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Rigged Configurations and Unimodality

To Professor Nikolai Reshetikhin on the occasion of 60th anniversary
with grateful and admiration from his friend anc co-author

By

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Abstract

For a given partition λ and a dominant sequence of rectangular shape partitions $\{R_a\}$, we give sufficient conditions which imply that the corresponding parabolic Kostka polynomial $K_{\lambda,\{R\}}(q)$ is symmetric and **unimodal**. Examples are found to satisfy these conditions include :

- Principal specialization of the internal product of Schur functions $s_\alpha * s_\beta(\frac{q-q^N}{1-q})$; in particular, q -Gaussian polynomials $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q$;
- Generalized q-Narayana numbers/polynomials $N_\ell((\lambda;\{R\});q)$ associated with a partition λ and a sequence of rectangular shape partitions $\{R\}=(R_1,\ldots,R_n)$;
- Symmetry and unimodality of the statistics charge generating function $\mathcal{K}^{[d]}_{\lambda,1^{|\lambda|}}(q)$ of the set of standard Young tableaux of a given shape λ and and fixed number of descents d;
- Schröder–Narayana $SchN_k(n.d;q)$, Kirkman–Caley $KC_d(n;q)$ and Motzkin sum $MS_d(n;q)$ polynomials;
- A (q,t)-deformation of the Euler polynomials $A_n(t)$;
- Reduced decomposition's polynomials $RD^{[\ell]}(n;q)$.

As a corollary of our general result, we prove symmetry and unmorality of

- Classical and rectangular q-Narayana numbers;
- A certain q-deformation of the odd Eulerian numbers $A(2n+1,k)$.

We also prove*strict log-concavity* of q-binomial coefficients. Also we introduce and initiate the study of double *Liskova polynomials* $L^{\gamma}_{\alpha,\beta}(q,t)$ which are natural generalization of the Kostka–Macdonald polynomials $K_{\alpha,\beta}(q,t)$.

2000 Mathematics Subject Classifications: 05E05, 05E10, 05A19.

1 Introduction

This paper presents brief an exposition of some results obtained by the author which is devoted various applications of Algebraic Bethe Ansatz associated with generalized Heisenberg chain, to Combinatorics of Young tableaux. The study of combinatorial aspects of Algebraic Bethe Ansatz was initiated by joint research with Nikolai Reshetikhin, [11]. For more details and proofs of statements which are formulated in our exposition, see [20, 16, 21, 22, 12, 15, 18, 14, 19].

1.1 A bit of History

The story of my collaboration with Nikolai Reshetikhin has been started in the early 80's of the last century from traditional at that time *Christmas Exercise* suggested to me by Professors L.D.Faddeev and L.A.Takhtajan, namely, to prove the following identity

$$(1+x)^n = \sum_{2\ell \leq n} \sum_{\nu} \prod_{j \geq 0} \binom{P_j^{(\ell)} + \nu_j}{\nu_j} \frac{x^\ell - x^{n-\ell+1}}{1-x}, \tag{1.1}$$

where the sum runs over set of non-negative integers $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$ such that $\sum_k k\nu_k = \ell$,

$$P_j^{(\ell)}(\nu) = n - 2\sum_k \min(j,k), \tag{1.2}$$

and for non-negative integers $\binom{P+m}{m} := \frac{(P+m)!}{P!\ m!}$.

It is clearly seen that the identity (1.1) on the level of characters of the Lie algebra $sl(2)$, describes the decomposition of the n-th tensor power of the spin 1/2 irreducible representation of Lie algebra $sl(2)$ into irreducible $sl(2)$-modules. In other words, the number

$$Z_n(1/2|\ell) = \sum_{\nu} \prod_{j \geq 0} \binom{P_j^{(\ell)} + \nu_j}{\nu_j} \tag{1.3}$$

is equal to the tensor product multiplicity

$$Z_n(1/2|\ell) = \mathrm{Mult}[V_{(n-2\ell+1)/2}(V_{1/2})^{\otimes n},$$

where V_j denotes the irreducible representation of $sl(2)$, $\dim V_j = 2j+1$.

For small n, say $n \leq 5$, the above formula for numbers $Z_n(1/2|\ell)$ can be proved by using the famous Kostant's formula for the triple tensor product multiplicities. But I still don't know how to deduce formula (1.3) by reference to Kostant's formula in general.

To prove the identity (1.1), I applied the so-called "inverse induction method". More precisely, I proved certain version of the original identity (1.1) for the case of infinite number of variables, and then, after a certain change of variables (related with solutions to the so-called Q-systems), one can come back to the identity (1.1). This method allows to prove very general identity, namely [12],

Proposition 1.1 *Let $b_1, b_2, \ldots$ be set of parameters. Then*

$$(1-x)\prod_{n\geq 1}(1-x^n)^{b_n} = 1+\sum_{\ell\geq 1} Z(\{b\}|\ell)\ x^{\ell}, \tag{1.4}$$

where $Z(\{b\}|\ell)(\nu)=\sum_{\nu_k}\prod_k \binom{P_k^{(\ell)}+\nu_k}{\nu_k}'$, and $P_k^{(\ell)}(\nu)=-\sum_{j=1}^{k}(k-j+1)b_j - 2\sum_k \min(j,k)$. Here $\binom{P+m}{m}' := P(P+1)\cdots(P+m-1)/m!$.

It is still an open **Problem** to describe integer sequences $(b_1, b_2, \ldots)$ and the set $C(\{b\}|\ell)$ consisting of compositions $\nu=(\nu_1,\nu_2,\ldots)$, $\sum_{k\geq 1} k\nu_k=\ell$, such that

(1) the coefficients $Z(\{b\}|\ell)(\nu)\in \mathbb{Z}_{\geq 0}$ for all $k, \ell, \nu \in C(\{b\})|\ell)$, and

2 $\sum_{\nu\notin C(\{b\})|\ell)} Z(\{b\}|\ell)(\nu)=0$.

I have proved that conditions (1) and (2) are satisfied for sequences $\{b_k\}$ of the following form: let $\{s_i\}_{1\leq i\leq n}$ be a sequence of positive integers, define

$$b_1=-n, b_k=\delta_{k,s_i}, k\geq 2.$$

I proved that such sequences satisfy the conditions (1) and (2) above and give rise to the following identity

$$(1-x)\prod_{j=1}^{n}\frac{1-x^{s_i}}{1-x}=\sum_{\ell} Z(\{s_i\}|\ell)\ (x^{\ell}-x^{n-2\ell+1}), \tag{1.5}$$

where

$$Z(\{s_i\}|\ell)=\sum_{\nu_k}\prod_{j\geq 1}\binom{\sum_a \min(j,s_a)-2\sum_a \min(j,a)\ \nu_a+\nu_j}{\nu_j},\quad \sum k\nu_k=\ell. \tag{1.6}$$

A bit later I met Leon Takhtajan and showed to him a proof of the identity (1.1), and asked about its origin and known generalizations. He told me that identity (1.1) had been proved by H.Bethe in his famous paper [1] in which he invented a method to diagonalize the Hamiltonian of the spin 1/2 Heisenberg chain. Nowadays this method is commonly called by *Algebraic Bethe Ansatz* and has many and varied applications in Physics. Leon also recommended me to ask Kolya Reshetikhin about the Bethe Ansatz, since Kolya is one of the leading specialist in this area.

I really was very enjoyed by discussions with Kolya ! He very clever and accessibly explained me what are the (generalizes) Heisenberg models,the Bethe Ansatz Equations, String Conjecture, Yang–Baxter equations and many other things from Mathematical Physics. And he always was friendly open for discussions and sharing ideas and results. I'm sure that everyone who had, has (and will have) a chance to collaborate or associate with Kolya was (will be !) always enjoyed by Kolya's clever lectures, conversations and collaboration. There are a plethora of evidence and examples which confirm this statement ! He made (and will do !) many important contribution to a wild variety areas of Mathematical Physics and Mathematics. In the present paper I want to overview some results concerning combinatorpal aspects of the Algebraic Bethe Ansatz (ABA) which I had learned from Kolya. The study combinatorial properties of ABA was initiated in our paper [11] and had led to discovery of Rigged Configuration Bijection, Fermionic formula for parabolic Kostka polynomials, Kirillov–Reshetikhin modules,combinatorial proof of unimodality of q-Gaussian polynomials and several other findings in Combinatorics. Other fields of our common interest and research included: the study of generalized XXX and XXZ models, Dilogarithm Identities,q-Orthogonal polynomials, $q6j$-symbols and related knot invariant, quantum Weyl groups, ... Professor Nikolai Reshetikhin made (and will do!) great contribution to Mathematical Physics and Mathematics !

1.2 Introduction

The main objective of the present paper is to prove unimodality of certain polynomials related to the parabolic Kostka polynomials. More precisely, we state and prove sufficient conditions for ℓ-partial parabolic Kostka polynomials to be symmetric and unimodal. As a corollary we prove unimodality of generalized q-Narayana numbers and generalized q-Gaussian polynomials, and give new proof of the strict unimodality of "general" q-binomial coefficients. All our proofs are based on the use of a fermionic formula for parabolic Kostka polynomials discovered by the author of the present paper in the middle of 80's of the last century. In essence, our fermionic formula gives rise to the decomposition of a given parabolic Kostka polynomial $K_{\lambda,\{R\}}(q)$ into the sum of symmetric and unimodal polynomials. Our main strategy is to find data $(\lambda,\{R\})$ such that a given symmetric polynomial can be identified [1] with some symmetric and unimodal part of the decomposition of certain parabolic Kostka polynomial of type $(\lambda\{R\})$. We illustrate this method by several examples, including, among others, the q-Gaussian polynomials, the Rectangular and Staircase Narayana, the Schröder–Narayana, a q-deformed Riordan and a (q,t)-deformed Euler polynomials.

Theorem 1.2 *Let λ be partition, the ℓ-part $\mathcal{K}^{[\ell]}_{\lambda,1^{|\lambda|}}(q)$ of the Kostka polynomial $K_{\lambda,1^{|\lambda|}}(q)$ is symmetric and unimodal polynomial for $\ell(\lambda)-1 \le \ell \le |\lambda|-\lambda_1'$.*

Theorem 1.3 *Let λ be partition and $R=\{R_a=(\mu_a^{\eta_a})\}$ be dominant sequence of rectangular shape partitions. If*
$\mu_a \ge (\nu^{\eta_a})_1'$ $\forall a$ and for all(admissible) configurations of type (λ,R),
then the parabolic Kostka polynomial $K_{\lambda,R}(q)$ is symmetric and unimodal.

For example, one can take $\mu_a \ge |\lambda|-\lambda_1,\ \ foralla \ge 1$. Under these assumptions the first row λ_1 of λ must be "very long" in comparison with other parts of λ, namely, $\lambda_1 \ge \lambda[1](-1+\sum_a \eta_a$, where we set $\lambda[1]=\sum_{j\ge 2}\lambda_j$. Examples satisfying conditions of Theorem include the principal specialization of Schur, i.e. q-Gaussian polynomials, and internal product of Schur functions.

Another objective of our paper is to draw attention of the readers to a natural generalization of the Kostka–Foulkes polynomials, which we named by *Liskova* and double Liskova polynomials , see [20, 16, 21]. Namely, let α,β,γ be three partitions of the same size n.

[1] Up to multiplication by some power of q

Definition 1.4 *Define Liskova polynomial $L^{\gamma}_{\alpha,\beta}(q)$ from the decomposition*

$$s_{\alpha} * s_{\beta}(x) = \sum_{\gamma} L^{\gamma}_{\alpha,\beta}(q) P_{\gamma}(x;q),$$

where $P_{\gamma}(x;q)$ denotes the Hall–Littlewood polynomial, [26], and $s_{al} * s_{\beta}(x)$ denotes the internal product of Schur auctions, Section 4 for details.

It is easy to see that the constant term of the Liskova polynomials $L^{\gamma}_{\alpha,\beta}(q)$ is equal to the multiplicity of the character χ^{γ} of the symmetric group $\mathbb{S}_n$ in the Kronecker product of characters χ^{α} and χ^{β}. In other words, if

$$\chi^{\alpha}\chi^{\beta} = \sum_{\gamma} g_{\alpha\beta\gamma}\chi^{\gamma} \Longrightarrow L^{\gamma}_{\alpha,\beta}(0) = g_{\alpha\beta\gamma}.$$

It is easy to see that one has also

$$L^{\gamma}_{\alpha,\beta}(q) = \sum_{\eta} g_{\alpha\beta\gamma} K_{\eta,\gamma}(q).$$

Note that if partition β consists of only one part, then

$$L^{\gamma}_{\alpha,\beta}(q) = K_{\alpha,\gamma}(q).$$

In other words, the Liskova polynomials are the natural generalization of the Kostka–Foulks polynomials.
The number $L^{\gamma}_{\alpha,\beta}(1)$ admits a combinatorial interpretation as the number of the Littlewood–Richardson sequences of tableaux $\mathcal{T} = (T_1, \ldots, T_{\ell(\gamma)})$ of type $(\alpha, \beta; \gamma)$, see [20, Definition 6.4]; cf. [26, pp. 184-185]. We denote by $\nu^{\gamma}(\alpha,\beta)$ the set of the Littlewood–Richardson sequences of tableaux defined in [20, Definition 6.4].

Problem 1.5 *Let $\mathcal{T} \in \nu^{\gamma}(\alpha,\beta)$ be a LR-sequence of type $(\alpha,\beta;\gamma)$. Define statistics $c(\mathcal{T})$ for a LR-sequence $\mathcal{T}$ such that*

$$L^{\gamma}_{\alpha,\beta}(q) = \sum_{\mathcal{T} \in \nu^{\gamma}(\alpha,\beta)} q^{c(\mathcal{T})}.$$

Therefore, $\quad g_{\alpha,\beta,\gamma} = \{\mathcal{T} \in \nu^{\gamma}(\alpha,\beta) \mid c(\mathcal{T}) = 0\}$.

Remark 1.6 *(***Charge and the Littlewood–Richardson rule***)*

Let λ, μ and ν be three partitions such that $|\lambda| + |\mu| = |\nu|$, then the Littlewood–Richardson number $c^{\nu}_{\lambda,\mu} := \mathrm{Mult}[V_{\nu} \colon V_{\lambda} \otimes V_{\mu}]$ is equal to the number of semistandard Young tableaux T of skew shape $\nu \setminus \mu$ and weight λ such that the right-to-left and top-to-bottom reading word $w(T)$ corresponding to tableaux T in question,is an Yamanuchi word. It is clearly seen that the word $w(T)$ obtained is an Yamanuchi word iff the charge of $w(T)$ is equal to zero. In other words,

$$c^{\nu}_{\lambda,\mu} = \{T \in SYT(\nu \setminus \mu, \lambda) | c(T) = 0\}.$$

We expect a similar rule for computation of the numbers $g_{\alpha,\beta,\gamma}$, as it was suggested in Problem 1.5.
Note also that if we set $\alpha_N := (N-|\lambda|, \lambda)$, $\beta_N := (N-|\mu|, \mu)$ and $\gamma_N := (N-|\nu|, \nu)$, and $N \geq |\nu| + \nu_1$, then [25]

$$g_{\alpha_N,\beta_N,\gamma_N} = c^{\nu}_{\lambda,\mu}.$$

- *(***Big Littlewood–Richardson numbers [21]***)*

Let α, β, γ be three partitions of the same size n. For integer N consider partitions $\alpha^{(N)} := (N+\alpha_1, \alpha[1])$, $\beta^{(N)} := (N+\beta_1, \beta[1])$ and $\gamma^{(N)} := (N+\gamma_1, \gamma[1])$, where for any partition $=(\lambda_1, \ldots, \lambda_r)$ and integer $1 \leq k < r$, we set $\lambda[k] := (\lambda_{k+1}, \ldots, \lambda_r)$.

Proposition 1.7 *The sequence of polynomials $\{L_{al^{\gamma_N}_{N,\beta_N}}(q)\}_N$ is stabilized to the polynomial $\mathcal{L}^{\nu}_{\lambda,\mu}(q)$, i.e. for N sufficiently large polynomials $L^{\gamma_N}_{\alpha_N,\beta_N}(q)$ doesn't depend on N and equals to $\mathcal{L}^{\nu}_{\lambda,\mu}(q)$.*

We call numbers $C^{\nu}_{\lambda,\mu} := \mathcal{L}^{\nu}_{\lambda,\mu}(0)$ by the big LR-coefficients and polynomials $\mathcal{L}^{\nu}_{\lambda,\mu}(q)$ by stable Liskova polynomials.

For example $\mathcal{L}^{21}_{21,21}(0) = 9$.Note that if $|\lambda| + |\mu| > |\nu|$, then $\mathcal{L}^{\nu}_{\lambda,\mu}(q) = 0$.

- *(***Domino, nspin and t-LR-numbers***)*

Let λ, μ and η be three partitions such that $|\lambda| + |\mu| = |\eta|$. Consider rectangular shape partition $(N^{\lambda_1'})$ such that $N \geq \mu_1$, and define partition $\Lambda_N := (N^{\lambda_1'}) + \lambda, \mu$ and dominant sequence of rectangular shape partitions $R_N := \{(N^{\lambda_1'}), \eta)^+$.

Theorem 1.8 (1) $K_{\Lambda_N,R}(q) \stackrel{\bullet}{=} c^{\eta}_{\lambda,\mu} + \ldots + q^{n(\eta)-n(\lambda)-n(\mu)}$;

(2)

$$K_{\Lambda_N,R}(q) \stackrel{\bullet}{=} \sum_{T \in STY^{(2)}((2\ \lambda)\ \vee\ (2\ \mu),\eta)} q^{spin(T)}$$

Finally, define polynomials

$$K^{\eta}_{\lambda,\mu}(q,t) := \sum_{\{\nu\}} t^{ns(\nu)} K_{\Lambda_N,R_N}(q),$$

where sum runs over the set of (admissible) configurations of type (Λ_N, R_N), and $ns(\nu)$ stands for the nips of configuration $\{\nu\}$.

Note that if partition β consists of only one part, then the set $\nu^{\gamma}(\alpha, \beta) = STY(\alpha, \gamma)$, that is a LR-sequence $\mathcal{T}$ of type $(\alpha, (|\beta|); \gamma)$ defines a semistandard tableau T of shape α and weight γ, and moreover $charge(\mathcal{T}) = charge(T)$.

Definition 1.9 *Define double Liskova polynomials as follows*

$$L^{\gamma}_{\alpha,\beta}(q,t) = \sum_{\eta} g_{\alpha\beta\eta} K_{\eta,\gamma}(q,t),$$

where $K_{\eta,\gamma}(q,t)$ denotes the Kostka–Macdonald polynomial.

It is easy to see [26] that

$$L^{\gamma}_{\alpha,\beta}(0,t) = L^{\gamma}_{\alpha,\beta}(t), \quad L^{\gamma}_{\alpha,\beta}(1,1) = f^{\alpha} f^{\beta} \quad L^{\gamma}_{\alpha,\beta}(0,0) = g_{\alpha\beta\gamma}.$$

Note also that if partition β consists only one part, then

$$L^{\gamma}_{\alpha,\beta}(q,t) = K_{\alpha,\gamma}(q,t).$$

Problem 1.10 *For a pair of standard Young tableaux $(T_1, T_2) \in STY(\alpha) \times STY(\beta)$ define pair of statistics $u_{\mu}(T_1,T_2)$ and $v_{\mu}(T_1,T_2)$ such that*

$$L^{\mu}_{\alpha,\beta}(q,t) = \sum_{(T_1,T_2) \in STY(\alpha) \times STY(\beta)} q^{u_{\mu}(T_1,T_2)} \ t^{v_{\mu}(T_1,T_2)}.$$

We also include a fermionic formula for generalized exponents polynomial $G_N(V_{\alpha} \otimes V^{*}_{\beta})(q)$ associated with mixed tensor representation $V_{\alpha} \otimes V^{*}_{\beta}$ of the Lie algebra $sl(N)$.

Let us say few words about the content of our paper.
In Section 2 we collect some basic definitions and notation we well use in our paper. Section 3 contains our main results. In Section 4 we introduce and study some properties of the internal product of Schur functions and define Liskova polynomials. We review our old result concerning a fermionic formula for principal specialization of the internal product of Schur functions. In Section 5 we state a fermionic formula for the mixed tensor product generalized exponents polynomial. In Section 6 for the reader's convenience we give brief review of basic facts about Rigged Configurations.

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2 Basic notation and definitions

To start with, let us recall definitions of symmetric, unimodal and strictly unimodal polynomials.

Definition 2.1

(1) *A sequence of non-negative real numbers $\mathbf{a} = \{a_k\}_{0 \le k \le n}$ is called unimodal if for some integer d,*

$$a_k \le a_{k+1}, \ 1 \le k \le d, \quad and \quad a_k \ge a_{k+1}, \ d \le k \le n;$$

.

(2) *A such sequence is called strictly unimodal if the inequalities in the above definition are all strict.*

(3) *A sequence* $\{a_k\}_{0\le k\le n}$ *is called symmetric if*

$$a_i = a_{n-i},\ i \le n/2.$$

For a symmetric sequence $\{a_k\}_{0\le k\le n}$ the number $n/2$ is called *centerline* or *unimodality index*, and denoted by $un(\mathbf{a})$.

Definition 2.2 *A polynomial* $p(x) = x^k(a_0+a_1x+\cdots+a_nx^n)$ *is called symmetric (and/or unimodal, or strictly unimodal) iff the sequence of its coefficients* $\{a_0,\ldots,a_n\}$ *is symmetric (and/or unimodal, or strictly unimodal).*

Definition 2.3 *(Strong log-concavity of q-binomial coefficients)*

The sequence of polynomials $\{r_n(q) \in \mathbb{N}[q]\}_{n\ge 0},\ r_0(q) = 1, r_n(q) = 0,\ if\ n \in \mathbb{Z}_{n<0}$, *is called strongly log-concative, if the difference*

$$r_n(q)r_m(q) - r_{n-k}(q)r_{m+k}(q)$$

is a unimodal polynomial with non-negative coefficients for all $n \ge m, andk \le m..$

Proposition 2.4 (1) *([33]) The product of symmetric and unimodal (resp. strictly unimodal) polynomials is again symmetric and unimodal (resp. strictly unimodal) polynomial.*

(2) *([7],[4]) Let* $k \ge 3$*, and* $n_1,\ldots,n_k$ *be a sequence of pairwise distinct positive integers.Then polynomial*

$$\prod_{a=1}^{k}\ [n_a+1]_q$$

is symmetric and strictly unimodal.

2.1 Polynomials associated with configurations of type (λ, R)

Now we are going to review some basic notation and definitions concerning the Rigged Configurations. For more details and examples see, e.g. Appendix, or [20, 22].

For a given data (λ, R) as before [2], let us define the set of admissible configurations of type (λ, R), and then introduce a few polynomials, which are the main object to study in our paper.

Thus, let λ be a partition and $R = (R_1,\ldots,R_p)$ be a sequence of rectangular shape partitions, say, $R = \{R_a := (\mu_a^{\eta_a}), a = 1,\ldots,p\}$. We assume that the size of λ is equal to $\sum_a \mu_a\eta_a$. With a such data (λ, R) we associate a set of admissible configurations of type (λ, R), which is denoted by $C(\lambda, R)$.

[2]That is λ is a partition and R is a sequence of rectangular shape partitions.

Definition 2.5 *An admissible configuration* $\{\nu\}$ *of type* (λ, R) *is a sequence of partitions* $\{\nu\} = (\nu^{(1)}, \nu^{(2)}, \ldots)$ *which satisfies the following conditions*

- *(Size of partition* $\nu^{(k)}$*)*

$$|\nu^{(k)}| = \sum_{j>k} \lambda_j - \sum_{a\geq 1} \mu_a \max(\eta_a - k, 0) = -\sum_{j\leq k} \lambda_j + \sum_{a\geq 1} \mu_a \min(k, \eta_a).$$

We set $\nu^{(0)} = \emptyset$*;*

- *(Nonnegativity constraints, or positivity of vacancy numbers* $P_j^{(k)}(\{\nu\})$*)*

$$P_j^{(k)}(\{\nu\}) := Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) + \sum_{a\geq 1} \min(\mu_a, j)\delta_{\eta_a,k} \quad \geq 0, \quad \forall k, j \geq 1,$$

where $\delta_{a,b}$ *stands for the Kronecker delta function; for any Young diagram* ν *we set*

$$Q_j(\nu) = \sum_{a\geq 1} \min(\nu_j, a) = \sum_{a\leq j} \nu_a^{'};$$

Note that if $k \geq l(\lambda)$ and $k \geq \eta_a$ for all a, then partition $\nu^{(k)} = \emptyset$. Therefore there are only finite number of admissible configurations of type (λ, R). It might be well to point out that one needs to check a non-negativity of a vacancy number $P_j^{(k)}(\{\nu\})$ even if the corresponding number of rows $m_j(\nu^{(k)})$ is equal to zero.

The main quantities which we associate with a configuration $\{\nu\} \in C(\lambda; R)$ we are planning to investigate, are

- $n(R) = \sum_{i,k\geq 1} \binom{\sum_a \theta(\eta_a - k)\theta(\mu_a - i)}{2} = \sum_{a<b} \ \min(\mu_a, \mu_b)\min(\eta_a, \eta_b)$,

- (Matrix $M(\{\nu\})$ associated with a configuration $\{\nu\}$)
  $M(\{\nu\}) := (m_{ij}(\{\nu\})$, where

$$(m_{ij}) = ((\nu^{(i-1)})_j^{'} - (\nu^{(i)})_j^{'}) + (\sum_{a\geq 1} \theta(\eta_a - i)\theta(\mu_a - j));$$

- (The umber of parts of size j in diagram/partition $\nu^{(k)}$)

$$m_j^{(k)}(\{\nu\}) := (\nu^{(k)})_j^{'} - (\nu^{(k)})_{j+1}^{'} = \sum_{a\geq 1} \min(\eta_a, k)\delta_{\mu_a,j} - \sum_{i\leq k}(m_{ij} - m_{i,j+1});$$

- (Vacancy numbers)

$$P_i^{(k)}(\{\nu\}) = \sum_{i\geq j}(m_{ki} - m_{k+1,i}),$$

- (Charge $c(\{\nu\})$ and cocharge $\overline{c}(\{\nu\})$ of configuration)

$$c(\{\nu\}) = \sum_{i,j\geq 1} \binom{m_{ij}(\{\nu\})}{2}, \qquad \overline{c}(\{\nu\}) = \sum_{j,k\geq 1} \binom{(\nu^{(k)})_j^{'} - (\nu^{(k+1)})_j^{'}}{2};$$

- (Centerline or unimodality index)

$$un(\nu) = n(R) - 1/2\Big(\sum_{a\geq 1} Q_{\mu_a}(\nu^{(\eta_a)})\Big);$$

in the case $\eta_a = 1, 1 \leq a \leq p$, our formula for $un(\{\nu\})$ can be rewritten as follows

$$2\ un(\{\nu\}) = 2\ n(\mu) - \sum_{a\geq 1} \mu_a^{'} \nu_a^{'},$$

where we set $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$.

- (Nips of configuration)

$$2\ ns(\{\nu\}) = \#\Big((ij)\ \Big| m_{ij}(\{\nu\}) \equiv 1 \mod 2\Big),$$

- (Descent index) $\quad des(\{\nu\}) = \nu_1^{'}$.

Definition 2.6 *A sequence of rectangular shape partitions $\{R_a = (\mu_a^{\eta_a})\}_{1\leq a\leq p}$ is called dominant, if $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p$.*

Now we are going to define ℓ-partial parabolic Kostka polynomials.
Let ℓ be a positive integer, using the quantities associated with a configurations $\{\nu\}$ of type (λ, R), we define the following polynomials

- (ℓ-partial $\mathcal{K}$-polynomials $\mathcal{K}_{\lambda,R}^{[\ell]}(q)$). Define

$$\mathcal{K}_{\lambda,R}^{[\ell]}(q) = \sum_{\substack{\{\nu\}\in C(\lambda,R)\\ un(\{\nu\})=\ell}} q^{c(\{\nu\})} \prod_{k,i\geq 1} \begin{bmatrix} P_i^{(k)}(\{\nu\}) \ + \ \mu_i^{(k)}(\{\nu\}) \\ \mu_i^{(k)}(\{\nu\}) \end{bmatrix}_q,$$

where the sum runs over the set of type (λ, R) configurations with the unimodality index equals ℓ,

- ($\mathcal{K}$-polynomials $\mathcal{K}_{\lambda,R}(q,t)$)

$$\mathcal{K}_{\lambda,R}(q,t) = \sum_{\{\nu\}\in C(\lambda,R)} q^{c(\{\nu\})}\ t^{un(\{\nu\})} \prod_{k,i\geq 1} \begin{bmatrix} P_i^{(k)}(\{\nu\}) \ + \ \mu_i^{(k)}(\{\nu\}) \\ \mu_i^{(k)}(\{\nu\}) \end{bmatrix}_q.$$

Therefore, $\quad \mathcal{K}_{\lambda,R}(q,t) = \sum_{\ell} \ t^{un(\{\nu\})} \mathcal{K}_{\lambda,R}^{(\ell)}(q), \quad$ and the polynomial $\mathcal{K}_{\lambda,R}(q,1)$ coincides with the parabolic Kostka polynomial corresponding to data (λ, R).

Definition 2.7 *Let $d \in \mathbb{N}$ and (λ, R) be a partition and a dominant sequence of rectangular shape partitions. Define generalized Narayana polynomials of type (λ, R) and degree d, denoted by $N_d(\lambda, R; q)$, as follows*

$$N_d(\lambda, R; q) := \mathcal{K}_{\lambda,R}^{(d)}(q).$$

2.2 Formulas for unimodality index

To state our main result, we need to compute the "centerline", that is the sum

$$un(\{\nu\}) := c(\{\nu\}) + 1/2 \sum_{i,k\geq 1} P_i^{(k)}(\{\nu\})\ m_i^{(k)}(\{\nu\})$$

for any admissible configuration $\{\nu\}$ of type $(\nu;R)$. First of all let us define an analog of the number $n(\lambda) := \sum_{i\geq 1}(i-1)\ \lambda_i$, where λ is a partition, for a sequence of rectangles $R=\{(\mu_a^{\eta_a})\}$. Namely let us set

$$n(R) := \sum_{i,k\geq 1} \binom{\sum_a \theta(\eta-k)\theta(\mu_a-i)}{2}.$$

It is not difficult to see that

$$n(R) = \sum_{a<b} \min(\mu_a,\mu_b)\ \min(\eta_a,\eta_b).$$

Proposition 2.8 *One has the following relations*

- $$c(\{\nu\}) = n(R) + \sum_{i,k\geq 1}\Big(((\nu^{(k)})_i')^2 - (\nu^{(k)})_i' \times (\nu^{(k+1)})_i'\Big) -$$
$$\sum_{k,i\geq 1}\Big(\sum_{a\geq 1} \theta(\mu_a-i)\ \delta_{\eta_a,k}\Big)\nu_i^{(k)}.$$

- $$\sum_{i,k\geq 1} P_i^{(k)}(\{\nu\})\ m_i^{(k)}(\{\nu\}) = 2\ \Big((\nu^{(k)})_i' \times (\nu^{(k+1)})_i' - ((\nu^{(k)})_i')^2\Big) +$$
$$\sum_{k,i\geq 1}\Big(\sum_{a} \theta(\mu_a-i)\ \delta_{\eta_a,k}\Big)\nu_i^{(k)}.$$

A proof is a bit lengthy work with definitions of the charge and vacancy numbers of a configuration $\{\nu\}$.

Therefore one has

Theorem 2.9

$$un(\{\nu;R\}) = n(R) - 1/2\Big(\sum_{i,k\geq 1}(\sum_a \theta(\mu_a-i)\ \delta_{\eta_a,k})\ \nu_i^{(k)}\Big) = n(R) - 1/2\sum_a\ Q_{\mu_a}(\nu^{(\eta_a)}).$$

Therefore,

$$un(\{\nu\}) = n(R) - 1/2\ \sum_a\ Q_{\mu_a}(\nu^{(\eta_a)}).$$

In the special case $\eta_a = 1,\ \forall a\geq 1,$ one has

Corollary 2.10 *Assume that $\eta_a = 1, \forall a,$ and denote by μ a unique partition corresponding to the sequence $\{\mu_1, \mu_2, ...\}$. Then for any admissible configuration $\{\nu\}$ of type $(\lambda; \mu)$ one has*

$$un(\{\nu\}) = n(\mu) - 1/2\Big(\sum_{a\geq 1} Q_{\mu_a}(\nu^{(1)}) = n(R) - 1/2(\sum_{i\geq 1} \mu_i^{'} \times (\nu^{(1)})_i^{'}\Big);$$

$$c(\{\nu\}) + \sum_{i,k\geq 1} P_i^{(k)}(\{\nu\})\ m_i^{(k)}(\{\nu\}) = n(R) - \sum_{i,k\geq 1} (\nu_i^{(k)})'\Big(\nu_i^{(k)})' - (\nu_i^{(k+1)})'\Big).$$

3 Main results

- (**Unimodality of q-Gaussian polynomials**)

Theorem 3.1 (1) *Let λ be a strict partition, $\ell(\lambda) \geq 3$. If $N > \ell(\lambda)$, then the q -Gaussian polynomial $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q$ is a symmetric and strictly unimodal.*

(2) *Let λ be a partition such that $\lambda_i^{'} - \lambda_{i+1}^{'} \geq 8$. If $N > \ell(\lambda) + 8$, then the q -Gaussian polynomial $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q$ is a symmetric and strictly unimodal.*

Indeed, first of all let us recall that

$$\begin{bmatrix} N \\ \lambda \end{bmatrix}_q \doteq K_{\Lambda,\mu}(q)$$

, where $\Lambda = (N|\lambda|, \lambda)$ and $\mu = (|\lambda|^{N+1})$.
Now let us consider the maximal configuration $\Delta := \Delta_{\Lambda,\mu}$ of type (Λ, μ), see Section 6. By definition $\Delta = (\lambda, \lambda[1], \ldots, \lambda[\ell(\lambda) - 1)]$, where $\lambda[k] := (\lambda_{k+1}, \ldots, \lambda_{\ell(\lambda)})$. By construction one can see that if $k \geq 2$, then the all vacancy numbers $P_j^{(k)}(\Delta)$ are equal to zero, whereas in the case (1)

$$P_j^{(1)}(\Delta) = jN - Q_j(\lambda) = \sum_{1\leq a\leq j} (N - \lambda_a) > 0.$$

Observe that if $a < b$, then

$$P_b^{(1)}(\Delta) - P_a^{(1)}(\Delta) = (b - a)N - \sum_{j=a+1}^{b} \lambda_j^{'} \geq (b - a)(N - \lambda_1^{'}) > 0.$$

Therefore it follows from Proposition, (2), that

$$K_{\Lambda,\mu}(\Delta) := \prod_{j\geq 1} [P_j^{(1)}(\Delta) + 1]_q$$

is strictly unimodal and symmetric. Since the polynomials $K_{\Lambda,\mu}(\{\nu\})$ have the same centre of symmetry and unimodal for all configurations $\{\nu\} \in C(\Lambda, \mu)$, and

$$\sum_{\{\nu\}} K_{\Lambda,\mu}(\{\nu\}) = K_{\Lambda,\mu}(q),$$

one concludes that the q- Gaussian polynomial $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q \doteq K_{\Lambda,\mu}(q)$ is also symmetric and strictly unimodal.
Similarly, in the second case one has $m_j^{(1)}(\Delta) := \lambda'_j - \lambda'_{J=1} \geq 8$, and $P_j^{(1)}(\Delta) \geq 8j$. Thus it follows from Proposition 2.4,(1), that the all q-binomial coefficients $\begin{bmatrix} P_j^{(1)}(\Delta)+m_j^{(1)}(\Delta) \\ m_j^{(1)}(\Delta) \end{bmatrix}$ are symmetric and unimodal. Therefore their product, i.e. $K_{\Lambda,\mu}(\Delta)$, also is symmetric and strictly unimodal. Hence, under the assumptions of Proposition 2.4,(2), the polynomial $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q$ is also symmetric and strictly unimodal. ■

It is still an open question to describe the all exceptions to Theorem3.1.
In the case when partition λ consists of one row (or one column), Theorem 3.1 has been proved first in [29]. The first combinatorial proof of unimodality of q-binomial coefficients $\begin{bmatrix} n+m \\ m \end{bmatrix}_q$ has been done in [28]. In [13] one can find rigged configuration interpretation of statistics introduced in K.O'Hara's combinatorial proof presented in [28]; see also [39].

Theorem 3.2 *(***Strong log-concavity of q-binomial coefficients***)*
For $0 \leq k \leq m \leq n < N-k$, polynomial

$$BS_{n,m}(N,k;q) := \begin{bmatrix} N \\ m \end{bmatrix}_q \begin{bmatrix} N \\ n \end{bmatrix}_q - q^{k(n-m+k)} \begin{bmatrix} N \\ m-k \end{bmatrix}_q \begin{bmatrix} N \\ n+k \end{bmatrix}_q$$

is a symmetric and unimodal polynomial in q with non-negative coefficients.

Indeed, one can show (see, e.g.,[3], Corollary 3.2, or [26],p.44, that

$$\sum_{j=0}^{k} s_{2^{m-j}1^{n-m+2j}}(1,q,\ldots,a^{N-1}) = q^{\binom{m}{2}+\binom{n}{2}} \Big(\begin{bmatrix} m \\ m \end{bmatrix}_q \begin{bmatrix} n \\ n \end{bmatrix}_q - q^{k(n-m+k)} \begin{bmatrix} N \\ m-k \end{bmatrix}_q \begin{bmatrix} N \\ n+k \end{bmatrix}_q \Big).$$

Now let us observe that the unimadality index of polynomial

$$s_{2^{n-j}1^{n-m+2j}}(1,q,\ldots,q^{N-1}), \quad 0 \leq j \leq k$$

is the same for all j and equals to $\frac{1}{2}(n+m)(N-1)$. According to our result that the principal specialization $s_\alpha(q,q^2,\ldots,q^N)$ of a Schur polynomial $s_\alpha(X)$ is symmetric and unimodal, see, e.g., [17], and as it's clearly sen that the polynomial from our Theorem 1.2 is the sum of symmetric and unimodal polynomials with the same symmetry and unimodality indexes, we conclude that the polynomial from Theorem 3.2 is symmetric and unimodal.

• (**Unimodality of** *Schröder–Narayna polynomials*)

Let d, N be positive integers, $N \geq 2d$. For each k, $1 \leq k \leq d$, define the kth Schröder–Narayana polynomial as follows $ScN_k(n,d;q) =$

$$\frac{1}{[d-k_q![d-k+1]_q!} \begin{bmatrix} N-2d+k-1 \\ k-1 \end{bmatrix}_q \prod_{j=0}^{d-k} [n-d-j]_q^{\min(2,j+1,d-k-j+1+\delta_{k,d})}.$$

Theorem 3.3 *The Schröder–Narayana polynomials are symmetric and unimodal for all N, d,$N \geq 2d \geq 2$.*

Our proof is based on observation that

$$ScN_k(n,d;q) = \mathcal{K}^{(d)}_{(d^2,1^{N-2d}),(1^N)}(q)$$

, and our result that for any partition λ the truncated Kostka–Foulkes polynomial $\mathcal{K}^{(d)}_{\lambda,(1^{|\lambda|})}(q)$ is symmetric and unimodal.

- (**Unimodality of almost rectangular Narayana polynomials**)

Let d,k,N be positive integers,$N \ge kd$. Defined almost rectangular Narayana polynomials $N_p(k,d,N;q)$ to be

$$N_p(k,d,N;q) = \mathcal{K}^{(p)}_{(d^k,1^{N-kd})}(q).$$

Theorem 3.4 (1) *Polynomials $N_p(k,d,N;q)$ are symmetric and unimodal.*

(2) *(Combinatorial formula)*

$$N_p(k,d,N;q) = \sum_{\substack{\{\nu\} \\ (\nu^{(1)})'_1 = p}} q^{c(\{\nu\})} \prod_{j,\ell} \begin{bmatrix} P_j^{(\ell)}(\{\nu\}) + m_j^{(\ell)} \\ m_j^{(\ell)} \end{bmatrix}_q,$$

where the sum runs over a sequence of partitions $\{\nu^{(0)} = \emptyset, \nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(k)} = 1^{N-kd}, \nu^{(k+1)} = 1^{N-kd-1}\}$, $|\nu^{(\ell)}| = (k-\ell)d,\ 1 \le \ell \le k-1$;

$$P_j^{(\ell)}(\{\nu\}) = min(N,j)\delta_{\ell,1} + Q_j(\nu^{(\ell-1)} - 2Q_j(\nu^{(\ell)}) + Q_j(\nu^{(\ell)+1}), \quad j \ge 1, \quad 1 \le \ell \le k;$$

$m_j(\nu^{(\ell)}) = (\nu^{(\ell)})')_j - (\nu^{(\ell)})')_{j+1}$; $c(\{\nu\})$ stands for the charge of configuration $\{\nu\}$.

In the case $k=2$ we have some explicit product formula for polynomials $N_p(k=2,d,N;q)$, see Theorem 3.4 stated above. As for the Kostka polynomials $K_{(d^k,1^{N-kd})}(q)$ thereof, they are not symmetric and unimodal in general. In fact the number $K_{(d^2,1^{N-2d})}(1)$ is equal to the so-called *Kirkman-Caley* number

$$KC(N,d) := \frac{1}{N-d+1}\binom{N}{d}\binom{N-d-1}{d-1}$$

and is equal to the number the so-called *short bushes* with N ends and d branching nodes, see [31, A108263] for more details and references. Moreover, one has

$$KC(n+d-1,d+1) = T(n,d),$$

where $T(n,d)$ denotes the number of dissections of a convex n-gon into $d+1$ regions, see, e.g., [31, A033282]. Note also that polynomial

$$MS_d(n;q) := \sum_{\substack{d=1 \\ 2d \le n}} K_{(d^2,1^{n-2d})}(q)$$

is a q deformation of the Motzkin sum (or Riordan) number $MS(n)$, see, e.g., [31, A005043] for definition, examples and references concerning the Motzkin sum numbers. For example,

$$MS_2(8;q) = (1,0,1,1,2,2,4,3,6,5,7,6,9,6,8,6,7,4,5,2,3,1,1,0,1)_q.$$

Finally we define rectangular Motzkin polynomials $M_d(n;q) := MS_d(n;q) + qMS_d(n+1;q)$.

Problem 3.5 *Define statistics on the set of Motzkin paths of length n with no horizontal steps at level 0 (i.e. the set of Riordan paths) such that the generating function of that statistics is equal to the polynomial $MS_d(n;q)$.*

For example, $M_2(8;q) =$
$(1,1,1,2,3,4,6,7,10,11,14,15,18,19,20,21,22,20,20,19,17,15,13,11,9,8,5,4,3,2,1,1)$.
We **expect** that the Motzkin polynomials $M_2(n;q)$ are unimodal for $n \ge 2$.

The case $N = dk$ corresponds to rectangular Narayana polynomials studied in [20, 36, 4].

- (q**-Deformed Eulerian polynomials**)

Definition 3.6 *Define a q-deformation of the Euler polynomials* [3] *$A_\ell(n,m;q)$, denoted by $A_\ell(n,m;q)$ as follows*

$$A_\ell(n,m;q) = \sum_{\{\nu\}} q^{c(\{\nu\})} \prod_{j,k \ge 1} \begin{bmatrix} P_j^{(k)}(\{\nu\}) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{bmatrix}_q ,$$

where summation runs over the set of configurations $\{\nu\}$ of type $(((n^n),1^m),((n-1)^{n-1},1^{n+m}))$, $(\nu^{(k)})' = n+m+k-1$, $k=1,\ldots,n+m-1$. Other notation are are the same as in Theorem 3.7 below.

Theorem 3.7 *If n is odd, then the polynomial $E(n;q) := K_{(n^n),((n-1)^{n-1},1^{n-1})}(q)$ is* **symmetric**, *and the polynomial $E(n;q) - q^n - q^{n^2}$ is* **unimodal**.
Moreover, set $E^{[d]}(n;q) := \mathcal{K}^{[d]}_{(n^n),((n-1)^{n-1},1^{n-1})}(q)$. Then

$$E^{[d]}(n;q) \doteq E^{[3n-3-d]}(n;q^{-1}, \quad n-1 \le d \le 2(n-1),$$

where for any two polynomials $P_1(q)$ and $P_2(q)$ the expression $P_1(q) \doteq P_2(q)$ means that the ratio $P_1(q)/P_2(q)$ is an integer power of q.

Conjecture 3.8 *Polynomials $\mathcal{K}^{[d]}_{(n^n),((n-1)^{n-1},1^{n-1})}(q)$ are unimodal for $n-1 \le d \le 2(n-1)$ and any n.*

Note that $K_{(n^n),((n-1)^{n-1},1^{n-1})}(q=1) = n!$ and $\mathcal{K}^{d}_{(n^n),((n-1)^{n-1},1^{n-1})}(q=1) = A(n,d)$.

$\blacksquare$

Example 3.9 *Take $n = 5$, so we deal with partition $\lambda = (5,5,5,5,5)$, and weight $\mu = (4,4,4,4,4,1,1,1,1,1)$. Clearly, $\mu' = (10,5,5,5)$. The corresponding Euler polynomial $A_5(q) = (1,26,66,26,1)_q$. On the other hand, there are 13 (admissible) configurations of type $((5^5),(4^5,1^5))$, namely,*
(1) $\nu[1]' = \{(4^5),(3^5),(2^5),(1^5)\}$ *with charge $c(\nu_{[1]} = 25$; all vacancy numbers equal to zero;*
(2) $\bullet$ $\nu[2]' = \{(5,4^3,3),(3^5),(2^5),(1^5)\}$, *with charge $c(\nu_{[2])} = 17$; the only non-zero vacancy*

[3] see,e.g., [34],[31, $A008292$] and the literature quoted theirin for definition, basic properties and applications of the Euler numbers $A9n,k)$ and Euler polynomials $E_n(t) = \sum_{k=0}^n A(n,k)\, t^k$.

numbers are $P_1^{(1)} = 3 = P_3^{(1)}$;
• $\nu_{[3]}\{(5^2,4^2,2),(3^5),(2^5),(1^5)\}$, *with charge* $c(\nu_{[3]}) = 16$; *the only non-zero vacancy numbers are* $P_2^{(1)} = 1 = P_4^{(1)}$;
• $\nu_{[4]}\{(5^2,4,3^2),(3^5),(2^5),(1^5)\}$, *with charge* $c(\nu_{[4]}) = 19$; *the only non-zero vacancy numbers are* $P_2^{(1)} = 1 = P_3^{(1)}$;
the contribution to the Kostka polynomial from these configurations is equal to

$$\mathcal{K}_{\lambda,\mu}^{[5]}(q) = q^{16}(1,3,4,5,6,4,2,1)_q, \quad \mathcal{K}_{\lambda,\mu}^{[5]}(1) = 26.$$

(3) • $\nu_{[5]}\{(6,4^3,3^2),(3^5),(2^5),(1^5)\}$, *with charge* $c(\nu_{[5]}) = 16$; *the only non-zero vacancy numbers are* $P_1^{(1)} = 1 = P_3^{(1)}$;
• $\nu_{[6]}\{(6,4^2,3^2),(3^5),(2^5),(1^5)\}$, *with charge* $c(\nu_{[6]}) = 13$; *the only non-zero vacancy numbers are* $P_1^{(1)} = 2 = P_4^{(1)}$;
• $\nu_{[7]}\{(6,4^3,2),(4,3^3,2),(2^5),(1^5)\}$, *with charge* $c(\nu_{[7]} = 11$; *the only non-zero vacancy numbers are* $P_1^{(1)} = 2 = P_4^{(1)}$;
• $\nu_{[8]}\{(6,5,4^2,1),(3^5),(2^5),(1^5)\}$, *with charge* $c(\nu_{[8]}) = 11$; *the only non-zero vacancy numbers are* $P_1^{(1)} = 2,\ P_4^{(2)} = 1$;
• $\nu_{[9]}\{(6,5,4,3,2),(4,3^3,2),(2^5),(1^5)\}$, *with charge* $c(\nu_{[9]}) = 13$; *the only non-zero vacancy numbers are* $P_1^{(1)} = 2 = P_4^{(1)}$;
the contribution to the Kostka polynomial from these configurations is equal to

$$\mathcal{K}_{\lambda,\mu}^{[6]}(q) = q^{11}(2,4,9,11,14,11,9,4,2)_q, \quad \mathcal{K}_{\lambda,\mu}^{[6]}(1) = 66.$$

(4) • $\nu_{[10]}\{(7,4^4,1),(5,3^3,1),(3,2^3,1),(1^5)\}$, *with charge* $c(\nu_{[10]}) = 7$; *the only non-zero vacancy numbers are* $P_1^{(1)} = 1 = P_4^{(1)}$;
• $\nu_{[11]}\{(7,4^2,3,2),(4,3^3,2),(3,2^3,1),(1^5)\}$, *with charge* $c(\nu_{[11]}) = 11$; *the only non-zero vacancy numbers are* $P_4^{(1)} = 2,\ P_1^{(2)} = 1$;
• $\nu_{[12]}\{(7,4^3,3,1),(4,3^3,2),(3,2^3,1),(1^5)\}$, *with charge* $c(\nu_{[12]}) = 9$; *the only non-zero vacancy numbers are* $P_1^{(2)} = 1 = P_4^{(2)}$;
the contribution to the Kostka polynomial from these configurations is equal to

$$\mathcal{K}_{\lambda,\mu}^{[7]}(q) = q^{7}(1,2,4,6,5,4,3,1)_q, \quad \mathcal{K}_{\lambda,\mu}^{[7]}(1) = 26.$$

(5) • $\nu_{[13]}\{(8,4^3),(6,3^3),(4,2^3),(2,1^3)\}$, *with charge* $c(\nu_{[12]}) = 5$; *the all vacancy numbers are equal to zero;*
Finally, one can easily check that $\sum_{d=4}^{8} \mathcal{K})\lambda,\mu^{[d]}(q) =$

$$q^5(1,0,2,4,6,7,8,12,12,14,12,12,8,7,6,4,2,0,1)_q = K_{(5^5),(4^5,1^5)}(q),$$

as well as to see that polynomial $K_{(5^5),(4^5,1^5)}(q) - q^5 - q^{25}$ *is symmetric and unimodal.*

Exercise 3.10 *Consider set of configurations* $C_n^{[d]} = \{\nu\} \in C((n^n),((n-1)^(n-1),1^{n-1})) \mid \nu_1' = d\}$ *Construct bijection between the sets* $C_n^{[d]}$ *and that* $C_n^{[3n-3-d]}$.

It is an interesting **task**to to find connections between our polynomials and a q-analog of Eulerian polynomials studied in [30].

Definition 3.11 *(Symmetric and unimodal polynomials associated with reduced decomposition's of the longest element $w_0^{(n)} \in \mathbb{S}_n$) Define polynomials*

$$RD^{[\ell]}(n;q) := \mathcal{K}^{[\ell]}_{\delta_n, 1^{\binom{n}{2}}}(q), \quad n-1 \le \ell \le 2(n-1),$$

where $\delta_n := (n-1, n-2, \ldots, 2, 1)$ denotes the staircase partition of size $\binom{n}{2}$. Clearly, $RD(n;q) := \sum_\ell RD^{[\ell]}(q) = K_{\delta_n, 1^{\binom{n}{2}}}(q)$. It is well-known, see,e.g., [34], that the number $K_{\delta_n, 1^{\binom{n}{2}}}(q=1)$ id equal to the number of standard Young tableaux of the staircase shape δ_n, and also is equal to the number of reduced decompositions of the longest permutation $w_0^{(n)} := [n, n-1, \ldots, 2, 1] \in \mathbb{S}_n$.

Proposition 3.12 • *Polynomials $RD^{[\ell]}(n;q) := \mathcal{K}^{[\ell]}_{\delta_n, 1^{\binom{n}{2}}}(q)$, $n-1 \le \ell \le 2(n-1)$, are all symmetric and unimodal;*

• *$RD^{[n-1]}(n;q) = \prod_{j=1}^{n-1}(1+q^j)^{n-j}$, and the ratio $RD^{[\ell]}(n;q)/RD^{[n-1]}(n;q)$ is a polynomial with nonnegative coefficients*

We **expect** that the ratio is a unimodal polynomial.

Example 3.13 *Take $n=4$. It is not difficult to check that there are 8 configurations, and*

$$RD(4,q) = q^{10}\ (1+q)^3(1+q^2)^2(1+q^3)(q^15, q^8\ [5]_q, q^3\ [5]_q, 1).$$

For $n=5$ one can check that $\{RD^{[\ell]}(5, q=1), \ell = 4, \ldots, 8\} = 2^{10}(1, 16, 252, 16, 1)$.

3.1 Generalized Catalan polynomials

Definition 3.14 *Define generalized Catalan polynomials $Cat(n,m;q,t)$ to be*

$$Cat(n,m;q,t) := \mathcal{K}_{(n^m, 1^{nm})}(q,t).$$

Note that for $n=2$ the polynomial $Cat(n,2;q,1)$ coincides with the so-called Carlitz–Riordan q-analog of the Catalan number C_n, as well as the rectangular Narayana polynomials $N_d(n,2;q,1)$ gives rise to a q-analog of the classical Narayana number. Mote that in the special case when $\lambda = (n^m)$ is a rectangular shape partition and $R = (1^{nm})$, the polynomials $N_d((n^m), 1^{nm}; q)$ coincide with the so-called *rectangular* Narayana numbers studied in[22], [36, 37].

Proposition 3.15 *Let λ be partition and R be dominant sequence of rectangular shape partitions. The generalized Narayana polynomials $N_d(\lambda, R; q)$ are symmetric and unimodal.*

Indeed, using the well-known fact that the product of two polynomials, which are both symmetric and unimodal, is again symmetric and unimodal one, see.e.g., [33], we conclude that any ℓ-partial $\mathcal{K}$-polynomial associated with (λ, R) is symmetric and unimodal. Therefore we proved Proposition 2.7 and

Corollary 3.16 *Let λ be a partition of size n and $\mu = (1^n)$. Then generalize q-Narayana numbers of type $(\lambda, 1^n)$ are symmetric and unimodal polynomials of q.*

Indeed, for any admissible configuration of type $(\lambda, 1^m)$ we have

$$un(\{\nu\}) = n(\mu) - \frac{1}{2}m\ (\nu)_1^{'},$$

that is the unimodality indices are the same for all configurations $\{\nu\}$ with the fixed length of the first column $(\nu^{(1)})_1^{'}$ of the first partition $\nu^{(1)}$. It is well-known that for a rigged configuration $(\{\nu\}, J)$ of type (λ, μ) the number $(\nu^{(1)})_1^{'}$ is equal to the number of descents of the semistandard Young tableau corresponding to that rigged configuration under the Rigged Configuration Bijection, see, e.g, [18, 19].

In a particular case when $\lambda = (n^m)$ and $\mu = 1^{nm}$, the q-Narayana numbers of type $(n^m, 1^{nm})$ coincide with the rectangular Narayana polynomials have been introduced and studied [4] in [22, 20, 13, 36, 37]. More combinatorial proof of Theorem has been done in [4].

Moreover, for a dominant sequence of rectangular shape partitions R such that the unimodality index $un(\{\nu\})$ is the same for *all* configurations $\{\nu\} \in C(\lambda, R)$, one can deduce that the parabolic Kostka polynomial $K_{\lambda,R}(q)$ is symmetric and unimodal. For example,........

Theorem 3.17 *Let λ be a partition and R be a dominant sequence of rectangular shape partitions. Assume that*

$$\mu_a \geq \left|\nu^{(\eta_a)}\right| = \left|\lambda[\eta_a]\right| - \sum_b \mu_b\ \max(\eta_b - \eta_a, 0), \forall a,$$

where for any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we have used notation $\lambda[k]$ for partition $(\lambda_{k+1}, \lambda_{k+2}, \ldots)$. Then the parabolic Kostka polynomial $K_{\lambda,R}(q)$ is symmetric and unimodal.

Indeed under the above assumptions, the unimodality index

$$un(\{\nu\}) = n(R) - 1/2\Big(\sum_a\ |\nu^{(\eta_a)}|\Big).$$

is the same for all configurations $\{\nu\} \in C(\lambda; R)$. Therefore under the assumptions stated in Theorem2.9, the parabolic Kostka polynomial $K_{\lambda,R}(q)$ is symmetric and unimodal.

[4]Erroneously it was stated as Conjecture 2.3 [22], even though more general statement has been proved in [22, 20].

3.2 Some remarks on strict unimodality

The first algebra-combinatorial proof of strict unimodality of q-binomial coefficients has been appear in [29]. A combinatorial proof of more general statement concerning the strict unimodality of q-binomial coefficients, has been done in [7]. Some results concerning strict unimodality of the product of at most two q-binomial coefficient has been done in [4]. The both of proofs given in [7, 4] use basically the following result

Proposition 3.18 *Let* $a_1 \geq a_2 \cdots \geq a_n$. *Then the product* $\prod_{j=1}^{n} [a_j+1]_q$ *is strictly unimodal iff*

$$a_1 \leq a_2 + \cdots + a_n + 1.$$

To investigate strict unimodality of q-binomial coefficients we apply the result of Proposition2.10 to the fermionic formula for parabolic Kostka polynomials discovered by the author in the middle of 80's of the 20th century. It was observed by the author that some special cases of that fermionic formula give rise to combinatorial/fermionic formulas for q-Gaussian polynomials $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q$, the principal specialization of internal product of Schur functions, general exponents polynomial of mixed tensor modules $G_N(V_\alpha \otimes V_\beta)(q)$, among several others, [22]. In the case of q-binomial coefficients, we have proved that in particular that the q-binomial $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ is equal to the Kostka–Foulkes polynomial

$$K_{m,n}(q) := q^{-n\binom{n+m}{2}} \ K_{((m+n)n,n),(n^{n+m+1})}(q).$$

Therefore a fermionic formula for q-binomial coefficients $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ has the following form

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = K_{m,n}(q) = \sum_{\nu \vdash n} q^{\hat{c}(\nu)} \prod_{j \geq 1} \begin{bmatrix} P_j(\nu) + (\nu)'_j - (\nu)'_{j+1} \\ (\nu)'_j - (\nu)'_{j+1} \end{bmatrix}_q,$$

where summation runs over all partitions on n with at most $[\frac{n+m+1}{2}]$ parts; $\hat{c}(\nu) = c(\nu) - n\binom{m+n}{2}$.

It follows from Proposition 2.10 that if among the partitions have been involved in the fermionic formula displayed above for q-binomial $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$, there exists at least one strict partition ν, $\ell(\nu) \leq [\frac{n+1}{2}]$ and with at least three non-zero vacancy numbers, then a such partition contributes *strict unimodality* summand in the fermionic formula for q-binomials, and therefore the total sum, i.e. the q-binomial in question, is also strict unimodal. As a corollary we get

Corollary 3.19 *Write* $\begin{bmatrix} m+n \\ n \end{bmatrix}_q := \sum_{d=0}^{mn} c_d(n,m)\ q^d$. *Then*

$$c_{[\frac{mn}{2}}(n,m) - c_{[\frac{mn-2}{2}}(n,m) \geq \widehat{p}([\frac{n+1}{2}], [\frac{n+1}{2}]),$$

where $\widehat{p}(m,d)$ *stands for the number of strict partitions of* m *with at most* d *parts and with positive (i.e.* ≥ 1*) vacancy numbers.*

The first non trivial case appears when $n = m = 8$. In this case $\widehat{p}(8,4) = 1$ and the corresponding configuration is $(4,3,1)$. Note that conditions $n, m \geq 8$ guarantee that the all vacancy numbers for configuration $\nu = ([\frac{n+1}{2}], [\frac{n-1}{2}], 1)$ are positive. Recall that by the vacancy numbers associated with partition/configuration ν we mean the following collection of numbers

$$\{(n+m+1)((\nu^{'})_j - (\nu^{'})_{j+1}) - 2\sum_{a \geq 1} \min\Big((\nu^{'})_j - (\nu^{'})_{j+1}, \nu_a\Big), \quad j = 1, \ldots, \nu_1.$$

Note that if $n, m \geq 8d$, then $\widehat{p}([\frac{n+1}{2}], [\frac{n+1}{2}]) \geq \widehat{p}(4d, 4d) \geq 2d - 1$. This observation is a generalization of a similar result has been proved first in [7]. Note that in [38] the author has used another configurations/partitions to find another lower bonds for n and m which guarantee the strict unimodality of q-binomial polynomials.

Comments 3.20 For a review of different proofs of the unimodality of q-binomial coefficients, and more generally that for q-Gaussian polynomials see, e.g.,[33]. The first combinatorial proof of unimodality of the q-Gaussian polynomials $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q$ has been done in [17]. The first prove of the unimodality of q-binomials goes back to J.Sylvester paper of 1878 year. In the modern notation it can be interpreted as a consequence of the following identity

$$\begin{bmatrix} m \\ \lambda' \end{bmatrix} = \sum_{\ell} g_{m-1,\ell}^{\lambda} \; q^{\frac{(m-1)|\lambda|}{2} - n(\lambda)} \; \frac{1 - q^{\ell+1}}{1 - q},$$

where $g_{m-1,\ell}^{\lambda}$ stands for the so-called $sl(2)$-*plethism* coefficients. see, e.g., [25, 26, 20, 21, 13]. Writing as before

$$\begin{bmatrix} N \\ \lambda' \end{bmatrix}_q = \sum_{k=0}^{m|\lambda|} c_k(m, \lambda) \; q^k,$$

we find that

$$c_j(m, \lambda) - c_{j-1}(m, \lambda) = g_{m-1,j}^{\lambda} \geq 0, \quad 2 \leq j \leq \frac{m|\lambda|}{2}.$$

In other words, strict unimodality of q-binomial coefficient is equivalent to non vanishing of certain $sl(2)$-plethysm coefficients. Recall that $sl(2)$-plethysm coefficients $c_j(m.\lambda)$ are defined from the decomposition of the Nth tensor power of $sl(2)$-irreducible representation V_m into the direct sum of irreducible $\mathbb{S}_N \times sl(2)$-models, namely

$$V_m^{\otimes N} = \bigoplus_{\lambda, \ell} \; g_{m,\ell}^{\lambda} \; \mathbb{S}_N^{\lambda} \times V_{\ell}^{sl(2)},$$

where $\mathbb{S}_N^{\lambda}$ denotes the irreducible representation of the symmetric group $\mathbb{S}_N$ corresponding to partition λ .

Similarly, if α and β are partitions of the same size and the both have length $\leq N$, then one can consider the decomposition

$$V_{\alpha}^{sl(N)} \otimes V_{\beta}^{sl(N)}|_{sl(2)} = \bigoplus_{\ell} \; g_{\alpha\beta\ell} \; V_{\ell}^{sl(2)}.$$

In terms of characters of representations involved, we came to identity

$$s_\alpha * s_\beta(q, q^2, \ldots, q^{N-1}) = \sum_{\ell} g_{\alpha\beta\ell} \, \frac{1-q^{\ell+1}}{1-q}.$$

Now let us write

$$s_\alpha * s_\beta(q, q^2, \ldots, q^{N-1} = \sum_{k=0}^{N|\alpha|} r_{\alpha\beta k}^{(n)} \, q^k.$$

Therefore,

$$g_{\alpha\beta k} = r_{\alpha\beta k}^{(N)} - r_{\alpha\beta(k-1)}^{(N)}, \quad 1 \le k \le \frac{N|\alpha|}{2} \ge 0.$$

Recall that for three partitions α, β, γ of the same size $|\alpha| = N$, the coefficient $g_{\alpha\beta\gamma}$ is equal to the multiplicity of the symmetric group $\mathbb{S}_N$ representation $\mathbb{S}_N^\gamma$ in the Kronecker product of representations $\mathbb{S}_N^\alpha$ and $\mathbb{S}_N^\beta$. In other words,

$$s_\alpha * s_\beta = \sum_{\gamma} g_{\alpha\beta\gamma} \, s_\gamma.$$

See Section 4 for definition and basic properties of the internal product of Schur functions. Note that as it has been proved in [17], the principal specialization of the Schur function, namely, $s_\alpha(q, q^2, \ldots, q^{N-1})$ is symmetric and unimodal (with unimodality index $\frac{N|\alpha|}{2}$) for any partition α and $N \ge 2$. Therefore the polynomial $s_\alpha * s_\beta(q, q^2, \ldots, q^{N-1})$ is also symmetric and unimodal.As a corollary we see that polynomial $s_\alpha * s_\beta(q, q^2, \ldots, q^{N-1})$ is strictly unimodal iff $g_{\alpha\beta k} > 0$ for $2 \le k \le \frac{N|\alpha|}{2}$.

3.3 Generalized q-Narayana and Carlitz–Riordan polynomials

Let λ be a partition and $R = (\{(\mu_a)^{\eta_a}\}_{1 \le a \le n}$ be a sequence of rectangular shape partitions such that $|\lambda| = \sum_a \mu_a \, \eta_a$.

Definition 3.21 *Define generalized q-Narayana numbers $N_\ell((\lambda; R); q)$ to be*

$$N_\ell((\lambda; R); q) = q^{c(\{\nu\})} \sum_{\substack{\{\nu\} \\ un(\{\nu\}))=\ell}} \prod_{i,k \ge 1} \begin{bmatrix} P_i^{(k)}(\{\nu\}) + m_i^{(k)}(\{\nu\}) \\ m_i^{(k)}(\{\nu\}) \end{bmatrix}_q.$$

Theorem 3.22 *The generalized q-Narayana numbers are symmetric and unimodal polynomials in q.*

Examples 3.23 (1) (Rectangular q-Narayana numbers)

Let λ be a partition. Denote by $SYT(\lambda)$ the set of standard Young tableaux of shape λ, and by $STY_\ell(\lambda)$ a subset of $STY(\lambda)$ consisting of Young tableaux which have the descent set of cordiality ℓ.

Proposition 3.24 *The polynomial*

$$D_\ell(\lambda;q) := \sum_{T \in STY_\ell(\lambda)} q^{charge(T)}$$

is symmetric and unimodal.

Indeed, if $|\lambda| = n$, then

$$un((\lambda; 1^{|\lambda|})) = \binom{n}{2} - n\ (\nu_1^{(1)})',$$

and it is well-known that $(\nu_1^{(1)})'$ is equal to the cordiality of the descent set of the corresponding Young tableau under the Rigged Configuration Bijection.

For example, take $\lambda = (n,n)$, then one can check that

$$N_\ell((n,n),1^{2n});q) \doteq \frac{1}{[\ell]_q} \begin{bmatrix} n \\ \ell \end{bmatrix}_q \begin{bmatrix} n \\ \ell-1 \end{bmatrix}_q = N(n,\ell;q),$$

where $N(n,\ell;q) = \frac{1}{[\ell]_q} \begin{bmatrix} n \\ \ell \end{bmatrix}_q \begin{bmatrix} n \\ \ell-1 \end{bmatrix}$ stands for a q-deformed Narayana number; we have used a notation $A(q) \doteq B(q)$ to stress that the ratio $A(q)/B(q)$ is equal to q^r for some $r \in \mathbb{Z}$.

More generally, for positive integers n and k define rectangular q-Narayana number $N_\ell(n,k;q)$ to be the generalized q-Narayana number $N_\ell(((n^k),1^{nk});q)$ of type $((n^k),1^{nk})$. It follows from Theorem 3.22 that rectangular q-Narayana numbers are symmetric and unimodal polynomials of q.

(2) (Generalized Gaussian polynomials)

Let λ be a partition of size nr. Assume that $r \geq \lambda_2$, and set $\mu = \underbrace{n \ldots, n}_{r}$. Then the Kostka polynomial $K_{\lambda,\mu}(q)$ is symmetric and unimodal.

Indeed, for any admissible configuration $\{\nu\}$ of type (λ,μ) one has $un(\{\nu\}) = r\binom{n}{2} - r\Big(\sum_a \nu_a^{(1)} = r\ |\nu^{(1)}| = r(nr-\lambda_1)$. That means that the unimodality centrelines are the same for all admissible configurations in question. Therefore, the Kostka polynomial $K_{\lambda,\mu}(q)$ is symmetric and unimodal. Since one can show that for any partition λ, the generalized q-Gaussian polynomial $\begin{bmatrix} \lambda' \\ N \end{bmatrix}$ (up to a power of q) is equal to $K_{\Lambda_N,\mu_N}(q)$, where $\Lambda_N = (N\ |\lambda|, \lambda)$ $\mu_N = \underbrace{|\lambda|, \ldots, |\lambda}_{N+1}$, we conclude that the generalized q-Gaussian coefficients are symmetric and unimodal polynomials of q.

Now let λ and μ be partitions such that $\lambda \geq \mu$ with respect to the dominant order on the set of partitions. Let $\{\nu\} = (\nu_i^{(k)})$ be an admissible configuration of type (λ,μ). We define *spin*, denoted by $spin(\{\nu\})$, of an admissible configuration $\{\nu\}$ to be

$$spin(\{\nu\}) = \#|\ \ (i,k):\ (\nu_i^{(k)})' - (\nu_i^{(k+1)})' \equiv 1\ (\mathrm{mod} 2)|.$$

Finally we define $q,p)$-deformation of Kostka polynomial $K_{\lambda,\mu}(q)$ to be

$$\mathcal{K}_{\lambda,\mu}(q,t,p) := \sum_{\{\nu\} \in C(\lambda,\mu)} q^{c(\{\nu\})}\ t^{nu(\{\nu\})}\ p^{spin(\{\nu\})} \prod_{i,k \geq 1} \begin{bmatrix} P_i^{(k)}(\{\nu\}) + m_i^{(k)}(\{\nu\}) \\ m_i^{(k)}(\{\nu\}) \end{bmatrix}_q.$$

Definition 3.25 *Let $\{\nu\}$ be an admissible configuration of type $C(\lambda,\mu)$. We say that it is Yamanuchi configuration of type (λ,μ) if*

$$0 \le (\nu_i^{(k-1)})' - (\nu_i^{(k)})' \le 2, \ \forall i,k \ge 1.$$

Recall that we set by definition $\nu_i^{(0)} := \mu$. We denote the set of Yamanuchi configurations of a type (λ,μ) by $Yam(\lambda,\mu)$. We define polynomial $Yam_{\lambda,\mu}(q,t,p)$ as follows

$$Yam_{\lambda,\mu}(q,t,p) := \sum_{\{\nu\}\in Yam(\lambda,\mu)} q^{c(\{\nu\})}\ t^{nu(\{\nu\})}\ p^{spin(\{\nu\})} \prod_{i,k\ge 1} \begin{bmatrix} P_i^{(k)}(\{\nu\})+m_i^{(k)}(\{\nu\}) \\ m_i^{(k)}(\{\nu\}) \end{bmatrix}_q.$$

It is clearly seen that $\mathcal{K}_{\lambda,\mu}(q,t=1,p=1) = K_{\lambda,\mu}(q)$, and if $core(\lambda)=\emptyset$, and $\mu=\beta\vee\beta$ for a composition β, then the polynomial $\mathcal{K}_{\lambda,\mu}(q=-1,t=1,p)$ is equal to the *spin* generating function for the set of semistandard domino tableaux of shape λ and weight β, see, e.g., [5].

For an Yamanuchi configuration $\{\nu\}$ of type (λ,μ) we set $\epsilon(\nu\}) = \epsilon^{c(\{\nu\})}$, for all others we set $\epsilon(\{\nu\}) = 1$.

Define 5-parameter deformation of the Kostka polynomial $K_{\lambda,\mu}(q)$ as follows

$$\mathcal{K}_{\lambda,\mu}(q,t,p,v,\epsilon) := \sum_{\{\nu\}\in C(\lambda,\mu)} q^{c(\{\nu\})}\ t^{(\nu^{(k)})'_1} v^{un(\{\nu\})}\ p^{spin(\{\nu\})}\ \epsilon(\{\nu\}) \prod_{i,k\ge 1} \begin{bmatrix} P_i^{(k)}(\{\nu\})+m_i^{(k)}(\{\nu\}) \\ m_i^{(k)}(\{\nu\}) \end{bmatrix}_q.$$

Consider deformed rectangular Narayana polynomials $\mathcal{K}_{(n^m),1^{nm}}(t,q,p)$ more closely. For example, $q^{-30}\ \mathcal{K}_{(6^2),(1^{12})}(t,q,p) =$

$$tq^{30}p^6 + t^2\Big(q^{20}\begin{bmatrix}9\\1\end{bmatrix}_q p^4 + q^{22}\begin{bmatrix}5\\1\end{bmatrix}_q p^2 + q^{24}\Big) + t^3\Big(q^{12}\begin{bmatrix}8\\2\end{bmatrix}_q p^4 + q^{14}\begin{bmatrix}7\\1\end{bmatrix}_q\begin{bmatrix}3\\1\end{bmatrix}_q p^2 + q^{18}p^2\Big) +$$

$$t^4\Big(q^6\begin{bmatrix}7\\3\end{bmatrix}_q p^2 + q^8\begin{bmatrix}6\\2\end{bmatrix}_q\Big) + t^5\ q^2\begin{bmatrix}6\\2\end{bmatrix}_q p^2 + t^6.$$

Clearly,

$$\mathcal{K}_{(6^2),(1^{12})}(t=1,q,p=1) = K_{(6^2),(1^{12})} = q^{30}\ \frac{\begin{bmatrix}12\\6\end{bmatrix}_q}{[7]_q} = q^{30}\ Cat_6^{CR}(q),$$

where $Cat_n^{CR}(q) := \frac{\begin{bmatrix}2n\\n\end{bmatrix}_q}{[n+1]_q}$ stands for the Carlitz–Riordan q-deformation of the Catalan number C_n.

$$\mathcal{K}_{(6^2),(1^{12})}(t,q=1,p=1) = t(1,10,50,50,10,1)_t,$$

and two variable polynomial $\mathcal{K}_{(n^2),(1^{2n})}(t,q,p=1)$ coincides with generating function for (normalized) q-Narayana numbers.

$$\mathcal{K}_{(6^2),(1^{12})}(t=1,q=-1,p) = (5,9,5,1)_p,$$

coincides with the *spin*-generating function for the number of standard *domino* tableaux of shape (6^2).

Theorem 3.26 *(a)* $\mathcal{K}_{((2n)^2),(1^{4n})}(t=1,q=-1,p=1)=\binom{2n}{n}$, *and*

$$\mathcal{K}_{((2n)^2),(1^{4n})}(t=1,q=-1,p)=\sum_{k=0}^{n}\#\Big|STY(n+k,n-k)\Big|\;p^k,$$

see also [31], A039599 for additional information concerning these polynomials.

(b)

$$\mathcal{K}_{((3)^n),(1^{3n})}(t=1,q=\zeta_3,p=1)=\frac{n!}{[n/3]!\;[(n+1)/3]!\;[(n+2)/3]!},$$

(c)

$$\mathcal{K}_{((n)^k),(1^{nk})}(t=1,q=\zeta_n,p=1)=$$

4 Internal product of Schur functions

The irreducible characters χ^λ of the symmetric group S_n are indexed in a natural way by partitions λ of n. If $w\in S_n$, then define $\rho(w)$ to be the partition of n whose parts are the cycle lengths of w. For any partition λ of m of length l, define the power–sum symmetric function $p_\lambda=p_{\lambda_1}\dots p_{\lambda_l}$, where $p_n(x)=\sum x_i^n$. For brevity write $p_w:=p_{\rho(w)}$. The Schur functions s_λ and power–sums p_μ are related by a famous result of Frobenius

$$s_\lambda=\frac{1}{n!}\sum_{w\in S_n}\chi^\lambda(w)p_w. \tag{4.1}$$

For a pair of partitions α and β, $|\alpha|=|\beta|=n$, let us define the internal product $s_\alpha * s_\beta$ of Schur functions s_α and s_β:

$$s_\alpha * s_\beta=\frac{1}{n!}\sum_{w\in S_n}\chi^\alpha(w)\chi^\beta(w)p_w. \tag{4.2}$$

It is well–known that $s_\alpha * s_{(n)}=s_\alpha$, $s_\alpha * s_{(1^n)}=s_{\alpha'}$, where α' denotes the conjugate partition to α.

Let α,β,γ be partitions of a natural number $n\ge 1$, consider the following numbers

$$g_{\alpha\beta\gamma}=\frac{1}{n!}\sum_{w\in S_n}\chi^\alpha(w)\chi^\beta(w)\chi^\gamma(w). \tag{4.3}$$

The numbers $g_{\alpha\beta\gamma}$ coincide with the structural constants for multiplication of the characters χ^α of the symmetric group S_n:

$$\chi^\alpha\chi^\beta=\sum_\gamma g_{\alpha\beta\gamma}\chi^\gamma. \tag{4.4}$$

Hence, $g_{\alpha\beta\gamma}$ are non–negative integers. It is clear that

$$s_\alpha * s_\beta=\sum_\gamma g_{\alpha\beta\gamma}s_\gamma. \tag{4.5}$$

Theorem 4.1 *(A.N. Kirillov) Let α and β be partitions of the same size, and $\ell(\alpha) = r$, then*

$$s_\alpha * s_\beta(q, q^2, \ldots, q^{N-1}) = \sum_{\{\nu\}} q^\nu \prod_{k,j\geq 1} \begin{bmatrix} P_j^{(k)}(\nu) + m_j(\nu^{(k)}) + N(k-1)\delta_{j,\beta_1}\theta(r-k) \\ P_j^{(k)(\nu)} \end{bmatrix}_q,$$

where ν runs over the set of all admissible configurations $\{\nu\}$ of the type $([\alpha,\beta]_N, \beta_1^N)$; $c(\{\nu\})$ stands for the charge of configuration $\{\nu\}$; $\ell(\alpha)$ denotes the length of partition α; for a real number x we set $\theta(x) = 1$, if $x \geq 0$ and $\theta(x) = 0$, if $x < 0$;
$P_j^{(k)}(\{\nu\}) = N\min(j,\beta_1)\delta_{k,1} + Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)} + Q_j(\nu^{(k+1)}$; $m_j(\nu^{(k)} = (\nu_j^{(k)} - \nu_{j+1}^{(k)}$; for the same size partitions α and β, we have used notation $s_\alpha * s_\beta(X)$ to denote the internal product of the corresponding Schur functions.

In particular, if $\beta = (|\alpha|)$ has only one part, when $s_\alpha * s_\beta = s_\alpha$, and Theorem1.3 gives the so-called *fermionic formula* for the the principal specialization of Schur function s_α, [11, 13, 18, 19, 20, 21]. For definition of admissible configurations of type $(\lambda; \{R\})$ see Section2.

4.1 Liskova polynomials

Let us introduce polynomials $L_{\alpha\beta}^\mu(q)$ via the decomposition of the internal product of Schur functions $s_\alpha * s_\beta(x)$ in terms of Hall–Littlewood functions:

$$s_\alpha * s_\beta(x) = \sum_\mu L_{\alpha\beta}^\mu(q) P_\mu(x;q). \tag{4.6}$$

It follows from and (4.5) and definition of Kostka polynomials that

$$L_{\alpha\beta}^\mu(q) = \sum_\gamma g_{\alpha\beta\gamma} K_{\gamma\mu}(q). \tag{4.7}$$

Thus, the polynomials $L_{\alpha\beta}^\mu(q)$ have non–negative integer coefficients, and $L_{\alpha\beta}^\mu(0) = g_{\alpha\beta\mu}$. The polynomials $L_{\alpha\beta}^\mu(q)$ can be considered as a generalization of the Kostka polynomials. Indeed, if partition β consists of one part, $\beta = (n)$, then

$$L_{\alpha\beta}^\mu(q) = K_{\alpha,\mu}(q).$$

Example 4.2 Take partitions $\alpha = (4,2)$ and $\beta = (3,2,1)$, then

$$s_\alpha * s_\beta = s_{51} + 2s_{42} + 2s_{41^2} + s_{3^2} + 3s_{321} + 2s_{31^3} + s_{2^3} + 2s_{2^21^2} + s_{21^4}.$$

Therefore,

$$L_{\alpha\beta}^{(51)}(q) = 1, \quad L_{\alpha\beta}^{(42)}(q) = (12), \quad L_{\alpha\beta}^{(411)}(q) = (132), \quad L_{\alpha\beta}^{(33)}(q) = (121),$$
$$L_{\alpha\beta}^{(321)}(q) = (1353), \quad L_{\alpha\beta}^{(31^3)}(q) = (135752),$$

$$L_{\alpha\beta}^{(2^3)}(q) = (135531), \quad L_{\alpha\beta}^{(2^21^2)} = (1369962),$$
$$L_{\alpha\beta}^{(21^4)}(q) = (1,3,6,10,13,14,12,8,4,1)$$
$$= (1+q)^2(1+q^2)(1+q+q^2)(1+q^2+q^3),$$
$$L_{\alpha\beta}^{(1^6)}(q) = q(1,3,6,10,14,18,20,20,18,14,10,6,3,1)$$
$$= q(1+q)^3(1+q^2)(1+q^2+q^4)^2.$$

Exercise 4.3 *Let $N \geq 1$, $M \geq 1$ be integer numbers. For any pair of partitions α and β of the same size n, consider the following polynomial*

$$S_{\alpha\beta;N,M}(q) = \tag{4.8}$$
$$\frac{1}{n!}\sum_{w\in S_n}\chi^{\alpha}(w)\chi^{\beta}(w)\prod_{k\geq 1}\left(\frac{q^k-(1+(-1)^k)q^{kN}+(-1)^k q^{k(N+M-1)}}{1-q^k}\right)^{\rho_k(w)}.$$

It is clear that

$$\begin{aligned} S_{\alpha\beta;N,1}(q) &= s_{\alpha} * s_{\beta}(q,\ldots,q^{N-1}), \\ S_{\alpha\beta;2,2}(q) &= (1+q)\sum_{\mu=(a|b)} g_{\alpha\beta\mu}q^b, \end{aligned}$$

summed over all hook partitions $\mu = (a+1, 1^b)$, $a+b = n-1$.

- *Show that $S_{\alpha\beta;N,M}(q)$ is a polynomial with non-negative integer coefficients.*

Example 4.4 Take partitions $\alpha = (31)$ and $\beta = (22)$. Using the character table for the symmetric group S_4, one can easily find the following expression for the internal product of Schur functions in question:

$$s_{\alpha} * s_{\beta} = \frac{1}{4!}(6p_1^4 - 6p_2^2),$$

and therefore,

$$\begin{aligned} S_{\alpha\beta;n,m}(q) &= q^5\begin{bmatrix} n+m-2 \\ 1 \end{bmatrix}_q \begin{bmatrix} 2n+2m-5 \\ 1 \end{bmatrix}_q + q^7\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \begin{bmatrix} n+m-1 \\ 4 \end{bmatrix}_q \\ &+ q^9\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \begin{bmatrix} n+m-2 \\ 4 \end{bmatrix}_q + q^{2n+2}\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_{q^2}\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_{q^2}. \end{aligned}$$

In particular,

$$S_{\alpha\beta;2,2}(q) = q^5(1+q+q^2) + q^6 = q^5(1+q)^2.$$

Problem 4.5 *Find a fermionic formula for polynomials $S_{\alpha\beta;N,M}(q)$ which generalizes that from Theorem 4.1.*

4.2 Two variable Liskova polynomials $L^{\mu}_{\alpha\beta}(q,t)$

Let α, β, μ be partitions, $|\alpha| = |\beta| = |\mu| = n$, define

$$L^{\mu}_{\alpha\beta}(q,t) = \sum_{\gamma} g_{\alpha\beta\gamma} K_{\gamma\mu}(q,t).$$

Polynomials $L^{\mu}_{\alpha\beta}(q,t)$ may be considered as a generalization of the double Kostka polynomials $K_{\alpha\mu}(q,t)$. Indeed, if $\beta = (n)$, then $L^{\mu}_{\alpha\beta}(q,t) = K_{\alpha\mu}(q,t)$, and $L^{\mu}_{\alpha(1^n)}(q,t) = K_{\alpha'\mu}(q,t)$. Polynomials $L^{\mu}_{\alpha\beta}(q,t)$ have properties similar to those of $K_{\alpha\mu}(q,t)$.

Exercises 4.6 *Show that*

i) $L^{\mu}_{\alpha\beta}(0,t) = L^{\mu}_{\alpha\beta}(t)$;

ii) $L^{\mu}_{\alpha\beta}(0,0) = g_{\alpha\beta\mu}$, $L^{\mu}_{\alpha\beta}(1,1) = f^{\alpha} f^{\beta}$, *where* f^{α} *denotes the number of standard (i.e. weight* $(1^{|\alpha|})$*) Young tableaux of shape* α;

iii) $L^{\mu}_{\alpha\beta}(q,t) = L^{\mu'}_{\alpha'\beta'}(t,q)$;

iv) $L^{\mu}_{\alpha\beta}(q,t) = q^{n(\mu')} t^{n(\mu)} L^{\mu}_{\alpha'\beta}(q^{-1}, t^{-1})$;

v) $L^{1^n}_{\alpha\beta}(q,t) = K_{\alpha'\beta}(t,t) \widetilde{K}_{\beta,(1^n)}(t) = K_{\beta'\alpha}(t,t) \widetilde{K}_{\alpha,(1^n)}(t)$.

vi) Let λ *and* μ *be partitions and* $\lambda \geq \mu$ *with respect to the dominance ordering on the set of partitions.*

- *Show that* $L^{\mu}_{\alpha\beta}(q) \geq L^{\lambda}_{\alpha\beta}(q)$, *i.e. the difference* $L^{\mu}_{\alpha\beta}(q) - L^{\lambda}_{\alpha\beta}(q)$ *is a polynomial with non–negative coefficients.*
- *Construct a natural embedding of the sets*

$$\boldsymbol{\nu}^{\mu}(\alpha,\beta) \hookrightarrow \boldsymbol{\nu}^{\lambda}(\alpha,\beta).$$

Hence, for any partition μ *there exists a natural embedding (standardization map)*

$$i_{\mu} \; : \; \boldsymbol{\nu}^{\mu}(\alpha,\beta) \hookrightarrow STY(\alpha) \times STY(\beta). \tag{4.9}$$

5 Generalized exponents and mixed tensor representations

Let $\mathfrak{g} = sl(N,\mathbb{C})$ denote the Lie algebra of all $N \times N$ complex matrices of trace 0, and $G = SL(N,\mathbb{C})$ denote the Lie group of all invertible $N \times N$ complex matrices. Let ad : $G \to \mathrm{Aut}(\mathfrak{g})$ denote the adjoint representation of G, defined by $(adX)(A) = XAX^{-1}$, where $X \in G$, and $A \in \mathfrak{g}$.

The adjoint action of $SL(N,\mathbb{C})$ extends to an action on the symmetric algebra $S^{\bullet}(\mathfrak{g}) = \oplus_{k\geq 0} S^k(\mathfrak{g})$, where $S^k(\mathfrak{g})$ denotes the k-th symmetric power. It is well known [24] that the ring $I = S^{\bullet}(\mathfrak{g})^G = \{f \in S^{\bullet}(\mathfrak{g}) | X \cdot f = f, \forall x \in G\}$ of invariants of this action is a polynomial ring in $N-1$ variables $f_2, \ldots, f_{N-1}$, where $f_i \in S^i(\mathfrak{g})^G$. By a theorem of Kostant [24], $S^{\bullet}(\mathfrak{g}) = I \otimes H$ is a free module over G–invariants I generated by harmonics H. Moreover, $H = \oplus_{p\geq 0} H^p$ is a graded (so $H^p = H \cap S^p(\mathfrak{g})$), locally finite $\mathfrak{g}$–representation. The graded character ch_q of the symmetric algebra of adjoint representation is given by the following formal power series

$$\mathrm{ch}_q(S^\bullet(\mathfrak{g})) = \sum_{k\geq 0} q^k \mathrm{ch}(S^k(\mathfrak{g})) = \prod_{1\leq i,j\leq n} (1-qx_i/x_j)^{-1}.$$

For any finite dimensional $\mathfrak{g}$–representation V let us define

$$G_N(V) := \sum_{p\geq 0} \langle G, H^p\rangle q^p, \tag{5.1}$$

where $\langle V_1, V_2\rangle = \dim_{\mathfrak{g}}\mathrm{Hom}(V_1, V_2)$ is the standard pairing on the representation ring of the Lie algebra $\mathfrak{g}$. By a theorem of Kostant [24], $G_N(V)|_{q=1} = \dim V(0)$, where $V(0)$ denotes the zero weight subspace of representation V. Hence, $G_N(V)$ is a polynomial in q with non–negative integer coefficients. Follow Kostant [24], the integers $e_1, \ldots, e_s$ with $G_n(V) = \sum_{i=1}^s q^{e_i}$ are called *generalized exponents* of the representation V. Kostant's problem [*ibid*] is to determine/compute these numbers for a given representation V.

Let $V_\lambda := V_\lambda^{[N]}$ denotes the irreducible highest weight λ representation of the Lie algebra $\mathfrak{g} := sl(N, \mathbb{C})$. Theorem 5.1 below together with the fermionic formula (6.6) for the Kostka–Foulkes polynomials, gives an effective method for computing the generalized exponents of *irreducible* representation of the Lie algebra $\mathfrak{g} = sl(N)$.

Theorem 5.1 ([10]) *Let λ be a partition, then*

$$G_N(V_\lambda) = \begin{cases} K_{\lambda, \left((\frac{|\lambda|}{N})^N \right)}(q), & \text{if } |\lambda| \equiv 0 (\mathrm{mod} N), \\ 0, & \text{otherwise.} \end{cases}$$

For an "elementary" proof of Theorem 5.1, which is based only on the theory of symmetric functions, see [8].

Using Theorem 5.1 one can compute – in principal – the generalized exponents for any finite dimensional $\mathfrak{gl}(N)$–module V. What seems to be very interesting is that for certain representations – see below – there exist alternative expressions for the generalized exponents polynomials which have independent interest and more convenient for computations. Before turning to our main results of this Section, it is useful to recall a few definitions and results from [10], [2] and [10].

Let α, β be partitions, and $V_\alpha^{(N)}$, $V_\beta^{(N)}$ be the highest weight α and β (respectively) irreducible representations of the Lie algebra $\mathfrak{gl}(N)$. For any finite dimensional $\mathfrak{gl}(N)$– module V let V^* denote its dual. If $l(\alpha) + l(\beta) \leq N$, denote by $V_{\alpha,\beta}^{(N)}$ the Cartan piece in the tensor product $V_\alpha^{(N)} \otimes V_\beta^{(N)*}$, i.e. the irreducible submodule generated by the tensor product of highest weight vectors of each component. Follow [10] we call a representation obtained in this way *a mixed tensor representation*. Clearly, $V_{\alpha,\beta}^{(N)}$ is the dual of $V_{\beta,\alpha}^{(N)}$.

Since it is irreducible, $V_{\alpha,\beta}^{(N)}$ is equal to $V_\lambda^{(N)}$ for a unique partition λ of less than N rows. Let us write $[\alpha, \beta]_N$ for this λ. It is well–known [10], and goes back to D. Littlewood [25], that

$$[\alpha,\beta]_N=(\alpha_1+\beta_1,\ldots,\alpha_s+\beta_1,\underbrace{\beta_1,\ldots,\beta_1}_{N-s-r},\beta_1-\beta_r,\ldots,\beta_2-\beta_1,0),$$

where $s=l(\alpha)$, $r=l(\beta)$, and we assume that $s+r\le N$. For example, $V_{0,0}^{(N)}=V_0^{(N)}\simeq\mathbb{C}$, $V_{(1),(1)}^{(N)}=V_{(21^{N-2})}^{(N)}=\mathfrak{g}$ is the adjoint representation.

Theorem 5.2 *Let α,β be partitions, $|\alpha|=|\beta|$, $l(\alpha)\le r$. Then*

$$G_N(V_\alpha\otimes V_\beta^*)\doteq K_{[\alpha,\beta]_{N+r},R_N}(q), \tag{5.2}$$

where $R_N=\{\underbrace{(\beta_1,\ldots,(\beta_1)}_{N},(\beta_1^r)\}$.

Corollary 5.3 (Fermionic formula for the generalized exponents polynomial $G_N(V_\alpha\otimes V_\beta^*)$) *Let α,β be partitions, $|\alpha|=|\beta|$, $l(\alpha)\le r$. Then*

$$q^{|\alpha|}G_N(V_\alpha\otimes V_\beta^*)=\sum_\nu q^{c(\nu)}\prod_{k,j\ge1}\left[\begin{matrix}P_j^{(k)}(\nu)+m_j(\nu^{(k)})+k\delta_{j,\beta_1}\theta(r-k)\\ P_j^{(k)}(\nu)\end{matrix}\right]_q, \tag{5.3}$$

summed over all admissible configurations ν of type $([\alpha,\beta]_N;(\beta_1^N))$.

Remark 5.4 Let α,β be partitions such that $l(\alpha)+l(\beta)\le N$, $l(\alpha)\le r$. Then $G_N(V_\alpha\otimes V_\beta^*)\neq0$ if and only if $|\alpha|\equiv|\beta|\mathrm{mod}N$ and $\widetilde{\beta}_1=\beta_1+\dfrac{|\alpha|-|\beta|}{N}\ge0$; if so, then

$$G_N(V_\alpha\otimes V_\beta^*)=K_{[\alpha,\beta]_{N+r},\widetilde{R}_N}(q), \tag{5.4}$$

where $\widetilde{R}_N=\{\underbrace{(\widetilde{\beta}_1),\ldots,(\widetilde{\beta}_1)}_{N},(\beta_1^r)\}$.

6 Rigged Configurations: a brief review

Let λ be a partition and $R=((\mu_a^{\eta_a}))_{a=1}^p$ be a sequence of rectangular shape partitions such that

$$|\lambda|=\sum_a|R_a|=\sum_a\mu_a\eta_a.$$

Definition 6.1

The configuration of type (λ,R) is a sequence of **partitions** $\{\nu\}=(\nu^{(1)},\nu^{(2)},\ldots)$ *such that*

$$|\nu^{(k)}|=\sum_{j>k}\lambda_j-\sum_{a\ge1}\mu_a\max(\eta_a-k,0)=-\sum_{j\le k}\lambda_j+\sum_{a\ge1}\mu_a\min(k,\eta_a)$$

for each $k\ge1$.

Note that if $k \geq l(\lambda)$ and $k \geq \eta_a$ for all a, then a partition $\nu^{(k)}$ is the empty one.

As in the previous Section, in the sequel we make the convention that $\nu^{(0)}$ is the empty partition [5].

For a partition μ and an integer $j \geq 1$ define the number

$$Q_j(\mu) = \mu'_1 + \cdots + \mu'_j,$$

which is equal to the number of cells in the first j columns of μ.

Definition 6.2 *The vacancy numbers* $P_j^{(k)}(\{\nu\}) := P_j^{(k)}(\nu; R)$ *of a configuration* $\{\nu\}$ *of type* (λ, R) *are defined by*

$$P_j^{(k)}(\{\nu\}) = Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) + \sum_{a \geq 1} \min(\mu_a, j)\delta_{\eta_a, k}$$

for $k, j \geq 1$, *where* $\delta_{a,b}$ *is the Kronecker delta.*

Definition 6.3 *The configuration* $\{\nu\}$ *of type* (λ, R) *is called* **admissible**, *if*

$$P_j^{(k)}(\nu; R) \geq 0 \text{ for all } k, j \geq 1.$$

We denote by $C(\lambda; R)$ the set of all admissible configurations of type (λ, R), and call the vacancy number $P_j^{(k)}(\nu, R)$ to be *essential*, if $m_j(\nu^{(k)}) > 0, \forall j \geq 1$.

Definition 6.4 *For a configuration* $\{\nu\}$ *of type* (λ, R) *let us define its* **charge**

$$c(\{\nu\}) = \sum_{k,j \geq 1} \left(\begin{array}{c} \alpha_j^{(k-1)} - \alpha_j^{(k)} + \sum_a \theta(\eta_a - k)\theta(\mu_a - j) \\ 2 \end{array} \right),$$

and **cocharge**

$$\overline{c}(\nu) = \sum_{k,j \geq 1} \left(\begin{array}{c} \alpha_j^{(k-1)} - \alpha_j^{(k)} \\ 2 \end{array} \right),$$

where $\alpha_j^{(k)} = (\nu^{(k)})'_j$ *denotes the size of the* j*-th column of the* k*-th partition* $\nu^{(k)}$ *of the configuration* $\{\nu\}$; *for any real number* $x \in \mathbb{R}$ *we put* $\theta(x) = 1$, *if* $x \geq 0$, *and* $\theta(x) = 0$, *if* $x < 0$.

Theorem 6.5 (**Fermionic formula for parabolic Kostka polynomials** [13])

Let λ *be a partition and* R *be a dominant* [6] *sequence of rectangular shape partitions. Then*

$$K_{\lambda R}(q) = \sum_{\nu} q^{c(\nu)} \prod_{k,j \geq 1} \left[\begin{array}{c} P_j^{(k)}(\nu; R) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{array} \right]_q, \tag{6.5}$$

summed over all admissible configurations ν *of type* $(\lambda; R)$; $m_j(\lambda)$ *denotes the number of parts of a partition* λ *of size* j.

---

[5] However, in the case when $\eta_{i_a} = 1, \forall a \geq 1$, it is more convenient to set $\nu^{(0)} = \mu := (\mu_1, \ldots, \mu_p)$.

[6] That is $m_1 \geq \mu_2 \geq \ldots \geq \mu_p$.

Corollary 6.6 (**Fermionic formula for Kostka–Foulkes polynomials [11]**)
Let λ and μ be partitions of the same size. Then

$$K_{\lambda\mu}(q) = \sum_{\nu} q^{c(\nu)} \prod_{k,j\geq 1} \left[\begin{array}{c} P_j^{(k)}(\nu,\mu) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{array} \right]_q , \tag{6.6}$$

summed over all sequences of partitions $\nu = \{\nu^{(1)}, \nu^{(2)}, \ldots\}$ such that

- $|\nu^{(k)}| = \sum_{j>k} \lambda_j,\ k = 1, 2, \ldots;$
- $P_j^{(k)}(\nu,\mu) := Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) \geq 0$ *for all* $k, j \geq 1$, *where by definition we* <u>put</u> $\nu^{(0)} = \mu$;
- $$c(\nu) = \sum_{k,j\geq 1} \binom{(\nu^{(k-1)})'_j - (\nu^{(k)})'_j}{2}. \tag{6.7}$$

It is frequently convenient to represent an admissible configuration $\{\nu\}$ by a matrix $m(\{\nu\}) = (m_{ij}),\ m_{ij} \in \mathbb{Z}, \forall i, j \geq 1$, which must meet certain conditions. Namely, starting from a collection of partitions $\{\nu\} = (\nu^{(1)}, \nu^{(2)}, \ldots, ...)$ corresponding to a configuration $\{\nu\}$ of type (λ, R), define matrix

$$m(\{\nu\}) := (m_{ij}),\quad m_{ij} = (\nu^{(i-1)})'_j - (\nu^{(i)})'_j + \sum_{a\geq 1} \theta(\eta_a - i)\theta(\mu_a - j),\quad \nu^{(0)} := \emptyset,$$

where, as before, we set by definition $\theta(x) = 1,\ if\ x \in \mathbb{R}_{\geq 0}$ and $\theta(x) = 0,\ x \in \mathbb{R}_{<0}$.
One can check that a configuration $\{\nu\}$ of type (λ, R) is **admissible** if and only if the matrix $m(\{\nu\})$ meets the following conditions

(0) $m_{ij} \in \mathbb{Z}$,
(1) $\sum_{i\geq 1} m_{ij} = \sum_{a\geq 1} \eta_a \theta(\mu_a - j)$,
(2) $\sum_{j\geq 1} m_{ij} = \lambda_i$,
(3) $\sum_{j\leq k} (m_{ij} - m_{i+1,j}) \geq 0$, for all i, j, k
(4) $\sum_{a\geq 1} \min(\eta_a, k)\delta_{\mu_a, j} \geq \sum_{i\leq k} (m_{ij} - m_{i,j+1})$, for all i, j, k.

One can check that if matrix (m_{ij}) satisfies the conditions $(0) - (4)$, then the set of partitions $\{\nu\} = (\nu^{(1)}, \nu^{(2)}, \ldots, ...)$, where

$$(\nu^{(k)})'_j := \sum_{i>k} m_{ij} - \sum_{a} \max(\eta_a - k, 0)\theta(\mu_a - j)$$

defines an admissible configuration of type $(\lambda, R = \{\mu_a^{\eta_a}\})$.

Example 6.7 *Take* $\lambda = (44332),\ \ R = \{(2^3), (2^2), (2^2), (1), (1)\}$, *so that*

$$\{\mu_a\} = (2, 2, 2, 1, 1)\ \ \textit{and}\ \ \{\eta_a\} = (3, 2, 2, 1, 1),\quad a = 1, \ldots, 5.$$

. *Therefore* $|\nu^{(1)}| = 12-2\times 2-2\times 1-2\times 1=4$, $|\nu^{(2)}|=8-2\times 1=6$, $|\nu^{(3)}|=5$, *and* $|\nu^{(4)}|=2$. *It is not hard to check that there exist* 6 *admissible configurations. They are:*

(1) $\{\nu^{(1)}=(3,1), \nu^{(2)}=(3,3), \nu^{(3)}=(3,2), \nu^{(4)}=(2)\}$,
(2) $\{\nu^{(1)}=(3,1), \nu^{(2)}=(3,2,1), \nu^{(3)}=(3,2), \nu^{(4)}=(2)\}$,
(3) $\{\nu^{(1)}=(2,2), \nu^{(2)}=(2,2,2), \nu^{(3)}=(3,2), \nu^{(4)}=(2)\}$,
(4) $\{\nu^{(1)}=(4), \ \nu^{(2)}=(3,3), \nu^{(3)}=(3,2), \nu^{(4)}=(2)\}$,
(5) $\{\nu^{(1)}=(3,1), \nu^{(2)}=(2,2,1,1), \nu^{(3)}=(2,2,1), \nu^{(4)}=(2)\}$,
(6) $\{\nu^{(1)}=(3,1), \nu^{(2)}=(2,2,1,1), \nu^{(3)}=(3,1,1), \nu^{(4)}=(2)\}$,

Let us compute the matrix $m(\{\nu\}) := (m_{ij})$ *corresponding to the configuration* (2). *Let us write,*

$$(m_{ij}) = ((\nu^{(i-1)})'_j - (\nu^{(i)})'_j) + (\textstyle\sum_{a\geq 1}\theta(\eta_a - i)\theta(\mu_a - j)) := U+W.$$

One can check that

$$U = \begin{pmatrix} -2 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 3 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Therefore,

$$m(\{\nu\}) = \begin{pmatrix} 3 & 2 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

One can read off directly from the matrix $m(\{\nu\}) = (m_{ij})$ the all additional quantities need to compute the parabolic Kostka polynomial corresponding to λ and a (dominant) sequence of rectangular shape partitions R. Namely,

$$P_j^{(k)}(\{\nu\}) = \sum_{i\geq j}(m_{ki} - m_{k+1,i}), \quad m_j(\nu^{(k)}) = \sum_{a\geq 1}\min(\eta_a, k)\delta_{\mu_a,j} - \sum_{i\leq k}(m_{ij} - m_{i,j+1}),$$

$$c(\{\nu\}) = \sum_{i,j\geq 1}\binom{m_{ij}}{2}.$$

For example, in our example, we have

$$c(\nu) = 8, \quad \textit{and} \ \ P_1^{(1)} = 1, P_2^{(2)} = 1, P_3^{(2)} = 1, P_2^{(3)} = 1,$$

are all non-zero vacancy numbers. Therefore the contribution of the configuration in question to the parabolic Kostka polynomial is equal to

$$q^8 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q^4.$$

Treating in a similar fashion the all other configurations, we come to a fermionic formula

$$K_{44332,\{(2^3),(2^2),(2^2),(1),(1)\}}(q) = q^{10}\begin{bmatrix}3\\1\end{bmatrix} + q^8\begin{bmatrix}2\\1\end{bmatrix}^4 + q^8\begin{bmatrix}3\\2\end{bmatrix} + q^{12} + q^6\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} + q^8.$$

■

Note that in the case when $\eta_a = 1,\ \forall a,$ one can show that $\sum_{a\geq 1}\eta_a\theta(\mu_a - j) = \mu_j^{'}$, and we set $\nu^{(0)} = \mu$. In this case one can rewrite the conditions $(1)-(4)$ as follows

$(1^{'})$ $\sum_{i\geq 1} m_{ij} = \mu_j^{'}$,
$(2^{'})$ $\sum_{j\geq 1} m_{ij} = \lambda_i$,
$(3^{'})$ $\sum_{j\leq k}(m_{ij} - m_{i+1,j}) \geq 0$, for all i, j, k,
$(4^{'})$ $\sum_{i>k}(m_{ij} - m_{i,j+1}) \geq 0$, for all i, j, k.

Let us remark that if $m_{ij} \in \mathbb{Z}_{\geq 0}, \forall i, j$, then the matrix $m(\{\nu\}) = (m_{ij})$ defines a *low-lattice plane partition* of shape λ. For example, take $\lambda = (6,4,2,2,1,1)$, $\mu = (2^8)$ and admissible configuration $\{\nu\} = \{(5,5),(4,2),(3,1),(2),(1)\}$. The corresponding matrix and low -lattice [7] plane partition of shape λ are

$$(m_{ij}) = \begin{pmatrix} 3&3&0&0&0&0\\ 1&3&0&0&0&0\\ 1&1&0&0&0&0\\ 1&1&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}, \qquad \text{and } \textit{plane partition} \quad \begin{matrix} 2&2&2&1&1&1\\ 2&2&2&1&&\\ 2&1&&&&\\ 2&1&&&&\\ 1&&&&&\\ 1&&&&& \end{matrix} .$$

The corresponding low-lattice word is 1.1.12.12.1222.111222.

In the case when $\eta_a = 1,\ \forall a$, there exists a unique admissible configuration of type (λ, μ), denoted by $\Delta(\lambda,\mu)$, such that $\max(c((\Delta(\lambda,\mu),J))) = n(\mu) - n(\lambda)$, where the maximum is taken over all rigged configurations associated with configuration $\Delta(\lambda,\mu)$. Recall that for any partition λ,

$$n(\lambda) = \sum_{j\geq 1}\binom{\lambda_a^{'}}{2}.$$

If $\lambda \geq \mu$ with respect to the *dominance order* on the set of partitions, then the degree of the Kostka polynomial $K_{\lambda,\mu}(q)$ is equal to $n(\mu) - n(\lambda)$, see, e.g. [26], Chapter 1, for details. Note that the maximal configuration $\Delta(\lambda,\mu)$ corresponds to the following matrix:

$$m_{1j} = \mu_j^{'} - \max(\lambda_j^{'} - 1, 0), j \geq 1, \quad m_{ij} = 1, \ \ if \ \ (i,j) \in \lambda, \ i \geq 2, \ \ m_{ij} = 0, \ if \ (i,j) \notin \lambda.$$

[7] Let π be a plane partition. We associate with π a word $w(\pi)$ as follows: let's begin reading of the entices of π right to left starting from the most right and bottom cell of π till the first cell of the last row of π. When do the same reading of the nest row above the previous one, and so on till the same reading of the first row of plane partition π. As a result we obtain a word $w(\pi)$. A plane partition is called *low-lattice*, if the corresponding word $w(\pi)$ is a lattice word.

In other words, the configuration $\Delta(\lambda,\mu)$ consists of the following partitions $(\lambda[1],\lambda[2],\ldots)$, where we set $\lambda[k]=(\lambda_{k+1},\lambda_{k+2},\ldots)$. It is not difficult to see that the contribution to the Kostka polynomial $K_{\lambda,\mu}(q)$ coming from the maximal configuration, is equal to

$$K_{max}(\Delta(\lambda,\mu)):=q^{c(\Delta(\lambda,\mu))}\prod_{j=1}^{\lambda_2}\begin{bmatrix}Q_j(\mu)-Q_j(\lambda)+\lambda_j^{'}-\lambda_{j+1}^{'}\\ \lambda_j^{'}-\lambda_{j+1}^{'}\end{bmatrix}_q,$$

where $c(\Delta(\lambda,\mu))=n(\lambda)+n(\mu)-\sum_{j\ge 1}\mu_j^{'}(\lambda_j^{'}-1)$. Therefore,

$$K_{\lambda,\mu}(q)\ge K_{max}(\Delta(\lambda,\mu)). \tag{6.8}$$

It is clearly seen that if $\lambda\ge\mu$, then $Q_j(\mu)\ge Q_j(\lambda),\ \forall\ j\ge 1$, and thus, $K_{q=1}(\Delta(\lambda,\mu))\ge 1$, and the inequality (2.4) can be considered as a "quantitative" generalization of the Gale–Ryser theorem, see, e.g. [26], Chapter I, Section 7, or [15] for details.

Now let us **stress** that for a fixed k, the all partitions $\nu^{(k)}$ which contribute to the set of admissible configurations of type (λ,μ) have the same size equals to $\sum_{j\ge k+1}\lambda_j$, and thus the size of each $\nu^{(k)}$ doesn't depend on μ. However the Rigged Configuration bijection

$$RC_{\lambda,\mu}:STY(\lambda,\mu)\longrightarrow RC(\lambda,\mu)$$

happens to be essentially depends on μ. One can check that the map $RC_{\lambda,\mu}$ is compatible with the familiar *Bender–Knuth* transformations on the set of semistandard Young tableaux of a fixed shape. More precisely, let $\mu=(\mu_1,\ldots,\mu_i,\mu_{i+1},\ldots)$ be a composition. We set $\mu^{(i)}=(\mu_1,\ldots,\mu_{i+1},\mu_i,\ldots)$, and denote by κ_i the Bender–Knuth transformation, namely a bijection κ_i: $STY(\lambda,\mu)\longrightarrow STY(\lambda,\mu^{(i)})$. Then $\kappa_i(T)=RC_{\lambda,\mu^{(i)}}(T)$, $T\in STY(\lambda,\mu)$. Let us stress again that $RC(\lambda,\mu)=RC(\lambda,\mu^{(i)})$.

As it was mentioned above, for a fixed k the all (admissible) configurations have the same size. Therefore, the set of admissible configurations admits a partial ordering denoted by "$\succcurlyeq$". Namely, if $\{\nu\}$ and $\{\xi\}$ are two admissible configurations of the same type (λ,μ), we will write $\{\nu\}\succcurlyeq\{\xi\}$, if either $\{\nu\}=\{\xi\}$ or there exists an integer ℓ such that $\nu^{(a)}=\xi^{(a)}$ if $1\le a\le\ell$, and $\nu^{(\ell+1)}>\xi^{(\ell+1)}$ with respect of the dominance order on the set of the same size partitions. It seems an interesting **Problem** to study poset structures on the set of admissible configurations of type (λ,μ), especially to investigate the posets of admissible configurations associated with the multidimensional Catalan numbers, (work in progress).

■

Theorem 6.8 *(***Duality theorem for parabolic Kostka polynomials** *[13])*

Let λ be partition and $R=\{(\mu_a^{\eta_a})\}$ be a dominant sequence of rectangular shape partitions. Denote by $\lambda^{'}$ the conjugate of λ, and by $R^{'}$ a dominant rearrangement of a sequence of rectangular shape partitions $\{(\eta_a^{\mu_a})\}$. Then

$$K_{\lambda,R}(q)=q^{n(R)}K_{\lambda^{'},R^{'}}(q^{-1}),$$

where

$$n(R) = \sum_{a<b} \min(\mu_a, \mu_b) \min(\eta_a, \eta_b).$$

A technical proof is based on checking of the statement that the map

$$\iota : m_{ij} \longrightarrow \hat{m}_{ij} = -m_{ji} + \theta(\lambda_j - i) + \sum_{a \geq 1} \theta(\mu_a - j)\theta(\eta_a - i)$$

establishes bijection between the sets of admissible configurations of type (λ, R) and that $(\lambda^{'}, R^{'})$, and the equality $\iota(c(m_{ij})) = c((\hat{m}_{ij}))$.

6.1 Example

Let $n = 6$, consider for example, a standard Young tableau

$$T = \begin{array}{cccccc} 1 & 2 & 3 & 6 & 8 & 9 \\ 4 & 5 & 7 & 10 & 11 & 12 \end{array}, \quad c(T) = 48.$$

The corresponding rigged configuration (ν, J) is

$$\nu = (321), \quad J = (J_3 = 0,\ J_2 = 2, J_1 = 6), \quad (m_{ij})(\nu) = \begin{pmatrix} 9 & -2 & -1 \\ 3 & 2 & 1 \end{pmatrix}, \quad c(\nu) = 44.$$

Recall that $c(T)$ and $c(\nu)$ denote the charge of tableau T and configuration ν correspondingly.

- One can see that $c(T) = c(\nu) + J_3 + J_2 + J_1$, as it should be in general.
- Now, the descent set and descent number of tableau T are $Des(T) = \{3, 6, 9\}$, $des(T) = 3$. One can see that $des(T) = 3 = \nu_1^{'}$, as it should be in general [8].
- One can check that our tableau T is invariant under the action of the Schützenberger involution [9] on the set of standard Young tableaux of a shape λ. It is clearly seen from the set of riggings J [10] that the rigged configuration (ν, J) corresponding to tableau T, is invariant under the Flip involution [11]

[8] In fact the shape of the first configuration $\nu^{(1)}$ of type (λ, μ) can be read off from the set of "secondary" descent sets $\{Des^{(1)}(T) = Des(T), Des^{(2)}(T), \ldots, ...)$, cf. [14].

[9] *http://en.wikipedia.org/wiki/Jeu_de_taquin*

[10] In our example $J = (0, 1, 3)$.

[11] Recall that a rigging of an admissible configuration ν is a collection of integers

$$J = (\{J_{s,r}^{(k)}\}\ ,\ 1 \leq s \leq m_r(\nu^{(k)})$$

such that for a given k, r one has

$$0 \leq J_{1,r}^{(k)} \leq J_{2,r}^{(k)} \leq \ldots \leq J_{m_r(\nu^{(k)}),r}^{(k)} \leq P_r^{(k)}(\nu).$$

The Flip involution κ is defined as follows:

$$\kappa(\nu, \{J_{s,r}^{(k)}\}) = (\nu, \{J_{m_r(\nu^{(k)})-s+1,r}^{(k)}\}).$$

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on the set of rigged configurations of type $(\lambda, 1^{|\lambda|})$, as it should be in general, see [19] for a complete proof of the statement that the action of the Schützenberger transformation on a Littlewood– Richardson tableau $T \in LR(\lambda, R)$, under the Rigged Configuration Bijection transforms tableau T to a Littlewood–Richardson tableau corresponding to the rigged configuration $\nu\kappa(J))$, where (νJ) is the rigged configuration corresponding to tableau T we are started with.

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