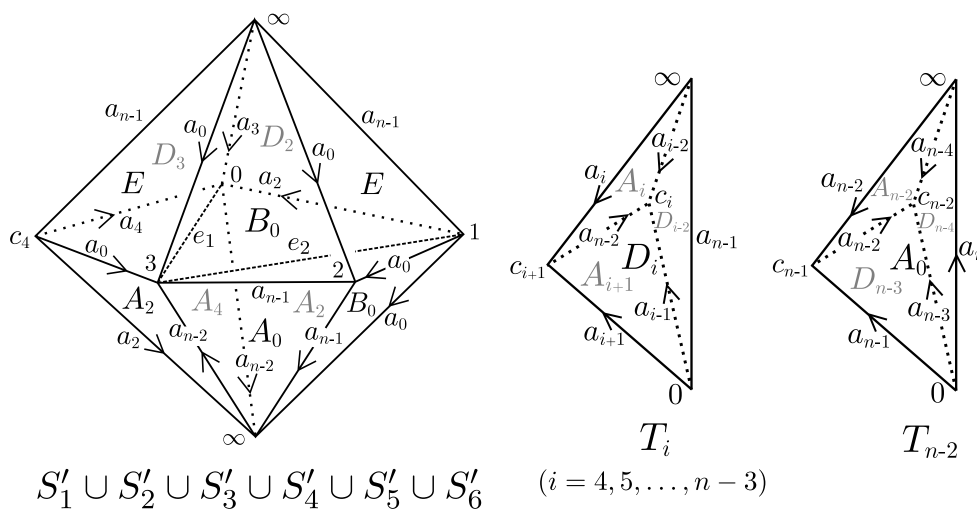

Figure 16

5 Proof of the theorems

In this section, we give proofs of Theorems 3.1, 3.2, 3.4. First, as the first half of the proof of the theorems, we describe that we can calculate the reduced DW invariant of

n-twist knot $\mathcal{T}_n$ by using the sequences in Appendix F. Next, as the second half of the proof of each Theorem 3.1, 3.4, we classify the sequences by Lemma 4.19, calculate the reduced DW invariant of $\mathcal{T}_n$ and a representation, and calculate the reduced DW invariant of $\mathcal{T}_n$ as the summation of them. Finally, we give the second half of the proof of Theorem 3.2 by using that the reduced DW invariant of $\mathcal{T}_n$ and $\mathbb{F}_{11}$ is equal to the number of the conjugacy classes of parabolic representations $\pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_p)$.

The first half of the proof of Theorem 3.1, 3.2, and 3.4 What we should calculate is $\widehat{\mathrm{DW}}(\mathcal{T}_n)$. This was defined in Section 2.3 as

$$\widehat{\mathrm{DW}}(\mathcal{T}_n) = \sum_{\rho} \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho). \tag{48}$$

Here, the summation on the right-hand side is over all the conjugacy classes of parabolic representations $\pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_p)$.

The conjugacy classes of parabolic representations $\pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_p)$ correspond one-to-one with elements of $\{u \in \mathbb{F}_p \mid F_n(u) = 0\}$ by Proposition 4.11. Under this correspondence, we denote the conjugacy class which correspond with $c \in \{u \in \mathbb{F}_p \mid F_n(u) = 0\}$ by ρ_{1-c}. Then (48) can be rewritten as

$$\widehat{\mathrm{DW}}(\mathcal{T}_n) = \sum_{c} \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{1-c}). \tag{49}$$

Here, the summation on the right-hand side is over all $c \in \{u \in \mathbb{F}_p \mid F_n(u) = 0\}$. The value of $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{1-c})$ can be calculated by using a sequence $(c_1, c_2, c_3, \dots)$ obtained from $c \in \{u \in \mathbb{F}_p \mid F_n(u) = 0\}$. The representation ρ_{1-c} can be denoted by ρ_{c_1} since c is related to c_1 in $(c_1, c_2, c_3, \dots)$ as $c_1 = F_1(c) = 1 - c$. Hence (49) can be rewritten as

$$\widehat{\mathrm{DW}}(\mathcal{T}_n) = \sum_{c_1} \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{c_1}). \tag{50}$$

Here the summation on the right-hand side is over all c_1 in $(c_1, c_2, c_3, \dots)$ satisfying that there is an n such that $c_n = 0$. We enumerate sequences $(c_1, c_2, c_3, \dots)$ satisfying that there is an n such that $c_n = 0$ in (87), and the sequences are classified into (i), (ii), and (iii) by Lemma 4.19. For each case, we gave an ideal triangulation of the complement of $\mathcal{T}_n$ such that the modulus of each ideal tetrahedron does not collapse; see Figure 15 and 16, and the summation of the moduli in $\check{\mathcal{P}}(\mathbb{F}_p)$ is equal to $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{c_1})$ by Proposition 4.8.

Proof (The second half of the proof of Theorem 3.1)**.** In the case where $p = 7$, since $c_n = 0$ for some n, the possible values of c_1 are $0, 2, 3, 4, 5, 6$ by (87). For each possible value of c_1, we calculate the reduced DW invariant of (K, ρ_{c_1}).

In the case where $c_1 = 3$ For the representation $\rho_3 \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_7)$ obtained from that $c_1 = 3$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_3)$. By (87), a sequence which we should consider is that $(c_1, c_2, c_3, \dots) = (3, 0, 1, \infty, 5, 6, 3, \dots)$(period 6). Hence, $n \equiv 2 \pmod 6$. Further, by Lemma 4.19,

$$\begin{cases} n = 2 & \text{when the sequence is of the form (i),} \\ n > 2 & \text{when the sequence is of the form (ii).} \end{cases}$$

By the classification in Lemma 4.19, the case where $n=2$ is classified as (i), and the case where $n>2$, $n \equiv 2 \pmod 6$ is classified as (ii).
In the case of (i). A sequence what we should consider is

$$(c_1, c_2) = (3, 0).$$

By (43), the reduced DW invariant of $(\mathcal{T}_n, \rho_3)$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_2, \rho_3) = [c_1] + [c_1] = 2[3] \in \check{\mathcal{P}}(\mathbb{F}_7).$$

Hence $\widehat{\mathrm{DW}}(\mathcal{T}_2, \rho_3)$ is mapped, by the isomorphism $\check{\mathcal{B}}(\mathbb{F}_7) \to \mathbb{Z}/2\mathbb{Z}$ in Lemma B.2, to $1 \in \mathbb{Z}/2\mathbb{Z}$. By identifying $\mathbb{Z}/2\mathbb{Z}$ and $\langle\, t \,|\, t^2 = 1\,\rangle$, this value is rewritten as,

$$\widehat{\mathrm{DW}}(\mathcal{T}_2, \rho_3) = t \in \langle\, t \,|\, t^2 = 1\,\rangle.$$

In the case of (ii). A sequence what we should consider is

$$(c_1, c_2, c_3, \dots, c_n) = (3, 0, 1, \infty, 5, 6, 3, \dots, 0, 1, \infty, 5, 6, 3, 0) \quad (n > 2,\ n \equiv 2 \bmod 6).$$

Since the sequence is of period 6, similarly as the case of (i), we calculate $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_3)$ by (45), as follows,

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_3) = 2[3] + \sum_{i=1}^{\frac{n-2}{6}} \left(\left[\frac{c_{6i}}{c_{6i-1}} \right] + \left[\frac{c_{6i+1}}{c_{6i}} \right] \right) \in \check{\mathcal{P}}(\mathbb{F}_7). \tag{51}$$

Since the sequence is of period 6, $\left[\frac{c_{6i}}{c_{6i-1}} \right]$ and $\left[\frac{c_{6i+1}}{c_{6i}} \right]$ are equal to $\left[\frac{c_6}{c_5} \right]$ and $\left[\frac{c_7}{c_6} \right]$ respectively. Then (51) is rewritten as

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_3) = 2[3] + \frac{n-2}{6} \left(\left[\frac{c_6}{c_5} \right] + \left[\frac{c_7}{c_6} \right] \right) = 2[3] + \frac{n-2}{6}(2[4]) = 2[3] \in \check{\mathcal{P}}(\mathbb{F}_7).$$

where we obtain the last equality since $2[4] = 0 \in \check{\mathcal{P}}(\mathbb{F}_7)$ by Lemma B.2. By the isomorphism $\check{\mathcal{B}}(\mathbb{F}_7) \to \mathbb{Z}/2\mathbb{Z}$ in Lemma B.2, this value is mapped to $1 \in \mathbb{Z}/2\mathbb{Z}$. Moreover, by identifying $\mathbb{Z}/2\mathbb{Z}$ and $\langle\, t \,|\, t^2 = 1\,\rangle$, this value is rewritten as

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_3) = t \in \langle\, t \,|\, t^2 = 1\,\rangle.$$

Therefore the reduced DW invariant of $(\mathcal{T}_n, \rho_3)$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_3) = \begin{cases} t & \text{if } n \equiv 2 \pmod 6, \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

In the case where $c_1 = 2$ For the representation $\rho_2 \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_7)$ obtained from that $c_1 = 2$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_2)$. By (87), we have that $(c_1, c_2, c_3, \dots) = (2, 3, 6, 5, 4, 0, 1, \infty, 2, \dots)$(period 8). Hence, $n \equiv 6 \pmod 8$. Further, by Lemma 4.19,

$$\begin{cases} n = 6 & \text{when the sequence is of the form (i),} \\ n > 6 & \text{when the sequence is of the form (ii).} \end{cases}$$

In the case of (i). Similarly as the case where $c_1 = 3$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_6, \rho_2) = t \in \langle\, t \,|\, t^2 = 1 \,\rangle.$$

In the case of (ii). Similarly as the case where $c_1 = 3$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) = \widehat{\mathrm{DW}}(\mathcal{T}_6, \rho_2) + \frac{n-6}{8} \sum_{i=9}^{12} \left[\frac{c_{i+1}}{c_i}\right] \in \check{\mathcal{P}}(\mathbb{F}_7).$$

By concrete calculation and the isomorphism between $\check{\mathcal{B}}(\mathbb{F}_7)$ and $\langle\, t \,|\, t^2 = 1 \,\rangle$,

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) = t^{(n+2)/8} \in \langle\, t \,|\, t^2 = 1 \,\rangle.$$

Therefore the reduced DW invariant of (T_n, ρ_2) is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) = \begin{cases} t^{(n+2)/8} & \text{if } n \equiv 6 \pmod 8, \\ 0 & \text{otherwise.} \end{cases} \tag{53}$$

In the case where $c_1 = 4$ For the representation $\rho_4 \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_7)$ obtained from that $c_1 = 4$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_4)$. By (87), we have that $(c_1, c_2, c_3, \dots) = (4, 6, 2, 5, 0, 1, \infty, 3, 4, \dots)$(period 8). Hence, $n \equiv 5 \pmod 8$. Further, by Lemma 4.19,

$$\begin{cases} n = 5 & \text{when the sequence is of the form (i),} \\ n > 5 & \text{when the sequence is of the form (ii).} \end{cases}$$

In the case of (i). Similarly as the case where $c_1 = 3$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_5, \rho_4) = 1 \in \langle\, t \,|\, t^2 = 1 \,\rangle.$$

In the case of (ii). Similarly as the case where $c_1 = 3$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_4) = \widehat{\mathrm{DW}}(\mathcal{T}_5, \rho_4) + \frac{n-5}{8} \sum_{i=8}^{11} \left[\frac{c_{i+1}}{c_i}\right] \in \check{\mathcal{P}}(\mathbb{F}_7).$$

By concrete calculation and the isomorphism between $\check{\mathcal{B}}(\mathbb{F}_7)$ and $\langle\, t \,|\, t^2 = 1 \,\rangle$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) = t^{(n-5)/8} \in \langle\, t \,|\, t^2 = 1 \,\rangle.$$

Therefore the reduced DW invariant of $(\mathcal{T}_n, \rho_4)$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_4) = \begin{cases} t^{(n-5)/8} & \text{if } n \equiv 5 \pmod 8, \\ 0 & \text{otherwise.} \end{cases} \tag{54}$$

In the case where $c_1 = 5$ For the representation $\rho_5 \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_7)$ obtained from that $c_1 = 5$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_5)$. By (87), we have that

$(c_1, c_2, c_3, \dots) = (5, 0, 1, \infty, 3, 4, 2, 6, 5, \dots)$(period 8). Hence, $n \equiv 2 \pmod 8$. Further, by Lemma 4.19,

$$\begin{cases} n = 2 & \text{when the sequence is of the form (i)}, \\ n > 2 & \text{when the sequence is of the form (ii)}. \end{cases}$$

In the case of (i). Similarly as the case where $c_1 = 3$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_2, \rho_5) = t \in \langle\, t \,|\, t^2 = 1 \,\rangle.$$

In the case of (ii). Similarly as the case where $c_1 = 3$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_5) = \widehat{\mathrm{DW}}(\mathcal{T}_2, \rho_5) + \frac{n-2}{8} \sum_{i=5}^{8} \left[\frac{c_{i+1}}{c_i} \right] \in \check{\mathcal{P}}(\mathbb{F}_7).$$

By concrete calculation and the isomorphism between $\check{\mathcal{B}}(\mathbb{F}_7)$ and $\langle\, t \,|\, t^2 = 1 \,\rangle$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) = t^{(n+6)/8} \in \langle\, t \,|\, t^2 = 1 \,\rangle.$$

Therefore the reduced DW invariant of $(\mathcal{T}_n, \rho_5)$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_5) = \begin{cases} t^{(n+6)/8} & \text{if } n \equiv 2 \pmod 8, \\ 0 & \text{otherwise}. \end{cases} \tag{55}$$

In the case where $c_1 = 6$ For the representation $\rho_6 \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_7)$ obtained from that $c_1 = 6$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_6)$. By (87), we have that $(c_1, c_2, c_3, \dots) = (6, 3, 0, 1, \infty, 5, 6, \dots)$(period 6). Hence, $n \equiv 3 \pmod 6$. Further, by Lemma 4.19,

$$\begin{cases} n = 3 & \text{when the sequence is of the form (i)}, \\ n > 3 & \text{when the sequence is of the form (ii)}. \end{cases}$$

In the case of (i). Similarly as the case where $c_1 = 3$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_3, \rho_6) = t \in \langle\, t \,|\, t^2 = 1 \,\rangle.$$

In the case of (ii). Similarly as the case where $c_1 = 3$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_6) = \widehat{\mathrm{DW}}(\mathcal{T}_3, \rho_6) + \frac{n-3}{6} \sum_{i=6}^{7} \left[\frac{c_{i+1}}{c_i} \right] \in \check{\mathcal{P}}(\mathbb{F}_7).$$

By concrete calculation and the isomorphism between $\check{\mathcal{B}}(\mathbb{F}_7)$ and $\langle\, t \,|\, t^2 = 1 \,\rangle$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_6) = t \in \langle\, t \,|\, t^2 = 1 \,\rangle.$$

Therefore the reduced DW invariant of $(\mathcal{T}_n, \rho_6)$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_6) = \begin{cases} t & \text{if } n \equiv 3 \,(\mathrm{mod}\, 6), \\ 0 & \text{otherwise}. \end{cases} \tag{56}$$

In the case where $c_1 = 0$ For the representation $\rho_0\colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_7)$ obtained from that $c_1 = 0$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_0)$. By (87), a sequence which we should consider is $(c_1, c_2, c_3, \dots) = (0, 1, \infty, 2, 3, 6, 5, 4, 0, \dots)$(period 8). Hence, $n \equiv 1 \pmod 8$. Further, by Lemma 4.19,

$$n > 1 \text{ when the sequence is of the form (iii).}$$

In the case of (iii). For the representation in the case where $n \equiv 1 \,(\mathrm{mod}\, 8)$, we calculate the reduced DW invariant of n-twist knots. A sequence which we should consider is

$$(c_1, c_2, c_3, \dots, c_n) = (0, 1, \infty, 2, 3, 6, 5, 4, 0, \dots, 2, 3, 6, 5, 4, 0) \ \ (n \equiv 1 \text{ mod} 8).$$

Since the sequence is of period 8, we calculate $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0)$ by (47), as follows,

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = \sum_{i=1}^{\frac{n-1}{8}} \left(\left[\frac{c_{8i-3}}{c_{8i-4}}\right] + \left[\frac{c_{8i-2}}{c_{8i-3}}\right] + \left[\frac{c_{8i-1}}{c_{8i-2}}\right] + \left[\frac{c_{8i}}{c_{8i-1}}\right] \right) \in \check{\mathcal{P}}(\mathbb{F}_7). \tag{57}$$

Since the sequence is of period 8, $\left[\frac{c_{8i-3}}{c_{8i-4}}\right]$, $\left[\frac{c_{8i-2}}{c_{8i-3}}\right]$, $\left[\frac{c_{8i-1}}{c_{8i-2}}\right]$, $\left[\frac{c_{8i}}{c_{8i-1}}\right]$ are equal to $\left[\frac{c_5}{c_4}\right]$, $\left[\frac{c_6}{c_5}\right]$, $\left[\frac{c_7}{c_6}\right]$, $\left[\frac{c_8}{c_7}\right]$ respectively. Then (57) is rewritten as

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = \frac{n-1}{8}\left(\left[\frac{c_5}{c_4}\right] + \left[\frac{c_6}{c_5}\right] + \left[\frac{c_7}{c_6}\right] + \left[\frac{c_8}{c_7}\right] \right) = \frac{n-1}{8}(2[2]+2[5]) = \frac{n-1}{8}(2[5]) \in \check{\mathcal{P}}(\mathbb{F}_7)$$

where we obtain the last equality since $2[2] = 0 \in \check{\mathcal{P}}(\mathbb{F}_7)$ by Lemma B.2 This value is mapped, by the isomorphism $\check{\mathcal{B}}(\mathbb{F}_7) \to \mathbb{Z}/2\mathbb{Z}$ in Lemma B.2, to $1 \in \mathbb{Z}/2\mathbb{Z}$. Moreover this value can be, by identifying $\mathbb{Z}/2\mathbb{Z}$ and $\langle\, t \,|\, t^2 = 1 \,\rangle$,

$$t \in \langle\, t \,|\, t^2 = 1 \,\rangle.$$

Hence we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = t^{(n-1)/8} \in \langle\, t \,|\, t^2 = 1 \,\rangle.$$

Therefore the reduced DW invariant of $(\mathcal{T}_n, \rho_0)$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = \begin{cases} t^{(n-1)/8} & \text{if } n \equiv 1 \,(\mathrm{mod}\, 8), \\ 0 & \text{otherwise.} \end{cases} \tag{58}$$

The reduced DW invariants of $(\mathcal{T}_n, \rho_{c_1})$ are given as (52), (53), (54), (55), (56), and (58). By (52) and (56), we put

$$A_n = \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_3) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_6) = \begin{cases} t & \text{if } n \equiv 2, 3 \,(\mathrm{mod}\, 6), \\ 0 & \text{otherwise.} \end{cases}$$

By (53), (54), (55), and (58), we put

$$B_n = \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_4) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_5) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = \begin{cases} t^{(n-1)/8} & \text{if } n \equiv 1 \,(\mathrm{mod}\, 8), \\ t^{(n+6)/8} & \text{if } n \equiv 2 \,(\mathrm{mod}\, 8), \\ t^{(n-5)/8} & \text{if } n \equiv 5 \,(\mathrm{mod}\, 8), \\ t^{(n+2)/8} & \text{if } n \equiv 6 \,(\mathrm{mod}\, 8), \\ 0 & \text{otherwise.} \end{cases}$$

By (50), we sum up these, and obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \mathbb{F}_7) = A_n + B_n \in \mathbb{Z}[\langle\, t \,|\, t^2 = 1\,\rangle].$$

Therefore the theorem holds. $\square$

Proof (The second half of the proof of Theorem 3.4)**.** In the case where $p = 13$, since $c_n = 0$ for some n, the possible values of c_1 are $0, 2, 4, 5, 10, 11$ by (89). For each possible value of c_1, we calculate the reduced DW invariant of (K, ρ_{c_1}).

In the case where $c_1 = 4$ The representation $\rho_4 \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_{13})$ obtained from that $c_1 = 4$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_4)$. By (89), a sequence which we should consider is $(c_1, c_2, c_3, \dots) = (4, 0, 1, \infty, 10, 11, 7, 5, 9, 3, 8, 2, 6, 4, \dots)$(period 13). Hence, $n \equiv 2 \pmod{13}$. Further, by Lemma 4.19,

$$\begin{cases} n = 2 & \text{when the sequence is of the form (i),} \\ n > 2 & \text{when the sequence is of the form (ii).} \end{cases}$$

In the case of (i). A sequence which we should consider is that

$$(c_1, c_2) = (4, 0).$$

By (43), the reduced DW invariant of $(\mathcal{T}_n, \rho_4)$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_2, \rho_4) = 2[c_1] = 2[4] \in \check{\mathcal{P}}(\mathbb{F}_{13}).$$

Hence, by the isomorphism $\check{\mathcal{B}}(\mathbb{F}_{13}) \to \mathbb{Z}/7\mathbb{Z}$ in Lemma B.6, $\widehat{\mathrm{DW}}(\mathcal{T}_2, \rho_4)$ is mapped to $4 \in \mathbb{Z}/7\mathbb{Z}$. by identifying $\mathbb{Z}/7\mathbb{Z}$ and $\langle\, t \,|\, t^7 = 1\,\rangle$, this value is rewritten as

$$t^4 \in \langle\, t \,|\, t^7 = 1\,\rangle.$$

In the case of (ii). A sequence which we should consider is

$$(c_1, c_2, c_3, \dots, c_n) = (4, 0, 1, \infty, \dots, 0, 1, \infty, 10, 11, 7, 5, 9, 3, 8, 2, 6, 4, 0) \ (n > 2,\ n \equiv 2 \bmod 13).$$

Since the sequence is of period 6, similarly as the case of (i), we calculate $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_4)$ by (45), as follows,

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_4) = 2[4] + \sum_{i=1}^{\frac{n-2}{13}} \sum_{j=0}^{8} \left[\frac{c_{13i-j+1}}{c_{13i-j}}\right] \in \check{\mathcal{P}}(\mathbb{F}_{13}). \tag{59}$$

Since the sequence is of period 6, $\left[\frac{c_{13i-j+1}}{c_{13i-j}}\right]$ $(j = 0, 1, \dots, 8)$ is equal to $\left[\frac{c_{13-j+1}}{c_{13-j}}\right]$ $(j = 0, 1, \dots, 8)$ respectively. Then (59) is rewritten as

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_4) = 2[4] + \frac{n-2}{13} \sum_{j=0}^{8} \left[\frac{c_{13-j+1}}{c_{13-j}}\right] = 2[4] + \frac{n-2}{13}(2[5] + 2[3] + 2[10] + 2[7] + [9]) \in \check{\mathcal{P}}(\mathbb{F}_{13}).$$

By the isomorphism $\check{\mathcal{B}}(\mathbb{F}_7) \to \mathbb{Z}/7\mathbb{Z}$ in Lemma B.6, $2[4]$ is mapped to $4 \in \mathbb{Z}/13\mathbb{Z}$, and $2[5]+2[3]+2[10]+2[7]+[9]$ is mapped to $0 \in \mathbb{Z}/7\mathbb{Z}$. Then we have that $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_4) = 4 \in \mathbb{Z}/7\mathbb{Z}$. Moreover, by identifying $\mathbb{Z}/7\mathbb{Z}$ and $\langle\, t \,|\, t^7 = 1 \,\rangle$, this value is rewritten as

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_4) = t^4 \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

Therefore the reduced DW invariant of $(\mathcal{T}_n, \rho_4)$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_4) = \begin{cases} t^4 & \text{if } n \equiv 2 \pmod{13}, \\ 0 & \text{otherwise.} \end{cases} \tag{60}$$

In the case where $c_1 = 2$ For the representation $\rho_2 \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_{13})$ obtained from that $c_1 = 2$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_2)$. By (89), we have that $(c_1, c_2, c_3, \dots) = (2, 3, 9, 7, 0, 1, \infty, 2, \dots)$(period 7). Hence, $n \equiv 5 \pmod 7$. Further, by Lemma 4.19,

$$\begin{cases} n = 5 & \text{when the sequence is of the form (i),} \\ n > 5 & \text{when the sequence is of the form (ii).} \end{cases}$$

In the case of (i). Similarly as the case where $c_1 = 4$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_5, \rho_2) = t \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

In the case of (ii). Similarly as the case where $c_1 = 4$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) = \widehat{\mathrm{DW}}(\mathcal{T}_5, \rho_2) + \frac{n-5}{7} \sum_{i=8}^{10} \left[\frac{c_{i+1}}{c_i} \right] \in \check{\mathcal{P}}(\mathbb{F}_{13}).$$

By concrete calculation and the isomorphism between $\check{\mathcal{B}}(\mathbb{F}_{13})$ and $\langle\, t \,|\, t^7 = 1 \,\rangle$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) = t^{(n+2)/7} \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

Hence the reduced DW invariant of (T_n, ρ_2) is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) = \begin{cases} t^{(n+2)/7} & \text{if } n \equiv 5 \pmod 7 \\ 0 & \text{otherwise.} \end{cases} \tag{61}$$

In the case where $c_1 = 5$ For the representation $\rho_5 \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_{13})$ obtained from that $c_1 = 5$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_5)$. By (89), we have that $(c_1, c_2, c_3, \dots) = (5, 8, 10, 0, 1, \infty, 4, , 5, \dots)$(period 7). Hence, $n \equiv 4 \pmod 7$. Further, by Lemma 4.19,

$$\begin{cases} n = 4 & \text{when the sequence is of the form (i),} \\ n > 4 & \text{when the sequence is of the form (ii).} \end{cases}$$

In the case of (i). Similarly as the case of $c_1 = 4$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_4, \rho_5) = t^6 \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

In the case of (ii). Similarly as the case where $c_1 = 4$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_5) = \widehat{\mathrm{DW}}(\mathcal{T}_4, \rho_5) + \frac{n-4}{7} \sum_{i=7}^{9} \left[\frac{c_{i+1}}{c_i}\right] \in \check{\mathcal{P}}(\mathbb{F}_7).$$

By concrete calculation and the isomorphism between $\check{\mathcal{B}}(\mathbb{F}_{13})$ and $\langle\, t \,|\, t^7 = 1 \,\rangle$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) = t^{(2n-15)/7} \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

Therefore the reduced DW invariant of $(\mathcal{T}_n, \rho_5)$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_5) = \begin{cases} t^{(2n-15)/7} & \text{if } n \equiv 4 \pmod 7 \\ 0 & \text{otherwise.} \end{cases} \tag{62}$$

In the case where $c_1 = 10$ For the representation $\rho_{10} \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_{13})$ obtained from that $c_1 = 10$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_{10})$. By (89), we have that $(c_1, c_2, c_3, \dots) = (10, 0, 1, \infty, 4, 5, 8, 10, \dots)$(period 7). Hence, $n \equiv 2 \pmod 7$. Further, by Lemma 4.19,

$$\begin{cases} n = 2 & \text{when the sequence is of the form (i),} \\ n > 2 & \text{when the sequence is of the form (ii).} \end{cases}$$

In the case of (i). Similarly as the case where $c_1 = 4$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_2, \rho_{10}) = t^3 \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

In the case of (ii). Similarly as the case where $c_1 = 4$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{10}) = \widehat{\mathrm{DW}}(\mathcal{T}_2, \rho_{10}) + \frac{n-2}{7} \sum_{i=5}^{7} \left[\frac{c_{i+1}}{c_i}\right] \in \check{\mathcal{P}}(\mathbb{F}_{13}).$$

By concrete calculation and the isomorphism between $\check{\mathcal{B}}(\mathbb{F}_{13})$ and $\langle\, t \,|\, t^7 = 1 \,\rangle$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{10}) = t^{(2n+17)/7} \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

Therefore the reduced DW invariant of $(\mathcal{T}_n, \rho_{10})$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{10}) = \begin{cases} t^{(2n+17)/7} & \text{if } n \equiv 2 \pmod 7 \\ 0 & \text{otherwise.} \end{cases} \tag{63}$$

In the case where $c_1 = 11$ For the representation $\rho_{11} \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_{13})$ obtained from that $c_1 = 11$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_{11})$. By (89), we have that $(c_1, c_2, c_3, \dots) = (11, 7, 5, 9, 3, 8, 2, 6, 4, 0, 1, \infty, 10, 11, \dots)$(period 13). Hence, $n \equiv 10$ (mod 13). Further, by Lemma 4.19,

$$\begin{cases} n = 10 & \text{when the sequence is of the form (i),} \\ n > 10 & \text{when the sequence is of the form (ii).} \end{cases}$$

In the case of (i). Similarly as the case where $c_1 = 4$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_{10}, \rho_{11}) = t \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

In the case of (ii). Similarly as the case where $c_1 = 4$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{11}) = \widehat{\mathrm{DW}}(\mathcal{T}_{10}, \rho_{11}) + \frac{n-10}{13}\sum_{i=13}^{21}\left[\frac{c_{i+1}}{c_i}\right] \in \check{\mathcal{P}}(\mathbb{F}_{13}).$$

By concrete calculation and the isomorphism between $\check{\mathcal{B}}(\mathbb{F}_{13})$ and $\langle\, t \,|\, t^7 = 1 \,\rangle$, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{11}) = t \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

Therefore the reduced DW invariant of $(\mathcal{T}_n, \rho_{11})$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{11}) = \begin{cases} t & \text{if } n \equiv 10 \pmod{13}, \\ 0 & \text{otherwise.} \end{cases} \tag{64}$$

In the case where $c_1 = 0$ For the representation $\rho_0 \colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2(\mathbb{F}_{13})$ obtained from that $c_1 = 0$, we calculate the reduced DW invariant of $(\mathcal{T}_n, \rho_0)$. By (89), a sequence which should consider is that $(c_1, c_2, c_3, \dots) = (0, 1, \infty, 2, 3, 9, 7, 0, \dots)$(period 7). Hence, $n \equiv 1 \pmod 7$. Further, by Lemma 4.19,

$$n > 1 \ \text{ when the sequence is of the form (i).}$$

In the case of (iii). For the representations in the case where $n \equiv 1 \,(\mathrm{mod}\, 7)$, we calculate the reduced DW invariant of n-twist knots. A sequence which we consider is

$$(c_1, c_2, c_3, \dots, c_n) = (0, 1, \infty, 2, 3, 9, 7, 0, \dots, 2, 3, 9, 7, 0) \ \ (\, n \equiv 1 \bmod 7).$$

Since the sequence is of period 7, we calculate $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0)$ by (47), as follows,

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = \sum_{i=1}^{\frac{n-1}{7}}\left(\left[\frac{c_{7i-2}}{c_{7i-3}}\right] + \left[\frac{c_{7i-1}}{c_{7i-2}}\right] + \left[\frac{c_{7i}}{c_{7i-1}}\right]\right) \in \check{\mathcal{P}}(\mathbb{F}_{13}). \tag{65}$$

Since the sequence is of period 7, $\left[\frac{c_{7i-2}}{c_{7i-3}}\right], \left[\frac{c_{7i-1}}{c_{7i-2}}\right], \left[\frac{c_{7i}}{c_{7i-1}}\right]$ are equal to $\left[\frac{c_5}{c_4}\right], \left[\frac{c_6}{c_5}\right], \left[\frac{c_7}{c_6}\right]$ respectively. Then (65) is rewritten as

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = \frac{n-1}{7}\left(\left[\frac{c_5}{c_4}\right] + \left[\frac{c_6}{c_5}\right] + \left[\frac{c_7}{c_6}\right]\right) = \frac{n-1}{7}(2[8] + [3]) \in \check{\mathcal{P}}(\mathbb{F}_{13}).$$

Hence, by the isomorphism $\check{\mathcal{B}}(\mathbb{F}_{13}) \to \mathbb{Z}/7\mathbb{Z}$ in Lemma B.6, $2[8] + [3]$ is mapped to $1 \in \mathbb{Z}/7\mathbb{Z}$. Moreover, by identifying $\mathbb{Z}/7\mathbb{Z}$ and $\langle\, t \,|\, t^7 = 1 \,\rangle$, this value is rewritten as

$$t \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

Hence we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = t^{(n-1)/7} \in \langle\, t \,|\, t^7 = 1 \,\rangle.$$

Therefore the reduced DW invariant of $(\mathcal{T}_n, \rho_0)$ is given as follows:

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = \begin{cases} t^{(n-1)/7} & \text{if } n \equiv 1 \pmod 7 \\ 0 & \text{otherwise.} \end{cases} \tag{66}$$

The reduced DW invariants of $(\mathcal{T}_n, \rho_{c_1})$ were given as (60),(61),(62),(63),(64),(66). By (60) and (64), we put

$$A_n = \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_4) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{11}) = \begin{cases} t & \text{if } n \equiv 10 \pmod{13}, \\ t^4 & \text{if } n \equiv 2 \pmod{13}, \\ 0 & \text{otherwise.} \end{cases}$$

By(61), (62), (63), and (66), we put

$$B_n = \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_5) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{10}) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = \begin{cases} t^{(n-1)/7} & \text{if } n \equiv 1 \pmod 7, \\ t^{(2n+17)/7} & \text{if } n \equiv 2 \pmod 7, \\ t^{(2n-15)/7} & \text{if } n \equiv 4 \pmod 7, \\ t^{(n+2)/7} & \text{if } n \equiv 5 \pmod 7, \\ 0 & \text{otherwise.} \end{cases}$$

By (50) and summing up these, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \mathbb{F}_{13}) = A_n + B_n \in \mathbb{Z}[\langle\, t \,|\, t^7 = 1 \,\rangle].$$

Therefore the theorem holds. $\square$

Proof (The second half of the proof of Theorem 3.2)**.** In the case where $p = 11$, since $c_n = 0$ for some n, the possible values of c_1 are $0, 2, 3, 5, 6, 7, 8, 10$ by (88). For each possible value of c_1, we calculate the reduced DW invariant of (K, ρ_{c_1}).

The reduced DW invariant of (K, ρ_{c_1}) can be, since $\check{\mathcal{B}}(\mathbb{F}_{11}) = \{0\}$ in Lemma B.4, $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{c_1}) = 0$. By identifying the trivial additive group $\{0\}$ and the trivial multiple group $\{1\}$, we have that $\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{c_1}) = 1$. Hence, when we obtain an $n \geq 2$ such that $c_n = 0$, we can calculate the reduced DW invariant of (K, ρ_{c_1}).

In the case where $c_1 = 0$ By (88), $n \geq 2$ such that $c_n = 0$ are $n = 10k + 1$ ($k = 1, 2, 3, \ldots$). Hence we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{10}, \\ 0 & \text{otherwise.} \end{cases} \tag{67}$$

In the case where $c_1 = 2$ By (88), $n \geq 2$ such that $c_n = 0$ are $n = 10k + 8$ ($k = 0, 1, 2, \ldots$). Hence we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) = \begin{cases} 1 & \text{if } n \equiv 8 \pmod{10}, \\ 0 & \text{otherwise.} \end{cases} \tag{68}$$

In the case where $c_1 = 3$ By (88), $n \geq 2$ such that $c_n = 0$ in $(c_1, c_2, c_3, \dots)$ are $n = 12k + 5$ $(k = 0, 1, 2, \dots)$. Hence we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_3) = \begin{cases} 1 & \text{if } n \equiv 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases} \tag{69}$$

In the case where $c_1 = 5$ By (88), $n \geq 2$ such that $c_n = 0$ are $n = 10k + 6$ $(k = 0, 1, 2, \dots)$. Hence we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_5) = \begin{cases} 1 & \text{if } n \equiv 6 \pmod{10}, \\ 0 & \text{otherwise.} \end{cases} \tag{70}$$

In the case where $c_1 = 6$ By (88), $n \geq 2$ such that $c_n = 0$ are $n = 12k + 4$ $(k = 0, 1, 2, \dots)$. Hence we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_6) = \begin{cases} 1 & \text{if } n \equiv 4 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases} \tag{71}$$

In the case where $c_1 = 7$ By (88), $n \geq 2$ such that $c_n = 0$ are $n = 12k + 7$ $(k = 0, 1, 2, \dots)$. Hence we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_7) = \begin{cases} 1 & \text{if } n \equiv 7 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases} \tag{72}$$

In the case where $c_1 = 8$ By (88), $n \geq 2$ such that $c_n = 0$ are $n = 10k + 3$ $(k = 0, 1, 2, \dots)$. Hence we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_8) = \begin{cases} 1 & \text{if } n \equiv 3 \pmod{10}, \\ 0 & \text{otherwise.} \end{cases} \tag{73}$$

In the case where $c_1 = 10$ By (88), $n \geq 2$ such that $c_n = 0$ are $n = 12k + 6$ $(k = 0, 1, 2, \dots)$. Hence we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{10}) = \begin{cases} 1 & \text{if } n \equiv 6 \pmod{12} \\ 0 & \text{otherwise.} \end{cases} \tag{74}$$

The reduced DW invariants of $(\mathcal{T}_n, \rho_{c_1})$ were given as (67), (68), (69), (70), (71), (72), (73), and (74). By (67), (68), (70), and (73), we put

$$A_n = \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_0) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_2) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_5) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_8) = \begin{cases} 1 & \text{if } n \equiv 1, 3 \pmod{5}, \\ 0 & \text{otherwise.} \end{cases}$$

By (69), (71), (72), and (74), we put

$$B_n = \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_3) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_6) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_7) + \widehat{\mathrm{DW}}(\mathcal{T}_n, \rho_{10}) = \begin{cases} 1 & \text{if } n \equiv 4, 5, 6, 7 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

By (50) and summing up these, we obtain that

$$\widehat{\mathrm{DW}}(\mathcal{T}_n, \mathbb{F}_{11}) = A_n + B_n \in \mathbb{Z}.$$

Therefore the theorem holds. □

A Invariance of the reduced DW invariant under some Pachner moves

In this section, we mention properties implying the existence of an invariant of knots whose value belongs to $\check{\mathcal{P}}(\mathbb{F}_p)$. It is known, see e.g. [7], that (ideal) triangulations of (cusped) homeomorphic 3-manifolds are related by Pachner moves. In the following, we review two lemmas implying partially the existence of an invariant whose value belongs to $\check{\mathcal{P}}(\mathbb{F}_p)$ by using Pachner moves for ideal triangulations of cusped 3-manifolds. These lemmas are already known, e.g. by Neumann [9], while they are not mentioned explicitly.

We assume that the modulus of each ideal tetrahedron does not collapse before and after applying a (0,2)-Pachner move. Since moduli does not collapse, we can regard moduli as elements of $\check{\mathcal{P}}(\mathbb{F}_p)$. We confirm that the summation of moduli in $\check{\mathcal{P}}(\mathbb{F}_p)$ does not change before and after applying a (0,2)-Pachner move.

Lemma A.1. We assume that, when we apply a (0,2)-Pachner move, the modulus of each tetrahedron appearing in the move does not collapse. Then the summation of moduli in $\check{\mathcal{P}}(\mathbb{F}_p)$ does not change before and after applying a (0,2)-Pachner move.

Proof. To prove the lemma, it is enough to prove that the summation of moduli of two tetrahedra in the right picture of Figure 17 in $\check{\mathcal{P}}(\mathbb{F}_p)$ is zero. A modulus does not depend on the action of $\mathrm{PGL}_2\mathbb{F}_p$ to ideal vertices of the ideal triangulation. Hence, without loss of generality we may assume that the label of three of ideal vertices are 0, 1, ∞ as in Figure 17.

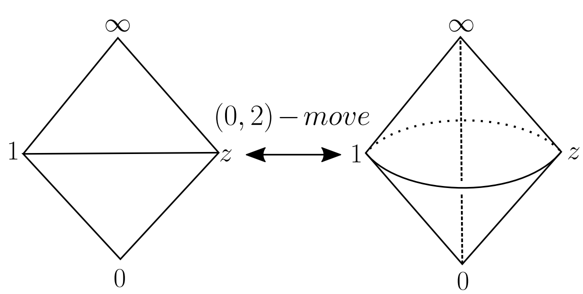

Figure 17: (0,2)-Pachner move

The modulus of tetrahedron in the front of the right picture is z, and the modulus of tetrahedron in the back of the right picture is $\frac{1}{z}$. Since

$$[z] = -\left[\frac{1}{z}\right]$$

by (2), the summation of moduli in $\check{\mathcal{P}}(\mathbb{F}_p)$ is zero. Therefore the lemma holds. $\square$

Lemma A.2. We assume that, when we apply a (2,3)-Pachner move, the modulus of each tetrahedron appearing in the move does not collapse. Then the summation of moduli in $\check{\mathcal{P}}(\mathbb{F}_p)$ does not change before and after applying a (2,3)-Pachner move.

Proof. To prove the lemma, it is enough to prove that the summation of moduli of two tetrahedra on the left-hand side of Figure 18 in $\check{\mathcal{P}}(\mathbb{F}_p)$ and that of right-hand side of Figure 18 are equal. By the same reason as in the proof of Lemma A.1, without loss of generality we may assume that the label of three of ideal vertices are 0, 1, ∞ as in Figure 18.

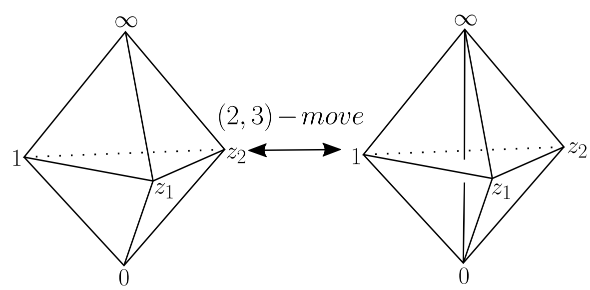

Figure 18: (2,3)-Pachner move

The moduli of tetrahedra on the left-hand side of Figure 18 are

$$\frac{1-z_2}{1-z_1},\ \frac{1-z_1^{-1}}{1-z_2^{-1}}.$$

The moduli of tetrahedra on the right-hand side of Figure 18 are

$$z_1,\ \frac{z_2}{z_1},\ \frac{1}{z_2}.$$

Since $[x] = -\left[\frac{1}{x}\right]$ by (1), the summations of moduli in $\check{\mathcal{P}}(\mathbb{F}_p)$ are equal each other. Therefore the lemma holds. $\square$

B How to obtain $\check{\mathcal{B}}(\mathbb{F}_p)$

In this section, we give concrete isomorphisms between $\check{\mathcal{B}}(\mathbb{F}_p)$ and standard abelian group for $7 = 7, 11, 13$.

For odd prime p, since $\mathbb{F}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$, we note that $\mathbb{F}_p^\times \wedge \mathbb{F}_p^\times \cong \mathbb{Z}/2\mathbb{Z}$.

Lemma B.1. We have that $\mathcal{P}(\mathbb{F}_7) \cong \mathbb{Z}/8\mathbb{Z}$, $\mathcal{B}(\mathbb{F}_7) \cong \mathbb{Z}/4\mathbb{Z}$. The former isomorphism is given as follows:

$$\varphi\colon \mathcal{P}(\mathbb{F}_7) \to \mathbb{Z}/8\mathbb{Z},$$
$$\varphi([2]) = \varphi([6]) = 2,\ \varphi([3]) = 1,\ \varphi([4]) = -2,\ \varphi([5]) = 3.$$

Proof. We enumerate all relations (1) in the case of $\mathbb{F}_7\backslash\{0,1\}$, we obtain

$$\begin{array}{lll}
[2]-[3]+[5]-[6]+[4]=0, & [3]+[2]-[5]=0, & 2[5]-[4]-[2]+[6]=0, \\
2[2]-[4]-[3]+[5]=0, & 2[4]-[2]-[5]+[3]=0, & 2[5]-2[6]+[4]=0, \\
2[2]-2[5]+[6]=0, & [4]-[3]+[6]-[2]+[5]=0, & [6]-[2]+2[5]-[4]=0, \\
-[6]+2[3]=0, & -[5]+[3]+[6]=0, & 2[6]-2[3]+[4]=0, \\
2[3]-[6]=0, & [4]-[6]+[5]-[3]+[2]=0, & [6]+[3]-[5]=0, \\
2[3]-2[4]+[6]=0, & [5]-[2]+[6]-[3]+[4]=0, & -[5]+[2]+[3]=0. \\
[3]-[5]+2[4]-[2]=0, & [5]-[3]+2[2]-[4]=0, &
\end{array}$$

The equations can be simplified into

$$[2]=[6]=2[3],\ [4]=6[3],\ [3]=-[5],\ 8[3]=0. \tag{75}$$

Then $\mathcal{P}(\mathbb{F}_7)$ is generated by [3], and all the other elements are multiples of [3], and $8[3]=0$, and there is no other relation. Hence we have

$$\mathcal{P}(\mathbb{F}_7) \cong \mathbb{Z}/8\mathbb{Z}.$$

A concrete isomorphism is given by putting $\varphi([3])=1$ as follows:

$$\varphi([2])=\varphi([6])=2,\ \varphi([3])=1,\ \varphi([4])=-2,\ \varphi([5])=3 \in \mathbb{Z}/8\mathbb{Z}.$$

We send elements of $\mathcal{P}(\mathbb{F}_7)$ by a homomorphism

$$[z] \mapsto z \wedge (1-z) \in \mathbb{F}_7^\times \wedge \mathbb{F}_7^\times \cong \mathbb{Z}/2\mathbb{Z}.$$

Then we have

$$[2]\mapsto 0,\ [3]\mapsto 1,\ [4]\mapsto 0,\ [5]\mapsto 1,\ [6]\mapsto 0.$$

Hence the kernel of the homomorphism is a subgroup of $\mathcal{P}(\mathbb{F}_7)$ generated by $[2],\ [4],\ [6]$, and $[3]-[5]$. Since

$$\varphi([2])=\varphi([6])=2,\ \varphi([4])=-2,\ \varphi([3]-[5])=-2,$$

We obtain that

$$\mathcal{B}(\mathbb{F}_7) \cong \{0,2,4,6\} \cong \mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/8\mathbb{Z}.$$

Therefore the lemma holds. $\square$

Lemma B.2. We have that $\check{\mathcal{P}}(\mathbb{F}_7) \cong \mathbb{Z}/4\mathbb{Z},\ \check{\mathcal{B}}(\mathbb{F}_7) \cong \mathbb{Z}/2\mathbb{Z}$. The former isomorphism

$$\psi\colon \check{\mathcal{P}}(\mathbb{F}_7) \to \mathbb{Z}/4\mathbb{Z}$$

is given by

$$\psi([2])=\psi([4])=\psi([6])=2,\ \psi([3])=1,\ \psi([5])=3,$$

the latter isomorphism

$$\phi\colon \check{\mathcal{B}}(\mathbb{F}_7) \to \mathbb{Z}/2\mathbb{Z}$$

is given by

$$\phi([2])=\phi([4])=\phi([6])=\phi([5]-[3])=1.$$

Proof. By enumerating concretely (2) and simplifying them, we have

$$[2] = [4] = [6], \ [3] = -[5]. \tag{76}$$

By (2.7) and (75), $\check{\mathcal{P}}(\mathbb{F}_7)$ is generated by [3], and all the other elements are multiples of [3], and $4[3] = 0$, and there is no other relation. Hence we have

$$\check{\mathcal{P}}(\mathbb{F}_7) \cong \mathbb{Z}/4\mathbb{Z}.$$

A concrete isomorphism is given by putting $\varphi([3]) = 1$, as follows:

$$\psi([2]) = \psi([4]) = \psi([6]) = 2, \ \psi([3]) = \psi(-[5]) = 1 \in \mathbb{Z}/4\mathbb{Z}.$$

The image of $\mathcal{B}(\mathbb{F}_7)$ by the projective homomorphism $\mathcal{P}(\mathbb{F}_7) \to \check{\mathcal{P}}(\mathbb{F}_7)$ obtained by (76) is an subgroup of $\check{\mathcal{P}}(\mathbb{F}_7)$ which is generated by the images of [2], [4], [6], and $[3]-[5]$. By

$$\psi([2]) = \psi([4]) = \psi([6]) = \psi([3]-[5]) = 2,$$

we have that

$$\check{\mathcal{B}}(\mathbb{F}_7) \cong \{0, 2\} \cong \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z}.$$

If we set ψ be the composite of the restriction of ψ to $\check{\mathcal{B}}(\mathbb{F}_7)$ and the natural homomorphism $\{0, 2\} \to \mathbb{Z}/2\mathbb{Z}$, then ϕ is what we require.

Therefore the lemma holds. □

Lemma B.3. We have that $\mathcal{P}(\mathbb{F}_{11}) \cong \mathbb{Z}/12\mathbb{Z}, \ \mathcal{B}(\mathbb{F}_{11}) \cong \mathbb{Z}/6\mathbb{Z}$. The former isomorphism

$$\varphi \colon \mathcal{P}(\mathbb{F}_{11}) \to \mathbb{Z}/12\mathbb{Z}$$

is given by

$$\varphi([2]) = 1, \ \varphi([3]) = 4, \ \varphi([4]) = -4, \ \varphi([5]) = 6, \ \varphi([6]) = 5,$$

$$\varphi([7]) = 4, \ \varphi([8]) = 2, \ \varphi([9]) = 6, \ \varphi([10]) = -3.$$

Proof. By enumerating concretely (1) in the case of $\mathbb{F}_{11}\backslash\{0, 1\}$ and simplifying, we obtain that

$$[3] = [7] = 4[2], \ [4] = -4[2], \ [5] = 8[2], \ [6] = 5[2], \ [8] = 2[2], \ [9] = 6[2], \ [10] = -3[2], \ 12[2] = 0. \tag{77}$$

Hence $\mathcal{P}(\mathbb{F}_{11})$ is generated by [2], and other elements are multiples of [2], and $12[2] = 0$, and there is no other relation. Hence we have that

$$\mathcal{P}(\mathbb{F}_{11}) \cong \mathbb{Z}/12\mathbb{Z}.$$

A concrete isomorphism is given by putting $\varphi([2]) = 1$, as follows:

$$\varphi([2]) = 1, \ \varphi([3]) = \varphi([7]) = 4, \ \varphi([4]) = -4, \ \varphi([5]) = \varphi([9]) = 6,$$

$$\varphi([6]) = 5, \ \varphi([8]) = 2, \ \varphi([10]) = -3 \in \mathbb{Z}/12\mathbb{Z}.$$

We send elements of $\mathcal{P}(\mathbb{F}_{11})$ by

$$[z] \mapsto z \wedge (1-z) \in \mathbb{F}_{11}^{\times} \wedge \mathbb{F}_{11}^{\times} \cong \mathbb{Z}/2\mathbb{Z}.$$

Then we have that

$$[2] \mapsto 1,\ [3] \mapsto 0,\ [4] \mapsto 0,\ [5] \mapsto 0,\ [6] \mapsto 1,\ [7] \mapsto 0,\ [8] \mapsto 0,\ [9] \mapsto 0,\ [10] \mapsto 1.$$

Hence the kernel of the homomorphism is the subgroup of $\mathcal{P}(\mathbb{F}_{11})$ which is generated by [3], [4], [5], [7], [8], [9], $[2]-[6]$, and $[6]-[10]$. Since

$$\varphi([3]) = \varphi([7]) = 4,\ \varphi([4]) = -4,\ \varphi([5]) = \varphi([9]) = 6,$$
$$\varphi([8]) = 2,\ \varphi([2]-[6]) = -4,\ \varphi([6]-[10]) = 8,$$

we obtain that

$$\mathcal{B}(\mathbb{F}_{11}) \cong \{0,2,4,6,8,10\} \cong \mathbb{Z}/6\mathbb{Z} \subset \mathbb{Z}/12\mathbb{Z}.$$

Therefore the lemma holds. □

Lemma B.4. We have that $\check{\mathcal{P}}(\mathbb{F}_{11}) \cong \mathbb{Z}/2\mathbb{Z},\ \check{\mathcal{B}}(\mathbb{F}_{11}) = \{0\}$. The former isomorphism

$$\psi \colon \check{\mathcal{P}}(\mathbb{F}_{11}) \to \mathbb{Z}/2\mathbb{Z}$$

is given by

$$\psi([2]) = \psi([6]) = \psi([10]) = 2,\ \psi([3]) = \psi([4]) = \psi([5]) = \psi([7]) = \psi([8]) = \psi([9]) = 1.$$

Proof. By enumerating all relations (2) in the case of $\mathbb{F}_{11}\backslash\{0,1\}$ and simplifying them, we obtain that

$$[2] = [6] = [10],\ [3] = -[4] = [5] = -[7] = [8] = -[9]. \tag{78}$$

By these and (77), $\check{\mathcal{P}}(\mathbb{F}_{11})$ is generated by [2], and the other elements are multiples of [2], and $2[2] = 0$, and there are no other relations. Hence we have that

$$\check{\mathcal{P}}(\mathbb{F}_{11}) \cong \mathbb{Z}/2\mathbb{Z}.$$

A concrete isomorphism is given by putting $\psi([2]) = 1$ as follows:

$$\psi([2]) = \psi([6]) = \psi([10]) = 1\,,\ \psi([3]) = \psi(-[4]) = \psi([5]) = \psi(-[7]) = \psi([8]) = \psi(-[9]) = 0 \in \mathbb{Z}/2\mathbb{Z}.$$

The image of $\mathcal{B}(\mathbb{F}_{11})$ by the homomorphism $\mathcal{P}(\mathbb{F}_{11}) \to \check{\mathcal{P}}(\mathbb{F}_{11})$ obtained by (78) is the subgroup of $\check{\mathcal{P}}(\mathbb{F}_{11})$ generated by the images of [3], [4], [5], [7], [8], [9], $[2]-[6]$, and $[6]-[10]$. Since

$$\psi([3]) = \psi(-[4]) = \psi([5]) = \psi(-[7]) = \psi([8]) = \psi(-[9]) = \psi([2]-[6]) = \psi([6]-[10]) = 0,$$

then we have that

$$\check{\mathcal{B}}(\mathbb{F}_{11}) = \{0\} \subset \mathbb{Z}/2\mathbb{Z}.$$

Therefore the lemma holds. □

Lemma B.5. We have that $\mathcal{P}(\mathbb{F}_{13}) \cong \mathbb{Z}/14\mathbb{Z}$, $\mathcal{B}(\mathbb{F}_{13}) \cong \mathbb{Z}/7\mathbb{Z}$. The former isomorphism

$$\varphi\colon \mathcal{P}(\mathbb{F}_{13}) \to \mathbb{Z}/14\mathbb{Z}$$

is given by

$$\varphi([2]) = \varphi([12]) = 0,\ \varphi([3]) = \varphi([5]) = 6,\ \varphi([4]) = 2,\ \varphi([6]) = -1$$
$$\varphi([7]) = 7,\ \varphi([8]) = 1,\ \varphi([9]) = \varphi([11]) = -6,\ \varphi([10]) = -2.$$

Proof. By enumerating relations (1) in the case of $\mathbb{F}_{13}\backslash\{0,1\}$ and simplifying them, we have that

$$[2] = [12] = 14[8] = 0,\ [3] = [5] = 6[8],\ [4] = 2[8],\ [9] = [11] = -6[8],$$
$$[6] = -[8],\ [7] = 7[8],\ [10] = -2[8]. \tag{79}$$

Hence $\mathcal{P}(\mathbb{F}_{13})$ is generated by [8], and other elements are multiples of [8], and $14[8] = 0$, there is no other relation. Hence we have that

$$\mathcal{P}(\mathbb{F}_{13}) \cong \mathbb{Z}/14\mathbb{Z}.$$

A concrete isomorphism is given by putting $\varphi([8]) = 1$, as follows:

$$\varphi([2]) = \varphi([12]) = 0,\ \varphi([3]) = \varphi([5]) = 6,\ \varphi([4]) = 2,\ \varphi([6]) = -1,\ \varphi([7]) = 7,$$
$$\varphi([8]) = 1,\ \varphi([9]) = \varphi([11]) = -6,\ \varphi([10]) = -2 \in \mathbb{Z}/14\mathbb{Z}.$$

We send elements of $\mathcal{P}(\mathbb{F}_{13})$ by the homomorphism

$$[z] \mapsto z \wedge (1-z) \in \mathbb{F}_{13}^\times \wedge \mathbb{F}_{13}^\times \cong \mathbb{Z}/2\mathbb{Z}.$$

Then we have that

$$[2] \mapsto 0,\ [3] \mapsto 0,\ [4] \mapsto 0,\ [5] \mapsto 0,\ [6] \mapsto 1,\ [7] \mapsto 1,$$
$$[8] \mapsto 1,\ [9] \mapsto 0,\ [10] \mapsto 0,\ [11] \mapsto 0,\ [12] \mapsto 0.$$

By these, the kernel of the homomorphism is the subgroup of $\mathcal{P}(\mathbb{F}_7)$ which is generated by $[2],\ [3],\ [4],\ [5],\ [9],\ [10],\ [11],\ [12],\ [6]-[7]$, and $[7]-[8]$. Since

$$\varphi([2]) = \varphi([12]) = 0,\ \varphi([3]) = \varphi([5]) = 6,\ \varphi([4]) = 2,\ \varphi([9]) = \varphi([11]) = -6,$$
$$\varphi([10]) = -2,\ \varphi([6]-[7]) = -8,\ \varphi([7]-[8]) = 6,$$

we obtain that

$$\mathcal{B}(\mathbb{F}_{13}) \cong \{0,2,4,6,8,10,12\} \cong \mathbb{Z}/7\mathbb{Z} \subset \mathbb{Z}/14\mathbb{Z}.$$

Therefore the lemma holds. □

Lemma B.6. We have that $\check{\mathcal{P}}(\mathbb{F}_{13}) \cong \mathbb{Z}/7\mathbb{Z}$, $\check{\mathcal{B}}(\mathbb{F}_{13}) \cong \mathbb{Z}/7\mathbb{Z}$. The former isomorphism

$$\psi\colon \mathcal{P}(\mathbb{F}_{13}) \to \mathbb{Z}/7\mathbb{Z}$$

is given by

$$\psi([2]) = \psi([7]) = \psi([12]) = 0,\ \psi([3]) = \psi([5]) = \psi([6]) = -1,\ \psi([4]) = 2,$$
$$\psi([8]) = \psi([9]) = \psi([11]) = 1,\ \psi([10]) = -2,$$

and, since $\check{\mathcal{P}}(\mathbb{F}_{13}) = \check{\mathcal{B}}(\mathbb{F}_{13})$ the letter isomorphism corresponds with the above.

Proof. By enumerating all relations (2) concretely and simplifying them, we have that

$$[2] = [7] = [12],\ -[3] = -[5] = -[6] = [8] = [9] = [11],\ [4] = -[10]. \tag{80}$$

By these and (79), $\check{\mathcal{P}}(\mathbb{F}_{13})$ is generated by [8], and all other elements are multiples of [8], and $7[8] = 0$, and there is no other relation. Hence we obtain that

$$\check{\mathcal{P}}(\mathbb{F}_{13}) \cong \mathbb{Z}/7\mathbb{Z}.$$

A concrete isomorphism is given by putting $\psi([8]) = 1$, as follows:

$$\psi([2]) = \psi([7]) = \psi([12]) = 0,\ \psi(-[3]) = \psi(-[5]) = \psi(-[6]) = \psi([8]) = \psi([9]) = \psi([11]) = 1,$$
$$\psi([4]) = \psi(-[10]) = 2 \in \mathbb{Z}/7\mathbb{Z}.$$

The image of $\mathcal{B}(\mathbb{F}_{13})$ by the projective homomorphism $\mathcal{P}(\mathbb{F}_{13}) \to \check{\mathcal{P}}(\mathbb{F}_{13})$ induced by (80) is the subgroup of $\check{\mathcal{P}}(\mathbb{F}_{13})$ which is generated by the images of [2], [3], [4], [5], [9], [10], [11], [12], $[6] - [7]$, and $[7] - [8]$. Since

$$\psi([2]) = \psi([7]) = \psi([12]) = 0,\ \psi(-[3]) = \psi(-[5]) = \psi(-[6]) = \psi([8]) = \psi([9]) = \psi([11]) = 1,$$
$$\psi([4]) = \psi(-[10]) = 2,\ \psi([6] - [7]) = \psi([7] - [8]) = -1,$$

we obtain that

$$\check{\mathcal{B}}(\mathbb{F}_{13}) \cong \{0, 1, 2, 3, 4, 5, 6\} = \mathbb{Z}/7\mathbb{Z}.$$

Therefore the lemma holds. □

C A proof of Lemma 4.10

In this section, we give a proof of Lemma 4.10.

We recall the statement of Lemma 4.10. "For a parabolic representation $\rho\colon \pi_1(S^3 \setminus \mathcal{T}_n) \to \mathrm{SL}_2\mathbb{F}_p$ and generators X and Y of $\pi_1(S^3 \setminus T_n)$ in Figure 3, there is a $P \in \mathrm{SL}_2\overline{\mathbb{F}_p}$ such that

$$P^{-1}\rho(X)P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ P^{-1}\rho(Y)P = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \tag{81}$$

where $u \in \mathbb{F}_p^\times$. Moreover u is $\phi([\rho])$." The above is the statement of Lemma 4.10.

We give a proof of Lemma 4.10 similarly as the proof of Lemma 7 of [13].

Proof (The proof of Lemma 4.10)**.** For $v \in \overline{\mathbb{F}_p} \cup \{\infty\}$, $\mathrm{SL}_2\mathbb{F}_p$ acts on $\overline{\mathbb{F}_p} \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \frac{av + b}{cv + d}.$$

We set $\widehat{X} = \rho(X)$ and $\widehat{Y} = \rho(Y)$. Since ρ is parabolic, each of $\widehat{X}$ and $\widehat{Y}$ has an only one fixed point, and we put such fixed points to be by $w,\ w'$ respectively.

We assume that $w = w'$. If we take conjugates of $\widehat{X}$ and $\widehat{Y}$ by a matrix which sends ∞ to w, since the determinants of $\widehat{X}$ and $\widehat{Y}$ are 1 and trace of them are 2, we can take simultaneous upper triangulation to matrices whose diagonal entries are 1. Then it contradicts that $\widehat{X}$ and $\widehat{Y}$ are non-commutative. Hence, $w \neq w'$.

We take $P_1 \in \mathrm{SL}_2\overline{\mathbb{F}_p}$ which satisfies that $P_1\infty = w$ and $P_1 0 = w'$. We put $\widehat{X}_1 = P_1^{-1}\widehat{X}P_1$ and $\widehat{Y}_1 = P_1^{-1}\widehat{Y}P_1$. Then, $\widehat{X}_1$ and $\widehat{Y}_1$ has fixed points ∞ and 0, and the determinants of $\widehat{X}$ and $\widehat{Y}$ are 1, and the traces of them are 2. Hence, we can present them as

$$\widehat{X}_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \widehat{Y}_1 = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix},$$

for some $a, b \in \overline{\mathbb{F}_p}$. By this presentation, we note that

$$\phi([\rho]) = \mathrm{trace}\widehat{X}\widehat{Y} - 2 = \mathrm{trace}\widehat{X}_1\widehat{Y}_1 - 2 = ab.$$

Since ρ is parabolic, then $a \neq 0$. Hence, by setting

$$P_2 = \begin{pmatrix} a^{\frac{1}{2}} & 0 \\ 0 & a^{-\frac{1}{2}} \end{pmatrix},$$

we obtain that

$$\widehat{X}_2 = P_2^{-1}\widehat{X}_1P_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \widehat{Y}_2 = P_2^{-1}\widehat{Y}_1P_2 = \begin{pmatrix} 1 & 0 \\ ab & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \phi([\rho]) & 1 \end{pmatrix}.$$

Hence,

$$(P_1P_2)^{-1}\rho(X)(P_1P_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ (P_1P_2)^{-1}\rho(X)(P_1P_2) = \begin{pmatrix} 1 & 0 \\ \phi([\rho]) & 1 \end{pmatrix}.$$

Therefore the lemma holds. □

D The reason why we can take (a, b)-curves whose a and b are coprime

In this section, we give the proof of Lemma D.1 below.

Let K be a knot. We consider a parabolic representation $\rho\colon \pi_1(S^3 \setminus N(K)) \to \mathrm{SL}_2\mathbb{F}_p$. Since $\partial(S^3 \setminus N(K))$ is homeomorphic to a torus, $\pi_1(\partial(S^3 \setminus N(K))) \cong \mathbb{Z} \oplus \mathbb{Z}$ when we fix this isomorphism by choosing meridian and longitude as generators. Then the following lemma holds.

Lemma D.1. One can take an $(a, b) \in \mathrm{kernel}\,\rho$ whose a and b are coprime.

We give the proof of Lemma D.1 in the last of this section.

In the following, to prove Lemma D.1, we describe $\mathrm{kernel}\,\rho$ as a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ concretely, and we prove how we can take an (a, b), whose a and b are coprime, from the subgroup.

Let m and l denote a meridian and a longitude on $\partial(S^3 \setminus N(K))$ respectively. Since $\pi_1(\partial(S^3 \setminus N(K))) \cong \mathbb{Z} \oplus \mathbb{Z}$ and m and l are in the image of $\pi_1(\partial(S^3 \setminus N(K))) \to \pi_1(S^3 \setminus N(K))$, $\rho(m)$ and $\rho(l)$ are commutative. In particular, since ρ is parabolic, the determinant of the image of a meridian by ρ is 1, and the eigenvalues of $\rho(l)$ are only 1's or -1's. Hence, by taking an appropriate conjugate ρ' of ρ, we obtain that

$$\rho'(m) = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix},\ \rho'(l) = \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix}.$$

Since ρ and ρ' are conjugate, it follows that $\operatorname{kernel}\rho = \operatorname{kernel}\rho'$. Moreover since ρ' is also parabolic, we have that $\mu \in \mathbb{F}_p^{\times}$.

We consider the case where $\rho'(l) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$. Then

$$\rho'(m)^a \rho'(l)^b = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}^a \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^b = \begin{pmatrix} 1 & a\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b\lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a\mu + b\lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence we obtain that

$$\operatorname{kernel}\rho \cong \{(a,b) \in \mathbb{Z}\oplus\mathbb{Z} \mid a\mu + b\lambda \equiv 0 \pmod p\}.$$

This set is rewritten, by using $1 \leq \mu' \leq p-1$ such that $\mu\mu' \equiv 1 \pmod p$, as follows:

$$\operatorname{kernel}\rho \cong \{(a,b) \in \mathbb{Z}\oplus\mathbb{Z} \mid a + b\mu'\lambda \equiv 0 \pmod p\}.$$

Since this subgroup of $\mathbb{Z}\oplus\mathbb{Z}$ is a free abelian group of rank two, and we can take $(-\mu'\lambda, 1)$, $(p, 0)$ as a generator of that, we obtain that

$$\operatorname{kernel}\rho \cong \{\alpha(-\mu'\lambda, 1) + \beta(p, 0) \mid \alpha, \beta \in \mathbb{Z}\}. \tag{82}$$

We consider the case where $\rho'(l) = \begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}$. Then it follows that

$$\begin{aligned} \rho'(m)^a \rho'(l)^b &= \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}^a \begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}^b = \begin{pmatrix} 1 & a\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (-1)^b & (-1)^{b-1} b\lambda \\ 0 & (-1)^b \end{pmatrix} \\ &= \begin{pmatrix} (-1)^b & (-1)^b(a\mu - b\lambda) \\ 0 & (-1)^b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence we obtain that

$$\operatorname{kernel}\rho \cong \{(a,b) \in \mathbb{Z}\oplus\mathbb{Z} \mid b \text{ is even and } a\mu - b\lambda \equiv 0 \pmod p\}.$$

This set is rewritten, by using $1 \leq \mu' \leq p-1$ such that $\mu\mu' \equiv 1 \pmod p$, as follows:

$$\operatorname{kernel}\rho \cong \{(a,b) \in \mathbb{Z}\oplus\mathbb{Z} \mid b \text{ is even and } a - b\mu'\lambda \equiv 0 \pmod p\}.$$

Since this subgroup of $\mathbb{Z}\oplus\mathbb{Z}$ is a free abelian group of rank two, and we can take $(-2\mu'\lambda, 2)$, $(p, 0)$ as a generator of that, we obtain that

$$\operatorname{kernel}\rho \cong \{\alpha(-2\mu'\lambda, 2) + \beta(p, 0) \mid \alpha, \beta \in \mathbb{Z}\}. \tag{83}$$

Proof (The proof of Lemma D.1)**.** We consider the case where $\rho'(l) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$. By (82), we can take $(a, b) = (-\mu'\lambda, 1)$ in the case where $\lambda \neq 0$, and take $(a, b) = (p, 1)$ in the case where $\lambda = 0$ as an (a, b)-curve whose a and b are coprime.

We consider the case where $\rho'(l) = \begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}$. By (83), we can take $(a, b) = (p - 2\mu'\lambda, 2)$ in the case where $\lambda \neq 0$, and take $(a, b) = (p, 2)$ in the case where $\lambda = 0$, as an (a, b)-curve whose a and b are coprime.

Therefore the lemma holds. □

E The proof of Lemma 4.19

In this section we give the proof of Lemma 4.19.

We fix a solution $c \in \mathbb{F}_p$ of $F_n(u) = 0$. We consider a sequence $(c_1, c_2, \ldots, c_n)$, where each c_k $(1 \leq k \leq n)$ is defined by (39). In this section we give the proof of Lemma 4.19. The former claim of Lemma 4.19 is that $(c_i, c_{i+1}, c_{i+2}) = (0, 1, \infty)$ when $c_i = 0$ $(1 \leq i \leq n-3)$. The letter claim of Lemma 4.19 is that $(c_1, c_2, \ldots, c_{n-1}, c_n)$ is classified into the following (i), (ii), and (iii):
(i) There exist no 0, 1, and ∞ in $(c_1, c_2, \ldots, c_{n-1})$, and $c_i \neq c_{i+1}$ $(1 \leq i \leq n-1)$,
(ii) There exist $0, 1, \infty$ in $(c_1, c_2, \ldots, c_{n-1})$ successively in order, and 3-tuples $0, 1, \infty$ does not appear successively, and $c_1 \neq 0$, and $c_i \neq c_{i+1}$ $(1 \leq i \leq n-1)$.
(iii) $(c_1, c_2, c_3) = (0, 1, \infty)$, and 3-tuples $0, 1, \infty$ does not appear successively in $(c_1, c_2, \ldots, c_{n-1})$, and $c_i \neq c_{i+1}$ $(1 \leq i \leq n-1)$.

To prove Lemma 4.19, we prove some lemmas.

Lemma E.1. (I) if $c_k = 0$ $(1 \leq k \leq n-3)$, then $(c_k, c_{k+1}, c_{k+2}) = (0, 1, \infty)$
(II) if $c_k = 1$ $(2 \leq k \leq n-2)$, then $(c_{k-1}, c_k, c_{k+1}) = (0, 1, \infty)$
(III) if $c_k = \infty$ $(3 \leq k \leq n-1)$, then $(c_{k-2}, c_{k-1}, c_k) = (0, 1, \infty)$.

Proof. We prove (I). We assume that $c_k = 0$, that is, $F_k(c) = 0$ and $F_{k-2}(c) \neq 0$. By the recursive formula of $F_k(u)$, we have that

$$F_{k+1}(c) = -cF_k(c) + F_{k-1}(c) = F_{k-1}(c),$$

and

$$c_{k+1} = \frac{F_{k+1}(c)}{F_{k-1}(c)} = 1.$$

Since $F_k(c) = 0$, we have that $c_{k+2} = \infty$. Hence $(c_k, c_{k+1}, c_{k+2}) = (0, 1, \infty)$, and (I) holds.

We prove (II). We assume that $c_k = 1$, that is, $F_k(c) = F_{k-2}(c)$. By the recursive formula of $F_k(u)$, we have that $F_k(c) = -cF_{k-1}(c) + F_{k-2}(c)$. Hence, we have that $cF_{k-1}(c) = 0$. If $c = 0$, then the first assumption contradicts that $F_n(0) = 1 \neq 0$. Then we have that $c \neq 0$, and $F_{k-1}(c) = 0$. Hence $(c_{k-1}, c_k, c_{k+1}) = (0, 1, \infty)$, and (II) holds.

We prove (III). We assume that $c_k = \infty$, that is, $F_{k-2}(c) = 0$. Then we have that $c_{k-2} = 0$. By the recursive formula (28) of $F_k(u)$, we have that $F_{k-1}(c) = F_{k-3}(c)$, that is, $c_{k-1} = 1$. Hence $(c_{k-2}, c_{k-1}, c_k) = (0, 1, \infty)$, and (III) holds.

Therefore the lemma holds. □

Lemma E.2. For $1 \leq k \leq n-1$, $c_k \neq c_{k+1}$.

Proof. We assume that $c_k = c_{k+1}$. Then, since $\frac{F_k(c)}{F_{k-2}(c)} = \frac{F_{k+1}(c)}{F_{k-1}(c)}$, we have that

$$0 = F_k(c)F_{k-1}(c) - F_{k+1}(c)F_{k-2}(c). \tag{84}$$

By the recursive formula of $F_k(u)$, we have that $F_{k+1}(c) = -cF_k(c) + F_{k-1}(c)$. By substituting this into (84), we obtain that

$$\begin{aligned} 0 &= F_k(c)F_{k-1}(c) - (-cF_k(c) + F_{k-1}(c))F_{k-2}(c) \\ &= F_k(c)(F_{k-1}(c) + cF_{k-2}(c)) - F_{k-1}(c)F_{k-2}(c)) \\ &= F_k(c)F_{k-3}(c) - F_{k-1}(c)F_{k-2}(c), \end{aligned}$$

where we obtain the last equality by $F_{k-1}(c) + cF_{k-2}(c) = F_{k-3}(c)$ which is from (28). Hence we have that $\frac{F_{k-1}(c)}{F_{k-3}(c)} = \frac{F_k(c)}{F_{k-2}(c)}$, that is, $c_{k-1} = c_k$. By repeating the above argument, we obtain that

$$1 - c = c_1 = c_2 = \cdots = c_{k+1}. \tag{85}$$

Similarly as above, we can obtain that $c_{k+1} = c_{k+2}$. By repeating this argument, we obtain that

$$c_k = c_{k+1} = \cdots = c_n. \tag{86}$$

Since $c_n = 0$, it follows from (85) and (86) that $c_i = 0$ $(1 \leq i \leq n)$. This contradicts (I) of Lemma E.1. Hence $c_k \neq c_{k+1}$. Therefore the lemma holds. $\square$

Lemma E.3. If $(c_{k+1}, c_{k+2}, c_{k+3}) = (0, 1, \infty)$, then $c_k \neq \infty$, $c_{k+4} \neq 0$, that is, the 3-tuples of 0, 1, and ∞ does not repeat successively in $(c_1, c_2, \ldots, c_{n-1})$.

Proof. We prove that $c_k \neq \infty$. In the following, we assume $c_k = \infty$, and show a contradiction. By $c_{k+1} = 0$ and the recursive formula of $F_k(u)$, we obtain that $cF_k(c) = F_{k-1}(c)$, and $F_k(c) = -cF_{k-1}(c) + F_{k-2}(c)$. Moreover since $c_k = \infty$, that is, $F_{k-2}(c) = 0$, we obtain that

$$(c^2 + 1)F_k(c) = F_k(c) + cF_{k-1}(c) = F_{k-2}(c) = 0.$$

By Remark 4.18, $F_k(c) \neq 0$, and hence, $c^2 + 1 = 0$. For any i, since $F_i(c) = -cF_{i-1}(c) + F_{i-2}(c)$ and $F_{i-1}(c) = -cF_{i-2}(c) + F_{i-3}(c)$, we obtain that $F_i(c) = (c^2 + 1)F_{i-2}(c) - cF_{i-3}(c) = -cF_{i-3}(c)$. Hence, if $F_i(c) = 0$, then $F_{i-3}(c) = 0$. Therefore, if $c_i = 0$, then $c_{i-3} = 0$. By repeating similar argument, it follows that

$$(c_1, \ldots, c_{k-2}, c_{k-1}, c_k, c_{k+1}, c_{k+2}, c_{k+3}) = (c_1, \ldots, 0, 1, \infty, 0, 1, \infty).$$

Hence $c_1 = 0,\ 1,$ or ∞. By definition of c_1, $c_1 = 0$. However. since $1 - c = F_1(c) = x_1 = 0$, we have $c = 1$, this contradicts that $c^2 + 1 = 0$. Hence we obtain that $c_k \neq \infty$.

One can prove that $c_{k+4} \neq 0$ by similar argument. $\square$

Proof (The proof of Lemma 4.19)**.** The former claim of Lemma 4.19 holds since it correspond with (I) of Lemma E.1.

The latter claim of Lemma 4.19 holds as follows. By Lemma E.2, for $1 \leq k \leq n-1$, we have that $c_k \neq c_{k+1}$. When there exist no zeros in $(c_1, c_2, \dots, c_{n-1})$, by Lemma E.1, there also exist no 1 and ∞. Hence this case is classified as (I).

When there exist zeros in $(c_1, c_2, \dots, c_{n-1})$, by Lemma E.1, There exist $0, 1, \infty$ successively in the order in $(c_1, c_2, \dots, c_{n-1})$. Moreover, by Lemma E.3, 3-tuples of 0, 1, and ∞ does not appear successively. When $c_1 \neq 0$, it is classified as (II), and when $c_1 = 0$, it is classified as (III).

Therefore the lemma holds. □

F Table of $(c_1, c_2, c_3, \dots)$

In this section, we specify possible values of $(c_1, c_2, c_3, \dots)$ when $p = 7$, 11, and 13. Here c_i $(i = 1, 2, 3, \dots)$ is an element of $\mathbb{P}^1(\mathbb{F}_p)$ defined by $c_1 = F_1(c)$, $c_k = \dfrac{F_k(c)}{F_{k-2}(c)}$ $(k \geq 2)$. (When $F_{k-2}(c) = 0$, we set $c_k = \infty$.)

In the following, we express a sequence such that $F_i(c) = F_{i+m}(c)$ $(i = 1, 2, 3, \dots)$ when "(period m)" is written after the sequence, and express a sequence such that $F_i(c) = rF_{i+m}(c)$ $(i = 1, 2, 3, \dots)$ when "(period m)(r multiple)" is written after the sequence,

In the case where $p = 7$ The sequences obtained by substituting $c = 0, \dots,$ and 6 for $(F_1(c), F_2(c), F_3(c), \dots)$ are given concretely as follows:

$$(F_1(c), F_2(c), F_3(c), \dots) = \begin{cases} (1, 1, 1, \dots)(\text{period } 1) & \text{if } c = 0 \\ (0, 1, 6, 2, 4, 5, 6, 6, 0, \dots)(\text{period } 8)(6 \text{ multiple}) & \text{if } c = 1, \\ (6, 3, 0, 3, 1, 1, 6, \dots)(\text{period } 6) & \text{if } c = 2, \\ (5, 0, 5, 6, 1, 3, 6, 6, 2, \dots)(\text{period } 8)(6 \text{ multiple}) & \text{if } c = 3, \\ (4, 6, 1, 2, 0, 2, 6, 6, 3, \dots)(\text{period } 8)(6 \text{ multiple}) & \text{if } c = 4, \\ (3, 0, 3, 6, 1, 1, 3, \dots)(\text{period } 6) & \text{if } c = 5, \\ (2, 3, 5, 1, 6, 0, 6, 6, 5, \dots)(\text{period } 8)(6 \text{ multiple}) & \text{if } c = 6. \end{cases}$$

Hence, by calculating concretely, we obtain that

$$(c_1, c_2, c_3, \dots) = \begin{cases} (0, 1, \infty, 2, 3, 6, 5, 4, 0, \dots)(\text{period } 8) & \text{if } c_1 = 0, \\ (2, 3, 6, 5, 4, 0, 1, \infty, 2, \dots)(\text{period } 8) & \text{if } c_1 = 2, \\ (3, 0, 1, \infty, 5, 6, 3, \dots)(\text{period } 6) & \text{if } c_1 = 3, \\ (4, 6, 2, 5, 0, 1, \infty, 3, 4, \dots)(\text{period } 8) & \text{if } c_1 = 4, \\ (5, 0, 1, \infty, 3, 4, 2, 6, 5, \dots)(\text{period } 8) & \text{if } c_1 = 5, \\ (6, 3, 0, 1, \infty, 5, 6, \dots)(\text{period } 6) & \text{if } c_1 = 6, \\ \text{The sequence does not contain } 0 & \text{otherwise.} \end{cases} \tag{87}$$

In the case where $p = 11$ The sequences obtained by substituting $c = 0, \dots,$ and 10 for $(F_1(c), F_2(c), F_3(c), \dots)$ are given concretely as follows:

$$(F_1(c), F_2(c), F_3(c), \dots) = \begin{cases} (1,1,1,\dots)(\text{period } 1) & \text{if } c = 0 \\ (0,1,10,2,8,5,3,2,1,1,0,\dots)(\text{period } 10) & \text{if } c = 1, \\ (10,3,4,6,3,0,3,5,4,8,10,10,1,\dots)(\text{period } 12)(-1 \text{ multiple}) & \text{if } c = 2, \\ (9,7,10,10,2,4,1,1,9,7,\dots)(\text{period } 8) & \text{if } c = 3, \\ (8,2,0,2,3,1,10,5,1,1,8,\dots)(\text{period } 10) & \text{if } c = 4, \\ (7,10,1,5,9,4,0,4,2,5,10,10,4,\dots)(\text{period } 12)(-1 \text{ multiple}) & \text{if } c = 5, \\ (6,9,7,0,7,2,6,10,1,4,10,10,5,\dots)(\text{period } 12)(-1 \text{ multiple}) & \text{if } c = 6, \\ (5,10,1,3,2,0,2,8,1,1,5,\dots)(\text{period } 10) & \text{if } c = 7, \\ (4,2,10,10,7,9,1,1,4,2,10,10,7,\dots)(\text{period } 8) & \text{if } c = 8, \\ (3,7,6,8,0,8,5,7,8,1,10,10,8,\dots)(\text{period } 12)(-1 \text{ multiple}) & \text{if } c = 9, \\ (2,3,5,8,2,10,1,0,1,1,2\dots)(\text{period } 10) & \text{if } c = 10. \end{cases}$$

Hence, by calculating concretely, we obtain that

$$(c_1, c_2, c_3, \dots) = \begin{cases} (0,1,\infty,2,3,8,10,7,4,6,0\dots)(\text{period } 10) & \text{if } c_1 = 0, \\ (2,3,8,10,7,4,6,0,1,\infty,2,\dots)(\text{period } 10) & \text{if } c_1 = 2, \\ (3,7,2,9,0,1,\infty,5,6,8,4,10,3,\dots)(\text{period } 12) & \text{if } c_1 = 3, \\ (5,10,9,8,2,0,1,\infty,6,7,5,\dots)(\text{period } 10) & \text{if } c_1 = 5, \\ (6,9,3,0,1,\infty,4,5,2,7,10,8,6,\dots)(\text{period } 12) & \text{if } c_1 = 6, \\ (7,10,8,6,9,3,0,1,\infty,4,5,2,7,\dots)(\text{period } 12) & \text{if } c_1 = 7, \\ (8,2,0,1,\infty,6,7,5,10,9,8,\dots)(\text{period } 10) & \text{if } c_1 = 8, \\ (10,3,7,2,9,0,1,\infty,5,6,8,4,10,\dots)(\text{period } 12) & \text{if } c_1 = 10, \\ \text{The sequence does not contain } 0 & \text{otherwise.} \end{cases} \tag{88}$$

In the case where $p = 13$ The sequences obtained by substituting $c = 0, \dots,$ and 12 for $(F_1(c), F_2(c), F_3(c), \dots)$ are given concretely as follows:

$$(F_1(c), F_2(c), F_3(c), \dots) = \begin{cases} (1,1,1,\dots)(\text{period } 1) & \text{if } c = 0, \\ (0,1,12,2,10,5,5,0,5,8,\dots)(\text{period } 7)(5 \text{ multiple}) & \text{if } c = 1, \\ (12,3,6,4,11,8,8,5,11,9,6\dots)(\text{period } 7)(8 \text{ multiple}) & \text{if } c = 2, \\ (11,7,3,11,9,10,5,8,7,0,7,5,5,3,\dots)(\text{period } 13)(5 \text{ multiple}) & \text{if } c = 3, \\ (10,0,10,12,1,8,8,2,\dots)(\text{period } 7)(8 \text{ multiple}) & \text{if } c = 4, \\ (9,8,8,7,\dots)(\text{period } 3)(8 \text{ multiple}) & \text{if } c = 5, \\ (8,5,4,7,1,1,8,\dots)(\text{period } 6) & \text{if } c = 6, \\ (7,4,5,8,1,1,7,\dots)(\text{period } 6) & \text{if } c = 7, \\ (6,5,5,4,\dots)(\text{period } 3)(5 \text{ multiple}) & \text{if } c = 8, \\ (5,8,11,0,11,5,5,12,\dots)(\text{period } 7)(5 \text{ multiple}) & \text{if } c = 9, \\ (4,0,4,12,1,2,7,10,11,4,10,8,8,6\dots)(\text{period } 13)(8 \text{ multiple}) & \text{if } c = 10, \\ (3,7,4,2,8,5,5,2,\dots)(\text{period } 7)(5 \text{ multiple}) & \text{if } c = 11, \\ (2,3,5,8,0,8,8,3,\dots)(\text{period } 7)(8 \text{ multiple}) & \text{if } c = 12. \end{cases}$$

Hence, by calculating concretely, we obtain that

$$(c_1, c_2, c_3, \dots) = \begin{cases} (0,1,\infty,2,3,9,7,0,\dots)(\text{period } 7) & \text{if } c_1 = 0, \\ (2,3,9,7,0,1,\infty,2,\dots)(\text{period } 7) & \text{if } c_1 = 2, \\ (4,0,1,\infty,10,11,7,5,9,3,8,2,6,4,\dots)(\text{period } 13) & \text{if } c_1 = 4, \\ (5,8,10,0,1,\infty,4,5,\dots)(\text{period } 7) & \text{if } c_1 = 5, \\ (10,0,1,\infty,4,5,8,10,\dots)(\text{period } 7) & \text{if } c_1 = 10, \\ (11,7,5,9,3,8,2,6,4,0,1,\infty,10,11,\dots)(\text{period } 13) & \text{if } c_1 = 11, \\ \text{The sequence does not contain } 0 & \text{otherwise.} \end{cases} \quad (89)$$

G Invariance of the reduced DW invariant

In this section, we show that the reduced DW invariant depend on only knots and finite fields in Lemma G.1 below.

Let K be a knot, and let $N(K)$ be a tubular neighborhood of K. We consider a parabolic representation $\rho\colon \pi_1(S^3 \setminus N(K)) \to \mathrm{SL}_2\mathbb{F}_p$. We take an $(a,b) \in \mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(\partial(S^3 \setminus N(K)))$ from the kernel of the restriction of ρ to $\pi_1(\partial(S^3 \setminus N(K)))$. In particular, by Lemma D.1, one can take such an (a,b) whose a and b are coprime. Then, by Dehn filling along the (a,b)-curve on $\partial(S^3 \setminus N(K))$, we obtain a closed 3-manifold $M_{a,b}(K)$. In general, the result 3-manifold depend on the choice of (a,b). However, for the reduced DW invariant, the following lemma holds.

Lemma G.1. $\widehat{\mathrm{DW}}(M_{a,b}(K), \rho)$ does not depend on the choice of the (a,b)-curve.

By Proposition 4.6, $\widehat{\mathrm{DW}}(M_{a,b}(K), \rho)$ is equal to the summation of moduli of the tetrahedra of a triangulation of $M_{a,b}(K)$ in $\check{\mathcal{P}}(\mathbb{F}_p)$. Hence, to prove the lemma, it is enough to prove that $\widehat{\mathrm{DW}}(M_{a,b}(K),\rho)$ is equal to the summation of the moduli of the ideal tetrahedra of an ideal triangulation of $S^3 \setminus K$ in $\check{\mathcal{P}}(\mathbb{F}_p)$.

Before we prove the lemma, we review Neumann's construction from (S^3, K) to $M_{(a,b)}(K)$; see [9] for details. We consider the singular 3-manifold obtained from S^3 by collapsing K to a point. This 3-manifold has only one singular point, and we let v denote the point. We give an ideal triangulation such that v is an ideal vertex of that, and remove all edges which connects ideal vertices by Pachner moves. We triangulate so that all of the vertices of the link of v are distinct. Then the link of v gives a triangulation of a torus. We assume that (a,b)-curve is an edge of a fundamental region of torus. This (a,b)-curve is a boundary of a disk D (a cone of the (a,b)-curve) embedded properly in $N(v)$. The boundary obtained by cutting the singular 3-manifold along D is a sphere obtained from two copies of D by gluing their boundaries. By gluing the above boundary and the boundary of a simplicial ball as in Figure 20, $\partial(N(v))$(on the right-hand side of Figure 19) is homeomorphic to S^2. Then we obtain an open 3-manifold. We note that an ideal triangulation of the obtained open 3-manifold has only one ideal vertex as v. Then we obtain a closed 3-manifold by filling v.

Moreover, Neumann [9] proved that the summation of moduli of a simplicial ball as in Figure 20 is 0 in $\check{\mathcal{P}}(\mathbb{C})$ as follows. The 3-ball in Figure 20 is the union of ideal tetrahedra glued around the edge vv' and the mirror image of the tetrahedra with respect to the horizontal plane (the faces which have no v as a vertex) in Figure 20. The new vertex v' is labeled by an element of $\mathbb{C} \cup \{\infty\}$ which is different from the labels of adjacent vertices. Then, for this simplicial sphere, The summation of the moduli of two tetrahedra which are mirror image of the other with respect to the horizontal plane is 0 in $\check{\mathcal{P}}(\mathbb{C})$. Hence the summation of all moduli of tetrahedra of the simplicial ball is 0 in $\check{\mathcal{P}}(\mathbb{C})$.

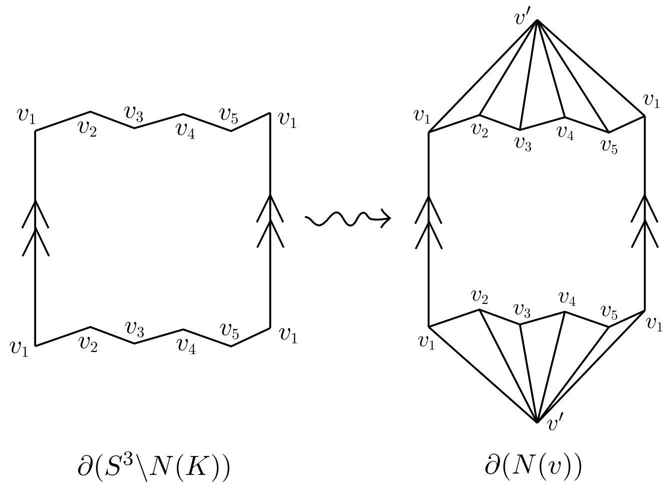

Figure 19: We depict $\partial(S^3 \setminus N(K))$ on the left-hand side. Since we identify the top and bottom edges, and left and right edges, it is homeomorphic to a torus. In particular, the path whose vertices are v_1, v_2, v_3, v_4, v_5 denotes an (a,b)-curve. We depict $\partial N(v)$ on the right-hand side. Since we identify left and right edges, it is homeomorphic to S^2.

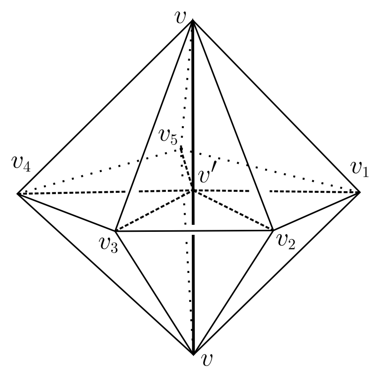

Figure 20

Proof (The proof of Lemma G.1)**.** We prove Lemma G.1 by dividing that into Step 1 and Step 2. In Step 1, we prove Lemma G.1 by assuming the following condition.

$$\begin{gathered}\text{we can subdivide the triangulation in such a way that we can take an } (a,b)-\text{curve} \\ \text{as a union of edges of the new triangulation of the torus and copies of such} \\ \text{an } (a,b)-\text{curve are parts of the boundary of a fundamental region of the torus.}\end{gathered} \tag{90}$$

In Step 2, we prove that (90) holds.

Step 1 We assume the condition (90) and prove Lemma G.1. We consider the case where the edge of a fundamental region of the torus $\partial(S^3 \setminus N(K))$ is (a,b)-curve. We prove that the reduced DW invariant of (K,ρ) does not depend on Neumann's construction from (S^3,K) to $M_{(a,b)}(K)$. Similarly as above, the summation of moduli of 3-ball in Figure 20 is 0. Therefore $\widehat{\mathrm{DW}}(M_{a,b}(K),\rho)$ does not depend on the choice of an (a,b)-curve. Hence, when the edge of a fundamental region of the torus $\partial(S^3 \setminus N(K))$ is (a,b)-curve, Lemma G.1 holds. In the following of the proof, it is enough to prove (90).

Step 2 We prove that the condition (90) satisfies. We consider an Euclidean metric on the torus $\partial(S^3 \setminus N(K))$. Then the triangulation of the torus can be linear. Namely, each segment on the torus can be PL-linear, and we obtain a triangulation of the torus by subdividing, and each edge is shortest on the Euclidean metric.

In Step 2-1, we prove that (90) holds by assuming that we can subdivide a triangulation of a torus so that the length of the longest edges are sufficiently small. To explain this by a concrete picture, we take a 5-twist knot $\mathcal{T}_5$ and $p = 13$ as an example. In Step 2-2, we prove that we can subdivide the triangulation of a torus so that the length of the longest edges are sufficiently small.

Step 2-1 For a knot K, a triangulation of the torus $\partial(S^3 \setminus N(K))$ is sufficiently subdivided if one can take an (a,b)-curve in the ribbon region on a fundamental region of the torus as a polygonal line which consists of edges of the triangulation. For example, for the 5-twist knot $\mathcal{T}_5$, what a triangulation of the torus $\partial(S^3 \setminus N(\mathcal{T}_5))$ is sufficiently subdivided is that we can take an (a,b)-curve in the dark gray fundamental region in Figure 22 which connect a_4's located in right upper side and left lower side as a polygonal line which consists of edges of the triangulation. Here, the dark gray region which connect a same vertex denotes, for a sufficiently small $\varepsilon > 0$, an ε-neighborhood of the segment which connect a_4's.

To replace the original fundamental region with the light gray fundamental region in Figure 22, it is enough to take an (a, b)-curve as a union of edges of the triangulation in the dark gray ribbon region.

Since the triangulation is sufficiently subdivided, by mapping each triangle by the action of $\pi_1(S^3 \setminus N(\mathcal{T}_5))$, one can replace the original fundamental region with the gray fundamental region (Figure 22) whose edge is an (a, b)-curve. Similarly as above, one can replace the original fundamental region with a fundamental region whose edge is an (a, b)-curve for general cases.

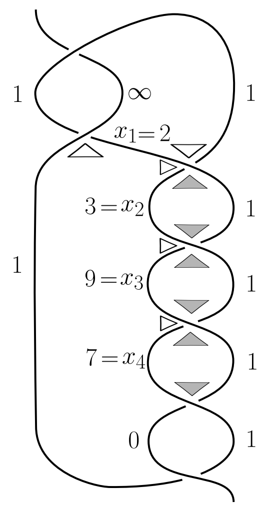

Figure 21: An ideal triangulation of the complement of $\mathcal{T}_5$ and $(c_1, c_2, c_3, c_4) = (2, 3, 9, 7)$

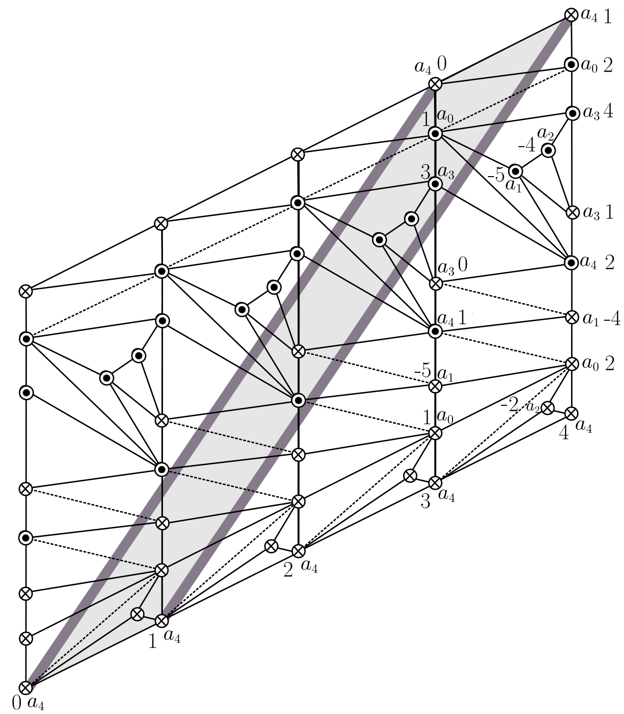

Figure 22: The union of 4 copies of a fundamental region in the universal cover of a torus

Step 2-2 For a Euclidean metric on a torus, we prove that one can change the maximal length to shorter length, where the maximal length is the maximum of length of edges of a triangulation of the torus. After removing all edges which connect ideal vertices, we consider the boundary of a neighborhood of v, which is homeomorphic to a torus. In particular, we consider the case where each vertex on the torus is connected to v by some edge. We consider a Euclidian metric on the torus. We fix the longest edge and let a and c denote the vertices of that. Since all vertices on the torus connects the ideal vertex v, there exists a face acv, and two ideal tetrahedra glued with respect to acv as in Figure 23. For each of these two ideal tetrahedra, we apply a (1,4)-Pachner move, then two ordinary vertices appear. We let x and y denote the ordinary vertices. For two tetrahedra glued with respect to acv, we apply a (2,3)-Pachner move. An edge obtained by this operation connects x and y. Figure 24 is a torus of a view from v. Since Pachner moves are topological operation, one can define a coordinate of x and y in $\mathbb{R}^2$ arbitrarily. When we take each of coordinates of x and y as the barycenter of the triangle, the length of the new edges is lower than 0.9 times the length of ac. By applying similar operation to all edges which is equal or greater than 0.9 times the maximum of length of edges (the number of such edges is finite), all length of edges is lower than the original maximum. By repeating similar operation, one can change the maximal length to sufficiently shorter length. $\square$

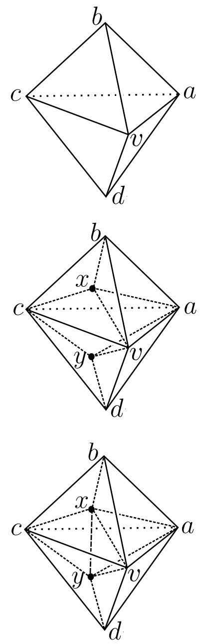

Figure 23: How to subdivide

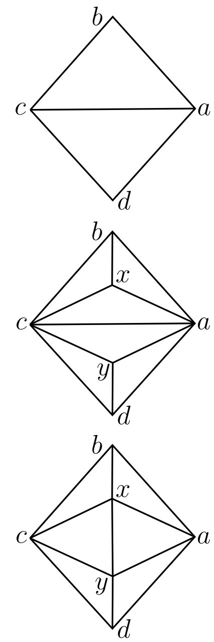

Figure 24: A view from v

Remark G.2. For the proof of Lemma G.1, the moduli of the ideal tetrahedra appeared anew by Dehn filling do not affect $\widehat{\mathrm{DW}}(M_{a,b}(K), \rho)$. Namely, $\widehat{\mathrm{DW}}(M_{a,b}(K), \rho)$ is equal to the summation of the moduli of the ideal tetrahedra of an ideal triangulation of $S^3 \setminus K$ in $\check{\mathcal{P}}(\mathbb{F}_p)$.

Remark G.3. We can obtain the example in Step 2-1, as follows. We fix $-1 \in \{u \in \mathbb{F}_{13} \mid F_5(u) = 0\}$. Then ideal vertices of an ideal triangulation of the complement of $\mathcal{T}_5$ is labeled by defined from that $c_1 = F_1(-1), c_i = \frac{F_i(-1)}{F_{i-2}(-1)}$ $(i = 2, 3, 4)$. Namely, Figure 21 correspond to an ideal triangulation in Figure 13 in Section 4.5, and each ideal vertex is labeled as in Figure 13.

References

[1] Brown, K.S., *Cohomology of groups*, Graduate Texts in Mathematics **87** (1982) Springer-Verlag.

[2] Burde, G., Zieschang, H., *Knots*, Second edition. De Gruyter Studies in Mathematics, **5** (2003) Walter de Gruyter.

[3] Dijkgraaf, R., Witten, E., *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990) 393–429.

[4] Dupont, J.L., Sah, C.H. *Scissors congruences. II*, J. Pure Appl. Algebra 25 (1982) 159–195.

[5] Dupont, J.L., Zickert, C.K., *A dilogarithmic formula for the Cheeger-Chern-Simons class*, Geom. Topol. **10** (2006) 1347–1372.

[6] Hutchinson, K., *A Bloch-Wigner complex for SL_2*, J. K-Theory **12** (2013) 15–68.

[7] Matveev, S.V., *Algorithmic topology and classification of 3-manifolds*, Algorithms and Computation in Mathematics **9** (2003) Springer-Verlag.

[8] Murakami, H., Yokota, Y., *Volume conjecture for knots*, SpringerBriefs in Mathematical Physics **30** (2018) Springer, Singapore.

[9] Neumann, W.D., *Extended Bloch group and the Cheeger-Chern-Simons class*, Geom. Topol. **8** (2004) 413–474.

[10] Neumann, W.D., Yang, J., *Bloch invariants of hyperbolic 3-manifolds*, Duke Math. J. **96** (1999) 29–59.

[11] Ohtsuki, T., *On the Bloch groups of finite fields and their quotients by the relation corresponding to a tetrahedral symmetry*, preprint, 2019,
`http://www.kurims.kyoto-u.ac.jp/~tomotada/paper/brff.pdf`.

[12] Ohtsuki, T., Takata, T., *On the Kashaev invariant and the twisted Reidemeister torsion of two-bridge knots*, Geom. Topol. **19** (2015) 853–952.

[13] Riley, R., *Holomorphically parameterized families of subgroups of SL(2,$\mathbb{C}$)*, Mathematika **32** (1985) 248–264.

[14] Riley, R., *Nonabelian representations of 2-bridge knot groups*, Quart. J. Math. Oxford Ser. (2) **35** (1984) 191-208.

[15] Suslin, A.A., *K_3 of a field, and the Bloch group*, Trudy Mat. Inst. Steklov. **183** (1990), 180–199,229, Translated in Proc. Steklov Inst. Math. **4** (1991), 217–239, Galois theory, rings, algebraic groups and their applications (Russian).

[16] Thurston, D.P., *Hyperbolic volume and the Jones polynomial*, Notes accompanying lectures at the summer school on quantum invariants of knots and three-manifolds (Joseph Fourier Institute, University of Grenoble, org. C. Lescop, July, 1999),
`http://www.math.columbia.edu/~dpt/speaking/Grenoble.pdf`.

[17] Thurston, W.P., *The geometry and topology of three-manifolds*, lecture notes, Princeton University (1979),
`http://library.msri.org/books/gt3m/`.

[18] Wakui, M., *On Dijkgraaf-Witten invariant for 3-manifolds*, Osaka J. Math. **29** (1992) 675–696.

[19] Weibel, C.A., *The K-book. An introduction to algebraic K-theory*, Graduate Studies in Mathematics **145** (2013) American Mathematical Society.

[20] Yokota, Y., *On the potential functions for the hyperbolic structures of a knot complement*, Invariants of knots and 3-manifolds (Kyoto, 2001), Geom. Topol. Monogr. **4**, Geom. Topol. Publ., Coventry, 2002.

[21] Yokota, Y., *On the complex volume of hyperbolic knots*, J. Knot Theory Ramifications **20** (2011) 955–976.

Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto, 606-8502, Japan
E-mail address: `karu@kurims.kyoto-u.ac.jp`